A Course in Model Theory I:

Introduction[1]

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[1]This **preliminary draft** is dated from August 27, 2022. The book will be published by Cambridge University Press. The book is approximately 99.908% complete, I expect that the final version will have about 810 pages, many sections of the current version will be revised and few will be added. I hope to have a stable version of this volume soon.
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