0.1 MORE EXERCISES

(1) Let T be a countable complete theory. Suppose that T is stable in $\aleph_0$. Prove that for every $\lambda \geq \aleph_0$ the theory T has a saturated model of cardinality λ.

(2) Let T be a countable complete theory. Suppose that T is stable in $\aleph_0$. If $R[\varphi(\mathbf{x};\mathbf{a})] = \alpha$ for an ordinal α then for every $\beta < \alpha$ there exists a formula $\psi_\beta(\mathbf{x};\mathbf{b}_\beta)$ such that $R[\psi_\beta(\mathbf{x};\mathbf{b}_\beta)] = \beta$.

(3) Let p be a type. Prove that for every automorphism f (of $\mathfrak{C}$) we have that $R[p] = R[f(p)]$.

(4) Use (2) and (3) to give an alternative proof to the fact that for countable theories, $\aleph_0$-stability implies $R[\mathbf{x} = \mathbf{x}] < \omega_1$.

(5) If T has the order property then T is unstable.

(6) Let λ be an infinite cardinal, show that $I(\lambda, PA) > 1$.

(7) Let T be a complete first order theory. Prove that the following are equivalent.

   (a) T is stable

   (b) every countable complete $T' \subseteq T$ is stable.

(8) Show that T_{ind} is unstable and is model complete.

(9) Show that the rank function $R[\cdot]$ defined in class and the function from page 211 of my book are equal.

(10) Let $\langle G, \cdot\rangle$ be an infinite group we say that G is *stable/superstable* iff $Th(\langle G, \cdot\rangle)$ is. By $H \leq_{def} G$ denote H *is a definable subgroup of* G.

   (a) G is $\aleph_0$ stable then

   $$\neg\exists\{H_n \leq_{def} G \mid n < \omega\} \quad with \quad 1 < [H_n : H_{n+1}] forall\ n < \omega.$$

   (b) G is superstable then

   $$\neg\exists\{H_n \leq_{def} G \mid n < \omega\} \quad with \quad [H_n : H_{n+1}] \geq \aleph_0 forall\ n < \omega.$$

   (c) G is stable then there is no $\{H_n \leq G \mid n < \omega\}$ *uniformly definable subgroups* with $H_{n+1} \lneq H_n foralln < \omega$.
   Where *uniformly definable subgroups* stands for: There are $\varphi(x;\mathbf{y};\mathbf{a})$ maybe with parameters from G and $\{\mathbf{b}_n \mid n < \omega\} \subseteq G$ such that $H_n = \varphi(G;\mathbf{b}_n;\mathbf{a})$ for all $n < \omega$.

(11) Use the previous exercise to prove the following:

Theorem 0.1.1 *Let $\langle G, \circ \rangle$ be a group and suppose that $h : G \to G$ is a non trivial definable (in the language of group theory) homomorphism. If $Th(\langle G, \circ, h \rangle)$ is superstable and the kernel of h is finite then h must be surjective.*

Moreover,

Theorem 0.1.2 *Let $\langle G, \circ \rangle$ be a group and suppose that $h : G \to G$ is a non trivial definable (in the language of group theory) homomorphism. If $Th(\langle G, \circ, h \rangle)$ is $\aleph_0$-stable then h must be surjective.*

Hint: Use Exercise 1.6.26.