SIMPLE-LIKE INDEPENDENCE RELATIONS IN ABSTRACT ELEMENTARY CLASSES

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Abstract. We introduce and study $*$-simple, simple and supersimple independence relations in the context of AECs with a monster model.

Theorem 0.1. *Let* $\mathbf{K}$ *be an AEC with a monster model.*

- *If* $\mathbf{K}$ *has a* $*$*-simple independence relation, then the relation is canonical,* $\mathbf{K}$ *is stable and* $\mathbf{K}$ *does not have the tree property.*
- *If* $\mathbf{K}$ *has a simple independence relation with* $(< \aleph_0)$*-witness property, then* $\mathbf{K}$ *does not have the tree property.*

The proof of both facts is done by finding cardinal bounds to classes of small Galois-types over a fixed model that are inconsistent for large subsets. We think this finer way of counting types is an interesting notion in itself.

We characterize supersimple independence relations by finiteness of the Lascar rank under locality assumptions on the independence relation.

Contents

1. Introduction

Simple theories were discovered by Shelah in the mid seventies, an early characterization from his 1978 book [Sh78] is Theorem III.7.7. Originally they were named *theories without the tree property*, Shelah's first paper on them was published in 1980 [Sh80]. Simple theories were ignored for more than a decade. In 1991 Hrushovski circulated [Hru02] (which was published in 2002), there he discovered that the first-order theory of an ultraproduct of finite fields while unstable is simple in the sense of Shelah and established an early version of the *type-amalgamation theorem*

Date: March 5, 2020.

AMS 2010 Subject Classification: Primary 03C48. Secondary: 03C45, 03C55. Key words and phrases. Abstract Elementary Classes; Stable independence; Simple independence; Tameness; Tree property.

(also known as the independence theorem). This work was extended later by Chatzidakis and Hrushovski in the mid nineties, eventually published as [ChHr99]. Influenced by these papers, Kim in [Kim98] and with Pillay in [KiPi97] managed to adapt the type-amalgamation theorem from the algebraic context to complete first-order theories and solved a technical difficulty Shelah had with forking. We recommend [GIL02] for some of the basic results, history (approved by Shelah) as well as some technical simplifications and the chain condition. The subject of simple theories and more generally studying various variants of forking-like relations for unstable first-order theories got much attention in the last 20 years as witnessed by three books dedicated to the subject: [Wag00], [Cas11] and [Kim14].

In 1976 and 1977 Shelah circulated preprints of [Sh87a], [Sh87b] and [Sh88] starting the far reaching program of extending his classification theory of first-order theories to several non-elementary classes. First classes axiomatizable by a theory in $L_{\omega_1,\omega}(\mathbf{Q})$ and later to the more general syntax-free context of *Abstract Elementary Classes (AECs for short)*. An elementary introduction to the theory of AECs can be found in [Gro02]. A more in depth introduction is the two volume book by Shelah [Sh09]. Another book is Baldwin's [Bal09]. For many years Shelah was the only person who managed to make progress in the field. Much of the early work was motivated by Shelah's categoricity conjecture (a generalization of Morley's categoricity theorem). Naturally the work was closely related to generalizing first-order $\aleph_0$-stability and superstability.

There is a very extensive literature about attempts to develop analogues to $\aleph_0$-stability, superstability and stability for various classes of AECs. Always under some extra assumptions on the AEC. This massive effort occupies thousands of pages and is impossible to summarize in this paper. A start can be found in the above mentioned books by Baldwin and Shelah, however in the last decade much was added. See in particular in the PhD theses of Boney [Bon14a] and Vasey [Vas17a].

The goal of this paper is to begin exploring analogues of simplicity in the context of AECs. A-priori it is unclear that there is a natural property (for AECs) that correspond directly to simplicity. It is plausible that there are several such properties. We introduce $*$-*simple*, *simple* and *supersimple* independence relations. The main difference between stable independence relations and the relations that we introduce is that we do not assume uniqueness of non-forking extensions and instead assume the type-amalgamation property. Although this may seem like a minor change, based on our knowledge of forking in first-order theories this is actually a significant one.

Simplicity in first-order theories can be approach from several points of view: using ranks, tree-property, axiomatic properties of forking (or independence properties in general) and counting families of types as in Shelah's Theorem III.7.7 of his first book [Sh78]. In this paper we too approach simplicity-like properties of AECs from various different directions.

We introduce the function $NT(\mu, \lambda, \kappa)$ to connect the existence of a simple-like independence relation with structural properties of the AEC. Our function generalizes $NT(\mu, \lambda)$ of [Cas99]. The function $NT(\mu, \lambda, \kappa)$ assigns to each $\mu \leq \lambda$ and κ cardinals the supremum of $|\Gamma|$ such that Γ is a subset of Galois-types over models of size less than μ which are contained in a fix model of size λ and such that any subset of Γ of cardinality greater than κ is inconsistent. Intuitively this function let us count types in a finer way than just calculating the number of types over a fix model.

We find the following bounds for the different kinds of independence relations studied in this paper.

Theorem. Let $\mathbf{K}$ be an AEC with a monster model.

(1) (Theorem 4.2) If $\overline{\downarrow}$ is a stable independence relation, then

$$NT(\mu, \lambda, \kappa) \leq \lambda^{\kappa_1(\overline{\downarrow})} + \kappa^{-}.$$

(2) (Theorem 5.13) If $\overline{\downarrow}$ is a $*$-simple independence relation, then

$$NT(\mu, \lambda, \kappa) \leq \lambda^{(2^{\kappa(\overline{\downarrow})}+2^{<\ell(\overline{\downarrow})})} + \kappa^{-}.$$

(3) (Theorem 6.7) If $\overline{\downarrow}$ is a simple independence relation, $\kappa(\overline{\downarrow}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\overline{\downarrow})} = \mu$, then

$$NT(\mu, \lambda, \aleph_0) \leq \lambda^{\kappa(\overline{\downarrow})} + 2^{\mu}.$$

(4) (Theorem 7.2, 8.6) If $\overline{\downarrow}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons or a supersimple independence relation, $\kappa(\overline{\downarrow}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\overline{\downarrow})} = \mu$, then

$$NT(\mu, \lambda, (2^{\mu})^{+}) \leq \lambda^{\kappa(\overline{\downarrow})} + 2^{\mu}.$$

We show that these bounds are useful as they imply that the AEC is stable or the failure of the tree property. The extension of the tree property to AECs is another of the contributions of the paper and the idea is that small types play the role of formulas (see Definition 3.4).

Corollary. Let $\mathbf{K}$ be an AEC with a monster model.

(1) (Corollaries 4.3, 4.4, 5.15, 5.14) If $\overline{\downarrow}$ is a stable independence relation or a $*$-simple independence relation, then $\mathbf{K}$ is stable and does not have the tree property.
(2) (Corollary 6.9) If $\overline{\downarrow}$ is a simple independence relation, then $\mathbf{K}$ does not have the 2-tree property.
(3) (Corollary 7.3, 8.6) If $\overline{\downarrow}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons or a supersimple independence relation, then $\mathbf{K}$ does not have the tree property.

We show that $*$-simple independence relations are canonical. This together with the first result of the above corollary can be used to show that for complete first-order theories an independence relation is $*$-simple if and only if it is stable (Lemma 5.20). Moreover, we obtain a new characterization of stable first-order theories assuming simplicity. We show that if first-order non-forking is contained in nonsplitting and T is simple then T is stable (Lemma 5.19).

In a different direction, we characterize supersimple independence relations via the Lascar rank (extended to AECs in [BoGr17]) under the $(< \aleph_0)$-witness property for singletons. This extends [Kim14, 2.5.16] to the AEC context.

Theorem 8.12. Assume $\mathbf{K}$ has a monster model. Let $\overline{\downarrow}$ a simple independence relationship with $(< \aleph_0)$-witness property for singletons. The following are equivalent.

(1) $\overline{\downarrow}$ is a supersimple independence relation.
(2) If $M \in \mathbf{K}$ and $p \in \mathbf{S}(M)$, then $U(p) < \infty$.

A natural question whenever encountering work in pure model theory is about applications. In this paper we do not deal with applications, we believe that it is premature to focus in applications as even for first-order simple theories the first significant applications were found more than 15 years after the basic results were discovered. Only recently some early applications were discovered of the much better understood theory of stable and superstable AECs. For this we refer the interested reader to recent results of the second author on classes of modules, among them: [KuMa], [Maz1] and [Maz2].

It is worth mentioning that there have been some efforts to extend the notion of simplicity to non-elementary settings. Buechler and Lessman introduced a notion of simplicity for a strongly homogeneous structure in [BuLe03], Ben-Yaccov introduced a notion of simplicity for compact abstract theories in [Ben03], Hyttinen and Kesälä introduced a notion of simplicity for $\aleph_0$-stable finitary AECs with disjoint amalgamation and a prime model in [HyKe06] and Shelah

and Vasey introduced a notion of supersimplicity for $\aleph_0$-nicely stable AECs in [ShVa18]. One major difference between our context and that of [BuLe03] is that in their context types can be identified with sets of first-order formulas. As for [Ben03], types in his setting have a strong finitary character built in. While in our context types are orbits of the monster model $\mathfrak{C}$ under the action of $\mathrm{Aut}_M(\mathfrak{C})$. As for [HyKe06] and [ShVa18], a major difference is that we do not assume any trace of stability.

On March 3rd, 2020, two days before posting this paper in the arXiv, Kamsma paper [Kam] was posted in the arXiv. In it, he introduced simple independence relations in AECat. The main difference between our setup and his is that he assumes that types are $(< \omega)$-tame and that a simple independence relation satisfies some analogue to finite character.

The paper is organized as follows. Section 2 presents necessary background. Section 3 introduces the function $NT(\text{-},\text{-},\text{-})$, which is the main technical device of the paper, and a tree property. Section 4 deals with stable independence relations, a bound for $NT(\mu, \lambda, \kappa)$ is found and it is shown that it implies stability and the failure of the tree property. Section 5 introduces $*$-simple independence relations, a bound for $NT(\mu, \lambda, \kappa)$ is found and it is shown that it implies stability and the failure of the tree property. Moreover, the canonicity of $*$-simple independence relations is obtained. Section 6 introduces simple independence relations, a bound for $NT(\mu, \lambda, \aleph_0)$ is found and it is shown that it implies the failure of the 2-tree property. Section 7 studies simple independence relations with locality assumptions. A bound for $NT(\mu, \lambda, (2^\mu)^+)$ is found and it is shown that it implies the failure of the tree property. Section 8 introduces supersimple independence relations and characterizes them by the Lascar rank. It is also shown that the existence of a supersimple independence relation in a class that admits intersections implies the $(< \aleph_0)$-witness property for singletons.

This paper was written while the second author was working on a Ph.D. under the direction of the first author at Carnegie Mellon University and the second author would like to thank the first author for his guidance and assistance in his research in general and in this work in particular. We thank Hanif Cheung for helpful conversations.

2. Preliminaries

We assume the reader has some familiarity with abstract elementary classes as presented for example in [Bal09, §4 - 8] and [Gro1X, §2, §4.4]. Familiarity with [BGKV16] and [LRV19] would be useful, but it is not required as we will recall the notions from [BGKV16] and [LRV19] that are used in this paper. We begin by quickly introducing the basic notions of AECs that we will use in this paper.

Since the main results of the paper assume joint embedding, amalgamation and no maximal models, we will assume these since the beginning.[2]

Hypothesis 2.1. *Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation and no maximal models.*

2.1. Basic concepts.

We begin by introducing some notation for AECs.

Notation 2.2.

- *If $M \in \mathbf{K}$, $|M|$ is the underlying set of M and $\|M\|$ is the cardinality of M.*
- *If λ is a cardinal, $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$ and $\mathbf{K}_{<\lambda} = \{M \in \mathbf{K} : \|M\| < \lambda\}$.*
- *If $M \in \mathbf{K}$ and $\lambda \leq \|M\|$, $[M]^\lambda = \{N : N \leq_{\mathbf{K}} M\} \cap \mathbf{K}_\lambda$ and $[M]^{<\lambda} = \{N : N \leq_{\mathbf{K}} M\} \cap \mathbf{K}_{<\lambda}$.*
- *Let $M, N \in \mathbf{K}$. If we write "$f : M \to N$" we assume that f is a $\mathbf{K}$-embedding, i.e., $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$.*

[2] Some of the definitions presented here make sense without these hypothesis.

We will also use the next set theoretic notation.

Notation 2.3.

- *For κ a cardinal, we define $\kappa^- = \theta$ if $\kappa = \theta^+$ and $\kappa^- = \kappa$ otherwise.*
- *For κ a cardinal and $\kappa \leq |A|$, let $\mathcal{P}_{<\kappa}(A) = \{B \subseteq A : |B| < \kappa\}$.*

Recall the following definitions due to Shelah.

Definition 2.4. *Let $M \in \mathbf{K}$.*

(1) *M is λ-universal if for every $N \in \mathbf{K}_{<\lambda}$, there exists $f : N \to M$.*
(2) *M is λ-model homogeneous if for every $M_0 \leq_{\mathbf{K}} N_0$ both in $\mathbf{K}_{<\lambda}$, if $M_0 \leq_{\mathbf{K}} M$ then there exists $f : N_0 \xrightarrow[M_0]{} N$.*

Remark 2.5. *Since $\mathbf{K}$ has joint embedding, amalgamation and no maximal models we can make use of a monster model $\mathfrak{C}$ (as in complete first-order theories). A monster model $\mathfrak{C}$ is large compared to all the models we consider and is universal and model homogeneous for small cardinals. As usual, we assume that all the elements and sets we consider are contained in the monster model $\mathfrak{C}$. The details of its construction can be consulted in [Gro1X, §4.4].*

Shelah introduced a notion of semantic type in [Sh300]. The original definition was refined and extended by many authors who following [Gro02] call these semantic types Galois-types (Shelah recently named them orbital types). We present here the modern definition and call them Galois-types throughout the text. We use the terminology of [MaVa18, 2.5] and introduce Galois-types without using the assumption of the existence of a monster model.

Definition 2.6.

(1) *Let $\mathbf{K}^3$ be the set of triples of the form $(\mathbf{b}, A, N)$, where $N \in \mathbf{K}$, $A \subseteq |N|$, and $\mathbf{b}$ is a sequence of elements from N.*
(2) *For $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$, we say $(\mathbf{b}_1, A_1, N_1) E (\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exists $f_\ell : N_\ell \xrightarrow[A]{} N$ such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$.*
(3) *Note that E is an equivalence relations on $\mathbf{K}^3$. It is transitive because $\mathbf{K}$ has amalgamation.*
(4) *For $(\mathbf{b}, A, N) \in \mathbf{K}^3$, let $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_E$. We call such an equivalence class a* Galois-type. *If $N = \mathfrak{C}$ (where $\mathfrak{C}$ is a monster model) we write $\mathbf{tp}(\mathbf{a}/A)$ instead of $\mathbf{tp}(\mathbf{a}/A; \mathfrak{C})$.*
(5) *For $N \in \mathbf{K}$, $A \subseteq N$ and I a non-empty set, $\mathbf{S}^I(A; N) = \{\mathbf{tp}(\mathbf{b}/A; N) : \mathbf{b} = \langle b_i \in N : i \in I \rangle\}$. Let $\mathbf{S}(M) := \mathbf{S}^1(M)$ and $\mathbf{S}^{<\infty}(M) := \bigcup_{\alpha<\infty} \mathbf{S}^\alpha(M)$.*
(6) *An AEC is* λ-Galois-stable *if for any $M \in \mathbf{K}_\lambda$ it holds that $|\mathbf{S}(M)| \leq \lambda$. An AEC is* stable *if there is $\lambda \geq \mathrm{LS}(\mathbf{K})$ such that $\mathbf{K}$ is λ-Galois-stable.*
(7) *For $p = \mathbf{tp}_{\mathbf{K}}((b_i)_{i\in I}/A; N) \in \mathbf{S}^I(A; N)$, $A' \subseteq A$ and $I_0 \subseteq I$, $p^{I_0} \restriction_{A'} := [((b_i)_{i\in I_0}, A', N)]_E$.*

The following fact shows that in the presence of a monster model, the Galois-type of $\mathbf{b}$ over a set A is simply the orbit of $\mathbf{b}$ under the action of the automorphisms of $\mathfrak{C}$ fixing A.

Fact 2.7. *Let $\mathfrak{C}$ be a monster model. $\mathbf{tp}(\mathbf{b}_1/A; \mathfrak{C}) = \mathbf{tp}(\mathbf{b}_2/A; \mathfrak{C})$ if and only if there exists $f \in Aut_A(\mathfrak{C})$ with $f(\mathbf{b}_1) = \mathbf{b}_2$.*

The notion of tameness was isolated by the first author and VanDieren in [GrVan06] and type-shortness by Boney in [Bon14b].

Definition 2.8.

- *$\mathbf{K}$ is* $(< \kappa)$-tame *for θ-types if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{S}^I(M)$ with $|I| = \theta$, there is $A \in \mathcal{P}_{<\kappa}(M)$ such that $p \restriction_A \neq q \restriction_A$.*

- $\mathbf{K}$ *is* κ-tame *for* θ*-types if it is* $(<\kappa^+)$-tame *for* θ*-types.*
- $\mathbf{K}$ *is* fully $(<\kappa)$-tame *if for every* θ *ordinal,* $\mathbf{K}$ *is* $(<\kappa)$*-tame for* θ*-types.*
- $\mathbf{K}$ *is* fully $(<\kappa)$-tame and -type-short *if for any* $M \in \mathbf{K}$ *and* $p \neq q \in \mathbf{S}^I(M)$*, there is* $A \in \mathcal{P}_{<\kappa}(M)$ *and* $I_0 \in \mathcal{P}_{<\kappa}(I)$ *such that* $p^{I_0} \restriction_A \neq q^{I_0} \restriction_A$.

2.2. **Independence relations and the witness property.** Global independence relations in the context of AECs and μ-AECs have been extensively studied in the last few years, see for example [BoGr17], [Vas16a] and [LRV19]. Below we introduce a weak independence notion. Our notation and choice of axioms is inspired by [LRV19] and the particular simple-like independence relations that we will study in this paper.

Definition 2.9. $\overline{\downarrow}$ *is a* independence relation *in an AEC* $\mathbf{K}$ *if the following properties hold:*

(1) $\overline{\downarrow} \subseteq \{(M, A, B) : M \leq_{\mathbf{K}} \mathfrak{C} \text{ and } A, B \subseteq \mathfrak{C}\}$. *We say that* $\mathbf{tp}(\bar{a}/B)$ *does not fork over* M *if* $ran(\bar{a}) \overline{\downarrow}_M B$. *This is well-defined by the next three properties.*

(2) *(Preservation under* $\mathbf{K}$*-embeddings) Given* $M_0 \leq_{\mathbf{K}} \mathfrak{C}$, $A, B \subseteq \mathfrak{C}$ *and* $f \in Aut(\mathfrak{C})$*, we have that* $A \overline{\downarrow}_{M_0} B$ *if and only if* $f[A] \overline{\downarrow}_{f[M_0]} f[B]$.

(3) *(Monotonicity) If* $A \overline{\downarrow}_{M_0} B$ *and* $A_0 \subseteq A$, $B_0 \subseteq B$, *then* $A_0 \overline{\downarrow}_{M_0} B_0$.

(4) *(Normality)* $A \overline{\downarrow}_{M_0} B$ *if and only if* $A \cup M_0 \overline{\downarrow}_{M_0} B \cup M_0$.

(5) *(Base monotonicity) If* $A \overline{\downarrow}_{M_0} B$, $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} \mathfrak{C}$ *and* $|M_1| \subseteq B$, *then* $A \overline{\downarrow}_{M_1} B$.

(6) *(Existence) If* $M \leq_{\mathbf{K}} N$ *and* $p \in \mathbf{S}^{<\infty}(M)$*, then there exists* $q \in \mathbf{S}^{<\infty}(N)$ *extending* p *such that* q *does not fork over* M.

(7) *(Transitivity) If* $M_0 \leq_{\mathbf{K}} M_1$, $A \overline{\downarrow}_{M_0} M_1$ *and* $A \overline{\downarrow}_{M_1} B$, *then* $A \overline{\downarrow}_{M_0} B$.

Let us introduce some notation.

Notation 2.10. *Given* $\overline{\downarrow}$ *an independence relation:*

- *For* α *a cardinal, let* $\kappa_\alpha(\overline{\downarrow})$ *be the minimum* λ *(or* ∞*) such that: If* $p \in \mathbf{S}^\alpha(M)$*, then there exists* $M_0 \leq_{\mathbf{K}} M$ *with* $||M_0|| \leq \lambda$ *and* p *does not fork over* M_0.
- *Let* $(\kappa(\overline{\downarrow}), \ell(\overline{\downarrow}))$ *be the minimum* (λ, θ) *(or* (∞, ∞)*) such that: If* $p \in \mathbf{S}^\alpha(M)$*, there exists* $M_0 \in \mathbf{K}$ *with* $M_0 \leq_{\mathbf{K}} M$, $\|M_0\| \leq \lambda + \alpha^{<\theta}$ *and* p *does not fork over* M_0.

The following notion is a locality notion for independence relations.

Definition 2.11 ([Vas16a, 3.19.(2)])**.** *Let* $\overline{\downarrow}$ *an independence relation.* $\overline{\downarrow}$ *has the* right $(<\theta)$-witness property *of length* α *if for all* $M \leq_{\mathbf{K}} N$ *and* $\mathbf{b} \in \mathfrak{C}^\alpha$*:* $\mathbf{b} \overline{\downarrow}_M N$ *if and only if* $\mathbf{b} \overline{\downarrow}_M A$ *for every* $A \in \mathcal{P}_{<\theta}(N)$*. We say that* $\overline{\downarrow}$ *has the right* $(<\theta)$*-witness property if and only if* $\overline{\downarrow}$ *has the right* $(<\theta)$*-witness property of length* α *for all* α.

Observe that since first-order non-forking has finite character, first-order non-forking has $(<\aleph_0)$-witness property. This might not be the case for independence relations as the next example shows. This example was first considered in [Adl05, 1.43].

Example 2.12. *Let* $L(\mathbf{K}) = \emptyset$ *and* $\mathbf{K} = (Sets, \subseteq)$*. Given* $M, A, B \in \mathbf{K}$ *let:*

$$A \overline{\downarrow}_M B \text{ if and only if } |(A \cap B) \backslash C| \leq \aleph_0$$

It is easy to show that $\overline{\downarrow}$ *is an independence relation.* $\overline{\downarrow}$ *has the* $(<\aleph_0)$*-witness property of length* α *for* α *countable, but not for* α *uncountable. Hence* $\overline{\downarrow}$ *does not have the* $(<\aleph_0)$*-witness property.*

In a few places in the paper we will assume that the independence relation under consideration has the witness property in order to be able to carry out some of the proofs (see for example Lemma 7.1 and Theorem 8.12).

The next lemma gives a natural condition that implies the witness property. It fixes a small gap in [Vas16a, 4.3]; the argument in [Vas16a, 4.3] seems to only work for M of cardinality less than or equal to $\kappa_\alpha(\overline{\downarrow})$ as we need $M_0 \leq_{\mathbf{K}} M_1$ in order to apply transitivity.

Lemma 2.13. *Let $\overline{\downarrow}$ an independence relation. If $\kappa_\alpha(\overline{\downarrow}) = \lambda$, then $\overline{\downarrow}$ has the $(< \lambda^+)$-witness property of length α.*

Proof. We prove the following by induction on the size $\|M\|$:

For all N and $\mathbf{a} \in \mathfrak{C}^\alpha$, if $M \leq_{\mathbf{K}} N$ and $\forall B \in \mathcal{P}_{\leq\lambda}(N)(\mathbf{a}\overline{\downarrow}_M B)$, then $\mathbf{a}\overline{\downarrow}_M N$.

Base: Assume $\|M\| = \text{LS}(\mathbf{K})$. Let $M \leq_{\mathbf{K}} N$ and $\mathbf{a} \in \mathfrak{C}^\alpha$, by $\kappa_\alpha(\overline{\downarrow}) = \lambda$ there is $N' \in [N]^\lambda$ such that $\mathbf{a}\overline{\downarrow}_{N'} N$. Since $\|M\| \leq \lambda$ and $M \leq_{\mathbf{K}} N$, we may assume without lost of generality that $M \leq_{\mathbf{K}} N'$. Moreover, by hypothesis $a\overline{\downarrow}_M N'$. Then by transitivity we conclude that $a\overline{\downarrow}_M N$.

Induction step: If $\|M\| \leq \lambda$, the same proof as the one presented in the base step works, so assume that $\|M\| > \lambda$. Let $M \leq_{\mathbf{K}} N$ and $\mathbf{a} \in \mathfrak{C}^\alpha$. Since $\kappa_\alpha(\overline{\downarrow}) = \lambda$ there is $M' \in [M]^\lambda$ such that $\mathbf{a}\overline{\downarrow}_{M'} M$. Using that $\forall B \in \mathcal{P}_{\leq\lambda}(N)(\mathbf{a}\overline{\downarrow}_M B)$ and transitivity, it follows that $\forall B \in \mathcal{P}_{\leq\lambda}(N)(\mathbf{a}\overline{\downarrow}_{M'} B)$. Then by induction hypothesis $\mathbf{a}\overline{\downarrow}_{M'} N$. Hence $\mathbf{a}\overline{\downarrow}_M N$ by base monotonicity. □

We will give a few other natural conditions that imply the witness property, see for example Fact 5.6 and Corollary 8.16.

3. The basic notions

In this section we introduce a way of counting Galois-types over small submodels and generalize the tree property to AECs. We think that this finer way of counting types is an interesting notion in itself. As mentioned in the preliminaries we are assuming Hypothesis 2.1.

In this paper Galois-types over submodels will play a central role.

Definition 3.1. *Let $M \in \mathbf{K}$ and $\mu \leq \|M\|$:*

$$\mathbf{S}(M, \leq \mu) = \{\mathbf{tp}(a/N) : N \leq_{\mathbf{K}} M \text{ and } \|N\| \leq \mu\}$$

The following notion generalizes [Cas99, 2.3] to the AEC setting.

Definition 3.2. *Let $\mu, \lambda \in [\text{LS}(\mathbf{K}), \infty)$ such that $\mu \leq \lambda$ and κ a cardinal (possibly finite). We define the following:*

$NT(\mu, \lambda, \kappa) = sup\{|\Gamma| : \exists M \in \mathbf{K}_\lambda (\Gamma \subseteq \mathbf{S}(M, \leq \mu)$ *and* $\forall \Delta \subseteq \Gamma (|\Delta| \geq \kappa \to \Delta$ *is inconsistent*$))\}$

If $\kappa = 2$ instead of writing $NT(\mu, \lambda, 2)$, we write $NT(\mu, \lambda)$ as in [Cas99].[3]

The following bounds are easy to calculate and hold in general. In what follows, see Theorems 4.2, 5.13, 6.7 and 7.2, we will find sharper bounds which will be the key to show stability or the failure of the tree property under additional hypothesis.

Proposition 3.3.

(1) *If $M \in \mathbf{K}_\lambda$, then $|\mathbf{S}(M)| \leq NT(\lambda, \lambda, 2)$.*
(2) *If $\mu_1 \leq \mu_2$, $\lambda_1 \leq \lambda_2$ and $\kappa_1 \leq \kappa_2$ then $NT(\mu_1, \lambda_1, \kappa_1) \leq NT(\mu_2, \lambda_2, \kappa_2)$.*
(3) *If $\mu \leq \lambda$, then the value of $NT(\mu, \lambda, \text{-})$ is bounded as follows:*
   (a) *If $\kappa \in [2, (\lambda^\mu)^+]$, then $NT(\mu, \lambda, \kappa) \leq \lambda^\mu$.*

[3]The definition given here does not fully match the definition of [Cas99] when $\mathbf{K} = (Mod(T), \preceq)$ for a complete first-theory T, since the bound μ on [Cas99] refers to the cardinality of the type (the number of formulas in it) while in our definition it refers to the cardinality of the domain of the type.

(b) *If* $\kappa \in ((\lambda^\mu)^+, (2^\lambda)^+]$, *then* $NT(\mu, \lambda, \kappa) \leq 2^\lambda$.
(c) *If* $\kappa \in ((2^\lambda)^+, 2^{\lambda^\mu}]$, *then* $NT(\mu, \lambda, \kappa) \leq 2^{\lambda^\mu}$.

(4) $\mathbf{K}$ *is* λ*-Galois stable if and only if* $NT(\mu, \lambda, \kappa) \leq \lambda$ *for every* $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda]$ *and* $\kappa \in [2, \lambda^+]$.

Proof.

(1) Let $\chi = |\mathbf{S}(M)|$ and $\{p_\alpha : \alpha < \chi\}$ an enumeration without repetitions of $\mathbf{S}(M)$. Observe $\{p_\alpha : \alpha < \chi\} \subseteq \mathbf{S}(M, \leq \lambda)$ and any set $\{p_\alpha, p_\beta\}$ is inconsistent if $\alpha \neq \beta$. Therefore, $|\mathbf{S}(M)| = \chi \leq NT(\lambda, \lambda, 2)$.

(2) Follows from the fact that if $\Gamma \subseteq \mathbf{S}(M, \leq \mu_1)$ for $M \in \mathbf{K}_{\lambda_1}$ and each subset of size greater or equal to κ_1 is inconsistent, then there is $M^* \in \mathbf{K}_{\lambda_2}$ with $M \leq_{\mathbf{K}} M^*$ and $\Gamma \subseteq \mathbf{S}(M^*, \leq \mu_2)$ such that any subset of size greater or equal to κ_2 is inconsistent.

(3) (a) Let $\kappa \in [2, (\lambda^\mu)^+]$, $\chi := \lambda^\mu$ and $\{p_\alpha : \alpha < \chi^+\} \subseteq \mathbf{S}(M, \leq \mu)$ for $M \in \mathbf{K}_\lambda$.
Let $\Phi : \chi^+ \to [M]^{\leq \mu}$ be defined as $\Phi(\alpha) = dom(p_\alpha)$, since $|[M]^{\leq \mu}| = \lambda^\mu$ by the pigeonhole principle there is $S \subseteq \chi^+$ of size χ^+ and $N \in [M]^{\leq \mu}$ such that $dom(p_\alpha) = N$ for each $\alpha \in S$. Let $\Psi : S \to \mathbf{S}(N)$ be defined as $\Psi(\alpha) = p_\alpha$, since $|\mathbf{S}(N)| \leq 2^\mu$ by the pigeonhole principle there is $S' \subseteq S$ of size χ^+ and $q \in \mathbf{S}(N)$ such that $p_\alpha = q$ for each $\alpha \in S'$. In particular $\{p_\alpha : \alpha \in S'\}$ is a consistent set of size χ^+. Hence $NT(\mu, \lambda, \kappa) \leq \lambda^\mu$.
(b) Let $\kappa \in ((\lambda^\mu)^+, (2^\lambda)^+]$, $\chi := 2^\lambda$ and $\{p_\alpha : \alpha < \chi^+\} \subseteq \mathbf{S}(M, \leq \mu)$ for $M \in \mathbf{K}_\lambda$.
Given $\alpha < \chi^+$, let $q_\alpha \in \mathbf{S}(M)$ such that $q_\alpha \geq p_\alpha$, it exists because we assumed that $\mathbf{K}$ has amalgamation. Let $\Phi : \chi^+ \to \mathbf{S}(M)$ be defined as $\Phi(\alpha) = q_\alpha$, since $|\mathbf{S}(M)| \leq 2^\lambda$ by the pigeonhole principle there is $S \subseteq \chi^+$ of size χ^+ and $q \in \mathbf{S}(M)$ with $q_\alpha = q$ for every $\alpha \in S$. In particular $\{p_\alpha : \alpha \in S'\}$ is a consistent set of size χ^+. Hence $NT(\mu, \lambda, \kappa) \leq 2^\lambda$.
(c) Similar to (b).

(4) The forward direction is similar to (3).(a) but using that for every $M \in \mathbf{K}_\lambda$ we have that $|\mathbf{S}(M)| \leq \lambda$ instead of only $|\mathbf{S}(M)| \leq 2^\lambda$. The backward direction follows from (1).

□

The next concept extends the tree property to the AEC context. The main idea is that Galois-types over finite sets in AECs play a similar roll as that of formulas in first-order theories. This correspondence is explored in [Vas16b].

Definition 3.4. *Let* $\mu, \lambda \in [\mathrm{LS}(\mathbf{K}), \infty)$ *and* $k < \omega$. $\mathbf{K}$ *has the* (μ, λ, k)*-tree property if there is* $\{(a_\eta, B_\eta) : \eta \in {}^{<\mu}\lambda\}$[4] *such that:*

(1) $\forall \eta \in {}^{<\mu}\lambda(\|B_\eta\| < \mathrm{LS}(\mathbf{K}))$.
(2) $\forall \nu \in {}^{\mu}\lambda(\{\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) : \alpha < \mu\}$ *is consistent*).
(3) $\forall \eta \in {}^{<\mu}\lambda(\{\mathbf{tp}(a_{\eta^\frown\alpha}/B_{\eta^\frown\alpha}) : \alpha < \lambda\}$ *is* k*-contradictory*).

We say that $\mathbf{K}$ *has the* k*-tree property if for all* $\mu, \lambda \in [\mathrm{LS}(\mathbf{K}), \infty)$ $\mathbf{K}$ *has the* (μ, λ, k)*-tree property and* $\mathbf{K}$ *has the tree property if there is a* $k < \omega$ *such that* $\mathbf{K}$ *has the* k*-tree property.*

The following lemma relates the two concepts we just introduced. A similar construction in the first-order context appears in [Cas99, 2.3].

Lemma 3.5. *Assume* $\lambda^{<\mu} = \lambda$ *and* $\mathrm{LS}(\mathbf{K}) \leq \mu \leq \lambda$. *If* $\mathbf{K}$ *has the* $(\mu, \lambda, 2)$*-tree property, then* $NT(\mu, \lambda, 2) = \lambda^\mu$. *Moreover,* $NT(\mu, \lambda, \kappa) \geq \lambda^\mu$ *for all* $\kappa \geq 2$.[5]

Proof. By the definition of the tree property we have $\{(a_\eta, B_\eta) : \eta \in {}^{<\mu}\lambda\}$ such that:

[4] As always we assume that $\forall \eta (a_\eta \in \mathfrak{C}$ and $B_\eta \subseteq \mathfrak{C})$.
[5] As usual we assume that λ, μ are cardinals way below the size of the monster model.

(1) $\forall \eta \in {}^{<\mu}\lambda(\|B_\eta\| < \text{LS}(\mathbf{K}))$.
(2) $\forall \nu \in {}^{\mu}\lambda(\{\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) : \alpha < \mu\}$ is consistent).
(3) $\forall \eta \in {}^{<\mu}\lambda(\{\mathbf{tp}(a_{\eta^\frown\alpha}/B_{\eta^\frown\alpha}) : \alpha < \lambda\}$ is 2-contradictory).

Let $A = \bigcup_{\eta\in{}^{<\mu}\lambda} B_\eta$. Since $\lambda^{<\mu} = \lambda$ and each B_η has cardinality less than $\text{LS}(\mathbf{K})$, we have that $|A| \leq \lambda$. So applying downward Löwenheim-Skolem in $\mathfrak{C}$ we obtain $M \in \mathbf{K}_\lambda$ such that $\forall \eta \in {}^{<\mu}\lambda(B_\eta \subseteq |M|)$.

For each $\nu \in {}^{\mu}\lambda$, pick $a_\nu \in \mathfrak{C}$ realizing $\{\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) : \alpha < \mu\}$ and apply downward Löwenheim-Skolem to $\bigcup_{\alpha<\mu} B_{\nu\restriction_\alpha}$ in M to get $M_\nu \in [M]^{\leq\mu}$. Then define $p_\nu := \mathbf{tp}(a_\nu/M_\nu)$.

Observe that $\{p_\nu : \nu \in {}^{\mu}\lambda\} \subseteq \mathbf{S}(M, \leq \mu)$ and using part (3) of the definition of the tree property it is easy to show that: if $\nu_1 \neq \nu_2$, then $p_{\nu_1} \neq p_{\nu_2}$. Therefore $|\{p_\nu : \nu \in {}^{\mu}\lambda\}| = \lambda^\mu$. Moreover, using part (3) of the definition of the tree property it follows that any pair of types is inconsistent. Hence $NT(\mu, \lambda, 2) \geq \lambda^\mu$.

The equality and moreover part follow from Proposition 3.3. □

As we will see later, if we only know that $\mathbf{K}$ has the tree property it becomes more complicated to obtain a lower bound on $NT(\text{-}, \text{-}, \text{-})$.

4. Stable independence

In this section we deal with stable independence relations. The definition given here for a stable independence relation is similar to the one given in [LRV19]. The properties given here are obtained by taking the "closure" of a stable independence relation in the sense of [LRV19]; this is formalized in [LRV19, 8.2].

Definition 4.1 ([LRV19, 8.4, 8.5, 8.6]). $\overline{\downarrow}$ *is a* stable independence relation *in* $\mathbf{K}$ *if the following properties hold:*

(1) $\overline{\downarrow}$ *is an independence relation.*
(2) *(Symmetry)* $A\overline{\downarrow}_{M_0}B$ *if and only if* $B\overline{\downarrow}_{M_0}A$.
(3) *(Uniqueness) Let* $p, q \in \mathbf{S}^{<\infty}(B; N)$ *with* $M \leq_{\mathbf{K}} N$ *and* $|M| \subseteq B \subseteq |N|$. *If* $p \restriction_M = q \restriction_M$ *and* p, q *do not fork over* M, *then* $p = q$.
(4) *(Local character) For each cardinal* α *there exists a cardinal* λ *(depending on* α*) such that: If* $p \in \mathbf{S}^\alpha(M)$, *then there exists* $M_0 \leq_{\mathbf{K}} M$ *with* $||M_0|| \leq \lambda$ *and* p *does not fork over* M_0.

We begin by bounding $NT(\text{-}, \text{-}, \text{-})$.

Theorem 4.2. *If* $\overline{\downarrow}$ *is a stable independence relation, then*

$$NT(\mu, \lambda, \kappa) \leq \lambda^{\kappa_1(\overline{\downarrow})} + \kappa^-.$$

In particular, we get that $NT(\mu, \lambda) \leq \lambda^{\kappa_1(\overline{\downarrow})}$.

Proof. Let $\lambda_0 = \kappa_1(\overline{\downarrow})$, $\chi = \lambda^{\lambda_0} + \kappa^-$ and $\{p_\alpha : \alpha < \chi^+\} \subseteq \mathbf{S}(M, \leq \mu)$ for $M \in \mathbf{K}_\lambda$.

By local character $\forall \alpha < \chi^+ \exists R_\alpha \in [M]^{\lambda_0}(p_\alpha$ does not fork over $R_\alpha)$, so we define $\Phi : \chi^+ \to [M]^{\lambda_0}$ as $\Phi(\alpha) = R_\alpha$. Then by the pigeonhole principle there is $R \in [M]^{\lambda_0}$ and $S \subseteq \chi^+$ of cardinality χ^+ such that p_α does not fork over R for every $\alpha \in S$. Now define $\Psi : S \to \mathbf{S}(R)$ as $\Psi(\alpha) = p_\alpha \restriction_R$, since $|\mathbf{S}(R)| \leq 2^{\lambda_0}$, by the pigeonhole principle there is $p \in \mathbf{S}(R)$ and $S' \subseteq S$ of size χ^+ such that $p_\alpha \restriction_R = p$ for every $\alpha \in S'$. Observe that $p_\alpha \geq p$ and p_α does not fork over R for every $\alpha \in S'$.

By the extension property and transitivity for each $\alpha \in S'$, there is $q_\alpha \in \mathbf{S}(M)$ extending p_α such that q_α does not fork over R. Then by uniqueness, using that for all $\alpha, \beta \in S'$ we have that $q_\alpha \restriction_R = p_\alpha \restriction_R = p = p_\beta \restriction_R = q_\beta \restriction_R$ and that both q_α, q_β do not fork over R, we conclude that

there is $q \in \mathbf{S}(M)$ such that $q_\alpha = q$ for every $\alpha \in S'$. In particular, $\{p_\alpha : \alpha \in S'\}$ is consistent and $|S'| \geq \kappa$. Hence $NT(\mu, \lambda, \kappa) \leq \lambda^{\lambda_0} + \kappa^-$.

□

The next corollary follows directly from Proposition 3.3 and the above theorem. A version of it already appears in [BGKV16, 5.17] and [LRV19, 8.15].

Corollary 4.3. *If $\overline{\downarrow}$ is a stable independence relation, then $\mathbf{K}$ is λ-Galois-stable for every λ such that $\lambda^{\kappa_1(\overline{\downarrow})} = \lambda$.*

We show that the existence of a stable independence relation implies the failure of the tree property.

Lemma 4.4. *If $\mathbf{K}$ has $\overline{\downarrow}$ a stable independence relation, then $\mathbf{K}$ does not have the tree property.*

Proof. Let $\kappa_1(\overline{\downarrow}) = \lambda_0$ and $k < \omega$ such that $\mathbf{K}$ has the k-tree property. Let $\mu = \lambda_0^+$ and $\lambda = \beth_\mu(\mu)$. By the definition of the (μ, λ, k)-tree property there are $\{(a_\eta, B_\eta) : \eta \in {}^{<\mu}\lambda\}$ such that:

(1) $\forall \eta \in {}^{<\mu}\lambda(\|B_\eta\| < \mathrm{LS}(\mathbf{K}))$.
(2) $\forall \nu \in {}^{\mu}\lambda(\{\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) : \alpha < \mu\}$ is consistent).
(3) $\forall \eta \in {}^{<\mu}\lambda(\{\mathbf{tp}(a_{\eta^\frown\alpha}/B_{\eta^\frown\alpha}) : \alpha < \lambda\}$ is k-contradictory).

Realize that $\lambda^{<\mu} = \lambda$, so doing a similar construction to that of Lemma 3.5 we have $M \in \mathbf{K}_\lambda$ and for each $\nu \in \lambda^\mu$ we fix $p_\nu = \mathbf{tp}(a_\nu/M_\nu)$ such that $M_\nu \in [M]^{\leq\mu}$ and $\forall \alpha < \mu(\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) \leq p_\nu)$.

Observe that if $A \subseteq {}^{\mu}\lambda$ and $\{p_\nu : \nu \in A\}$ is consistent then the tree $\{\nu \restriction_\alpha : \alpha < \mu, \nu \in A\}$ is finitely branching by condition (3) of the tree property, hence $|A| \leq 2^\mu$. Therefore we can conclude that for all $\Delta \subseteq \{p_\nu : \nu \in \lambda^\mu\}$, if $|\Delta| \geq (2^\mu)^+$, then Δ is inconsistent.

Since $cf(\lambda) = \mu$, by König Lemma, we have that $\lambda^\mu = \beth_\mu(\mu)^\mu \geq \beth_\mu(\mu)^+ = \lambda^+$. We claim that $|\{p_\nu : \nu \in \lambda^\mu\}| \geq \lambda^+$. If it was not the case, then there would be $S \subseteq \lambda^\mu$ with $|S| = \lambda^+$ and $\{p_\mu : \nu \in S\}$ consistent; but this would contradict the previous paragraph since $(2^\mu)^+ < \beth_\mu(\mu)^+ = \lambda^+$. Hence

$$\lambda^+ \leq NT(\mu, \lambda, (2^\mu)^+). \tag{1}$$

On the other hand, by Theorem 4.2, we have that $NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{\lambda_0} + 2^\mu$. Moreover, one can show that $\lambda^{\lambda_0} = \lambda$ and that $2^\mu \leq \lambda$, hence

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda. \tag{2}$$

The last two equations gives us a contradiction.

□

The above proof can also be carried out in Shelah's context of good frames, see [Sh09, §II] or [Maz20, §3] for a definition.

Corollary 4.5. *Let $\mathbf{K}$ be an AEC. If $\mathbf{K}$ has a type-full good $[\lambda_0, \infty)$-frame, then $\mathbf{K}$ does not have the tree property.*

Proof sketch. Using local character (in the sense of a good frame) it is easy to show by induction on $\|M\|$ that $\forall p \in \mathbf{S}(M) \exists N \in [M]^{\lambda_0}(p$ does not fork over $N)$. Using this fact together with the properties of type-full good $[\lambda_0, \infty)$-frame one can show that the proofs of Theorem 4.2 and Lemma 4.4 go through. □

Remark 4.6. *The above corollary goes through in the weaker setting of a type-full good$^-$ $[\lambda_0, \infty)$-frame (see [Maz20, 3.5.(4)]). We do not know if it still goes through in the even weaker setting of w-good frames (see [Maz20, 3.7]).*

5. $*$-Simple independence

In this section we introduce $*$-simple independence relations. These are independence relations that are not stable because there is not a unique non-forking extension, but which are very close to being stable. This is the case as the existence of a $*$-simple independence relation implies stability of the AEC (Lemma 5.15) and the existence of a subμ-AEC with a stable independence relation (Lemma 5.16). Moreover, for first-order theories $*$-simple independence relations and stable independence relations are the same. A similar notion is studied in [ShVa18, §6] under stability assumptions.

Before we introduce $*$-simple independence relations, let us recall the following generalization of nonsplitting that was introduced in [BGKV16].

Definition 5.1 ([BGKV16, 3.14]). *We say that A* does not explicitly split *from B over M, denoted by $A \overset{(nes)}{\downarrow}_M B$, if and only if for every $C_1, C_2 \subseteq C$, if $\mathbf{tp}(C_1/M) = \mathbf{tp}(C_2/M)$ then $\mathbf{tp}(AC_1/M) = \mathbf{tp}(AC_2/M)$.*

Let us introduce our new notion.

Definition 5.2. *$\overline{\downarrow}$ is a* $*$-simple independence relation *in $\mathbf{K}$ if the following hold:*

1. *$\overline{\downarrow}$ is an independence relation.*
2. *(Symmetry) $A \overline{\downarrow}_{M_0} B$ if and only if $B \overline{\downarrow}_{M_0} A$.*
3. *(Type-amalgamation) If $p \in \mathbf{S}^{<\infty}(M)$, $M \subseteq A, B \subseteq N$ and $A \overline{\downarrow}_M B$, then for all $q_1 \in \mathbf{S}^{<\infty}(A; \mathfrak{C}), q_2 \in \mathbf{S}^{<\infty}(B; \mathfrak{C})$ and $N^* \supseteq A, B$ such that $q_1, q_2 \geq p$ and q_1, q_2 do not fork over M, there exists $q \in \mathbf{S}^{<\infty}(N^*)$ such that $q \geq q_1, q_2$ and q does not fork over M.*
4. *(Uniform local character) There exists θ and λ cardinals such that: If $p \in \mathbf{S}^{\alpha}(M)$, then there exists $M_0 \leq_{\mathbf{K}} M$ with $||M_0|| \leq \lambda + \alpha^{<\theta}$ and p does not fork over M_0. Recall that $(\kappa(\overline{\downarrow}), \ell(\overline{\downarrow}))$ are the least (λ, θ) with such a property.*
5. *$\overline{\downarrow} \subseteq \overset{(nes)}{\downarrow}$.*

Remark 5.3. *The only difference between stable independence relations and $*$-simple independence relations are conditions (3), (4) and (5). As for (3), while we assume uniqueness in stable independence relations, we only assume type-amalgamation in $*$-simple independence relations. Although this may seem like a minor change, based on our knowledge of forking in first-order theories this is actually a significant one. As for (4), this is a minor change and we give natural condition under which local character implies uniform local character (see Fact 5.4 and Corollary 5.7). As for (5), we will show that a stable independence relation satisfies it and it will be used throughout the section.*

The proof of the following fact is the same as that of [LRV19, 8.10], since the hypothesis are slightly different and the proof is short we repeat the argument for the convenience of the reader.

Fact 5.4. *Let $\overline{\downarrow}$ an independence relation. If $\overline{\downarrow}$ has local character and $(< \theta)$-witness property, then $\overline{\downarrow}$ has uniform local character.*

Proof. Since $\overline{\downarrow}$ has local character, for each $\alpha < \theta$ we have that $\kappa_\alpha(\overline{\downarrow}) < \infty$. Let $\lambda_0 = sup\{\kappa_\alpha(\overline{\downarrow}) : \alpha < \theta\}$. We show that the pair (λ_0, θ) is a witness for uniform local character.

Let $M \in \mathbf{K}$ and $p = \mathbf{tp}(\mathbf{b}/M; N) \in \mathbf{S}^{\beta}(M)$. For each $I \subseteq \beta$ with $|I| < \theta$, let $M_I \in [M]^{\lambda_0}$ such that $\mathbf{b} \restriction_I \overline{\downarrow}_{M_I} M$, this exists by the choice of λ_0. Let $M_0 = \bigcup_{I \subseteq \beta, |I| < \theta} M_I$. Observe that $\|M_0\| \leq \lambda_0 + \beta^{<\theta}$ and $(< \theta)$-witness property together with monotonicity imply that $\mathbf{b} \overline{\downarrow}_{M_0} M$. □

The next lemma gives a condition under which a stable independence relation is a $*$-simple independence relation.

Lemma 5.5. *If $\overline{\downarrow}$ is a stable independence relation that has $(< \theta)$-witness property, then $\overline{\downarrow}$ is a $*$-simple independence relation.*

Proof. We only need to check properties (3), (4) and (5). As for (4), this follows from Fact 5.4. (5) is basically [BGKV16, 4.2]. So we only need to show the type-amalgamation property.

Let $p \in \mathbf{S}^{<\infty}(M)$, $M \subseteq A, B \subseteq \mathfrak{C}$, $A \overline{\downarrow}_M B$, $q_1 \in \mathbf{S}^{<\infty}(A; \mathfrak{C})$ and $q_2 \in \mathbf{S}^{<\infty}(B; \mathfrak{C})$ and $N^* \supseteq A, B$ such that $q_1, q_2 \geq p$ and q_1, q_2 do not fork over M. Since $q_1 \restriction_M \in \mathbf{S}^{<\infty}(M)$ and $M \leq_{\mathbf{K}} N^*$, by the extension property there is $q \in \mathbf{S}^{<\infty}(N^*)$ such that $q \geq q_1 \restriction_M$ and q does not fork over M.

Observe that $q \restriction_A, q_1 \in \mathbf{S}^{<\infty}(A, \mathfrak{C})$, $q \restriction_A, q_1$ do not fork over M and $(q \restriction_A) \restriction_M = p = q_1 \restriction_M$, then by the uniqueness property ((3) of Definition 4.1) we have that $q \restriction_A = q_1$. Hence $q_1 \leq q$. One can similarly show that $q \restriction_B = q_2$.

Therefore, $q \geq q_1, q_2$ and q does not fork over M. □

The next fact gives a natural assumption on $\mathbf{K}$ that implies $(< \theta)$-witness property.

Fact 5.6 ([LRV19, 8.8])**.** *If $\mathbf{K}$ is fully $(< \theta)$-tame and $\overline{\downarrow}$ is a stable independence relation, then $\overline{\downarrow}$ has $(< \theta)$-witness property.*

Corollary 5.7. *If $\mathbf{K}$ is fully $(< \theta)$-tame and $\overline{\downarrow}$ is a stable independence relation, then $\overline{\downarrow}$ is a $*$-simple independence relation.*

We begin by showing that a class with a $*$-simple independence relation is tame. This extends [LRV19, 8.16] as they prove it for stable independence relations.

Lemma 5.8. *If $\overline{\downarrow}$ is a $*$-simple independence relation, then $\mathbf{K}$ is $(\kappa(\overline{\downarrow}) + (2\alpha)^{<(\ell(\overline{\downarrow})})$-tame for types of length α.*

Proof. Let $N \in \mathbf{K}$ and $p, q \in \mathbf{S}^{\alpha}(N)$ such that $p \restriction_D = q \restriction_D$ for every $D \in \mathcal{P}_{\leq \kappa(\overline{\downarrow}) + (2\alpha)^{<(\ell(\overline{\downarrow})}}(N)$. Assume that $p = \mathbf{tp}(\mathbf{a}/N)$ and $q = \mathbf{tp}(\mathbf{b}/N)$ for $\mathbf{a}, \mathbf{b} \in \mathfrak{C}^{\alpha}$.

Consider $\mathbf{tp}(\mathbf{ab}/N)$, then by local character there is $N_0 \leq_{\mathbf{K}} N$ such that $\mathbf{tp}(\mathbf{ab}/N)$ does not fork over N_0 and $\|N_0\| \leq \kappa(\overline{\downarrow}) + (2\alpha)^{<(\ell(\overline{\downarrow})}$. By symmetry and the hypothesis that $\overline{\downarrow} \subseteq \overline{\downarrow}^{(nes)}$ we have that:

$$N \overline{\downarrow}^{(nes)}_{N_0} \mathbf{ab}.$$

Since $\mathbf{tp}(\mathbf{a}/N_0) = p \restriction_{N_0} = q \restriction_{N_0} = \mathbf{tp}(\mathbf{b}/N_0)$ because N_0 is small, we have by the definition of explicitly nonsplitting that $\mathbf{tp}(\mathbf{a}N/N_0) = \mathbf{tp}(\mathbf{b}N/N_0)$. Hence $p = q$. □

The next result is the key result for many of the arguments given in this section. The idea of the proof is similar to that of the proof of weak uniqueness given in [Van06, Theorem I.4.12].

Lemma 5.9. *Let μ, κ infinite cardinals. Assume $\overline{\downarrow}$ is a $*$-simple independence relation, $\mu \geq \kappa(\overline{\downarrow}) + \kappa^{<\ell(\overline{\downarrow})}$. If M is μ^+-model homogeneous, $M \leq_{\mathbf{K}} N$, $p, q \in \mathbf{S}^{<\infty}(N)$, p, q do not fork over M and $p \restriction_M = q \restriction_M$, then $p^{I_0} \restriction_A = q^{I_0} \restriction_A$ for every $A \in \mathcal{P}_{<\kappa}(N)$ and $I_0 \in \mathcal{P}_{<\kappa}(|p|)$.*

Proof. Let A, I_0 as required and assume that $p = \mathbf{tp}(\mathbf{a}/N)$, $q = \mathbf{tp}(\mathbf{b}/N)$ for $\mathbf{a}, \mathbf{b} \in \mathfrak{C}^{\alpha}$ and α an ordinal.

Consider $p^{I_0} \restriction_M$ and $q^{I_0} \restriction_M$ then by local character, base monotonicity and using that $|I_0| < \kappa$ there is $L \leq_{\mathbf{K}} M$ such that $p^{I_0} \restriction_M, q^{I_0} \restriction_M$ do not fork over L and $\|L\| \leq \kappa(\overline{\downarrow}) + \kappa^{<\ell(\overline{\downarrow})} \leq \mu$.

Let L' the structure obtained by applying downward Löwenheim-Skolem to $L \cup A$ in N, observe that $\|L'\| \leq \mu$. Since M is μ^+-model homogeneous, there is $f : L' \underset{L}{\to} M$.

Then by base monotonicity, monotonicity, transitivity and the fact that $\overline{\downarrow} \subseteq \overline{\downarrow}^{(nes)}$, we obtain that:

$$\mathbf{a} \restriction_{I_0} \overline{\downarrow}^{(nes)}_L N \text{ and } \mathbf{b} \restriction_{I_0} \overline{\downarrow}^{(nes)}_L N.$$

Let $C_1 = L'$ and $C_2 = f[L']$. Realize that $L \subseteq C_1, C_2 \subseteq N$ and $\mathbf{tp}(C_1/L) = \mathbf{tp}(C_2/L)$, then by the above equations, the definition of explicitly nonsplitting and the choice of C_1, C_2 we obtain that:

$$\mathbf{tp}(\mathbf{a} \restriction_{I_0} L'/L) = \mathbf{tp}(\mathbf{a} \restriction_{I_0} f[L']/L) \text{ and } \mathbf{tp}(\mathbf{b} \restriction_{I_0} L'/L) = \mathbf{tp}(\mathbf{b} \restriction_{I_0} f[L']/L).$$

Since by hypothesis $p \restriction_M = q \restriction_M$ and $f[L'] \leq_{\mathbf{K}} M$, we have that $\mathbf{tp}(\mathbf{a} \restriction_{I_0} /f[L']) = \mathbf{tp}(\mathbf{b} \restriction_{I_0} /f[L'])$. Then it follows that $\mathbf{tp}(\mathbf{a} \restriction_{I_0} f[L']/L) = \mathbf{tp}(\mathbf{b} \restriction_{I_0} f[L']/L)$. Therefore, by the above equation and using that $A \subseteq L'$, we conclude that $p^{I_0} \restriction_A = q^{I_0} \restriction_A$. □

The following two corollaries are straightforward, we record them as we will use them in what follows.

Corollary 5.10. *Let μ an infinite cardinal. Assume $\overline{\downarrow}$ is a $*$-simple independence relation, $\mu \geq \kappa(\overline{\downarrow})$ and $\mathbf{K}$ is μ-tame. If M is μ^+-model homogeneous, $M \leq_{\mathbf{K}} N$, $p, q \in \mathbf{S}(N)$, p, q do not fork over M and $p \restriction_M = q \restriction_M$, then $p = q$.*

Corollary 5.11. *Let μ, κ infinite cardinals. Assume $\overline{\downarrow}$ is a $*$-simple independence relation, $\mu \geq \kappa(\overline{\downarrow}) + \kappa^{<\ell(\overline{\downarrow})}$ and $\mathbf{K}$ is fully $(< \kappa)$-tame and type-short. If M is μ^+-model homogeneous, $M \leq_{\mathbf{K}} N$, $p, q \in \mathbf{S}^{<\infty}(N)$, p, q do not fork over M and $p \restriction_M = q \restriction_M$, then $p = q$.*

Remark 5.12. *For $\mathbf{K}$ an AEC with joint embedding, amalgamation and no maximal models, one can show as in first-order that if $\lambda \geq \kappa > \mathrm{LS}(\mathbf{K})$, $M \in \mathbf{K}_{\leq \lambda}$ and $\lambda^{<\kappa} = \lambda$, then there is $N \in \mathbf{K}_\lambda$ such that N is κ-Galois-saturated. Moreover, N is κ-model homogeneous as Shelah showed the equivalence between saturation and model homogeneity in [Sh09, §II.1.14].*

We obtain a bound for $*$-simple independence relations.

Theorem 5.13. *If $\overline{\downarrow}$ is a $*$-simple independence relation, then*

$$NT(\mu, \lambda, \kappa) \leq \lambda^{(2^{\kappa(\overline{\downarrow})} + 2^{<\ell(\overline{\downarrow})})} + \kappa^-.$$

Proof. Let $\lambda_0 = \kappa(\overline{\downarrow}) + 2^{<\ell(\overline{\downarrow})}$, $\chi = \lambda^{2^{\lambda_0}} + \kappa^-$ and $\{p_\alpha : \alpha < \chi^+\} \subseteq \mathbf{S}(M, \leq \mu)$ for $M \in \mathbf{K}_\lambda$.

Observe that by the above remark there is M' extending M such that M' is $(2^{\lambda_0})^+$-model homogeneous and $\|M'\| = \lambda^{2^{\lambda_0}}$. For each $\alpha < \chi^+$, fix $q_\alpha \in \mathbf{S}(M')$ such that $p_\alpha \leq q_\alpha$, this exist by amalgamation. Moreover, given $\alpha < \chi^+$, by local character there is $N \in K_{\lambda_0}$ such that q_α does not fork over N. Since $(2^{\lambda_0})^{\lambda_0} = 2^{\lambda_0}$, by the remark above there is N' extending N such that N' is (λ_0^+)-model homogeneous and $\|N'\| = 2^{\lambda_0}$. Since M' is $(2^{\lambda_0})^+$-model homogeneous, there is $f : N' \underset{N}{\to} M'$. So fix $N_\alpha = f[N']$, realize $N_\alpha \in \mathbf{K}_{2^{\lambda_0}}$, N_α is (λ_0^+)-model homogeneous and q_α does not fork over N_α by base monotonicity.

Define $\Phi : \chi^+ \to [M']^{2^{\lambda_0}}$ as $\Phi(\alpha) = N_\alpha$. Then by the pigeonhole principle there is $N \in [M']^{2^{\lambda_0}}$ and $S \subseteq \chi^+$ of cardinality χ^+ such that q_α does not fork over N for every $\alpha \in S$. Now define

$\Psi : S \to \mathbf{S}(N)$ as $\Psi(\alpha) = q_\alpha \restriction_N$, since $|\mathbf{S}(N)| \leq 2^{2^{\lambda_0}}$, by the pigeonhole principle there is $q \in \mathbf{S}(N)$ and $S' \subseteq S$ of size χ^+ such that $q_\alpha \restriction_N = q$ for every $\alpha \in S'$

Observe that $q_\alpha \geq q$ and q_α does not fork over N for every $\alpha \in S'$. Then since N is (λ_0^+)-model homogeneous and $\mathbf{K}$ is $\lambda_0 = \kappa(\overline{\downarrow}) + 2^{<\ell(\overline{\downarrow})}$-tame (by Lemma 5.8), it follows from Corollary 5.10 that $q_\alpha = q_\beta$ for every $\alpha, \beta \in S'$. In particular, $\{p_\alpha : \alpha \in S'\}$ is consistent and $|S'| \geq \kappa$. Hence $NT(\mu, \lambda, \kappa) \leq \lambda^{2^{\lambda_0}} + \kappa^-$. □

The next results show that having a $*$-simple independence relation implies that $\mathbf{K}$ is stable and that $\mathbf{K}$ does not have the tree property.

Corollary 5.14. *If $\overline{\downarrow}$ is a $*$-simple independence relation, then $\mathbf{K}$ does not have the tree property.*

Proof. Let $\mu = (2^{\kappa(\overline{\downarrow}) + 2^{<\ell(\overline{\downarrow})}})^+$ and $\lambda = \beth_\mu(\mu)$. Since $\lambda^{<\mu} = \lambda$, the same construction as that of Lemma 4.4 gives us that:

$$\lambda^+ \leq NT(\mu, \lambda, (2^\mu)^+).$$

On the other hand, by the previous theorem we have that:

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{2^{\kappa(\overline{\downarrow}) + 2^{<\ell(\overline{\downarrow})}}} + 2^\mu = \lambda.$$

Putting together the last two equation we get that $\lambda^+ \leq \lambda$, which is clearly a contradiction. □

Lemma 5.15. *If $\overline{\downarrow}$ is a $*$-simple independence relation, then $\mathbf{K}$ is a stable.*

Proof. Let $\lambda = 2^{2^{\lambda_0}}$ and $M \in \mathbf{K}_\lambda$. By Proposition 3.3.(1) $|\mathbf{S}(M)| \leq NT(\lambda, \lambda, 2)$. Then by the previous theorem we have that $|\mathbf{S}(M)| \leq \lambda$. □

The next result shows that a $*$-simple independence relation is close to being a stable independence relation. Recall that $\mathbf{K}^{\mu^+\text{-mh}}$ is the μ^+-AEC (see [BGLRV16]) which models are the μ^+-model homogeneous models of $\mathbf{K}$ and which order is the same as that of $\mathbf{K}$.

Lemma 5.16. *Assume $\mathbf{K}$ is fully $(< \kappa)$-tame and type-short. If $\overline{\downarrow}$ is $*$-simple independence relation and $\mu \geq \kappa(\overline{\downarrow}) + \kappa^{<\ell(\overline{\downarrow})}$, then $\mathbf{K}^{\mu^+\text{-}mh}$ has a stable independence relation. This is precisely the restriction of $\overline{\downarrow}$ to μ^+-model homogeneous models.*

Proof. A big monster model of $\mathbf{K}$ is a monster model of $\mathbf{K}^{\mu^+\text{-mh}}$. For $M \in \mathbf{K}^{\mu^+\text{-mh}}$, $A, B \subseteq \mathfrak{C}$ define:

$$A \overline{\overset{(*)}{\downarrow}}_M B \text{ if and only if } A \overline{\downarrow}_M B.$$

We claim that $\overline{\overset{(*)}{\downarrow}}$ is a stable independence relation in $\mathbf{K}^{\mu^+\text{-mh}}$. It is straightforward to show that it is an independence relation that satisfies symmetry. Uniqueness follows from Corollary 5.11. As for local character, we have that given α and $p \in \mathbf{S}^\alpha(M)$ with $M \in \mathbf{K}^{\mu^+\text{-mh}}$ there is $N \in \mathbf{K}^{\mu^+\text{-mh}}$ such that p does not $\overline{\overset{(*)}{\downarrow}}$-forks over N and $\|N\| \leq \kappa(\overline{\downarrow}) + \alpha^{<\ell(\overline{\downarrow})} + \text{LS}(\mathbf{K})^\mu$. □

Remark 5.17. *The existence of a stable independence relation in a sub-μ-AEC of $\mathbf{K}$ for $\mathbf{K}$ a class with an $*$-simple independence relation and fully tame, also follows from Lemma 5.15, [Vas16b, 4.15] and [LRV19, 10.1].*

The next lemma shows that $*$-simple independence relations are canonical. It extends [LRV19, 9.1] as in [LRV19, 9.1] is shown (based on [BGKV16]) that if an AEC has a stable independence relation then this is canonical. The proof relies heavily on [BGKV16] so we will only sketch it. The proof uses in a nontrivial way the type-amalgamation property, specifically Proposition 6.4.

Lemma 5.18. *If $\overset{\overline{(1)}}{\downarrow}$ and $\overset{\overline{(2)}}{\downarrow}$ are $*$-simple independence relations, then $\overset{\overline{(1)}}{\downarrow} = \overset{\overline{(2)}}{\downarrow}$.*

Proof sketch. The arguments given in [BGKV16, 4.10, 4.11, 4.13] can be carried out in our setting changing nonsplitting for explicitly nonsplitting to obtain the hypothesis of [BGKV16, 4.7]. Then by applying [BGKV16, 4.7] (but changing nonsplitting for explicitly nonsplitting) twice, it follows that $\overset{\overline{(1)}}{\downarrow} = \overset{\overline{(2)}}{\downarrow}$. One of the hypothesis of [BGKV16, 4.7] is that the relation is contained in explicitly nonsplitting, it is in this step that it is crucial that $*$-simple independence relations are contained in explicitly nonsplitting. □

We finish this section by showing that the results in this section can be used to obtain a new characterization of stability assuming simplicity for first-order theories. In order to present it, let us recall the notion of nonsplitting for first-order theories. A complete type p in $\bar{x}$ does not split over A a subset of the monster model if and only if for every $\bar{a}, \bar{b} \in Dom(p)$ and $\phi(\bar{x}, \bar{y})$ first-order formula, if $tp(\bar{a}/A) = tp(\bar{b}/A)$, then $\phi(\bar{x}, \bar{a}) \in p$ if and only if $\phi(\bar{x}, \bar{b}) \in p$. This notion was introduced by Shelah in Definition 2.2 of [Sh3].

Lemma 5.19. *Let T a simple complete first-order theory. The following are equivalent.*

1. $\underset{M}{\downarrow} \subseteq \overset{(ns)}{\underset{M}{\downarrow}}$ *for every M model of T, where $\downarrow$ denotes first-order non-forking and $\overset{(ns)}{\downarrow}$ denotes first-order nonsplitting.*
2. *T is stable*

Proof. $\boxed{\rightarrow}$ Lemma 5.9, Theorem 5.13 and Lemma 5.15 can be carried out if one replaces explicitly nonsplitting for nonsplitting in complete first-order theories.

$\boxed{\leftarrow}$ Since T is stable, non-forking has uniqueness (stationarity) over models. Under this hypothesis it is easy to show that $\underset{M}{\downarrow} \subseteq \overset{(ns)}{\underset{M}{\downarrow}}$ for every M model of T (a proof is given in [BGKV16, 4.2]). □

We can also show that for complete first-order theories the notion of a $*$-simple independence relation and stable independence relation coincide.

Lemma 5.20. *Let T a complete first-order theory. If $\overset{\overline{|}}{\downarrow}$ is a $*$-simple independence relation on $(Mod(T), \preceq)$, then $\overset{\overline{|}}{\downarrow}$ is a stable independence relation on $(Mod(T), \preceq)$.*

Proof. Let $\overset{\overline{|}}{\downarrow}$ a $*$-simple independence relation on $(Mod(T), \preceq)$. By Lemma 5.15 T is an stable theory, so first-order non-forking is a stable independence relation. We denote first-order non-forking by $\downarrow$. Since $(Mod(T), \preceq)$ is fully $(< \aleph_0)$-tame, it follows from Corollary 5.7 that $\downarrow$ is a $*$-simple independence relation. Then by the canonicity of $*$-simple independence relations (Lemma 5.18) we conclude that $\overset{\overline{|}}{\downarrow} = \downarrow$. Therefore, $\overset{\overline{|}}{\downarrow}$ is a stable independence relation. □

6. Simple independence

We introduce simple independence relations and begin their study. We bound the possible values of $NT(\text{-}, \text{-}, \text{-})$ under the existence of a simple independence relation and as a corollary we are able to show the failure of the 2-tree property. As in the previous section we are assuming Hypothesis 2.1.

Definition 6.1. $\overline{\downarrow}$ *is a* simple independence relation *in* $\mathbf{K}$ *if the following properties hold:*

(1) $\overline{\downarrow}$ *is an independence relation.*
(2) *(Symmetry)* $A\overline{\downarrow}_{M_0}B$ *if and only if* $B\overline{\downarrow}_{M_0}A$.
(3) *(Type-amalgamation) If* $p \in \mathbf{S}^{<\infty}(M)$, $M \subseteq A, B \subseteq N$ *and* $A\overline{\downarrow}_M B$, *then for all* $q_1 \in \mathbf{S}^{<\infty}(A; \mathfrak{C}), q_2 \in \mathbf{S}^{<\infty}(B; \mathfrak{C})$ *and* $N^* \supseteq A, B$ *such that* $q_1, q_2 \geq p$ *and* q_1, q_2 *do not fork over* M, *there exists* $q \in \mathbf{S}^{<\infty}(N^*)$ *such that* $q \geq q_1, q_2$ *and* q *does not fork over* M.
(4) *(Uniform local character) There exists* θ *and* λ *cardinals such that: If* $p \in \mathbf{S}^{\alpha}(M)$, *then there exists* $M_0 \leq_{\mathbf{K}} M$ *with* $||M_0|| \leq \lambda + \alpha^{<\theta}$ *and* p *does not fork over* M_0. *Recall that* $(\kappa(\overline{\downarrow}), \ell(\overline{\downarrow}))$ *are the least* (λ, θ) *with such a property.*

Remark 6.2. *Let* T *be a complete first-order theory. If* T *is simple and* $\overline{\downarrow}$ *is first-order non-forking, then* $\overline{\downarrow}$ *is a simple independence relation.*

Remark 6.3. *It is clear that a* $*$*-simple independence relation is a simple independence relation as the only difference between both definitions is that in simple independence relations we do not assume that* $\overline{\downarrow} \subseteq \overline{\overset{(nes)}{\downarrow}}$. *Moreover, by Corollary 5.7 it follows that in fully* $(< \theta)$*-tame AECs, every stable independence relation is simple.*

The next technical proposition is important as it shows that even when we are considering independence relations over sets in some sense models are ubiquitous

Proposition 6.4. *Let* $\overline{\downarrow}$ *a simple independence relation. If* $A\overline{\downarrow}_M B$, *then there is* $M^* \in \mathbf{K}$ *with* $B \cup M \subseteq M^*$ *and* $A\overline{\downarrow}_M M^*$.

Proof. Assume $A\overline{\downarrow}_M B$. By normality and monotonicity we can conclude that $A\overline{\downarrow}_M B \cup M$. Let $M' \in \mathbf{K}$ the structure obtained by applying downward Löwenheim-Skolem in $\mathfrak{C}$ to $M \cup B \subseteq M'$

Consider $p = \mathbf{tp}(A/M)$, $q_1 = \mathbf{tp}(A/M \cup B)$ and $q_2 = \mathbf{tp}(A/M)$. Observe that $p \leq q_1, q_2$, $q_1 \in \mathbf{S}^{<\infty}(M \cup B; \mathfrak{C})$ does not fork over M, $q_2 \in \mathbf{S}^{<\infty}(M)$ does not fork over M, $M \subseteq M \cup B, M \subseteq M'$ and $M \cup B\overline{\downarrow}_M M$. Recognize that p, q_1, q_2 and $M \subseteq M, M \cup B \subseteq M'$ satisfy the hypothesis of the type-amalgamation property, then there is $r \in \mathbf{S}^{<\infty}(M') \geq q_1, q_2$ such that r does not fork over M.

Suppose that $r = tp(A'/M')$, since $r \geq q_1$ there is $f \in Aut_{M \cup B}(\mathfrak{C})$ such that $f[A'] = A$. Since r does not fork over M, we have that $A'\overline{\downarrow}_M M'$. Then by invariance $f[A']\overline{\downarrow}_{f[M]} f[M']$. Observe $f[A'] = A$, $f[M] = M$, so $A\overline{\downarrow}_M f[M']$. Finally, realize that $M \cup B \subseteq f[M']$, hence $M^* := f[M']$ satisfies what is needed. □

The following notion generalizes the chain condition introduced in [Les00, 2.3].

Definition 6.5. *We say* $\overline{\downarrow}$ *has the* ι*-bound condition if:* $\forall \lambda \in [\mathrm{LS}(\mathbf{K}), \infty) \forall M \in \mathbf{K}_\lambda \forall \kappa \in [\mathrm{LS}(\mathbf{K}), \lambda] \forall p \in \mathbf{S}(M, \kappa) \forall \mu \in [\kappa(\overline{\downarrow}) + \kappa, \lambda]$*(If* $\mu^{<\ell(\overline{\downarrow})} = \mu$ *and* $\{p_\alpha : \alpha < (2^\mu)^+\} \subseteq \mathbf{S}(M, \leq \mu)$ *are such that* $\forall \alpha < (2^\mu)^+ (p_\alpha$ *is a non-forking extension of* $p)$, *then there are* $A \subseteq (2^\mu)^+$ *and* q *a type such that* $|A| = \iota$ *and* $\forall \alpha \in A (p_\alpha \leq q)$ *). Moreover, we say that* $\overline{\downarrow}$ *has the strong* ι*-bound condition if the type* q *is a non-forking extension of* p.

The following is a generalization of [Les00, 2.4], which is based on an argument of Shelah which appeared in [GIL02, 4.9]. Compared to [Les00, 2.4], instead of showing that two types are comparable we show that countably many types are comparable, [Les00, 2.5] mentions that this can be done in the first-order case. We have decided to present the argument to show that it does come through in this more general setting and because we will extend it in Lemma 7.1.

Lemma 6.6. *If $\overline{\downarrow}$ is a simple independence relation, then $\overline{\downarrow}$ has the $\aleph_0$-bound condition.*[6]

Proof. Let $\lambda, \mu, \kappa \in Car$, $M \in \mathbf{K}_\lambda$, $R \in [M]^\kappa$, $p \in \mathbf{S}(R)$ and $\{p_\alpha \in \mathbf{S}(N_\alpha) : \alpha < (2^\mu)^+\} \subseteq \mathbf{S}(M, \leq \mu)$ as in the definition of the $\aleph_0$-bound condition. By the extension property we may assume that all N_α have size μ.

We build $\{M_\alpha : \alpha < (2^\mu)^+\}$ strictly increasing and continuous chain such that:

(1) $\forall \alpha \in (2^\mu)^+ (M_\alpha \in \mathbf{K}_{2^\mu})$.
(2) $R \leq_{\mathbf{K}} M_0$.
(3) $\forall \alpha \in (2^\mu)^+ (N_\alpha \leq_{\mathbf{K}} M_{\alpha+1})$

Let $S = \{\alpha < (2^\mu)^+ : cf(\alpha) = \mu^+\}$ and $\Phi : S \to (2^\mu)^+$ be the function where $\Phi(\alpha) = min\{\beta : \mathbf{tp}(N_\alpha/M_\alpha)$ does not fork over $M_\beta\}$. Observe that Φ is regressive by local character and the fact that $\mu^{<\ell(\overline{\downarrow})} = \mu$. Then by Fodor's lemma there is $S^* \subseteq S$ stationary and $\alpha^* < (2^\mu)^+$ such that $\forall \alpha \in S^* (\mathbf{tp}(N_\alpha/M_\alpha)$ does not fork over $M_{\alpha^*})$. We may assume without loss of generality that $S = S^*$ and $\alpha^* = 0$. Hence,

$$\forall \alpha \in S(\mathbf{tp}(N_\alpha/M_\alpha) \text{ does not fork over } M_0). \tag{3}$$

By local character and using again that $\mu^{<\ell(\overline{\downarrow})} = \mu$ we have that $\forall \alpha \in S \exists R_\alpha \in [M_0]^\mu (\mathbf{tp}(N_\alpha/M_\alpha) \restriction_{M_0}$ does not fork over $R_\alpha)$. Define $\Psi : S \to [M_0]^\mu$ as $\Psi(\alpha) = R_\alpha$. Then by the pigeonhole principle, since $|[M_0]^\mu| = 2^\mu$, we may assume that there is a $R^* \in [M_0]^\mu$ such that:

$$\forall \alpha \in S(\mathbf{tp}(N_\alpha/M_\alpha) \restriction_{M_0} \text{ does not fork over } R^*). \tag{4}$$

By base monotonicity we may further assume that $R \leq_{\mathbf{K}} R^*$. Then applying transitivity to the previous two equations we obtain that:

$$\forall \alpha \in S(N_\alpha \overline{\downarrow}_{R^*} M_\alpha). \tag{5}$$

Moreover, given $\alpha \in S$ $p_\alpha \in \mathbf{S}(N_\alpha)$ does not fork over R and $N_\alpha \leq_{\mathbf{K}} M_{\alpha+1}$. Applying extension and transitivity, there is $q_\alpha \in \mathbf{S}(M_{\alpha+1})$ extending p_α and q_α does not fork over R. By base monotonicity, since $R \leq_{\mathbf{K}} R^* \leq_{\mathbf{K}} M_{\alpha+1}$, we also have that q_α does not fork over R^*.

Let $\Upsilon : S \to \mathbf{S}(R^*)$ be defined as $\Upsilon(\alpha) = q_\alpha \restriction_{R^*}$, by the pigeonhole principle we may assume that there is $q \in \mathbf{S}(R^*)$ such that:

$$\forall \alpha \in S(q_\alpha \geq q \text{ and } q_\alpha \text{ does not fork over } R^*). \tag{6}$$

Let $\{\alpha_n : n \in \omega\} \subseteq S$ increasing set of ordinals. We build $\{r_n : n \in \omega\}$ such that:

(1) $r_0 = q_{\alpha_0}$.
(2) $r_{n+1} \geq r_n, p_{\alpha_{n+1}}$.
(3) $r_n \in \mathbf{S}(M_{\alpha_n+1})$.
(4) r_n does not fork over R.

The base step is given so let us do the induction step. By equation (5) $N_{\alpha_{n+1}} \overline{\downarrow}_{R^*} M_{\alpha_{n+1}}$. Since $\alpha_n + 1 \leq \alpha_{n+1} \in S$, we have that $M_{\alpha_n+1} \leq_{\mathbf{K}} M_{\alpha_{n+1}}$, so by monotonicity $N_{\alpha_{n+1}} \overline{\downarrow}_{R^*} M_{\alpha_n+1}$ and by normality we have that $N_{\alpha_{n+1}} \cup R^* \overline{\downarrow}_{R^*} M_{\alpha_n+1}$. Realize that $q \in \mathbf{S}(R^*)$, $q_{\alpha_{n+1}} \restriction_{N_{\alpha_{n+1}} \cup R^*} \in \mathbf{S}(N_{\alpha_{n+1}} \cup R^*; \mathfrak{C})$, $r_n \in \mathbf{S}(M_{\alpha_n+1})$ and $M_{\alpha_{n+1}+1}$ substituted by p, q_1, q_2 and N^* satisfy the hypothesis of the type-amalgamation property. Therefore there is $r_{n+1} \in \mathbf{S}(M_{\alpha_{n+1}+1})$ such that $r_{n+1} \geq q_{\alpha_{n+1}} \restriction_{N_{\alpha_{n+1}} \cup R^*}, r_n$ and r_{n+1} does not fork over R^*.

[6]Symmetry is not used to obtain this result.

In particular we have that $r_{n+1} \geq r_n, p_{\alpha_{n+1}}$ (since $q_{\alpha_{n+1}} \geq p_{\alpha_{n+1}}$) and by transitivity (since $r_{n+1} \geq r_n$, $R^* \leq M_{\alpha_n+1}$ and r_n does not fork over R) we have that r_{n+1} does not for over R. This finishes the construction.

Finally $\{r_n \in \mathbf{S}(M_{\alpha_n+1}) : n \in \omega\}$ is an increasing chain of types so by [Bal09, 11.3], there is $r^* \in \mathbf{S}(\bigcup_{n\in\omega} M_{\alpha_n+1})$ such that $r^* \geq r_n$ for each $n \in \omega$. In particular, by clause (2) of the construction, we have that $\forall n < \omega(p_{\alpha_n} \leq r^*)$, which is precisely what we need to show. □

The following generalizes [Les00, A] to the AEC context. The proof is similar to that of Theorem 4.2, but using the $\aleph_0$-bound condition instead of the uniqueness property.

Theorem 6.7. *If $\overline{\downarrow}$ is a simple independence relation, $\kappa(\overline{\downarrow}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\overline{\downarrow})} = \mu$, then*

$$NT(\mu, \lambda, \aleph_0) \leq \lambda^{\kappa(\overline{\downarrow})} + 2^\mu.$$

In particular, $NT(\mu, \lambda) \leq \lambda^{\kappa(\overline{\downarrow})} + 2^\mu$

Proof. Let $\lambda_0 = \kappa(\overline{\downarrow})$, $\chi = \lambda^{\lambda_0} + 2^\mu$ and $\{p_\alpha \in \mathbf{S}(N_\alpha) : \alpha < \chi^+\} \subseteq \mathbf{S}(M, \leq \mu)$ where $M \in \mathbf{K}_\lambda$. Observe that by the extension property we may assume that each $N_\alpha \in \mathbf{K}_\mu$. As in the proof of Theorem 4.2 there are $S \subseteq \chi^+$ of size χ^+, $R \in [M]^{\lambda_0}$ and $p \in \mathbf{S}(R)$ such that $\forall \alpha \in S(p_\alpha \geq p$ and p_α does not fork over $R)$.

By the $\aleph_0$-bound condition, where the cardinal parameters are as in the definition except that $\kappa := \lambda_0$ and all the model theoretic parameters are the same with $\{p_\alpha : \alpha \in S\}$ being the collection of types and $dom(p) = R$, we obtain that there are countable $A \subseteq S$ and q a type such that $q \geq q_\alpha$ for each $\alpha \in A$. In particular $\{q_\alpha : \alpha \in A\}$ is consistent. Hence $NT(\mu, \lambda, \aleph_0) \leq \lambda^{\lambda_0} + 2^\mu$. □

Remark 6.8. *Observe that when $\overline{\downarrow}$ is a stable or $*$-simple independence relation Theorem 4.2 and 5.13 give us a better bound. Moreover, Theorem 4.2 and 5.13 give us a bound for each $\kappa \in Car$ while the above corollary only gives us a bound when κ is countable, as we will see in Theorem 7.2 more can be said if we assume the $(< \aleph_0)$-witness property.*

The following result shows that we can not have the 2-tree property if $\mathbf{K}$ has a $\overline{\downarrow}$ simple independence relation.

Corollary 6.9. *If $\overline{\downarrow}$ is a simple independent relation, then $\mathbf{K}$ does not have the 2-tree property.*

Proof. Suppose for the sake of contradiction that $\mathbf{K}$ has the 2-tree property.

Let $\lambda_0 = \kappa(\overline{\downarrow})$, $\mu = (\beth_{\ell(\overline{\downarrow})^+}(\lambda_0^+))^+$ and $\lambda = \beth_\mu(\mu)$. Observe that the following cardinal arithmetic equalities hold:

(1) $\mu^{<\ell(\overline{\downarrow})} = \mu$, using that $cf(\beth_{\ell(\overline{\downarrow})^+}(\lambda_0^+)) = \ell(\overline{\downarrow})^+$ and Hausdorff formula.
(2) $\lambda^{\lambda_0} + 2^\mu = \lambda$, using that $cf(\lambda) = \mu > \lambda_0$ and that $\beth_\mu(\mu) > 2^\mu$.
(3) $\lambda^{<\mu} = \lambda$, using that $cf(\lambda) = \mu$.

Applying Theorem 6.7, this is possible by the first cardinal arithmetic equality, and by the second cardinal arithmetic equality we get that:

$$NT(\mu, \lambda) \leq \lambda^{\lambda_0} + 2^\mu = \lambda. \tag{7}$$

Applying Lemma 3.5, this is possible by the third cardinal arithmetic equality, we get that

$$\lambda^\mu \leq NT(\mu, \lambda). \tag{8}$$

So putting inequalities (7) and (8) we obtain that $\lambda^\mu \leq \lambda$, but this is a contradiction to König's Lemma since $cf(\lambda) = \mu$. □

Remark 6.10. *In the result above, instead of showing the failure of the* 2*-tree property, we would have liked to obtain the failure of the tree property. We will show in Corollary 7.3 that this is the case if* $\overline{\downarrow}$ *has the* $(< \aleph_0)$*-witness property for singletons.*

The next result follows trivially from the results of this section.

Corollary 6.11. (1) $\rightarrow$ (2) $\rightarrow$ (3) *where:*

(1) $\overline{\downarrow}$ *a simple independence relation.*
(2) $\exists \lambda_0 \exists \theta \forall \mu, \lambda($ *If* $\lambda_0 \leq \mu \leq \lambda$ *and* $\mu^{<\theta} = \mu$*, then* $NT(\mu, \lambda, \aleph_0) \leq \lambda^{\lambda_0} + 2^{\mu})$.
(3) $\mathbf{K}$ *does not have the 2-tree property.*

Proof. The first implication is Theorem 6.7 and the second one is Theorem 6.9. □

7. Simple independent relations with the witness property

In this section we continue the study of simple independence relations under locality assumptions. We begin by showing the failure of the tree property under the existence of a simple independence relation with $(< \aleph_0)$-witness property. Then we study simple independence relations with $(< \mathrm{LS}(\mathbf{K})^+)$-witness property and obtain some basic results.

7.1. **Failure of the tree property.** The next argument extends the one presented in Lemma 6.6.

Lemma 7.1. *If* $\overline{\downarrow}$ *is a simple independence relation with the* $(< \aleph_0)$*-witness property for singletons, then* $\overline{\downarrow}$ *has the strong* $(2^{\mu})^+$*-bound condition.*

Proof sketch . Everything is the same as the proof of Lemma 6.6 until equation (6), but in this case instead of building only countably many $r'_n s$ we will build $(2^{\mu})^+$ many of them.

Let $\{\alpha_i : i < (2^{\mu})^+\} \subseteq S$ an increasing set of ordinals. We build $\{r_i : i < (2^{\mu})^+\}$, $\{a_i : i < (2^{\mu})^+\}$ and $\{f_{j,i} : j < i < (2^{\mu})^+\}$ such that:

(1) $r_0 = q_{\alpha_0} = \mathbf{tp}(a_0/M_{\alpha_0+1})$.
(2) If $k < j < i < (2^{\mu})^+$, then $f_{k,i} = f_{j,i} \circ f_{k,j}$.
(3) $\forall j < i (f_{j,i} \restriction_{M_{\alpha_j+1}} = \mathrm{id}_{M_{\alpha_j+1}}, f_{j,i}(a_j) = a_i$ and $f_{j,i} \in Aut(\mathfrak{C}))$.
(4) $r_i = \mathbf{tp}(a_i/M_{\alpha_i+1})$ does not fork over R.
(5) $r_i \geq p_{\alpha_i}$.
(6) $\forall j < i (r_j \leq r_i)$.

The construction in the successor step is similar to that of Lemma 6.6, so we only show how to do the the step when i is a limit ordinal. Since $\{r_j : j < i\}$, $\{a_j : j < i\}$ and $\{f_{k,j} : k < j < i\}$ is a directed system, by [Bal09, 11.3], there is $p^* = \mathbf{tp}(a^*/\bigcup_{j<i} M_{\alpha_j+1})$ upper bound for $\{r_j : j < i\}$ and $\{f^*_{j,i} : j < i\}$ satisfying (2) and (3) but with a^* substituted for a_i.

Using $(< \aleph_0)$-witness property, invariance and monotonicity it is easy to show that p^* does not fork over R. Observe that $\bigcup_{j<i} M_{\alpha_j+1} \subseteq M_{\alpha_i}$, $N_{\alpha_i} \overline{\downarrow}_{R^*} M_{\alpha_i}$ (by equation (5) of Lemma 6.6) and $p^* \geq r_0$. Using these, one can show that $q \in \mathbf{S}(R^*)$, $q_{\alpha_i} \restriction_{N_{\alpha_i} \cup R^*} \in \mathbf{S}(N_{\alpha_i} \cup R^*; \mathfrak{C})$, $p^* \in \mathbf{S}(\bigcup_{j<i} M_{\alpha_j+1})$ and M_{α_i+1} substituted for p, q_1, q_2 and N^* satisfy the hypothesis of the type-amalgamation property. Therefore, there is $r_i \in \mathbf{S}(M_{\alpha_i+1})$ such that $r_i \geq q_{\alpha_i} \restriction_{N_{\alpha_i} \cup R^*}, p^*$ and r_i does not fork over R^*.

Let $r_i := \mathbf{tp}(a_i/M_{\alpha_i+1})$. Since $r_i \restriction_{\bigcup_{j<i} M_{\alpha_j+1}} = p^*$, there is $g \in Aut(\mathfrak{C})$ such that $g(a^*) = a_i$ and $g \restriction_{\bigcup_{j<i} M_{\alpha_j+1}} = id_{\bigcup_{j<i} M_{\alpha_j+1}}$. For each $j < i$, let $f_{j,i} := g \circ f^*_{j,i}$. It is easy to show that $r_i, a_i, \{f_{j,i} : j < i\}$ satisfy (1) through (6), for conditions (4)-(6) see the explanation given in Lemma 6.6 . This finishes the construction.

We have constructed $\{(r_i, a_i, \{f_{k,j} : k < j < i\}) : i < (2^\mu)^+\}$ a coherent sequence of types, then by [Bal09, 11.3] there is $r^* \in \mathbf{S}(\bigcup_{i<(2^\mu)^+} M_{\alpha_i+1})$ such that $\forall i < (2^\mu)^+ (r_i \leq r^*)$. In particular, $p_{\alpha_i} \leq r^*$ for every $i < (2^\mu)^+$, since by condition (5) $p_{\alpha_i} \leq r_i$ for each $i < (2^\mu)^+$. Moreover, using the $(< \aleph_0)$-witness property it follows that r^* does not fork over R. □

Using the above result instead of Lemma 6.6 we are able to extend Theorem 6.7 to uncountable cardinals. As the proof is similar to that of Theorem 6.7 we omit it.

Theorem 7.2. *If $\overline{\downarrow}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons, $\kappa(\overline{\downarrow}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\overline{\downarrow})} = \mu$, then*

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{\kappa(\overline{\downarrow})} + 2^\mu.$$

As a corollary we obtain the failure of the tree property.

Corollary 7.3. *If $\overline{\downarrow}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons, then $\mathbf{K}$ does not have the tree property.*

Proof sketch . Let $\lambda_0 = \kappa(\overline{\downarrow})$. Let μ and λ as in Theorem 6.9, i.e., $\mu = (\beth_{\ell(\overline{\downarrow})^+}(\lambda_0^+))^+$ and $\lambda = \beth_\mu(\mu)$. Then doing a similar construction to that of Lemma 4.4 we get that:

$$\lambda^+ \leq NT(\mu, \lambda, (2^\mu)^+). \tag{9}$$

But by Theorem 7.2 we have that $NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{\lambda_0} + 2^\mu$, then by choice of μ and λ we have that $\lambda^{\lambda_0} + 2^\mu = \lambda$, so:

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda. \tag{10}$$

Observe that equations (9) and (10) give us a contradiction.

□

Remark 7.4. *It is natural to ask which relations satisfy the hypothesis of this subsection, we give two classes of examples:*

- *Let T be a complete first-order theory. If T is simple and $\overline{\downarrow}$ is first-order non-forking, then $\overline{\downarrow}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons. This follows from the fact that forking has finite character.*
- *If $\overline{\downarrow}$ is stable independence relation and $\mathbf{K}$ is fully $(< \aleph_0)$-tame, then $\overline{\downarrow}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons. This follows from Corollary 5.7, Fact 5.4 and Lemma 5.5.*

7.2. **Simple independence relations with $(< \mathrm{LS}(\mathbf{K})^+)$-witness property.** We continue the study of simple independence relations but with the additional hypothesis of the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons. Recall that we have shown that if $\kappa_1(\overline{\downarrow}) = \mathrm{LS}(\mathbf{K})$. then $\overline{\downarrow}$ has the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons (Lemma 2.13).

The following simple proposition will be used to study the Lascar rank in the next section.

Proposition 7.5. *Let $\overline{\downarrow}$ a simple independence relation with $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons. If $M \leq_{\mathbf{K}} N$, $p \in \mathbf{S}(M)$, $q \in \mathbf{S}(N)$ and q is a forking extension of p, then there is $M^* \leq_{\mathbf{K}} N$ with $\|M^*\| = \|M\|$, $M \leq_{\mathbf{K}} M^*$ and $q \restriction_{M^*}$ a forking extension of p.*

Proof. Assume that $q = \mathbf{tp}(b/N)$. Suppose for the sake of contradiction that it is not the case, hence for every $M^* \leq_{\mathbf{K}} N$ with $\|M^*\| = \|M\|$ and $M \leq_{\mathbf{K}} M^*$ it holds that $q \restriction_{M^*}$ does not fork over M. We will show, using the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons, that $b \overline{\downarrow}_M N$.

Let $A \subseteq N$ and $|A| \leq \mathrm{LS}(\mathbf{K})$, then apply downward Löwenheim-Skolem to $A \cup M$ inside N to get $M^* \in \mathbf{K}_{\|M\|}$ such that $A \cup M \subseteq M^* \leq_{\mathbf{K}} N$. Then by assumption $b \overline{\downarrow}_M M^*$. So by

monotonicity $b \overline{\downarrow}_M A$. Therefore, by ($< \text{LS}(\mathbf{K})^+$)-witness property for singletons, we conclude that $b \overline{\downarrow}_M N$, which contradicts the hypothesis that q forks over M. □

The next lemma generalizes [Kim14, 2.3.7].

Lemma 7.6. *Let $\overline{\downarrow}$ a simple independence relation that has ($< \text{LS}(\mathbf{K})^+$)-witness property for singletons and without uniform local character. The following are equivalent.*

1. $\kappa_1(\overline{\downarrow}) = \lambda$.
2. *There are no $\{M_i : i \leq \lambda^+\}$ and $p \in \mathbf{S}(M_{\lambda^+})$ such that $\{M_i : i \leq \lambda^+\}$ is strictly increasing and continuous chain and p forks over M_i for every $i < \lambda^+$.*[7]

Proof. $\boxed{\rightarrow}$ Assume for the sake of contradiction that there is $\{M_i : i \leq \lambda^+\}$ a strictly increasing and continuous chain and $p \in \mathbf{S}(M_{\lambda^+})$ such that p forks over M_i for every $i < \lambda^+$. Then by hypothesis there is $M' \in [M_{\lambda^+}]^\lambda$ such that p does not fork over M'. Then by regularity of λ^+ and base monotonicity there is $i < \lambda^+$ such that p_{λ^+} does not fork over M_i. This is a contradiction.

$\boxed{\leftarrow}$ Assume for the sake of contradiction that $\kappa_1(\overline{\downarrow}) \neq \lambda$, then there is $q = \mathbf{tp}(a/N) \in \mathbf{S}(N)$ such that q forks over M for every $M \in [N]^\lambda$. Realize that $\|N\| \geq \lambda^+$ as q does not fork over N.

We build $\{M_i : i < \lambda^+\}$ strictly increasing and continuous chain such that:

1. For every $i < \lambda^+$, $M_i \in \mathbf{K}_\lambda$ and $M_i \leq_{\mathbf{K}} N$.
2. For every $j > i$, $q \restriction_{M_j}$ forks over M_i.

Before we do the construction observe that this is enough by taking $M_{\lambda^+} = \bigcup_{i<\lambda^+} M_i$, $\{M_i : i \leq \lambda^+\}$ and $p = q \restriction_{M_{\lambda^+}}$.

In the base step, just take any $M_0 \in [N]^\lambda$. If $i < \lambda^+$ limit take unions and and it works by monotonicity, so the only interesting case is when $i = j + 1$. Then by ($< \text{LS}(\mathbf{K})^+$)-witness property there is $B \subseteq N$ of size $\text{LS}(\mathbf{K})$ such that $p \restriction_B$ forks over M_j and pick $c \in N \backslash M_j$. Let M_{j+1} be the structure obtained by applying downward Löwenheim-Skolem to $B \cup M_j \cup \{c\}$ in N. This works by the choice of B and monotonicity. □

Realize that even simple assertions as the ones above become very hard to prove or perhaps even false if the independence relation does not have some locality assumptions.

8. Supersimple independence and the U-rank

In this section we introduce supersimple independence relations and show that they can be characterized by the Lascar rank under a locality assumption on the independence relation. We also show that the existence of a supersimple independence relation implies the ($< \aleph_0$)-witness property for singletons in classes with intersections.

Let us introduce the notion of a supersimple independence relation.

Definition 8.1. *$\overline{\downarrow}$ is a* supersimple independence relation *if the following properties hold:*

1. *$\overline{\downarrow}$ is a simple independence relation.*
2. *(Finite local character) For every δ limit ordinal, $\{M_i : i \leq \delta\}$ increasing and continouos chain and $p \in \mathbf{S}(M_\delta)$, there is $i < \delta$ such that p does not fork over M_i.*

Remark 8.2. *Let T be a complete first-order theory. If T is supersimple and $\overline{\downarrow}$ is first-order non-forking, then $\overline{\downarrow}$ is a supersimple independence relation.*

The following is straightforward but will be useful.

Lemma 8.3. *If $\overline{\downarrow}$ is a supersimple independence relation, then $\kappa_1(\overline{\downarrow}) = \text{LS}(\mathbf{K})$.*

[7] This generalizes the first-order notion of a forking chain.

Proof sketch. The proof can be done by induction on the cardinality of the domain of the type. The base step is clear because types do not over their domain and for the induction step use that $\overline{\downarrow}$ has finite local character. □

The above lemma together with Lemma 2.13 can be used to obtain the next result.

Corollary 8.4. *If $\overline{\downarrow}$ is a supersimple independence relation, then $\overline{\downarrow}$ has the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons.*

The next lemma shows that supersimplicty and stability imply superstability.

Lemma 8.5. *If $\overline{\downarrow}$ is a stable and supersimple independence relation, then $\mathbf{K}$ is Galois-stable in a tail of cardinals*[8].

Proof. Since $\overline{\downarrow}$ is a stable independence relation, by Corollary 4.3 $\mathbf{K}$ is a Galois-stable AEC, so let λ_0 be the first stability cardinal. We show by induction on $\mu \geq \lambda_0$ that $\mathbf{K}$ is μ-Galois-stable.

The base step is clear, so let us do the induction step. We proceed by contradiction, let $M \in \mathbf{K}_\mu$ and $\{p_i : i < \mu^+\} \subseteq \mathbf{S}(M)$ an enumeration of different Galois-types. Let $\{M_\alpha : \alpha < \mu\} \subseteq \mathbf{K}_{<\mu}$ an increasing chain of submodels of M such that $\bigcup_{\alpha<\mu} M_\alpha = M$. Then by supersimplicity for every $i < \mu^+$ there is $\alpha_i < \mu$ such that p does not fork over M_{α_i}. Then by the pigeonhole principle and using that $\overline{\downarrow}$ has uniqueness, one can show (as in Theorem 4.2) that there are $i \neq j < \mu^+$ such that $p_i = p_j$- This is clearly a contradiction. Therefore, $\mathbf{K}$ is μ-Galois-stable. □

It is worth noticing that Lemma 7.1 can be carried out with the finite local character assumption instead of the $(< \aleph_0)$-witness property for singletons. The idea is that by applying finite local character and transitivity in limit stages one can show that the type constructed does not fork over R (where R is the one introduced in condition (4) of Lemma 7.1).

Corollary 8.6. *If $\overline{\downarrow}$ is a supersimple independence relation, then*

- *if $\kappa(\overline{\downarrow}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\overline{\downarrow})} = \mu$, then*
$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{\kappa(\overline{\downarrow})} + 2^\mu.$$
- **K** *does not have the tree property.*

8.1. **Lascar rank.** The Lascar rank was extended to the AEC context by Boney and the first author in [BoGr17].

Definition 8.7 ([BoGr17, 7.2])**.** *We define U with domain a type and range an ordinal or ∞ by, for any $p \in \mathbf{S}(M)$*

(1) *$U(p) \geq 0$.*
(2) *$U(p) \geq \alpha$ if and only if $U(p) \geq \beta$ for each $\beta < \alpha$.*
(3) *$U(p) \geq \beta + 1$ if and only if there are $M' \geq_{\mathbf{K}} M$ and $p' \in \mathbf{S}(M')$ with $\|M'\| = \|M\|$, p' a forking extension of p and $U(p') \geq \beta$.*
(4) *$U(p) = \alpha$ if and only if $U(p) \geq \alpha$ and it is not the case that $U(p) \geq \alpha + 1$.*
(5) *$U(p) = \infty$ if and only if $U(p) \geq \alpha$ for each α ordinal.*

The next couple of results show that U is a well-behaved rank. The proofs are similar to the ones presented in [BoGr17, §7], but we fix a minor mistake of [BoGr17, §7]. The arguments of [BoGr17, §7] only work when the models under consideration are all of the same size, we are

[8]This is equivalent to any notion of superstability in the context of AECs if one assume that the AEC has a monster model and is tame by [GrVas17] and [Vas18].

able to extend the arguments for models of different sizes by using ($< \mathrm{LS}(\mathbf{K})^+$)-witness property, specifically Proposition 7.5.

Lemma 8.8. *Let $\overline{\downarrow}$ a simple independence relation with ($< \mathrm{LS}(\mathbf{K})^+$)-witness property for singletons, then the U-rank satisfies:*

(1) *([BoGr17, 7.4]) Invariance: If $p \in \mathbf{S}(M)$ and $f : M \cong M'$, then $U(p) = U(f(p))$.*
(2) *Monotonicity: If $M \leq_{\mathbf{K}} N$, $p \in \mathbf{S}(M)$, $q \in \mathbf{S}(N)$ and $p \leq q$, then $U(q) \leq U(p)$.*

Proof. We provide a proof for (2) based on [BoGr17, 7.3]. We prove induction on α that: if $p \leq q$, then if $U(q) \geq \alpha$, then $U(p) \geq \alpha$. The base step and limit step are trivial so assume that $\alpha = \beta + 1$ and that $U(q) \geq \beta + 1$. By definition there is $N' \geq_{\mathbf{K}} N$ and $q' \in \mathbf{S}(N')$ with $\|N'\| = \|N\|$, $q' \geq q$, q' forks over N and $U(q') \geq \beta$. Observe that by monotonicity q' forks over M and clearly $q' \geq p$. Then by Proposition 7.5 there is $M' \geq_{\mathbf{K}} M$ with $\|M'\| = \|M\|$, $q' \restriction_{M'} \geq p$ and $q' \restriction_{M'}$ forks over M. Since $q' \restriction_{M'} \leq q'$, by induction hypothesis $U(q' \restriction_{M'}) \geq \beta$. Therefore, by the definition of the U-rank $U(p) \geq \beta + 1$. □

Lemma 8.9. *Let $\overline{\downarrow}$ a simple independence relation with ($< \mathrm{LS}(\mathbf{K})^+$)-witness property for singletons. Let $M \leq_{\mathbf{K}} N$, $p \in \mathbf{S}(M)$ and $q \in \mathbf{S}(N)$ with $p \leq q$ and $U(p), U(q) < \infty$. Then:*

$$U(p) = U(q) \text{ if and only if } q \text{ is a non-forking extension of } p.$$

Proof. $\boxed{\rightarrow}$ Assume for a sake of contradiction that q forks over p. Then by Proposition 7.5 there is $M^* \in \mathbf{K}$ with $\|M^*\| = \|M\|$, $q \restriction_{M^*} \geq p$ and $q \restriction_{M^*}$ forks over M. Then from monotonicity of the rank and the definition of the U-rank, we can conclude that $U(p) \geq U(q) + 1$, which clearly contradicts our hypothesis.

$\boxed{\leftarrow}$ The same argument given in [BoGr17, 7.7] can be carried out in our context due to Proposition 6.4. □

Fact 8.10. *([BoGr17, 7.8]) Let $\overline{\downarrow}$ a simple independence relation with ($< \mathrm{LS}(\mathbf{K})^+$)-witness property for singletons. For each $\mu \geq \mathrm{LS}(\mathbf{K})$, there is some $\alpha_{\mathbf{K},\mu} < (2^\mu)^+$ such that for any $M \in \mathbf{K}_\mu$, if $U(p) \geq \alpha_{\mathbf{K},\mu}$, then $U(p) = \infty$.*

The proof of the following lemma is similar to that of [BoGr17, 7.9].

Lemma 8.11. *Let $\overline{\downarrow}$ a simple independence relation with ($< \mathrm{LS}(\mathbf{K})^+$)-witness property for singletons. Let $M \in \mathbf{K}_\mu$ and $p \in \mathbf{S}(M)$. The following are equivalent.*

(1) *$U(p) = \infty$*
(2) *There is an increasing chain of types $\{p_n : n < \omega\}$ such that $p_0 = p$ and p_{n+1} is a forking extension of p_n for each $n < \omega$.*

Proof. $\boxed{\rightarrow}$ Let $\alpha_{\mathbf{K},\mu}$ the ordinal given by Fact 8.10. We build $\{M_n : n < \omega\}$ and $\{p_n \in \mathbf{S}(M_n) : n < \omega\}$ by induction such that:

(1) $p_0 = p$.
(2) $M_n \in \mathbf{K}_\mu$.
(3) p_{n+1} is a forking extension of p_n for every $n < \omega$.
(4) $U(p_n) \geq \alpha_{K,\mu} + 1$.

The base step is given by condition (1). As for the induction step, we have by induction that $U(p_n) \geq \alpha_{K,\mu} + 1$. Then by definition of the U-rank there is $M_{n+1} \geq M_n$ and $p_{n+1} \in \mathbf{S}(M_{n+1})$ a forking extension of p_n such that $\|M_{n+1}\| = \|M_n\| = \mu$ and $U(p_{n+1}) \geq \alpha_{\mathbf{K},\mu}$. Observe that since $U(p_{n+1}) \geq \alpha_{\mathbf{K},\mu}$ and $M_{n+1} \in \mathbf{K}_\mu$, we have that $U(p_{n+1}) = \infty$, so $U(p_{n+1}) \geq \alpha_{\mathbf{K},\mu} + 1$.

$\boxed{\leftarrow}$ Let $\{p_n : n < \omega\}$ an increasing chain of types such that $p_0 = p$ and p_{n+1} is a forking extension of p_n for each $n < \omega$. We prove by induction that for each $\alpha \in Or(\forall n \in \omega(U(p_n) \geq \alpha))$. The base step and limit case are trivial so assume that $\alpha = \beta + 1$ and take $n \in \omega$. By

induction hypothesis $U(p_{n+1}) \geq \beta$ and by hypothesis p_{n+1} is a forking extension of p_n. Then by Proposition 7.5 there is $M^* \in \mathbf{K}$ with $\|M^*\| = \|dom(p_n)\|$, $p_{n+1} \restriction_{M^*} \geq p_n$ and $p_{n+1} \restriction_{M^*}$ forks over $dom(p_n)$. Then by monotonicity of the rank and the definition of the U-rank we can conclude that $U(p_n) \geq \beta + 1 = \alpha$. □

With this we obtain our main result regarding the relationship between a supersimple independence relations and the U-rank . This generalizes a characterization of supersimplicity for first-order theories [Kim14, 2.5.16].

Theorem 8.12. *Let $\overline{\downarrow}$ a simple independence relationship with $(< \aleph_0)$-witness property for singletons. The following are equivalent.*

(1) *$\overline{\downarrow}$ is a supersimple independence relation.*
(2) *If $M \in \mathbf{K}$ and $p \in \mathbf{S}(M)$, then $U(p) < \infty$.*

Proof. $\boxed{\rightarrow}$ Suppose there is $M \in \mathbf{K}$ and $p \in \mathbf{S}(M)$ such that $U(p) = \infty$. Then, by Lemma 8.11, there is an increasing chain of types $\{p_n : n < \omega\}$ such that $p_0 = p$ and p_{n+1} is a forking extension of p_n for every $n < \omega$.

Since we have that $\{p_n : n < \omega\}$ is an increasing chain of types, by [Bal09, 11.3], there is $p_\omega \in \mathbf{S}(\bigcup_{n<\omega} dom(p_n))$ such that $p_\omega \geq p_n$ for each $n < \omega$. Then, by the definition of supersimplicty, there is $n < \omega$ such that p_ω does not fork over $dom(p_n)$. Hence by monotonicity $p_\omega \restriction_{dom(p_{n+1})} = p_{n+1}$ does not fork over $dom(p_n)$, which contradicts the fact that p_{n+1} is a forking extension of p_n.

$\boxed{\leftarrow}$ Assume for the sake of contradiction that $\overline{\downarrow}$ is not a supersimple independence relation, then there are δ limit ordinal and $\{N_i : i \leq \delta\}$ increasing and continuous chain and $p \in \mathbf{S}(N_\delta)$, such that p forks over N_i for every $i < \delta$.

We first show that for every $i < \delta$ there is $j_i \in (i, \delta)$ such that p_{j_i} forks over N_i. Let $i < \delta$ and suppose for the sake of contradiction that p_j does not fork over N_i for each $j \in (i, \delta)$. Then using the $(< \aleph_0)$-witness property for singletons, as in Proposition 7.5, one can show that p does not fork over N_i, contradicting the hypothesis that p forks over N_i.

Then one can build by induction $\{i_n : n < \omega\} \subseteq \delta$ increasing such that $\{p_{i_n} : n < \omega\}$ is an increasing chain of types with $p_{i_{n+1}}$ a forking extension of p_{i_n} for each $n < \omega$ where $p_{i_n} = p \restriction_{N_{i_n}}$. Therefore by Lemma 8.11 we can conclude that $U(p_{i_0}) = \infty$, this contradicts the fact that $U(p_{i_0}) < \infty$ by hypothesis. □

8.2. **A familiy of classes with the $(< \aleph_0)$-witness property.** In this subsection we show that in classes that admit intersections one obtains the $(< \aleph_0)$-witness property for singletons from supersimplicity. We begin by recalling the definition of classes that admit intersections, these were introduced by Shelah and Baldwin.

Definition 8.13 ([BaSh08, 1.2]). *An AEC* admits intersections *if for every $N \in \mathbf{K}$ and $A \subseteq |N|$ there is $M_0 \leq_{\mathbf{K}} N$ such that $|M_0| = \bigcap\{M \leq_{\mathbf{K}} N : A \subseteq |M|\}$. For $N \in \mathbf{K}$ and $A \subseteq |N|$, we denote by $cl^N_{\mathbf{K}}(A) = \bigcap\{M \leq_{\mathbf{K}} N : A \subseteq |M|\}$, if it is clear from the context we will drop the $\mathbf{K}$. We write $cl(A)$ for $cl^{\mathfrak{C}}_{\mathbf{K}}(A)$ where $\mathfrak{C}$ is a monster model of $\mathbf{K}$ and $\mathbf{K}$ is clear from the context.*

Below we provide the properties of AECs that admit intersections that we will use, for a more detailed introduction to AECs that admit intersections the reader can consult [Vas17b, §2].

Fact 8.14. *Let $\mathbf{K}$ be an AEC that admits intersections.*

(1) *If $A \subseteq B \subseteq N$, then $cl^N(A) \leq_{\mathbf{K}} cl^N(B)$.*
(2) *If $A \subseteq M$ and $M \in \mathbf{K}$, then $cl(A) \leq_{\mathbf{K}} M$.*
(3) *(Finite character) Let $M \in \mathbf{K}$ and $a \in cl^M(B)$, then there is $B_0 \subseteq_{fin} B$ such that $a \in cl^M(B_0)$.*

Proof. (1) and (2) are trivial and (3) is [Vas17b, 2.14]. □

We show that finite local character is actually witnessed by a finite set in classes with intersections.

Lemma 8.15. *Let* $\mathbf{K}$ *be an AEC with a monster model that admits intersections and* $\overline{\downarrow}$ *a simple independence relation. The following are equivalent.*

(1) *(Finite local character) For every* δ *limit ordinal,* $\{M_i : i \leq \delta\}$ *increasing and continouos chain and* $p \in \mathbf{S}(M_\delta)$*, there is* $i < \delta$ *such that* p *does not fork over* M_i*.*
(2) *For every* $M \in \mathbf{K}$ *and* $p \in \mathbf{S}(M)$*, there is* $D \subseteq_{fin} M$ *such that* p *does not fork over* $cl(D)$*.*

Proof. The backward direction follows trivially using monotonicity, so we show the forward direction.

Let $M \in \mathbf{K}$ and $p \in \mathbf{S}(M)$, we show by induction on $\lambda \leq \|M\|$ the following:

$(*)_\lambda$: For every $A \in \mathcal{P}_\lambda(M)$ and $p \in \mathbf{S}(cl(A))$, there is $D \subseteq_{\text{fin}} M$ s.t. p does not fork over $cl(D)$.

Observe that this is enough as $cl(M) = M$. So let us do the proof.

Base: If λ is finite $(*)_\lambda$ is clear because given $p \in \mathbf{S}(cl(A))$, p does not fork over $cl(A)$. So let us do the case when $\lambda = \aleph_0$. Let $A = \{a_i : i < \omega\}$ an enumeration without repetitions and $p \in \mathbf{S}(cl(A))$. Let $M_i = cl(\{a_j : j < i\})$ for every $i < \omega$ and $M_\omega = \bigcup_{i<\omega} M_i$. Observe that $\{M_i : i \leq \omega\}$ is an increasing and continuous chain and $\bigcup_{i<\omega} M_i = cl(A)$ by the finite character of the closure operator. Then by (1) there is $i < \omega$ such that p does not fork over $M_i = cl(\{a_j : j < i\})$. So $D = \{a_j : j < i\}$ is as needed.

Induction step: Let λ an uncountable cardinal and suppose that $(*)_\mu$ holds for every $\mu < \lambda$. In this case the proof is similar to that of the base step when $\lambda = \aleph_0$. The only difference is that on top of using (1), one uses the induction hypothesis and transitivity of the independence relation.

□

Corollary 8.16. *Let* $\mathbf{K}$ *be a class that admits intersections. If* $\overline{\downarrow}$ *is a supersimple independence relation, then* $\overline{\downarrow}$ *has the* $(< \aleph_0)$*-witness property for singletons.*

Proof. Let $M \leq_{\mathbf{K}} N$ and $a \in \mathfrak{C}$ such that $a \overline{\downarrow}_M B$ for every $B \subseteq_{\text{fin}} N$.

By the previous theorem there is $D \subseteq_{\text{fin}} N$ such that $a \overline{\downarrow}_{cl(D)} N$, then by base monotonicity $a \overline{\downarrow}_{cl(DM)} N$. On the other hand, by hypothesis $a \overline{\downarrow}_M D$, then by normality, monotonicity and Proposition 6.4 it follows that $a \overline{\downarrow}_M cl(DM)$. Therefore, applying transitivity to $a \overline{\downarrow}_M cl(DM)$ and $a \overline{\downarrow}_{cl(DM)} N$ we obtain that $a \overline{\downarrow}_M N$. □

9. Future work

In [KiPi97, 4.2] it is shown that if a complete first-order theory is simple, then there is a canonical independence relation satisfying the type-amalgamation property. In [BGKV16] it is shown that stable independence relations are canonical and in Lemma 5.18 we showed that $*$-simple independence relations are canonical. So it is natural to ask if the same holds true for simple and supersimple independence relations.

Question 9.1. *If* $\mathbf{K}$ *has* $\overline{\downarrow}$ *a simple or supersimple independence relation, is* $\overline{\downarrow}$ *canonical?*

It is known that for a complete first-order theory T, T is simple if and only if T does not have the tree property (see for example [GIL02, 3.10]). In Sections 6 and 7 we showed some instances of the forward direction for simple independence relations (Corollary 6.9 and Corollary 7.3). So we ask the following:

Question 9.2. *If* $\mathbf{K}$ *does not have the tree property, does* $\mathbf{K}$ *have* $\overline{\downarrow}$ *a simple independence relation?*

Another notion that we studied in this paper is that of the witness property for independence relations. This seems to be a very strong hypothesis that can be taken for granted in first-order theories as forking has finite character. Regarding it we ask:

Question 9.3. *Can Fact 5.6 be extended to simple independence relations? More precisely, if* $\mathbf{K}$ *is fully* $(< \theta)$*-tame and* $\overline{\downarrow}$ *is a simple independence relation, does* $\overline{\downarrow}$ *have the* $(< \theta)$*-witness property?*

A related question is the following:

Question 9.4. *Is Corollary 8.16 true for all AECs with a monster model?*

Moreover, we used the witness properties a few times in this paper, see for example Lemma 7.1 and Theorem 8.12. An interesting question would if the use of the witness property is necessary in those arguments where we use it.

Finally, in [LRV19, 8.16] it is shown that the existence of a stable independence relation implies that the AEC is tame. We extended this result for $*$-simple independence relations in Lemma 5.8, so a natural question to ask is:

Question 9.5. *If* $\mathbf{K}$ *has* $\overline{\downarrow}$ *a simple or supersimple independence relation, is* $\mathbf{K}$ *tame?*

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