Classification theory for abstract elementary classes

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ABSTRACT. In this paper some of the basics of classification theory for abstract elementary classes are discussed. Instead of working with types which are sets of formulas (in the first-order case) we deal instead with *Galois types* which are essentially orbits of automorphism groups acting on the structure.

Some of the most basic results in classification theory for non elementary classes are presented. The motivating point of view is Shelah's categoricity conjecture for $L_{\omega_1,\omega}$.

While only very basic theorems are proved, an effort is made to present number of different technologies: Flavors of weak diamond, models of weak set theories, and commutative diagrams. We focus in issues involving existence of Galois types, extensions of types and *Galois-stability*.

Introduction

In recent years the view that *stability theory* has wider applicability than the originally limited context (i.e. first-order stable theories) is getting increasing recognition among model theorists. The current interest in simple (first-order) theories and beyond signifies a shift in the opinion of many that similar tools and concepts to those of basic stability theory can be developed and are relevant in a wider context. Much of Shelah's effort in model theory in the last 18-19 years is directed toward development of *classification theory for non elementary classes*. I feel that the study of *classification theory for non elementary classes* will not only provide us with a better understanding of classical stability (and simplicity) theory but also will develop new tools and concepts that will be useful in projecting new light on classical problems of "main stream" mathematics.

The purpose of this article is to present some of the basics of classification theory for non elementary classes. For several reasons I will concentrate in what I consider to be the most important and challenging framework: *Abstract Elementary Classes*, however this is not the only important framework for such a classification theory.

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I made an effort to keep most technical details to a minimum, while still having some deep ideas of the subject explained and part of the overall picture described. In the last section I discuss direction for future developments and some open problems.

An interested reader can find more details in Chapter 13 of [**Gr**].

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1. definitions and examples

DEFINITION 1.1. *Let $\mathcal{K}$ be a class of structures all of the same similarity type* $\mathrm{L}(\mathcal{K})$. *$\mathcal{K}$ is an* Elementary class *if there exists a first-order T in* $\mathrm{L}(\mathcal{K})$ *such that* $\mathcal{K} = \mathrm{Mod}(T)$.

There are *very many* natural classes that are not elementary:

- Archimedean ordered fields,
- locally-finite groups,
- well-ordered sets,
- Noetherian rings and
- the class of algebraically closed fields with infinite transcendence degree.

Extensions of predicate calculus permit a model-theoretic treatment of the above:

1. *The basic infinitary languages:* $L_{\omega_1,\omega} \subseteq L_{\lambda^+,\omega} \subseteq L_{\lambda^+,\mu}$ and $L_{\infty,\omega}$. The earliest work on infinitary logic was published in 1931 by Ernst Zermelo [**Z**].

   Later pioneering work was done by: Novikoff (1939, 1943), Bochvar (1940) ([**Bo**]) and from the late forties to the sixties the main contributors were: Tarski, Hanf, Erdős, Henkin, Chang, Scott, Karp, Lopez-Escobar, Morley, Makkai, Kueker and Keisler.

   Recall

   DEFINITION 1.2.
   - *$L_{\omega_1,\omega}$ contains all the first-order formulas in the language L and is closed under propositional connectives, first-order quantification and the rule:*
     If $\{\varphi_n(\mathbf{x}) \mid n < \omega\} \subseteq L_{\omega_1,\omega}$ *then*
     $$\bigvee_{n<\omega} \varphi_n(\mathbf{x}) \in L_{\omega_1,\omega}, \quad \ell(\mathbf{x}) < \omega.$$
     If $\{\varphi_\alpha(\mathbf{x}) \mid \alpha < \lambda\} \subseteq L_{\lambda^+,\omega}$ *then*
     $$\bigvee_{\alpha<\lambda} \varphi_\alpha(\mathbf{x}) \in L_{\lambda^+,\omega}.$$
   - *If* $\{\varphi_\alpha(\mathbf{x}) \mid \alpha < \lambda\} \subseteq L_{\lambda^+,\mu}$ *with* $\ell(\mathbf{x}) < \mu$ *then*
     $$\bigvee_{\alpha<\lambda} \varphi_\alpha(\mathbf{x}) \in L_{\lambda^+,\mu}$$

and for every sequence of variables $\langle x_\beta \mid \beta < \alpha < \mu \rangle$

$$\mathbf{Q}x_0\mathbf{Q}x_1 \cdots \mathbf{Q}x_\beta \cdots \varphi(\langle x_\beta \mid \beta < \alpha \rangle) \in L_{\lambda^+,\mu}$$

for $\mathbf{Q} \in \{\forall, \exists\}$.

- *For a limit cardinal* χ*:*

$$L_{\chi,\mu} := \bigcup_{\lambda<\chi} L_{\lambda^+,\mu}.$$

- $L_{\infty,\mu}$ *is the proper class obtained by letting* λ *vary over all cardinals. I.e. it is*

$$L_{\infty,\mu} := \bigcup_{\lambda \in \text{Card}} L_{\lambda^+,\mu}.$$

Basic references: For $L_{\omega_1,\omega}$ see H. J. Keisler [**Ke2**] and for $L_{\lambda^+,\mu}$ see M. A. Dickmann's [**Di**] books.

2. *Cardinality quantifiers:* Andrzej Mostowski in 1957 (see [**Mos**]) introduced several cardinality quantifiers. The most popular among them was studied extensively in the sixties by Gerhard Fuhrken [**Fur**] and Jerry Keisler [**Ke1**]. It is the $\aleph_1$-interpretation: $L(\mathbf{Q})$ and extensions like $L_{\omega_1,\omega}(\mathbf{Q})$ where

$$M \models \mathbf{Q}x\varphi(x) \iff \{a \in |M| \ : \ M \models \varphi[a]\} \quad \text{is uncountable.}$$

3. *Other second order quantifiers:* Measure-theoretic quantifiers, and topological logics. Barwise, Makkai, Kaufmann, Flum, Ebbinghaus and Ziegler.
4. *Other logics:* Monadic logics. Rabin, Gurevich, Büchi, Shelah.

There are many more logics, see the volume edited by Jon Barwise and Solomon Feferman [**BaFe**].

There is a particularly rich model theory for $L(\mathbf{Q})$. In the last 20 years, this theory took on a set-theoretic flavor, see: Fuhrken [**Fur**], Keisler [**Ke1**], [**Sh 82**], [**Gr2**] and [**HLSh**]. For an overview of some of this I recommend Wilfrid Hodges's book [**Ho1**].

A common feature of all the above extensions of first order logic is the failure of the compactness theorem.

An ordered field $\langle F, +, \cdot, \leq \rangle$ is archimedean iff

$$\langle F, +, \cdot, \leq \rangle \models \forall x \Big[x > 0 \to \bigvee_{n<\omega} [\underbrace{x + \ldots + x}_{n\text{-times}} \geq 1] \Big].$$

A group G is periodic iff

$$G \models \forall x \bigvee_{n<\omega} [x^n = 1].$$

DEFINITION 1.3. *Let* M *and* N *both be* L*-structures. Suppose that* $A \subseteq |M|$*, and* $B \subseteq |N|$*. A function* $f : A \to B$ *is called a* partial isomorphism *iff it is a bijection and for every relation symbol* $R(\mathbf{x})$ *and every* $\mathbf{a} \in A$ *we have that* $\mathbf{a} \in R^M \iff f(\mathbf{a}) \in R^N$*, for every function symbol* $F(\mathbf{x})$ *we have that*

$f(F^M(\mathbf{a})) = F^N(f(\mathbf{a}))$ *and for every constant symbol* c *if* $c^M \in \mathrm{Dom}(f)$ *then* $f(c^M) = c^N$.

A family $\mathcal{F}$ *of partial isomorphisms from* M *into* N *has the* back and forth property *iff*
(forward) for every $a \in |M|$ *and every* $g \in \mathcal{F}$ *there exists* $h \in \mathcal{F}$ *such that* $g \subseteq h$ *and* $a \in \mathrm{Dom}(h)$ *and*
(back) for every $b \in |N|$ *for every* $g \in \mathcal{F}$ *there exists* $h \in \mathcal{F}$ *such that* $g \subseteq h$ *and* $b \in \mathrm{Rang}(h)$.

Denote this by $\mathcal{F} : M \cong^p N$ *. We write* $M \cong^p N$ *for "there exists a non empty* $\mathcal{F}$ *such that* $\mathcal{F} : M \cong^p N$".

At first glance the following special case of a theorem of Carol Karp may look a bit surprising:

FACT 1.4 (Karp's test). *Let* M *and* N *be* L*-structures*.

$$M \cong^p N \iff M \equiv_{\infty,\omega} N.$$

Shelah's plenary talk at the International Congress of Mathematicians in Berkeley in 1986 (see [**Sh 299**] and [**Sh tape**]) was dedicated to classification theory for non elementary classes and in particular universal classes.

There are several frameworks for classification theory for non elementary classes. With the exception of the last item they are listed in (more or less) increasing level of generality:

- *Homogeneous model theory* (this was formerly called *Finite diagrams stable in power*). In this context we have a fixed (sequentially) homogeneous monster model $\mathbf{M}$ and limit the study to elementary submodels of $\mathbf{M}$ and its subsets. The subject was started by Shelah in 1970 with [**Sh3**] and continued by him with [**Sh54**]. In the last 10-15 years the subject was revived in several publications. See [**Gr3**], [**Gr4**],[**HySh1**], [**HySh2**], [**GrLe1**], [**GrLe3**], [**Le1**], [**Le2**], [**BuLe**], [**Be**] and [**Kov1**].
- *Submodels of a given structure*. This is a generalization of homogeneous model theory. Start with a given model $\mathbf{M}$ (*not necessarily homogeneous*) and limit the study to submodels of $\mathbf{M}$ and its subsets. Started in Grossberg [**Gr3**], [**Gr4**]. Further progress is in [**GrLe2**].
- *Excellent classes*. An excellent class consists of the atomic models of a first-order countable theory which satisfy a very strong amalgamation property: The $(\aleph_0, n)$-goodness for every $n < \omega$. Shelah introduced them in [**Sh87a**], [**Sh87b**] as a tool to analyze $\mathrm{Mod}(\psi)$ when $\psi \in L_{\omega_1,\omega}$ under the model theoretic assumption that $I(\aleph_n, \psi) < 2^{\aleph_n}$ holds for every positive integer n. Later in [**Sh c**] Shelah used excellent classes at a crucial point in his proof of the main gap for first-order theories, $(\aleph_0, n)$-goodness is renamed in [**Sh c**], page 616 as $(\aleph_0, n)$-existence property. In section 5 of chapter XII Shelah essentially show that for countable and superstable T without the dop, notop is equivalent to the $(\aleph_0, 2)$-existence property which using his resolution technique from [**Sh87b**] implies excellence. Lately Kolesnikov [**Kov2**] announced an n-dimensional analysis to get new results for simple unstable

theories. Grossberg and Hart in [**GrHa**] developed orthogonality calculus to the level that permitted deriving the main gap for excellent classes. This was the first time that a main gap was proved for non elementary classes.

- *Universal classes*. The prototypical example is $\mathrm{Mod}(\psi)$ when ψ is an $L_{\omega_1,\omega}$ sentence of the form $\bigwedge_{n<\omega}\psi_n$ where the ψ_n are Π_1-first-order sentences. This work began in [**Sh 300**]. Shelah is currently writing a book [**Sh h**] that, among other things, will include a "main gap"-style of theorem for universal classes.
- *Abstract elementary classes*. See Definition 1.5 below. This is in my opinion the deepest direction. It is the focal point of this article. Already in the fifties model theorists studied non elementary classes of structures (e.g. Jónsson [**Jo1**], [**Jo2**] and Fra¨issé [**Fre**]). In [**Sh88**], Shelah introduced the framework of abstract elementary classes and embarked on the ambitious program of developing a *classification theory* for abstract elementary classes. This work was continued in many publications of Shelah (totaling more than 700 pages) and members of his school.
- *Primal framework*. This is a generalization of abstract elementary classes obtained by relaxing the chain axioms (A4 from Definition 1.5 below). See Baldwin and Shelah's papers [**BS1**], [**BS2**], [**BS3**] and [**Gr5**].
- *Classification theory over a predicate*. Unlike the other frameworks this is really an extension of first-order model theory, when the notion of isomorphism is replaced by a stronger one. While in my opinion this framework does not precisely fit into what I call classification theory for non elementary classes, many of the methods are common. The fact that many years ago Shelah announced a solution for the main gap in this context, while for AECs such a theorem is not even on the horizon, indicates to me that this framework is much easier. I suggest to the reader to start with Wilfrid Hodges's survey article [**Ho2**]. Pillay and Shelah's article [**PiSh**] is the beginning. Further work includes [**Sh 234**] and [**Sh 322**]. Shelah has written several hundreds of (unpublished) pages that continue this up to a Main gap. This work is not available to me. I suggest that interested people will contact Shelah directly.

The focus of this article is the framework of *abstract elementary classes*. This framework, in my opinion, has the best balance of generality, a rich and sophisticated theory. The context of AECs is much more general than that of homogeneous model theory, model theory for $L_{\omega_1,\omega}$ or even the framework of submodels of a given structure.

DEFINITION 1.5. *Let $\mathcal{K}$ be a class of structures all in the same similarity type* $\mathrm{L}(\mathcal{K})$*, and let $\prec_{\mathcal{K}}$ be a partial order on $\mathcal{K}$. The ordered pair $\langle\mathcal{K},\prec_{\mathcal{K}}\rangle$ is an* abstract elementary class, AEC for short *iff*

A0 (Closure under isomorphism)
  (a) For every $M\in\mathcal{K}$ and every $L(\mathcal{K})$-structure N if $M\cong N$ then $N\in\mathcal{K}$.

(b) *Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.*

A1 *For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.*

A2 *Let M, N, M^* be $L(\mathcal{K})$-structures. If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$.*

A3 *(Downward Löwenheim-Skolem) There exists a cardinal $LS(\mathcal{K}) \geq \aleph_0 + |\operatorname{L}(\mathcal{K})|$ such that for every $M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \supseteq A$ and $\|N\| \leq |A| + LS(\mathcal{K})$.*

A4 *(Tarski-Vaught Chain)*

(a) *For every regular cardinal μ and every $N \in \mathcal{K}$ if $\{M_i \prec_{\mathcal{K}} N : i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing (i.e. $i < j \Longrightarrow M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $\bigcup_{i<\mu} M_i \prec_{\mathcal{K}} N$.*

(b) *For every regular μ, if $\{M_i : i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i<\mu} M_i$.*

REMARK 1.6. *I invite the reader to verify the following important basic observation: Given $M, N_1, N_2 \in \mathcal{K}$ such that $N_1 \prec_{\mathcal{K}} N_2$ and $f : N_1 \cong M$ then there are $N \in \mathcal{K}$ and $g : N_2 \cong N$ such that $M \prec_{\mathcal{K}} N$ and g extends f.*

EXAMPLE 1.7 (elementary classes). *Let T be a first-order theory, $\mathcal{K} = \operatorname{Mod}(T)$ and $\prec_{\mathcal{K}}$ the usual notion of elementary submodel. Then $\langle \mathcal{K} \prec_{\mathcal{K}} \rangle$ is an AEC with $LS(\mathcal{K}) = |\operatorname{L}(T)| + \aleph_0$.*

EXAMPLE 1.8 ($\aleph_1$-saturated models of a f.o. theory). *Let T be a complete countable superstable and not $\aleph_0$-stable, $\prec_{\mathcal{K}}$ be the elementary submodel relation and $LS(\mathcal{K}) = 2^{\aleph_0}$. $\mathcal{K} := \{M \models T : M$ is $\aleph_1$-saturated$\}$. By Theorem III.3.12 of* [**Sh c**] *$\mathcal{K}$ is closed under unions (recall that superstability implies $\kappa(T) = \aleph_0$).*

Recall (Keisler [**Ke2**]): for $\psi \in L_{\omega_1,\omega}$ a subset $L_{\mathcal{A}}$ of $L_{\omega_1,\omega}$ is a *fragment containing* ψ iff $\psi \in L_{\mathcal{A}}$, $L_{\mathcal{A}}$ is closed under: Taking subformulas, first-order connectives and quantifiers.

DEFINITION 1.9. *Let M and N be L-structures. Suppose $L_{\mathcal{A}}$ is an L-fragment. $M \subseteq_{TV,L_{\mathcal{A}}} N$ iff*

1. *$M \subseteq N$ and*
2. *for every $\mathbf{a} \in |M|$ and every $\varphi(y; \mathbf{x}) \in L_{\mathcal{A}}$ if $N \models \exists y \varphi(y; \mathbf{a})$ then there exists $b \in |M|$ such that $N \models \varphi[b; \mathbf{a}]$.*

When $L_{\mathcal{A}}$ consists of only the first-order formulas from the language of the structure we omit it.

It is the contents of the Tarski-Vaught test that M is an elementary submodel of N iff $M \subseteq_{TV} N$.

EXAMPLE 1.10 ($L_{\omega_1,\omega}$). *Let $\psi \in L_{\omega_1,\omega}$ be a sentence in a countable language and suppose that $L_{\mathcal{A}} \subseteq L_{\omega_1,\omega}$ is a countable fragment containing ψ. Take $\prec_{\mathcal{K}}$ to be defined by $M \prec_{\mathcal{K}} N \iff M \subseteq_{TV,L_{\mathcal{A}}} N$.*

Let $\mathcal{K} := \mathrm{Mod}(T)$. *Clearly* $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *is an abstract elementary class with* $LS(\mathcal{K}) = \aleph_0$.

EXAMPLE 1.11 ($L_{\omega_1,\omega}(\mathbf{Q})$). $T \subseteq L_{\omega_1,\omega}(\mathbf{Q})$ *be a countable theory in a countable language and suppose that* $L_{\mathcal{A}} \subseteq L_{\omega_1,\omega}(\mathbf{Q})$ *is a countable fragment containing* T. $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *is an abstract elementary class, when* $\mathcal{K} = \mathrm{Mod}(T)$ *and* $M \prec_{\mathcal{K}} N$ *iff* $M \subseteq_{TV,L_{\mathcal{A}}} N$

1. *if* $\mathbf{a} \in |M|$ *and* $M \models \mathbf{Q}x\varphi(x;\mathbf{a})$ *then there exists* $b \in |N| - |M|$ *such that* $N \models \varphi[b;\mathbf{a}]$ *and*
2. *if* $\mathbf{a} \in |M|$ *and* $M \models \neg\mathbf{Q}x\varphi(x;\mathbf{a})$ *then* $\varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a})$.

In fact AECs are more general than $L_{\omega_1,\omega}(\mathbf{Q})$.

The contents of the following Theorem is that in the Chain Axioms ($A4$) it is possible to replace the regular cardinal μ by an arbitrary directed set.

THEOREM 1.12 (Theorem 21.4 of [**Gra**]). *Let* $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *be an AEC and* $\langle M_s : s \in I \rangle$ *be a directed system. Then* $\bigcup_{s\in I} M_s \in \mathcal{K}$. *Moreover*

(a) If $M_s \prec_{\mathcal{K}} N$ *for every* $s \in I$, *then* $\bigcup_{s\in I} M_s \prec_{\mathcal{K}} N$.
(b) $M_s \prec_{\mathcal{K}} \bigcup_{t\in I} M_t$, *for every* $s \in I$.

PROOF. Show directly for finite and countable I. For uncountable I by induction on $|I|$ using the following:

FACT 1.13. *For every* I *uncountable directed set. There exists* $\{I_\alpha \mid \alpha < |I|\}$ *increasing such that each* I_α *is a directed subset of* I *of cardinality* $|\alpha| + \aleph_0$ *and* $I = \bigcup_{\alpha<|I|} I_\alpha$.

□

An early version of 1.12 can be found in Tsurane Iwamura's paper from 1944 ([**Iw**]).

NOTATION 1.14. *Denote by* $K_{<\lambda}$ *the class* $\{M \in \mathcal{K} : \|M\| < \lambda\}$ *and by* $\mathcal{K}_\lambda$ *the class* $\{M \in \mathcal{K} : \|M\| = \lambda\}$. $I(\lambda, \mathcal{K})$ *is the cardinality of* $\mathcal{K}_\lambda / \cong$.

Some examples from "main stream mathematics":

EXAMPLE 1.15 (Normed fields). *Let*

$$\mathcal{K} := \{\langle F, +, \cdot, |\ | \rangle \mid F \text{ is an algebraically closed field } \}$$

$$\langle F_1, +, \cdot, |\ |_1 \rangle \prec_{\mathcal{K}} \langle F_2, +, \cdot, |\ |_2 \rangle \iff F_1 \leq F_2,\ (\forall a \in F_1)[|a|_1 = |a|_2]$$
and the value groups are the equal.

EXAMPLE 1.16 (Local fields). *Let* $\langle F, +, \cdot, |\ | \rangle$ *be a non archimedean normed field (i.e.* $|a + b| \leq \mathrm{Max}\{|a|, |b|\}$ *for all* $a, b \in F$*). It is well known that (*[**Cass**]*)* $R := \{a \in F : |a| \leq 1_{\mathbb{R}}\}$ *is a subring of* F *and* $I := \{a \in F : |a| < 1_{\mathbb{R}}\}$ *is a maximal ideal of* R. *The field* R/I *is called the* residue field of F. *A field is* local *iff its residue field is finite*.

The class $\mathcal{K} := \{\langle F, +, \cdot, |\ | \rangle : F \text{ is local}\}$ *is an AEC when* $\langle F_1, +, \cdot, |\ |_1 \rangle \prec_{\mathcal{K}} \langle F_2, +, \cdot, |\ |_2 \rangle$ *is as in the previous example. Another relation* $\mathcal{K}$ *is defined by dropping the requirement of equal value groups*.

EXAMPLE 1.17 (Noetherian rings). *Define $R \prec_{\mathcal{K}} S$ iff R is a subring of S and $R \equiv_{\infty,\omega} S$. Notice that by Karp's test when R is Noetherian then also S is.*

EXAMPLE 1.18 (Rings of finite dimension). *Let $\mathcal{K}$ be a class of rings. Define $R \prec_{\mathcal{K}} S$ iff R is a subring of S and if $I \subsetneq J$ are ideals of R and $I' \subsetneq J'$ are ideals of S such that*

1. *$I' \cap R = I$,*
2. *$J' \cap R = J$ and*
3. *there is no ideal of R strictly between I and J*

then there is no ideal of S strictly between I' and J'.

EXAMPLE 1.19 (coordinate rings). *Let F be a finite field or the rationals. Suppose $p \in F[x_1, \dots, x_n]$ is such that $\{\mathbf{a} \in \overline{F}^n \mid p(\mathbf{a}) = 0\}$ is an irreducible variety. For an algebraically closed field $K \geq F$ let K_p be the coordinate ring of $\{\mathbf{a} \in K^n \mid p(\mathbf{a}) = 0\}$. The following is an AEC:*

$$\mathcal{K}_p := \{K_p \mid K \text{ is an algebraically closed field extending } F\}.$$

The relation $\prec_{\mathcal{K}}$ is the subring relation.

Recently motivated by a problem in transcendental number theory, Boris Zilber discovered the following most interesting example:

1.1. Zilber's Schanuel structures.

Some of the most intractable problems of number theory involve trascendental numbers. E.g. it is conjectured but unknown that the number $\mathbf{e}+\pi$ is trascendental. Schanuel's conjecture (see 1.20 below) is a far reaching conjecture that implies the above (using the identity $\mathbf{e}^{\mathbf{i}\pi} = -1$) and several other (very difficult) conjectures.

Let

$$\mathcal{K}_e := \big\{\langle F, +, \cdot, exp\rangle \mid F \text{ is an algebraically closed field of characteristic zero, } \forall x \forall y [exp(x+y) = exp(x) \cdot exp(y)]\big\}$$

$$\mathcal{K}_{pexp} := \big\{\langle F, +, \cdot, exp\rangle \in K_e \mid \ker(exp) = \pi\mathbb{Z}\big\}$$

Zilber introduced the following class:

$$\mathcal{K}_{exp} := \big\{\langle F, +, \cdot, exp\rangle \in K_{pexp} \mid \text{ Satisfying EC, CC and SCH}\big\}$$

EC is the essentially the requirement that the class is existentially closed.
CC stands for *countable closure property* - every analytic subset of F^n of dimension 0 is essentially countable.
SCH stands for

CONJECTURE 1.20 (Schanuel). *For every $x_1, \dots, x_n \in F$ if $\{x_1, \dots, x_n\}$ are linearly independent over $\mathbb{Q}$ then*

$$tr.deg_{F/\mathbb{Q}}\{x_1, \dots, x_n, exp(x_1), \dots, exp(x_n)\} \geq n.$$

For $M, N \in \mathcal{K}_{exp}$ the relation $M \prec_{\mathcal{K}_{exp}} N$ is defined as $M \subseteq N$ and

$$tr.deg_{M/\mathbb{Q}}\{x_1, \dots, x_n\} = tr.deg_{N/\mathbb{Q}}\{x_1, \dots, x_n\}$$
for every $n < \omega$ and for every $x_1, \dots, x_n \in M$.

Zilber noticed that $\langle \mathcal{K}_{exp}, \prec_{\mathcal{K}_{exp}} \rangle$ is axiomatizable by a sentence of $L_{\omega_1,\omega}(Q)$ when the similarity type is that of $\mathrm{L}(\mathcal{K}_e)$.

Zilber managed to show using Fraissé's- type of construction:

FACT 1.21 (Zilber). *$\mathcal{K}_{exp}$ is categorical in $\aleph_1$*

Using difficult field arithmetic (using the theories of fractional ideals in number fields as well as Weil divisors and the normalization theorem) Zilber managed to prove the following:

FACT 1.22 (Zilber [**Zi1**]). *Given algebraically closed fields $L_1, \dots, L_n \subseteq \mathbb{C}$ and elements $a_1, \dots, a_n \in \mathbb{C}^\times$ the multiplicative group of the field*

$$\mathbb{Q}(L_1, \dots, L_n, a_1, \dots, a_n)$$

can be presented as

$$A \cdot T \cdot L_1^\times \cdots L_n^\times,$$

where A is a free abelian, T is the torsion subgroup (when $n = 0$).

Recently Zilber observed that the model-theoretic contents of Fact 1.22 is that $\mathcal{K}_{exp}$ is an excellent class (in the sense of Shelah [**Sh87b**]). Recall that an AEC $\mathcal{K}$ is excellent if $\mathcal{K}_{\aleph_0}$ satisfies a strong amalgamation property.

Using two results of Shelah that were discovered in the late seventies (in the course of setting the fundations for classification theory for non elementary classes).[1]

FACT 1.23 (Shelah [**Sh87b**]). *An excellent class has arbitrarly large models.*

FACT 1.24 (Shelah [**Sh87b**]). *Let $\mathcal{K}$ be excellent. If $\mathcal{K}$ is categorical in an uncountable cardinal then $\mathcal{K}$ is categorical in every uncountable cardinal.*

Combining the last two together with Fact 1.22 Zilber concluded

COROLLARY 1.25 (Zilber). *$\mathcal{K}_{exp}$ has a unique object of cardinality $2^{\aleph_0}$. Thus in order to prove Schanuel's conjecture it suffices to show that the function $exp(x)$ defined on $\mathbb{C}$ (obtained from the categorical structure) is indeed $\mathbf{e}^x$.*

Zilber also noticed that the structures of $\mathcal{K}_{\exp}$ are not sequentially homogeneous, thus this class does not fit into the framework of *homogenous model theory* (previously known as *finite diagrams stable in power*). Thus it is a more "mathematical" example than Marcus's example of a categorical class that is not homogeneous from [**Ma**]. This is also the first example known to me for a categorical AEC very different from the previous ones.

I think that it is astonishing that model theoretic work that in the past was viewed by many to be detached from the main body of mathematics which depends heavily on methods of combinatorial set theory can have connections with

[1] At first Zilber rediscovered in [**Zi2**] a special case of Shelah's 1.24.

"concrete" mathematics, especially transcedental number theory. I am confident that in the future many other concrete examples will be found.

2. what is the purpose of this?

Develop a classification theory for non elementary classes for $\mathcal{K} = \mathrm{Mod}(T)$ when T is a complete $L_{\omega_1,\omega}$-theory (or maybe some other non first-order theory).

We try to imitate the rich and productive theory for elementary classes.

Eventually this will have applications in "mainstream" mathematics.

What is classification theory?

We want to be able to answer every question about $\mathcal{K}_\lambda / \cong$.

Basic test questions:

1. Given $\lambda \geq LS(\mathcal{K})$ is $\mathcal{K}_\lambda \neq \emptyset$?
2. Does $\mathcal{K}_\lambda \neq \emptyset$ imply $\mathcal{K}_{\lambda^+} \neq \emptyset$?
3. Does $I(\lambda^+, \mathcal{K}) = 1$ imply that $I(\lambda, \mathcal{K}) = 1$?
4. What are the possible functions $\lambda \mapsto I(\lambda, \mathcal{K})$?
5. Under what condition on $\mathcal{K}$ is it possible to find a nice dependence relation on subsets of every $M \in \mathcal{K}$?

In pure model theory the Łoś Conjecture (see [**Lo**]) was a major driving force:

CONJECTURE 2.1 (Łoś conjecture 1954 and Morley 1965). *Let T be a first-order theory. If there exists $\lambda > |T| + \aleph_0$ such that $I(\lambda, T) = 1$ then $I(\mu, T) = 1$ holds for every $\mu > |T| + \aleph_0$.*

In 1965 Morley [**Mo1**] confirmed the conjecture for theories in a countable language. For his proof Morley discovered the notions of Morley rank, prime model over a set and implicitly strongly minimal formulas. Important progress was made earlier by Ehrenfeucht and Mostowski as well as by Vaught. At the end of his article [**Mo1**] Morley raised the question whether his categoricity theorem holds for theories in uncountable languages. Several people recognized it as an important problem. Fredrick Rowbottom [**Ro**] and J. P. Ressayre [**Re**] made important progress toward a complete solution. In 1970, in addition to building on earlier work, Shelah invented superstable theories, weakly minimal formulas and local rank to prove the conjecture for all first-order theories.

There is a need for good test-questions to measure progress in classification theory for non-elementary classes. Around 1977 Shelah proposed a conjecture that would serve as a benchmark for progress of the theory and may serve as a guide for future developments:

CONJECTURE 2.2 (Shelah's conjecture). *Let T be a countable theory in $L_{\omega_1,\omega}$. If there exists $\lambda \geq \beth_{\omega_1}$ such that $I(\lambda, T) = 1$ then $I(\mu, T) = 1$ holds for every $\mu \geq \beth_{\omega_1}$.*

There is a similar conjecture for AEC generalizing the above conjecture. See Conjecture 3.6 in the next section.

Based on experience with the first-order version it is likely that any attempt to prove 2.2 will produce a rich and powerful machinery. Indeed the partial results obtained so far indicate that this is the case.

A word about examples: Essentially all known examples for categorical classes are derived from first-order ones.

Generic example: Let T be a countable first-order theory categorical in $\aleph_1$. Suppose $M \models T$ of cardinality $\aleph_1$, pick a countable A subset of $|M|$.
Let $L' := L(T) \cup \{P\}$ when P is a new unary predicate.

$$\psi_{T,A} := \bigwedge T \wedge \forall x[P(x) \rightarrow \bigvee_{a \in A} x = a]$$

By the proof of Morley's theorem, $\aleph_1$-categoricity of T implies that also $Th(\langle M, a\rangle_{a\in A})$ is categorical in every $\lambda \geq \aleph_1$. This is an example of an $L_{\omega_1,\omega}$-theory categorical in all uncountable cardinals.

The only exceptions are the example of coordinate rings (Example 1.19) which is categorical in every uncountable cardinal, Marcus's example and Ziber's Shanuel structures.

I think it is too early at this stage of the theory to formulate a Zilber-like conjecture of the nature of all categorical AECs or even the categorical $L_{\omega_1,\omega}$-theories, but perhaps there is a classifcation of the classes categorical classes above the Hanf number.

Conjecture 2.2 is very open. There are more than 500 published pages dedicated for partial results. Among them are:

1. Keisler (1971), using a two cardinal theorem that improves Vaught's theorem, has shown that under the additional assumption of the existence of sequentially homogeneous model the conjecture is true. Unfortunately, Shelah observed that using a construction of Leo Marcus [**Ma**], Keisler's additional assumption does not follow from categoricity. Thus Keisler's strategy, while being very elegant, is a dead end.
2. Shelah (1978) in [**Sh87a**] and [**Sh87b**] building on [**Sh48**] proved a form of the conjecture under the additional assumption of $I(\aleph_{n+1}, T) < 2^{\aleph_{n+1}}$ for every $n < \omega$. Such a class of structures is called an *excellent class*. Grossberg and Hart [**GrHa**] proved the main gap for excellent classes.
3. Lessmann ([**Le1**]) proved the conjecture for countable finite diagrams, using a Baldwin-Lachlan style argument by introducing the necessary pregeometries via a new rank function.
4. Makkai and Shelah [Sh285] proved a downward version of the conjecture under the additional assumption that both λ and μ are above a strongly compact cardinal, and $\lambda = \chi^+$. It is a major open problem of [**Sh 702**] to get rid of the assumption that λ is a successor. There are nicely behaved forking and orthogonality calculi for this.
5. Kolman-Shelah [**KoSh**] and Shelah [**Sh472**] contain partial going down results for λ above a measurable cardinal with the additional assumptions that $\mathcal{K}$ has the amalgamation property and λ is a successor cardinal.
6. [**Sh 394**] deals with classes that satisfy the amalgamation property. Several important concepts are introduced and a downward categoricity is concluded without using a large cardinal assumption as in [**KoSh**].

7. In [**ShVi**] Shelah and Andrés Villaveces embarked on an even more ambitious program: Deal with the categoricity conjecture for classes with slightly weaker model and set-theoretic assumptions. Namely, they assume GCH and work with classes that have no maximal models (this is a weakening of the amalgamation property). Recently more progress on this direction was made by Monica VanDieren [**Va**].
8. Inspired by a question of mine, whether it is possible to generalize the results of [**Sh48**] and [**Sh88**] for uncountable cardinals (or for PC_κ classes for uncountable κ). E.g. generalize [**Sh48**] from $\aleph_1$ to arbitrary λ. the problem is that both in [**Sh48**] and [**Sh88**] the assumption that the categoricity is the successor of $\aleph_0$ was used heavily in a form of applying a weak compactness phenomenon (in the form of undefinability of a well ordering in $L_{\omega_1,\omega}$). I suspected that this attempt must produce new model-theoretic machinery. It turns out that I was right. In several massive papers Shelah answered that question and more. This appears in [**Sh 576**] (125 pages), [**Sh 600**] (82 pages), and [**Sh 603**] (20 pages). In particular he has shown (under weak GCH and no large cardinals) that if $\mathcal{K}$ is categorical in both λ and λ^+ then $I(\lambda^{++}, \mathcal{K}) < 2^{\lambda^{++}} \implies \mathcal{K}_{\lambda^{+++}} \neq \emptyset$.
9. Depending heavily on [**Sh 576**] Shelah in [**Sh 705**] is developing the analog theory of excellent classes (from [**Sh87b**]) for AECs. The current draft of the paper has more than 120 pages in it.

In 1986 I proposed

CONJECTURE 2.3 (Intermediate Łoś conjectures). *Let $\psi \in L_{\omega_1,\omega}$*

1. *If ψ is categorical in some $\lambda \geq \beth_{\omega_1}$, then $\mathrm{Mod}(\psi)$ has the amalgamation property in every $\mu \geq \beth_{\omega_1}$.*
2. *If $\mathrm{Mod}(\psi)$ has the amalgamation property for all $\mu \geq \beth_{\omega_1}$, then ψ categorical in some $\lambda \geq \beth_{\omega_1}$ implies that ψ categorical in χ for every $\chi \geq \beth_{\omega_1}$.*

In section 6 it will be shown that the amalgamation property permits a nice theory of types.

In [**KoSh**] Oren Kolman and Shelah derive the amalgamation property from the assumption that ψ is categorical above a measurable cardinal. Lately Shelah and Villaveces in [**ShVi**] have shown that if $\mathcal{K}$ has no maximal models (i.e. every $M \in \mathcal{K}$ has a proper $\prec_{\mathcal{K}}$-extension in $\mathcal{K}$) the weak GCH implies that every model can be extended to an amalgamation base.

[**Sh 394**] is dedicated to progress toward #2. It contains a proof of a downward version of the categoricity conjecture under the assumption that the class has the amalgamation property.

WHY?

What could be the benefits of such a theory? Looking at the first-order example the notions of independence, several model-theoretic rank functions, forking, orthogonality calculus, regular types, pre-weight, prime models etc. all have found concrete applications in algebra, generalizing Krull's dimension theory from commutative algebra.

1. Clarify the above notions:

> "Studying only the model theory of first-order logic would be analogous to the study of real analysis never knowing of any but the polynomial functions; core concepts like continuity, differentiability, analyticity, and their relations would remain at best vaguely perceived. It is only the study of more general functions that one sees the importance of these notions, and their different roles, even for the simple case." -Jon Barwise, page 15–16 of [**BaFe**].

2. It is beautiful and difficult mathematics.
3. Effect on model theory for first-order theories. Already [**Sh87a**] and [**Sh87b**] had a profound effect on the proof of main gap for first-order theories. Especially good sets and stable systems. See the last 5 sections of Chapter XII in [**Sh c**] and what Shelah named the book's main theorem-Theorem XII.6.1. I expect that a similar n-dimensional analysis will be used to better our understanding of simple unstable first-order theories. Also classification theory over a predicate benefited much from this.
4. Many interesting concepts of classical mathematics are not first-order (Archimedean, Noetherian and any chain conditions. etc).
5. Potential applications in classification theory of finite models. See Baldwin and Lessmann [**BaLe**] and Lessmann [**Le3**].

3. basic facts

The following is the notion of pseudo-elementary (or projective) class.

DEFINITION 3.1. *Let $L_1 \supseteq L$ and let T_1 be a first-order theory in L_1, suppose that Γ is a set of T_1-types without parameters. We denote by*

$$\mathrm{PC}(T_1, \Gamma, L) = \{ M \restriction L \,:\, M \models T_1 \text{ and } M \text{ omits all types from } \Gamma \}.$$

DEFINITION 3.2. *A class $\mathcal{K}$ of structures is called a* PC-class *if there exists an expansion L_1 of $L(\mathcal{K})$, a first-order theory T_1 in L_1 and a set of T_1-types Γ such that $\mathcal{K} = \mathrm{PC}(T_1, \Gamma, L(\mathcal{K}))$. When $|T_1| + |\Gamma| + \aleph_0 = \mu$ we say that the class is PC_μ.*

In the special case when $L_1 = L(\mathcal{K})$ we write $EC(T, \Gamma)$ for $PC(T, \Gamma, L(\mathcal{K}))$ and say that the class $\mathcal{K}$ is an EC*-class.*

THEOREM 3.3 (C.C. Chang 1968). *If T is a theory in $L_{\mu^+,\omega}$ of cardinality $\leq \mu$ then $\mathrm{Mod}(T)$ is PC_μ.*

Similarly to Birkhoff's presentation theorem for varieties/equational classes, there is a syntactic presentation theorem which generalizes Theorem 3.3:

THEOREM 3.4 (Shelah's presentation theorem).
If $(\mathcal{K}, \prec_{\mathcal{K}})$ is an AEC, then there exists $\mu \leq 2^{LS(\mathcal{K})}$ such that $\mathcal{K}$ is a PC_μ class. Also $\mathcal{K}^{\prec_{\mathcal{K}}} := \{(N, M) \mid M \prec_{\mathcal{K}} N\}$ is a PC_μ class, for $L(\mathcal{K}^{\prec_{\mathcal{K}}})$ consisting of a single unary predicate symbol.

A proof uses Theorem 1.12, details can be found in [**Sh88**] or in Chapter 13 of [**Gr**].

Using Morley's result on the bound on computing Hanf numbers this immediately gives a corollary that is surprising and difficult to prove directly from the definitions, but is a trivial consequence of Shelah's presentation theorem.

COROLLARY 3.5. *Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC. If $\mathcal{K}_\lambda \neq \emptyset$ for some $\lambda \geq \beth_{(2^{2^{LS(\mathcal{K})}})^+}$ then $\mathcal{K}_\chi \neq \emptyset$ holds for all $\chi \geq LS(\mathcal{K})$.*

A generalization of Conjecture 2.2 to AECs appears in [**Sh 702**]:

CONJECTURE 3.6 (Shelah's conjecture for AEC). *Let $\mathcal{K}$ be an AEC. If there exists $\lambda \geq \beth_{(2^{LS(\mathcal{K})})^+}$ such that $I(\lambda, \mathcal{K}) = 1$ then $I(\mu, \mathcal{K}) = 1$ holds for every $\mu \geq \beth_{(2^{LS(\mathcal{K})})^+}$.*

Now we introduce the appropriate generalization of elementary embedding.

DEFINITION 3.7. *For $M, N \in \mathcal{K}$ a monomorphism $f : M \to N$ is called a* $\mathcal{K}$-embedding *iff $f[M] \prec_{\mathcal{K}} N$. Denote this by writing $f : M \xrightarrow{\mathcal{K}} N$. When the identity of $\mathcal{K}$ is clear it is omitted and we write $f : M \hookrightarrow N$.*

DEFINITION 3.8. *Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC, and let $\mu, \kappa \geq \lambda \geq LS(\mathcal{K})$ be cardinals.*

1. *We say that a model $M \in \mathcal{K}_\lambda$ is a* (μ, κ)-amalgamation base *iff for every $M_1 \in \mathcal{K}_\kappa$, $M_2 \in \mathcal{K}_\mu$ and $f_\ell : M \xrightarrow{\mathcal{K}} M_\ell$ for $\ell = 1, 2$, there exists $N \in \mathcal{K}$ so that there are $g_1 : M_1 \xrightarrow{\mathcal{K}} N$ and $g_2 : M_2 \xrightarrow{\mathcal{K}} N$ satisfying $g_1 \circ f_1 = g_2 \circ f_2$. Namely, the following diagram is commutative:*

$$\begin{array}{ccc} M_1 & \xrightarrow{g_1} & N \\ {\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle g_2} \\ M & \xrightarrow[f_2]{} & M_2 \end{array}$$

   N is called an amalgam *of M, M_1, M_2, f_1, f_2.*

   When $\kappa = \mu = \lambda$ we say that M is an amalgamation base.
2. *$M \in \mathcal{K}_\lambda$ is an* $(< \mu, \leq \kappa)$-amalgamation base *iff for every $\lambda \leq \mu_1 < \mu$, $\lambda \leq \kappa_1 \leq \kappa$, $M_1 \in \mathcal{K}_{\kappa_1}$, $M_2 \in \mathcal{K}_{\mu_1}$ and $f_\ell : M \xrightarrow{\mathcal{K}} M_\ell$ for $\ell = 1, 2$, there exists an amalgam N of M, N_1, N_2, f_1 and f_2.*
3. *$\mathcal{K}$* satisfies the (λ, μ, κ)-amalgamation property *iff every $M \in \mathcal{K}_\lambda$ is an (μ, κ)-amalgamation base.*
4. *$\mathcal{K}$* satisfies the λ-amalgamation property *iff every $M \in \mathcal{K}_\lambda$ is an amalgamation base.*

It is a corollary of the Robinson consistency theorem that if $\mathcal{K} = \mathrm{Mod}(T)$ for some complete first-order theory then $\mathcal{K}$ has the λ-amalgamation property for all $\lambda \geq \aleph_0 + |L(T)|$.

$\mathcal{K}_\lambda$ has the JEP iff for every $M_1, M_2 \in \mathcal{K}_\lambda$ there are $N \in \mathcal{K}$ and $\mathcal{K}$-embeddings $f_\ell : M_\ell \xrightarrow{\mathcal{K}} N$.

REMARK 3.9. *Note that for first-order theories an even stronger amalgamation property holds. Every model is an amalgamation base for arbitrarily large models. I.e. remove the requirements that the models N_ℓ have the same cardinality as M. The stronger amalgamation property is used in* [**Va**]*, but we will work here with that from the previous definition (when both extensions are of the same cardinality as the base.)*

Already in 1956 B. Jónsson studied abstract elementary classes with the amalgamation and JEP properties. This influenced Morley and Vaught to introduce saturated models ([**MoV**]). Shelah was the first in [**Sh88**] to consider AEC in the context of *classification theory*. Implicitly indicating that the methods of stability/classification theory are not limited to elementary classes only and potentially have a broader applicability than first order.

In the proof of Morley's categoricity theorem (as well as in Shelah's generalization to uncountable theories) saturated models play a central role. At first one shows that saturated models exists (at least at the categoricity cardinal λ) and using the uniqueness of them it suffices to show that having an uncountable non-saturated model implies the existence of a non-saturated model in λ, contradicting categoricity. Non elementary classes in their very nature are connected to omitting types, so working with saturated models is not reasonable.

It turns out that *model homogeneity* is a good replacement for saturation. In that it generalizes saturation (for elementary classes) and we have the analog existence and uniqueness theorems.

DEFINITION 3.10. *Let $\mathcal{K}$ be an AEC.*

1. *Let $\lambda > LS(\mathcal{K})$. We say that M* is λ-model homogeneous *iff for all $N \prec_{\mathcal{K}} N' \in \mathcal{K}_{<\lambda}$ such that $N \prec_{\mathcal{K}} M$ there exists $f : N' \xrightarrow[\mathcal{K}]{} M$ such that $f \restriction |N| = id_{|N|}$.*
2. *M is said to be* model homogeneous *iff M is $\|M\|$-model homogeneous.*

It is an exercise (using the compactness theorem) to show:

PROPOSITION 3.11. *Let $\mathcal{K}$ be an elementary class. For $\lambda > LS(\mathcal{K})$ and $M \in \mathcal{K}_\lambda$.*
M is saturated iff M is model homogeneous.

By imitating the argument of the proof of existence of saturated models one can show:

THEOREM 3.12 (existence). *Let $\mathcal{K}$ is an AEC and $\lambda \geq LS(\mathcal{K})$. Suppose that $2^\lambda = \lambda^+$. Further assume that $\mathcal{K}_{\lambda^+}$ is not empty. If $\mathcal{K}$ has the λ-amalgamation property, then there exists a model homogeneous $M \in \mathcal{K}_{\lambda^+}$.*

THEOREM 3.13 (better existence). *Suppose that $\mathcal{K}$ be an AEC such that $\mu > LS(\mathcal{K})$ satisfies $\mu = \mu^{<\mu}$ and $\mathcal{K}_\mu$ is not empty. If $\mathcal{K}$ has the $(< \mu, \mu)$-amalgamation property, then for all $N \in \mathcal{K}_\mu$ there exists $M \succ_{\mathcal{K}} N, M \in \mathcal{K}_\mu$ which is model homogeneous.*

THEOREM 3.14 (uniqueness). *Let $\mathcal{K}$ be an AEC. Suppose that $\mathcal{K}_\lambda$ is categorical for some $\lambda \geq LS(\mathcal{K})$. If M and N are model homogeneous members of $\mathcal{K}_{\lambda^+}$, then $M \cong N$.*

There are uniqueness theorems that follow from weaker assumptions.

While model homogeneity is very nice, model homogeneous models are natural to consider only when $\mathcal{K}$ has the amalgamation property. I.e. Suppose that $\mathcal{K}$ is categorical in λ and $\mathcal{K}$ has no maximal models. The model of cardinality λ is model homogeneous iff $\mathcal{K}$ has the $(< \mu, \mu)$-amalgamation property for every $\mu < \lambda$.

A substitute called (μ, σ)-limit model was introduced in [**KoSh**] and where used in a substantial way to obtain the amalgamation property (from categoricity above a measurable cardinal), limit models reappeared also in [**Sh 394**] under the assumption that $\mathcal{K}$ has the amalgamation property. Further study of limit models (without requiring the amalgamation property) is in Shelah and Villaveces in [**ShVi**]. The uniqueness: Any two (μ, σ)-limit models are isomorphic for different σ's was proved only lately by Monica VanDieren who has introduced and offered a characterization of the correct notion of model homogeneity for classes not requiring amalgamation in [**Va**]. Since most of this article deals with classes that have the amalgamation property I will not discuss (μ, σ)-limit models here.

4. the beginning of classification theory for AEC

In his JSL list of open problems from 1975 Harvey Friedman reproduced a question that started classification theory for non elementary classes.

QUESTION 4.1 (Baldwin's problem 1975). *Does there exists a countable similarity type and a countable $T \subseteq L(\mathbf{Q})$ (in the $\aleph_1$ interpretation) such that T has a unique uncountable model (up to isomorphism)?*

Since the Downward Löwenheim Skolem theorem holds for $L(\mathbf{Q})$, Baldwin's question is equivalent to "Does there exists a countable similarity type and a countable $T \subseteq L(\mathbf{Q})$ such that T is categorical in $\aleph_1$ but does not have a model of cardinality $\aleph_2$?"

The question is important since it suggested for the first time a connection between categoricity in a cardinal and existence of models in its successor.

A natural extension (generalizing $L(\mathbf{Q})$ by an AEC and more importantly replacing $\aleph_0$ by an arbitrary λ):

QUESTION 4.2. *Let $\mathcal{K}$ be an AEC and $\lambda \geq LS(\mathcal{K})$. Does* categoricity *in λ^+ of $\mathcal{K}$ imply* existence *of a model of cardinality λ^{++}?*

The following is a relatively simple example of a family of deep results that was motivated by Baldwin's question.

THEOREM 4.3. *Suppose $2^\lambda < 2^{\lambda^+}$. For an AEC $\mathcal{K}$ which fails to have the λ-amalgamation property. If $I(\lambda, \mathcal{K}) = 1$ and $\lambda \geq LS(\mathcal{K})$ then $I(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$.*

Section 8 of this paper is dedicated to the a proof of Theorem 4.3.

The set-theoretic assumption $2^\lambda < 2^{\lambda^+}$ is known as the *weak continuum hypothesis* since it follows from $2^\lambda = \lambda^+$. However instead of using cardinal arithmetic we use a "diamond-like" combinatorial principle known as the Devlin-Shelah's weak diamond:

FACT 4.4 (Devlin-Shelah). *If $2^\lambda < 2^{\lambda^+}$ then there are λ^+ pairwise disjoint stationary subsets of λ^+ such that for any of these sets S the principle $\Phi^2_{\lambda^+}(S)$ holds, where $\Phi^2_{\lambda^+}(S)$ is:*
For all $F : {}^{\lambda^+ >}2 \to 2$ there exists $g : \lambda^+ \to 2$ so that for every $f : \lambda^+ \to 2$ the set $\{\delta \in S \mid F(f \restriction \delta) = g(\delta)\}$ is stationary.

For more details see [**DS**] or section 13.5 of [**Gr**] or Chapter 13 of [**Sh f**].

The use of the weak diamond is essential, the statement of Theorem 4.3 is false under $MA_{\aleph_1} + 2^{\aleph_0} > \aleph_1$:

FACT 4.5. *There exists an AEC, $\mathcal{K}$ with $LS(\mathcal{K}) = \aleph_0$ such that*

1. *$\mathcal{K}$ is categorical in $\aleph_0$,*
2. *$\mathcal{K}_{\aleph_1} \neq \emptyset$ and*
3. *The amalgamation property fails in $\mathcal{K}_{\aleph_0}$.*

We have that

$$MA_{\aleph_1} + 2^{\aleph_0} > \aleph_1 \implies I(\aleph_1, \mathcal{K}) = 1.$$

This class is obtained by essentially considering the countable substructures of the random bipartite graph whose left side is ω and right side is ω_1. The categoricity proof is similar to Baumgartner's ([**Bau**]) proof of the uniqueness of $\aleph_1$-dense orders.

I will conclude this section with a typical application of the weak diamond to AEC, this is not a particular case of Theorem 4.3 but rather a different theorem. The following theorem is a simple prototype of several more sophisticated results (e.g. [**ShVi**] and [**Va**]). A structure $M \in \mathcal{K}_\mu$ is called *universal model* iff for every $N \in \mathcal{K}_\mu$ there exists a $\mathcal{K}$-embedding from N into M.

THEOREM 4.6. *Suppose $2^\lambda < 2^{\lambda^+}$. For an AEC $\mathcal{K}$ which fails to have the λ-amalgamation property. If $I(\lambda, \mathcal{K}) = 1$ and $\lambda \geq LS(\mathcal{K})$ then $\mathcal{K}_{\lambda^+}$ does not have a universal model.*

In this section, as well as in sections 5 and 8, I will use some elementary facts about stationary sets of ordinals (all can be found in Kunen's book [**Ku**] or in section 1.8 of [**Gr**]): Let λ be an uncountable regular cardinal. A set C of ordinals all less than λ is called a *closed unbounded set (club)* iff for every $\alpha < \lambda$ there exists $\beta \in C$ such that $\beta > \alpha$ and for every bounded $A \subseteq C$ we have that $\bigcup A \in C$. A set $S \subseteq \lambda$ is *stationary* iff $S \cap C \neq \emptyset$ for every club $C \subseteq \lambda$.

FACT 4.7. *Let λ be an uncountable regular cardinal. Let M be a structure of cardinality λ in a countable language.*

1. *If $\{M_i \prec M \mid i < \lambda\}$ and $\{N_i \prec M \mid i < \lambda\}$ are elementary chains which are increasing and continuous such that $M = \bigcup_{i<\lambda} M_i = \bigcup_{i<\lambda} N_i$ and $\|M_i\| + \|N_i\| < \lambda$, for all $i < \lambda$, then the set $\{\delta < \lambda \mid N_\delta = M_\delta\}$ contains a closed unbounded subset of λ.*

2. *Suppose that $L(M)$ contains a unary predicate P such that P^M is the set of ordinals less than λ. Then for every continuous increasing elementary chain $\{M_i \prec M \mid i < \lambda\}$ such that $M = \bigcup_{i<\lambda} M_i$ and $\|M_i\| < \lambda$ for all $i < \lambda$ the set $\{\delta < \lambda \mid P^{M_\delta} = \delta\}$ contains a club of λ.*

Instead of using the principle $\Phi^2_{\lambda^+}$ directly we use another combinatorial principle:

DEFINITION 4.8. *Θ_{λ^+} is said to hold if and only if for all $\{f_\eta : \eta \in {}^{\lambda^+}2$ where $f_\eta : \lambda^+ \to \lambda^+\}$ and for every club $C \subseteq \lambda^+$, there exists $\eta \neq \nu \in {}^{\lambda^+}2$ and there exists a $\delta \in C$ such that $\eta \restriction \delta = \nu \restriction \delta$, $f_\eta \restriction \delta = f_\nu \restriction \delta$ and $\eta[\delta] \neq \nu[\delta]$.*

In [**DS**] Devlin and Shelah have shown that Θ_{λ^+} follows from $\Phi^2_{\lambda^+}$ (see also Chapter 13 of [**Gr**]).

Now to the proof of Theorem 4.6:

PROOF. By assumption, we may take $N_0, N_1, N_2 \in \mathcal{K}_\lambda$, that can not be amalgamated.

For $\rho \in {}^{\lambda^+ >}2$ define a family of $M_\rho \in \mathcal{K}_\lambda$ so that the following hold:

1. $|M_\rho| = \lambda(1 + \ell(\rho))$,
2. $\nu \lhd \eta \to M_\nu \prec_{\mathcal{K}} M_\eta$,
3. when $\ell(\rho)$ is a limit ordinal, $M_\rho = \bigcup_{\alpha<\ell(\rho)} M_{\rho\restriction\alpha}$,
4. $M_{\rho^\frown 0}$ and $M_{\rho^\frown 1}$ cannot be amalgamated over M_ρ.

Using λ-categoricity and the triple N_0, N_1, N_2 the construction is possible.

For $\eta \in {}^{\lambda^+ >}2$ let $M_\eta := \bigcup_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$.

Now suppose that $M \in \mathcal{K}_{\lambda^+}$ is universal. Without loss of generality we may assume that $|M| = \lambda^+$. By universality for every $\eta \in {}^{\lambda^+}2$ there is a $\mathcal{K}$-embedding $f_\eta : M_\eta \to M$.

Now consider $C := \{\delta < \lambda^+ \mid \delta = \lambda(1 + \delta)\}$, it contains a club. Using Θ_{λ^+} there are $\eta \neq \nu \in {}^{\lambda^+}2$ and $\delta \in C$ as in Θ_{λ^+}. Denote by ρ the largest common initial segment of η and ν (it is $\eta \restriction \delta$). Since $\eta[\delta] \neq \nu[\delta]$ we assume that $\eta[\delta] = 0$ and $\nu[\delta] = 1$. Pick $M^* \prec_{\mathcal{K}} M$ of cardinality λ containing the set $f_\eta[M_{\rho^\frown 0}] \cup f_\nu[M_{\rho^\frown 1}]$.

Note that the diagram

$$\begin{array}{ccc} M_{\rho^\frown 0} & \xrightarrow[f_\eta]{} & M^* \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle f_\nu} \\ M_\rho & \xrightarrow[\mathrm{id}]{} & M_{\rho^\frown 1} \end{array}$$

is commutative in contradiction to requirement 4 in the construction. □

The argument used in the proof of Theorem 4.6 can be used to prove the following useful:

COROLLARY 4.9. *Suppose* $2^\lambda < 2^{\lambda^+}$ *and* $\mathcal{K}$ *is an AEC with* $LS(\mathcal{K}) \leq \lambda$. *If* $I(\lambda, \mathcal{K}) = 1$ *and there exists* $\mu > \lambda$ *such that* $I(\mu, \mathcal{K}) = 1$ *then* $\mathcal{K}$ *has the* λ*-amalgamation property.*

5. Solution to Baldwin's question and $\mathcal{K}^3_\lambda$

Suppose $T \subseteq L(\mathbf{Q})$. Given $M \models T$ of cardinality $\aleph_1$ pick a $\psi \in L_{\omega_1,\omega}$ a Scott sentence of a countable substructure of M. Now let $L_{\mathcal{A}} \subseteq L_{\omega_1,\omega}(\mathbf{Q})$ be a countable fragment containing T and ψ. Let $\mathcal{K} := \mathrm{Mod}(\bigwedge T \wedge \psi)$ and let $N_1 \prec_{\mathcal{K}} N_2$ iff $N_1 \subseteq_{TV,L_{\mathcal{A}}} N_2$. It is enough to prove:

THEOREM 5.1. *Suppose* $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *is an AEC which is* $\mathrm{PC}_{\aleph_0}$. *If* $\mathcal{K}$ *is categorical both in* $\aleph_0$ *and* $\aleph_1$ *then* $\mathcal{K}_{\aleph_2} \neq \emptyset$.

PROOF. Since $\mathcal{K}$ is categorical in $\aleph_1$ and closed under union it is enough to show that

$$(*) \quad \text{There are } N_0 \neq N_1 \in \mathcal{K}_{\aleph_1} \text{ such that } N_0 \prec_{\mathcal{K}} N_1.$$

The following concept is central to the theory:

DEFINITION 5.2.

$$\mathcal{K}^3_\lambda := \{(M, N, a) \ : M, N \in \mathcal{K}_\lambda,\ M \prec_{\mathcal{K}} N,\ a \in |N| - |M|\}.$$

On $\mathcal{K}^3_\lambda$ *define a partial ordering by*

$$(M, N, a) < (M', N', a') \quad \stackrel{def}{\iff} \quad M \prec_{\mathcal{K}} M' \wedge M \neq M' \wedge N \prec_{\mathcal{K}} N' \wedge a = a'.$$

When $\mathcal{K}$ is categorical in λ then the assumption $\mathcal{K}_{\lambda^+} \neq \emptyset$ implies that $\mathcal{K}^3_\lambda \neq \emptyset$.

REMARK 5.3. *In the next section we will show that the element* (M, N, a) *of* $\mathcal{K}^3_\lambda$ *plays a similar role to that of* $\mathrm{tp}(a/M, N)$ *in first-order logic.*

Using $(*)$ and the assumption that $I(\aleph_1, \mathcal{K}) = 1$ once more, Theorem 5.1 follows from:

THEOREM 5.4. *Suppose* $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *is an AEC which is* $\mathrm{PC}_{\aleph_0}$. *If* $\mathcal{K}$ *is categorical in* $\aleph_0$ *and the poset* $\langle \mathcal{K}^3_{\aleph_0}, < \rangle$ *is not empty and has a maximal element then* $I(\aleph_1, \mathcal{K}) = 2^{\aleph_1}$.

Why enough? If every element of $\mathcal{K}^3_{\aleph_0}$ has an extension define $\{(M_\alpha, N_\alpha, a) \mid \alpha < \omega_1\} \subseteq \mathcal{K}_{\aleph_0}$, strictly increasing and continuous. Then the following is a witness for $(*)$:

$$\bigcup_{\alpha<\omega_1} M_\alpha \prec \bigcup_{\alpha<\omega_1} N_\alpha, \text{ since } a \notin \bigcup_{\alpha<\omega_1} M_\alpha.$$

It is time to recall a fact from the mid sixties that was discovered independently by Lopez-Escobar (using proof-theoretic methods) and Morley (using model-theoretic techniques):

FACT 5.5 (undefinability of well-ordering). *Let $\mathcal{K}^*$ be a $PC_{\aleph_0}$-class such that $L(\mathcal{K}^*)$ contains a binary relation $<$ and a unary predicate P. If there exists $\mathfrak{A} \in \mathcal{K}^*$ with $\langle P^{\mathfrak{A}}, <^{\mathfrak{A}} \rangle = \langle \omega_1, \in \rangle$, then there exists $\mathfrak{B} \in \mathcal{K}^*$ such that there is $\{\alpha_n \in P^{\mathfrak{B}} \mid n < \omega\}$ satisfying*

$$\mathfrak{B} \models \alpha_{n+1} < \alpha_n \quad \textit{for all } n < \omega.$$

The main step is the following:

LEMMA 5.6 (technical lemma). *Suppose $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an AEC which is $\mathrm{PC}_{\aleph_0}$, and $\mathcal{K}$ is categorical both in $\aleph_0$ and $\aleph_1$. Then for every $M \in \mathcal{K}_{\aleph_0}$ there are $\{M_n \in \mathcal{K}_{\aleph_0} \quad n < \omega\}$ such that for every $n < \omega$ we have that $M_{n+1} \prec_{\mathcal{K}} M_n$ and $M = \bigcap_{n<\omega} M_n$.*

PROOF. By categoricity in $\aleph_0$ it is sufficient to show that there exists a model M as in the statement.

Let $T_1, \Gamma_1, T_2, \Gamma_2$ be at most countable such that $\mathcal{K} = \mathrm{PC}(T_1, \Gamma_1, L(\mathcal{K}))$ and $\{\langle |N|, |M| \rangle \ : \ M \prec_{\mathcal{K}} N\} = \mathrm{PC}(T_2, \Gamma_2, \{P(x)\})$. Since $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an abstract elementary class and $\mathcal{K}_{\omega_1} \neq \emptyset$ we can fix an $\prec_{\mathcal{K}}$-increasing continuous chain of models $\{M_\alpha \in \mathcal{K}_{\aleph_0} \ : \ \alpha < \omega_1\}$. Denote by $f : \omega_1 \to \mathcal{K}_{\aleph_0}$ the mapping $\alpha \mapsto M_\alpha$. Let $M := \bigcup_{\alpha<\omega_1} M_\alpha$. For every $a \in |M|$ let $h(a) := \mathrm{Min}\{\xi < \omega_1 \ : \ a \in |M_\xi|\}$.

By the reflection principle there exists a regular cardinal χ sufficiently large so that

$H(\chi) \supseteq \{\lambda^+, f, h, M, T_1, \Gamma_1, T_2, \Gamma_2, L(K), ...\}$ and $\langle H(\chi), \in \rangle$ reflects all relevant information. Namely

$$\begin{aligned} \langle H(\chi), \in \rangle \models & \forall a \in |M| [h(a) \in \omega_1] \wedge \forall \alpha < \gamma \in \omega_1 \\ & [f(\alpha) \in \mathrm{PC}(T_1, \Gamma_1, L(\mathcal{K})) \wedge \\ & \langle f(\gamma), f(\alpha) \rangle \in \mathrm{PC}(T_2, \Gamma_2, \{P(x)\})] \end{aligned} \tag{1}$$

$$\langle H(\chi), \in \rangle \models \forall \alpha < \gamma \in \omega_1 [\alpha < \gamma \to f(\alpha) \prec_{\mathcal{K}} f(\gamma)] \tag{2}$$

$$\begin{aligned} \langle H(\chi), \in \rangle \models & \forall \alpha \in \omega_1 [\forall \gamma [\gamma < \alpha \to \gamma + 1 < \alpha]] \to \\ & \forall M \in \mathrm{Rang}(f) [M \prec_{\mathcal{K}} f(\alpha) \wedge M \neq f(\alpha)] \to \\ & \exists i < \alpha [M \prec_{\mathcal{K}} f(i)]] \end{aligned} \tag{3}$$

$$\langle H(\chi), \in \rangle \models \text{“} \langle \mathcal{K}, \prec_{\mathcal{K}} \rangle \quad \text{is an AEC”} \tag{4}$$

$$\langle H(\chi), \in \rangle \models \text{“} \langle \mathcal{K}, \prec_{\mathcal{K}} \rangle \quad \text{satisfies Theorem 1.12”} \tag{5}$$

$$\begin{aligned} \langle H(\chi), \in \rangle \models & M = \bigcup_{\alpha<\omega_1} f(\alpha) \\ & \langle H(\chi), \in \rangle \models \forall a \in |M| [a \in f(h(a))] \\ & \wedge (\forall \alpha \in \omega_1) [\alpha < h(a) \to a \notin f(\alpha)]. \end{aligned} \tag{6}$$

Let

$$\mathfrak{A} := \langle H(\chi), \in, |M|, \omega, f, h, Q, \bar{T}_1, \bar{T}_2 \bar{\Gamma}_1, \bar{\Gamma}_2, \varphi, p, \zeta \rangle_{\zeta<\lambda, \varphi \in T_1 \cup T_2, p \in \Gamma_1 \cup \Gamma_2}.$$

Where $\bar{\Gamma}_1, \bar{\Gamma}_2$ are unary predicates interpreted by the corresponding sets of types, similarly $\bar{T}_l$ are unary predicates interpreting T_l, Q is a unary predicate interpreted by the set of ordinals ω_1, ω is a unary predicate interpreted by the set of natural numbers and f and h are unary function symbols interpreted by the corresponding functions. ζ, φ and p are individual constants interpreted by the corresponding elements. Now let $p_1(y) := \{\bar{\Gamma}_1(y) \wedge y \neq q \ : \ q \in \Gamma_1\}$,
$p_2(y) := \{\bar{\Gamma}_2(y) \wedge y \neq q \ : \ q \in \Gamma_2\}$ and
$p_3(\varphi) := \{\bar{T}_1(\varphi) \wedge \varphi \neq \psi \ : \ \psi \in T_1\}$,
$p_4(\varphi) := \{\bar{T}_2(\varphi) \wedge \varphi \neq \psi \ : \ \psi \in T_2\}$, finally let
$p_5(j) := \{\lambda(j) \wedge j \neq \zeta \ : \ \zeta < \lambda\}$.

Denote by T_3 the theory of $\mathfrak{A}$. Clearly $\mathfrak{A} \in \mathrm{EC}(T_3, \{p_1, p_2, p_3, p_4, p_5\})$ and for all $\mathfrak{B} \in \mathrm{EC}(T_3, \{p_1, p_2, p_3, p_4, p_5\})$ since $T_l^{\mathfrak{B}} = T_l$, $\Gamma_l^{\mathfrak{B}} = \Gamma_l$ and we have (using $(*)_0$ and $(*)_1$) that

$$\text{For all } \alpha < \gamma \in Q^{\mathfrak{B}} \quad [f(\alpha) \prec_{\mathcal{K}} f(\gamma)] \wedge [f(\alpha) \in \mathrm{PC}(T_1, \Gamma_1, L(\mathcal{K}))].$$

Since $\mathfrak{B}$ omits p_5 we get that $\lambda^{\mathfrak{B}} = \lambda$, namely we have that $\mathcal{K}_\lambda^{\mathfrak{B}} / \cong = \mathcal{K}_\lambda / \cong$.

Since $\langle Q^{\mathfrak{A}}, \in \rangle$ has order type λ^+ and from the assumption on λ an application of Fact 5.5 to $\mathrm{EC}(T_3, \{p_1, p_2, p_3, p_4, p_5\})$ produces a model
$\mathfrak{B} \in \mathrm{EC}(T_3, \{p_1, p_2, p_3, p_4, p_5\})$ such that there exists $\{\alpha_n \ : \ n < \omega\} \subseteq Q^{\mathfrak{B}}$ such that for every $n < \omega$ we have that $\mathfrak{B} \models \alpha_{n+1} < \alpha_n$. For $n < \omega$ let $M_n := f(\alpha_n)$. We conclude with showing that

CLAIM 5.7. *There exists $N \in \mathcal{K}_\lambda^{\mathfrak{B}}$ such that $N = \bigcap_{n<\omega} M_n$.*

PROOF. Since $\omega \subseteq Q^{\mathfrak{B}}$ and for all $k < \omega$ and for every $n < \omega$ we have

$$\mathfrak{B} \models [k < \alpha_n],$$

the set $I := \{\beta \in Q^{\mathfrak{B}} \ : \ \forall n < \omega [\beta < \alpha_n]\}$ is nonempty and directed. Since $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle^{\mathfrak{B}}$ is an abstract elementary class (by $(*)_3$) we have that there exists $N := \bigcup_{s \in I} f(s)$. By the definition of N we have that for all $n < \omega$ $N \prec_{\mathcal{K}} M_n$. So clearly $N \subseteq \bigcap_{n<\omega} M_n$. Using the function h we show that the last two sets are equal. Suppose that there exists $a \in \bigcap_{n<\omega} M_n - |N|$. By $(*)_5$ there exists a first $\gamma \in Q$ such that $a \in f(\gamma)$. Since $a \in M_n$ for every n by minimality we get that $\gamma \leq \alpha_n$ for all $n < \omega$. Since f is order-preserving we get that $f(\gamma) \prec_{\mathcal{K}} f(\alpha_n)$, namely $f(\gamma) \subseteq M_n$ for all n which is a contradiction to the choice of the element a. □

Suppose $(M, N, a) \in \mathcal{K}^3_{\aleph_0}$ is maximal. Given any $S \subseteq \omega_1$, define $\{M_\alpha^S \ : \ \alpha < \omega_1\} \subseteq \mathcal{K}_{\aleph_0}$ as follows:

1. $|M_\alpha^S| = \omega(1 + \alpha)$,
2. For α limit, $M_\alpha^S = \bigcup_{\beta<\alpha} M_\beta^S$,
3. for $\alpha = \beta + 1$, there are two cases:
   (a) if $\beta \in S$, using $\aleph_0$-categoricity take $M_{\beta+1}^S \gneq_{\mathcal{K}} M_\beta^S$ and $a_\beta^S \in |M_{\beta+1}^S| - |M_\beta^S|$ so that $(M_\beta^S, M_{\beta+1}^S, a_\beta^S) \cong (M, N, a)$

(b) if $\beta \notin S$, then take a descending decomposition. I. e. apply Lemma 5.6 to the model M^S_β to get a descending $\{M_n \mid n < \omega\} \subseteq \mathcal{K}_{\aleph_0}$ such that $M^S_\beta = \bigcap_{n<\omega} M_n$. Now define $M^S_{\beta+1}$ to be M_0.

Notice that $\alpha < \beta \implies M^S_\alpha \precneqq_{\mathcal{K}} M^S_\beta$.

For each set S let $M_S := \bigcup_{\alpha<\omega_1} M^S_\alpha$.

CLAIM 5.8. *If* $S_1 \not\equiv S_2 \mod D_{\omega_1}$ *then* $M^{S_1} \ncong M^{S_2}$.

PROOF. Let $S_1 \not\equiv S_2 \mod D_{\omega_1}$ and $f : M^{S_1} \cong M^{S_2}$ be given. By requirement (1) of the construction $|M^{S_1}| = |M^{S_2}| = \omega_1$. Observe that

$$C_1 := \{\delta < \omega_1 \ : \ (\forall \alpha < \delta)[f(\alpha) < \delta]\} \quad \text{is a club.}$$

Using continuity of the chains we get that

$$C_2 := \{\delta < \omega_1 \ : \ f \restriction \delta : M^{S_1}_\delta \cong M^{S_2}_\delta\} \quad \text{is also a club.}$$

Take $C := C_1 \cap C_2$. Since C is a club we may assume without loss of generality that there exists $\delta \in C \cap (S_1 - S_2)$. Since $\delta \in S_1$ by the construction we have that $(M^{S_1}_\delta, M^{S_1}_{\delta+1}, a^{S_1}_\delta)$ is a maximal element of $\mathcal{K}^3_{\aleph_0}$. Since

$$f(a^{S_1}_\delta) \notin M^{S_2}_\delta,$$

by the assumption $\delta \notin S_2$ there exists $n < \omega$ such that

$$f(a^{S_1}_\delta) \notin M_n.$$

Let $N := f^{-1}[M_n]$. Since $M^{S_2}_\delta$ is a proper substructure of M_n we have that $M^{S_1}_\delta \subsetneq N$. Since $|N|$ is a countable subset of $|M^{S_1}|$ there exists $\gamma < \omega_1$ such that $N \prec M^{S_1}_\gamma$. Since $C \cap (S_1 - S_2)$ is unbounded there exists $\xi \in C \cap (S_1 - S_2)$ greater than $\max\{\delta, \gamma\}$. Thus we have $(N, M^{S_1}_\xi, a^{S_1}_\delta)$ is a proper $\mathcal{K}^3_{\aleph_0}$-extension of $(M^{S_1}_\delta, M^{S_1}_{\delta+1}, a^{S_1}_\delta)$ contradicting its maximality.

By Ulam's theorem the claim gives $I(\aleph_1, \mathcal{K}) = 2^{\aleph_1}$. □

□

6. Galois types

Recall the following basic result of elementary model theory:

FACT 6.1. *Let* $\mathfrak{C}$ *be an uncountable saturated model of cardinality greater than* $|L(\mathfrak{C})|$. *For* $a, b \in |\mathfrak{C}|$ *and* $A \subseteq |\mathfrak{C}|$ *such that* $|A| < \|\mathfrak{C}\|$.

$$\mathrm{tp}(a/A) = \mathrm{tp}(b/A) \iff \exists f \in \mathrm{Aut}_A(\mathfrak{C}) \text{ such that } f(a) = b.$$

Namely the orbit of the element a under the group action of $\mathrm{Aut}_A(\mathfrak{C})$ on $\mathfrak{C}$ can be identified with $\mathrm{tp}(a/A)$. The set of $L(\mathcal{K})$-formulas satisfied by a does not have the corresponding property for abstract elementary classes. Thus we need a replacement. A replacement introduced by Shelah in [**Sh 300**] and since takes a prominent role in model theory of AECs is to work directly with orbits instead of set of formulas! This is the notion of *Galois type* to be defined below. Unfortunately not having formulas (in any logic) creates many technical difficulties.

DEFINITION 6.2. *Let $M \in \mathcal{K}_\lambda$, We say that (M, N_1, a_1) and $(M, N_2, a_2) \in \mathcal{K}^3_\lambda$, are* $\sim$-related*, written*

$$(M, N_1, a_1) \ \sim \ (M, N_2, a_2),$$

if there exists $N \in \mathcal{K}_\lambda$ and $\mathcal{K}$-embeddings

$$h_1 : N_1 \to N \quad \textit{and} \quad h_2 : N_2 \to N,$$

such that

$$h_1(a_1) = h_2(a_2)$$

and the following diagram commutes:

$$\begin{array}{ccc} N_1 & \xrightarrow{h_1} & N \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle h_2} \\ M & \xrightarrow[\mathrm{id}]{} & N_2 \end{array}$$

LEMMA 6.3. *Suppose that $\mathcal{K}_\lambda$ has the amalgamation property. For $M \in \mathcal{K}_\lambda$ the relation $\sim$ is an equivalence relation on $\mathcal{K}^3_\lambda$.*

PROOF. Exercise, notice that we use that M *and* N_ℓ are amalgamation bases. □

DEFINITION 6.4 (Galois types).

1. *For $(M, N, a) \in \mathcal{K}^3_\lambda$, we let*

$$\text{ga-tp}(a/M, N) = (M, N, a)/\sim .$$

   This is the type of a over M in N.

2. *For $M \in \mathcal{K}_\lambda$, we let*

$$\text{ga-S}(M) = \{ \text{ ga-tp}(a'/M, N') \ : \ (M, N', a') \in \mathcal{K}^3_\lambda \}.$$

3. *Given $p \in \text{ga-S}(M)$ and $N \in \mathcal{K}_{\geq\lambda}$, we say that p is* realized *by $a \in N$, if there exists $N' \prec_{\mathcal{K}} N$ of cardinality λ containing $|M| \cup a$ such that*

$$(M, N', a) \in \mathcal{K}^3_\lambda \quad \textit{and} \quad p = \text{ga-tp}(a/M, N').$$

Assuming that $\mathcal{K}_{\aleph_0}$ has the amalgamation property, the content of Theorem 5.4 is a weak replacement of the compactness theorem:

THEOREM 6.5 (extension property of types). *Let $\mathcal{K}$ be an AEC which is* $\mathrm{PC}_{\aleph_0}$. *Then the assumption $I(\aleph_1, \mathcal{K}) = 1$ implies that for every $M \prec_{\mathcal{K}} N \in \mathcal{K}_{\aleph_0}$ and every $p \in \text{ga-S}(M)$ there exists $q \in \text{ga-S}(N)$ extending p.*

Thus the assumptions of categoricity in $\aleph_1$ together with $\mathcal{K}_{\aleph_0}$ has the amalgamation property are a replacement of an easy fact in first-order logic which is a corollary of the compactness theorem. This is a typical example of "compactness regained" which appears also in much more complicated results.

DEFINITION 6.6. *Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an abstract elementary class and suppose that $\lambda > LS(\mathcal{K})$. For $N \in \mathcal{K}_{\geq\lambda}$ the model N is* λ-Galois saturated *iff for every $M \prec_{\mathcal{K}} N$ of cardinality less than λ and every $p \in \text{ga-S}(M)$ is realized in N.*

The contents of the following theorem that for AEC classes that have the amalgamation property, model homogeneity and saturation are equivalent properties. It is further evidence that the notion of Galois type makes sense.

THEOREM 6.7. *Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an abstract elementary class and suppose that $\lambda > LS(\mathcal{K})$. Suppose that $\mathcal{K}$ has the $(< \lambda, < \lambda)$-amalgamation property then for $M \in \mathcal{K}_{\geq \lambda}$ we have that*
M is λ-Galois saturated iff M is λ-model homogeneous.

PROOF. It is easy to show that homogeneity implies saturation.

Let $\mu \geq LS(\mathcal{K})$ be such that $\mu < \lambda$.

Let $N_1 \prec N_2 \in \mathcal{K}_{\leq \mu}$ be given such that $N_1 \prec M$.

We may assume that μ is the cardinality of N_2. Fix $\langle a_i \mid i < \mu \rangle$ an enumeration of N_2. Now by induction on $i < \mu$ define two increasing continuous chains of models $\langle N_l^i, f_i \mid i \leq \mu,\ l \in \{1, 2\} \rangle$ and mappings $\langle f_i \mid i < \mu \rangle$ satisfying:

1. $N_l^0 = N_l,\ f_0 = id_{N_1},\ \|N_2^i\| = \mu$,
2. $N_1^i \prec_{\mathcal{K}} N_2^i$,
3. $f_i : N_1^i \hookrightarrow M$ and
4. $a_i \in N_1^{i+1}$ for every $i < \mu$.

Since the chains are continuous we only have to define two models and an embedding for $i = j + 1$:

Let $M_1^j := f_j[N_1^j]$. Using Remark 1.6 let M_2^j be an amalgam of N_2^j and M_1^j over N_1^j such that $M_2^j \succ M_1^j$ and let $g_j : N_2^j \cong M_2^j$ be an extension of f_j. Namely the diagram

$$\begin{array}{ccccc} N_2^j & \xrightarrow{g_j} & M_2^j & & \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\ N_1^j & \xrightarrow[f_j]{} & M_1^j & \xrightarrow[\mathrm{id}]{} & M \end{array}$$

commutes.

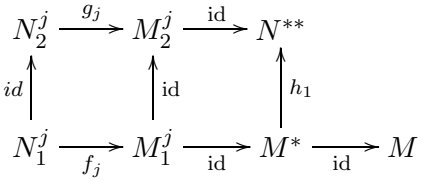
If $g_j(a_j) \in M_1^j$ then do nothing.

Otherwise consider $p := \text{ga-tp}(g_j(a_j), M_1^j, M_2^j)$ and use the hypothesis that M is μ_2^+-saturated to get $M^* \in \mathcal{K}_{\mu_2}$ and to find $b \in M^*$ such that $M_1^j \prec M^* \prec M$ and $(M_1^j, M_2^j, g_j(a_j)) \sim (M_1^j, M^*, b)$.

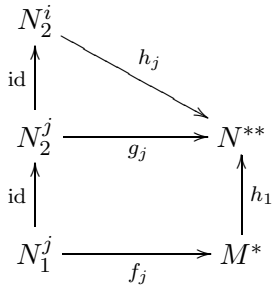
Unwinding the definition of $\sim$ gives: There exists
$N^{**} \in \mathcal{K}_\mu$ and mappings h_1, h_2 such that the diagram

$$\begin{array}{ccc} M_2^j & \xrightarrow{h_2} & N^{**} \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle h_1} \\ M_1^j & \xrightarrow[\mathrm{id}]{} & M^* \end{array}$$

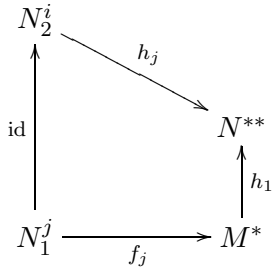
commutes and in addition we have
$h_2(g_j(a_j)) = h_1(b)$. By renaming the elements of N^{**} and changing the mapping h_1 accordingly we may assume that h_2 is the identity. Thus by gluing the last two diagrams together we get that the diagram



commute. Now pick $N_2^i \succ N_2^j$ and $h_j \supseteq g_j$ such that $h_j : N_2^i \cong N^{**}$.
So we have that



Commutes. Let $N_1^i := h_j^{-1}[h_1[M^*]]$.
Since $g_j(a_j) = h_1(b)$ (using also that $b \in M^*$) we get that $a_j \in N_1^{j+1}$ and from $h_j \supseteq g_j$ and the fact that



commutes we get that $N_1^i \supseteq N_1^j$. Now we are ready to define the mapping f_i, let $f_i := h_1^{-1} \circ (h_j \restriction N_1^j)$. It is a $\mathcal{K}$-embedding that extends f_j as required. Verify using Axiom $A2$ that $\bigcup_{i<\mu} f_i$ is an embedding of N_2 over N_1 into M.

□

7. Stability-like properties

REMARK 7.1. *Notice that $\mathcal{K}^3_\lambda$ has no maximal element corresponds to "every type over M can be extended to a type over M' for all $M \subsetneq M' \in \mathcal{K}_\lambda$". The contents of Theorem 5.4 is that if $\mathcal{K}$ is an $PC_{\aleph_0}$ class categorical in $\aleph_0$ such that $I(\aleph_1, \mathcal{K}) < 2^{\aleph_1}$ every type over a countable model has a proper extension.*

THEOREM 7.2 ([**Sh 576**]). *Suppose $\mathcal{K}$ is a PC_μ-class. Let $\lambda \geq \mu$.*
If $I(\lambda, \mathcal{K}) = I(\lambda^+, \mathcal{K}) = 1$ and $\mathcal{K}_{\lambda^{++}} \neq \emptyset$, then K^3_λ has no maximal element.

DEFINITION 7.3. $(M_0, M_1, a) \in \mathcal{K}^3_\lambda$ is minimal *iff for every*

$$(M'_0, M'_\ell, a_\ell) \geq_{h_\ell} (M_0, M_1, a) \quad \ell = 1, 2$$

we have that $\text{ga-tp}(a_1, M'_0, M_1) = \text{ga-tp}(a_2, M'_0, M_2)$.
$(M'_0, M'_1, a') \geq_h (M_0, M_1, a)$ *stands for "h is an embedding of M_1 into M'_1 taking M_0 into M'_0 and $h(a) = a'$".*

The above notion of minimal triple is a generalization of strongly minimal non algebraic type. We are actually generalizing the *stationarity* property of strongnly minimal types. I.e. if a_1, a_2 realize extensions (to $|M'_0|$) of the minimal type of a then they types of a_1 and a_2 are equal.

There are several basic existence results of strongly minimal types. I.e.

FACT 7.4. *(W. Marsh) If T is an $\aleph_0$-stable first-order then there exists a strongly minimal formula.*

It is natural at this stage to introduce the following "obvious" concept:

DEFINITION 7.5. $\mathcal{K}$ is stable in λ *iff* $|\text{ga-S}(M)| \leq \lambda$ *for every* $M \in \mathcal{K}_\lambda$.

An analog to Marsh'e theorem is the following result:

THEOREM 7.6. *Let $\lambda \geq LS(\mathcal{K})$. Suppose $\mathcal{K}$ has the λ-amalgamation property and $\mathcal{K}^3_\lambda$ does not have a maximal triple. If there is no minimal $(M_0, M_1, a) \in \mathcal{K}^3_\lambda$, let $p := \text{ga-tp}(a/M_0, M_1)$ then the following holds:*

1. *There is $M'_0 \in \mathcal{K}_\lambda$ such that $M_0 \preceq M'_0$ and*
   $|\{p' \in \text{ga-S}(M_0) \ : \ p' \geq p\}| \geq 2^{log(\lambda)}$;
2. *There exists $N \in \mathcal{K}_{\lambda^+}$ such that $M'_0 \preceq N$ and*
   $|\{p \in \text{ga-S}(M'_0) \ : \ p$ is realized in $N\}| = \lambda^+$.
   Namely,

$$|\{c \in N \ : \exists N' \succeq N, M'_0 \preceq N', N' \in \mathcal{K}_\lambda, \text{ such that } \text{ga-tp}(c/M'_0, N') \geq \text{ga-tp}(a/M_0, M_1)\}| = \lambda^+.$$

The previous theorem is further evidence that the notion of Galois types is useful.

The notion of minimal element of $\mathcal{K}^3_\lambda$ plays a central role in proving categoricity results. A key to categoricity: Under strong assumptions on $\mathcal{K}$ we have that if $p \in \text{ga-S}(M)$ is minimal then for every $M_2 \succ M_1 \succ M$ if $M_1 \subsetneq M_2$ then there exists $a \in |M_2| - |M_1|$ realizing p.

At the current state of affairs there is no nice forking-like relation for AECs (even under the assumption that they are categorical above the hanf number).

However there are several approximations. Since not even the parallel to Morley's theorem is available for AECs one can investigate one of the coarser notions from the days of stability theory before forking. Below it is shown that a notion parallel to splitting of types is moderately nicely behaved. For this we make the following

HYPOTHESIS 7.7. *$\mathcal{K}$ has the $(\lambda, < \kappa, \mu)$-amalgamation property for every λ, κ and μ, we may assume that there exists a monster model $\mathfrak{C}$. By* monster model *we mean a very large model homogeneous model.*

Notice that existence of large Galois-saturated models is also desireable, but we need more than just amalgamation to prove their existence.

DEFINITION 7.8. *Suppose $N \prec_{\mathcal{K}} M \in \mathcal{K}$, $\lambda \geq \|N\|$ a cardinal number.* The type $p \in \text{ga-S}(M)$ λ-splits over N *iff there exists N_1, N_2, h of cardinality λ such that $N \prec N_\ell \prec M$, $h : N_1 \cong_N N_2$ and the types $p \restriction N_2$ and $h(p \restriction N_1)$ are contradictory (= there if N^* is an extension of M then there is no $a \in |N^*|$ such that $a \models p \restriction N_2 \cup h(p \restriction N_1)$).*

It is tempting to call the previoous notion Galois-splitting. However I feel that doing so will make certain passages unreadable. It is important to recognize the similarity of Galois-splitting to the ususal first-order splitting as well as the differences. The key difference that here we don't have a formula wittnessing the splititng, moreover the splitting is evidenced by *models* rather than a finite sequence of parameters.

THEOREM 7.9. *Let $\lambda \geq LS(\mathcal{K})$. Suppose $\mathcal{K}$ is stable in λ. For every $M \in \mathcal{K}_{\geq \lambda}$ and every $p \in \text{ga-S}(M)$ there exists $M_0 \prec M$ of cardinality $\leq \lambda$ such that p does not split over M_0.*

PROOF. Suppose $N \succ_{\mathcal{K}} M$, $a \in N$ such that $p = \text{ga-tp}(a/M, N)$ and p splits over N_0, for every $N_0 \prec_{\mathcal{K}} M$ of cardinality λ.

Let $\chi := \min\{\chi \mid 2^\chi > \lambda\}$. Notice that $\chi \leq \lambda$ and $2^{<\chi} \leq \lambda$.

We'll define $\{M_\alpha \prec M \mid \alpha < \chi\} \subseteq \mathcal{K}_\lambda$ increasing and continuous $\prec_{\mathcal{K}}$-chain which will be used to construct $M^*_\chi \in \mathcal{K}_\lambda$ such that

$$|\text{ga-S}(M^*_\chi)| \geq 2^\chi > \lambda \text{ obtaining a contradiction to } \lambda\text{-stability.}$$

Pick $M_0 \prec M$ any model of cardinality λ.

For $\alpha = \beta + 1$; since p splits over M_β there are $N_{\beta,\ell} \prec_{\mathcal{K}} M$ of cardinality λ for $\ell = 1, 2$ and there is $h_\beta : N_{\beta,1} \cong_{M_\beta} N_{\beta,2}$ such that
$h_\beta(p \restriction N_{\beta,1}) \neq p \restriction N_{\beta,2}$. Pick $M_\beta \prec_{\mathcal{K}} M$ of cardinality λ containing the set $|N_{\beta,1}| \cup |N_{\beta,2}|$.

Now for $\alpha < \chi$ define $M^*_\alpha \in \mathcal{K}_\lambda$ and for $\eta \in {}^\alpha 2$ define a $\mathcal{K}$-embedding h_η such that

1. $\beta < \alpha \implies M^*_\beta \prec_{\mathcal{K}} M^*_\alpha$,
2. for α limit let $M^*_\alpha = \bigcup_{\beta<\alpha} M^*_\beta$,
3. $\beta < \alpha \wedge \eta \in {}^\alpha 2 \implies h_{\eta\restriction\beta} \subseteq h_\eta$,
4. $\eta \in {}^\alpha 2 \implies h_\eta : M_\alpha \xrightarrow{\mathcal{K}} M^*_\alpha$ and
5. $\alpha = \beta + 1 \wedge \eta \in {}^\alpha 2 \implies h_{\eta^\frown 0}(N_{\beta,1}) = h_{\eta^\frown 1}(N_{\beta,2})$.

The construction is possible by using the λ-amalgamation property at $\alpha = \beta + 1$ several times. Given $\eta \in {}^\beta 2$ let N^* be of cardinality λ and f_0 be such that

the diagram

$$\begin{array}{ccc} M_{\beta+1} & \xrightarrow{f_0} & N^* \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_\beta & \xrightarrow[h_\eta]{} & M^*_\beta \end{array}$$

commutes. Denote by N_2 the model $f_0(N_{\beta,2})$. Since $h_\beta : N_{\beta,1} \cong_{M_\beta} N_{\beta,2}$ there is a $\mathcal{K}$-mapping g fixing M_β such that $g(N_{\beta,1}) = N_2$. Using the amalgamation property now pick $N^{**} \in \mathcal{K}_\lambda$ and a mapping f_1 such that the diagram

$$\begin{array}{ccc} M_{\beta+1} & \xrightarrow{f_1} & N^{**} \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ N_{\beta,1} & \xrightarrow[g]{} & N_2 \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_\beta & \xrightarrow[h_\eta]{} & M^*_\beta \end{array}$$

Finally apply the amalgamation property to find $M^*_{\beta+1} \in \mathcal{K}_\lambda$ and mappings e_0, e_1 such that

$$\begin{array}{ccc} N^{**} & \xrightarrow{e_1} & M^*_{\beta+1} \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle e_0} \\ M^*_\beta & \xrightarrow[\mathrm{id}]{} & N^* \end{array}$$

commutes. After renaming some of the elements of $M^*_{\beta+1}$ and changing e_1 we may assume that $e_0 = \mathrm{id}_{N^*}$.

Let $h_{\eta^\frown 0} := f_0$ and $h_{\eta^\frown 1} := e_1 \circ f_1$.

Now for $\eta \in {}^{\chi}2$ let

$$M^*_\chi := \bigcup_{\alpha<\chi} M^*_\alpha \quad \text{and} \quad H_\eta := \bigcup_{\alpha<\chi} h_{\eta\restriction\alpha}.$$

Take $N^*_\eta \succ_{\mathcal{K}} M^*_\chi$ from $\mathcal{K}_\lambda$, an amalgam of N and M^*_χ over M_χ such that

$$\begin{array}{ccc} N & \xrightarrow{H_\eta} & N^*_\eta \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_\chi & \xrightarrow[h_\eta]{} & M^*_\chi \end{array}$$

commutes.

Notice that

$$\eta \neq \nu \in {}^{\chi}2 \implies \text{ga-tp}(H_\eta(a)/M^*_\chi, N^*_\eta) \neq \text{ga-tp}(H_\nu(a)/M^*_\chi, N^*_\nu).$$

Thus $|\text{ga-S}(M^*_\chi)| \geq 2^\chi > \lambda$. □

QUESTION 7.10. *Is it possible to replace $\aleph_0$ in Remark 7.1 with an uncountable λ?*

This is important since a positive answer will require developing a replacement for the undefinability of well ordering in $L_{\omega_1,\omega}$. The fact that undefinability of well ordering does not have a natural generalization for uncountable cardinals follows from an independence result of Jon Barwise and Ken Kunen [**BaKu**].

Here is a very simple example of a result from [**Sh 576**] in this direction:

THEOREM 7.11. *Let $\lambda \geq \mu$. Suppose $\mathcal{K}$ is an AEC such that $\mathcal{K}$ is PC_μ, $I(\lambda, \mathcal{K}) = I(\lambda^+, \mathcal{K}) = 1$ and $\mathcal{K}_{\lambda^{++}} \neq \emptyset$. Then $\mathcal{K}^3_\lambda$ does not have a maximal triple.*

PROOF. ($\mathcal{K}^3_\lambda$ has no max element). Since $I(\lambda, \mathcal{K}) = 1$ and $\mathcal{K}_{\lambda^+} \neq 0$, it is enough to show that
if $(M_0, M_1, a) \in \mathcal{K}^3_\lambda$, then there exists

$$(M'_0, M'_1, a') \geq (M_0, M_1, a)$$

such that $M_0 \neq M'_0$.

Suppose for contradiction that $(M_0, M_1, a) \in \mathcal{K}^3_\lambda$ is a maximal triple.
Define $\{N_i : i < \lambda^+\} \subseteq \mathcal{K}_\lambda$ increasing continuous
and $\{h_i : M_1 \cong N_{i+1} : i < \lambda^+\}$ such that $N_i = h_i[M_0]$. Since we want these sets to be continuous, it is enough to define them at successor stages. Given N_i, where $i < \lambda^+$, by $I(\lambda, \mathcal{K}) = 1$ there exists an isomorphism $\bar{h} : M_0 \cong N_i$. Since $M_0 \prec_{\mathcal{K}} M_1$, we can find N_{i+1} and h_i such that $\bar{h} \subseteq h_i$ and
$h : M_1 \cong N_{i+1}$. Note that $h_i(a) \in N_{i+1} - N_i$. From the construction we get that that $(N_i, N_{i+1}, h_i(a)) \in \mathcal{K}^3_\lambda$ is a maximal triple, for $i < \lambda^+$.

Now let $N = \bigcup_{i<\lambda^+} N_i$.
Then, since $h_i(a) \in N_{i+1} - N_i$ for all $i < \lambda^+$ and since $\{N_i : i < \lambda^+\}$ is a chain, $N \in \mathcal{K}_{\lambda^+}$. Since $I(\lambda^+, \mathcal{K}) = 1$ and $\mathcal{K}_{\lambda^{++}} \neq 0$, there is $N^1 \neq N$ in $\mathcal{K}_{\lambda^+}$ such that $N \prec_{\mathcal{K}} N^1$.

Pick $\{N^1_\beta \prec_{\mathcal{K}} N^1 : \beta < \lambda^+\} \subseteq \mathcal{K}_\lambda$ increasing and continuous such that for all $\beta < \lambda^+, N_\beta \subset N^1_\beta$ but $N_\beta \neq N^1_\beta$. (This is possible since $|N^1| \neq |N|$).

Define a function $g : |N^1| \to \lambda^+$ as follows:

$$g(b) := \begin{cases} i & \text{if } h_i(a) = b \\ 0 & \text{Otherwise.} \end{cases}$$

Notice that the relation $g(b) = i$ is a function since we have seen above that $i \neq j \implies h_i(a) \neq h_j(a)$.

Applying the reflection theorem from set theory, let α be large enough such that V_α contains the set

$$\{N, N^1, M_0, M_1, a, \langle N_i, N^1_i, h_i : i < \lambda^+\rangle, g, \lambda^+, \text{“the PC}_\mu \text{ definition of } \mathcal{K}\text{”}\}$$

and such that the model

$$\mathcal{B}^* = \langle V_\alpha, \epsilon, N, N^1, M_0, M_1, Q, \mathcal{K}, g, i \mapsto (N_i, N^1_i, h_i), a\rangle,$$

where $Q = \lambda^+$, reflects the following sentences:
(i) "$\{N_i : i < \lambda\}$ is increasing and continuous in $\mathcal{K}_{\lambda+}$"
(ii) $(\forall i \in Q)[N_i \subset N_i^1 \wedge N_i \neq N_i^1]$
(iii) "for all $i \in Q$, $(N_i, N_{i+1}, h_i(a))$ is a maximal element in $\mathcal{K}^3_\lambda$"
(iv) $g : |N^1| \to Q$
(v) $(\forall b \in N^1)(\exists i \in Q)((h_i(a) = b) \to g(b) = i)$
(vi) $N = \bigcup_{i \in Q} N_i$
(vii) $N^1 = \bigcup_{i \in Q} N_i^1$.

Let $\mathcal{B} \prec_{\mathcal{K}} \mathcal{B}^*$ of cardinality λ such that $Q^{\mathcal{B}} = \delta$, a limit ordinal. (For example, take $\{\mathcal{B}_i \prec_{\mathcal{K}} \mathcal{B}^* : i < \lambda^*\}$ increasing continuous such that $||\mathcal{B}_i|| = \lambda$ and $Q^{\mathcal{B}_i} \supseteq i$. Then use the fact that $\{\delta < \lambda^+ : \delta = Q^{\mathcal{B}_\delta}\}$ is a club.)

Now, $N^{\mathcal{B}} = \bigcup_{\alpha<\delta} N_\alpha = \bigcup_{\alpha<\delta} N_\alpha = N_\delta$. Similarly, $N^{1\mathcal{B}} = N^1_\delta$.

Denote $a_\delta := h_\delta(a)$.

Claim $N_{\delta+1} \cap (N^1)^{\mathcal{B}} = N_\delta$. In particular, $a_\delta \notin N^1_\delta$.

PROOF. If $a_\delta \in N^1_\delta$ then $a_\delta \in \mathcal{B}$. Since $\mathcal{B}$ is closed under g, $g(a_\delta) \in \mathcal{B}$. But recall that

$$\mathcal{B} \models (g : |N^1| \to Q) \wedge (\forall b \in N^1) \\ ((\exists i \in Q)(h_i = b) \to (g(b) = i)).$$

So $g(a_\delta) \in Q$. Compute $g(a_\delta) = \delta$.
Since $\mathcal{B} \models ZF^-$, also $\delta + 1 \in Q^{\mathcal{B}}$, which contradicts the fact that $\delta = Q^{\mathcal{B}}$.

If $N_{\delta+1} \cap N^1_\delta$ properly contains N_δ, then since $a_\delta \in N_{\delta+1} \subseteq N^1_{\delta+1}$, by the fact that $a \notin N^1_\delta$ we get that $(N^1_\delta, N^1_{\delta+1}, a_\delta) \in \mathcal{K}^3_\lambda$. And since then

$$(N_\delta, N_{\delta+1}, a_\delta) < (N^1_\delta, N^1_{\delta+1}, a_\delta),$$

we get a contradiction to the assumption that $(N_\delta, N_{\delta+1}, a_\delta)$ is a maximal triple. □

□

In [**GrVa**] Grossberg and VanDieren have shown that for categorical AEC Morley-sequences exist when the dependence relation is non splitting.

Denote by $M \prec^{univ}_{\mathcal{K}} N$ the statement N *is universal over* M i.e. for every $M' \succ M$ of cardinality $\|M\|$ there exists a $\mathcal{K}$-embedding from M' into N over M.

DEFINITION 7.12 (from [**GrVa**]). *Let $\mathcal{K}$ be an AEC. The class is called* μ-superstable *iff there exists $\mu > LS(\mathcal{K})$ satisfying*

1. *for every $M \in \mathcal{K}_\mu$, there exists $M \prec^{univ}_{\mathcal{K}} M' \in \mathcal{K}_\mu$ and*
2. *for every $\kappa = \mathrm{cf}(\kappa) < \mu^+$ whenever $\langle M_i \in \mathcal{K}_\mu \mid i \leq \kappa\rangle$ is $\prec^{univ}_{\mathcal{K}}$-increasing and continuous and $p \in \text{ga-S}(M_\kappa)$, there exists $i < \kappa$ such that p does not μ-split over M_i.*

THEOREM 7.13 (from [**GrVa**]). *Suppose $\mathcal{K}$ is μ-superstable for some $\mu \geq LS(\mathcal{K})$ and $\mathcal{K}$ has the amalgamation property. Let $M \in \mathcal{K}_{>\mu}$, $A, I \subset M$ be*

given such that $|I| \geq \mu^+ > |A|$. *Then there exists* $J \subset I$ *of cardinality* μ^+, *indiscernible over* A. *Moreover* J *can be chosen to be a Morley sequence over* A.

Where

DEFINITION 7.14 ([**GrVa**]). $\{\mathbf{a}_i \in \mu^+\}$ *is a* Morley sequence over M_0 *iff there exists an* $\prec_{\mathcal{K}}$*-chain* $\{M_i \mid i < \mu^+\}$ *such that* $p_i := \text{ga-tp}(\mathbf{a}_i/M_i, M_{i+1})$ *does not* μ*-split over* M_0 *and* $\langle \mathbf{a}_i \mid i \in \lambda \rangle$ *is an indiscernible sequence over* M_0.

8. proof of Theorem 4.3

Before starting the proof let me point out that we will be using eth Devlin-Shelah weak diamond. Instead of using the principle Φ_{λ^+} we use a principle that may look little stronger but using a pairing function together with with Fact 4.7 one can show that it follows from Φ_{λ^+}:

We will be using:

There exists a stationary subset S of λ^+ such that for every

$$F : {}^{<\lambda^+}\lambda^+ \times {}^{<\lambda^+}\lambda^+ \times {}^{<\lambda^+}\lambda^+ \to 2,$$

there exists a guess $g : \lambda^+ \to 2$, such that for every $\eta, \nu, h : \lambda^+ \to \lambda^+$, the set

$$\{\delta \in S \mid F(\eta \restriction \delta, \nu \restriction \delta, h \restriction \delta) = g(\delta)\} \text{ is stationary.}$$

PROOF OF THEORM 4.3. Recall that $I(\lambda, \mathcal{K}) = 1$ and $\mathcal{K}$ fails to have the λ-amalgamation property implies that $\mathcal{K}_{\lambda^+}$ is nonempty.

By assumption, we may take $N_0, N_1, N_2 \in \mathcal{K}_\lambda$, that can not be amalgamated.

For $\eta \in {}^{\lambda^+ >}2$ define a family of $M_\eta \in \mathcal{K}_\lambda$ so that the following hold:

1. $|M_\eta| = \lambda(1 + \ell(\eta))$,
2. $\nu \lhd \eta \to M_\nu \prec_{\mathcal{K}} M_\eta$,
3. when $\ell(\eta)$ is a limit ordinal, $M_\eta = \bigcup_{\alpha < \ell(\eta)} M_{\eta \restriction \alpha}$,
4. $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ cannot be amalgamated over M_η.

Using λ-categoricity and the triple N_0, N_1, N_2 the construction is possible.

Divide the proof into two cases, in the first case assume a stronger failure of the amalgamation and the second is the negation of the first.

Case A: Suppose that there exist $N \prec M \in \mathcal{K}_\lambda$ so that for every M' extending M in $\mathcal{K}_\lambda$, there is a pair M^0 and M^1 extending M' so that M^0 and M^1 cannot be amalgamated over N.

To requirements 1-4 we add
$M_{\langle\rangle} = N$ and replace (4) by

$(4)'$ $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ cannot be amalgamated over N.

For $\eta \in {}^{\lambda^+}2$, let $M_\eta := \bigcup_{\alpha < \lambda^+} M_{\eta \restriction \alpha}$.

CLAIM 8.1. $\eta \neq \nu \implies \langle M_\eta, a \rangle_{a \in |N|} \ncong \langle M_\nu, a \rangle_{a \in |N|}$.

PROOF. Let ρ be the meet of η and ν. If there was an isomorphism between $\langle M_\eta, a \rangle_{a \in |N|}$ and $\langle M_\nu, a \rangle_{a \in |N|}$, we would have that M_ν is an amalgam of $M_{\rho^\frown 0}$ and $M_{\rho^\frown 1}$ over N, a contradiction to requirement (4). $\square$

Suppose for the sake of contradiction that

$$\mu := I(\lambda^+, \mathcal{K}) < 2^{\lambda^+}.$$

Take $\{M_i \mid i < \mu\}$ to be a complete set of representatives for $\mathcal{K}_{\lambda^+}$.

Clearly,

$$\begin{aligned} |\{\langle M_\eta, a\rangle_{a\in|N|}/\cong \ : \ \eta \in {}^{\lambda^+}2\}| \\ &\leq |\{\langle M_i, a\rangle_{a\in|N|} \ : \ i<\mu\}| \\ &\leq \mu\|M_i\|^\lambda = \mu(\lambda^+)^\lambda = \mu\lambda^+ 2^\lambda = \\ &\mu 2^\lambda < 2^{\lambda^+}. \end{aligned}$$

But this is a contradiction to the fact we have many pairwise non-embeddable models. Notice how we used the assumption $2^\lambda < 2^{\lambda^+}$ in the final step.

Case B: Suppose for all N and every $M \in \mathcal{K}_\lambda$ there is an M' extending M so that for any extensions M_0 and M_1 of M', M_0 and M_1 can be amalgamated over N.

Again, we tweak requirement (4) and replace it by:

$(4)'$ if M^0 and M^1 are extensions of $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$, then M_η, M^0 and M^1 can be amalgamated.

To perform the construction at successors, we do the following, to define $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ from M_η:

Apply λ categoricity to fix an isomorphism $f : M_\eta \cong N_0$ (the unamalgamable triple we chose at the beginning).

Using N_1, N_2 and their preimages pick M^* and M^{**} extensions of M_η which cannot be amalgamated over M_η.

By the assumption of case B, we can take $M_{\eta^\frown 0}$ to be an extension of M^* so that any extensions of $M_{\eta^\frown 0}$ can be amalgamated over M_η. Take $M_{\eta^\frown 1}$ similarly for M^{**}.

Let $C := \{\delta < \lambda \ : \ \delta = \lambda(1+\delta)\}$, notice that it is a club. By Ulam's theorem there are $\{S_\gamma \subseteq C \ : \ \gamma < \lambda^+\}$ stationary sets such that
$\gamma_1 \neq \gamma_2 \Longrightarrow S_{\gamma_1} \cap S_{\gamma_2} = \emptyset$ and for all $\gamma < \lambda^+$ we have that $\Phi_{\lambda^+}(S_\gamma)$ holds.

For every $\delta < \lambda^+$ such that $\delta = \lambda(1+\delta)$, $h : \delta \to \delta$ and $\eta, \nu \in {}^\delta 2$ let

$$F(\eta,\nu,h) := \begin{cases} 1 & \text{if } h : M_\eta \hookrightarrow M_\nu \text{ and} \\ & M_{\eta^\frown 0} \\ & \uparrow \text{id} \\ & M_\eta \xrightarrow[h]{} M_{\nu^\frown 0} \\ & \text{can be amalgamated.} \\ 0 & \text{Otherwise.} \end{cases}$$

By $\Phi_{\lambda^+}(S_\gamma)$ pick $g_\gamma : \lambda^+ \to 2$ such that
for all $\eta, \nu \in {}^{\lambda^+}2$ and every $h : \lambda^+ \to \lambda^+$ we have that

$$S'_\gamma := \{\delta \in S_\gamma \ : \ F(\eta \restriction \delta, \nu \restriction \delta, h \restriction \delta) = g_\gamma(\delta)\}$$

is stationary. For $X \subseteq \lambda^+$ and every $\delta < \lambda^+$ let

$$\eta_X[\delta] := \begin{cases} g_\gamma(\delta) & \text{if } \delta \in S_\gamma \text{ and } \delta \in X \\ \\ 0 & \text{Otherwise.} \end{cases}$$

Notice that since $\{S_\gamma \ : \ \gamma < \lambda^+\}$ are pairwise disjoint for any $\delta \in X$ there is at most one γ such that $\delta \in S_\gamma$ (maybe none), so η_X is well defined. We finish by showing:

CLAIM 8.2. *For every $X \neq Y \subseteq \lambda^+$ we have that $M_{\eta_X} \ncong M_{\eta_Y}$.*

PROOF. Suppose $X \neq Y \subseteq \lambda^+$, $h : M_{\eta_X} \hookrightarrow M_{\eta_Y}$. An application of $\Phi_{\lambda^+}(S_\gamma)$ to η_X, η_Y and h yields S'_γ as above.
Let $D := \{\delta < \lambda^+ \mid (h \restriction \delta) : \delta \to \delta\}$ be a club. For $\gamma < \lambda^+$ let $S''_\gamma := S'_\gamma \cap D$. Without loss of generality there exists $\gamma \in X - Y$. Pick $\delta \in S''_\gamma$.

Denote by η the sequence $\eta_X \restriction \delta$ and by ν the sequence $\eta_Y \restriction \delta$. Since $\gamma \notin Y$ by the definition of the sequence η_Y we have that $\eta_Y[\delta] = 0$, namely $\nu \lessdot \nu^\frown 0 \lessdot \eta_Y$.

Now consider $\eta_X[\delta]$. There are two possibilities (according to the value of $\eta_X[\delta]$):

1. If $\eta_X[\delta] = 1$, by the definition of $\eta_X[\delta]$ necessarily $\eta_X[\delta] = g_\gamma(\delta)$, since $\delta \in S'_\gamma$ we have that $F(\eta, \nu, h) = 1$. From the definition of F we get that $M_{\eta^\frown 0}$ and $M_{\nu^\frown 0}$ can be amalgamated over M_η. Denote by M^1 the amalgam and let f and g be such that the following diagram commutes:

$$(*)_1 \qquad \begin{array}{ccc} M_{\eta^\frown 0} & \xrightarrow[g]{} & M^1 \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle f} \\ M_\eta & \xrightarrow[h]{} & M_{\nu^\frown 0} \end{array}$$

   Since $\eta^\frown 1 \lessdot \eta_X$, $\nu^\frown 0 \lessdot \eta_Y$ and from the assumption that $h : M_{\eta_X} \hookrightarrow M_{\eta_Y}$, the following diagram

$$\begin{array}{ccc} M_{\eta^\frown 1} & \xrightarrow[h]{} & M_{\eta_Y} \\ \uparrow & & {\scriptstyle \mathrm{id}}\uparrow \\ M_\eta & \xrightarrow[h]{} & M_{\nu^\frown 0} \end{array}$$

   must commute. By axiom $A4$ there exists $M^2 \prec_{\mathcal{K}} M_{\eta_Y}$ of cardinality λ such that $h(M_{\eta^\frown 1}) \prec_{\mathcal{K}} M^2$ and $M_{\nu^\frown 0} \prec_{\mathcal{K}} M^2$.

Namely the following diagram commutes:

$(*)_2$
$$\begin{array}{ccc} M_{\eta^\frown 1} & \xrightarrow[h]{} & M^2 \\ \uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_\eta & \xrightarrow[h]{} & M_{\nu^\frown 0} \end{array}$$

Since

$$\begin{array}{ccc} M^1 & & \\ \uparrow{\scriptstyle f} & & \\ M_{\nu^\frown 0} & \xrightarrow[\mathrm{id}]{} & M^2 \end{array}$$

by the second half of requirement $(4)'$ there are $M^3 \in K_\lambda$ and $\mathcal{K}$-mappings $e_l : M^l \hookrightarrow M^3$ (for $l = 1, 2$) such that

$(*)_3$
$$\begin{array}{ccc} M^1 & \xrightarrow[e_1]{} & M^3 \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle e_2} \\ M_\nu & \xrightarrow[\mathrm{id}]{} & M^2 \end{array}$$

is commutative. Combining $(*)_1$, $(*)_2$ and $(*)_3$ together we get

$(*)_4$
$$\begin{array}{ccccc} M_{\eta^\frown 0} & \xrightarrow[g]{} & M^1 & \xrightarrow[e_1]{} & M^3 \\ \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle e_2} \\ M_\eta & \xrightarrow[h]{} & M_\nu & \xrightarrow[\mathrm{id}]{} & M^2 \\ \downarrow{\scriptstyle \mathrm{id}} & & & \nearrow{\scriptstyle h} & \\ M_{\eta^\frown 1} & & & & \end{array}$$

Thus we have that

$$\begin{array}{ccc} M_{\eta^\frown 1} & \xrightarrow{h \circ e_2} & M^3 \\ \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle g \circ e_1} \\ M_\eta & \xrightarrow[\mathrm{id}]{} & M_{\eta^\frown 0} \end{array}$$

is commutative, which is a contradiction to the first half of requirement $(4)'$.

2. If $\eta_X[\delta] = 0$ then we have that $\eta^\frown 0 \lhd \eta_X$. Since $h : M_{\eta_X} \hookrightarrow M_{\eta_Y}$ we get that M_{η_Y} is an amalgam of $M_{\eta^\frown 0}$ and $M_{\nu^\frown 0}$ over M_η so by the definition of F we get that $F(\eta, \nu, h \restriction \delta) = 1$. Since $\delta \in S''_\gamma$ we have that $g_\gamma(\delta) = 1$ which by the definition of η_X gives $\eta_X[\delta] = 1$ and this contradicts the assumption of this case.

□

□

9. The main problems

In my opinion most of the future results will come from two separate lines of research, that many years from now will merge.

1. *Stability theory for AECs*. The most ambitious form of this program can be stated as follows: Let MT stand for Morley's categoricity theorem, f.o. stability for the contents of Shelah's book ([**Sh c**]) and let n.e.categ stand for the partial results toward Shelah's categoricity conjecture (the contents of [**Ke2**],[**Sh48**], [**Sh87a**], [**Sh87b**], [**MakSh**], [**KoSh**], [**Sh 394**], [**Sh472**], [**Le1**]).

   By developing stability theory for AECs I mean solving the equation

   $$(*) \quad \frac{\text{f.o. stability}}{\text{MT}} = \frac{x}{\text{n.e.categ}}.$$

   Examples of progress toward this direction can be found in [**Sh3**], [**Sh88**], [**Sh 300**], [**GrHa**], [**Gr3**],[**Gr4**], [**GrHa**],[**GrLe1**],[**GrLe2**], [**GrLe4**], [**GrSh1**], [**GrSh2**], [**GrSh3**], [**GrSh4**],[**Le1**], [**Le2**],[**Sh 576**], [**Sh 600**],[**ShVi**], [**GrVa**],[**Kov1**] and [**Va**].

2. *Geometric model theory for AECs*. This program can be described as an attempt to solve the following equation:

   $$(\#) \quad \frac{\text{HZ}}{\text{f.o. stability}} = \frac{?}{x}$$

   where HZ stands for Hrushovski's extension of Zilber's geometric ideas, and x is a (partial) solution for $(*)$. There very few results in this direction. Among them are [**Le1**], [**GrLe3**] and [**Le3**].

I expect that several interactions of $(*)$ and $(\#)$ will eventually yield among other things a solution for Shelah's categoricity conjecture.

While I am convinced that eventually the theory will have more applications to main stream mathematics via commutative algebra, algebraic geometry or analytic structures than model theory of first-order logic, so far there are no applications in sight.

The greater potential is due to the ability to axiomatize local finiteness and structures satisfying various chain conditions. It is too early to predict what exactly these applications will be. It is natural to expect that studying AECs of some concrete structures (rings and groups) may produce valuable results.

There is one conjecture that may eventually be solved using non elementary methods. This is Zilber's conjecture (stronger than Schanuel's conjecture) concerning analytic structure from [**Zi**]. Namely that $\mathbb{C}_{\exp}$ is the canonical structure of cardinality $2^{\aleph_0}$ in the class $\mathcal{H}(\text{ex}/\text{st})$.

The following are major concrete problems in AEC:

1. In Makkai-Shelah [**MakSh**] as well as in [**Sh 394**] and [**Sh472**] partial results toward a categoricity theorem are presented. In both cases the hypothesis is that the class is categorical in a successor cardinal. It is not clear at all if this is just a technical limitation or a central problem. I suspect it is central. Probably replacing the assumption that the class is categorical in λ^+ with the assumption of categoricity in λ is of similar difficulty.
2. The categoricity theorem in [**Sh 394**] is a going down theorem. Is there a going up theorem? The simplest instance of this is:

   CONJECTURE 9.1. *Let $\mathcal{K}$ be an AEC. If there exists $\lambda \geq Hanf(\mathcal{K})$ such that $\mathcal{K}$ is categorical in λ then $\mathcal{K}$ is categorical in λ^+.*

3. It is a major open problem to find a nice (forking-like) dependence notion for AEC. In fact even under the assumption that $\mathcal{K}$ has the amalgamation property and the class is categorical in a cardinal above $Hanf(\mathcal{K})$ this is open (see Remark 4.10(1) in [**Sh 394**]).
4. One of the technical problems of working in AEC without the amalgamation property is the inexistence of monster models and therefore types over models can not always be extended to global types. To deal with this Shelah and Andres Villaveces have introduced in ([**ShVi**]) an interpolant, which is the framework of AEC without maximal models, under GCH. They have managed to show that categoricity implies that every small model (below the categoricity cardinal) can be extended to an amalgamation base and several other basic facts. Some extensions of this work can be found in [**Va**]. As of today there are no categoricity results in this context. It is natural to expect the following:

   CONJECTURE 9.2. *Let $\mathcal{K}$ be an AEC without maximal models and suppose that $\lambda \geq Hanf(\mathcal{K})$. If $\mathcal{K}$ is categorical in λ^+ then $\mathcal{K}$ is categorical in every $\mu \leq \lambda$.*

5. Probably the appropriate name for this is *existence of Hanf number for amalgamation*:

   CONJECTURE 9.3. *Let $\mathcal{K}$ be an AEC. Suppose (for simplicity?) that $\mathcal{K}$ does not have maximal models. There exists a cardinal number $\mu(\mathcal{K})$ such that if $\mathcal{K}$ has the $\mu(\mathcal{K})$-amalgamation property then $\mathcal{K}$ has the λ-amalgamation property for all $\lambda \geq \mu(\mathcal{K})$.*

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