SUPERSTABILITY FROM CATEGORICITY IN ABSTRACT ELEMENTARY CLASSES

WILL BONEY, RAMI GROSSBERG, MONICA M. VANDIEREN, AND SEBASTIEN VASEY

Abstract. Starting from an abstract elementary class with no maximal models, Shelah and Villaveces have shown (assuming instances of diamond) that categoricity implies a superstability-like property for a certain independence relation called nonsplitting. We generalize their result as follows: given an abstract notion of independence for Galois (orbital) types over models, we derive that the notion satisfies a superstability property provided that the class is categorical and satisfies a weakening of amalgamation. This extends the Shelah-Villaveces result (the independence notion there was splitting) as well as a result of the first and second author where the independence notion was coheir. The argument is in ZFC and fills a gap in the Shelah-Villaveces proof.

1. Introduction

1.1. **General motivation and history.** Forking is one of the central notions of model theory, discovered and developed by Shelah in the seventies for stable and NIP theories [She78] In the mid-nineties, Kim [Kim98] proved that Shelah's theory of forking can be extended to the class of so-called unstable simple theories. The work of Kim influenced many people to further explore properties of forking-like relations in various classes of unstable theories. For a modern summary see Adler's [Adl09].

Another way to extend Shelah's first-order stability theory is to move beyond first-order: in the mid seventies Shelah also started the program of *classification theory for non-elementary classes* focusing first on classes axiomatizable in $\mathbb{L}_{\aleph_1,\aleph_0}(\mathbf{Q})$ [She75] and later on the more general abstract elementary classes (AECs) [She87a]. Roughly, an AEC is a pair $\mathcal{K} = (K, \prec_{\mathcal{K}})$ satisfying some of the basic category-theoretic properties of $(\mathrm{Mod}(T), \prec)$ (but not the compactness theorem). Among the central problems, there are the decades-old categoricity and eventual

Date: September 22, 2016
AMS 2010 Subject Classification: Primary 03C48. Secondary: 03C45, 03C52, 03C55, 03C75, 03E05.
Key words and phrases. Abstract elementary classes; Categoricity; Superstability; Splitting; Coheir; Independence; Forking.
This material is based upon work done while the first author was supported by the National Science Foundation under Grant No. DMS-1402191.

categoricity conjectures of Shelah. In this paper, we assume that the reader has a basic knowledge of AECs, see for example [Gro02] or [Bal09].

A lesson learned from the proof of Morley's categoricty theorem [Mor65] is that the identification of $\text{tp}(\mathbf{a}/A)$ with the orbit of $\mathbf{a}$ under the group $\text{Aut}_A(\mathfrak{C})$ is important. In the early days of classification theory for non elementary classes (see [She72, GS86b, GS86a]) the notion of types studied in $\mathbb{L}_{\lambda^+,\omega}$-axiomatizable classes was $\text{tp}_{\lambda^+,\omega}(\mathbf{b}/A; M) := \{\phi(\mathbf{x}; \mathbf{a}) : \phi(\mathbf{x}; \mathbf{y}) \text{ is a formula from } \mathbb{L}_{\lambda^+,\omega}, \mathbf{a} \in {}^{<\omega}A, M \models \phi[\mathbf{b}; \mathbf{a}]\}$.

In [She87b] and [She09] a better nameless notion was introduced, Grossberg [Gro02] named it *Galois type* (Shelah uses the name *orbital types* in later papers). This has an easy definition when the class $\mathcal{K}$ has amalgamation, joint embedding and no maximal models, as these properties allow us to assume that all the elements of $\mathcal{K}$ we would like to discuss are substructures of a "monster" model $\mathfrak{C} \in \mathcal{K}$. In that case, $\text{gtp}(\mathbf{b}/A)$ is defined as the orbit of $\mathbf{b}$ under the action of the group $\text{Aut}_A(\mathfrak{C})$ on $\mathfrak{C}$. One can develop the notion of Galois type also without assuming amalgamation, joint embedding, or no maximal models, however then the definition is more technical.

1.2. **Independence, superstability, and no long splitting chains in AECs.** In [She99] a first candidate for an independence relation was introduced: the notion of splitting (roughly, $p \in \text{gS}(M)$ splits over $M_0 \prec_{\mathcal{K}} M$ provided there are $M_0 \prec_{\mathcal{K}} M_\ell \prec_{\mathcal{K}} M$, $\ell = 1, 2$ and $f : M_1 \cong_{M_0} M_2$ such that $f(p \upharpoonright M_1) \neq p \upharpoonright M_2$).

This notion was used by Shelah to establish a downward version of his categoricity conjecture from a successor for classes having the amalgamation property. Later similar arguments [GV06] were used to derive a strong upward version of Shelah's conjecture for classes satisfying the additional locality property of (Galois) types called tameness.

In Chapter II of [She09], Shelah introduced good λ-frames: an axiomatic definition of forking on Galois types over models of size λ. The notion is, by definition, required to satisfy basic properties of forking in superstable first-order theories (e.g. symmetry, extension, uniqueness, and local character). The theory of good λ-frames is well-developed and has had several applications to the categoricity conjecture (see Chapters III and IV of [She09] and recent work of the fourth author [Vasd, Vase, Vasc, Vasa]).

Constructions of good frames rely on weaker independence notions such as the aforementioned splitting, see e.g. [Vas16b, VV]: a key property there is the so-called no long splitting chains in $\mathcal{K}_\mu$: If $\langle M_i : i \leq \alpha \rangle$ is an increasing continuous chain in $\mathcal{K}_\mu$ (so $\alpha < \mu^+$ is a limit ordinal) and M_{i+1} is universal over M_i for each $i < \alpha$, then for any $p \in \text{gS}(M_\alpha)$ there exists $i < \alpha$ so that p does not split over M_i (this is called *strong universal local character at α* in the present paper, see Definition 3). This can be seen as a replacement for the statement "every type does not fork over a finite set". The property is already studied in [She99], and has several nontrivial consequences: for example (assuming amalgamation, joint embedding, no maximal models, stability in μ, and tameness), no long splitting chains in $\mathcal{K}_\mu$ implies that $\mathcal{K}$ is stable everywhere above μ [Vas16b, Theorem 5.6] and has a good μ^+-frame (on the subclass of saturated models of cardinality μ^+) [VV, Corollary 6.14]. No long splitting chains has consequences on the uniqueness of limit models, another superstability-like property akin to "the union of an increasing chain of μ-saturated models is μ-saturated" (see for example [SV99, Van06, Van13, Van]).

Boney and Grossberg explore another approach to independence; they adapted the notion of coheir to AECs. They show that for classes satisfying amalgamation which are also tame and short (a strengthening of tameness, using the variables of a type instead of its parameters), a little bit more than stability implies that coheir has some basic properties of forking from a stable first-order theory. There the "no long coheir chain" property also has strong consequences (for example on the uniqueness of limit models [BG, Corollary 6.10]).

1.3. **No long splitting chains from categoricity.** It is natural to ask whether no long splitting chains (or no long coheir chains) in $\mathcal{K}_\mu$ follows from categoricity above[1] μ. Shelah shows that this holds for splitting (assuming amalgamation and no maximal models) if the categoricity cardinal has cofinality greater than μ [She99, Lemma 6.3]. Without any cofinality restriction, a breakthrough was made in a paper of Shelah and Villaveces when they proved no long splitting chains assuming no maximal models and instances of diamond [SV99, Theorem 2.2.1]. Later Boney and Grossberg used the Shelah-Villaveces argument to derive the result in their context also for coheir [BG, Theorem 6.6]. It was also observed that the Shelah-Villaveces argument does not need diamond if one assumes full amalgamation [GV, Theorem 6.1]. In conclusion we have:

Fact 1. *Let $\mathcal{K}$ be an AEC with no maximal models. Let* $\mathrm{LS}(\mathcal{K}) \leq \mu < \lambda$ *and assume that $\mathcal{K}$ is categorical in λ.*

1. *[SV99, Theorem 2.2.1] If $\diamondsuit_{S^{\mu^+}_{\mathrm{cf}\,\mu}}$ holds then $\mathcal{K}$ has no long splitting chains in $\mathcal{K}_\mu$.*
2. *[BG, Theorem 6.6] If $\mathcal{K}$ has amalgamation, $\kappa \in (\mathrm{LS}(\mathcal{K}), \mu)$, $\mathcal{K}$ has no weak κ-order property and is fully $(< \kappa)$-tame and short, then $\mathcal{K}$ has no long coheir chains in $\mathcal{K}_\mu$.*
3. *[GV, Theorem 6.1] If $\mathcal{K}$ has amalgamation, then $\mathcal{K}$ has no long splitting chains in $\mathcal{K}_\mu$.*

Remark 2. *Fact 1 has applications to more "concrete" frameworks than AECs. One can deduce from it (and the aforementioned fact that no long splitting chains implies stability on a tail in the presence of tameness) an alternate proof that a first-order theory T categorical above $|T|$ is superstable. More generally, one*

[1]If one has categoricity at or below μ, one can try to derive no long splitting chains below the categoricity cardinal and then transfer it upward assuming tameness [Vas16a, Proposition 10.10], but one cannot hope to do this in general without tameness [HS90, BK09].

obtains the same statement for the class K of models of a homogeneous diagram in T [She70]. The later was open for $|T|$ uncountable and K categorical in $\aleph_\omega(|T|)$ (see [Vasc, Section 4]).

1.4. **Statement and discussion of the main theorem.** In this paper, we prove a generalization of Fact 1. In order to state it we introduce a rather weak notion of independence.

Definition 3. *Let $\mathcal{K}^*$ be an abstract class*[2] *and $\overset{*}{\mathop{\smile}\limits^{\vert}}$ be a 4-ary relation such that if $a \overset{*}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}}{}^N M$ holds, then $M_0 \prec_{\mathcal{K}^*} M \prec_{\mathcal{K}^*} N$ are all in $\mathcal{K}^*$ and $a \in |N|$.*

(1) *The following are several properties we will assume about $\overset{*}{\mathop{\smile}\limits^{\vert}}$ (but we will always mention when we assume them).*
   (a) *$\overset{*}{\mathop{\smile}\limits^{\vert}}$ has* invariance *(I) if it is preserved under isomorphisms: if $a \overset{*}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}}{}^N M$ and $f : N \cong N'$, then $f(a) \overset{*}{\underset{f[M_0]}{\mathop{\smile}\limits^{\vert}}}{}^{N'} f[M]$.*
   (b) *$\overset{*}{\mathop{\smile}\limits^{\vert}}$ has* monotonicity *(M) if:*
      (i) *If $a \overset{*}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}}{}^N M$, $M_0 \prec_{\mathcal{K}^*} M_0' \prec_{\mathcal{K}^*} M' \prec_{\mathcal{K}^*} M$, and $N \prec_{\mathcal{K}^*} N'$, then $a \underset{M_0'}{\mathop{\smile}\limits^{\vert}}{}^{N'} M'$; and:*
      (ii) *If $a \overset{*}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}}{}^N M$, $N' \prec_{\mathcal{K}^*} N$ is such that $M \prec_{\mathcal{K}^*} N'$ and $a \in |N'|$, then $a \overset{*}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}}{}^{N'} M$.*

(2) *(I) and (M) mean that this relation is really about Galois types, so we write* $\mathrm{gtp}(a/M; N)$ does not $*$-fork over M_0 *for $a \overset{*}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}}{}^N M$.*

(3) *For a limit ordinal α, $\overset{*}{\mathop{\smile}\limits^{\vert}}$ has* weak universal local character at α *if for any increasing continuous sequence $\langle M_i \in \mathcal{K}^* \mid i \leq \alpha \rangle$ and any type $p \in \mathrm{gS}(M_\alpha)$, if M_{i+1} is universal over M_i for each $i < \alpha$, then there is some $i_0 < \alpha$ such that $p \restriction M_{i_0+1}$ does not $*$-fork over M_{i_0}.*

(4) *For a limit ordinal α, $\overset{*}{\mathop{\smile}\limits^{\vert}}$ has* strong universal local character at α *if for any increasing continuous sequence $\langle M_i \in \mathcal{K}^* \mid i \leq \alpha \rangle$ and any type $p \in \mathrm{gS}(M_\alpha)$, if M_{i+1} is universal over M_i for each $i < \alpha$, then there is some $i_0 < \alpha$ such that p does not $*$-fork over M_{i_0}.*

Remark 4.

[2]That is, a partial order $(K^*, \prec_{\mathcal{K}^*})$ such that K^* is a class of structures in a fixed vocabulary closed under isomorphisms, $\prec_{\mathcal{K}^*}$ is invariant under isomorphisms, and $M \prec_{\mathcal{K}^*} N$ implies that M is a substructure of N.

(1) *In the setup of Fact 1.(1), non-μ-splitting on the class $\mathcal{K}^*$ of amalgamation bases of cardinality μ will have (I), (M), see Fact 7.*
(2) *If $\alpha < \beta$ are limit ordinals and $\overset{*}{\downarrow}$ has weak universal local character at α, then $\overset{*}{\downarrow}$ has weak universal local character at β, but this need not hold for strong universal local character (if say* $\operatorname{cf}\beta < \operatorname{cf}\alpha$*).*
(3) *If $\overset{*}{\downarrow}$ has (M) and $\overset{*}{\downarrow}$ has strong universal local character at* $\operatorname{cf}\alpha$*, then $\overset{*}{\downarrow}$ has strong universal local character at α.*
(4) *If $\overset{*}{\downarrow}$ has (M), strong universal local character at α implies weak universal local character at α.*
(5) *If (as will be the case in this note) $\mathcal{K}^*$ is a class of structures of a fixed size μ, then we only care about the properties when $\alpha < \mu^+$.*

Theorem 5 (Main Theorem)**.** If*:*
(1) *$\mathcal{K}$ is an AEC.*
(2) *$\mu \geq \operatorname{LS}(\mathcal{K})$.*
(3) *For every $M \in \mathcal{K}_\mu$, there exists an amalgamation base $M' \in \mathcal{K}_\mu$ such that $M \prec_{\mathcal{K}} M'$.*
(4) *For every amalgamation base $M \in \mathcal{K}_\mu$, there exists an amalgamation base $M' \in \mathcal{K}_\mu$ such that M' is universal over M.*
(5) *Every limit model in $\mathcal{K}_\mu$ is an amalgamation base.*
(6) *$\overset{*}{\downarrow}$ is as in Definition 3 with $\mathcal{K}^*$ the class of amalgamation bases in $\mathcal{K}_\mu$ (ordered with the strong substructure relation inherited from $\mathcal{K}$).*
(7) *$\overset{*}{\downarrow}$ satisfies invariance (I) and monotonicity (M).*
(8) *$\overset{*}{\downarrow}$ has weak universal local character at* some *cardinal $\sigma < \mu^+$.*
(9) *$\mathcal{K}$ has an Ehrenfeucht-Mostowski (EM) blueprint Φ with $|\tau(\Phi)| \leq \mu$ such that every $M \in \mathcal{K}_{[\mu,\mu^+]}$ embeds inside $\operatorname{EM}_\tau(\mu^+, \Phi)$ (where we write $\tau := \tau(\mathcal{K})$).*

Then $\overset{*}{\downarrow}$ *has* strong *universal local character at* all *limit ordinals $\alpha < \mu^+$.*

Remark 6. *As in [SV99], when we say that M is an* amalgamation base *we mean that it is an amalgamation base in the class $\mathcal{K}_{\|M\|}$, i.e. we do* not *require that larger models can be amalgamated over M.*

Before proving Theorem 5, we give several contexts in which its hypotheses hold. This shows in particular that Fact 1 follows from Theorem 5.

Corollary 7. *Let $\mathcal{K}$ be an AEC with arbitrarily large models. Let $\operatorname{LS}(\mathcal{K}) \leq \mu < \lambda$ and assume that $\mathcal{K}$ is categorical in λ and $\mathcal{K}_{<\lambda}$ has no maximal models. Then:*

(1) *If $\diamondsuit_{S^{\mu^+}_{\operatorname{cf}\mu}}$ holds, then the hypotheses of Theorem 5 hold with $\overset{*}{\downarrow}$ being non-μ-splitting.*

(2) *If $\mathcal{K}_\mu$ has amalgamation, then:*
   (a) *The hypotheses of Theorem 5 hold with $\overset{*}{\mathop{\smile\hspace{-0.9em}\vert}}$ being non-μ-splitting.*
   (b) *If $\kappa \in (\text{LS}(\mathcal{K}), \mu)$ is such that $\mathcal{K}$ does not have the weak κ-order property, then the hypotheses of Theorem 5 hold with $\overset{*}{\mathop{\smile\hspace{-0.9em}\vert}}$ being $(< \kappa)$-coheir (see [BG]).*

Proof. Fix an EM blueprint Ψ for $\mathcal{K}$ (with $|\tau(\Psi)| \leq \mu$). We first show that there exists an EM blueprint Φ with $|\tau(\Phi)| \leq \mu$ such that any $M \in \mathcal{K}_{[\mu,\mu^+]}$ embeds inside $\text{EM}_\tau(\mu^+, \Phi)$. Let $M \in \mathcal{K}_{[\mu,\mu^+]}$. Using no maximal models and categoricity, M embeds inside $\text{EM}_\tau(\lambda, \Psi)$, and hence inside $\text{EM}_\tau(S, \Psi)$ for some $S \subseteq \lambda$ with $|S| \leq \mu^+$. Therefore M also embeds inside $\text{EM}_\tau(\alpha, \Psi)$, where $\alpha := \text{otp}(S) < \mu^{++}$. Now it is well known (see e.g. [Bal09, Claim 15.5]) that α embeds inside $\text{EM}_\tau({}^{<\omega}\mu^+, \Phi)$. The class $\{{}^{<\omega}I \mid I \text{ is a linear order}\}$ is an AEC, therefore by composing EM blueprints there exists an EM blueprint Φ for $\mathcal{K}$ such that $|\tau(\Phi)| \leq \mu$ and $\text{EM}_\tau(I, \Phi) = \text{EM}_\tau({}^{<\omega}I, \Psi)$ for any linear order I. In particular, M embeds inside $\text{EM}_\tau(\mu^+, \Phi)$, as desired.

As for the hypotheses on density of amalgamation bases, existence of universal extension, and limit models being an amalgamation base, in the first context this is proven in [SV99] (note that $\diamondsuit_{S^{\mu^+}_{\text{cf}\,\mu}}$ implies $2^\mu = \mu^+$). When $\mathcal{K}_\mu$ has full amalgamation, only existence of universal extension is nontrivial. It is stated for example as [She99, Lemma 2.2]; see [GV06, Claim 2.9] for a proof.

In all the contexts given, it is trivial that $\overset{*}{\mathop{\smile\hspace{-0.9em}\vert}}$ satisfies (I) and (M). In the first context, it can be shown that non μ-splitting has weak universal local character at any $\sigma < \mu^+$ such that $2^\sigma > \mu$ (see the proof of case (c) in [SV99, Theorem 2.2.1] or [Bal09, Lemma 12.2]). Of course, this also holds when $\mathcal{K}_\mu$ has full amalgamation. As for $(< \kappa)$-coheir, it has weak universal local character at any $\sigma < \mu^+$ such that $2^\sigma > \kappa$. This is given by the proof of [BG, Theorem 6.6] (note that using a back and forth argument, one can assume without loss of generality that any M_{i+1} in the chain is κ-saturated). $\square$

Some of the hypotheses of Theorem 5 may appear technical. Let us give a little more motivation. Hypotheses (3-5) are the statements that Shelah and Villaveces derive (assuming instances of diamond) from categoricity and no maximal models. It is well known that they hold in AECs with amalgamation. Note that (4) implies stability in μ. As for (8) it can be seen as a consequence of stability (akin to "every type does not fork over a set of size less than μ"). We have seen that (9) is implied by categoricity but it is really weaker: it is a weak version of solvability (a property that Shelah [She09, Chapter IV] has introduced as a potential definition of superstability in AECs). It can be shown that (8) holds in any superstable first-order theory, see [GV, Section 5]). See also [Vasb] for more applications of solvability.

1.5. **Gaps in the Shelah-Villaveces proof.** The proof of the main theorem follows the proof of [SV99, Theorem 2.2.1], but some changes had to be made. In a preliminary version of [BG], the proof of Theorem 6.8 referred to the argument used in [SV99, Theorem 2.2.1]. The referee of [BG] insisted that the full argument necessary for Theorem 6.8 be included. After looking closely at the argument in [SV99], we concluded that there was a small gap in the division of cases and a need to specify the exact use of the club guessing principle that they imply.

More specifically, Shelah and Villaveces [SV99, Theorem 2.2.1] assume for a contradiction that no long splitting chains fails and can divide the situation into three cases, (a), (b), and (c). In the division into cases [SV99, Claim 2.2.3], just after the statement of property $\otimes_i$, Shelah and Villaveces claim that they can "repeat the procedure above" on a certain chain of models of length μ. However the "procedure above" was used on a chain of length σ, where σ is a *regular* cardinal and regularity was used in the proof. As μ is a potentially singular cardinal, there is a problem (this is addressed here in Lemma 9.(5)).

Once the division of cases is done, Shelah and Villaveces prove that cases (a), (b), (c) contradict categoricity. When proving this for (b), they use a club-guessing principle for μ^+ on the stationary set of points of cofinality σ (see Fact 11). The principle only holds when $\sigma < \mu$, so the case $\sigma = \mu$ is missing (this is addressed here by a division into cases in step (3) of the proof of Theorem 5 at the end of this paper).

1.6. **Other advantages of the main theorem.** The discussion above gave many results that rely on the Shelah-Villaveces theorem, hence in our opinion it is a very important result. This is why we give a detailed, corrected, and generalized proof of the main theorem that does not rely on any of the material in [SV99]. As should be clear from Corollary 7, another advantage of our main theorem is that it separates the combinatorial set theory from the model theory (it holds in ZFC) and also shows that there is nothing special about splitting in the Shelah-Villaveces paper.

Some results here have independent interest. For example, any independence relation with invariance and monotonicity has (assuming categoricity) a certain continuity property (Lemma 10). Variants of this have recently been used by the fourth author to study stable (not necessarily superstable) AECs [Vasf].

1.7. **Acknowledgments.** This paper was written while the fourth author was working on a Ph.D. thesis under the direction of the second author at Carnegie Mellon University and he would like to thank Professor Grossberg for his guidance and assistance in his research in general and in this work specifically.

2. Proof of the main theorem

The proof of Theorem 5 has two steps. First, we study two more variations on local character: continuity and absence of alternations. We show that if strong local character fails but enough weak local character holds, then there must be some failure of continuity, or some alternation. Second, we show that neither continuity nor alternation can happen if the categoricity (or more precisely existence of a universal EM model in μ^+) hypothesis holds. The first step uses the weak local character (but not categoricity, it is essentially forking calculus) but the second does not (but does use categoricity).

The precise definitions of continuity and alternations are as follows.

Definition 8. *Let $\mathcal{K}^*$ and $\overset{*}{\mathop{\smile}\limits^{|}}$ be as in Definition 3 and let α be a limit ordinal.*

(1) $\overset{*}{\mathop{\smile}\limits^{|}}$ *has* universal continuity at α *if for any increasing continuous sequence $\langle M_i \in \mathcal{K}^* \mid i \le \alpha \rangle$ and any type $p \in \mathrm{gS}(M_\alpha)$, if for each $i < \alpha$ M_{i+1} is universal over M_i and $p \restriction M_i$ does not $*$-fork over M_0, then p does not $*$-fork over M_0.*

(2) *For $\delta < \mu^+$ a limit,* $\overset{*}{\mathop{\smile}\limits^{|}}$ *has* no δ-limit alternations at α *if for any increasing continuous sequence $\langle M_i \in \mathcal{K}^* \mid i \le \alpha \rangle$ with M_{i+1} (μ, δ)-limit over M_i for all $i < \alpha$ and any type $p \in \mathrm{gS}(M_\alpha)$, there exists $i < \alpha$ such that the following fails: $p \restriction M_{2i+1}$ $*$-forks over M_{2i} and $p \restriction M_{2i+2}$ does not $*$-fork over M_{2i+1}. If this fails, we say that* $\overset{*}{\mathop{\smile}\limits^{|}}$ has δ-limit alternations at α.

Note that the failure of universal continuity and no δ-limit alternation correspond respectively to cases (a) and (b) in the proof of [SV99, Theorem 2.2.1]. Case (c) there corresponds to failure of weak universal local character at μ (which is assumed to hold here, see (8) of Theorem 5). The following lemma implements the first step described at the beginning of this section. In particular, (7) below says that if we can prove weak local character at some σ, continuity and no alternations at *all* α, then strong local character at all α follows.

Lemma 9. *Assume (1)-(7) from the statement of Theorem 5. Let $\alpha < \mu^+$ be a regular cardinal, $\sigma < \mu^+$ be a (not necessarily regular) cardinal, and $\delta < \mu^+$ be a limit ordinal.*

(1) *If $\overset{*}{\mathop{\smile}\limits^{|}}$ has weak universal local character at σ, then $\overset{*}{\mathop{\smile}\limits^{|}}$ has no δ-limit alternations at σ.*

(2) *If $\overset{*}{\mathop{\smile}\limits^{|}}$ has universal continuity at α and weak universal local character at α, then $\overset{*}{\mathop{\smile}\limits^{|}}$ has strong universal local character at α.*

(3) *We obtain an equivalent definition of weak universal local character (or strong local character) at σ, if in the statement we ask in addition that "M_{i+1} is (μ, δ)-limit over M_i" for all $i < \sigma$.*

(4) *Assume that* $\overset{*}{\perp}$ *has weak universal local character at* σ. *Let* $\langle M_i : i \leq \sigma \rangle$ *be increasing continuous in* $\mathcal{K}^*$ *with* M_{i+1} *universal over* M_i *for all* $i < \sigma$. *For any* $p \in \text{gS}(M_\sigma)$ *there exists a* successor $i < \sigma$ *such that* $p \restriction M_{i+1}$ *does not* $*$*-fork over* M_i.

(5) *If* $\overset{*}{\perp}$ *has universal continuity at* σ, *weak universal local character at* σ, *and no* δ*-limit alternations at* ω, *then* $\overset{*}{\perp}$ *has strong universal local character at* σ.

(6) *Assume that* $\overset{*}{\perp}$ *has strong universal local character at* σ. *If* $\overset{*}{\perp}$ *does* not *have weak universal local character at* α, *then* $\overset{*}{\perp}$ *has* σ*-limit alternations at* α.

(7) *Assume that* $\overset{*}{\perp}$ *has* weak *universal local character at* σ. *If* $\overset{*}{\perp}$ *has universal continuity at* α *and* σ, $\overset{*}{\perp}$ *has no* σ*-limit alternations at* ω, *and* $\overset{*}{\perp}$ *has no* σ*-limit alternations at* α, *then* $\overset{*}{\perp}$ *has strong universal local character at* α.

Proof.

(1) Fix $\langle M_i : i \leq \alpha \rangle$, δ, p as in the definition of having no δ-limit alternations. Apply weak universal local character to the chain $\langle M_{2i} : i \leq \alpha \rangle$.

(2) Suppose that $\langle M_i : i \leq \alpha \rangle$, p is a counterexample.

Claim: For each $i < \alpha$, there exists $j_i \in (i, \alpha)$ such that $p \restriction M_{j_i}$ $*$-forks over M_i.

Proof of Claim: If $i < \alpha$ is such that for all $j \in (i, \alpha)$, $p \restriction M_j$ does not $*$-fork over M_i, then applying universal continuity at α on the chain $\langle M_k : k \in [i, \alpha] \rangle$ we would get that p does not $*$-fork over M_i, contradicting the choice of $\langle M_i : i \leq \alpha \rangle$, p. $\dagger_{\text{Claim}}$

Now define inductively for $i \leq \alpha$, $k_0 := 0$, $k_{i+1} := j_{k_i}$, $k_i := \sup_{j<i} k_j$. Note that $\langle k_i : i \leq \alpha \rangle$ is strictly increasing continuous and $i < \alpha$ implies $k_i < \alpha$ (this uses regularity of α; when α is singular, see (5)).

Apply weak universal local character to the chain $\langle M_{k_i} : i \leq \alpha \rangle$ and the type p. We get that there exists $i < \alpha$ such that $p \restriction M_{k_{i+1}}$ does not $*$-fork over M_{k_i}. This is a contradiction since $k_{i+1} = j_{k_i}$ and we chose j_{k_i} so that $p \restriction M_{j_{k_i}}$ $*$-forks over M_{k_i}.

(3) We do it for weak universal local character, and the proof for the strong version is similar. Fix $\langle M_i^0 : i \leq \sigma \rangle$, p witnessing failure of weak universal local character at σ. We build a witness of failure $\langle M_i : i \leq \sigma \rangle$, p such that $M_\sigma = M_\sigma^0$, and M_{i+1} is (μ, δ)-limit over M_i for each $i < \alpha$. Using existence of universal extensions, we can extend each M_i^0 to M_i^* that is (μ, δ)-limit over M_i^0. Since M_{i+1}^0 is universal over M_i^0, we can find $f_i : M_{i+1}^* \rightarrow_{M_i^0} M_{i+1}^0$.

Since limit models are amalgamation bases, $f_i(M^*_{i+1})$ is an amalgamation base. Now set $M^1_i := M^0_i$ for $i \leq \sigma$ limit or 0 and $M^1_{i+1} := f_i(M^*_{i+1})$. This is an increasing continuous chain of amalgamation bases with M^1_{i+1} (μ, δ)-limit over M^1_i. Let $M_i := M^1_{2i}$.

This works: if there was an $i < \sigma$ such that $p \restriction M_{i+1}$ does not $*$-fork over M_i, this would mean that $p \restriction M^1_{2i+2}$ does not $*$-fork over M^1_{2i}, but since $M^1_{2i} \prec_{\mathcal{K}^*} M^0_{2i+1} \prec_{\mathcal{K}^*} M^0_{2i+2} \prec_{\mathcal{K}^*} M^1_{2i+2}$, we have by (M) that $p \restriction M^0_{2i+2}$ does not $*$-fork over M^0_{2i+1}, a contradiction.

(4) Apply weak universal local character to the chain $\langle M_{2i} : i < \sigma \rangle$ to get $j < \sigma$ such that $p \restriction M_{2j+2}$ does not $*$-fork over M_{2j}. By monotonicity, this implies that $p \restriction M_{2j+2}$ does not $*$-fork over M_{2j+1}. Let $i := 2j + 1$.

(5) Suppose not, and let $\langle M_i : i \leq \sigma \rangle, p$ be a counterexample. By (3), without loss of generality M_{i+1} is (μ, δ)-limit over M_i for all $i < \delta$. As in the proof of (2), for each $i < \sigma$, there exists $j_i \in [i, \sigma)$ such that $p \restriction M_{j_i}$ $*$-forks over M_i. On the other hand, applying (4) to the chain $\langle M_j : j \in [j_i, \sigma] \rangle$, for each $i < \sigma$, there exists a *successor* ordinal $k_i \geq j_i$ such that $p \restriction M_{k_i+1}$ does not $*$-fork over M_{k_i}. Define by induction on $n \leq \omega$, $m_0 := 0$, $m_{2n+1} := k_{m_{2n}}$, $m_{2n+2} := k_{m_{2n}} + 1$, and $m_\omega := \sup_{n<\omega} m_n$. By construction, the sequence $\langle M_{m_n} : n \leq \omega \rangle$ witnesses that $\overset{*}{\perp}$ has δ-limit alternations at ω, a contradiction.

(6) Let $\gamma := \sigma \cdot \sigma$. By (3), there exists $\langle M_i : i \leq \alpha \rangle$, p witnessing failure of weak universal local character at α such that for all $i < \alpha$, M_{i+1} is (μ, γ)-limit over M_i. Let $\langle M_{i,j} : j \leq \gamma \rangle$ witness that M_{i+1} is (μ, γ)-limit over M_i (i.e. it is increasing continuous with $M_{i,j+1}$ universal over $M_{i,j}$ for all $j < \gamma$, $M_{i,0} = M_i$, and $M_{i,\delta} = M_{i+1}$). By strong universal local character at σ, for all $i < \alpha$, there exists $j_i < \gamma$ such that $p \restriction M_{i+1}$ does not $*$-fork over M_{i,j_i}. By replacing j_i by $j_i + \sigma$ if necessary we can assume without loss of generality that $\operatorname{cf} j_i = \operatorname{cf} \sigma$.

Observe also that for any $i < \alpha$, $p \restriction M_{i+1,j_i}$ $*$-forks over M_i (using (M) and since by assumption $p \restriction M_{i+1}$ $*$-forks over M_i). Therefore $\langle M_0, M_{1,j_1}, M_2, M_{3,j_3}, \ldots \rangle$, p witness that $\overset{*}{\perp}$ has σ-limit alternations at α.

(7) By (5), $\overset{*}{\perp}$ has strong universal local character at σ. By the contrapositive of (6), $\overset{*}{\perp}$ has weak universal local character at α. By (2), $\overset{*}{\perp}$ has strong universal local character at α.

□

The next lemma corresponds to the second step outlined at the beginning of this section. Note that in contrast to Lemma 9 we are assuming (9) from Theorem 5.

Lemma 10. *Assume (1)-(7) and (9) in Theorem 5. Let $\alpha < \mu^+$ be a regular cardinal. Then:*

(1) $\overset{*}{\unionsq}$ *has universal continuity at α.*
(2) *If in addition $\alpha < \mu$, then for any limit $\gamma < \mu^+$, $\overset{*}{\unionsq}$ has no γ-limit alternations at α.*

Proof. Let $\langle M_i \mid i \leq \alpha \rangle$ and p be as in the definition of universal continuity or γ-limit alternations. Let $S^{\mu^+}_\alpha := \{\delta < \mu^+ \mid \operatorname{cf}\delta = \alpha\}$. We say that $\bar{C} = \langle C_\delta \mid \delta \in S^{\mu^+}_\alpha \rangle$ is an $S^{\mu^+}_\alpha$*-club sequence* if each $C_\delta \subseteq \delta$ is club. Clearly, club sequences exist: just take $C_\delta := \delta$ (this will be enough for proving universal continuity). Shelah [She94] proves the existence of club-guessing club sequences in ZFC under various hypotheses. We will describe a construction of a sequence of models $\bar{N}(\bar{C})$ based on a club sequence and then plug in the necessary club sequence in each case.

Given an $S^{\mu^+}_\alpha$-club sequence $\bar{C}$, enumerate $C_\delta \cup \{\delta\}$ in increasing order as $\langle \beta_{\delta,j} \mid j \leq \alpha \rangle$.

Claim: Let $\gamma < \mu^+$ be a limit ordinal. We can build increasing, continuous $\bar{N}(\bar{C}) = \langle N_i \in \mathcal{K}^* \mid i < \mu^+ \rangle$ such that for all $i < \mu^+$:

(1) N_{i+1} is (μ, γ)-limit over N_i;
(2) when $i \in S^{\mu^+}_\alpha$, there is $g_i : M_\alpha \cong N_i$ such that $g_i(M_j) = N_{\beta_{i,j}}$ for all $j \leq \alpha$; and:
(3) when $i \in S^{\mu^+}_\alpha$, there is $a_i \in N_{i+1}$ that realizes $g_i(p)$.

Proof of Claim: Build the increasing continuous chain of models as follows: start with an amalgamation base N_0 (which exists as we are assuming (3) of Theorem 5). Given an amalgamation base N_i, build N_{i+1} to be (μ, γ)-limit over it. This exists by (4) of Theorem 5), and N_{i+1} is an amalgamation base by (5) there. At limits, it also guarantees we have an amalgamation base.

At limits i of cofinality α, use the uniqueness of (μ, γ)-limits models to find the desired isomorphisms: the weak version gives $M_0 \cong M_{\beta_{i,0}}$, and the strong (over the base) version allows this isomorphism to be extended to get an isomorphism g_i between $\langle M_j \mid j \leq \alpha \rangle$ and $\langle N_{\beta_{i,j}} \mid j \leq \alpha \rangle$ as described. Since N_{i+1} is universal over N_i, we there is some $a_i \in N_{i+1}$ that realizes $g_i(p)$. $\dagger_{\text{Claim}}$

By (9) of Theorem 5, we may assume that $N := \bigcup_{i<\mu^+} N_i \prec_{\mathcal{K}^*} \mathrm{EM}_\tau(\mu^+, \Phi)$. Thus, we can write $a_i = \rho_i(\gamma^i_1, \ldots, \gamma^i_{n(i)})$ with:

$$\gamma^i_1 < \cdots < \gamma^i_{m(i)} < i \leq \gamma^i_{m(i)+1} < \cdots < \gamma^i_{n(i)} < \mu^+$$

Now we begin to prove each part of the lemma. In each, we will find $i_1 < i_2 \in S^{\mu^+}_\alpha$ such that $\operatorname{gtp}(a_{i_1}/N_{i_1}; N)$ and $\operatorname{gtp}(a_{i_2}/N_{i_1}; N)$ are both the same (because of the EM structure) and different (because they exhibit different $*$-forking behavior), which is our contradiction.

(1) Assume that $p \restriction M_j$ does not fork over M_0, for all $j < \alpha$.

Let $\bar{C}$ be an $S^{\mu^+}_\alpha$-club sequence, and set $\langle N_i \in \mathcal{K}^* \mid i < \mu^+ \rangle = \bar{N}(\bar{C})$ as in the Claim (the value of γ doesn't matter here, e.g. take $\gamma := \omega$). By Fodor's Lemma, there is a stationary subset $S^* \subseteq S^{\mu^+}_\alpha$, a term ρ_*, $m_*, n_* < \omega$ and ordinals $\gamma^*_0, \dots \gamma_{n_*}$, $\beta_{*,0}$ such that:

For every $i \in S^*$, we have $\rho_i = \rho_*$; $n(i) = n_*$; $m(i) = m_*$; $\gamma^i_j = \gamma^*_j$ for $j \leq m_*$; and $\beta_{i,0} = \beta_{*,0}$.

Set $E := \{\delta < \mu^+ \mid \delta \text{ is limit and } \mathrm{EM}_\tau(\delta, \Phi) \cap N = N_\delta\}$. This is a club. Let $i_1 < i_2$ both be in $S^* \cap E$. Then we have:

$$\begin{aligned} \mathrm{gtp}\,(a_{i_1}/N_{i_1}) &= \mathrm{gtp}\,\big(\rho_*(\gamma^*_1, \dots, \gamma^*_{m_*}, \gamma^{i_1}_{m_*+1}, \dots, \gamma^{i_1}_{n_*})/N \cap \mathrm{EM}_\tau(i_1, \Phi)\big) \\ &= \mathrm{gtp}\,\big(\rho_*(\gamma^*_1, \dots, \gamma^*_{m_*}, \gamma^{i_2}_{m_*+1}, \dots, \gamma^{i_2}_{n_*})/N \cap \mathrm{EM}_\tau(i_1, \Phi)\big) \\ &= \mathrm{gtp}\,(a_{i_2}/N_{i_1}) \end{aligned}$$

where all the types are computed inside N. This is because the only differences between a_{i_1} and a_{i_2} lie entirely above i_1.

We have that $g_{i_1} : (N_{i_1}, N_{\beta_{*,0}}) \cong (M_\alpha, M_0)$ and that p $*$-forks over M_0. Thus, $\mathrm{gtp}(a_{i_1}/N_{i_1}) = g_{i_1}(p)$ $*$-forks over $N_{\beta_{*,0}}$. On the other hand, C_{i_2} is cofinal in i_2, so there is $j < \alpha$ such that $\beta_{i_2,j} > i_1$ and, thus, $N_{i_1} \prec_{\mathcal{K}^*} N_{\beta_{i_2,j}}$. Again, $g_{i_2} : (N_{\beta_{i_2,j}}, N_{\beta_{*,0}}) \cong (M_j, M_0)$ and $p \restriction M_j$ does not $*$-fork over M_0 by assumption. Thus, $\mathrm{gtp}(a_{i_2}/N_{\beta_{i_2,j}}) = g_{i_2}(p \restriction M_j)$ does not $*$-fork over $N_{\beta_{*,0}}$. By monotonicity (M), $\mathrm{gtp}(a_{i_2}/N_{i_1})$ does not $*$-fork over $N_{\beta_{*,0}}$. Thus, $\mathrm{gtp}(a_{i_1}/N_{i_1}) \neq \mathrm{gtp}(a_{i_2}/N_{i_2})$, a contradiction.

(2) Let χ be a big-enough cardinal and create an increasing, continuous elementary chain of models of set theory $\langle \mathfrak{B}_i \mid i < \mu^+ \rangle$ such that for all $i < \mu^+$:
   (a) $\mathfrak{B}_i \prec (H(\chi), \in)$;
   (b) $\|\mathfrak{B}_i\| = \mu$;
   (c) $\mathfrak{B}_0$ contains, as elements[3], Φ, $\mathrm{EM}(\mu^+, \Phi)$, h, μ^+, $\langle N_i \mid i < \mu^+ \rangle$, $S^{\mu^+}_\alpha$, $\langle a_i \mid i \in S^{\mu^+}_\alpha \rangle$, and each $f \in \tau(\Phi)$; and
   (d) $\mathfrak{B}_i \cap \mu^+$ is an ordinal.

We will use the following fact which was originally proven in [She94, III.2] (or see [AM10, Theorem 2.17, Exercise 2.18.2]).

Fact 11. *Let λ be a cardinal such that $\mathrm{cf}\,\lambda \geq \theta^{++}$ for some regular θ and let $S \subseteq S^\lambda_\theta$ be stationary. Then there is a S-club sequence $\langle C_\delta \mid \delta \in S \rangle$ such that, if $E \subseteq \lambda$ is club, then there are stationarily many $\delta \in S$ such that $C_\delta \subseteq E$.*

[3]When we say that $\mathfrak{B}_0$ contains a sequence as an element, we mean that it contains the function that maps an index to its sequence element.

We have that $\alpha < \mu$, so we can apply Fact 11 with λ, θ, S there standing for $\mu^+, \alpha, S^{\mu^+}_\alpha$ here. Let $\bar{C}$ be the $S^{\mu^+}_\alpha$-club sequence that the fact gives. Let $\langle N_i \in K_\mu \mid i < \mu^+ \rangle = \bar{N}(\bar{C})$ be as in the Claim. Note that $E := \{i < \mu^+ \mid \mathfrak{B}_i \cap \mu^+ = i\}$ is a club. By the conclusion of Fact 11, there is some $i_2 \in S^{\mu^+}_\alpha$ such that $C_{i_2} \subseteq E$. We have $a_{i_2} = \rho_{i_2}(\gamma_1^{i_2}, \dots, \gamma^{i_2}_{n(i_2)})$, with:

$$\gamma_1^{i_2} < \dots < \gamma^{i_2}_{m(i_2)} < i_2 \leq \gamma^{i_2}_{m(i_2)+1} < \dots < \gamma^{i_2}_{n(i_2)}$$

Since the $\beta_{i_2,j}$'s enumerates a cofinal sequence in i_2, we can find $j < \alpha$ such that $\gamma^{i_2}_{m(i_2)} < \beta_{i_2,2j+1} < i$. Recall that we have $p \restriction M_{2j+2}$ does not $*$-fork over M_{2j+1} by assumption. Then $(H(\chi), \in)$ satisfies the following formulas with parameters exactly the objects listed in item (2c) above and ordinals below $\beta_{i_2,2j+2}$:

$$\begin{aligned}\exists x, y_{m(i_2)+1}, \dots, y_{n(i)}.(&\text{“}x \in S^{\mu^+}_\alpha\text{”}\\ &\wedge \text{“}x > \beta_{i_2,2j+1}\text{”} \wedge \text{“}y_k \in (x, \mu^+) \text{ are increasing ordinals”}\\ &\wedge \text{“}a_x = \rho_{i_2}(\gamma_1^{i_2}, \dots, \gamma^{i_2}_{m(i_2)}, y_{m(i_2)+1}, \dots, y_{n(i_2)})\text{”}\\ &\wedge \text{“}N_x \subset \mathrm{EM}(x, \Phi)\text{”})\end{aligned}$$

This is witnessed by $x = i_2$ and $y_k = \gamma_k^{i_2}$. By elementarity, $\mathfrak{B}_{\beta_{2,2j+2}}$ satisfies this formula as it contains all the parameters. Let $i_1 \in (\beta_{i_2,2j+1}, \mu^+) \cap \mathfrak{B}_{\beta_{i_2,2j+2}} = (\beta_{i_2,2j+1}, \beta_{i_2,2j+2})$[4] witness this, along with $\gamma'_{m(i_2)+1} < \dots < \gamma'_{n(i_2)} < \mu^+$. Then we have:

$$a_{i_1} = \rho_{i_2}(\gamma_1^{i_2}, \dots, \gamma^{i_2}_{m(i_2)}, \gamma'_{m(i_2)+1}, \dots, \gamma'_{n(i_2)})$$

with $\beta_{i_2,2j+1} < \gamma_{m(i_2)+1}$. We want to compare $\mathrm{gtp}(a_{i_2}/N_{i_1})$ and $\mathrm{gtp}(a_{i'}/N_{i_1})$.

- From the elementarity, we get that $N_{i_1} \subseteq \mathrm{EM}_\tau(i_1, \Phi)$. We also know that $i_1 < \beta_{i_2,2j+2} < \gamma^{i_2}_{m(i_2)+1}, \gamma'_{m(i_2)+1}$. Thus, as before, the types are equal.
- We know that $p \restriction M_{2j+2}$ does not $*$-fork over M_{2j+1}. Thus, $\mathrm{gtp}(a_{i_2}/N_{\beta_{i_2,2j+2}})$ does not $*$-fork over $N_{\beta_{i_2,2j+1}}$. Since we have $N_{\beta_{i_2,2j+1}} \prec_{\mathcal{K}^*} N_{i_1} \precneqq_{\mathcal{K}^*} N_{\beta_{i_2,2j+2}}$, this gives $\mathrm{gtp}(a_{i_2}/N_{i_1})$ does not $*$-fork over $N_{\beta_{i_2,2j+1}}$.
- We have $\beta_{i_2,2j+1} < i_1$, so there is some $k < \alpha$ such that $\beta_{i_2,2j+1} < \beta_{i_1,k} < i'$. By assumption, p $*$-forks over M_k. Thus $g_{i_1}(p)$ $*$-forks over $N_{\beta_{i_1,k}}$. Thus, $\mathrm{gtp}(a_{i_1}/N_{i_1})$ $*$-forks over $N_{\beta_{i_2,2j+1}} \prec_{\mathcal{K}^*} N_{\beta_{i_1,k}}$.

As before, these three statements contradict each other.

□

[4]The equality here is the key use of club guessing.

Proof of Theorem 5. By Remark 4, it suffices to show that $\overset{*}{\perp}$ has strong universal local character at any *regular* $\alpha < \mu^+$. Pick a cardinal $\sigma < \mu^+$ such that $\overset{*}{\perp}$ has weak universal local character at σ (exists by assumption (8)). We proceed in several steps.

(1) $\overset{*}{\perp}$ has weak universal local character at any limit $\sigma' \in [\sigma, \mu^+)$. [By Remark 4].
(2) $\overset{*}{\perp}$ has universal continuity at any regular $\alpha < \mu^+$. [By Lemma 10].
(3) For any limit $\gamma < \mu^+$ and any regular $\alpha < \mu^+$, $\overset{*}{\perp}$ has no γ-limit alternations at α. [By Lemma 10 when $\alpha < \sigma$. When $\alpha \geq \sigma$, combine (1) and Lemma 9.(1).].
(4) For any regular $\alpha < \mu^+$, $\overset{*}{\perp}$ has strong universal local character at α. [By (2), (3), and Lemma 9.(7)].

□

References

[Adl09] Hans Adler, *A geometric introduction to forking and thorn-forking*, Journal of Mathematical Logic **9** (2009), no. 1, 1–20.

[AM10] Uri Abraham and Menachem Magidor, *Cardinal arithmetic*, Handbook of set theory (Matthew Foreman and Akihiro Kanamori, eds.), Springer, 2010, pp. 1149–1227.

[Bal09] John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BG] Will Boney and Rami Grossberg, *Forking in short and tame AECs*, Preprint. URL: `http://arxiv.org/abs/1306.6562v9`.

[BK09] John T. Baldwin and Alexei Kolesnikov, *Categoricity, amalgamation, and tameness*, Israel Journal of Mathematics **170** (2009), 411–443.

[Gro02] Rami Grossberg, *Classification theory for abstract elementary classes*, Contemporary Mathematics **302** (2002), 165–204.

[GS86a] Rami Grossberg and Saharon Shelah, *A nonstructure theorem for an infinitary theory which has the unsuperstability property*, Illinois Journal of Mathematics **30** (1986), no. 2, 364–390.

[GS86b] ———, *On the number of nonisomorphic models of an infinitary theory which has the infinitary order property. Part A*, The Journal of Symbolic Logic **51** (1986), no. 2, 302–322.

[GV] Rami Grossberg and Sebastien Vasey, *Superstability in abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1507.04223v3`.

[GV06] Rami Grossberg and Monica VanDieren, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[HS90] Bradd Hart and Saharon Shelah, *Categoricity over P for first order T or categoricity for $\phi \in L_{\omega_1,\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \dots, \aleph_{k-1}$*, Israel Journal of Mathematics **70** (1990), 219–235.

[Kim98] Byunghan Kim, *Forking in simple unstable theories*, Journal of the London Mathematical Society **57** (1998), no. 2, 257–267.

[Mor65] Michael Morley, *Categoricity in power*, Transactions of the American Mathematical Society **114** (1965), 514–538.

[She70] Saharon Shelah, *Finite diagrams stable in power*, Annals of Mathematical Logic **2** (1970), no. 1, 69–118.

[She72] ______, *A combinatorial problem; stability and order for models and theories in infinitary languages*, Pacific Journal of Mathematics **41** (1972), no. 1, 247–261.

[She75] ______, *Categoricity in* $\aleph_1$ *of sentences in* $L_{\omega_1,\omega}(Q)$, Israel Journal of Mathematics **20** (1975), no. 2, 127–148.

[She78] ______, *Classification theory and the number of non-isomorphic models*, Studies in logic and the foundations of mathematics, vol. 92, North-Holland, 1978.

[She87a] ______, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.

[She87b] ______, *Universal classes*, Classification theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She94] ______, *Cardinal arithmetic*, Oxford Logic Guides, no. 29, Clarendon Press, 1994.

[She99] ______, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), no. 1, 261–294.

[She09] ______, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[SV99] Saharon Shelah and Andrés Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1999), 1–25.

[Van] Monica VanDieren, *Superstability and symmetry*, Annals of Pure and Applied Logic, To appear. URL: `http://arxiv.org/abs/1507.01990v4`.

[Van06] ______, *Categoricity in abstract elementary classes with no maximal models*, Annals of Pure and Applied Logic **141** (2006), 108–147.

[Van13] ______, *Erratum to "Categoricity in abstract elementary classes with no maximal models" [Ann. Pure Appl. Logic 141 (2006) 108-147]*, Annals of Pure and Applied Logic **164** (2013), no. 2, 131–133.

[Vasa] Sebastien Vasey, *Downward categoricity from a successor inside a good frame*, Preprint. URL: `http://arxiv.org/abs/1510.03780v5`.

[Vasb] ______, *Saturation and solvability in abstract elementary classes with amalgamation*, Preprint. URL: `http://arxiv.org/abs/1604.07743v2`.

[Vasc] ______, *Shelah's eventual categoricity conjecture in tame AECs with primes*, Preprint. URL: `http://arxiv.org/abs/1509.04102v3`.

[Vasd] ______, *Shelah's eventual categoricity conjecture in universal classes. Part I*, Preprint. URL: `http://arxiv.org/abs/1506.07024v9`.

[Vase] ______, *Shelah's eventual categoricity conjecture in universal classes. Part II*, Preprint. URL: `http://arxiv.org/abs/1602.02633v2`.

[Vasf] ______, *Toward a stability theory of tame abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1609.03252`.

[Vas16a] ______, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 11, 1029–1092.

[Vas16b] ______, *Forking and superstability in tame AECs*, The Journal of Symbolic Logic **81** (2016), no. 1, 357–383.

[VV] Monica VanDieren and Sebastien Vasey, *Symmetry in abstract elementary classes with amalgamation*, Preprint. URL: `http://arxiv.org/abs/1508.03252v3`.

Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA
E-mail address: wboney@math.harvard.edu

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA
E-mail address: rami@cmu.edu

Department of Mathematics, Robert Morris University, Moon Township Pennsylvania USA
E-mail address: vandieren@rmu.edu

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA
E-mail address: sebv@cmu.edu