A Course in Model Theory I:

Introduction[1]

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[1]This **preliminary draft** is dated from October 30, 2021. The book will be published by Cambridge University Press. The book is approximately 99.853% complete, I expect that the final version will have about 810 pages, many sections of the current version will be revised and few will be added. I hope to have a stable version of this volume soon.
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Preface

1. About this book

> I hope that my errors this time will not be so numerous and/or obtrusive that you will be discouraged from reading the material carefully. I did try to make it both interesting and authoritative, as far as it goes. But the field is so vast, I cannot hope to have surrounded it enough to corral it completely. Therefore I beg you to let me know about any deficiencies you discover.
>
> Donald E. Knuth 2001[1]

DISCLAIMER: This book is a *work in progress*, soon I will have a significantly different version. I expect the final version to have about 800 pages and to be different from the current one.

This version is distributed for the convenience of students who are currently taking my course. Only Chapters 1 - 7 are near final form (some sections, theorems and proofs are still missing). Some of the chapters are outdated and will be replaced with different (Chapter 8 and 9 are among them). A measure of the maturity of a section is the number of exercises. The rest will change significantly in the future.

This book at the current form is not intended for general distribution.

If your copy is more than a month old, please contact me for the current version.

I very much appreciate any comments you may have. If you are interested in receiving revisions, please let me know.

[1] Knuth offers paying $2.56 for any discovery of an error. Taking into consideration the current stage of the book and my income, I afford 2c per error.

What is Model Theory? I will not attempt to give a short and concise definition, however it is interesting to note that in two important books such compact definitions were attempted.

In their 1973 book [**CK**], Chang and Keisler presented the following equation:

universal algebra+logic = model theory.

In his 1997 book [**Ho3**] Hodges included

algebraic geometry-field theory = model theory.

In my opinion both definitions were very good and captured what most model theorists considered at the time to be central.

Marker in his 2002 Introduction [**Ma**] writes:

> Model Theory is a branch of mathematical logic where we study mathematical structures by considering the first-order sentences true in those structures and the sets definable by first-order formulas.

In my opinion Marker's description, while correctly describe the contents of his book, is overly narrow and was an incorrect description of the subject.

I decided not to attempt such a simple definition of the subject, my views on the subject are reflected by the contents of this book.

In my opinion, the state of our current understanding of model theory is far from being satisfactory. I strongly believe that we are not unlike the prisoners in Plato's cave allegory in *The Republic*, book VII (reproduced in the Appendix). Some of the structure of this book is an example of Plato's allegory, having gradual refinements and generalizations of the concepts.

Observing developments in model theory in the last 40 years, it seems to me that at present, several central notions, properties and concepts still need to be discovered and what we know now are just shadows of yet to be discovered objects.

This book contains little more than what in my opinion should be covered in an introduction to model theory. It is designed to be used as a textbook, I have included enough material for little more than a couple of one-semester courses in model theory. The material was presented in courses given at Carnegie Mellon University from 1989 to 2019. Some of the material was discussed at the Berkeley model theory seminar during the academic year 2002/2003. My main goal is to present some of the important ideas of model theory in an accessible way to

advanced undergraduate and beginning graduate students. I attempt at times to describe how some of the ideas were developed. There are also comments at the end of the book on the historical evolution of the field[2].

The typical audience of the first semester of my first course is usually a mixture of undergraduate and first year graduate students. The audience for the more advanced courses consisted of mainly graduate students. The choice of material for the courses was guided by my opinion of the scientific value as well as personal likes and dislikes.

I found it difficult to write a non-technical mathematical introduction, an attempt is in the next section. A further introduction is in the last section of Chapter 3. Since it uses some of the basic model-theoretic concepts, it seems to me to be better to present it after learning little model theory. The reader is encouraged to attempt reading that section any time.

As it can be seen from the table of contents, I focus almost entirely in what is known as "pure model theory", in the sixties and seventies it was known in the USA by the name of "west coast model theory" as opposed to "algebraic model theory" that was also known as "east coast model theory". The first was promoted and developed by Alfred Tarski and his students at Berkeley. The latter initiated in North America by Abraham Robinson (and his students in Yale), its focus are applications of model theory to other branches of mathematics (outside logic), originally most applications were limited to algebra and elementary analysis. The development of pure model theory was motivated by internal problems of the field rather than by specific applications in mind. Already in the mid sixties it was clear that the techniques and results of the two branches of model theory are different. A typical result of the east coast school have used very little machinery from model theory or logic (not much more than use of the compactness theorem or some other basic facts) and the bulk of the proof was typically an argument borrowed from algebra or sometimes from analysis. The logical depth and the set theoretic complexity of west coast arguments were significantly greater. Echoing some of their early childhood arguments often model theorists (even when sober) claimed that what they are doing is more important and more difficult than the other camp. One of the leaders of the east-coast school in the late eighties managed to get a reputation of claiming that "the only important theorem of model theory is the compactness theorem". I have started writing this book with the intention of refuting this stupid and ignorant view.

In the early nineties Ehud Hrushovski managed to shake the east-coast school by discovering several applications of model theory to algebraic geometry, diophantine geometry and differential algebra. See [], []. These applications at

[2]Unfortunately often a complete description of the history interferes with the mathematical presentation, which is the primary goal of this book. In order to minimize the interruption to the mathematical flow of ideas, I limited the historical comments in the text. Chapter 14 is dedicated to additional but incomplete historical information.

present are the deepest applications of mathematical logic to fields outside logic. One of the most interesting philosophical aspects of Hrushovski's work is that he was using *pure model theoretic concepts and methodology*, he applied directly some of the more abstract ideas and techniques to specific problems involving abelian varieties (solving the Lang-Mordel conjecture for positive characteristics) and obtaining a new proof to Manin's theorem.

When several model theorists published a collection of survey articles (See [**Bou**]) aimed to outline a short path to Hrushovski's proof of the Lang-Mordel conjecture. In the first paragraph to the introduction for that volume, the editor Elisabeth Bouscaren wrote:

> Model theorists have often joked in recent years that the part of mathematical logic known as "pure model theory" (or stability theory), as opposed to the older and more traditional "model theory applied to algebra", turns out to have more and more to do with other subjects of mathematics and to yield genuine applications to combinatorial geometry, differential algebra and algebraic geometry.

My book has three goals:

The book was written from the point of view of pure model theory, I hope that it will help to open a gate to this exciting branch of mathematics.

A secondary goal is to offer a preparation for reading more advanced works such as Shelah's [**Sh c**], [**Sh e**], [**Sh h**], [**Sh i**], [**Sh j**] and Pillay's [**Pi2**]. I hope that this book will not be considered only as a background to read Shelah's [**Sh c**] and [**Sh i**] but will bring the interested student closer to some of the research frontiers of model theory while keeping everything as simple as possible[3].

My third goal is to stress the connections with set theory and algebra. This is not limited only to applications but in inspiring the theory. It is important to recognize that some of the central and sophisticated concepts of model theory were discovered by a set-theoretic analysis. I hope to be able to convince the reader of the importance of the set theoretic point of view in discovering and studying many of the fundamental concepts of model theory. While I decided to restrain myself from preaching this philosophy I feel that from the pragmatic point of view it is important to recognize this. I am confident that many of the future developments in model theory will continue to be influenced by such a set theoretic analysis.

I made an effort to present some of the ideas and concepts of model theory in the simplest form, often special cases of more general theorems. Thus like the prisoners in Plato's allegory, slowly and gradually I expose the reader to more and more details of reality as I understand it at present.

---

[3]My wife suggested to name this book "Classification Theory for Dummies". For several reasons I did not adopt her suggestion.

While I think that in mathematics and model theory in particular, we have to deal with general situations. I resent the style of some authors who drop on the reader the most general version of a theorem known to them[4], with many sub cases, while the proof of the main and most important case can be explained in much clearer way with fewer details. Some may call this approach "anti-Bourbaki" I view this as "student friendly".

Since I wanted to write a real text-book (as opposed to an encyclopedia), I had to compromise and leave out many topics that some experts may consider to be essential. Among the topics left out, some are dear to me. The criteria as of what to include and exclude is difficult to state. I made an effort to include everything that I consider to be basic and important. Certain topics were omitted due to one (or more) of the following: It is too advanced to be included here, I don't know it, I don't like it, this book is finite or there is already an excellent exposition in a textbook.

Often I have chosen more powerful arguments than necessary for stated results. The benefit of this may not be clear at first reading, however I hope that the reason for added "power" to some of the proofs will become clearer as the reader progress.

The reader should realize that the exercises are an integral part of the text. Some of the exercises are used in later parts. It is strongly recommended to attempt doing at least 60% of them before going to later sections. The reader as making progress in the book may benefit from returning to earlier sections and attempting most of the remaining unsolved exercises from previous sections. There are four kinds of exercises: Some test (and develop) basic technical skills, some present extensions of theorems in the text, some are to make the reader to fill simple gaps that were left in the text (they are there to force you to do them), and the last kind (less than 5%) are difficult, some students need several hours just to understand them and less than half of my students were able to do them in a reasonable amount of time. Some of these exercises are marked by the word *Hard*, since difficulty is often a subjective term, you may find some of the marked exercises not difficult and may have difficulties with few unmarked exercises.

In *The emperor's handbook*, 12.6 Marcus Aurelius gives the reader the following advice:

> Practice also the things you don't expect to master.

In my opinion this advice is good also for the exercises in this book.

The very last Chapter is a guide to the literature.

[4]To be replaced 10 years later by a little more general statement.

PREREQUISITES: Elementary logic, complete set of connectives, basic facts like $\neg(\forall x)\varphi(x) \leftrightarrow (\exists y)\neg\varphi(y)$. See Goldstern and Ihoda [**GJ**]. It is assumed that the reader is familiar with elementary naive set theory. The expected level of proficiency is no more than what is required in a course in General topology. We will be working in the most commonly accepted set theory known as *Zermelo Fraenkel Set Theory* (formally described in Definition 10.1 and denoted by ZF). ZFC stands for *Zermelo Fraenkel Set Theory with Choice*. In recent years, especially the last four decades many logicians treat ZF and sometimes even ZFC with reverence I have seen given only to the bible. For several reasons I am far from a believer in ZF, however I selected this framework as it is at present the most efficient precise framework available to present model theory. Chapter [**?**] contains a crash course in elementary set theory. If everything there is new to you I suggest reading only the first two or three pages and return to it as needed.

I hope that the reader have seen a couple of applications of the axiom of choice, transfinite induction as well as some basic facts from cardinal-arithmetic: Like the theorem of ZFC that states that for infinite cardinals λ and κ we have that $\lambda \cdot \kappa = \lambda + \kappa = Max\{\lambda, \kappa\}$ and what is known as the Cantor-Bernsetin theorem that asserts if there exists $f : A \to B$ and $g : B \to A$ injections then there exists a bisection $h : A \to B$. All the set theory used beyond the above is developed in the text. My favorite book of elementary set theory is the book "Introduction to set theory" by Karel Hrbacek and Thomas Jech, [**HJ**]. Azriel Levy's "Basic Set Theory" [**Lev**] is also an excellent introduction aimed at the more advanced student. The volumes [**JW96**], [**JW97**] by Winfried Just and Martin Weese are aimed at a little more advanced audience and contains a very good coverage and emphasize some of the *applicable* aspects of set theory.

Since applications to algebra are important in the development of model theory, I have included several basic algebraic examples that hopefully will illuminate the material. It is desirable (but not necessary) that the reader is familiar with the basic concepts of abstract algebra or at least is taking concurrently such a course (covering most of Herstein's classic [**He**], Dummit & Foote's [**DF**] or the basics of Hungerford [**Hu**]). In case the reader is willing to skip several examples it is possible to follow the material with virtually no knowledge of algebra.

NOTATION Everything is standard. We will treat functions as set-theoretic objects and not objects of category-theory. Namely a function $f : A \to B$ is a subset of $A \times B$ with certain properties.

Recall

$$\begin{aligned}
&\mathrm{Rang}(f) := \{b \in B \mid \exists a \in A[f(a) = b]\},\\
&\mathrm{Dom}(f) := \{a \in A \mid f(a) \in B\},\\
&f[X] := \{b \in B \mid \exists a \in X[f(a) = b]\} \text{ for } X \subseteq A,\\
&f^{-1}[Y] := \{a \in A \mid \exists b \in Y[f(a) = b]\} \text{ for } Y \subseteq B,\\
&\text{For } f : A \to B \text{ by } f \restriction X \text{ denote the set } f \cap (X \times B).
\end{aligned}$$

When f and g are functions, $f \subseteq g$ is equivalent to

$$\mathrm{Dom}(f) \subseteq \mathrm{Dom}(g) \text{ and } f(x) = g(x) \text{ for all } x \in \mathrm{Dom}(f).$$

ω stands for the set of non-negative integers (the natural numbers).

$\alpha, \beta, \gamma, \delta$ and ξ stands for ordinals, On is the (proper) class of ordinals, since the ordinal α is the set

$$\{\beta \in \mathrm{On} \mid \beta < \alpha\}$$

we often identify ordinals with the set of its smaller ordinals, when n is a natural number then $n = \{0, \ldots, n-1\}$, so in particular 2 stands for the set $\{0, 1\}$. The class of ordinals is linearly ordered by $\in$, so at times instead of $\alpha \in \beta$ we will write $\alpha < \beta$ for ordinals α and β

An ordinal λ is an *initial ordinal* which means that if $\alpha < \lambda$ then there is no injective function from λ to α. The first infinite ordinal is denoted by ω and at times by $\aleph_0$.

λ, κ and μ are cardinal numbers, Card is the class of cardinal numbers.

φ and ψ stand for formulas,

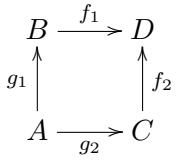
k, l, m and n natural numbers. The first infinite ordinal is ω, it stand also for the set of natural numbers.

Often we deal with *sequences* that are functions whose domain is an ordinal number. The letters η, ν and ρ will usually be reserved for sequences. By $\ell(\eta)$ denote the *length of* η which is the domain of the function η. For sequences η and ν use $\eta \lhd \nu$ to denote that η is an *initial segment* of η, namely $\eta[i] = \nu[i]$ for all $i < \ell(\eta)$.

For ordinals α and β denote by ${}^{\alpha}\beta$ the set of all sequences of length α of ordinals less than β. ${}^{\alpha>}\beta := \bigcup_{\gamma<\alpha} {}^{\gamma}\beta$ and ${}^{\alpha\geq}\beta := \bigcup_{\gamma\leq\alpha} {}^{\gamma}\beta$. E.g. ${}^{\omega>}2$ denotes the set of all finite sequence of zeros and ones.

Commutative Diagrams. Let A, B and C be sets. Suppose $g_1 : A \to B$, $f_1 : B \to C$, $f_2 : C \to D$ and $g_2 : A \to C$ we say that the following *diagram*

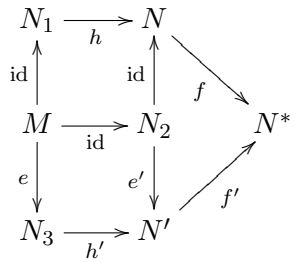
commutes



is a short hand for the following statement:

$$\forall a \in A[f_1(g_1(a)) = f_2(g_2(a))].$$

The statement that the diagram



commutes, stands for the following statements:

$$\forall a \in M[[h(a) = a] \wedge [f(h(a)) = f'(h'(a))] \wedge [e'(a) = h'(e(a))]]$$

and

$$\forall a \in N_2[f(a) = f'(e'(a))].$$

Acknowledgments.

THIS IS NOT FINAL

Mathematical....

The treatment of the rank functions in Chapter **??** benefited much(NOT DONE YET) from the article [**GKTV**] coauthored with Alexei Kolesnikov, Ivan Tomašić, and Monica VanDieren.

Part of Chapter 9 is based on [**GIL**] coauthored with José Iovino and Olivier Lessmann that was inspired by my lectures.

Chapter **??** started as lecture notes taken by my students in short course dedicated to the classification theory for non-elementary classes. Without the efforts and dedication of students in that class Chapter **??** would not exist. It is based on lecture notes taken by John Kruger, Olivier Lessmann, Nate Segerlind and Monica VanDieren.

For some of the material (especially the parts about stability) I have used lecture notes from my classes taken by my former students: Olivier Lessmann and Monica VanDieren.

Many serious errors and omissions in preliminary versions were discovered by Bart Kastermans and Alexei Kolesnikov.

I would like to express my gratitude to Andres Villaveces for experimenting in using this text in courses given on several occasions at the National University of Colombia at Bogotá and at Carnegie Mellon University, also for many constructive discussions. Angela Weiss and ...

2. A mathematical introduction to the book

The conjecture that set the direction for research in pure model theory from the mid fifties to 1970 was:

CONJECTURE 2.1 (Jerzy Loś [**Lo54**]). *Let T be a countable first-order theory. If there exists a cardinal number $\lambda > \aleph_0$ such that T is categorical in λ then T is μ-categorical for every $\mu > \aleph_0$.*

Vaught's theorems that categoricity implies completeness of the theory (see Theorem 3.26) and decidability (of theories with recursive axiomatization) as well as his theorem asserting that no countable complete first-order theory can have exactly two countable models (see Theorem II.9.40) are byproducts of early attempts to make progress toward Loś's conjecture.

The next landmark development occurred in 1962, when Michael Morley proved in his Ph.D. thesis that if a countable theory has a unique model (up to isomorphism) in some uncountable cardinal, then it has a unique model in every uncountable cardinal.

THEOREM 2.2 (Morley [**Mo1**]). *Let T be a first-order theory in a countable language. If there exists a cardinal $\lambda > \aleph_0$ such that T is categorical in λ then T is μ-categorical for every $\mu > \aleph_0$.*

Morley for his proof identified an important class of first-order theories he named *totally transcendental* (for countable T they coincide with the class of $\aleph_0$-*stable theories*). For his proof he introduced several important concepts to model theory. Among them are the notions of a *rank* and *prime model over a set*.

Chapter VI is dedicated to the proof of Morley's theorem (I present three different proofs) and for some of its extensions.

On page 537 of [**Mo1**] Morley raised the question whether in his theorem the countability assumption of the language of T can be removed, he asked the following:

QUESTION 2.3. Let T be a first-order theory. If there exists a cardinal number $\lambda > |\,\mathrm{L}(T)| + \aleph_0$ such that T is categorical in λ then T is μ-categorical for every $\mu > |\,\mathrm{L}(T)| + \aleph_0$.

Several people in the sixties obtained significant partial positive results on Morley's question: The most important work was carried out by Frank Rowbottom, J. P. Ressayer and Shelah. The question was eventually answered affirmatively by Saharon Shelah in 1970 (See [**Sh31**]). For his proof he introduced the class of *superstable theories* (see Definition 4.1.2 below) and the notion of *weakly minimal formula* generalizing *strongly minimal formulas* (see Definition 3.**??**).

In the early seventies Shelah set his goal at the a very ambitious target: Given an arbitrary first-order theory T what are the elements of $\mathrm{Mod}(T)$ up to isomorphism[5]? Shelah realized that the above vague question must be made precise in order to be able to measure progress. Realizing that there is a need for well defined test-questions, he suggested several conjectures (in [**Sh 10**] and [**Sh31**]) concerning the behavior of several cardinal-valued functions, among them is the function $\lambda \mapsto \mathrm{I}(\lambda, T)$ as important mile-stones in measuring progress toward complete understanding of $\mathrm{Mod}(T)/\cong$. In a process that lasted about 15 years, yielded many papers and culminated in his book [**Sh c**], he identified finitely many classes of theories (with the independence property, strict-order property, simple, order property, stable, dimensional-order property, omitting types-order property, super stable, multidimensional and unidimensional) that effects the asymptotic behavior of the function $\lambda \mapsto \mathrm{I}(\lambda, T)$. For this elaborate analysis he introduced several rank functions that among other things generalizes Krull's dimension from commutative algebra (Definition 3.1.2) as well as the important dependence relation called *forking* (see Definition **??**.1.8) which simultaneously generalizes linear dependence in a vector space and algebraic dependence in a field.

The book is organized as follows:

Chapter 1: Is dedicated to the Fundamentals. There are several sections marked with an asterisk where results that will be not used in later sections are presented. Among them is the celebrated Paris-Harrington theorem (a Ramsey-like theorem which is true for the integers but not provable from Peano's axioms).

Chapter 2: An Abstract Elementary Class is a class of structures all of the same similarity type together with a notion of embedding satisfying some natural axioms. We introduce the basic notions to this topic that is almost category-theoretic approach of model theory rather than the syntactic one that depends of language and formulas in some logic. This is a unifying approach to model theory that allows us to discover additional structure among class of models of a first-order theory and more importantly deal with several extensions of first-order logic.

Chapter 3:

Chapter 4: We introduce types, which are sets of formulas that together with saturated models provides us with very important technical tools by diverting attention from definable sets to orbits of the automorphism group in analog way to what is done in Galois theory.

Several two-cardinal theorems are presented. The important omitting types theorems of Henkin, Vaught and Morley are presented. We connect omitting types theorems with two cardinal theorems (in Chapter 3 they are connected with categoricity).

Chapter 7: We present three proofs to Morley's theorem, it was the first deep result in model theory and had great impact on the field. Show that

[5]In the late seventies Shelah named this subfield of pure model theory *Classification Theory*.

a theory categorical in some $\lambda > |T| + \aleph_0$ is stable in every μ where $|T| \leq \mu < \lambda$. Introduce an ordinal-valued rank function $R[\cdot]$ from type definable subsets of the monster model which unlike Morley's rank can be extended to $L_{\omega_1,\omega}$. Several basic properties of this function are presented Among them:

Theorem. Let T be a complete first-order theory.

(1) If T is in a countable language then the following are equivalent:
   (a) $R[\mathfrak{C}] < \infty$,
   (b) $R[\mathfrak{C}] < \omega_1$,
   (c) T is $\aleph_0$-stable.

(2) If $R[\mathfrak{C}] < \infty$ and $S \subseteq |\mathfrak{C}|$ then we have the following important properties for a type-definable sets. Suppose p is a type (maybe with parameters from $\mathfrak{C}$) such that $S = p(\mathfrak{C})$ then
   invariance: Let $B \subseteq \mathfrak{C}$ be the set of parameters of formulas in p, if $f \in \mathrm{Aut}_B(\mathfrak{C})$ then $R[S] = R[f[S]]$.
   finite character: There exists a finite $q \subseteq p$ such that $R[q(\mathfrak{C})] = R[p(\mathfrak{C})]$. Moreover, when p is a complete type there exists a formula $\varphi(\mathbf{x}; \mathbf{a}) \in p$ such that $R[\varphi(\mathfrak{C}; \mathbf{a})] = R[q(\mathfrak{C})]$
   monotonicity: $R[S] \geq R[S']$ for every $S' \subseteq S$
   uniqueness of extension: Suppose $p_1, p_2 \in S(A) \supseteq q$ such that $R[p_1(\mathfrak{C})] = R[p_2(\mathfrak{C})] = R[q(\mathfrak{C})]$ and is an ordinal then $p_1 = p_2$.

We introduce the first (of many) notions of independence: Let $p := \mathrm{tp}(\mathbf{c}/B)$ for $A \subseteq B$ we say that *p does not split over A* provided $\mathrm{orbit}_B(\mathbf{c}) = \mathrm{orbit}_A(\mathbf{c})$, where $\mathrm{orbit}_X(\mathbf{c})$ is the orbit of $\mathbf{c}$ under the action of $\mathrm{Aut}_X(\mathfrak{C})$ on the set ${}^{\ell(\mathbf{c})}|\mathfrak{C}|$. We use the rank to establish that $\aleph_0$-stability implies that every type does not split over a finite subset of its own parameters. This is used to find long sequence of indiscernible in arbitrary sets. Another application of the rank is to manufacture prime models.

Chapter 8: Contains basic facts about stable theories. Several characterizations of stability are presented: In terms of the negation of the order property, definability of types and boundness of a rank function. Also it is shown that unstable theory has either the independence property or the strict-order property. We introduce several families of rank functions show that one of them can be used to characterize superstable theories. Have a general result about existence of indiscernible sequences. For a given a complete first-order theory T the following problem that was motivated by combinatorial set theory:

PROBLEM 2.4 (stability spectrum). *Characterize the class*

$$\{\lambda \in \mathrm{Card} \mid T \textit{ is stable in } \lambda\}.$$

The stability spectrum question had great impact on the development of classification theory. We present in this chapter a complete characterization of the stability spectrum of a *countable* theory.

Chapter 9: Forking is the most important model-theoretic concept introduced since Morley's work in the early sixties. Forking is a relationship between three sets A, B, C we denote by $A \underset{B}{\downarrow} C$. This relation is defined as: $\mathrm{tp}(\mathbf{a}/C)$ is *determined* by $\mathrm{tp}(\mathbf{a}/B)$ for all finite $\mathbf{a} \in A$, where *determined* has a precise technical formulation (in fact several equivalent definitions). We will establish that for $\aleph_0$-stable theories the relation $A \underset{B}{\downarrow} C$ satisfy some very nice properties and is a generalization of linear dependence in vector spaces and also of algebraic dependence in field theory. It turns out that also for stable theories $A \underset{B}{\downarrow} C$ satisfy some nice properties and even for several interesting classes $A \underset{B}{\downarrow} C$ still has some weak properties. It turns out that in almost all known cases of existence of a *nice* dependence relation it is the relation $A \underset{B}{\downarrow} C$. This has far reaching effect as it allowed model theorists to discover structure and order in mathematical objects where such a theory was not known to exist.

Chapter ??: The area of *stable groups* have started with the following strong converse to Steinitz's theorem:

THEOREM 2.5 (Macintyre). *Let F be an infinite field. If $\mathrm{Th}(\langle F, +, \cdot \rangle)$ is categorical in an uncountable cardinal then F is algebraically closed.*

Instead of proving Macintyre's theorem, I present the more general result due to Cherlin and Shelah (Theorem 5.**??**) that weaken the hypothesis on the theory of the field to super-stability.

Some of the results are presented as study of stable and super-stable groups, this serve three different goals: It is a concrete exercise involving abstract concepts, prepares the necessary tools for the geometric theory, and is used for several applications.

Chapter ??: This chapter is dedicated to *forking* (the notion that generalizes linear dependence). While studying the basic properties of forking we also introduce the class of *simple theories* (a generalization of the class of stable theories that includes also some unstable theories e.g. the theory of the random graph –T_{ind}). We derive the finite equivalence relations theorem from the independence theorem.

The relationship between the classes of theories is summarized in the diagram above. The bolder lines stands for the more important properties. So stability and simplicity are considered to be more important than categoricity or even superstability. The diagram does not include everything is known, there many more interesting properties. I have included only the main classes of theories. Notice that this diagram is correct only for theories in a *countable language*. For theories in uncountable languages you should delete the t.t. property and replace *categoricity in an uncountable cardinal* with T is categorical in some cardinal greater than $|T| + \aleph_0$.

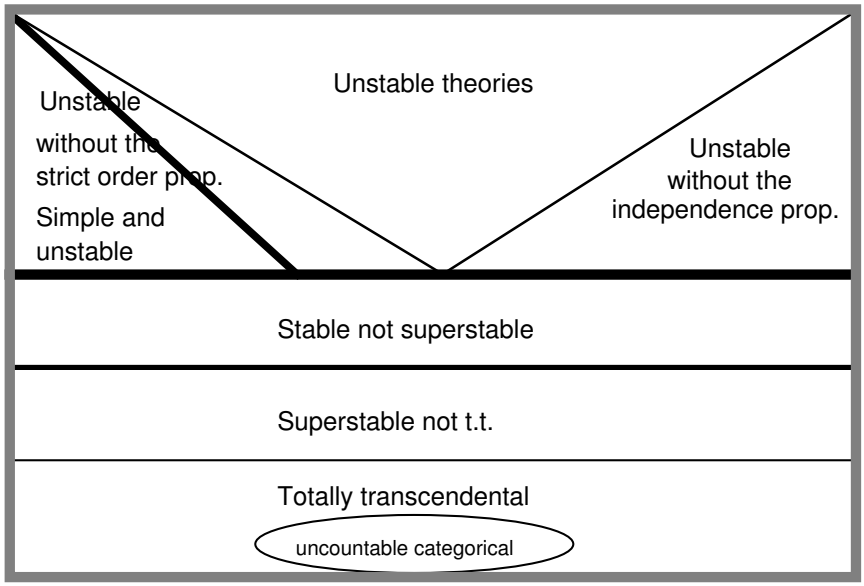


FIGURE 1. Classes of theories in a countable language

Chapter ??: We use the finite equivalence relations theorem to complete the stability spectrum theorem for all theories (including uncountable theories). This was the main reason for Shelah to introduce forking (i.e. computing the stability spectrum also for cardinals below $2^{|T|}$). As byproduct we get a proof of Alistair Lachlan's celebrated theorem (Lachlan's original proof of this theorem was a precursor of the finite equivalence relations theorem):

THEOREM 2.6 (Lachlan). *Let T be an $\aleph_0$-categorical theory. If T is superstable then T is $\aleph_0$-stable.*

We also solve the question when a union of saturated models is saturated? Also characterize λ-stable theories as the theories that have saturated models of cardinality λ for λ satisfying $\lambda^{|T|} > \lambda$. Finally we discuss canonical bases.

Chapter ??: Is about existence and uniqueness of prime models. We present

THEOREM 2.7 (Shelah). *Suppose T is $\aleph_0$-stable theory in a countable language. For every $A \subseteq \mathfrak{C}$ if M_ℓ are both prime models over A then there exists an isomorphism $f : M_1 \cong M_2$ which is the identity on A.*

The theorem is derived from a more general uniqueness fact. The theorem was the first application of stability theory to algebra. Together with Lenore Blums' result that the theory of a differentially closed field is $\aleph_0$-stable the uniqueness of differential closure follows answering an

old question of Ritt(?). This established an important link between model theory and differential algebra.

Chapter ??: Here we use the already established machinery to complete the proof of Loś conjecture (-Morley's theorem for uncountable languages).

Chapter ??:

Chapter ??: We present the basics of geometric model theory.

Chapter ??: The basics of classification theory for non-elementary classes.

The central test problem in the field (in its simplest form) is:

CONJECTURE 2.8 (Shelah's conjecture. (about 1977)). *Let T be a countable theory in $L_{\omega_1,\omega}$. If there exists $\lambda \geq \beth_{\omega_1}$ such that $\mathrm{I}(\lambda, T) = 1$ then $\mathrm{I}(\mu, T) = 1$ holds for every $\mu \geq \beth_{\omega_1}$.*

There are many partial results toward this conjecture. The total number of published pages (by Shelah) is more than 1000 pages and it seems that we are still far from a complete solution.

In this chapter we introduce some of the tools that where developed in an effort to solve Shelah's conjecture as well as extensions of stability theory to the context of Abstract Elementary Classes.

Chapter ??: While this chapter has a clear mathematical goal, the undelining thesis of this chapter and its sequal is more than the stated mathematical results.

THESIS 2.9. For several classes of first-order theories and some families of non-elementary classes it is possible to analyze precisely how information on countable models control uncountable models.

The main mathematical concept of this chapter is the notion of *excellent class*. An application is (a partial solution of Conjecture 2.8):

THEOREM 2.10 (Shelah). *Suppose $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$. For any sentence $\psi \in L_{\omega_1,\omega}$ in a countable similarity type. If*

$$\mathrm{I}(\aleph_{n+1}, \psi) = 1 \quad \textit{for every } n < \omega$$

Then ψ has models in all cardinals and is categorical in all uncountable cardinals.

Chapter ??: Is dedicated to prove what Shelah called in his 1990 book ([**Sh c**]) "The main theorem of this book". While this is not the most important result of this book (in my opinion it is not the most important problem of Shelah's book), its proof combines many of the methods introduced earlier and I think that its statement is very surprising:

THEOREM 2.11 (The Main Gap). *Let T be a countable first-order complete theory. There are exactly two possibilities:*

(1) $\mathrm{I}(\aleph_\alpha, T) = 2^{\aleph_\alpha}$ *for every* $\alpha > 0$ *or*

(2) *there exists a countable ordinal $Dp(T)$ such that*

$$\mathrm{I}(\aleph_\alpha, T) \leq \beth_{Dp(T)+1}(\aleph_0 + |\alpha|) \quad \textit{for every } \alpha.$$

Chapter ??: Is dedicated to what Shelah calls "non-structure theory". Among the results we will see several methods to construct models for unstable theories and some complicated Boolean algebras.
Chapter 11: Survey of further topics
Appendix 16: Is dedicated for basic facts about pregeometries (combinatorial geometries).
Chapter 12: Is a guide to the literature.

omitting-types theorems

Morley's theorem, Shelah's categoricity Uniqueness of prime models

structure/non-structure,

Shelah's/Church's thesis applications to algebra (Macintyre),

new applications to algebraic geometry Lang-Mordel (Hrushovski),

pregeometries, forking relation

In 1996 Hart-Hrushovski-Laskowski completed the computation of all the spectrum functions [**HHL**]

infinitary logics,

non-elementary classes (see [**Gr1**]).

Applications of fields see Fried-Yarden, vdDries, Marker-Messmer-Pillay.

Course outlines

DUE TO RECENT REARRANGEMENT OF MATERIAL THIS PAGE SHOULD BE IGNORED, I EXPECT TO UPDATE THIS PAGE AROUND JUNE 2012.

Sections marked by * are dead end, this does not necessarily mean that they are not important or esoteric, it just an indication in the sense that rarely there are references in later sections to stared sections. The point of view I usually take in my teaching does not permit coverage of more than one stared section per semester. In my opinion usually the unstared sections are of greater importance and often are more challenging.

(1) The following are suggestions for a one semester introduction:
   (a) 1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.12,
   2.1,2.2,2.3, 2.5, 2.6, 2.8, 2.9,
   3.1,3.2,3.3,3.4,3.5,3.6,3.7,
   4.1,4.2, 4.3.
   (b) Alternatively: 1.1,1.2,1.3,1.4, 1.5,1.6, 1.7,1.8,1.9,1.10,
   2.1,2.2,2.3,2.5,2.8
   3.1,3.2,3.3,3.4, 3.5,3.6,
   4.1,4.2, 4.3,4.5,4.6,4.9.
(2) The following is a suggestion for a two semester course:
   (a) The first semester:
   1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8, 1.10,
   2.1,2.2,2.4. 2.5, 2.6, 2.7,2.8, 2.9,2.11,
   3.1,3.2,3.3,3.4,3.5,3.6,
   4.1,4.2, 4.3, 4.4,
   5.1.
   (b) Second semester: 2.10,2.11,3.7,3.8,3.9,
   4.5,4.6,4.7,4.8,4.9,4.10,
   2.12,
   6.1,6.2,6.3,6.4,6.5.
(3) A three semester course:
   (a) First semester:
   1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.10,
   2.1,2.2,2.3. 2.5, 2.6, 2.7,2.8, 2.9,2.10,2.11,
   3.1,3.2,3.3,3.4,3.5,3.6,
   4.1,4.2, 4.3, 4.4.
   (b) Second semester: 2.7,
   3.7,3.8,3.9,
   4.5,4.6,4.7,4.8,4.9,4.10,
   2.12,
   6.1,6.2,6.3,6.4,6.5,6.6,
   5.1,5.2,5.3,5.4.

(c) Third semester: 7.1,7.2, 7.3, 7.4, 7.5, 7.6, 8,9, Chapter 10.
(d) Alternative third semester: 7.1,7.2, 7.3, 7.4, Chapter 15.

Part 1

Definability

CHAPTER 1

Fundamentals

Introduction

In this chapter the basic concepts of model theory are introduced and elementary connections between them are explored. Since it is assumed that the reader had some familiarity with elementary logic, the treatment of some of the basic concepts from predicate calculus is incomplete.

Some of the fundamental ideas of model theory are introduced in this chapter, several only implicitly.

Overview of the structure of this chapter:

Section 1: In order to establish a common background I introduce the basic notions of first-order formulas, models are introduced as generalizations of the known algebraic structures and the satisfaction relation is introduced.

Section 2: The concepts of elementary equivalence and isomorphism of models are presented as well as the most important cardinals to cardinals function in model theory, named the *spectrum function*: $\lambda \mapsto \mathrm{I}(\lambda, T)$, it is the number of pairwise non-isomorphic models of cardinality λ of the theory T. An effort is made to describe the notion of *structure theory* for the class $\mathrm{Mod}(T)$.

Section 3: The basic existence theorems are discussed among them are the completeness, compactness and the Löwenheim-Skolem-Tarski theorems are presented as well as the important notion of elementary submodel. The Tarski-Vaught theorems about elementary chains establish that the category $\mathrm{Mod}(T)$ of models of a first-order theory T is closed under direct (projective) limits and has other nice properties. There several classes of morphisms of the classs $\mathrm{Mod}(T)$ making it into a category. The most important morphisms are *elementary embeddings*.

Section 4: The Erdős-Rado theorem is a generalization of Ramsey's theorem for uncountable cardinals. This section is central to the book. The proof is used as an excuse to introduce several important concepts. The second Tarski-Vaught chain theorem (Th 3.31) is used to analyze a given uncountable model. As an application we prove the Erős-Rado theorem. The primary goal of this section is not this particular application, but it is

here to introduce the notions of $\mathrm{tp}(\mathbf{a}/A, M)$ and the notion of *relatively saturated submodel*.

Section 5: Several applications of the compactness theorem are presented. It turns out that first-order logic is unable the capture many mathematically interesting objects, among them are the classes of *locally-finite groups*, *periodic groups*, *Archimedean ordered-fields*, *algebraically-closed fields with infinite transcendence degree* and the structure $\langle \omega, +, \cdot \rangle$:

Theorem. True arithmetic and in particular Peano's arithmetic has $2^{\aleph_0}$ pairwise non-isomorphic countable models.

Several of the applications of the compactness theorem are evidence to limitations of the expressive power of first-order logic. Notions like non Archimedean ordered-field, being periodic group and well ordering are properties that can not be described by first-order sentences in the relevant language. However while this point of view is true, it is not completely correct. We will use the compactness theorem to derive positive results: E.g. the finite Ramsey theorem from the Erdős-Rado theorem. An even more remarkable application (discovered by James Ax) of the compactness theorem that does not mention models or logic is the following:

Theorem. Let $p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ be given, and suppose $f(\mathbf{x}) := \langle p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}) \rangle$. If the function $f : \mathbb{C}^n \to \mathbb{C}^n$ is injective then f is surjective.

Section 6: Abraham Robinson discovered that when T is a complete first-order theory then $\langle \mathrm{Mod}(T), \preceq \rangle$ satisfies both the *Joint Embedding Property* and the *Amalgamation Property*. One of the results of this section is a fundamental property of first-order logic. A related result is

Robinson's Consistency Lemma. Let L_0, L_1, L_2 be similarity types such that $L_0 \subseteq L_\ell$ ($\ell = 1, 2$) and $L_0 = L_1 \cap L_2$. Let T_ℓ be theories in L_ℓ ($\ell = 1, 2$). If $T_1 \cap T_2$ is a complete theory in L_0 then $T_1 \cup T_2$ has a model.

An immediate conclusion is that for a complete T first-order the class $\mathrm{Mod}(T)$ has the amalgamation property over sets.

We also discuss Craig's interpolation Theorem and Beth's definability Theorem. Both are classical theorems from the fifties that play interesting role in modern developments of non-first-order logics and also in applications to computer science.

Section 7: Types and the diagram of T. A type is a set of formulas, basically is a generalization of a (usually infinite) set of equations. There is an unexpected connection between types and Galois theory. In a sense

that will be made precise in Corollary 4.2.34, we will be identify a type with an orbit of $\mathrm{Aut}(M)$ acting on the set $|M|$ when M is sufficiently saturated.

Section 8: There are many mathematically interesting properties that can't be axiomatized using first-order sentences. Many such examples are presented in Section 5. In this section we present several logics stronger than first-order, examples for infinitary languages. Some of the basic theorems are presented and much deeper facts are explored in later sections. One of the fundamental facts is that given ψ an $L_{\omega_1,\omega}$-sentence then there exists a countable expansion L_1 of the language of ψ, a first-order theory T_1 in L_1 and a type $q(y)$ such that

$$\mathrm{Mod}(\psi) = \{M \upharpoonright \mathrm{L}(\psi) \ : \ M \models T_1, M \text{ omits the type } q\}.$$

Thus, given a first-order theory T, a cardinal number $\lambda \geq |\mathrm{L}(T_1)| + \aleph_0$ and a T_1-type q, the question of existence of $M \models T$ omitting q of cardinality λ is of central importance.

Section 9: Countable models and Henkin's omitting types theorem. We describe one of the most powerful and important methods for constructing a countable model for a theory in a countable language with an additional condition. Given a theory T whose language is at most countable and a T-type $p(x)$, we say that p *is not isolated* provided for every $\varphi(x) \in \mathrm{Fml}(\mathrm{L}(T))$ if $T \cup \{\exists x \varphi(x)\}$ is consistent then there exists $\psi(x) \in p$ such that $T \cup \ \{\exists x[\varphi(x) \wedge \neg\psi(x)]\}$ is consistent. Henkin's theorem asserts that for every T and p as above there exists a countable $M \models T$ omitting the type p.

In section 12 we will see some powerful applications of Henkin's theorem.

Section 10 : Models of weak set theory. While from Gödel's second incompleteness theorem we know that from ZFC alone it is not possible to derive existence of a model of ZFC. Let ZFC^- stands for ZFC$-\{$power set axiom$\}$, i.e. the axioms of ZFC without the powerset axiom. It is provable from ZFC that there are sets that serve as models for ZFC^-. The full power of ZFC is rarely needed, many results of mathematics and model theory in particular are theorems of ZFC^-. In this section we study elementary chains of models of weak set theory. This is one of the most powerful tools of modern logic.

We'll revisit the Erdős-Rado Theorem and outline several other applications.

Section 11: Absoluteness.

Section 12: Two cardinal theorems, are a generalization of of the Löwenheim-Skolem theorems. Suppose T is a theory such that $\mathrm{L}(T)$ contains a unary predicate P. For pairs of cardinal numbers $\mu \leq \lambda$ and $\mu_1 \leq \lambda_1$ it is reasonable to ask: Does the existence of $M \models T$ of cardinality λ with $|P^M| = \mu$ implies the existence of $N \models T$ such that $\|N\| = \lambda_1$ and

$|P^N| = \mu_1$? There are several examples that indicates that this implication is false in general. At certain cases such an implication is not provable in ZFC as one can code Kurepa and ronszajn trees tress. However there are certain general positive results due to Vaught, Chang, Keisler and Morley.

Section 13: One of the fundamental ideas of Abraham Robinson was the introduction of the notion of model-complete theory. A theory T is called *model complete* iff for every $M, N \models T$

$$M \subseteq N \implies M \preceq N.$$

It turns out that several natural theories are model-complete (e.g. the theory of the field of complex numbers). We study several characterizations of model complete theories and discuss briefly the question of when a theory T has an extension (without changing the similarity type) to a model complete theory?

Section 14: Skolemization of a theory, is an expansion that to a theory that allows us like in algebra to assign to a subset of a model a well defined notion of closure which inherits all the first-order properties of the original structure. The goal of Skolemization can be viewed as adding new function symbols such that for $A \subseteq |M|$ the closure of A under the functions is an elementary submodel of M. This simple device is of great technical value.

1. Structures and languages

From a naive point of view, model theory can be viewed as an attempt to generalize the classical algebraic structures. Let's take a look at the concepts of *group*, *ring*, *semigroup*, *field*, *vector space*, *graph*, *matroid*, and that of an *ordered field*. We can see that the feature common to all is that in every case we have a nonempty underlying set with several operations defined on the elements of the set, and sometimes (in the case of a partially ordered set and an ordered field) we have a relation defined and/or operation(s) on the elements of the structure. Loosely speaking, a model is a generalization of the classical algebraic structures and that of the classical combinatorial objects like a graph and a partially ordered set. Namely it consists of a nonempty set (which we call the universe) and several functions and relations defined on that universe.

DEFINITION 1.1. Let $\mathbf{R}$, $\mathbf{F}$, and $\mathbf{C}$ be three pairwise disjoint sets (they may be empty). We call $\mathbf{R}$ the set of *relation symbols*, $\mathbf{F}$ is the set of *function symbols*, and $\mathbf{C}$ the set of *constant symbols*. The sets $\mathbf{R}$, and $\mathbf{F}$ are equipped with an integer-valued function. Namely we have $\mathbf{n} : \mathbf{F} \to \omega$, and $\mathbf{n} : \mathbf{R} \to \omega$ which we call the *arity function*. A *similarity-type*[1] is denoted by L it is the quadruple $\langle \mathbf{R}, \mathbf{F}, \mathbf{n}, \mathbf{C} \rangle$.

By $|L|$ we denote the *cardinality of* L it is $|\mathbf{R}| + |\mathbf{F} + |\mathbf{C}|$.

EXAMPLES 1.2. The following are examples for some useful languages:

(1) Let $\mathbf{R} = \{<\}$, $\mathbf{F} = \{+, \cdot\}$, $\mathbf{C} = \{0, 1\}$, and $\mathbf{n}(<) = \mathbf{n}(+) = \mathbf{n}(\cdot) = 2$. For future reference we will denote the above similarity-type by L_{orf} . This is a natural language to formalize the theory of ordered fields.

(2) Let $\mathbf{R} = \emptyset$, $\mathbf{F} = \{\cdot, ^{-1}\}$, $\mathbf{C} = \{e\}$, and $\mathbf{n}(\cdot) = 2$, $\mathbf{n}(^{-1}) = 1$. We denote this similarity type by L_{gr} it is a natural similarity-type to formalize the theory of groups.

(3) Another similarity type that is used to deal with groups is the following $\mathbf{R} = \emptyset$, $\mathbf{F} = \{\cdot\}$, $\mathbf{C} = \{e\}$, and $\mathbf{n}(\cdot) = 2$. We denote this similarity type by L_g

(4) Let $\mathbf{R} = \{L, P, in\}$, $\mathbf{F} = \emptyset$, $\mathbf{C} = \emptyset$, and $\mathbf{n}(P) = \mathbf{n}(L) = 1$, $\mathbf{n}(in) = 2$. This is a natural language to formulate plane geometry, when L and P are unary predicates for lines and points (respectively), and in is the incidence relation. For future reference we denote this language by L_{geo}.

NOTATION 1.3. Let L be a similarity type. By $\mathbf{R}(L)$ we denote the set of relation symbols in L, by $\mathbf{F}(L)$ the set of function symbols in L, and by $\mathbf{C}(L)$ the set of constant symbols of L, when the identity of L is clear we will omit the parameter L.

[1]Some authors call this a *signature* or a *language*, in accordance with common convention I will use all three.

Below we introduce several definitions whose aim is to define the notions of first-order formula and sentences. In order to do that we need several preliminary definitions.

DEFINITION 1.4. Let L be a similarity type, and let $X = \{x_n \ : \ n < \omega\}$ be an infinite set of variables.

(1) The set of *L-terms* is denoted by $\mathrm{Term}(L)$ it is the minimal set of finite strings of symbols from the set $\mathbf{F}(L) \bigcup X \bigcup \mathbf{C}(L) \bigcup \{(,), [,]\}$ containing the set $X \bigcup \mathbf{C}(L)$ and is closed under the following rule:
For every $H(x_1, \dots, x_{\mathbf{n}(H)}) \in \mathbf{F}(L)$ and every $\tau_1(\mathbf{x}_1), \dots, \tau_{\mathbf{n}(H)}(\mathbf{x}_{\mathbf{n}(H)}) \in \mathrm{Term}(L)$ the string $H(\tau_1(\mathbf{x}_1), \dots, \tau_{\mathbf{n}(H)}(\mathbf{x}_{\mathbf{n}(H)}))$ is a member of $\mathrm{Term}(L)$.

(2) An L-term is an element of $\mathrm{Term}(L)$, usually denoted by the letter τ perhaps with a subscript. When $\mathbf{x}$ is a finite sequence of variables from X containing the variables of τ we write $\tau(\mathbf{x})$. By $\ell(\mathbf{x})$ we denote the number of distinct variables in the sequence $\mathbf{x}$.

REMARK 1.5. Notice that when L is infinite then $|\mathrm{Term}(L)| = |L|$, see Exercise 1.

EXAMPLES 1.6. Some terms:

(1) The following are terms of L_{orf}:

$$x_1,$$
$$(1 + 1) \cdot x_1 \cdot x_1 \cdot x_8, \text{ and}$$
$$(x_1 + ((1 + 1 + 1) \cdot (1 + 1))) \cdot (x_2 + (1 + 1) \cdot x_6).$$

(2) Some terms in L_{gr}:

$$e,$$
$$e \cdot e \cdot e,$$
$$x_1 \cdot x_1 \cdot x_1, \text{ and}$$
$$(x_1 \cdot x_7)^{-1} \cdot x_5.$$

(3) The set of terms of L_{geo} is the set of variables X.

DEFINITION 1.7. Let L be a similarity type, and let X be an infinite set of variables.

(1) The set of *L-atomic formulas* is denoted by $\mathrm{AFml}(L)$ it is the minimal set of finite strings of symbols from the set $\mathbf{R}(L) \bigcup \mathbf{F}(L) \bigcup X \bigcup \mathbf{C}(L) \bigcup \{(,), =\}$ that is closed under the following rules:
   (a) for every $\tau_1(\mathbf{x}) \in \mathrm{Term}(L)$, and every $\tau_2(\mathbf{x}) \in \mathrm{Term}(L)$, $\tau_1(\mathbf{x}) = \tau_2(\mathbf{x})$ is a member of $\mathrm{AFml}(L)$;

(b) for every relation symbol $R(y_1, \ldots, y_{\mathbf{n}(R)})$ and any terms $\tau_1(\mathbf{x}), \ldots, \tau_{\mathbf{n}(R)}(\mathbf{x})$, the string $R(\tau_1(\mathbf{x}), \ldots, \tau_{\mathbf{n}(R)}(\mathbf{x}))$ is an element of $\mathrm{AFml}(L)$.

(2) An *L-atomic formula* is an element of $\mathrm{AFml}(L)$. When $\mathbf{x}$ is a finite sequence of variables from the set X containing all the variables of the formula φ we will write $\varphi(\mathbf{x})$.

NOTATION 1.8. *We will work with a fixed set of variables. The letters v, w, x, y and z will denote variables. By $\mathbf{x}$ we denote a finite sequence of variables.*

EXAMPLES 1.9. Atomic formulas:

The following are atomic formulas of L_{orf}:

$$1 = (1 + 1),$$
$$1 = x_1 \cdot x_2$$
$$x_1 + x_2 = ((x_1 + x_2) \cdot x_1 + 1).$$

Here are some atomic formulas in L_{gr}:

$$x_1 = x_1,$$
$$e^{-1} = x_7 \cdot e,$$
$$x_2 \cdot x_1 \cdot x_2 \cdot (x_1)^{-1} = x_1 \cdot x_2.$$

L_{geo} has exactly four kinds of atomic formulas they are: $\mathrm{L}(x)$, $P(y)$, $in(p, l)$, and $x = y$, when $x, y, p, l \in X$.

DEFINITION 1.10. Let L be a similarity type, and let $X = \{x_n \ : \ n < \omega\}$ be a set of variables.

(1) The set of *first-order formulas over L* is denoted by $\mathrm{Fml}(L)$ is the minimal set of finite strings of symbols from the set $\mathbf{R}(L) \bigcup \mathbf{F}(L) \bigcup X \bigcup \mathbf{C}(L) \bigcup \{(,), =, \neg, \wedge, \vee, \forall, \exists, \rightarrow, \leftrightarrow\}$ containing $\mathrm{AFml}(L)$ and closed under the following rules:
   (a) if $\varphi \in \mathrm{Fml}(L)$ then $\neg\varphi \in \mathrm{Fml}(L)$
   (b) if φ and ψ are elements of $\mathrm{Fml}(L)$ then $\varphi \wedge \psi \in \mathrm{Fml}(L)$
   (c) if φ and ψ are elements of $\mathrm{Fml}(L)$ then $\varphi \vee \psi \in \mathrm{Fml}(L)$
   (d) if φ and ψ are elements of $\mathrm{Fml}(L)$ then $\varphi \rightarrow \psi \in \mathrm{Fml}(L)$
   (e) if φ and ψ are elements of $\mathrm{Fml}(L)$ then $\varphi \leftrightarrow \psi \in \mathrm{Fml}(L)$
   (f) if $\varphi \in \mathrm{Fml}(L)$ and $y \in X$ then $\forall y\varphi \in \mathrm{Fml}(L)$
   (g) if $\varphi \in \mathrm{Fml}(L)$ and $y \in X$ then $\exists y\varphi \in \mathrm{Fml}(L)$.

(2) By a *first-order formula* we mean an element of the set $\mathrm{Fml}(L)$. Often the letters φ, ψ, χ will denote first-order formulas.

(3) For every L-formula φ we denote by $\mathrm{FV}(\varphi)$ *the set of free variables of the formula* φ. It is defined by induction on the length of the formula φ:
   (a) if φ is atomic then $\mathrm{FV}(\varphi)$ is the set of variables appearing in φ.

(b) if φ is $\neg\psi$ then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi)$.
(c) if φ is $\psi \# \chi$ when $\#$ is one of the binary propositional connectives then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi) \bigcup \mathrm{FV}(\chi)$.
(d) if φ is $(\mathbf{Q}y)\psi$ when $\mathbf{Q}$ is one of the quantifiers $\forall$ or $\exists$ then $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi) - \{y\}$.

(4) $\mathrm{FV}(\varphi)$ is called the set of *free variables of* φ.
(5) A formula φ without free variables (i.e. $\mathrm{FV}(\varphi) = \emptyset$) is called a *sentence*.
(6) Denote by $\mathrm{Sent}(L)$ the set $\{\varphi \in \mathrm{Fml}(L) \ : \ \mathrm{FV}(\varphi) = \emptyset\}$.
(7) A *theory in* L is a set of sentences in L.
(8) When T is a theory we denote by $\mathrm{L}(T)$ the smallest language such that T is an $\mathrm{L}(T)$-theory. By $|T|$ we denote $|\mathrm{L}(T)|$.

REMARK 1.11. Notice that $|T| = |\mathrm{Fml}(\mathrm{L}(T))|$, see Exercise 1.

Here are some formulas in L_{orf}:

$\neg(0 < 1)$,
$\forall x_1(1 = x_2)$,
$\forall x_1 \forall x_2((x_1 + x_2 = x_2 + x_1) \wedge (x_1 \cdot x_2 = x_2 \cdot x_1))$ and
$\forall x_1 \forall x_2 \exists x_4(x_1 \cdot x_4 \cdot x_4 + x_2 \cdot x_4 + x_3 = 0)$.

A sentence in L_{gr}: $\forall x_1 \exists x_2(x_1 \cdot x_2 = e \to x_2 = (x_1)^{-1})$.

The following are sentences in L_{geo}:

$$\forall x(P(x) \vee \mathrm{L}(x)),$$
$$\forall x \forall y(in(x, y) \to (P(x) \wedge \mathrm{L}(x))).$$

It is a common practice to mix parenthesis and brackets to enhance the readability of formulas.

NOTATION 1.12. *For a formula* $\varphi \in \mathrm{Fml}(L)$ *and a finite sequence of variables* $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$ *we write* $\varphi(\mathbf{x})$ *to reflect the fact that* $\mathrm{FV}(\varphi) \subseteq \{x_1, \ldots, x_n\}$. *It is customary to write also* $\varphi(\mathbf{x})$ *even when* φ *is a sentence (i.e. has no free variables).*

EXAMPLE 1.13 (Linear Order). Let $L_{ord} = \{<\}$ when $<$ is a binary relation symbol, T_{lo} is the following set of sentences:

$$\forall x[\neg(x < x)],$$
$$\forall x \forall y \forall z[x < y \wedge y < z \to x < z],$$
$$\forall x \forall y[x < y \vee x = y \vee y < x].$$

EXAMPLE 1.14 (Dense Linear Order). T_{ord} is the set T_{lo} together with following set of sentences:

$$\forall x \exists y [x < y],$$
$$\forall x \exists y [y < x],$$
$$\forall x \forall y [x < y \to \exists z [x < z \wedge z < y]].$$

EXAMPLE 1.15 (Group Theory). Denote by T_g the following theory in L_g:

$$\forall x \forall y \forall z [x \cdot (y \cdot z) = (x \cdot y) \cdot z],$$
$$\forall x [x \cdot e = e \cdot x = x],$$
$$\forall x \exists y [x \cdot y = y \cdot x = e].$$

Due to obvious reasons T_{ord} is called the theory of *dense linear order without endpoints*, and T_g is called *group theory*. There are other equivalent axiomatizations of group theory (E.g. Exercise **??**.4).

NOTATION 1.16. From now on we assume that we have a fixed countable set of variables, we denote variables by the letters x, y, z or v perhaps with a subscript. By $\mathbf{x}$ or $\mathbf{y}$ we denote finite sequences of variables, by $\ell(\mathbf{x})$ we denote the length of the sequence $\mathbf{x}$. When $\mathbf{x}$ and $\mathbf{y}$ are finite sequences $\mathbf{x}^\frown \mathbf{y}$ stands for the concatenation of the sequences (namely it is a sequence of length $\ell(\mathbf{x}) + \ell(\mathbf{y})$ such that at the first $\ell(\mathbf{x})$-places there is a copy of the sequence $\mathbf{x}$ and at the $\ell(\mathbf{x}) + 1$-th place of $\mathbf{x}^\frown \mathbf{y}$ is the first element of $\mathbf{y}$ and so on). We follow the usual usage in mathematics of placing parenthesis to enhance readability and will delete them in cases formulas looks more pretty without parenthesis. Quite often when R is a binary relation symbol (i.e. $\mathbf{n}(R) = 2$, and $R \in \mathbf{R}(L)$) we write xRy instead of writing $R(x, y)$; this convention was already used in the last example of L_{orf} by writing $0 < 1$ instead of writing $< (0, 1)$ that probably looks quite strange to the normal (ordinary?) mathematician.

In the following definition the central concept of the subject is introduced. It is the concept of a structure (or a model), it can be viewed as an attempt to generalize the fundamental algebraic structures; I.e. groups, rings, fields, and ordered fields. A good way to think of groups is to imagine a nonempty set of elements together with a binary function (multiplication), a unary function (the multiplicative inverse of an element), and a specified element (the identity element). The definition below is a generalization of exactly this.

DEFINITION 1.17. Let L be a similarity type.

(1) An L-*structure* is a quadruple $M := \big\langle |M|, \{G^M : G \in \mathbf{F}(L)\}, \{P^M : P \in \mathbf{R}(L)\}, \{c^M : c \in \mathbf{C}(L)\} \big\rangle$, where $|M|$ is a nonempty set.

$$G^M : \underbrace{|M| \times |M| \times \ldots \times |M|}_{n(G)\text{-times}} \to |M| \quad \text{for all } G \in \mathbf{F}(L),$$

$$P^M \text{ is a subset of } \underbrace{|M| \times |M| \times \ldots \times |M|}_{n(P)\text{-times}} \quad \text{for every } P \in \mathbf{R}(L),$$

and

$$c^M \in |M| \quad \text{for every } c \in \mathbf{C}(L).$$

(2) The set $|M|$ is called *the universe* of the structure M. We will write $a \in M$ instead of writing $a \in |M|$, by abuse of notation we will write $\mathbf{a} \in M$ instead of writing "$\mathbf{a}$ is a finite sequence of elements from the universe of the structure M".
(3) *The cardinality* of the structure M is denoted by $\|M\|$ and it is the cardinality of its universe. We say that M is countable when $\|M\| = \aleph_0$.
(4) Let M be an L-structure. Note that the similarity type L is part of the specification of M, so given a structure M we can read the similarity type from M. Sometimes we refer to the *similarity type of* M (or *signature*) and denote it by $\mathrm{L}(M)$.
(5) In case that the intention is clear we do not distinguish between the function symbol $F(\mathbf{x})$ and the corresponding function $F^M(\mathbf{x})$ as interpreted in M; when convenient the superscript M will be deleted.

EXAMPLES 1.18. The following are important examples of structures:

(1) *The field of ordered reals*. Let $\mathbb{R}$ stand for the real numbers, $+, \cdot$ the usual addition and multiplication of real numbers, $<$ the usual order, and the interpretation of the constants 0 and 1 are the numbers $0, 1$ (note that we purposely use the same symbols in two different contexts!) $\langle \mathbb{R}, +, \cdot, <, 0, 1 \rangle$ is an L_{orf} structure.

(2) *The additive group of integers*. Let $\mathbb{Z}$ be the set of integers, and let $+$ be the usual addition of integers. $\langle \mathbb{Z}, +, 0 \rangle$ is an L_{gr} structure, when the interpretation of the constant symbol e is the number 0.

(3) *The finite symmetric group*. Let n be a positive integer, S_n is the group of permutations on the set $\{1, \ldots, n\}$ let id_n be the identity mapping, and let "$\circ$" be composition of functions then $\langle S_n, \circ, \mathrm{id}_n \rangle$ is another L_{gr} structure.

(4) *incidence relation on the Euclidean plane* . Let $\mathbb{R}$ be the field of real numbers. Let $P := \mathbb{R} \times \mathbb{R}$, for every $a, b \in \mathbb{R}$,
let $l_{a,b} := \{(x, y) \subseteq \mathbb{R} \times \mathbb{R} \ : \ y = ax + b\}$, now define
$L := \{l_{a,b} \ : \ (a, b) \in \mathbb{R} \times \mathbb{R}\}$ (think of L as the set of lines in the plane $\mathbb{R} \times \mathbb{R}$), for $p \in P$ and $l \in L$ define the following binary relation $in(p, l)$ to hold iff $p \in l$. Now let $G := P \cup L$. The structure $\langle G, P, L, in \rangle$ is an L_{geo}-structure. It is called the *affine plane on* $\mathbb{R}$. Notice that we have made no essential use of the real numbers in the above construction. A generalization of the above when $\mathbb{R}$ is replaced by an arbitrary field (or even a division-ring) is very interesting.

In the following definition we relate the notions of formula with that of a structure.

DEFINITION 1.19. Let L be a similarity type, and let M be an L-structure. For every $\mathbf{a} \in M$ and every term $\tau(\mathbf{x}) \in \mathrm{Term}(L)$ such that $\ell(\mathbf{x}) = \ell(\mathbf{a})$

(1) Define $\mathrm{Val}_M(\tau, \mathbf{a}) \in |M|$ by induction on the structure of τ:
Suppose $\mathbf{a} = \langle a_1, \ldots, a_n \rangle$, and that $\tau = \tau(x_1, \ldots, x_n)$.
   (a) if τ is the constant symbol c then $\mathrm{Val}_M(\tau, \mathbf{a}) := c^M$.
   (b) when τ is the variable x_i then $\mathrm{Val}_M(\tau, \mathbf{a}) := a_i$.
   (c) when τ is $F(\tau_1, \ldots, \tau_n)$ when F is a function symbol and $\tau_1, \ldots, \tau_n$ are terms then let
   $$\mathrm{Val}_M(\tau, \mathbf{a}) := F^M(\mathrm{Val}_M(\tau_1, \mathbf{a}), \ldots, \mathrm{Val}_M(\tau_n, \mathbf{a})).$$
(2) When the identity of M is fixed instead of writing $\mathrm{Val}_M(\tau, \mathbf{a})$ we write $\tau[\mathbf{a}]$.

To illustrate the last definition, think of the language of rings L_r (containing the binary function symbols: $+, \cdot$ and the constant symbols 0, 1). It is easy to see that a term $\tau(\mathbf{x})$ in the variables $x_1, \ldots, x_n$ is a polynomial in $x_1, \ldots, x_n$ with coefficients from $\mathbf{N}$. Now given a ring R (which is an L_r-structure with the usual interpretations) and given a sequence of elements $\mathbf{a}$ from R, $\tau[\mathbf{a}]$ is the element (of R) we get after evaluating the polynomial τ at the parameters $\mathbf{a}$.

DEFINITION 1.20. Let L be a similarity type, and let M be an L-structure. For every $\mathbf{a} \in M$ and every $\varphi(\mathbf{x}) \in \mathrm{Fml}(L)$ such that $\ell(\mathbf{a}) = \ell(\mathbf{x})$ define $M \models \varphi[\mathbf{a}]$ by induction on the structure of the formula φ:

(1) if φ is atomic then define by cases:
   (a) when $\varphi(\mathbf{x})$ is $\tau_1(\mathbf{x}) = \tau_2(\mathbf{x})$ then
   $$M \models \varphi[\mathbf{a}] \text{ holds} \quad \text{iff} \quad \tau_1[\mathbf{a}] = \tau_2[\mathbf{a}].$$
   (b) when $\varphi(\mathbf{x})$ is $R(\tau_1(\mathbf{x}), \ldots, \tau_n(\mathbf{x}))$ then
   $$M \models \varphi[\mathbf{a}] \text{ holds} \quad \text{iff} \quad \langle \tau_1[\mathbf{a}], \ldots, \tau_n[\mathbf{a}] \rangle \in R^M.$$
(2) if φ is not atomic then:
   (a) when $\varphi(\mathbf{x})$ is $\neg\psi(\mathbf{x})$ then
   $$M \models \varphi[\mathbf{a}] \text{ holds} \quad \text{iff} \quad M \models \psi[\mathbf{a}] \text{ is false.}$$
   (b) when $\varphi(\mathbf{x})$ is $\psi_1(\mathbf{x}) \vee \psi_2(\mathbf{x})$ then
   $$M \models \varphi[\mathbf{a}] \text{ holds} \quad \text{iff} \quad M \models \psi_1[\mathbf{a}] \text{ or } M \models \psi_2[\mathbf{a}].$$
   (c) when $\varphi(\mathbf{x})$ is $\psi_1(\mathbf{x}) \wedge \psi_2(\mathbf{x})$ then
   $$M \models \varphi[\mathbf{a}] \text{ holds} \quad \text{iff} \quad M \models \psi_1[\mathbf{a}] \text{ and } M \models \psi_2[\mathbf{a}].$$
   (d) when $\varphi(\mathbf{x})$ is $\psi_1(\mathbf{x}) \to \psi_2(\mathbf{x})$ then
   $$M \models \varphi[\mathbf{a}] \text{ holds} \quad \text{iff} \quad M \models \psi_1[\mathbf{a}] \implies M \models \psi_2[\mathbf{a}].$$
   (e) when $\varphi(\mathbf{x})$ is $\psi_1(\mathbf{x}) \leftrightarrow \psi_2(\mathbf{x})$ then
   $$M \models \varphi[\mathbf{a}] \text{ holds} \quad \text{iff} \quad M \models \psi_1[\mathbf{a}] \iff M \models \psi_2[\mathbf{a}].$$

(f) when $\varphi(\mathbf{x})$ is $\exists y\chi(y,\mathbf{x})$ then

$M \models \varphi[\mathbf{a}]$ holds iff there exists an element $b \in |M|$ such that $M \models \chi[b,\mathbf{a}]$ holds.

(g) when $\varphi(\mathbf{x})$ is $\forall y\chi(y,\mathbf{x})$ then

$M \models \varphi[\mathbf{a}]$ holds iff for every element $b \in |M|$ we have that $M \models \chi[b,\mathbf{a}]$ holds.

Now to the notion of *satisfaction*:

DEFINITION 1.21. Let L be a similarity type, and let M be an L-structure. For every $\mathbf{a} \in M$ and every $\varphi(\mathbf{x}) \in \mathrm{Fml}(L)$ such that $\ell(\mathbf{x}) = \ell(\mathbf{a})$.

We say that M *satisfies* $\varphi[\mathbf{a}]$ iff $M \models \varphi[\mathbf{a}]$ is true. When $M \models \varphi[\mathbf{a}]$ is false we write $M \not\models \varphi[\mathbf{a}]$.

Notice that from the definition of satisfaction it follows that given a structure M for any $\varphi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(M))$ and every $\mathbf{a} \in |M|$ exactly one of the following possibilities holds $M \models \varphi[\mathbf{a}]$ or $M \models \neg\varphi[\mathbf{a}]$. It is an elementary fact of propositional calculus that the sets of connectives $\{\neg, \vee\}$ and $\{\neg, \wedge\}$ are *complete sets of connectives*. That is to say every propositional connective can be expressed using connectives from the complete set of connectives. Since in model theory usually the number of symbols used to form a formula is not an issue, often we will assume that we have only the connectives $\neg$ and $\wedge$. Using DeMorgan's law together with the definition of satisfaction it is easy to see that
$M \models \exists y\chi(y,\mathbf{x}) \iff M \models \neg\forall y\neg\chi(y,\mathbf{x})$, and
$M \models \forall y\chi(y,\mathbf{x}) \iff M \models \neg\exists y\neg\chi(y,\mathbf{x})$. These observations enables us to work with one quantifier only. Instead of using both the universal and existential quantifiers we will use only one. However since one can be interpreted using the other (and negation) we occasionally switch between the quantifiers.

EXAMPLE 1.22. Let $M := \langle \mathbb{R}, +, \cdot, <, 0, 1\rangle$ be the ordered field of real numbers. It is easy to check that $M \models \forall x\exists y[x < y]$, and $M \models \neg(1 < 0)$ are both true.

EXAMPLE 1.23. Let n be an integer greater than 2, and let S_n be the symmetric group on $\{1, \ldots, n\}$ let id_n be the identity mapping, and let $\circ$ be composition of permutations. It is a well known fact that $\langle S_n, \circ, \mathrm{id}_n\rangle$ is an L_g-structure and satisfy the sentence $\exists x\exists y[x \cdot y \neq y \cdot x]$.

EXAMPLE 1.24. Let $M := \langle G, P, L, in\rangle$ be the affine plane over the real numbers (see Example 1.18(4)).
Clearly we have $M \models \forall x[P(x) \vee \mathrm{L}(x)]$, and
$M \models \forall p\forall q[P(p) \wedge P(q) \wedge p \neq q \rightarrow$
$\exists l[\mathrm{L}(l) \wedge in(p,l) \wedge in(q,l) \wedge \exists r[P(r) \wedge \neg in(r,l)]]$.
The second statement is a formalization of the fact that for every pair of distinct points there exists a line containing them and there is a point not on that line.

DEFINITION 1.25. Let L be a similarity type, let M be an L-structure and let T be a theory in L.

(1) We say that M is a *model of the theory* T iff for every sentence $\varphi \in T$ we have that $M \models \varphi$ holds. We denote this by writing $M \models T$.
(2) By $\mathrm{Th}(M)$ denote the following theory $\{\varphi \in \mathrm{Sent}(\mathrm{L}(M)) \ : \ M \models \varphi\}$ we call it *the theory of* M .

EXAMPLE 1.26. Let $M := \langle \mathbb{R}, +, \cdot, <, 0, 1\rangle$ be the ordered field of real numbers. Using basic properties of the rationals and the reals it is easy to verify that both $\langle \mathbb{Q}, <\rangle$ and M are models of the theory T_{ord}.

EXAMPLE 1.27. Let $N := \langle \mathbb{Z}, +, 0\rangle$ be the integers with addition, the theory $\mathrm{Th}(N)$ is quite interesting and is called *Presburger's arithmetic* .

EXAMPLE 1.28. Let $L_f := \{+, \cdot, 0, 1\}$ when $+$ and $\cdot$ are binary function symbols, and $0, 1$ constant symbols. Let $M := \langle \mathbb{R}, +, \cdot, 0, 1\rangle$ be the field of real numbers and let $N := \langle \mathbb{C}, +, \cdot, 0, 1\rangle$ be the field of complex numbers.
It is not difficult to see that $\mathrm{Th}(M) \neq \mathrm{Th}(N)$. In fact $\mathrm{Th}(M) \not\subseteq \mathrm{Th}(N)$ and $\mathrm{Th}(N) \not\subseteq \mathrm{Th}(M)$ since $M \models \psi$ and $N \models \neg\psi$ where ψ is the sentence $\exists x[x \neq 0 \wedge \neg\exists y[x = y \cdot y]]$. The other proper inclusion can be shown by considering the sentence $\exists x[x \cdot x + 1 = 0]$.

EXAMPLE 1.29. In the similarity type $\{+, \cdot, 0, 1\}$ the theory $\mathrm{Th}(\langle \omega, +, \cdot, 0, 1\rangle)$ is called *true arithmetic*. Note that despite of its simple definition it is an extremely complex object. If we were able to look into the theory and answer the question whether $\varphi \in \mathrm{Th}(\langle \omega, +, \cdot, 0, 1\rangle)$ for every sentence φ we could answer all the famous open problems of number theory.

In 1900 at the International Congress of Mathematics in held Paris. David Hilbert, the greatest mathematician of the time presented 23 open problems that were very influential on 20th century mathematics. Two of the problems were in logic. The first problem on Hilbert's list was the continuum hypothesis, the other logic related problem was Hilbert's 10th problem. Using current terminology it is the following statement

PROBLEM 1.30 (Hilbert's tenth problem). *Does there exists a computer program* $P(\cdot)$ *such that for every* $q(\mathbf{x}) \in \mathbb{Z}[x_0, \ldots, x_{n-1}]$*:*

$$P(q(\mathbf{x})) := \begin{cases} \textit{Yes} & \textit{if } \langle \mathbb{Z}, +, \cdot\rangle \models \exists \mathbf{x} q(\mathbf{x}) = 0 \\ \\ \textit{No} & \textit{otherwise.} \end{cases}$$

Since digital computers were invented only in the mid forties, obviously Hilbert did not formulate his problem using the language above.

The mathematical notion of computable function was introduced and discovered independently by Alonzo Church [**Chu**], A.A. Markov and Alan Turing

[**?**, Tu] They came up with different and mathematically equivalent formulations of computation. Below I present a 4th formalization (also equivalent to the above) that has the advantage of being intuitively closer to modern programming languages.

This is Shepherdson's notion of *Register Machine*.

DEFINITION 1.31. A *register machine program* P is a finite sequence $\langle \sigma_1, \ldots, \sigma_n \rangle$ of *instructions*. An *instruction* must have one of the following forms:

(1) $label \;\; R_i \leftarrow R_j$
(2) $label \;\; R_i \leftarrow R_j + R_k$
(3) $label \;\; R_i \leftarrow R_j - R_k$
(4) $label1$ GO TO $label2$
(5) $label1$ If $R_j = 0$ THEN GO TO $label2$
(6) n END

When $label$ stands for a positive integer that is no more than n. The first instruction is

$$1 \;\; R_1 \leftarrow \text{INPUT}$$

The last instruction is

$$n \;\; \text{END}$$

The R_i are indexed by positive integers and are called *registers*, we have an unlimited supply of them.

At the begining of a *computation* of the program P all registers are set to zero. The *output* of the program P is the contents of Register R_1 when the last instruction END is reached.

Often we write $P(x)$ to denote the contents of register R_1 after a run of the program P on input x.

It can be shown via tedious exercise in elementary computer programming that many functions involving integers can be computed by a register machine program. E.g. $n!$, n^n, the function

$$f(n) := \begin{cases} 1 & \text{if } n \text{ is a prime number} \\ \\ 0 & \text{otherwise.} \end{cases}$$

Also there is a program that produces the nth prime number for input n.

Notice that it is possible to code a finite sequence of natural numbers by a single integer. One obvious way to do that is to use powers of primes i.e.

$$\text{Code} \quad \langle e_1, e_2, e_3, e_4, e_5, \ldots \rangle \quad \text{by} \quad 2^{e_1} 3^{e_2} 5^{e_3} 7^{e_4} 11^{e_5} \cdots$$

Thus despite the fact that the programs use only one register for input we can write programs that compute several variable functions. Standard examples are $x \cdot y$, $\lfloor x/y \rfloor$, x^y and $\gcd(x, y)$.

Now we are ready to introduce the notion of computable function:

DEFINITION 1.32. A set S of integers is called *recursive* iff there exists a register machine program P_S such that

$$x \in S \iff P(x) = 1.$$

A function $f : \omega \to \omega$ is called a *recursive function* iff the set $\{2^x 3^y \mid f(x) = y\}$ is a recursive set of integers.

A set of integers S is *recursively enumerable* iff there exists a recursive function f such that $S = \{f(n) \mid n \in \omega\}$.

The branch of mathematical logic dealing with recursive functions is called *recursion theory* classical introduction to the subject is Hartley Rogers's book [**Rog**]. In recent years several recursion theorists in order to emphasize the connections of the subject with theoretical computer science decided to rename the above concepts as *computable set*, *computable function* and *computable enumerable set*. A more recent text than Rogers's book is Piergiorgio Odifreddi's book [**Od**].

There are few basic fundamental facts about recursive and recusivly enumerable sets that will be left as exercises for interested readers.

THEOREM 1.33.

(1) *If $S \subseteq \omega$ is recursive also $\omega - S$ is recursive. If S_1, S_2 are recursive sets then also $S_1 \cup S_2$ is recursive. Any boolean combination of finitely many recursive sets is recursive.*
(2) *Not every set of integers is recursively enumerable.*
(3) *Let $S \subseteq \omega$. If both S and $\omega - S$ are r.e. sets then S is recursive.*

PROOF. The first part is trivial. The second part is true since there are only countable many recursive functions. □

There are several standard methods to code letters of the alphabeth by integers *ASCII code* is the most famous method. Thus by the above any text in English can be coded by a single integer. This obesrvation adds much power to the notion of recursive set.

In particular it follows that a register machine program can be coded by an integer. Thus the following function is a well defined function from ω in to $\{0, 1\}$:

$$f_{halt}(x) := \begin{cases} 1 & \text{if } x \text{ is a code for a program that stops} \\ \\ 0 & \text{otherwise.} \end{cases}$$

When M is a structure in a recursive language (note than if the language is finite it is recursive) then the *theory of M is decidable* iff $\mathrm{Th}(M)$ is a recursive set.

It turns out that the answer to Problem 1.30 is no.

Martin Davis **??**, Hillary Putnam **??** and

Julia Robinson in 1968 the bounded quantifier theorem [**JRob**].

Yuri Matiyasevich in 1970 using Fibonnaci sequence [**Mati1**], later Davis using Pell equations.

See [**Mati3**] and [**Da**].

13 variables are enough [**MaRo**], in fact 9 variables are enough see [**Mati2**].

1.1. Multi sorted structures.

DEFINITION 1.34.

DEFINITION 1.35.

Exercises 1.1.

(1) Let L be a similarity type. Using an argument similar to Cantor's proof that the set of polynomials with rational coefficients is countable establish that $|\operatorname{Term}(L)| \leq |\operatorname{Fml}(L)| + \aleph_0$. Thus for infinite L we have that $|\operatorname{Term}(L)| = |\operatorname{Fml}(L)|$.
(2) Let L be a similarity type, let $\mathbf{x}$ be a sequence of variables and let M be an L-structure. Suppose $\varphi(\mathbf{x})$ is an L-formula, by $\varphi(M)$ denote the set $\{\mathbf{a} \in |M| \ : \ M \models \varphi[\mathbf{a}]\}$, in the future we refer to this set as *the set defined by the formula* φ. Show that for every formula $\varphi(\mathbf{x})$ there exists an L sentence ψ such that
$M \models \psi \iff M \models \exists!\mathbf{x}\varphi(\mathbf{x})$[2].
(3) Let L be a similarity type, and let M be an L-structure. Suppose $\varphi(x)$ and $\psi(x)$ are L formulas. Verify the following
   (a) $M \models \forall x[\varphi(x) \to \psi(x)]$ iff $\varphi(M) \subseteq \psi(M)$
   (b) $M \models \forall x[\varphi(x) \leftrightarrow \psi(x)]$ iff $\varphi(M) = \psi(M)$
   (c) $M \models \exists x[\varphi(x) \wedge \psi(x)]$ iff $\varphi(M) \cap \psi(M) \neq \emptyset$
   (d) $M \models \exists x[\varphi(x) \wedge \neg\psi(x)]$ iff $\varphi(M) - \psi(M) \neq \emptyset$
   (e) $M \models \forall x[\varphi(x) \vee \psi(x)]$ iff $\varphi(M) \cup \psi(M) = |M|$.
(4) Let L be a similarity type, and let M be an infinite L-structure. Suppose that $\varphi(x)$ is an L-formula and let n be a positive natural number.
   (a) We denote by $M \models \exists^n x\varphi(x)$ the statement $|\varphi(M)| = n$. Show that for every formula $\varphi(x)$ and every positive integer n there exist a sentence $\psi_{\varphi,n}$ such that $M \models \exists^n x\varphi(x)$ iff $M \models \psi_{\varphi,n}$.
   (b) Denote by $M \models \exists^{\leq n} x\varphi(x)$ the statement that $1 \leq |\varphi(M)| \leq n$ show that for every φ and every n there is a sentence $\chi_{\varphi,n}$ such that for every L-structure M we have that $M \models \exists^{\leq n} x\varphi(x)$ iff $M \models \chi_{\varphi,n}$.
(5) Let L be a similarity type, let n be a positive integer and let M be an L-structure. Let $\mathbf{x}$ be an n-tuple of variables, denote by $\operatorname{Fml}(\mathbf{x}, L)$ the set of all L formulas which have the free variables $\mathbf{x}$. On the above set define the following binary relation:
For $\varphi(\mathbf{x}), \psi(\mathbf{x}) \in \operatorname{Fml}(\mathbf{x}, L)$ let $\varphi \approx_M \psi$ iff $\varphi(M) = \psi(M)$. Prove that $\approx_M$ is an equivalence relation on the set $\operatorname{Fml}(\mathbf{x}, L)$.
(6) For every $m < \omega$ there exists a group of cardinality m.
(7) For every infinite cardinal λ there exists a group of cardinality λ.
(8) For every infinite cardinal λ there exists a field of cardinality λ.
(9) For every infinite cardinal λ there exists an ordered field of cardinality λ.
(10) In the following group of exercises we develop several basic facts about Boolean algebras. Some of these facts will be used in several forthcoming sections and chapters.
   (a) A *Boolean Algebra* is a structure $\langle A, \cup, \cap, -, 0, 1\rangle$ such that
      (i) $0 \neq 1$

[2] $M \models \exists!\mathbf{x}\varphi(\mathbf{x})$ is a short hand for $|\varphi(M)| = 1$

(ii) for every $a \in A$ we have $a \cup (1-a) = 1$ and $a \cap (1-a) = 0$
(iii) for every $a \in A$ we have that $a \cap 1 = a$ and $a \cup 0 = a$
(iv) for every $a \in A$ we have that $a \cup a = a$ and $a \cap a = a$
(v) for every $a, b \in A$ we have $a \cup b = b \cup a$ and $a \cap b = b \cap a$
(vi) for every $a, b, c \in A$ we have that $a \cap (b \cap c) = (a \cap b) \cap c$ and $a \cup (b \cup c) = (a \cup b) \cup c$
(vii) for every $a, b, c \in A$ we have that $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ and $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$
(viii) for every $a, b \in A$ we have that $1 - (a \cap b) = (1-a) \cup (1-b)$ and $1 - (a \cup b) = (1-a) \cap (1-b)$

Verify that the following are all examples of Boolean Algebras

(i) Let $A := \{T, F\}$ let 0 stand for F and 1 for T and the operations $\cup, \cap$ and $-$ are the usual logical operations.
(ii) Let S be a nonempty set, define $A := \mathcal{P}(S)$ let 0 stand for $\emptyset$ and 1 for the set A the operations $\cup, \cap$ and $-$ are the usual set-theoretic operations.
(iii) Let $S := \mathcal{P}(\omega)$, on the set S consider the following equivalence relation $\approx$ defined by

$$B \approx C \overset{def}{\iff} |(B - C) \cup (C - B)| < \aleph_0.$$

Let $A := S/\approx$, let 0 stand for the set of all finite subsets of ω, let 1 be the $\approx$-equivalence class of ω (it is the set of all co-finite subsets of ω), for $a, b \in S$ denote the corresponding equivalence classes by $[a]$ and $[b]$ let

$$\begin{aligned} [a] \cap [b] &:= [a \cap b], \\ [a] \cup [b] &:= [a \cup b] \text{ and} \\ [a] - [b] &:= [a - b]. \end{aligned}$$

This Boolean Algebra is often denoted by $\mathcal{P}(\omega)/finite$.
(iv) Let A be the set of all Borel subsets of $\mathbb{R}$. By interpreting 0 by $\emptyset$ and 1 by $\mathbb{R}$ together with the usual set-theoretic operations we get a Boolean algebra.
(v) Let A be the set of Lebesgue-measurable subsets of the interval $[0, 1]$, the operations are the usual set-theoretic operations and 0 is interpreted by $\emptyset$ while 1 is interpreted by $[0, 1]$.
(vi) Let X be a topological space, let

$$A := \{S \subseteq X \ : \ S \text{ is closed and open in } X\},$$

the operations are the usual set-theoretic operations and 0 is interpreted by $\emptyset$ while 1 is interpreted by X.

(b) Suppose $\langle A, \cup, \cap, -, 0, 1\rangle$ is a boolean algebra. For $a, b \in A$ define $a \leq b$ iff $a \cap b = a$. Prove that $\leq$ is an antisymmetric partial order on the set A and prove that $a \leq b$ iff $a \cup b = b$. The partial order $\leq$ is called *The partial order of the algebra*.

(c) If $\langle A, \cup, \cap, -, 0, 1\rangle$ is a boolean algebra and $|A|$ is finite then there exists a positive integer n such that $|A| = 2^n$. Hint: Let n be the number of atoms, where $a \in A$ is called an *atom* iff $a > 0$ and $A \models \neg\exists x[x \neq 0 \wedge x \leq a \wedge x \neq a]$.

(d) Construct a countable boolean algebra.

(e) For every infinite cardinal λ there exists a boolean algebra of cardinality λ.

(f) Let n be a positive integer, let $\mathbf{x} = \langle x_1, \ldots, x_n\rangle$. Denote by $\mathrm{BA}^n(M)$ the set of equivalence classes $\{\varphi(\mathbf{x})/\approx_M : \varphi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(M))\}$, let $\cap, \cup$ be two operations defined on $\mathrm{BA}^n(M)$ as follows: Given $a, b \in \mathrm{BA}^n(M)$, pick $\varphi(\mathbf{x}), \psi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(M))$ such that $a = \varphi/\approx$ and $b = \psi/\approx$. We define $a \cup b$ to be $[\varphi(\mathbf{x}) \vee \psi(\mathbf{x})]/\approx$, and $a \cap b$ to be $[\varphi(\mathbf{x}) \wedge \psi(\mathbf{x})]/\approx$. Let $a - b := [\varphi(\mathbf{x}) \wedge \neg\psi(\mathbf{x})]/\approx$. Prove that the above are well defined operations (namely the definition is independent of the choice of representatives of the equivalence classes). Prove that $\langle \mathrm{BA}^n(M), \cup, \cap\rangle$ is a boolean algebra where 0 is $[\mathbf{x} \neq \mathbf{x}]/\approx$ and 1 is $[\mathbf{x} = \mathbf{x}]/\approx$. The algebra $\mathrm{BA}^n(M)$ is called
The Lindenbaum Boolean Algebra of definable subsets in M .

(g) For every $n < m$ let $\pi_{m,n} : \mathrm{BA}^m(M) \to \mathrm{BA}^n(M)$ be the natural projections induced by

$$[\varphi(x_0, \ldots, x_{n-1}, x_n, \ldots, x_{m-1})] \mapsto [\exists x_n \cdots \exists x_{m-1}\varphi(x_0, \ldots, x_{n-1}, x_n, \ldots, x_{m-1})].$$

Show that $\pi_{m,n}$ is a surjective homomorphism of Boolean algebras.

(h) This exercise generalizes Exercise 10f: Let α be an ordinal. Let $V := \{x_\beta \mid \beta < \alpha\}$ be a set of variables. Denote by $\mathrm{BA}^\alpha(M)$ the set $\varphi(\mathbf{x})/\sim$, where for $\mathbf{x} = \langle x_{\beta_0}, \ldots, x_{\beta_{n-1}}\rangle$ we define

$$\varphi_1(\mathbf{x}) \sim \varphi(\mathbf{x}) \overset{def}{\iff} M \models \forall x_{\beta_0} \ldots \forall x_{\beta_{n-1}}[\varphi_1(\mathbf{x}) \leftrightarrow \varphi(\mathbf{x})].$$

Show that $\mathrm{BA}^\alpha(M)$ is a distributive lattice.

(i) Let $\langle A, \cup, \cap, -, 0, 1\rangle$ be a boolean algebra, and let $\leq$ be its partial order.

DEFINITION 1.36. (i) Let $I \subseteq A$. The set I is called an *ideal* of the algebra (sometimes called a *proper ideal*) iff
(A) $1 \notin I$,
(B) $\forall a \in I\ \forall b \in A$ we have that $a \cup b = a \to b \in I$, and
(C) $\forall a, b \in I$ we have that $a \cup b \in I$.

(ii) A subset F of A is called a *filter* (sometimes called a *proper filter*) iff
(A) $0 \notin F$,
(B) for every $a, b \in F$ we have that also $a \cap b \in F$,
(C) for every $a, b \in A$ the following implication is true: if $a \in F$ and $a = a \cap b$ then $b \in F$.

Show that

(i) $I := \{S \subseteq \omega \ : \ S \text{ is finite}\}$ is an ideal in $\mathcal{P}(\omega)$.

(ii) Let λ be an infinite cardinality, and let $\mathcal{P}(\lambda)$ the powerset algebra. Show that $F := \{S \subseteq \lambda \ : \ |\lambda - S| < \lambda\}$ is a filter of the algebra.

(iii) $I := \{X \subseteq [0,1] \ : \ \mu(X) = 0\}$ is an ideal of the algebra in Exercise 10a(e).

(iv) Consider the powerset algebra of the reals (i.e. take in Exercise 10a(b) the set A to be $\mathbb{R}$).
The following set is an ideal: $\{S \subseteq \mathbb{R} \ : \ S \text{ is nowhere dense}\}$.

(v) Let λ be a regular uncountable cardinality. Consider the algebra ... it is λ-complete.

(vi) measure zero

(vii) nowhere dense REFERENCE TO OXTOBY.

(j) (Hard)

DEFINITION 1.37. Let $\langle A, \cup, \cap, -, 0, 1\rangle$ be a boolean algebra.
A filter F in A is called an *ultrafilter* iff for every $a \in A$ exactly one of the following conditions holds: $a \in F$ or $1 - a \in F$.

Let M be a countable structure in a countable language. Using the notation of Exercise 10f prove the following
Theorem Let $\mathcal{F}$ be the family of all ultrafilters in the algebra $\mathrm{BA}^1(M)$.
If $|\mathcal{F}| > \aleph_0$ then $|\mathcal{F}| = 2^{\aleph_0}$.

2. The basic concepts

The following definition is the direct generalization of the corresponding notions of subgroup, subring, and subfield from algebra as well as a generalization of the notion of subgraph from combinatorics.

DEFINITION 2.1. Let L be a similarity type, and let M and N be L-structures. We say that N *is a submodel of* M iff

(1) $|N| \subseteq |M|$, and
(2) (a) for every constant symbol c we have that $c^M = c^N$,
    (b) for every relation symbol R we have that $R^N = R^M \cap (\underbrace{|N| \times \cdots \times |N|}_{n(R)\text{-many times}})$, and
    (c) for every function symbol $F(\mathbf{x})$ and every $\mathbf{a} \in |N|$ we have that $F^N(\mathbf{a}) = F^M(\mathbf{a})$.

When N is a submodel of M we denote this by $N \subseteq M$.

EXAMPLE 2.2. Let $N := \langle \omega, +, 0\rangle$, $\quad M := \langle \mathbb{Z}, +, 0\rangle$ both are $\{+, 0\}$-structures. Notice that N is a submodel of M, while M is a group, N is not a group. Thus a submodel of a group does not have to be a subgroup.

EXAMPLE 2.3. Let $N := \langle \omega, +, {}^{-1}, 0\rangle$, $\quad M := \langle \mathbb{Z}, +, {}^{-1}, 0\rangle$ be L_{gr}-structures while the interpretation of $^{-1}$ in N is the usual group-theoretic inverse (i.e. a^{-1} is the additive inverse $-a$) for every $b \in \omega$ define b^{-1} to be 1. Notice that in this case N is not a submodel of M.

EXAMPLE 2.4. It is interesting to observe that for L_{orf} structures M and N, the relation $N \subseteq M$ does not imply that N is a subfield of M. However, for M and N L_{gr}-structures we have that $N \subseteq M$ iff N is a subgroup of the group M.

EXAMPLE 2.5. Let $S := \omega - \{0\}$ (= the positive integers), $M := \langle S, +, 1\rangle$ when $+$ is the usual addition and the number 1 interprets the symbol 1. It is interesting to observe that $N \subseteq M \implies N = M$. Namely the structure M does not have proper submodels.

In the following fact a basic connection between the notion of a submodel and that of satisfaction is explained. The proof is typical of many proofs to come. The proof is by induction on terms and formulas. We present full details here, later less and less details of similar proofs will be presented.

THEOREM 2.6. *Let L be a similarity type, and let N and M be two L-structures. Suppose that $N \subseteq M$. For every quantifier-free formula $\varphi(\mathbf{x})$ and every $\mathbf{a} \in |N|$ we have that $N \models \varphi[\mathbf{a}] \iff M \models \varphi[\mathbf{a}]$.*

PROOF. First we prove a

LEMMA 2.7.

(1) *For every* $\mathbf{a} \in |N|$ *and every* $\tau(\mathbf{x}) \in \mathrm{Term}(L)$ *we have that* $\tau^N[\mathbf{a}] = \tau^M[\mathbf{a}]$.
(2) *For every* $\varphi(\mathbf{x}) \in \mathrm{AFml}(L)$ *and every* $\mathbf{a} \in |N|$ *we have that* $N \models \varphi[\mathbf{a}] \iff M \models \varphi[\mathbf{a}]$ *holds.*

PROOF.

(1) The proof is by induction on the definition of term and follows closely Definitions 1.4 and 1.19.
   (a) Suppose $\tau(\mathbf{x})$ is the constant symbol c. By Definition 1.19(1) and the assumption that $N \subseteq M$ we have that

$$\begin{aligned}\tau^N[\mathbf{a}] &= \mathrm{Val}_N(\tau, \mathbf{a}) = \mathrm{Val}_N(c, \mathbf{a}) = c^N = c^M \\ &= \mathrm{Val}_M(c, \mathbf{a}) = \mathrm{Val}_M(\tau, \mathbf{a}) = \tau^M[\mathbf{a}].\end{aligned}$$

   (b) Suppose τ is the variable x_k. By Definition 1.19(2) and the assumption that N is a submodel of M, we have that $\tau^N[\mathbf{a}] = \mathrm{Val}_N(\tau, \mathbf{a}) = \mathrm{Val}_N(x_k, \mathbf{a}) = a_k = \mathrm{Val}_M(x_k, \mathbf{a}) = \mathrm{Val}_M(\tau, \mathbf{a}) = \tau^M[\mathbf{a}]$.
   (c) Suppose that τ is the term $F(\tau_1, \ldots, \tau_n)$. By Definition 1.19(3) and the induction hypothesis and the hypothesis that $N \subseteq M$ we have that

$$\begin{aligned}\tau^N[\mathbf{a}] &= \mathrm{Val}_N(\tau, \mathbf{a}) = \mathrm{Val}_N(F(\tau_1, \ldots, \tau_n), \mathbf{a}) \\ &= F^N(\tau_1^N[\mathbf{a}], \ldots, \tau_n^N[\mathbf{a}]) = F^N(\tau_1^M[\mathbf{a}], \ldots, \tau_n^M[\mathbf{a}]) \\ &= F^M(\tau_1^M[\mathbf{a}], \ldots, \tau_n^M[\mathbf{a}]) = \mathrm{Val}_M(F(\tau_1, \ldots, \tau_n), \mathbf{a}) \\ &= \mathrm{Val}_M(\tau, \mathbf{a}) = \tau^M[\mathbf{a}].\end{aligned}$$

(2) (a) Suppose that $\varphi(\mathbf{x})$ is the formula $\tau_1(\mathbf{x}) = \tau_2(\mathbf{x})$; then we have by the assumption on φ we have $N \models \varphi[\mathbf{a}] \iff \tau_1^N[\mathbf{a}] = \tau_2^N[\mathbf{a}]$. By part (1) we have $\tau_l^N[\mathbf{a}] = \tau_l^M[\mathbf{a}]$ for $l \in \{1, 2\}$. Combining the last two statements gives $\tau_1^M[\mathbf{a}] = \tau_2^M[\mathbf{a}]$ which by the assumption on φ yields that $M \models \varphi[\mathbf{a}]$ holds.
   (b) Suppose that φ is the formula $R(\tau_1, \ldots, \tau_n)$; then we have $N \models \varphi[\mathbf{a}] \iff$ (by definition of satisfaction) $N \models R(\tau_1[\mathbf{a}], \ldots, \tau_n[\mathbf{a}]) \iff \langle \tau_1^N[\mathbf{a}], \ldots, \tau_n^N[\mathbf{a}] \rangle \in R^N$ (by part (1)) $\iff \langle \tau_1^M[\mathbf{a}], \ldots, \tau_n^M[\mathbf{a}] \rangle \in R^N$ since $N \subseteq M$ $\iff \langle \tau_1^M[\mathbf{a}], \ldots, \tau_n^M[\mathbf{a}] \rangle \in R^M$ (by the assumption on φ) $\iff M \models \varphi[\mathbf{a}]$.

□

We left to show that for every quantifier-free formula $\varphi(\mathbf{x})$ and every $\mathbf{a} \in N$ we have that

$$N \models \varphi[\mathbf{a}] \iff M \models \varphi[\mathbf{a}].$$

Prove that by induction on the complexity of the quantifier-free formula φ:
For atomic φ this is by Lemma 2.7(2).

(1) Suppose that $\varphi(\mathbf{x})$ is the formula $\psi(\mathbf{x}) \wedge \chi(\mathbf{x})$. By the definition of satisfaction we have that

$$M \models \varphi[\mathbf{a}] \iff M \models \psi[\mathbf{a}] \text{ and } M \models \chi[\mathbf{a}].$$

Now by the induction hypothesis (applied twice) we have

$$N \models \psi[\mathbf{a}] \iff M \models \psi[\mathbf{a}] \text{ and } N \models \chi[\mathbf{a}] \iff M \models \chi[\mathbf{a}].$$

So we have $M \models \varphi[\mathbf{a}] \iff M \models \psi[\mathbf{a}]$ and $M \models \chi[\mathbf{a}] \iff N \models \psi[\mathbf{a}]$ and $N \models \chi[\mathbf{a}] \iff N \models \varphi[\mathbf{a}]$.

(2) Suppose that φ is $\neg\chi$. By definition of satisfaction we have:

$$M \models \varphi[\mathbf{a}] \iff M \not\models \chi[\mathbf{a}]$$

by the inductive hypothesis the last statement is equivalent to $N \not\models \chi[\mathbf{a}]$ which by the definition of satisfaction is equivalent to $N \models \varphi[\mathbf{a}]$.

□

The notion of submodel is not as important in model theory as in algebra, it turns out that the key concept is a stronger notion than that of a submodel. The stronger notion is called elementary submodel. It will be discussed in the following section. The next theorem generalizes also for elementary substructures. It answers affirmatively the question: Whether given an L-structure M does there exists a submodel N of M of smaller cardinality? In the proof of the following simple theorem we present an important idea of model theory.

THEOREM 2.8 (Existence of submodels). *Let L be a similarity type. Suppose that M is an L-structure of cardinality at least $|L|$. For every $A \subseteq |M|$ there exists $N \subseteq M$ such that $|N| \supseteq A$ and $\|N\| \leq |L| + \aleph_0 + |A|$.*

PROOF. By induction on $n < \omega$ define $A_n \subseteq |M|$ with the following properties:

(1) $A_0 = A \cup \{c^M \ : \ c \in \mathbf{C}(L)\}$,
(2) for every $n < \omega \quad A_n \subseteq A_{n+1}$,
(3) for every $n < \omega$ we have that $|A_n| \leq |L| + |A|$, and
(4) for every $n < \omega$, every function symbol $F(\mathbf{x})$ and every finite sequence $\mathbf{a} \in A_n$ we have that $F^M(\mathbf{a}) \in A_{n+1}$.

This is enough: Suppose that such a sequence exists. Let $S := \bigcup_{n<\omega} A_n$. It is easy to verify that S is a universe of a submodel of M containing the set A and the inequality about the cardinality also holds since (using (3)):

$$|S| = |\bigcup_{n<\omega} A_n| \leq \sum_{n<\omega} |A_n| \leq \aleph_0 \cdot (|L| + |A| + \aleph_0) = |L| + |A| + \aleph_0.$$

This is possible: We have to show that there exists a sequence of sets satisfying conditions (1)-(4). The definition is by induction on $n < \omega$. For $n = 0$ define A_0 as the set $A \cup \{c^M \ : \ c \in \mathbf{C}(L)\}$. Suppose A_n is defined and we have to define A_{n+1}. Consider

$$B := \{F^M(\mathbf{a}) \ : \ \mathbf{a} \in A_n, \ F(\mathbf{x}) \text{ is an } L\text{-function symbol}\}.$$

Define now A_{n+1} as the set $B \cup A_n$. From the definition it is clear that requirements (1),(2) and (4) holds. We need to verify only (3): By the inductive hypothesis it is enough to show that $|B| \leq |L| + |A| + \aleph_0$ [3]. From the definition of B we see that $B = \bigcup_{F(\mathbf{x})\in L}\{F^M(\mathbf{a}) \ : \ \mathbf{a} \in A_n\}$. Thus using cardinal arithmetic we have

$$\begin{aligned} |B| =& |\bigcup_{F(\mathbf{x})\in L} \{F^M(\mathbf{a}) \ : \ \mathbf{a} \in A_n\}| \leq \sum_{F(\mathbf{x})\in L} |\{F^M(\mathbf{a}) \ : \ \mathbf{a} \in A_n\}| \\ &\leq \sum_{F(\mathbf{x})\in L} |\{\mathbf{a} \ : \ \mathbf{a} \in A_n\}| \leq \sum_{F(\mathbf{x})\in L} \sum_{k<\omega} |A_n|^k. \end{aligned}$$

By the inductive hypothesis and cardinal arithmetic we have

$$\begin{aligned} \sum_{F(\mathbf{x})\in L} \sum_{k<\omega} |A_n|^k &\leq \sum_{F(\mathbf{x})\in L} \sum_{k<\omega} (|A| + \aleph_0)^k \\ &= \sum_{F(\mathbf{x})\in L} \sum_{k<\omega} \aleph_0 \cdot (|A| + \aleph_0) = \sum_{F(\mathbf{x})\in L} \sum_{k<\omega} (|A| + \aleph_0) \\ &= \sum_{F(\mathbf{x})\in L} (|A| + \aleph_0) \leq |L| \cdot (|A| \cdot \aleph_0) = |A| + |L| + \aleph_0. \end{aligned}$$

□

The next definition is the direct generalization of the concept of isomorphism of algebraic structures and graphs.

DEFINITION 2.9. Let L be a similarity type, and let M and N be L-structures.

(1) We say that M is *isomorphic* to N iff there exists a bijection $f : |M| \to |N|$ with the following properties:

[3] This is the first (and definitely not the last) place that we use the axiom of choice, we are using the fact that for every infinite cardinalities λ and κ we have that $\lambda + \kappa = \lambda \cdot \kappa = Max\{\lambda, \kappa\}$ which is essentially an equivalent statement to AC.
By a theorem of Tarski from 1924 ZF $\vdash$ [[$\forall\lambda$ infinite cardinal $\lambda \cdot \lambda = \lambda$] $\leftrightarrow$ AC]. See Proposition V.1.14 in Levy's Basic Set Theory [**Lev**].

(a) for every constant symbol c we have that $f(c^M) = c^N$,
(b) for every relation symbol $R(\mathbf{x})$ and every $\mathbf{a} \in |M|$ of the same arity as $\mathbf{x}$ we have $\mathbf{a} \in R^M \iff f(\mathbf{a}) \in R^N$, when by $f(\mathbf{a})$ we denote $\langle f(a_0), \dots, f(a_{n-1})\rangle$ where $\mathbf{a} = \langle a_0, \dots, a_{n-1}\rangle$, and
(c) for every function symbol $F(\mathbf{x})$ and every $\mathbf{a} \in |M|$ we have that $f(F^M(\mathbf{a})) = F^N(f(\mathbf{a}))$ holds.

A bijection f as above is called *an isomorphism from M onto N*. We denote this by writing $f : M \cong N$.

(2) If there exists an isomorphism from M onto N we denote this by writing $M \cong N$.
(3) A function $f : |M| \to |N|$ satisfying (a)-(c) of (1) is called a *homomorphism* when f is injective we say that f is a *monomorphism*.

REMARK 2.10. Notice that if M and N are both L-structures then $f : |M| \to |N|$ is a monomorphism iff there exists $N^* \subseteq N$ such that $f : M \cong N^*$

REMARK 2.11. Suppose that M is an L-structure, there exists an L-structure N such that $\exists f : M \cong N$ and $|M| \cap |N| = \emptyset$. Moreover, A be a set of cardinality $\|M\|$ and disjoint to $|M|$ we can define N as above such that $|N| = A$.

A fundamental question we will face is given an L-structure how to find an L-structure M^* such that $M \subseteq M^*$? Usually we will have some additional requirements on the relationship between M and M^*. We will often use the following

LEMMA 2.12 (Fundamental extension Lemma). *Let M be an L-structure. If there are an L-structure N and a monomorphism $f : M \to N$ then there are $M^* \supseteq M$ and an isomorphism f^* from M^* onto N such that $f^* \supseteq f$.*

PROOF. Using Remark 2.11 we may assume that $|M| \cap |N| = \emptyset$. Let $|M^*| := |M| \cup (|N| - f[|M|])$. We need to specify the interpretations of the symbols of L in M^*.

Given c a constant symbol of L let $c^{M^*} := c^M$.

Given $F(x_0, \dots, x_{n-1})$ a function symbol and $\mathbf{a} = \langle a_0, \dots, a_{n-1}\rangle$ sequence of elements of M^* we need to define $F^{M^*}(\mathbf{a})$:

Denote by $\mathbf{b_a}$ the sequence of elements of $|N|$ obtained from $\mathbf{a}$ as follows: if $a_i \in |M|$ then $b_i := f(a_i)$ and in the other case i.e. when $a_i \in |N| - f[|M|]$, let $b_i := a_i$.

Now let

$$F^{M^*}(\mathbf{a}) := \begin{cases} F^N(\mathbf{b_a}) & \text{if } F^N(\mathbf{b_a}) \notin f[|M|] \\ \\ f^{-1}(F^N(\mathbf{b_a})) & \text{if } F^N(\mathbf{b_a}) \in f[|M|]. \end{cases}$$

Given $R(x_0, \ldots, x_{n-1})$ a relation symbol and $\mathbf{a} \in {}^n|M|$ define

$$R^{M^*}(\mathbf{a}) \iff R^N(\mathbf{b}_\mathbf{a}).$$

Verify that $f^* := f \cup \mathrm{id}_{|M^*|-|M|}$ is an isomorphism from M^* onto N. Remark **??** gives us that $f[|M|]$ induces induces a universe of a submodel of N, check that this implies that $M \subseteq M^*$. □

The following theorem is the celebrated Theorem of Georg Cantor that characterizes the order-type of the rational numbers, dressed up in a fancy notation.

THEOREM 2.13. *Suppose $M \models T_{ord}$. If M is countable then $M \cong \langle \mathbb{Q}, < \rangle$.*

PROOF. Since M is countable we may assume that $\{a_n : n < \omega\} = |M|$, and let $\{q_n : n < \omega\}$ be an enumeration of $\mathbb{Q}$.

Consider

$$F := \{f \mid \exists A \subseteq_{finite} |M| \; \exists B \subseteq_{finite} \mathbb{Q} \quad f : \langle A, <^M \rangle \cong \langle B, <^{\mathbb{Q}} \rangle\}.$$

We will define a sequence $\{f_n : n < \omega\} \subseteq F$ such that

(1) for every $n < \omega$ $f_n \subseteq f_{n+1}$,
(2) for every $n < \omega$ we have that $a_n \in \mathrm{Dom}(f_{2n+1})$,
(3) for every $n < \omega$ we have that $q_n \in \mathrm{Rang}(f_{2n+2})$.

This is enough: Suppose that $\{f_n : n < \omega\}$ satisfies (1)-(3).
Let $G := \bigcup_{n<\omega} f_n$. Show that $G : M \cong \langle \mathbb{Q}, < \rangle$: Since $\{f_n : n < \omega\}$ is an increasing sequence of functions, G is a function. G is injective since $G(x) = G(y)$ implies that there are positive integers n_x, n_y such that $x \in \mathrm{Dom}(f_{n_x})$ and $y \in \mathrm{Dom}(f_{n_y})$. Using property (1) without loss of generality we may assume that $n_x \geq n_y$.
By the definition of G we have that

$$f_{n_x}(x) = G(x) = G(y) = f_{n_y}(y) = f_{n_x}(y).$$

Namely we have shown that $f_{n_x}(x) = f_{n_x}(y)$ which by injectivity of f_{n_x} implies that $x = y$. To show that $\mathrm{Dom}(G) = |M|$ it suffices to show that for every $a \in |M|$ there exists $k < \omega$ such that $a \in f_k$. Since $\{a_n : n < \omega\}$ is an enumeration of $|M|$ there exists $n_0 < \omega$ such that $a = a_{n_0}$. By condition (2) we have that $a = a_{n_0} \in \mathrm{Dom}(f_{2n_0+1})$. Since $\mathrm{Dom}(f_{2n_0+1}) \subseteq G$ we have shown that $a = a_{n_0} \in G$. The argument that G is onto $\mathbb{Q}$ is similar (just use (3) in place of (2)).

This is possible: We have to prove that a sequence $\{f_n : n < \omega\}$ satisfying (1)-(3) exists. We do that by defining it by induction on $n < \omega$ given the sequence $\{f_k : k < n\}$ we define f_n:

For $n = 0$: Take $f_0 := \{\langle a_0, q_0 \rangle\}$.

For $n > 0$: Suppose that $\{f_k \ : \ k < n\}$ is defined, now there are two cases:

(1) n is odd; Suppose $n = 2\ell + 1$, if $a_l \in \mathrm{Dom}(f_{n-1})$ then define the function f_n to be f_{n-1}. In case $a_\ell \notin \mathrm{Dom}(f_{n-1})$ let $D := \mathrm{Dom}(f_{n-1})$. Since D is finite and $|M|$ is linearly ordered we may assume that $\{b_1, \ldots, b_k\}$ is an enumeration of the elements of D (in increasing order with respect to $<^M$). Now consider the relationship of the element a_l with respect to the elements of D. There are exactly three possibilities:

(1) $a_\ell <^M b_1$, or
(2) $a_\ell >^M b_k$, or
(3) there exists i such that $1 \leq i < k$ and $b_i < a < b_{i+1}$.

If $a_\ell <^M b_1$, let $m := \min\{m < \omega \ : \ q_m < f_{n-1}(b_1)\}$ define f_n to be $f_{n-1} \cup \{\langle a_\ell, q_m\rangle\}$. Now it is easy to check that $\{f_k \ : \ k \leq n\}$ satisfies requirements (1) and (2).

If $a_\ell >^M b_k$, let $m := \min\{m < \omega \ : \ q_m > f_{n-1}(b_k)\}$ define f_n to be $f_{n-1} \cup \{\langle a_\ell, q_m\rangle\}$. Now it is easy to check that $\{f_k \ : \ k \leq n\}$ satisfies requirements (1) and (2).

There exists i such that $1 \leq i < k$ and $b_i < a < b_{i+1}$.
Let $m := \min\{m < \omega \ : \ f_{n-1}(b_1) < q_m < f_{n-1}(b_k)\}$ define f_n to be $f_{n-1} \cup \{\langle a_l, q_m\rangle\}$. Now it is easy to check that $\{f_k \ : \ k \leq n\}$ is as required.

(2) n is even; Suppose $n = 2\ell$, if $q_l \in \mathrm{Rang}((f_{n-1}))$ define f_n to be f_{n-1}. In case $q_v \notin R := \mathrm{Rang}((f_{n-1}))$ let $\{r_1, \ldots, r_k\}$ be an enumeration of R increasing in the order of the rationals. Consider the relationship of q_ℓ with respect to $\{r_1, \ldots, r_k\}$. As above there are three possibilities. Since they are all similar we deal only with the parallel to the third case. Suppose that there exists $i < k$ such that $r_i < q_\ell < r_{i+1}$. Let $a := f_{n-1}^{-1}(r_i)$ and $b := f_{n-1}^{-1}(r_{i+1})$. Since f_{n-1}^{-1} is also order-preserving we have that $M \models a < b$. Since $M \models T_{ord}$ we have that $M \models \forall x \forall y[x < y \to \exists z[x < z \wedge z < y]]$, the set $\{m < \omega \ : \ a < a_m < b\}$ is not empty. Let m be the first element of this set. define f_n to be $f_{n-1} \cup \{\langle a_m, q_l\rangle\}$. Now it is easy to check that $\{f_k \ : \ k \leq n\}$ satisfy (1) and (3). □

DEFINITION 2.14. Let L be a similarity type.

(1) Let M and N be L-structures. We say that M and N are *elementary equivalent* iff $\mathrm{Th}(M) = \mathrm{Th}(N)$. We denote this by writing $M \equiv N$.
(2) Let T be an L-theory. The class of models of T is the following:

$$\mathrm{Mod}(T) = \{M \ : \ M \models T\}.$$

(3) When $\mathcal{K} = \mathrm{Mod}(T)$ for some theory T we say that T *is an axiomatization of* $\mathcal{K}$. The sentences in T are *axioms of* $\mathcal{K}$. When $\mathrm{Mod}(T) = \mathrm{Mod}(T^*)$ we say that T^* *is an axiomatization of* T.

$\mathrm{Mod}(T_g)$ is the class of groups. Within few pages we will present the theory T_{ACF}. It will have the property that $\mathrm{Mod}(T_{ACF})$ is the class of all algebraically closed fields.

There are some obvious properties of elementary equivalence and isomorphism.

PROPOSITION 2.15. *Let L be a similarity type, and let M_1, M_2, M_3 be L-structures.*

(1) $M_1 \equiv M_1$, $M_1 \equiv M_2 \Longrightarrow M_2 \equiv M_1$,
$M_1 \equiv M_2$ *and* $M_2 \equiv M_3 \Longrightarrow M_1 \equiv M_3$.
(2) $M_1 \cong M_1$, $M_1 \cong M_2 \Longrightarrow M_2 \cong M_1$,
$M_1 \cong M_2$ *and* $M_2 \cong M_3 \Longrightarrow M_1 \cong M_3$.

PROOF. Exercise. □

A natural question to ask is: What is the relationship between the notions of elementary equivalence and that of isomorphism? The answer is that they are different and based on experience from algebra one can guess the obvious connection. Namely that always $M \cong N \Longrightarrow M \equiv N$ but often the converse is false.

THEOREM 2.16. *Let L be a similarity type, suppose that M and N are L-structures. If $M \cong N$ then $M \equiv N$.*

PROOF. Since we have to show that for every $\varphi \in \mathrm{Sent}(L)$ we have that $M \models \varphi$ iff $N \models \varphi$ it is natural to try to prove this by induction on the complexity of sentences. This is one of the cases where instead of proving a statement directly by induction we prove a strengthening of the statement (that later will be used extensively).

CLAIM 2.17. *Suppose that $g : M \cong N$ is a given isomorphism. For every $\mathbf{a} \in |M|$ and every $\varphi(\mathbf{x}) \in \mathrm{Fml}(L)$ we have that $M \models \varphi[\mathbf{a}]$ iff $N \models \varphi[g(\mathbf{a})]$.*

PROOF. To start we need a trivial fact

SUB CLAIM 2.18. *For every term $\tau(\mathbf{x})$ and every $\mathbf{a} \in |M|$ we have that $g(\tau^M[\mathbf{a}]) = \tau^N[g(\mathbf{a})]$.*

PROOF. By induction on the structure of τ:

(1) Suppose that τ is the constant symbol c, by Definition 1.19(1)(a) and the definition of isomorphism we have that
$g(\tau^M[\mathbf{a}]) = g(c^M) = c^N = \tau^N[g(\mathbf{a})]$.
(2) Suppose that τ is the variable x_k, by Definition 1.19(1)(b) and the definition of isomorphism we have that $g(\tau^M[\mathbf{a}]) = g(a_k) = \tau^N[g(\mathbf{a})]$.
(3) Suppose that τ is $F(\tau_1(\mathbf{x}), \ldots, \tau_n(\mathbf{x}))$, by Definition 1.19(1)(c) and the definition of isomorphism we have that

$$\begin{aligned} g(\tau^M[\mathbf{a}]) =& g(F^M(\tau_1^M(\mathbf{a}), \ldots, \tau_n^M(\mathbf{a}))) \\ =& F^N(g(\tau_1^M(\mathbf{a})), \ldots, g(\tau_n^M(\mathbf{a}))). \end{aligned}$$

Applying the induction hypothesis n-times gives that

$$F^N(g(\tau_1^M(\mathbf{a})), \ldots, g(\tau_n^M(\mathbf{a}))) = F^N(\tau_1^N(g(\mathbf{a})), \ldots, \tau_n^N(g(\mathbf{a}))).$$

Using Definition 1.19(1)(c) once more we get that the last term is equal to $\tau^N[g(\mathbf{a})]$.

□

By induction on formulas we show that for every $\varphi(\mathbf{x}) \in \mathrm{Fml}(L)$ and for every $\mathbf{a} \in |M|$ we have that

$$M \models \varphi[\mathbf{a}] \text{ iff } N \models \varphi[g(\mathbf{a})].$$

(1) For atomic φ there are two possibilities:
  (a) Suppose that φ is $\tau_1(\mathbf{x}) = \tau_2(\mathbf{x})$; By the Definition of satisfaction we have that $M \models \varphi[\mathbf{a}]$ iff $M \models \tau_1(\mathbf{a}) = \tau_2(\mathbf{a})$. By the last Sub Claim we have that $g(\tau_l^M(\mathbf{a})) = \tau_l^N(g(\mathbf{a}))$. Thus we have that $M \models \tau_1(\mathbf{a}) = \tau_2(\mathbf{a})$ iff $N \models \tau_1(g(\mathbf{a})) = \tau_2(g(\mathbf{a}))$. Using the definition of satisfaction the last statement is equivalent to
  $N \models \varphi[g(\mathbf{a})]$.
  (b) Suppose that φ is $R(\tau_1(\mathbf{x}), \ldots, \tau_n(\mathbf{x}))$; By the definition of satisfaction and the Sub Claim we have
  $M \models \varphi[\mathbf{a}]$ iff $\langle \tau_1^M(\mathbf{a}), \ldots, \tau_n^M(\mathbf{a})\rangle \in R^M$ iff
  $\langle g(\tau_1^M(\mathbf{a})), \ldots, g(\tau_n^M(\mathbf{a}))\rangle \in R^N$ iff
  $\langle \tau_1^N(g(\mathbf{a})), \ldots, \tau_n^N(g(\mathbf{a}))\rangle \in R^N$ iff $N \models \varphi[g(\mathbf{a})]$.
(2) (a) Suppose that $\varphi(\mathbf{x})$ is $\neg\psi(\mathbf{x})$; By the definition of satisfaction and the induction hypothesis we have that
  $M \models \varphi[\mathbf{a}]$ iff $M \not\models \psi[\mathbf{a}]$ iff $N \not\models \psi[g(\mathbf{a})]$ iff $N \models \varphi[g(\mathbf{a})]$.
  (b) Suppose that $\varphi(\mathbf{x})$ is $\psi(\mathbf{x}) \wedge \chi(\mathbf{x})$;

$$\begin{aligned} M \models \varphi[\mathbf{a}] &\underset{\text{by definition of } \models}{\iff} M \models \psi[\mathbf{a}] \text{ and } M \models \chi[\mathbf{a}] \\ &\underset{\text{by induction hypothesis}}{\iff} N \models \psi[g(\mathbf{a})] \text{ and } N \models \chi[g(\mathbf{a})] \\ &\underset{\text{by definition of } \models}{\iff} N \models \varphi[g(\mathbf{a})]. \end{aligned}$$

(3) Suppose that $\varphi(\mathbf{x})$ is $\exists y\psi(y,\mathbf{x})$; By the definition of satisfaction and the induction hypothesis we have that

$$M \models \varphi[\mathbf{a}] \iff \text{ there is } b \in |M| \text{ such that } M \models \psi[b,\mathbf{a}].$$

The last statement (using the induction hypothesis) implies that $N \models \psi[g(b), g(\mathbf{a})]$ using the definition of satisfaction we get that there exists $c \in |N|$ such that $N \models \psi[c, g(\mathbf{a})]$ which is equivalent to $N \models \exists y\psi(y, g(\mathbf{a}))$. For the converse, suppose that $N \models \exists y\psi(y, g(\mathbf{a}))$, namely there exists $c \in |N|$ such that $N \models \psi[c, g(\mathbf{a})]$. Since g is surjective there exists $b \in |M|$ such that $g(b) = c$. So by the induction hypothesis we get "there exists $b \in |M|$ such that $M \models \psi[b,\mathbf{a}]$. However the last statement is equivalent to $M \models \varphi[\mathbf{a}]$.

□

Let L be a similarity type, let T be a theory, and let φ be an L-sentence. We write $T \vdash \varphi$ to denote that φ *is provable from* T. The notion of provability is discussed and studied with great detail in any elementary course in logic. As far as model-theory is concerned we do not need much of the proof theory of predicate calculus. All we need to know is following: For any similarity type there is a set of sentences called the *logical axioms* containing all the obviously true statements (for example for every formula $\varphi(x)$ the following are logical axioms: $\forall x\varphi(x) \to \exists x\varphi(x)$, $\forall x\varphi(x) \to \forall y\varphi(y)$ (provided y does not appear bounded in φ), $\forall x[\varphi(x) \vee \neg\varphi(x)]$ are all logical axioms). There is a set of rules called *inference rules* (e.g. from the sentences φ and $\varphi \to \psi$ we can deduce ψ). Now the exact meaning of $T \vdash \varphi$ is that there exists a finite sequence of L-sentences $\langle\varphi_0, \ldots, \varphi_n\rangle$ such that $\varphi = \varphi_n$ and for every $k \leq n$ we have that φ_k is an element of T, a logical axiom or obtained by an application of an inference rule to sentences from $\{\varphi_l \,:\, l < k\}$. The sequence $\langle\varphi_0, \ldots, \varphi_n\rangle$ is called a *proof of* φ *from* T. For T and φ as above when there is no proof of φ from T we denote this by $T \nvdash \varphi$.

Notice that the following relationship between $\vdash$ and $\models$: Suppose that φ is a sentence, T and $M \models T$. If $T \vdash \varphi$ then necessarily $M \models \varphi$ (see Exercise 50). Let me recall some fundamental concepts from predicate calculus:

DEFINITION 2.19. Let T be a theory (- just a set of sentences).

(1) T is *a consistent theory* if there is no $\varphi \in \mathrm{Sent}(\mathrm{L}(T))$ such that

$$T \vdash \varphi \text{ and } T \vdash \neg\varphi.$$

T is called *inconsistent* if T is not consistent.

(2) A consistent theory T is called *complete theory* iff

$$T \vdash \varphi \text{ or } T \vdash \neg\varphi \quad \text{ for every } \varphi \in \mathrm{Sent}(\mathrm{L}(T)).$$

(3) A consistent theory T is called *deductively closed* provided

$$T \vdash \varphi \implies \varphi \in T \quad \text{for every } \varphi \in \mathrm{Sent}(\mathrm{L}(T))$$

(4) A consistent theory T is called *maximally consistent* iff

$T \cup \{\varphi\}$ is consistent implies that $\varphi \in T$ for every $\varphi \in \mathrm{Sent}(\mathrm{L}(T))$.

(5) For a consistent theory T we say that the sentence φ *is independent from* T iff

$$T \nvdash \varphi \text{ and } T \nvdash \neg\varphi.$$

REMARK 2.20. Notice that it follows automatically from the definitions that if $T \vdash \varphi$ then there exists a finite $T_0 \subseteq T$ such that $T_0 \vdash \varphi$. Thus if T is an inconsistent theory then there exists a finite $T_0 \subseteq T$ which is inconsistent. We will refer to this property as the *finite character of consistency* .

PROPOSITION 2.21. *Let T be a consistent theory.*

(1) *T is maximally consistent iff T is deductively closed.*
(2) *If T is maximally consistent then T is complete.*
(3) *There exists a maximally consistent theory $T^* \supseteq T$ in $\mathrm{L}(T)$.*

PROOF. Exercise. □

THEOREM 2.22 (Lindenbaum). *If T is a consistent theory in L then there exists a maximally consistent L-theory $T^* \supseteq T$.*

PROOF. Consider

$$P := \{T_1 \subseteq \mathrm{Sent}(L) \mid T_1 \text{ consistent and containing } T\}.$$

Viewing P is partially ordered by $\subseteq$ the subset relation clearly, any maximal element of P is as required. We establish existence of a maximal element using Zorn's lemma. It sufices to show that every chain $C \subseteq P$ has an upper bound.

Let C be a chain. Take b to be $\bigcup C$. Clearly b is a set of sentences satisfying $b \supseteq T$. As $b \supseteq T'$ for every $T' \in C$ it is enough to show that b is consistent: Otherwise, using Remark 2.20 there exists a finite inconsistent $T_0 \subseteq b$. Let $\{\varphi_1, \ldots, \varphi_n\} = T_0$. There are $T'_k \in C$ such that $\varphi_k \in T'_k$ for all k. Since C is a chain there exists $T" \in P$ such that $T" \supseteq T'_k$ for all k. Now since $T_0 \subseteq T"$ we get that $T"$ must be inconsistent, which is a contradiction to $T" \in P$.

□

EXAMPLE 2.23. Let M be an L-structure. The theory $\mathrm{Th}(M)$ is consistent and complete.

EXAMPLE 2.24. Let L be the similarity type consisting a single binary relation symbol $<$. Consider the *theory of linear order* :

$$\begin{aligned} T_{lo} := \Big\{ & \forall x[\neg(x < x)], \\ & \forall x \forall y \forall z[x < y \wedge y < z \to x < z], \\ & \forall x \forall y[x < y \vee y < x \vee x = y] \Big\}. \end{aligned}$$

T_{lo} is incomplete since the sentence $\varphi := \exists x \forall y [x < y \vee x = y]$ is independent from T_{lo}. Clearly $\mathrm{Th}(\langle \omega, < \rangle) \supseteq T_{lo}$ and $\neg\varphi \in \mathrm{Th}(\langle \mathbb{Q}, < \rangle)$, so

$$T_{lo} \nvdash \varphi.$$

Since $\mathrm{Th}(\langle \omega, < \rangle) \supseteq T_{lo}$ and $\varphi \in \mathrm{Th}(\langle \omega, < \rangle)$, thus

$$T_{lo} \nvdash \neg\varphi.$$

EXAMPLE 2.25. The sentence $\psi := \forall x \forall y [xy = yx]$ is independent from T_g (group theory, see Example **??**).
Since $\langle \mathbb{Z}, + \rangle \models T_g \cup \{\psi\}$ while $\langle S_3, \circ \rangle \models T_g \cup \{\neg\psi\}$.

EXAMPLE 2.26. In 1963 Paul Cohen proved that ZFC $\nvdash$ CH where CH stands for the continuum hypothesis. In the mid thirties Kurt Gödel has shown that ZFC $\nvdash \neg$CH. Both results put together proves that CH is independent from ZFC.

EXAMPLE 2.27. Shortly we will show that T_{ord} is a complete theory. We will accomplish this via a general sufficient condition for the completeness of a theory.

Recall from Exercise 1.1.2 the notion of definable subset in a structure. The following generalization is particularly important:

DEFINITION 2.28. Let M be a structure, A a subset of its universe and n a positive integer. A set $S \subseteq {}^{n}|M|$ is *definable over* A iff there are a formula $\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(M))$ and a squence $\mathbf{b} \in {}^{\ell(\mathbf{y})}\mathrm{A}$ with $\ell(\mathbf{x}) = n$ such that

$$S = \{\mathbf{a} \in {}^{n}|M| \ : \ M \models \varphi[\mathbf{a}; \mathbf{b}]\}.$$

Often we write $\varphi(M; \mathbf{b})$ to denote the above set. When $A = \emptyset$ we say that S is *definable over the empty set* in M, or simply S is *definable in M without parameters.* S is *definable with parameters* provided S is $|M|$-definable.

In general there are several ways to define the same set.

EXAMPLES 2.29.

(1) Let M be the field of real numbers. The set $\{2, 3\}$ is definable by

$$\varphi(x) := [x = 1 + 1 \vee x = 1 + 1 + 1] \quad \text{and also by}$$

$$\psi(x) := [x \cdot x - (1 + 1 + 1 + 1 + 1) \cdot x + 1 + 1 + 1 + 1 + 1 + 1 = 0].$$

(2) Let $M := \langle \mathbb{R}, +, \cdot, \sin, 0, 1 \rangle$ be the structure, were $\sin$ is the usual trigonometric function and $+, \cdot$ are the usual operations on the real numbers. The set of integers $\mathbb{Z}$ is definable over $\{\pi\}$ by

$$\varphi(x) := [\sin(x \cdot (1 + 1) \cdot \pi) = 0], \quad \text{and also by}$$

$$\psi(x) := \exists z \big[(1 + 1) \cdot z = 1 \wedge [\sin(x \cdot z \cdot \pi) = 1 \vee 1 + \sin(x \cdot z \cdot \pi) = 0]\big].$$

One can find a formula without parameters that defines the integers.

(3) The set of even integers $2\mathbb{Z}$ is definable in the ring $\langle \mathbb{Z}, +, \cdot, 0, 1\rangle$ by the formula $\exists z[z+z = x]$. There are infinitely many other formulas defininig the same set.

For a model theorist the actual formula that defines a set is much less important than it is for a proof theorist. If M is an infinite structure whose cardinality is at least the cardinality of its language there are always subsets that are not definable. This follows from Cantor's Theorem 15.2.4: Since the number of definable subsets is at most $\|M\|$ while the total number of distinct subsets is $2^{\|M\|}$ there exists $S \subseteq |M|$ not definable.

DEFINITION 2.30. Let M be an L-structure. $f : |M| \to |M|$ is called an *automorphism* iff f is an isomorphism from M onto M.

$$\mathrm{Aut}(M) := \{f \mid f \text{ is an automorphism of } M\}.$$

For $A \subseteq |M|$ let

$$\mathrm{Aut}_A(M) := \{f \in \mathrm{Aut}(M) \ : \ f \restriction A = \mathrm{id}_A\}.$$

To identify whether a given subset of M is definable over A is little harder. The answer is connected to the study of the action of the group $\mathrm{Aut}_A(M)$ on the set $|M|$ (see exercise 66 below). At first we present a necessary condition and in the next chapter (see Theorem 2.1.13) we will prove that if M is rich enough then the condition is also sufficient.

PROPOSITION 2.31. *Let $A \subseteq |M|$ and $S \subseteq {}^n|M|$ for some integer n. If S is definable over A then for every $f \in \mathrm{Aut}_A(M)$ we have that $f[S] = S$.*

PROOF. Otherwise there are $\mathbf{c} \in S$ and $f \in \mathrm{Aut}_A(M)$ such that $f(\mathbf{c}) \notin S$. Let $\varphi(\mathbf{x}; \mathbf{y})$ be a formula in the language of M such that there is $\mathbf{b} \in {}^{\ell(\mathbf{y})}\mathrm{A}$ satisfying $S = \varphi(M; \mathbf{b})$. By the assumption on $\mathbf{c}$ we have

$$M \models \varphi[\mathbf{c}; \mathbf{b}] \quad \text{and} \quad M \models \neg\varphi[f(\mathbf{c}); \mathbf{b}].$$

Since $f(\mathbf{b}) = \mathbf{b}$ this is a contradiction to Claim 2.17. □

REMARK 2.32. We will show in Chapter 4 that under the assumptions that M is saturated and $\|M\| > |A|$ also the converse of the last proposition holds, see Theorem 4.1.13.

DEFINITION 2.33. Let T be a consistent theory, and let λ be an infinite cardinal such that $\lambda \geq |\mathrm{L}(T)|$.

(1) We say that T is λ-*categorical* iff for every pair of $\mathrm{L}(T)$-structures M and N of cardinality λ we have that $M, N \models T \Longrightarrow M \cong N$.
(2) T is *categorical* if there exists a cardinal $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ such that T is λ-categorical.

(3) T is *totally categorical* iff T is categorical in λ for every $\lambda \geq |\operatorname{L}(T)|+\aleph_0$.

EXAMPLE 2.34. It is the contents of Theorem 2.13 that T_{ord} is an $\aleph_0$-categorical theory.

EXAMPLE 2.35 (Theory of infinite models). Let L be the empty similarity type (yes no symbols). Let T be the following infinite set of axioms: $\{\varphi_n \ : \ n < \omega\}$ when

$$\varphi_n := \exists x_0 \dots \exists x_{n-1} \bigwedge_{i<j<n} [x_i \neq x_j].$$

Notice that any infinite set is a model of T. Since the language is empty for every $M, N \models T$ we have that $M \cong N$ iff $\|M\| = \|N\|$. Thus the theory T is totally categorical.

EXAMPLE 2.36 (The theory of the countable random graph). Let L consists of a single binary relation R. Let T_{ind} be the following theory $\{\forall x \forall y[xRy \rightarrow yRx] \wedge \forall x[\neg(xRx)]\} \cup \{\varphi_{n,k} \ : \ n, k < \omega\}$ where $\varphi_{n,k}$ stands for the sentence

$$\forall x_0 \dots \forall x_{n-1} \forall y_0 \dots \forall y_{k-1} \Big[[\bigwedge_{i<j<n} x_i \neq x_j \wedge \bigwedge_{i<j<k} y_i \neq y_j \wedge \bigwedge_{\substack{i<n \\ j<k}} x_i \neq y_j] \rightarrow \exists z [\bigwedge_{\substack{i<n \\ j<k}} zRx_i \wedge \neg(zRy_j) \wedge z \neq x_i \wedge z \neq y_j] \Big].$$

T_{ind} is an $\aleph_0$-categorical theory (see Exercise 61).

REMARK 2.37. It is interesting to notice that the countable model of T_{ind} was discovered in 1960 by Erdös and Renyi in a different context. It is known to combinatorialists as *the countable random graph*. The random graph is the (undirected) graph whose set of vertices is the set of natural numbers and the decision whether $\{i, j\}$ is an edge is determined by flipping a fair coin, the graph G obtained in this way is called *the random graph*. It is a theorem that any two graphs constructed in the above process are isomorphic, moreover $\operatorname{Th}(G) = T_{ind}$. For more details see [**ES**].

Now I introduce one of the more important theories that is rooted in algebra. This theory had a central role in the development of the subject, both in the applied and the purest directions. The theory is known as the theory of *algebraically closed fields*, the theory is denoted by T_{ACF}. The similarity type is that

of fields, namely L_f introduced above. T_{ACF} is the following set of sentences:

$$\Big\{0 \neq 1,$$
$$\forall x \forall y \forall z[x + (y + z) = (x + y) + z \wedge$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z],$$
$$\forall x \exists y[x + y = 0],$$
$$\forall x[x \neq 0 \to \exists y[x \cdot y = 1]],$$
$$\forall x \forall y[x + y = y + x \wedge x \cdot y = y \cdot x],$$
$$\forall x \forall y \forall z[x \cdot (y + z) = x \cdot y + x \cdot z]\Big\} \cup$$
$$\{\varphi_n \ : \ 1 \leq n < \omega\}.$$

Where for $1 \leq n < \omega$ the sentence φ_n is the statement that every polynomial of degree n has a root, in symbols it is- $\forall a_0 \ldots \forall a_n[a_n \neq 0 \to \exists x[\sum_{i=0}^{n} a_i \cdot x^i = 0]]$ where by x^i we denote the term $x \cdot x \cdots x$ (i times) and $\sum_{i=0}^{n} b_i$ is a shorthand to the term $b_0 + b_1 + \cdots + b_n$. When working in the language L_f for a positive integer n instead of writing out the term $\underbrace{x + x + \cdots + x}_{n\text{-times}}$ we write $n \cdot x$.

The set of the first seven axioms in T_{ACF} (without the φ_n's) is denoted by T_f and is called *field theory*.

Let p be a prime number, by T_{ACF_p} denote the theory

$$T_{ACF} \cup \big\{[p \cdot 1 = 0] \wedge \bigwedge_{1 \leq k < p} [k \cdot 1 \neq 0]\big\}.$$

T_{ACF_p} is called the *theory of algebraically closed fields of characteristics* p. The theory T_{ACF_0} is called the *theory of algebraically closed fields of characteristics* 0 and it is

$$T_{ACF} \cup \big\{[n \cdot 1 \neq 0] \ : \ 2 \leq n < \omega\big\}.$$

While from the fundamental theorem of algebra (for a purely algebraic proof see, Theorem V 3.19 in Hungerford [**Hu**]) we know that the field of complex numbers is a model of T_{ACF_0}, apriori it is not clear at all that the theory T_{ACF_0} captures all the first-order properties of the structure $\langle \mathbb{C}, +, \cdot \rangle$. In section 3 we will show that this indeed is the case. Namely, T_{ACF_0} is an axiomatization of $\mathrm{Th}(\langle \mathbb{C}, +, \cdot \rangle)$.

The following result which is a corollary of Steinitz's theorem that characterizes the isomorphism-type of an algebraically closed field by the transcendence degree (for a proof see Theorem VI.1.12 in [**Hu**]) is one of the precursors of modern model theory.

THEOREM 2.38 (Steinitz's theorem). *Let p be a prime number or zero. For every uncountable cardinal number λ, the theory T_{ACF_p} is categorical in λ.*

Moreover, suppose $F_\ell \leq E_\ell$ (for $\ell = 0, 1$) are algebraically closed fields such that E_ℓ has cardinality $\lambda > \aleph_0$. If $f : F_0 \cong F_1$, $|F_0| < \lambda$, $c \in E_0$ transcendental over F_0 and $d \in E_1$ transcendental over F_1 then there exists $g : E_0 \cong E_1$ such that $g \supseteq f$ and $g(c) = d$.

Before starting the proof of Theorem 2.38 let me review some basic concepts and facts from field theory:

DEFINITION 2.39. Suppose that $F \subseteq E$ are fields [4]. An element $a \in E$ is *algebraic over F* iff there exists a non-zero polynomial $p(x) \in F[x]$ such that $E \models [p(a) = 0]$.

An element $a \in E$ is *transcendental over F* iff it is not algebraic over F.

For fields $F \subseteq E$ and a subset A of E denote by $F(A)$ the *subfield of E generated by A over F* it is the set $\{\frac{a}{b} \mid a, b \in F[A] \text{ such that } b \neq 0\}$, where $F[A]$ is the ring of polynomials with coefficients from F and variables from the set A.

The following is a summary of some basic facts from field theory, this can be found in every algebra text book (E.g. see section 5.1 in [**Hu**]):

FACT 2.40.

(1) *Let F be a field and F_p be its prime field (- its minimal subfield). Let $p = CharF$, if $p = 0$ then $F_p \cong \langle \mathbb{Q}, +, \cdot \rangle$ and if $p > 0$ then it is a prime number and $F_p \cong \mathbb{Z}/p\mathbb{Z}$*[5].
(2) *Suppose that $F \subseteq E$ are fields. If $a \in E$ is transcendental over F then $F(a) \cong F(x)$. By $F(x)$ we denote the* field of rational functions over F, *namely $F(x) := \{\frac{p(x)}{q(x)} \mid p, q \in F[x]$ and $q(x) \neq 0\}$.*
(3) *Suppose that $F \subseteq E$ are fields. If $a \in E$ is algebraic over F then there exists a unique monic (-with leading coeffieicient $= 1$) polynomial with minimal degree $p_a \in F[x]$ such that $F \models p_a(a) = 0$ and $F(a) \cong F[x]/(p_a)$.*

Now we are ready to prove Theorem 2.38. I hope that the remarkable similarity to the proof of Theorem 2.13 will not escape the reader.

PROOF OF THEOREM 2.38. Suppose $E_\ell \geq F_\ell$ are both fields as in the moreover part. algebraically closed fields of characteristic of an uncountable cardinality λ. Fix enumerations of both fields: $\{a_i \mid i < \lambda\} = E_0 - F_0$ and $\{b_i \mid i < \lambda\} = E_1 - F_1$ such that a_0 is c and b_0 is d.

[4] notice that we assume that F is a subfield of E.

[5] The standard notation for the field with p elements is $\mathbb{F}_p$.

It suffices to define $A_i := F_0(\{a_j \mid j < i\})$, $B_i := F_1(\{b_j \mid j < i\})$ and functions $\{f_i : X_i \to Y_i \mid A_i \subseteq X_i \subseteq |F|,\ B_i \subseteq Y_i \subseteq |E|, i < \lambda\}$ such that $f_i \restriction A_i : A_i \cong B_i$ such that f_0 is f and $f_i \subseteq f_j$ for all $i < j < \lambda$.

The definition is by induction on $i < \lambda$:

Let $i < \lambda$, suppose that f_i is defined. Now define sets X_{i+1}, Y_{i+1} and a function f_{i+1} as above. If a_i is algebraic over A_i let $p_{a_i} \in A_i[x]$ be its monic polynomial with minimal degree. Suppose that $p_{a_i} = \sum c_n x^n$ Since E is algebraically closed the polynomial $q_{a_i} := \sum f_i(c_n)x^n$ has a root b in E. Using Fact 2.40 twice we get:

$$A_i(a_i) \cong A_i[x]/(p_{a_i}) \overset{f_i}{\cong} B_i[x]/(q_{a_i}) \cong B_i(b).$$

Denote by g_i the above isomorphism from $A_i(a_i)$ onto $B_i(b)$

In case a_i is transcendental over A_i the cardinality of the set $\{b \in E \mid b \text{ is algebraic over } B_i\}$ is at most $|B_i| + \aleph_0$ and since

$$|B_i| + \aleph_0 \leq |i| + \aleph_0 \underset{\text{since } \lambda \text{ is uncountable}}{<} \lambda$$

there exists a transcendental element b in E over B_i. Using Fact 2.40 we get

$$A_i(a_i) \cong A_i(x) \overset{f_i}{\cong} B_i(x) \cong B_i(b)$$

Denote by g_i the above isomorphism from $A_i(a_i)$ onto $B_i(b)$.

Notice that in both cases (a_i algebraic or not) $g_i \supseteq f_i$.

Now consider the element b_i with respect to the field $B_i(b)$. A similar argument will produce an element $a \in F$ and an extension f_{i+1} of g_i such that $f_{i+1} : A_i(a_i)(a) \cong B_i(b)(b_i)$. □

Using the argument from the proof of Theorem 2.38 one can show:

THEOREM 2.41. *Let E be an uncountable algebraically closed field and let A be a subset such that $|A| < |E|$. If $b, c \in E$ are transcendental over A (both elements are transcendental over the extension of the prime field with the set A) then there exists $f \in \mathrm{Aut}_A(E)$ such that $f(b) = c$.*

PROOF. Exercise. □

The following cardinal-valued function is the most important function in model theory. It is called the *spectrum function of a theory*. See more in the mathematical introduction (section 2).

DEFINITION 2.42. Let $\mathcal{K}$ be a class of L-structures and λ a cardinal number.

By $\mathrm{I}(\lambda, \mathcal{K})$ denote the number of non-isomorphic models of the class $\mathcal{K}$ of cardinality λ. In symbols:

$$\mathrm{I}(\lambda, \mathcal{K}) := |\{M/\cong : \ M \in \mathcal{K}, \ \|M\| = \lambda\}|.$$

Let T be a first-order theory, by $\mathrm{I}(\lambda, T)$ we denote $\mathrm{I}(\lambda, \mathcal{K})$ where $\mathcal{K}$ is $\mathrm{Mod}(T)$.

Fixing a theory T, we get a natural function $\lambda \mapsto \mathrm{I}(\lambda, T)$ which is a function from the class of cardinal numbers Card into Card. This is called the *spectrum function of* T.

PROPOSITION 2.43. *Let T be a theory and suppose that λ is a cardinal greater or equal to $|\mathrm{L}(T)| + \aleph_0$. Then $\mathrm{I}(\lambda, T) \leq 2^\lambda$.*

PROOF. Exercise 80. □

The following theorem follows from Fact 2.38:

THEOREM 2.44. *Let p be a prime number or zero. If λ is an uncountable cardinal number then $\mathrm{I}(\lambda, T_{ACF_p}) = 1$. While $\mathrm{I}(\aleph_0, T_{ACF_p}) = \aleph_0$.*

PROOF. Exercise 82. □

Another important theory that shares some of the properties of the T_{ACF_p} (see Exercise 81) while being quite different is the theory of vector spaces over a specified field F:

EXAMPLE 2.45 (The theory of vector spaces). Let F be a field. Denote by L the following similarity type $\{+, 0\} \cup \{f_s \ : \ s \in F\}$ when f_s are unary function symbols (they stand for multiplication by the scalar s) and $+$ is a binary function that stands for addition of vectors.

The *theory of vector spaces over F* is denoted by $T_{V,F}$ and it is the following set of sentences:

$$\left\{\forall x \forall y \forall z[x + (y + z) = (x + y) + z],\right.$$
$$\forall x \forall y[x + y = y + x],$$
$$\forall x[x + 0 = x],$$
$$\left.\forall x \exists y[x + y = 0]\right\} \bigcup$$
$$\{\forall x \forall y[f_s(x + y) = f_s(x) + f_s(y)] \ : \ s \in F\} \bigcup$$
$$\big\{\forall x[f_{s \cdot t}(x) = f_s(f_t(x))] \wedge \forall x[f_{s+t}(x) = f_s(x) + f_t(x)] \ : \ s, t \in F\big\}.$$

Below are several additional examples of first-order theories. At this stage all we want to do is to get some experience in computing several values of the spectrum function. Probably the deepest theorem in model theory can be stated as:

THEOREM 2.46 (Shelah's Main Gap). *Let T be a complete theory in a countable language. If there exists an uncountable cardinal λ such that $\mathrm{I}(\lambda, T) < 2^{\lambda}$ then $\mathrm{I}(\aleph_\alpha, T) < \beth_{\omega_1}(\aleph_0 + |\alpha|)$ for every ordinal α.*

Where $\beth_{\omega_1}(\lambda)$ stands for the cardinal number obtained after iterating the power set operation ω_1-many times. See Definition 4.18.

A related result is the following:

THEOREM 2.47. *Let T be a complete theory in a countable language. If there exists an uncountable cardinal number λ such that $\mathrm{I}(\lambda, T) < 2^{\lambda}$ then there exists a theory T^* which is one of the listed below such that for every uncountable cardinal μ we have $\mathrm{I}(\mu, T^*) = \mathrm{I}(\mu, T)$*

Both theorems are derived from a much stronger result. For the necessary machinary to state that stronger result (see Theorem **??.??**) one need to use the notion of *forking* to be introduced in a later chapter.

So the very "innocent looking" hypothesis that there exists an uncountable cardinal λ such that $\mathrm{I}(\lambda, T) < 2^{\lambda}$ implies that as far as the computation of the spectrum function is involved it suffices to look at only *finitely* many theories.

EXAMPLE 2.48. Let S be an infinite set and suppose that $P \subseteq S$. Let $T := \mathrm{Th}(\langle S, P\rangle)$. There are several possibilities:

(1) Suppose that S is infinite and also $S - P$ is infinite. We have that $\mathrm{I}(\aleph_0, T) = 1,\ \mathrm{I}(\aleph_1, T) = 3$ and in general for $n < \omega$ $\mathrm{I}(\aleph_n, T) = 2n + 1$ while for infinite α we have $\mathrm{I}(\aleph_\alpha, T) = |\alpha|$.

Why? Suppose $\langle S_\ell, P_\ell\rangle \models T$ for $\ell = 1, 2$. If $g : S_1 - P_1 \to S_2 - P_2$ and $h : P_1 \to P_2$ are bijections then $g \cup h$ is an isomorphism from $\langle S_1, P_1\rangle$ onto $\langle S_2, P_2\rangle$.

(2) Suppose that P is finite and $S - P$ is infinite. The theory T is totally categorical.

Let $n := |P|$ and let

$$\psi_{n,P} := \exists x_0 \cdots \exists x_{n-1} \Big[\bigwedge_{i<n} P(x_i) \wedge [\bigwedge_{i<j<n} [x_i \neq x_j] \to \forall y [P(y) \to \bigvee_{i<n} [y = x_i]] \Big].$$

Let $T^* := \{\varphi_k \mid k < \omega\} \cup \{\psi_{n,P}\}$. When φ_k are from Example 2.35. Check that $\mathrm{Mod}(T) = \mathrm{Mod}(T^*)$.

(3) Suppose that S is infinite and $S - P$ is finite. Also in this case T is totally categorical, it is the dual to the theory in the previous example.

EXAMPLE 2.49. Let S be a countable set and suppose that $P \subseteq S$ is infinite and also coinfinite. Let $f : (S - P) \to P$ be a bijection. Pick $a \in P$ and define $\bar{f} := f \cup P \times \{a\}$. The theory $\mathrm{Th}(\langle S, P, \bar{f}\rangle)$ is totally categorical.

EXAMPLE 2.50. Let $2\mathbb{Z}$ be the set of even integers. Fix a bijection $f : \mathbb{Z} \to 2\mathbb{Z}$ and a bijection g from $\mathbb{Z}$ onto the odd integers. Consider $M := \langle \mathbb{Z}, 2\mathbb{Z}, f, g\rangle$, verify that $\mathrm{Th}(M)$ has the same spectrum function as $\mathrm{Th}(N)$ when $N := \langle \mathbb{Z}, 2\mathbb{Z}, h\rangle$ and

$$h(x) := \begin{cases} x+1 & \text{if } x \in 2\mathbb{Z} \\ \\ x & \text{if } x \text{ is odd.} \end{cases}$$

EXAMPLE 2.51. Let S be an infinite set and suppose that $P_2 \subseteq P_1 \subseteq S$. Consider the theory $T := \mathrm{Th}(\langle S, P_1, P_2\rangle)$. Let

$$\mu_\ell := |S - P_\ell| \text{ for } \ell = 1, 2.$$

There are several possibilities:

(1) If μ_1 and μ_2 are both finite then T is totally categorical and

$$T^* := \{\varphi_n \mid n < \omega\} \cup \{\psi_{\mu_1, \neg P_1}, \psi_{\mu_2, \neg P_2}\}$$

is an alternative axiomatization for the class $\mathrm{Mod}(T)$. In this case also T is totally categorical.

(2) If μ_1 is finite but μ_2 infinite then the spectrum function of T is exactly the same as that of the theory from Example 2.48(1).

(3) If both μ_1 and μ_2 are infinite then T is $\aleph_0$-categorical and $\mathrm{I}(\aleph_1, T) = 4$ and in general for $n < \omega$, notice that $\mathrm{I}(\aleph_n, T) = (n+1)^3 - n^3$, while for infinite α we have $\mathrm{I}(\aleph_\alpha, T) = |\alpha|$. ?

?!

Let $\mu_3 := |P_1 - P_2|$ when P_2 is infinite and $\mu_3 \geq \aleph_0$ then for $\alpha \geq \omega$ we have $\mathrm{I}(\aleph_\alpha, T) = 2^{|\alpha|}$.

EXAMPLE 2.52. Let S be an infinite set and suppose that $P_2, P_1 \subseteq S$. Consider the theory $T := \mathrm{Th}(\langle S, P_1, P_2\rangle)$. There are several possibilities: $P_2 \subseteq P_1 \subseteq S$ or $P_1 \subseteq P_2 \subseteq S$ which reduces to the previous example. The remaining case is $P_1 \not\subseteq P_2 \subseteq S \wedge P_2 \not\subseteq P_1 \subseteq S$. This brakes down to several cases. Let $\mu_\ell := |S - P_\ell|$ for $\ell = 1, 2$; $\mu_3 := |P_1 - P_2|$, $\mu_4 := |P_2 - P_1|$, $\mu_5 := |S - P_1|$, $\mu_6 := |S - P_2|$. For different assignments for $\langle \mu_\ell \mid 1 \leq \ell \leq 6\rangle$ we get different theories with different spectrum functions. Verify that always for each of the theories we have that $\mathrm{I}(\aleph_1, T) < \omega$ holds.

EXAMPLE 2.53 (The theory of ω-many independent predicates). Let S be the set of finite sequences consisting of 0 and 1.

For every $n < \omega$ let $P_n := \{\eta \in S \mid \eta[n] = 1\}$, namely P_n is the set of sequences having 1 at the n-th place. Notice that the following holds

$$\bigcap_{i\in w} P_i \cap \bigcap_{j\in m-w} (S - P_j) \neq \emptyset \quad \text{for all } m < \omega \text{ and for all } w \subseteq m.$$

The theory $T := \mathrm{Th}(\langle S, P_n\rangle_{n<\omega})$ is called the *theory of ω-many independent predicates*. Notice that also this theory is $\aleph_0$-categorical. Verify that for uncountable λ we have that $\mathrm{I}(\lambda, T) = 2^\lambda$.

EXAMPLE 2.54 (The random directed graph). Let $V := \omega$ for every $n \neq k \in \omega$ let nEk iff in the binary expansion of k the n-th digit is 1.

The theory $T := \mathrm{Th}(\langle V, E\rangle)$ is called the *The random directed graph*. Notice that also this theory is $\aleph_0$-categorical. Verify that for uncountable λ we have that $\mathrm{I}(\lambda, T) = 2^\lambda$.

This graph is known also by the name *Rado's graph* see [**Rado 1964**].

EXAMPLE 2.55. Let S be an infinite set, let E be an equivalence relation defined on S and let $M := \langle S, E\rangle$, denote by T the theory of M. By considering $|S/E|$-the number of equivalence classes we get different first-order theories:

(1) Suppose that E has only two equivalence classes, one is finite and the other infinite, then $\mathrm{I}(\lambda, T) = 1$ for every infinite λ.

(2) Suppose that E has only two equivalence classes, and both are infinite, then $\mathrm{I}(\aleph_0, T) = 1$ while $\mathrm{I}(\aleph_1, T) = 2$ and $\mathrm{I}(\aleph_2, T) = 3$.

Notice that the theory T^* defined below is an alternative axiomatization of T.

$$T^* := \Big\{\exists x_1\exists x_2\big[[\neg x_1 E x_2] \wedge \forall y[yEx_1 \vee yEx_2]\big]\Big\} \cup \big\{\psi_n \mid n < \omega\big\},$$

where

$$\psi_n := \exists x_1\exists x_2\big[[\neg x_1 E x_2] \wedge \exists y_0 \cdots \exists y_{n-1} \bigwedge_{i<j<n} [y_i \neq y_j \wedge y_j E x_1] \wedge$$
$$\exists z_0 \cdots \exists z_{n-1} \bigwedge_{i<j<n} [z_i \neq z_j \wedge z_j E x_2]\big].$$

(3) Suppose that E has infinitely many equivalence classes, and all are infinite, then $\mathrm{I}(\aleph_0, T) = 1$ while in $\aleph_1$ we have the following possibilities for every $n < \omega$ there is $M_n \models T$ of cardinality $\aleph_1$ such that exactly n of the equivalence classes are countable (and the rest are all uncountable), $M_{\omega+n}$ the structure with exactly n uncountable equivalence classes, M_ω is the model with countably many of the classes are countable, $M_{2\omega+1}$ the structure with countably many of the classes are uncountable and finally $M_{2\omega+2}$ the structure with uncountable many of the classes are uncountable and uncountable many of the classes are uncountable.

Check that for every $\alpha, \beta \leq 2\omega + 2$ we have that $\alpha \neq \beta$ implies that $M_\alpha \ncong M_\beta$ and for every $M \models T$ of cardinality $\aleph_1$ there exists $\alpha \leq 2\omega + 2$ such that $M \cong M_\alpha$. Thus $\mathrm{I}(\aleph_1, T) = \aleph_0$. Show that for every ordinal α we have that $\mathrm{I}(\aleph_\alpha, T) = \aleph_0 + |\alpha|$.

EXAMPLE 2.56. Let S be an infinite set, let E_1 and E_2 be equivalence relations defined on S suppose that

$$aE_2b \implies aE_1b, \quad \text{for every } a, b \in S.$$

Now let $M := \langle S, E_1, E_2 \rangle$ and $T = \mathrm{Th}(M)$.

(1) Suppose that both E_1 and E_2 have two infinite equivalence classes.
We have $\mathrm{I}(\aleph_0, T) = 1$ while in $\aleph_1$ there are the following possibilities:
   (a) Both classes of E_2 are uncountable (thus also the classes of E_1)
   (b) one of the classes of E_2 is countable (the other must be uncountable) now we have two possibilities: the class of E_1 containing the countable class of E_2 is countable or not.

Thus we have shown that $\mathrm{I}(\aleph_1, T) = 3$.

EXAMPLE 2.57. Let S be an infinite set, let $n < \omega$ and suppose $\{E_k : k < n\}$ are equivalence relations defined on S such that for every $k_1 < k_2 < n$ and every $a, b \in S$ we have that $aE_{k_2}b$ implies that $aE_{k_1}b$, let $M := \langle S, E_1, \ldots, E_{n-1} \rangle$ and $T = \mathrm{Th}(M)$.

(1)
(2)
(3)
(4)

EXAMPLE 2.58 (Cantor's set). Let $S = {}^{\omega}2$, namely S is the set of functions from ω into $\{0, 1\}$.

For every $n < \omega$ and every $\eta, \nu \in S$ define $\eta E_n \nu$ as $\eta \restriction n = \nu \restriction n$.

Let $M := \langle S, E_n \rangle_{n<\omega}$ and $T = \mathrm{Th}(M)$. We have for every $\mu \leq 2^{\aleph_0}$ that $\mathrm{I}(\mu, T) = 2^\mu$ and in general for every ordinal α we have that
$\mathrm{I}(\aleph_\alpha, T) = \min\{2^{\aleph_\alpha}, 2^{2^{\aleph_0}} + |\alpha|\}$

EXAMPLE 2.59 (Polish space). Let $S = {}^{\omega}\omega$, namely S is the set of functions from ω into ω. For every $n < \omega$ and every $\eta, \nu \in S$ define $\eta E_n \nu$ as $\eta \restriction n = \nu \restriction n$. Let $M := \langle S, E_n \rangle_{n<\omega}$ and $T = \mathrm{Th}(M)$.
For every μ uncountable we have $\mathrm{I}(\mu, T) = 2^\mu$

EXAMPLE 2.60. Let α be an ordinal, suppose that S is a set such that $|S| > |\alpha| + \aleph_0$. Let $\{E_\beta : \beta \leq \alpha\}$ be equivalence relations such that if $\gamma_1 < \gamma_2 \leq \alpha$ then for every $a, b \in S$ we have that $aE_{\gamma_1}b \implies aE_{\gamma_2}b$, suppose that E_{γ_1} divides every equivalence class of E_{γ_2} into infinitely many pieces and E_α has infinitely many equivalence classes.

EXAMPLE 2.61. While in Theorem 2.13 we have seen that T_{ord} is categorical in $\aleph_0$, for uncountable cardinalities the situation is drastically different. Namely:

PROPOSITION 2.62 (Many uncountable models of T_{ord}). $\mathrm{I}(\lambda, T_{ord}) = 2^\lambda$ *for every uncountable* λ.

PROOF. Let λ be an uncountable cardinal. Recall that by λ we denote the well ordered set of ordinals of cardinality less than λ. By λ^* denote the *reverse ordering* of λ, that is $\lambda^* := \langle \lambda, < \rangle$ where $\alpha < \beta \iff \beta \in \alpha$ for all $\alpha, \beta \in \lambda$.

Let η be be the order-type of the rationals. Recall that $\lambda^* + \lambda$ denotes the order-type of a set of order type λ^* followed by a set of order-type λ, e.g. $\omega^* + \omega$ is the order type of the integers, with usual ordering. The order type $\eta \cdot (\lambda^* + \lambda)$ is the order type of a linearly-ordered set obtained from $\lambda^* + \lambda$ by replacing every $x \in \lambda^* + \lambda$ with a copy of the rationals. Since $\lambda + 1$ has a last element and λ does not the order types $\eta \cdot (\lambda^* + \lambda)$, $\eta \cdot (\lambda^* + \lambda + 1)$ are different.

Now for every sequence $\nu \in {}^\lambda 2$ let M_ν be a linearly-ordered set of order-type $\sum_{i<\lambda} \eta \cdot (\lambda^* + \lambda + \nu[i])$. Using the fact that any order preserving function from $\langle \lambda, \in \rangle$ into itself must be the identity function show that for $\nu_1 \neq \nu_2 \in {}^\lambda 2$ we have that $M_{\nu_1} \ncong M_{\nu_2}$. □

EXAMPLE 2.63 (loop-free successor function). Let $L := \{0, s\}$ where 0 is an individual constant and s is a unary function. Let T_s be the following set of axioms:

$$\{\neg\exists x[s(x) = 0] \wedge \forall y[y \neq 0 \to \exists x[s(x) = y]] \\ \wedge \forall x \forall y[s(x) = s(y) \to x = y]\} \cup \{\varphi_n \mid n < \omega\},$$

where

$$\varphi_n := \neg\exists x_0 \cdots \exists x_{n-1}[\bigwedge_{k<n-1} [s(x_k) = x_{k+1}] \wedge s(x_{n-1}) = x_0].$$

Let λ be a cardinal (can be finite, even zero) and let M_λ be L-structure whose universe is $\omega \cup \{a_{i,n} \mid i < \lambda, n < \omega\}$. The interpretation of the function symbol is defined as: $0^{M_\lambda} := 0$, $s^{M_\lambda}(n) := n + 1$ for all $n \in \omega$ and $s^{M_\lambda}(a_{i,n}) := a_{i,n+1}$ for all $i < \lambda$.

Clearly $M_\lambda \models T_s$. Verify that for every $M \models T_s$ there exists a unique cardinal λ such that $M \cong M_\lambda$. Thus T_s is complete, categorical in all uncountable cardinals and $\mathrm{I}(\aleph_0, T_s) = \aleph_0$.

A generalization of the concept of isomorphism is the notion of *monomorphism*. Just take Definition 2.9 and relax the requirement that the function is surjective (i.e require the function to be only injective and preserving the structure). Contrary to the situation in algebra where often monomorphism is the natural functor the notion of model monomorphism is not as important. The natural functor in

model theory is notion of elementary embedding to be introduced in the following section. However in some applications of model theory to algebra monomorphism of models plays a role. They will be used in section 13 to characterized model complete theories.

2.1. Coding set-theoretic objects. In future chapters we will use the fact that the existence of certain interesting set-theoretic objects like successor of cardinals, power-set as well as Aro and Kurepa trees whose existence may be independent of the axioms of ZF can be coded into first-order theories.

While these examples are good to develop a better intuition for what is expressible in first-order logic, they have little value for the developments of the basics of model theory I suggest the reader to skip these examples at first reading and to return to this subsection when these examples will be called on.

EXAMPLE 2.64 (Coding the successor of a cardinal). Let $L_1 := \{P, <, F\}$ where P is a unary predicate, $<$ a binary predicate, F a binary function symbol. Let T_1 be the theory

$$\begin{aligned} &< \text{ is a linear order } \wedge \\ &\forall x \forall y [P(F(x,y)0] \wedge \\ &\forall x [\neg P(x) \to \forall y_1 \forall y_2 [y_1 \neq y_2 < x \to F(x,y_1) \neq F(x,y_2)]]. \end{aligned}$$

Let λ be an infinite cardinal. Notice that if we fix for every $\alpha \in \lambda^+ - \lambda$ a bijection $f_\alpha : \alpha \to \lambda$ then letting $F(\alpha, \beta) := f_\alpha(\beta)$ for all $\beta < \alpha$ we have that $M := \langle \lambda^+, \lambda, \in, F \rangle \models T_1$.

Also notice that $N \models T_1 \implies \|N\| \leq |P^N|^+$ for infinite N.

EXAMPLE 2.65 (Coding the power set). Let P be a unary predicate and E a binary predicate. Let

$$L := \{P, E\}.$$

Let T be a first-order theory in L consisting of the following axioms:

(1) $\forall x \forall y [xEy \to P(x) \wedge \neg P(y)]$,
(2) $\forall x [P(x) \to \exists y [\neg P(y) \wedge xEy \wedge \forall z [P(z) \wedge zEy \to z = x]]$,
(3) $\forall y_1 \forall y_2 [\neg P(y_1) \wedge \neg P(y_2) \to [y_1 = y_2 \leftrightarrow \forall x [P(x) \to [xEy_1 \leftrightarrow xEy_2]]]$.

The idea is that if $M \models T$ then its universe consists of two disjoint perts P^M - the objects and its complement is a collection of subsets of P^M where E is the membership relation. Notice that the third axiom is an extensionality axiom for E.

Notice that for any set S if $\mathcal{P}(S)$ is its powerset then $\langle \mathcal{P}(S), S, \in \rangle \models T$.

Observe that

FACT 2.66. *If $M \models T$ then $\|M\| \leq 2^{|P^M|}$.*

In order to establish independence results, a significant simplification is obtained by translated existence of a two cardinal theorem into a specific theory of a combinatorial object that can more easlily be manipulated than models of arbitrary theories.

In the sixties Rowbottom and Silver discovered a connetcion between the property of existence of a $\langle \lambda^+, \lambda \rangle$-model and existence of special λ^+-Aronszajn trees.

DEFINITION 2.67 (special λ^+-Aronszajn trees).

(1) A *tree* is a partially ordered set $\langle T \leq \rangle$ such that

$$\{y \in T \mid y \leq x\} \quad \text{is well ordered for all } x \in T.$$

We say that T is a μ-tree iff $|T| = \mu$.

(2) When $\langle T, \leq \rangle$ is a tree for $x \in T$ let

$$\text{level}(x) := \text{ the order type of } \{y \in T \mid y \leq x\}.$$

By T_α denote $\{x \in T \mid \text{level}(x) = \alpha\}$. The set $T \restriction \alpha$ stands for $\bigcup_{\beta<\alpha} T_\beta$.

$B \subseteq T$ is a *branch* iff $\langle B, \leq \rangle$ is linearly ordered and

$$\langle T, \leq \rangle \models \forall x \forall y [y \in b \wedge x \leq y \to x \in B].$$

A branch is a *maximal branch* iff it is not a poroper subset of another branch.

(3) For a regular μ we say that T is a *μ-Aronzjan tree* iff T is a μ-tree, every for every α we have that $|T_\alpha| < \mu$ and every branch of T has cardinality $< \mu$.

(4) A λ^+-Aronzjan tree T is called *special* iff there exists

$$f : T \to \lambda \quad \text{such that } T \models \forall x \forall y [x <_T y \implies f(x) \neq f(y)].$$

THEOREM 2.68 (Aronszjan). *There exists an $\aleph_1$-special Aronszjan tree.*

PROOF. First we constrruct $\aleph_1$-special tree: We will construct a tree such that

$$T \subseteq \{\overline{x} \in {}^{\alpha}\mathbb{Q} \mid \alpha < \omega_1, \overline{x}[i] < \overline{x}[j] \text{ for all } i < j < \alpha\}.$$

For $\alpha = 0$ let $T_0 := \{\langle\rangle\}$, for $\alpha = 1$ take $T_\alpha := \{\langle q \rangle \mid q \in \mathbb{Q}\}$.

Inductive assumption: For every $\alpha < \omega_1$ every $\beta < \gamma < \alpha$ and every $\overline{x} \in T_\beta$ and every $q \in \mathbb{Q}$ such that $q > \lim \overline{x}$, there is $\overline{y} \in T_\gamma$ such that $\overline{x}$ is an initial segment of $\overline{y}$ and $\lim \overline{y} = q$.

For successor:

$$T_{\alpha+1} := \{\overline{x}^\frown \langle q \rangle \mid \overline{x} \in T_\alpha \wedge q \in \mathbb{Q} \wedge q > \sup \overline{x}\}.$$

Clearly we have that $|T_{\alpha+1}| = |T_\alpha| \cdot \aleph_0 = \aleph_0$.

For limit α, for every $\overline{x} \in T \restriction \alpha$ and every rational $q > \sup \overline{x}$ we construct an increasing $\overline{y}_{\overline{x},q} \in {}^{\alpha}\mathbb{Q}$ extending $\overline{x}$ such that

$$\sup_i \overline{y}_{\overline{x},q}[i] = q.$$

Pick $\{\beta_n \mid n < \omega\}$ increasing sequence of ordinals converging to α such that $\overline{x} \in T_{\beta_0}$, also let $\{q_n \mid n < \omega\}$ an increasing sequence of rational numbers converging to q.

Take $\overline{x}_0 := \overline{x}$. By the inductive assumption for every positive integer $n < \omega$ there is $\overline{x}_{n+1} \in T^{\beta_n}$ extending $\overline{x}_n$ such that $\sup_i \overline{x}_n[i] = q_n$.

Let $\overline{y}_{\overline{x},q} := \bigcup_{n<\omega} \overline{x}_n$. Now, let

$$T_\alpha := \{\overline{y}_{\overline{x},q} \mid \overline{x} \in T \restriction \alpha, q \in \mathbb{Q}, \sup_i \overline{x}[i] < q\}.$$

Again it is clear that T_α is countable.

To verify that $T := \bigcup_{\alpha<\omega_1} T_\alpha$ is an Aronszjan tree it suffices to show that it does not have an ω_1 branch. Notice that existence of such a branch will produce an increasing sequence of $\aleph_1$ many rationals which is not possible.

Notice that

$$g : T \to \mathbb{Q} \quad \text{defined by } g(\overline{x}) = \sup_i \overline{x}[i] \text{ is order preserving.}$$

Fix a bijection $h : \mathbb{Q} \to \omega$. Now the function $f : T \to \omega$ defined as $f := h \circ g$ witness that T is special.

□

The previous proof can be generalized to show that if $\lambda = \lambda^{<\lambda}$ then there exists a λ^+-special Aronszjan tree. This can be derived also from a two carinal theorem and is left as an exercise.

COROLLARY 2.69. *Under* GCH*, every successor of regular carry a special Aronszjan tree.*

EXAMPLE 2.70 (Rowbottom and Silver). There is a finite language L^A and $\varphi^A \in \mathrm{Sent}(L^A)$ such that

$$\varphi^A \text{ has a } \langle \lambda^+, \lambda \rangle\text{-model} \iff \text{there is a special } \lambda^+\text{-Aronszajn tree.}$$

DEFINITION 2.71. For a regular μ we say that T is a *μ-Kurepa tree* iff T is a μ-tree, every level has cardinality $< \mu$ and T has at least μ^+-many maximal branches.

DEFINITION 2.72. *Kurepa hypothesis* KH_λ stands for the statement "there exists a λ^+-Kurepa tree".

EXAMPLE 2.73 (Vaught 1965). There is a finite language L^K and $\varphi^K \in \mathrm{Sent}(L^K)$ such that

$$\varphi^K \text{ has a } \langle \lambda^{++}, \lambda \rangle\text{-model} \iff \text{there is a } \lambda^+\text{-Kurepa tree.}$$

Exercises 1.2.

(1) Suppose that $f : M \cong N$ prove that f^{-1} is an isomorphism from N onto M.
(2) Let $L := \{P\}$, when P is a unary relation symbol and let n be a positive integer. How many L-structure are there up to isomorphism of cardinality n? What is the number up to isomorphism of countable L-structures?
(3) Let $L := \{P, Q\}$, when P and Q are unary relation symbols and let n be a positive integer. How many L-structures are there up to isomorphism of cardinality n?
(4) Let M and N be L-structures. Suppose $f : |M| \to |N|$ is a monomorphism. Let $S \subseteq |N|$ be the image of the set $|M|$ under f. Let $M^* := \langle S, F^*, R^*, c^N \rangle_{F,R,c \in L}$. Verify that $M^* \subseteq N$ and $f : M \cong M^*$.
(5) Prove the statement made in Remark 2.11.
(6) Show that Theorem 2.6 can not be generalized to first order sentences. Take $M := \langle \mathbb{Q}, < \rangle$ be the rationals with the usual ordering and let $N := \langle \mathbb{N}, < \rangle$. Find a sentence φ in $\mathrm{L}(M)$ such that $M \models \varphi$ but $N \models \neg\varphi$.
(7) Prove Proposition 2.15
(8) Prove the converse to Proposition YY
(9) Let n be a positive integer. Write a sentence ψ_n (over the empty language) such that $M \models \psi_n \iff \|M\| \geq n$. Observe that

$$M \models T_\omega \iff \|M\| \geq \aleph_0, \quad \text{where } T_\omega := \{\psi_n \mid n < \omega\}.$$

Can T_ω be replaced by a single sentence? Namely is there a sentence ψ_∞ over the empty language such that $M \models \psi_\infty \iff \|M\| \geq \aleph_0$.
(10) Suppose that T is finite. Show that there exists $\psi_T \in \mathrm{Sent}(\mathrm{L}(T))$ such that $\mathrm{Mod}(T) = \mathrm{Mod}(\psi_T)$.
(11) Let L be a similarity type, suppose that M and N are L-structures and let $\mathbf{a}$ be a finite sequence of elements of $|M|$ and $\mathbf{b}$ a finite sequence of elements of $|N|$. Show that if

$$M \models \varphi[\mathbf{a}] \iff N \models \varphi[\mathbf{b}] \quad \text{for all atomic } \varphi$$

then

$$M \models \varphi[\mathbf{a}] \iff N \models \varphi[\mathbf{b}] \quad \text{for all quantifier free } \varphi.$$

(12) If $M \equiv N$ then for every positive integer n there exists an isomorphism from the Boolean Algebra $\mathrm{BA}^n(M)$ onto the Boolean Algebra $\mathrm{BA}^n(N)$.
(13) Let $\mathbb{Q}$ be the set of rational numbers, let $a, b \in \mathbb{Q}$ be such that $a < b$ we will use the usual notation of intervals, i.e. denote by
$(a, b) := \{q \in \mathbb{Q} \ : \ a < q < b\}$, and $[a, b] := \{q \in \mathbb{Q} \ : \ a \leq q \leq b\}$. Suppose that $a, b, c, d \in \mathbb{Q}$ are such that $a < b$ and $c < d$. Prove the following claims
   (a) $\langle (a, b), < \rangle \cong \langle (c, d), < \rangle$
   (b) $\langle [a, b], < \rangle \cong \langle [c, d], < \rangle$
   (c) $\langle (a, b], < \rangle \cong \langle (c, d], < \rangle$
   (d) $\langle [a, b), < \rangle \ncong \langle (c, d], < \rangle$, and

(e) for every pair of open, closed or half open intervals with the usual order relation state whether they are elementarily equivalent or not.

(14) Consider $M := \langle \mathbb{Q}, <, c_n \rangle_{n \in \omega - \{0\}}$ where $<$ is the usual order on the rational numbers and c_n^M is $\frac{1}{n}$. Let $T := \mathrm{Th}(M)$.
Show that $\mathrm{I}(\aleph_0, T) = 1$.

(15) Consider $T := Th(\langle \mathbb{Q}, < \rangle) \cup \{c_n < c_{n+1} \mid n < \omega\}$ where $<$ is the usual order on the rational numbers. Show that $\mathrm{I}(\aleph_0, T) = 3$.

(16) Let $\mathbf{a}, \mathbf{b} \in \mathbb{Q}$ be such that $\mathbf{a} = \langle a_0, \ldots, a_{n-1} \rangle$ and $\mathbf{b} = \langle b_0, \ldots, b_{n-1} \rangle$. Let $\mathbf{a} \equiv \mathbf{b}$ stand for

$$a_i < a_j \iff b_i < b_j \text{ and } a_i = a_j \iff b_i = b_j \text{ for all } i, j < n.$$

Prove that $\mathbf{a} \equiv \mathbf{b}$ if and only if there exists $f \in \mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ such that $f(\mathbf{a}) = \mathbf{b}$, namely $f(a_i) = b_i$ for all $i < n$.

(17) Show that for every formula $\varphi(\mathbf{x})$ in L_{ord} there exists a quantifier free formula $\psi_\varphi(\mathbf{x})$ such that

$$\langle \mathbb{Q}, < \rangle \models [\varphi(\mathbf{a}) \leftrightarrow \psi_\varphi(\mathbf{a})] \text{ for all } \mathbf{a} \in \mathbb{Q}.$$

Hint: Use the previous exercise.

(18) Verify that this is an alternative proof of Theorem 2.8: Let A_0 be as in the beginning of the proof. Show that

$$S := \{\tau^M(\mathbf{a}) \mid \tau(\mathbf{x}) \in \mathrm{Term}(L), \mathbf{a} \in A_0\}$$

determines a universe of a sub model as required.

(19) Let $G := \langle \mathbb{Z}, + \rangle$ and let $H := \langle \mathbb{Q}, + \rangle$ with the usual addition. Show that $G \not\equiv H$.

(20) Let L be $\{<\}$. Let $N := \langle \omega, < \rangle$ and $M := \langle \omega + \omega, < \rangle$, is N an elementary equivalent to M? Are they isomorphic?

(21) Let $M \models T_{ord}$ be of cardinality $2^{\aleph_0}$. Is necessarily $M \cong \langle \mathbb{R}, < \rangle$? Justify your answer without using Proposition 2.62.

(22) Let G be an uncountable group, show that G has a countable subgroup.

(23) Let $M := \langle \mathbb{R}, +, \cdot, <, 0, 1 \rangle$ be the real ordered field. Show that there exists a countable $N \subseteq M$ containing $\sqrt{2}$.

(24) Prove that T_{ind} is $\aleph_0$-categorical.

(25) Let T be a maximally consistent theory. Show that for every $L_0 \subseteq \mathrm{L}(T)$ there exists $T_0 \subseteq T$ complete in L_0.

(26) Let T be a maximally consistent theory in an uncountable language of cardinality μ. Show that there exists $\{L_i \mid i < \mu\}$ such that $\mathrm{L}(T) = \bigcup_{i<\mu} L_i$, $|L_i| < \mu$ for every $i < \mu$, $L_i \subsetneqq L_j$ for all $i < j < \mu$ and there are complete L_i-theories $T_i \subseteq T$ such that $T = \bigcup_{i<\mu} T_i$.

(27) Given a countable linearly ordered set $\langle P <_P \rangle$, show that there exists an order preserving function from $\langle P <_P \rangle$ into the rationals.

(28) Let M be a structure and $\mathbf{a}$ a sequence of elements from $|M|$. Suppose that $\mathbf{c}$ is a sequence of constant symbols not in $\mathrm{L}(M)$. Show that

$$\{\varphi(\mathbf{c}) \mid \varphi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(M)),\ M \models \varphi[\mathbf{a}]\}$$

is a complete theory in $\mathrm{L}(M) \cup \{\mathbf{c}\}$.

(29) Let M and N be finite L-structures, show that $M \equiv N$ implies that $\|M\| = \|N\|$.

(30) For finite L-structures $M \equiv N \implies M \cong N$.

(31) Let M be the group $\langle \mathbb{R}, + \rangle$. Verify that for every $a \in \mathbb{R} - \{0\}$ the function $f(x) := ax$ is an automorphism of M.

(32) Prove that while the usual order of the real numbers is definable in $\langle \mathbb{R}, +, \cdot \rangle$, it is not definable in the structure $\langle \mathbb{R}, + \rangle$.

(33) Prove that in the field of complex numbers the set $\{i, -i\}$ is definable, while none of its elements is definable.

(34) Prove that multiplication of the real numbers is not definable in the structure M from the previous exercise. A binary operation $\cdot$ is *definable* iff there exists a first-order formula $\varphi(x, y, z)$ such that

$$M \models \varphi[a, b, c] \iff a \cdot b = c \quad \text{for all } a, b, c \in |M|.$$

(35) What is the cardinality of $\mathrm{Aut}(\langle \mathbb{R}, + \rangle)$?

(36) Let $A \subseteq \mathbb{R}$ be such that $|A| < 2^{\aleph_0}$. What is the cardinality of $\mathrm{Aut}_A(\langle \mathbb{R}, + \rangle)$? Hint: View $\langle \mathbb{R}, + \rangle$ as a vector space over $\mathbb{Q}(A)$. Conclude that multiplication is not definable in $\langle \mathbb{R}, + \rangle$ even by formulas with parameters.

(37) The set of real numbers is not a definable subset of the field of complex numbers.

(38) Suppose that M is an L-structure such that $\|M\| \geq \aleph_0 + |L|$ and let $S \subseteq {}^n|M|$ for some $n < \omega$.
If $|\{f[S] \ : \ f \in \mathrm{Aut}(M)\}| > \|M\|$ then S is not definable in M.

(39) Let M, L, n and S as in the previous problem. Prove that if S is not definable with parameters then S must be infinite.

(40) Let M be a countable model of T_{ind}. Show that for every countable graph $\langle V, E \rangle$ there is a submodel N of M isomorphic to $\langle V, E \rangle$.

(41) A Boolean algebra A is *atomless* iff it satisfies

$$\varphi := \forall a[a \neq 0 \to \exists b[b \neq 0 \wedge b \neq a \wedge b \cap a = b]].$$

Prove that there exists a countable atomless Boolean algebra.

(42) Prove:
Theorem If A_1 and A_2 are countable atomless boolean algebras then $B_1 \cong B_2$.

(43) Is the class of atom less boolean algebras $\aleph_1$-categorical?

(44) The following group of exercises deals with an *important* generalization of Theorem 2.13:

(a) Let M and N be both L-structures. Suppose that $A \subseteq |M|$, and $B \subseteq |N|$. A function $f : A \to B$ is called a *partial isomorphism* iff it is a bijection and for every relation symbol $R(\mathbf{x})$ and every $\mathbf{a} \in A$ we have that
$\mathbf{a} \in R^M \iff f(\mathbf{a}) \in R^N$, for every function symbol $F(\mathbf{x})$ we have that

$f(F^M(\mathbf{a})) = F^N(f(\mathbf{a}))$ and for every constant symbol c if $c^M \in \mathrm{Dom}(f)$ then $f(c^M) = c^N$.
We say that a family $\mathcal{F}$ of partial isomorphisms from M into N has the *back and forth property* iff
(forward) for every $a \in |M|$ and every $g \in \mathcal{F}$ there exists $h \in \mathcal{F}$ such that $g \subseteq h$ and $a \in \mathrm{Dom}(h)$ and
(back) for every $b \in |N|$ for every $g \in \mathcal{F}$ there exists $h \in \mathcal{F}$ such that $g \subseteq h$ and $b \in \mathrm{Rang}(h)$.
We denote this by $\mathcal{F} : M \cong^p N$. We write $M \cong^p N$ for "there exists a non=empty $\mathcal{F}$ such that $\mathcal{F} : M \cong^p N$".
Prove that: If M and N are countable (in the same similarity type) and $M \cong^p N$ then $M \cong N$. Moreover, if $\mathcal{F} : M \cong^p N$ then for every $g \in \mathcal{F}$ there exists $\bar{g} : M \cong N$ extending g.

(b) Show that for every finite sequence $\mathbf{a} \in \mathbb{Q}$ there exists $\mathcal{F}$ satisfying $\langle \mathbb{Q}, <, \mathbf{a}\rangle \cong_p \langle \mathbb{Q}, <, \mathbf{a}\rangle$ such that there exists $g \in \mathcal{F}$ with $g \neq \mathrm{id}_{\mathrm{dom}\, g}$.

(c) Show that the relation $\cong^p$ is transitive.

(d) Let L be a similarity type. Suppose that M and N are infinite L-structures. Does $M \cong^p N$ imply that $M \cong N$?

(45) Let $+, \cdot$ be the usual operation on $\mathbb{R}$. Verify the following claims:
(a) $\langle \mathbb{Q}, +, \cdot\rangle \subseteq \langle \mathbb{R}, +, \cdot\rangle$
(b) $\langle \mathbb{Z}, +, \cdot\rangle \subseteq \langle \mathbb{Q}, +, \cdot\rangle$,
(c) $\langle \omega, +, \cdot\rangle \subseteq \langle \mathbb{Z}, +, \cdot\rangle$,
(d) Let $^{-1}$ be the unary operation of taking multiplicative inverse (defined for every $a \in \mathbb{R}-\{0\}$). Show that $\langle \mathbb{Q}, +, \cdot, ^{-1}\rangle \subseteq \langle \mathbb{R}, +, \cdot, ^{-1}\rangle$, but $\langle \mathbb{Z}, +, \cdot, ^{-1}\rangle \not\subseteq \langle \mathbb{Q}, +, \cdot, ^{-1}\rangle$ why?

(46) Let N_1, N_2, N_3 be L structures, show that if $N_1 \subseteq N_2$ and $N_2 \subseteq N_3$ then $N_1 \subseteq N_3$.

(47) Let I be a linearly ordered set and suppose that $\{N_i : i \in I\}$ are all L-structures such that $i <_I j \Longrightarrow N_i \subseteq N_j$. Define an L-structure N such that $|N| = \bigcup_{i\in I} |N_i|$ prove that for every $i \in I$ we have that $N_i \subseteq N$.

(48) Define a similarity type L and find two L-structures M and N such that $N \equiv M$ but $N \ncong M$. Can you find two structures such that in addition to the above they have also the same cardinality?

(49) Let L be a similarity type, and let M be an L-structure. Often when L has no relation symbols M is called an *algebra* . An algebra M is called *Jónsson Algebra* iff M does not have proper subalgebras of the same cardinality. I.e. if $N \subseteq M$ and $\|N\| = \|M\|$ then $N = M$.
(a) Define a similarity type L which contains finitely many symbols, and define a countable Jónsson Algebra for L.
(b) Find an infinite group that is a Jónsson Algebra.

(50) Let T be a theory, let M be an $\mathrm{L}(T)$-structure and let φ be a sentence. Suppose $T \vdash \varphi$ and $M \models T$. Show that $M \models \varphi$ (hint- Argue by induction on the length of the proof of φ).

(51) Let T be a theory, let M be an $\mathrm{L}(T)$-structure. Suppose that T is a deductively complete theory. Show that if $M \models T$ then $\mathrm{Th}(M) = T$.
(52) Let $\mathbb{C}$ be the field of complex numbers, and let F be the algebraic closure of $\mathbb{C}(x)$ -the field of rational functions in the variable x. Prove that $\mathbb{C} \equiv F$ (in the language $\{+, \cdot, 0, 1\}$).
(53) Show that the fields $\langle \mathbb{R}, +, \cdot, 0, 1\rangle$ and $\langle \mathbb{C}, +, \cdot, 0, 1\rangle$ can not be isomorphic.
(54) Let $M := \langle \mathbb{Z}, +, \cdot\rangle$. Show that there exists a first-order formula $\varphi(x, y)$ in $\mathrm{L}(M)$ such that it defines a linear order on $\mathbb{Z}$. Moreover

$$M \models \varphi[a, b] \iff a <_{\mathbb{Z}} b \quad \text{for all } a, b \in \mathbb{Z}.$$

Hint: You may use the following fact of elementary number theory: **Lagrange's Theorem**. The set of non-negative integers is equal to $\{n \in \mathbb{Z} \ : \ M \models \exists x_1 \exists x_2 \exists x_3 \exists x_4 [n = x_1^2 + x_2^2 + x_3^2 + x_4^2]\}$.
(55) The following exercise is much harder than the previous one. Show that there is no first-order formula in $\mathrm{L}(N)$ that defines a linear order on an infinite set in the structure $N := \langle \mathbb{Z}, +\rangle$.
(56) The pupose of the following exercises is to show that $\mathbb{Z}$ is definable without parameters in $\langle \mathbb{R}, +, \cdot, \sin, 0, 1\rangle$.
  (a) The usual linear ordering on $\mathbb{R}$ is definable by a fomula without parameters in $\langle \mathbb{R}, +, \cdot, 0, 1\rangle$.
  (b) The number π is definable in $\langle \mathbb{R}, +, \cdot, \sin, 0, 1\rangle$ by a formula without parameters.
  (c) Show that in $\langle \mathbb{R}, +, \cdot, \sin, 0, 1\rangle$ the integers are definable by a formula without parameters.
(57) Let T be a set of sentences. Show that if T is inconsistent then there exists a finite set $T' \subseteq T$ which is inconsistent.
(58) Let T be a consistent theory. Show that there exists a complete theory T' in the language of $\mathrm{L}(T)$ containing T.
  (a) Use the completeness theorem to prove the last statement.
  (b) Find another proof, you may use the following hint: Apply Zorn's lemma to the partially ordered set $P := \{S \subseteq \mathrm{Sent}(\mathrm{L}(T)) \ : \ S \supseteq T$, and S is consistent$\}$ and use the previous exercise.
(59) Let p be a prime number or zero. Show that T_{ACF_p} is a complete theory.
(60) The following is an exercise in elementary algebra rather then in logic: Let p be a prime number, prove that

$$T_{ACF} \cup \{[p \cdot 1 = 0]\} \vdash \bigwedge_{1 \leq k < p} [k \cdot 1 \neq 0].$$

(61) Construct explicitly a model for T_{ind}. Hint: define by induction an increasing chain of finite graphs so that the union will be a countable model for T_{ind}.

(62) In 1964 Richard Rado suggested the following structure $\langle \omega, E\rangle$, for $n \neq k \in \omega$, suppose both numbers are in written in binary let

$$nEk := \begin{cases} \text{True} & \text{if the } n\text{th digit in } k \text{ is one or the } k\text{th digit in } n \text{ is one} \\ \\ \text{False} & \text{otherwise.} \end{cases}$$

Show that $\langle \omega, E\rangle \models T_{ind}$. This structure is known by the name *The Rado graph* and also known as *Erdő & Rényi graph*

(63) Suppose λ is a cardinal number satisfying $2^{<\lambda} = \lambda$. Let S be the set of sequences of 0's and 1's of length $< \lambda$, i.e. $S := {}^{\lambda>}2$, fix a bijection $f : \lambda \to S$.

For every $\alpha \neq \beta < \lambda$ let

$$\alpha E \beta := \begin{cases} \text{True} & \text{if } f(\alpha)[\beta] = 1 \vee f(\beta)[\alpha] = 1 \\ \\ \text{False} & \text{otherwise.} \end{cases}$$

Show that $\langle \lambda, E\rangle$ is a model of T_{ind}.

(64) For every $n < \omega$ define a finite language including a unary predicate P and a sentence ψ_n in that language such that there exists $N \models \psi_n$ with $\|N\| = \aleph_n$ and $|P^N| = \aleph_0$, show that if $M \models \psi_n$ is infinite then $\|M\| \leq \aleph_n(|P^M|)$.

(65) For every $n < \omega$ define a finite language including a unary predicate P and a sentence φ_n in that language such that there exists $N \models \psi_n$ satisfying $\|N\| = \beth_n$ and $|P^N| = \aleph_0$, show that if $M \models \varphi_n$ is infinite then $\|M\| \leq \beth_n(|P^M|)$.

(66) Let M be an L-structure, let $\mathrm{Aut}(M) := \{f \mid f : M \cong M\}$.

(a) Show that $\mathrm{Aut}(M)$ forms a group with respect to composition of mappings. $\mathrm{Aut}(M)$ is called *group of automorphisms of* M.

(b) Let $A \subseteq |M|$, and let

$$\mathrm{Aut}_A(M) := \{f \in \mathrm{Aut}(M) \ : \ (\forall a \in A)[f(a) = a]\},$$

show that $\mathrm{Aut}_A(M)$ is a subgroup of $\mathrm{Aut}(M)$.

(c) Let $A \subseteq |M|$ and $f \in \mathrm{Aut}(M)$ we say that
f fixes the set A iff $f(a) \in A \iff a \in A$ for all $a \in |M|$.
Let $\mathrm{Aut}_{\{A\}}(M) := \{f \in \mathrm{Aut}(M) \ : \ f$ fixes the set $A\}$, show that $\mathrm{Aut}_{\{A\}}(M)$ is a subgroup of $\mathrm{Aut}(M)$.

(d) Check that $f \mapsto f(\mathbf{a})$ is a *group action* of $\mathrm{Aut}(M)$ on the set of finite sequences of elements of $|M|$. For $\mathbf{a}, \mathbf{b} \in {}^{<\omega}|M|$ such that $\ell(\mathbf{a}) = \ell(\mathbf{b})$ the relation $\mathbf{a} \sim_A \mathbf{b}$ defined by $\exists f \in \mathrm{Aut}_A(M)[f(\mathbf{a}) = \mathbf{b}]$ is an equivalence relation and the equivalence classes are called *orbits*.

(67) Prove Theorem 2.41.

(68) Show that $\mathrm{Aut}_A(M)$ is a normal subgroup of $\mathrm{Aut}_{\{A\}}(M)$.

(69) In this exercise we introduce an interesting subgroup of the group $\mathrm{Aut}_A(M)$. Since the definition requires the notion of elementary submodel to be introduced in the next section we suggest to readers not familiar with that

notion to skip this exercise at present. Let A be a subset of a model M. An automorphism $g \in \mathrm{Aut}_A(M)$ is a *strong automorphism* iff there exists a model $N \preceq M$ such that $|N| \supseteq A$ and $g \in \mathrm{Aut}_{|N|}(M)$. The subgroup of $\mathrm{Aut}_A(M)$ generated by

$$\{g \in \mathrm{Aut}(M) \mid g \text{ is a strong automorphism}\}$$

is called the *group of strong automorphisms over* A and is denoted by $\mathrm{SAut}_A(M)$. Prove that $\mathrm{SAut}_A(M)$ is a normal subgroup of $\mathrm{Aut}_A(M)$.

(70) What is the group $\mathrm{Aut}(\langle \mathbb{Z}, < \rangle)$?

(71) For each of the following groups compute its cardinality:
$\mathrm{Aut}(\langle \omega, < \rangle)$, $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$, $\mathrm{Aut}(\langle \mathbb{Q}, +, \cdot, 0, 1 \rangle)$, $\mathrm{Aut}(\langle \mathbb{R}, +, \cdot, 0, 1 \rangle)$,

(72) Show that $|\mathrm{Aut}(\langle \mathbb{C}, +, \cdot, 0, 1 \rangle)| = 2^{2^{\aleph_0}}$. Hint: Analyze the proof of Theorem 2.41.

(73) (Hard) The purpose of this group of three exercises is to verify that $|\mathrm{Aut}(M)|$ satisfies the continuum hypothesis when M is a countable structure. I.e. our goal is to prove

THEOREM 2.74 (Kueker). *Let M be a countable structure. If $\mathrm{Aut}(M)$ is uncountable then $|\mathrm{Aut}(M)| = 2^{\aleph_0}$.*

(a) Let M be a countable structure. If there exists a finite $A \subseteq |M|$ such that $\mathrm{Aut}_A(M) = \{\mathrm{id}_M\}$ then $\mathrm{Aut}(M)$ is at most countable.

(b) Let M be a countable structure and suppose that $\{b_n \mid n < \omega\}$ is an enumeration of $|M|$ such that for every $n < \omega$ there is $\sigma_n \in \mathrm{Aut}(M)$ satisfying $\sigma_n \restriction \{b_k \mid k < n\} = \mathrm{id}_{\{b_k | k<n\}}$ and $\sigma_n(b_n) \neq b_n$. For $\eta \in {}^{\omega}2$ let $\rho_\eta \in \mathrm{Aut}(M)$ be defined as

$$\rho_\eta(b_n) := \prod_{k \leq n} \sigma_k^{\eta[k]}(b_n) \quad \text{for every } n < \omega.$$

Verify that indeed $\rho_\eta \in \mathrm{Aut}(M)$. Show that $\rho_\eta \neq \rho_\nu$ for every $\eta \neq \nu$

(c) Derive Kueker's theorem from the previous two exercises.

(74) Prove that the reals is not a definable subset of the field of complex numbers. Hint: Use the facts that the field of complex numbers has many more automorphisms than the field of real numbers.

(75) Produce an alternative proof to the fact that the set of real numbers is not definable in the field of complex numbers. This time use the facts that there are $a \in \mathbb{R}, b \in \mathbb{C} - \mathbb{R}$ both transcendental over $\mathbb{Q}$ (why this is true?) and Proposition 2.31.

(76) Prove the following strengthening of Theorem 2.8:
Theorem 2.8' Let L be a similarity type, and let M be an L-structure of cardinality at least $|L| + \aleph_0$. For every cardinal number λ such that $|L| + \aleph_0 \leq \lambda \leq \|M\|$ and every $A \subseteq |M|$ of cardinality less or equal to λ there exists $N \subseteq M$ such that $|N| \supseteq A$ and $\|N\| = \lambda$.

(77) Let T_{dtfa} be the *theory of divisible, torsion-free abelian groups*. Namely, the laguage is $\{+, 0\}$ and T_{dtfa} consists of the axioms o Abelain groups, together with

$$\{\forall y \exists x [\underbrace{x + x + \cdots + x}_{n\text{-times}} = y] \mid n \text{ is a positive integer}\}$$

$$\cup \{\forall x [x \neq 0 \to \underbrace{x + x + \cdots + x}_{n\text{-times}} \neq 0 \mid n \text{ is a positive integer}\}.$$

Prove that T_{dtfa} is categorical in every infinite cardinal. Hint: Identify $M \models T_{dtfa}$ with a vector space over $\mathbb{Q}$.

(78) Define a finite similarity type L such that there exists $\varphi \in \mathrm{Sent}(L)$ satisfying: For every finite L-structure M we have that

$$M \models \varphi \quad \Longrightarrow \quad \|M\| \text{ is an even number.}$$

(79) For a integer $n \geq 2$ and an integer $0 \leq r < n$ there are $L_{n,r}$ and $\varphi \in \mathrm{Sent}(L_{n,r})$ such that for every finite $L_{n,r}$-structure M we have

$$M \models \varphi \quad \Longrightarrow \quad \|M\| \equiv r \mod n.$$

(80) Prove that for every λ and T (as in Definition 2.42) we have that $\mathrm{I}(\lambda, T) \leq 2^{\lambda}$.

(81) Let F be a field and let λ be an infinite cardinal greater than $|F|$. Prove that $\mathrm{I}(\lambda, T_{V,F}) = 1$.

(82) Prove Theorem 2.44. Hint: Given an algebraically closed field F of characteristics p let E_p be the algebraic closure of the prime subfield of F and consider the transcendence degree of F over E_p.

(83) Let T be a consistent theory and let n be a positive integer. For every pair of formulas $\varphi(\mathbf{x}), \psi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T))$ in the variables $\{x_0, \ldots, x_{n-1}\}$ define $\varphi \approx_T \psi$ to hold iff $T \vdash \forall x_0 \cdots \forall x_{n-1} [\varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x})]$.
   (a) Show that $\approx_T$ is an equivalence relation.
   (b) For $\varphi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T))$ let $[\varphi]$ be the $\approx_T$-equivalence class of φ (i.e. $[\varphi] := \{\psi(\mathbf{x}) \ : \ \varphi \approx_T \psi\}$).
   Denote by $\mathrm{BA}^n(T)$ the set $\{[\varphi] \ : \ \varphi(x_0, \ldots, x_{n-1}) \in \mathrm{Fml}(\mathrm{L}(T))\}$. Similarly to Exercise 2.2.10f define operations $\cup, \cap$ and $-$. Show that $\mathrm{BA}^n(T)$ with the above operations is a boolean algebra.
   It is called the *Lindenbaum boolean algebra of the theory* T .

(84) Suppose T is a complete theory, and let $M \models T$. Show that for every positive integer n we have that $\mathrm{BA}^n(T) = \mathrm{BA}^n(M)$.

(85) Present an explicit axiomatization for the theory of the structure from Example 2.49.

(86) Let S be an infinite set and let $P_3 \subseteq P_2 \subseteq P_1 \subseteq S$ be infinite subsets such that $S - P_1$, $P_1 - P_2$ and $P_2 - P_3$ are infinite. Let $M := \langle S, P_1, P_2, P_3 \rangle$ and let T be the theory of M. Find an explicit function $f : \omega \to \omega$ such that for every $n < \omega$ we have that $\mathrm{I}(\aleph_n, T) = f(n)$. What is $\mathrm{I}(\aleph_\omega, T)$?

(87) Present an explicit axiomatization for the theory of the structure from Example 2.51(3).

(88) Present an explicit axiomatization for the theory of ω-many independent predicates (Example 2.53).

(89) Show that each of the theories from Example 2.50 have recursive axiomatizations.

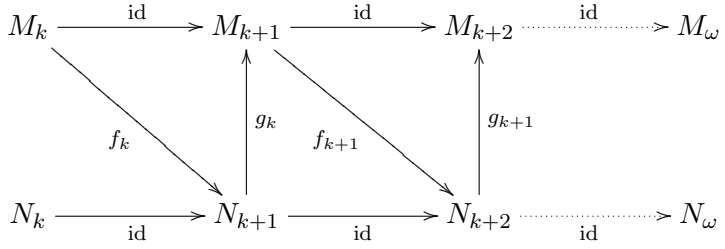
(90) Suppose that $\{M_\alpha, N_\alpha \mid \alpha \leq \omega\}$ are all L-structures satisfying:

(1) $M_k \subseteq M_{k+1} \subseteq M_\omega$ and $N_k \subseteq N_{k+1} \subseteq N_\omega$ for every $k < \omega$.
and

(2) $M_\omega = \bigcup_{k<\omega} M_k$ and $N_\omega = \bigcup_{k<\omega} N_k$.

If $\{f_k, g_k \mid k < \omega\}$ are L-homomorphisms satisfying $f_k : M_k \to N_{k+1}$, $g_k : N_{k+1} \to M_{k+2}$ such that $f_k \subseteq f_{k+1}$, $g_{k+1} \subseteq g_{k+2}$ and $g_k(f_k(a)) = a$ for all $a \in M_k$, $f_{k+1}(g_k(b)) = b$ for all $b \in N_{k+1}$ and for every $k < \omega$ then $\bigcup_{k<\omega} f_k : M_\omega \cong N_\omega$.

Note that the above requirements can be also written using the language of commutative diagrams, we require that the diagram



commutes for all $k < \omega$.

3. On existence of models and elementary submodels

The following result is a central result of predicate calculus (often it is the main result proved in an introductory logic course). It is known as Gödel's completeness theorem, for a proof consult any elementary book in logic like Goldstern and Ihoda's [**GJ**] or D. Monk's [**M**].

THEOREM 3.1 (Gödel's completeness theorem). *Let L be a similarity type, and let T be an L-theory. The following conditions are equivalent*

(1) *T is consistent*
(2) *There exists an L-structure M such that $M \models T$.*

The direction (1) implies (2) is the hard one. The statement that (2) implies (1) is easier and is called *the soundness theorem*.

Below we present two applications of the completeness theorem. The first is a simple test for the completeness of a theory:

PROPOSITION 3.2. *Let T be a consistent theory.*
If for every pair of $\mathrm{L}(T)$-structures M and N we have that $M \models T$ and $N \models T$ implies that $M \equiv N$ then T is complete.

PROOF. Suppose T is not complete. Let $\varphi \in \mathrm{Sent}(\mathrm{L}(T))$ be a witness for T's incompleteness, i.e. $T \not\vdash \varphi$ and $T \not\vdash \neg\varphi$. It follows that $T_1 := T \cup \{\neg\varphi\}$ and $T_2 := T \cup \{\varphi\}$ are both consistent theories. By Theorem 3.1 there are $M_\ell \models T_l$ for $\ell = 1, 2$. Since $M_1, M_2 \models T$ by the hypothesis of the proposition we have that $M_1 \equiv M_2$ which is a contradiction. □

The next result is a most useful corollary of the completeness theorem. It is the one of the two most important tools of elementary model theory[6].

THEOREM 3.3 (The compactness theorem). [7] *Let L be a similarity type, and let T be a set of sentences in L. If for every finite $T' \subseteq T$ there exists an L-structure $M' \models T'$ then there exists an L-structure N such that $N \models T$.*

PROOF. By theorem 3.1 it suffices to show that T is consistent. For the sake of contradiction suppose that T is inconsistent. Namely there exists a sentence φ such that both $T \vdash \varphi$ and $T \vdash \neg\varphi$ holds. By Remark 2.20 there are $T_\ell \subseteq_{finite} T$ (for $\ell = 1, 2$) such that $T_1 \vdash \varphi$ and $T_2 \vdash \neg\varphi$. Now let $T' := T_1 \cup T_2$. Since T' is an inconsistent theory by the easy direction of Theorem 3.1 T' does not have a model. Thus we produced a finite subset of T that does not have a model, this is a contradiction to the assumption on T. □

[6] The other is the Downward Löwenheim-Skolem-Tarski theorem (Theorem 3.20).

[7] Later in Theorem 4.8.10 it will be shown that the statement of the compactness theorem is the assertion that a certain topological space is compact.

Later (see, Theorem 2.19) we will present a purely algebraic proof of the compactness theorem that does not depend on the notion of consistency, in particular the algebraic proof does not use the completeness theorem.

A set-theoretic note: It turns out that ZF $\nvdash$"completeness theorem". More precisely, even the statement of the compactness theorem is not provable from ZF alone. The status of the compactness theorem is quite surprising. It is equivalent to several other principles that came up in different areas. Among them are: TyH which stands for Tychonoff's theorem for compact Hausdorff spaces, BPI that stands for the Boolean Prime Ideal theorem. It has been shown by Lauchli and Levi (for an exposition see [**Je73**]) that ZF $\nvdash$ BPI, while it turns out that ZF $\vdash$ CT $\leftrightarrow$ TyH $\leftrightarrow$ BPI where CT stands for the compactness theorem. It follows from the above that ZFC $\vdash$ CT however ZF $\nvdash$ CT. Due to the fact that also the Downward Löwenheim-Skolem-Tarski theorem depends on the axiom of choice and the above comments about the compactness theorem there is virtually no model theory without some form of choice. In fact all the work in model theory that is known to me is carried out in ZFC.

In order to motivate the next definition let's consider the following example: Consider L_f and let L' be $\{+, 0\}$. Let $M := \langle \mathbb{Q}, +, \cdot, 0, 1\rangle$ be the field of rational numbers. Let $N := \langle \mathbb{Q}, +, 0\rangle$ be the additive group of the rationals (it is an L'-structure). We call N a reduct of M, and M is an expansion of N.

DEFINITION 3.4. Let L and L' be languages such that $L \supseteq L'$.
Let $M := \langle |M|, G^M, R^M, c^M\rangle_{G\in\mathbf{F}(L),\, R\in\mathrm{R}(L),\, c\in\mathbf{C}(L)}$.
The *reduct of M to L'* is the following L'-structure-

$$N := \langle |M|, G^M, R^M, c^M\rangle_{G\in\mathbf{F}(L'),\, R\in\mathbf{R}(L'),\, c\in\mathbf{C}(L')},$$

we denote the model N by $M \restriction L'$. We say that *N is the reduct of M to L'* and *M is an expansion of N to L.*

LEMMA 3.5. *Let M be an L-structure and $L_1 \supseteq L$. If N is an L_1-structure such that there exists $g : N \restriction L \cong M$ then there is an expansion M_1 of M to L_1 such that $g : N \cong M_1$.*

PROOF. For a constant symbol $c \in L_1 - L$ let $c^{M_1} := g(c^N)$. For a function symbol $F(\mathbf{x}) \in L_1 - L$ and $\mathbf{a}$ a finite sequence of elements of $|M|$ let $F^{M_1}(\mathbf{a}) := g(F^N(g^{-1}(\mathbf{a}))$. For relation symbols do something similar. □

Notice that always the model and its reduct have the same universe.

Given an L-structure M and a subset of its universe A we often will consider a special expansion of M to $\{c_a \mid a \in A\}$:

NOTATION 3.6. Let M be an L-structure and $A \subseteq |M|$ be given. Suppose that $\{c_a \mid a \in A\}$ is a new set of constants and let L' be the expansion of L by the constants $\{c_a \mid a \in A\}$.

By M_A we denote the expansion to L' of M where $c_a^{M_A} = a$ for every a.

The following result is a classical result in our subject.

PROPOSITION 3.7 (Upward Löwenheim-Skolem theorem). *Let L be a similarity type, and let T be an L-theory. If T has an infinite model then for every infinite cardinal number λ the theory T has a model of cardinality at least λ.*

PROOF. Let λ be given, and let $C := \{c_i \ : \ i < \lambda\}$ be a set of constant symbols disjoint to $\mathbf{C}(L)$. Form $L_1 := \langle \mathbf{R}(L), \mathbf{F}(L), \mathbf{n}(L), \mathbf{C}(L) \cup C \rangle$, i.e. $\mathbf{R}(L_1) = \mathbf{R}(L); \mathbf{F}(L_1) = \mathbf{F}(L)$, and $\mathbf{C}(L_1) = \mathbf{C}(L) \cup C$. Let $T_1 := T \cup \{c_i \neq c_j \ : \ i < j < \lambda\}$. It is enough to show that T_1 has a model. Suppose $M \models T_1$ then by the definition of T_1 it is clear that $\|M\| \geq \lambda$. Let $N := M \restriction L$, certainly $N \models T$ and N has cardinality $\|M\|$ (since they have the same universes).

We use the compactness theorem to show that T_1 has a model. It suffices to show that for every $T^* \subseteq_{finite} T_1$ the theory T^* has a model. Let M be an infinite model of T from the assumption of the proposition. We may assume that $\{d_1, \ldots, d_n\}$ are the elements of C appearing in T^*. Since $|M|$ is an infinite set there exist $a_1, \ldots, a_n$ all distinct elements of $|M|$ that are different from interpretations of the constant symbols of L that appear in the finite theory T^*. Now define M^* be the expansion of M to $L \cup \{d_1, \ldots, d_n\}$ such that $d_i^{M^*} = a_i$. Check that $M^* \models T^*$. □

The idea used in the proof of the last proposition is the simplest application of a method that will be used very often. The abstract idea is the following: We are given a consistent theory T and want to find a model $N \models T$ satisfying some additional condition (in the proof of Proposition 3.7 the extra condition was that $\|N\| \geq \lambda$). We pick an appropriate expansion L_1 of $\mathrm{L}(T)$, define a theory T_1 in L_1 such that $T_1 \supseteq T$, use the compactness theorem to find an L_1-structure $N_1 \models T_1$, define $N := N_1 \restriction \mathrm{L}(T)$. N is a model of T as required. In the above argument the language of $\mathrm{L}(T)$ was expanded by λ-many constant symbols, in later applications occasionally we will expand by relation and function symbols.

DEFINITION 3.8. Let L be a similarity type, let $\Phi \subseteq \mathrm{Fml}(L)$, and let N be an L-structure.

(1) For nonempty subsets A and B of the universe of N we say that *A and B satisfy the Tarski-Vaught condition for* Φ iff
   (a) $A \subseteq B$, and
   (b) for every formula $\varphi(x; \mathbf{y}) \in \Phi$ and every $\mathbf{a} \in A$ if there exists $b \in B$ such that $N \models \varphi[b; \mathbf{a}]$ then there exists $b' \in A$ such that $N \models \varphi[b'; \mathbf{a}]$.
(2) When A and B satisfy the above condition we denote this by $A \subseteq_{TV,\Phi} B$ often we will use also the notation $A \prec_\Phi B$.

(3) In the particular case when $B = |N|$ and A is a universe of a submodel M of N and Φ is the set of all formulas in L then we denote $|M| \subseteq_{TV,\mathrm{Fml}(L)} |N|$ by $M \preceq N$ and say that *M is an elementary submodel of N* we also say that *N is an elementary extension of M*.
(4) When $\Phi = \mathrm{Fml}(L)$ we will not mention Φ.

The following example may clarify the above notion of $M \preceq N$. Suppose that L is the language of fields (what we denoted earlier by L_f), suppose M, N are fields then $M \preceq N$ implies that M is a subfield of the field N and in addition for every finite system of equations (and inequalities) with coefficients from M if there is a solution to the system in N then there is a solution in M. Thus in particular, if N is an algebraically closed field then also M is algebraically closed.

PROPOSITION 3.9. *Let L be a similarity type, and let M be an L-structure. If $A \subseteq_{TV} |M|$ and $A \supseteq \{c^M \mid c$ is a constant symbol of $L\}$ then A is a universe of an elementary submodel of M.*

PROOF. It is enough to show that A is closed under the functions of M. Let $F(\mathbf{x})$ be a given function of M, and let $\mathbf{a} \in A$. Let $b \in |M|$ be such that $F(\mathbf{a}) = b$. Clearly $M \models \exists x[x = F(\mathbf{a})]$. By Definition 3.8(1)(b) there exists $b' \in A$ such that $M \models F(\mathbf{a}) = b'$. Hence A is closed under the functions of M. □

Notice that in the above proof we actually get that $b = b'$.

LEMMA 3.10. *Let L be a similarity type, and let M_ℓ be L-structures (for $\ell = 1, 2, 3$).*

(1) *If $M_1 \preceq M_2$ and $M_2 \preceq M_3$ then $M_1 \preceq M_3$.*
(2) *If $M_1 \preceq M_3$, $M_2 \preceq M_3$ and $M_1 \subseteq M_2$ then $M_1 \preceq M_2$.*

PROOF. Exercise 5. □

Property (2) of the last lemma is unusually important, it is called *coherence*.

Before stating the next lemma let me make few remarks on notation (or rather an abuse of notation): Consider the polynomial

$$y_0 + y_1 x + \ldots + y_{m-1} x^{m-1}.$$

In everyday mathematics often the variables that stand for the coefficients (or parameters) are treated differently than the x-the unknown (or the variable). Influenced by this convention we denote by $\varphi(x; \mathbf{y})$ the formula

$$y_0 + y_1 x + \ldots + y_{m-1} x^{m-1} = 0.$$

Semicolon is used to separate the variables from the parameters or more generally: Given an L- structure M and a formula $\varphi(\mathbf{x};\mathbf{y})$ from L, for $\mathbf{a} \in {}^{\ell(\mathbf{y})}|M|$ often we consider statements of the form $M \models \exists\mathbf{x}\varphi(\mathbf{x};\mathbf{a})$. The last can be viewed as a relation between the three objects M, φ and $\mathbf{a}$. Often we treat the object $\varphi(\mathbf{x};\mathbf{a})$ as a formula; The formal justification for that is the following: Define $L' = L \cup \{c_1, \ldots, c_\ell\}$ (the c_i's are constants not in L) where $\ell = \ell(\mathbf{a})$ and denote by M' the expansion of M to L' when $c_i^{M'} := a_i$. In L' the object $\varphi(\mathbf{x};\mathbf{a})$ is a shorthand for the formula $\varphi(\mathbf{x};\mathbf{c})$ (substitute for the variables $\mathbf{y}$ the constants $\mathbf{c}$). When $A \subseteq |M|$ and $\mathbf{a} \in A$ we say that $\varphi(\mathbf{x};\mathbf{a})$ is a *formula with parameters from the set* A. Notice that the above is consistent with the usual practice of substituting constants for some variable. E.g. when $p(x,y,z)$ is a polynomial in the variables x, y and z by substituting constants for y and z we get a polynomial in x.

DEFINITION 3.11. Let M be an L-structure, and suppose $A \subseteq |M|$. Denote by L_A the language $L \cup \{c_a \ : \ a \in A\}$, let M_A be the expansion of M to L_A when $c_a^{M_A} := a$ for all $a \in A$. We denote the last structure by $\langle M, a\rangle_{a\in A}$.

For $A \subseteq |M|$ and a natural number n let $\mathrm{BA}^n_A(M)$ be the *Lindenbaum Boolean Algebra of* $\langle M, a\rangle_{a\in A}$, namely it is $\mathrm{BA}^n(\langle M, a\rangle_{a\in A})$.

Our definition of the relation of elementary substructure ($\prec$) is different than that of most authors. The more common definition is the statement of Lemma 3.12(3) below. The condition presented in Definition 3.8 is often called the *Tarski-Vaught test* named so after the pioneering work of Alfred Tarski and Robert Vaught presented in their paper from 1956 ([**TV**]).

LEMMA 3.12 (Tarski-Vaught test). *Let M and N be L-structures. The following are equivalent*

(1) $M \preceq N$
(2) $M \subseteq N$ *and* $\langle M, a\rangle_{a\in|M|} \equiv \langle N, a\rangle_{a\in|M|}$.
(3) $M \subseteq N$ *and for every L formula $\varphi(\mathbf{x})$ and every $\mathbf{a} \in |M|$ we have that*

$$M \models \varphi[\mathbf{a}] \iff N \models \varphi[\mathbf{a}].$$

PROOF. $(1) \implies (2)$: By induction on formulas show that for every L-formula $\varphi(\mathbf{x})$

$$(*) \quad M \models \varphi[\mathbf{a}] \iff N \models \varphi[\mathbf{a}] \quad \text{for every } \mathbf{a} \in |M|.$$

For $\varphi(\mathbf{x})$ atomic use Theorem 2.6.

For non-atomic formulas[8]

[8]While by Theorem 2.6 (*), holds for all quantifier free formulas we still have to consider the cases that φ is obtained from sub formulas using logical connectives as the sub formulas might contain quantifiers.

If $\varphi(\mathbf{x})$ is $\neg\psi(\mathbf{x})$ then

$$M \models \varphi[\mathbf{a}] \iff M \models \neg\psi[\mathbf{a}] \iff M \not\models \psi[\mathbf{a}]$$
$$\iff \text{(by induction hypothesis) } N \not\models \psi[\mathbf{a}] \iff N \models \varphi[\mathbf{a}].$$

If $\varphi(\mathbf{x})$ is $\psi(\mathbf{x}) \wedge \chi(\mathbf{x})$

$$M \models \varphi[\mathbf{a}] \iff M \models \psi[\mathbf{a}] \wedge \chi[\mathbf{a}]$$
$$\iff M \models \psi[\mathbf{a}] \text{ and } M \models \chi[\mathbf{a}]$$
$$\iff \text{(by induction hypothesis) } N \models \psi[\mathbf{a}] \text{ and } N \models \chi[\mathbf{a}]$$
$$\iff N \models \varphi[\mathbf{a}].$$

If $\varphi(\mathbf{x})$ is $\exists y\psi(y;\mathbf{x})$

Suppose $M \models \varphi[\mathbf{a}]$ by the definition of satisfaction there exists $b \in |M|$ such that $M \models \psi[b;\mathbf{a}]$, by the induction hypothesis we have that $N \models \psi[b;\mathbf{a}]$ thus by the definition of satisfaction $N \models \exists y\psi[y;\mathbf{a}]$.

Suppose $N \models \varphi[\mathbf{a}]$, namely $N \models \exists y\psi[y;\mathbf{a}]$, using the definition of satisfaction find $b \in |N|$ such that $N \models \psi[b;\mathbf{a}]$ by the hypothesis that $M \preceq N$ there exists $b' \in |M|$ such that $N \models \psi[b';\mathbf{a}]$. Since $b'^\frown\mathbf{a} \in |M|$ by the induction hypothesis the latter fact is equivalent to $M \models \psi[b';\mathbf{a}]$ thus we have shown that $M \models \exists y\psi[y;\mathbf{a}]$.

(2)$\iff$(3): Immediate.
(3)$\implies$(1): Let $\varphi(y;\mathbf{x})$ and $\mathbf{a} \in |M|$ be given. Suppose $N \models \exists y\varphi(y;\mathbf{a})$. By the hypothesis $M \models \exists y\varphi(y;\mathbf{a})$, by definition of satisfaction we get that there exists $b \in |M|$ such that $M \models \varphi[b;\mathbf{a}]$, another application of the hypothesis gives $N \models \varphi[b;\mathbf{a}]$. □

COROLLARY 3.13. *If $M \preceq N$ then $M \equiv N$.*

PROOF. By $(1) \implies (3)$ in Lemma 3.12. □

PROPOSITION 3.14 (Another characterization of elementary equivalence). *Let M and N be L-structures. The following are equivalent*

(1) $M \preceq N$
(2) $M \subseteq N$ *and for every L formula $\varphi(\mathbf{x};\mathbf{y})$ and every $\mathbf{a} \in {}^{\ell(\mathbf{y})}|M|$ we have that*

$$\varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a}) \cap {}^{\ell(\mathbf{x})}|M|.$$

PROOF. Exercise using the Tarski-Vaught test. □

A natural question to ask here: Is the converse of the last corollary true? Namely let M and N be L-structures. Is the following implication true?

$$M \equiv N \text{ and } M \subseteq N \implies M \preceq N.$$

The answer is no (see Exercise 6).

The following idea is extracted from the structure appearing in Definition 3.11:

DEFINITION 3.15. Let M be an L-structure.

(1) Let N be $\langle M, a\rangle_{a\in|M|}$. Its theory $\mathrm{Th}(N)$ is denoted by $\mathrm{CD}(M)$ is called the *complete diagram of* M.
(2) The *elementary diagram of* M is denoted by $\mathrm{ED}(M)$ it is the set

$$\{\varphi \in \mathrm{CD}(M) \mid \varphi \text{ is a quantifier free sentence}\}.$$

Notice that for a group G the elementary diagram of G is a generalization of the multiplication table of G and the complete diagram of G is the multiplication table together with all statements involving quantifiers.

It appears that contrary to the situation in algebra (in group theory, or ring theory) the useful notion of embedding is not that of homomorphism but the one introduced below:

DEFINITION 3.16 (Elementary embedding). Let L be a similarity type, and suppose that M and N are L-structures. A function $f : |M| \to |N|$ is called an *elementary embedding* iff there exists $N^* \preceq N$ such that f is an isomorphism from M onto N^*.

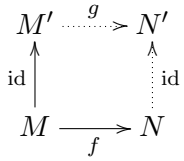
Often elementary embedding is also called *elementary mapping* I will use both notations and will not distinguish between the two.

REMARK 3.17. Notice the analogy of the definition of elementary embedding with that of monomorphism (see Remark 2.10)

The following Lemma is a main tool in constructing elementary embeddings (and elementary extensions of given models). It permits to reduce the question of existence of an elementary embedding to that of consistency of a first-order theory that contain a complete diagram.

LEMMA 3.18 (The fundamental extension Lemma).

(1) *Suppose $f : M \cong N$ and $N' \supseteq N$ then there exists $M' \supseteq M$ and $g \supseteq f$ such that $g : M' \cong N'$. Namely the following diagram commutes:*

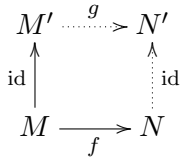


(2) *Let M be a given structure, if $N \models \mathrm{ED}(M)$ then there exists an injective homomorphism from M onto a submodel of $N \restriction \mathrm{L}(M)$ and there exists $N^* \cong N$ such that $M \subseteq N^*$.*

(3) *Let M be a given structure, if $N \models \mathrm{CD}(M)$ then there exists an elementary mapping of M into $N \restriction \mathrm{L}(M)$ and there exists $N^* \cong N$ such that $M \preceq N^*$*

(4) *Suppose that M and N are L-structures. If there exists an elementary embedding from M into N then there is an expansion N' of N to $\mathrm{L}(\mathrm{CD}(M))$ such that $N' \models \mathrm{CD}(M)$.*

(5) *Let M, N, M' be L-structures. If $M \preceq M'$ and $f : M \cong N$ then there are $N' \succeq N$ and $g : M' \cong N'$ satisfying $g \supseteq f$. Namely the following diagram commutes:*



PROOF. Notice that (1) is a restatement of Lemma 2.12.

(2) Let $f(a) := c_a^N$ be a function from $|M|$ into $|N|$. Verify that f is a monomorphism from $M \restriction L$ into N. Apply part (1) to f.

(3) Following the notation of Lemma 3.12 let $\{c_a : a \in |M|\}$ be new constants (not in $\mathrm{L}(M)$). Define a function $f : |M| \to |N|$ by $f(a) := c_a^N$ for every $a \in |M|$.

Let $B := \mathrm{Range}(f)$. It suffices to show that B is a universe of an elementary submodel of N and f is an isomorphism from M onto B. By Proposition 3.9 in order to show that $B \preceq N$ it is enough to show that $B \subseteq_{TV} N$. Let $\varphi(x, \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(M))$ and $\mathbf{b} \in B$ be given. Suppose that there exists $d \in |N|$ such that $N \models \varphi[d, \mathbf{b}]$. By the definition of satisfaction we have that $N \models \exists x \varphi[x, \mathbf{b}]$. Thus we have $N \models \exists x \varphi[x, \mathbf{b}]$, since $\mathbf{b} \in \mathrm{Range}(f)$ we can pick $a_1, \ldots, a_n \in |M|$ such that $b_k = f(a_k)$ which translate to $N \models \exists x \varphi[x, f(\mathbf{a})]$. Substituting the definition of f gives $N \models \exists x \varphi(x, c_{a_1}, \ldots, c_{a_n})$ holds. Since $\exists x \varphi(x, c_{a_1}, \ldots, c_{a_n})$ is a sentence in the language of $\mathrm{CD}(M)$ and $N \models \mathrm{CD}(M)$ necessarily we have that $\exists x \varphi(x, c_{a_1}, \ldots, c_{a_n}) \in \mathrm{CD}(M)$. Namely
$\langle M, a \rangle_{a \in |M|} \models \exists x \varphi(x, c_{a_1}, \ldots, c_{a_n})$. By the definition of satisfaction there exists

$d \in |M|$ such that $\langle M, a\rangle_{a\in|M|} \models \varphi[d, c_{a_1}, \ldots, c_{a_n}]$. Since we have constant symbols for every element of $|M|$ also
$\langle M, a\rangle_{a\in|M|} \models \varphi(c_d, c_{a_1}, \ldots, c_{a_n})$ holds.

Now $\varphi(c_d, c_{a_1}, \ldots, c_{a_n}) \in \mathrm{CD}(M)$ which (using the hypothesis that $N \models \mathrm{CD}(M)$) gives that $N \models \varphi(c_d, c_{a_1}, \ldots, c_{a_n})$, by the definition of f we have that $d' := f(c_d) \in B$ satisfy $N \models \varphi[d', c_{a_1}, \ldots, c_{a_n}]$, using our choice of the elements $a_1, \ldots, a_n$ we can rewrite the above as $N \models \varphi[d', \mathbf{b}]$.

We left to show that f is an isomorphism. Clearly it is surjective. injectivity follows from the following observation: Let $a \neq b \in |M|$ be given. By the definition of $\mathrm{CD}(M)$ we have that the sentence $c_a \neq c_b$ belong to $\mathrm{CD}(M)$. Since $N \models \mathrm{CD}(M)$ also $N \models c_a \neq c_b$ which by the definition of f translate to $f(a) \neq f(b)$. To show that relations are preserved by f: Let $R(x_1, \ldots, x_n)$ be a given relation symbol in $\mathrm{L}(M)$ and suppose that $M \models R(a_1, \ldots, a_n)$ for some $\{a_1, \ldots, a_n\} \in |M|$ by the definition of $\mathrm{CD}(M)$ we have that $R(c_{a_1}, \ldots, c_{a_n}) \in \mathrm{CD}(M)$ since $N \models \mathrm{CD}(M)$ we have that $N \models R(c_{a_1}, \ldots, c_{a_n})$ using the definition of f we can rewrite this as $N \models R(f(a_1), \ldots, f(a_n))$. The verification that f preserves function symbols is similar.

(4) See Exercise 15.

(5) ADD □

COROLLARY 3.19 (Upward Löwenheim-Skolem-Tarski). *Let M be a structure with an infinite universe. For every cardinal $\lambda \geq \|M\| + |\mathrm{L}(M)|$ there exists $N \succ M$ of cardinality at least λ.*

PROOF. Let $T := \mathrm{CD}(M)$. Apply Proposition 3.7 to T to get model N' of cardinality at least λ, by Lemma 3.18 there exists an elementary embedding $g : M \to N' \restriction \mathrm{L}(M)$. Use N' to construct a model that is an elementary extension of M.

Let $|N| := (|N'| - g[|M|]) \cup |M|$.

For a constant symbol c let $c^N := c^M$.

For a function symbol $F(x_0, \ldots, x_{n-1})$ and a given $\mathbf{a}' \in {}^n|N'|$ there exists $b \leq n-1$ such that for $i \leq k$ we have that $a_i \in |M|$ and for $i > k$ we have that $a_i \in |N'| - g[|M|]$. Now for $i \leq k$ denote by a'_i the element $g(a_i)$. Denote by $\mathbf{a}'$ the sequence $\langle a'_0, \ldots, a'_{k-1}, a_k, \ldots, a_{n-1}\rangle$. Define

$$F^N(\mathbf{a}) := \begin{cases} g^{-1}(F^{N'}(\mathbf{a}')) & \text{if } F^{N'}(\mathbf{a}') \in g[|M|] \\ F^{N'}(\mathbf{a}') & \text{if } F^{N'}(\mathbf{a}') \in |N'| - g[|M|]. \end{cases}$$

We use the same idea to define the relation symbols of N.
Let $R(x_0, \ldots, x_{n-1})$ be given. Define $R^N := \{\mathbf{a} \in {}^n|N| \ : \ \mathbf{a}' \in R^{N'}\}$. The verification that the structure $\langle |N|, F^N, R^N, c^N \rangle_{R,F,c \in \mathrm{L}(M)}$ is an elementary extension of M is left to the reader. □

THEOREM 3.20 (Downward Löwenheim-Skolem-Tarski). *Let N be an infinite structure. For every $A \subseteq |N|$ there exists $M \preceq N$ such that $|M| \supseteq A$ and $\|M\| \leq |A| + |\mathrm{L}(N)| + \aleph_0$.*

PROOF. Fix a well-ordering $<$ of the set $|N|$, denote by b_0 its first element. For every formula $\varphi(x, \mathbf{y}) \in \mathrm{L}(N)$ and every $\mathbf{a} \in |N|$ define the following function:

$$G_\varphi(\mathbf{a}) := \begin{cases} Min\{b \in |N| \ : \ N \models \varphi[b, \mathbf{a}]\} & \text{if } N \models \exists x \varphi[x, \mathbf{a}] \\ \\ b_0 & \text{otherwise.} \end{cases}$$

By induction of $n < \omega$ define $A_n \subseteq |N|$ with the following properties:

(1) $A_0 = A$,
(2) for every $n < \omega \quad A_n \subseteq A_{n+1}$,
(3) for every $n < \omega$ we have that $|A_n| \leq |\mathrm{L}(N)| + |A|$, and
(4) for every $n < \omega$, every formula $\varphi(x, \mathbf{y}) \in \mathrm{L}(N)$ and every finite sequence $\mathbf{a} \in A_n$ we have that $G_\varphi(\mathbf{a}) \in A_{n+1}$.

This is enough: Suppose that such a sequence exists. Let $S := \bigcup_{n<\omega} A_n$. We will show that S determines a universe of an elementary submodel M of N as required. By requirements (1) and (3) it is clear that it is enough to show that $S \subseteq_{TV} N$. By Proposition 3.9 S determines a universe of a substructure. All we have to worry about is verifying the Tarski-Vaught test: Let $\mathbf{a} \in S$ and $\varphi(x; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(N))$ be given. Suppose that $N \models \exists x \varphi[x; \mathbf{a}]$, since the sequence $\{A_n \ : \ n < \omega\}$ is increasing there exists $n_0 < \omega$ such that all the elements of the sequence $\mathbf{a}$ appear in A_{n_0}. By the definition of the function G_φ we have that $N \models \varphi[G_\varphi(\mathbf{a}); \mathbf{a}]$ holds. But by requirement (4) we have that $G_\varphi(\mathbf{a}) \in A_{n_0+1} \subseteq S$.

This is possible: We have to show that there exists a sequence of sets satisfying conditions (1)-(4)[9]. The definition is by induction on $n < \omega$. For $n = 0$ define A_0 as the set A. Suppose A_n is defined and we have to define A_{n+1}.

Consider $B_n := \{G_\varphi(\mathbf{a}) \ : \ \mathbf{a} \in A_n, \ \varphi(x; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(N))\}$. Define now A_{n+1} as the set $A_n \cup B_n$. From the definition it is clear that requirements (1),(2) and (4) holds. We need to verify only (3): From the definition of B_n, the inductive hypothesis and cardinal arithmetic show that $|B_n| \leq |A| + |\mathrm{L}(N)|$. Using cardinal arithmetic again conclude that $|A_{n+1}| \leq |A| + |\mathrm{L}(N)|$. □

[9]Compare with the argument of Theorem 2.8

REMARK 3.21 (UPLS+DLS $\implies$ AC). It is interesting to note that while the Upward Löwenheim-Skolem Theorem is weaker than the Axiom of Choice (as it is equivalent to BPI) together with the Downward Löwenheim-Skolem Theorem it implies the axiom of Choice! See Exercise 9.

COROLLARY 3.22. *Let N be an infinite structure of cardinality at least $|\operatorname{L}(N)|$.*

(1) *For every $A \subseteq |N|$ and every λ such that $|A| + |\operatorname{L}(N)| \leq \lambda \leq \|N\|$ there exists $M \preceq N$ such that $|M| \supseteq A$ and $\|M\| = \lambda$.*
(2) *For every cardinal $\lambda \geq \|N\|$ there exists $M \succ N$ of cardinality λ.*
(3) *N has a proper elementary extension of the same cardinality. Namely there exist $M \succ N$ such that $\|N\| = \|M\|$ and $|N| \neq |M|$.*

PROOF. (1) Given $A \subseteq |N|$ and λ as above. Pick $B \subseteq |N|$ of cardinality λ, define $C := A \cup B$, and apply now Theorem 3.20 to N and C. Check that the cardinality of M is exactly λ.

(2) Given $\lambda \geq \|N\|$, using Corollary 3.7 get a model $N^* \succ N$ of cardinality at least λ. Pick $B \subseteq |N^*|$ of cardinality λ. Now apply part (1) to the model N^* and the set $A := B \cup |N|$ to get M as required.

(3) Let $\mu > \|N\|$, Using (2) get an elementary extension N^* of N of cardinality μ. Pick $a \in |N^*| - |N|$. Apply now (1) to N^*, $A := |N| \cup \{a\}$ and $\lambda := \|N\|$ to get $M \preceq N^*$ of cardinality $\|N\|$ containing $\|N\|$. Since $M \preceq N^*$, $N \subseteq M$ and $N \preceq N^*$ by Lemma 3.10(2) we have that $M \succ N$. □

A natural question one can ask in light of the last corollary is whether there is a dual to part (3). Namely let N be a given structure of cardinality greater than $|\operatorname{L}(M)| + \aleph_0$ does there exists a proper elementary submodel of N of the same cardinality? A related (and easier) question is: Let N be a given structure of cardinality greater than $|\operatorname{L}(M)|$ does there exists a proper submodel of N of the same cardinality? Inspection of Example 2.3.2(4) should show why we are assuming above that $\|M\| > |\operatorname{L}(M)|$. These questions motivated the following:

The following is an easy corollary of the compactness theorem and the downward Löwenheim-Skolem theorems applied to theories:

THEOREM 3.23. *[Löwenheim-Skolem for theories] Suppose that T is a first-order theory in an at most countable language. If T has an infinite model then for any set A there exists $M \models T$ such that $\|M\| = |A| + |\operatorname{L}(T)| + \aleph_0$.*

PROOF. Let $N \models T$ be infinite. $\{c_i \mid i < \lambda\}$ be new constant symbols (not in $\operatorname{L}(T)$). Consider

$$T^* := T \cup \{c_i \neq c_j \mid i < j < \lambda\}.$$

Given $T_0 \subseteq T^*$ finite, let $i_0 < \cdots i_{n-1} < \lambda$ be such that

$$T_0 \subseteq T \cup \{c_{i_k} \neq c_{i_j} \mid k < j < n\}.$$

Pick any $\{a_i \mid i < n\} \subseteq |N|$. Let N_0 be an expansion to N to $\mathrm{L}(T^*)$ such that $c_i^{N_0} = a_i$ for all $i < n$. Clearly $N_0 \models T_0$. By the compactness theorem there exists $M^* \models T^*$. An application of the DLST theorem gives $M_1 \preceq M^*$ of cardinality λ containing the set A. Check that $M := M_1 \restriction \mathrm{L}(T)$ is as required. □

DEFINITION 3.24. Let L be a similarity type, and let M be an L-structure of cardinality greater than $|L|$.

(1) N is called a *Jónsson model* iff $M \preceq N$ and $|M| \neq |N|$ implies that $\|M\| < \|N\|$.
(2) N is called a *Jónsson algebra* iff N has no relation symbols and

$$M \subseteq N \text{ and } |M| \neq |N| \implies \|M\| < \|N\|.$$

It is not difficult to show that there exists a Jónsson algebra of cardinality $\aleph_1$ (in a finite language), see Exercise 38. The construction can be generalized to show that for every positive integer n there exists a Jónsson algebra of cardinality $\aleph_n$. Assuming the axiom $\mathbf{V} = \mathbf{L}$ of constructibility one can show the existence of Jonsson algebras in all cardinalities. However in ZFC for other cardinal numbers the situation is more complicated (see Theorem 2.**??** and section 13 of Chapter 4).

In 194XXX Kurosh posed the following problem, known as *Kurosh's problem*:

PROBLEM 3.25. *Does there exists a Jónsson group? Namely, is there a group G of cardinality $\aleph_1$ such that every proper subgroup H of G is at most countable?*

The problem can be restated as: Does there exists a Jónsson algebra of cardinality $\aleph_1$ that is a group? It appears that the answer is much harder than the general question for arbitrary algebras. It is much harder to construct groups than algebras. Only in 1976 S. Shelah managed to answer (affirmatively) Kurosh's question. See [**Sh69**]. For more recent results see section 8 of Kanamori's book [**Ka94**] (for Jónsson algebras) and E. Rips [**Ri**] on Jónsson groups[10].

The following result (together with its corollary) is my personal candidate for the title of "the most surprising result of elementary model theory".

THEOREM 3.26 (Vaught 1954). *Suppose T is a first-order theory that has no finite models. If there exists $\lambda \geq \aleph_0 + |\mathrm{L}(T)|$ such that T is λ-categorical lthen T is complete.*

[10]Rips has a general device to construct groups from a closure operation, in 1981 I pointed out to him that his machinery can be used to translate any Jónsson algebra into a Jónsson group.

PROOF. For the sake of contradiction suppose that T is incomplete, namely there exists a sentence φ such that $T \nvdash \varphi$ and $T \nvdash \neg\varphi$. By the definition of consistency of a theory we have that both T_1 and T_2 are consistent theories where $T_1 := T \cup \{\neg\varphi\}$ and $T_2 := T \cup \{\varphi\}$. By the completeness theorem there are models $M_l \models T_\ell$ (for $\ell = 1, 2$), since T does not have finite models both models are infinite.

Using Theorem 3.23 there are $N_\ell \models T_\ell$ for $\ell = 1, 2$ of cardinality λ.

Since $N_\ell \models T$ by the categoricity of T in λ we have that $N_1 \cong N_2$. Now by Theorem 2.16 we have that $N_1 \equiv N_2$. This contradict the fact that the models were chosen so that $N_1 \models \neg\varphi$ and $N_2 \models \varphi$. □

The following is the great surprise

COROLLARY 3.27. *Suppose T is a first-order theory without finite models with a recursive axiomatization. If T is categorical then T is decidable.*

Consider T_{ACF_0}, using Steinitz's theorem on transcendence degree the theory is categorical in $\aleph_1$ (but not in $\aleph_0$). Thus T_{ACF_0} is complete and decidable. Now to the proof:

PROOF. Use the silly (lexicographic) theorem prover in parallel running to find a proof of $T \vdash \varphi$ and $T \vdash \neg\varphi$ by completeness of T the algorithm will stop. □

It is surprising since decidability of a theory is a statement of existence of a computer program (or a Turing machine) which is an assertion of existence on a natural number with certain properties (the Gödel number that codes the decision procedure) that decides whether a given sentence is a theorem of the theory or not. We have managed to show existence of such a program using non-constructive mathematics. For T_{ACF_0} for example Steinitz's theorem depends on the axiom of choice and its proof uses highly non-constructive mathematics.

COROLLARY 3.28. *T_{ord}, T_{ind} and T_{ACF_p} are all decidable theories.*

Combining Theorem 3.26 together with the categoricity of the theories T_{ord} and T_{ACF_p} we get the following important corollary:

COROLLARY 3.29.

(1) *T_{ord} is an axiomatization of $\mathrm{Th}(\langle \mathbb{Q}, < \rangle)$, and*
(2) *T_{ACF_0} is an axiomatization of the first-order theory of the field of complex numbers.*

PROOF. Exercise. □

Looking at the notions $\prec$ and existence of elementary embedding from the category theoretic point of view one can show that the class of L-structures is closed under direct limits (when morphisms are elementary embedding). This follows from the following fundamental result of Tarski and Vaught.

THEOREM 3.30 (Tarski-Vaught first chain theorem). *Let L be a similarity type, and let α be an ordinal. Suppose that $\{M_i \ : \ i < \alpha\}$ is an $\prec$-*increasing chain*, namely for every $i < j < \alpha$ we have that $M_i \preceq M_j$.*
Then $M := \bigcup_{i<\alpha} M_i$ is an L structure and $M_i \preceq M$ holds for every $i < \alpha$.

PROOF. It is easy to check (this is Exercise 3.47) that M is an L structure and that for every $i < \alpha \ \ M_i \subseteq M$. All we have to do is to show that for every $i < \alpha$ and every $\varphi(\mathbf{x}) \in \mathrm{Fml}(L)$ and all $\mathbf{a} \in |M_i|$ we have that
$M_i \models \varphi[\mathbf{a}] \iff M \models \varphi[\mathbf{a}]$ holds.

The above is proved by induction on φ for all $i < \alpha$. By Theorem 2.6 the above holds for all quantifier-free formulas.

Suppose that $\varphi(\mathbf{x})$ is the formula $\neg\psi(\mathbf{x})$; $M_i \models \varphi[\mathbf{a}]$ iff $M_i \not\models \psi[\mathbf{a}]$ (by inductive hypothesis) iff $M \not\models \psi[\mathbf{a}]$ (by definition of satisfaction) iff $M \models \varphi[\mathbf{a}]$.

Suppose that $\varphi(\mathbf{x})$ is the formula $\chi(\mathbf{x}) \wedge \psi(\mathbf{x})$;

$$\begin{aligned} M_i \models \varphi[\mathbf{a}] &\iff M_i \models \chi[\mathbf{a}] \text{ and } M_i \models \psi[\mathbf{a}] \qquad \text{(by induction)} \\ &\iff M \models \chi[\mathbf{a}] \text{ and } M \models \psi[\mathbf{a}] \qquad \text{(by definition of satisfaction)} \\ &\iff M \models \varphi[\mathbf{a}]. \end{aligned}$$

Suppose that $\varphi(\mathbf{x})$ is the formula $\exists y \psi(y; \mathbf{x})$;

$M_i \models \varphi[\mathbf{a}]$ implies (by definition of satisfaction) that there exists $b \in |M_i|$ such that $M_i \models \psi[b; \mathbf{a}]$. Now by the induction hypothesis we have that there exists $b \in |M_i|$ such that $M \models \psi[b; \mathbf{a}]$ which by the definition of satisfaction gives that $M \models \varphi[\mathbf{a}]$.

$M \models \varphi[\mathbf{a}]$ implies that $b \in |M|$ such that $M \models \psi[b; \mathbf{a}]$. Since $|M|$ is a union of an increasing chain of sets there exists $j < \alpha$ such that $b \in |M_j|$. Let $\xi := \max\{i, j\}$ since $\xi \geq j$ we have that $b \in |M_\xi|$ an application of induction hypothesis gives $M_\xi \models \psi[b; \mathbf{a}]$ which by definition of satisfaction gives that $M_\xi \models \varphi[\mathbf{a}]$. Since $i \leq \xi$ by the hypothesis we have $M_i \preceq M_\xi$ thus $M_i \models \varphi[\mathbf{a}]$. □

The following result will be used in the next section, it has an almost identical proof to that of Theorem 3.30.

THEOREM 3.31 (Tarski-Vaught second chain theorem). *indexTarski-Vaught second chain theorem Let L be a similarity type, let α be an ordinal and N an L-structure. Suppose that $\{M_i \preceq N \ : \ i < \alpha\}$ is an $\prec$-*increasing chain*, namely for*

every $i < j < \alpha$ we have that $M_i \preceq M_j$.
Then $M := \bigcup_{i<\alpha} M_i$ is an elementary submodel of N.

PROOF. Exercise 46. □

In Theorems 3.30 and 3.31 the assumption that $\{M_i : i < \alpha\}$ in an $\prec$-increasing chain and indexed by an ordinal can be replaced by a weaker condition that the system is directed. To state it we need a definition:

DEFINITION 3.32.

(1) A partially ordered set $\langle P, \leq \rangle$ is a *directed poset* iff for every $a, b \in P$ there exists $c \in P$ such that $c \geq a$ and $c \geq b$.
(2) Let $\langle I, \leq \rangle$ be a directed poset. A system $\{M_i \mid i \in I\}$ of L-structures is called *directed system* iff $i \leq j \implies M_i \preceq M_j$ for all $i, j \in I$.

The following is a generalization of Theorems 3.30 and 3.31 to directed systems of models:

THEOREM 3.33. *Let L be a similarity type, $\langle I, \leq \rangle$ be a directed poset.*

(1) *If $\{M_i \mid i \in I\}$ is a directed system. Then $M := \bigcup_{i\in I} M_i$ is an L structure and $M_i \preceq M$ for every $i \in I$.*
(2) *If $\{M_i \preceq N \mid i \in I\}$ is a directed system. Then $M := \bigcup_{i\in I} M_i$ is an elementary substructure of N.*

PROOF. Exercise 27 □

A further generalization of Theorem 3.33 is possible (see Theorem 2.7 below).

DEFINITION 3.34. Let M be an L-structure such that $\lambda := \|M\| > |L| + \aleph_0$. An elementary chain $\{M_\alpha \mid \alpha < \lambda\}$. Satisfying:

(1) $\alpha < \beta \implies M_\alpha \precneqq M_\beta$,
(2) $\|M_\alpha\| < \lambda$ for all α,
(3) if $\alpha < \lambda$ is a limit ordinal then $M_\alpha = \bigcup_{i<\alpha} M_i$ and
(4) $M = \bigcup_{\alpha<\lambda} M_\alpha$.

is called *a resolution of M*

THEOREM 3.35. *If M is an L-structure such that $\|M\| > |L| + \aleph_0$ then it has a resolution.*

PROOF. Exercise. □

Exercises 1.3.

(1) Suppose that $M \preceq N$, $n < \omega$, $\varphi(x, \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(N))$ and $\mathbf{a} \in |M|$. If $M \models \exists^{=n} x \varphi(x, \mathbf{a})$ then $\varphi(M, \mathbf{a}) = \varphi(N, \mathbf{a})$.
(2) Let L be a similarity type, and let M be an L-structure. A formula $\varphi(\mathbf{x}, \mathbf{a})$ with parameters from M is called *algebraic for* M iff there exists an integer $n \geq 1$ such that $M \models \exists^{\leq n} \mathbf{x} \varphi[\mathbf{x}, \mathbf{a}]$.
   (a) Show that if $N \succ M$ then for every $\mathbf{a} \in |M|$ and every $\varphi(\mathbf{x}, \mathbf{a})$ algebraic over M we have that $\varphi(M, \mathbf{a}) = \varphi(N, \mathbf{a})$.
   (b) Suppose $M \preceq N$. Show that for every $\varphi(\mathbf{x}; \mathbf{y})$ and every $\mathbf{a} \in |M|$ we have that $\varphi(\mathbf{x}; \mathbf{a})$ is algebraic for M iff $\varphi(\mathbf{x}; \mathbf{a})$ is algebraic for N.
   (c) Let T be a complete first-order theory and suppose that $\varphi(\mathbf{x})$ is an $\mathrm{L}(T)$-formula (without parameters). Show that $\varphi(\mathbf{x})$ is algebraic for some $M \models T$ iff $\varphi(\mathbf{x})$ is algebraic for every $N \models T$.
(3) Let M be the interval $[0, 1]$ of real numbers, with $\mathrm{L}(M) = \{<\}$ the usual order. Suppose N is the interval $[-1, 2]$. Is $M \prec N$?
(4) Construct an example of a structure M and a formula $\varphi(x) \in \mathrm{Fml}(\mathrm{L}(M))$ such that if $N \supsetneq M$ then $\varphi(N) \supsetneq \varphi(M)$.
(5) Prove Lemma 3.10.
(6) Let L be $\{<\}$. Let $N := \langle \omega, < \rangle$ and $M := \langle \omega - \{0\}, < \rangle$. Show that $M \subseteq N$ and $M \equiv N$ (in fact $M \cong N$) but M is not an elementary submodel of N.
(7) Let L be $\{<\}$. Let $N := \langle \omega, < \rangle$ and $M := \langle \omega + \omega, < \rangle$, is N an elementary submodel of M?
(8) Prove Theorem 3.23.
(9) Prove (in ZF) that the Upward Löwenheim Skolem Theorem together with Downward Löwenheim Skolem Theorem implies the axiom of choice. Hint: Use both of the following facts
   (a) By a theorem of Tarski from 1924
   ZF $\vdash$ [[$\forall a$ infinite set $|a \times a| = |a|$] $\rightarrow$ AC]. (See Proposition V.1.14 in Levy's Basic Set Theory [**Lev**]) and
   (b) There is a bijection $f : \omega \times \omega \to \omega$.
(10) Let M_1, M_2 and N be L-structures. Suppose that $f_\ell : M_\ell \to N$are injective homomorphisms. Show that there are an L-structure N^* and an injective homomorphism f^* such that $M_1 \subseteq N^*$ and $f^* : M_2 \to N^*$.
(11) If in the previous question we have that $f_1[M_1] \cap f_2[M_2] = \emptyset$ then there are N^* and f^* such that in addition to the above requirement we have also that $f^*[M_1] \cap M_2 = \emptyset$.
(12) Let M be a given structure and suppose that N is a structure in the language of $\mathrm{CD}(M)$. If there exists an elementary mapping of M into $N \restriction \mathrm{L}(M)$ then $N \models \mathrm{CD}(M)$.
(13) In the last paragraph of the proof of the Downward Löwenheim-Skolem-Tarski theorem (Theorem 3.20) we could define A_{n+1} as B_n instead of taking A_{n+1} to be $A_n \cup B_n$, why?

(14) The goal of this exercise is to prove a generalization of Theorem 3.20.

THEOREM 3.36. *Let $N_1 \preceq N$ be given. For any $A \subseteq |N|$ there exists $M \preceq N$ of cardinality $\leq |A| + |\mathrm{L}(N)| + \aleph_0$ such that $|M| \supseteq A$ and $|M| \cap |N_1|$ is a universe of an elementary submodel of N.*

(15) Prove Lemma 3.18(2).

(16) The purpose of this exercise is to show that in Theorem 3.30 the hypothesis that $\{M_i \mid i < \alpha\}$ is an elementary chain can not be replaced by the following weaker assumption: $M_i \subseteq M_j$ and $M_i \equiv M_j$ for all $i < j < \alpha$. Find a similarity type L and a family $\{M_n \mid n < \omega\}$ of L-structures such that $M_n \subseteq M_{n+1}$ and $M_n \cong M_{n+1}$ for all $n < \omega$, but $M_0 \not\equiv M$ when $M := \bigcup_{n<\omega} M_n$. Hint: Solve Exercise 6.

(17) Let T be a first-order theory, prove that if T has only finite models then there exists a positive integer n such that $M \models T \implies \|M\| < n$.

(18) The purpose of this exercise is to establish that the Cantor-Bernstein theorem fails for elementary embbedings. Show that there are structures M and N such that there are $g : M \hookrightarrow N$ and $h : N \hookrightarrow M$ elementary embeddings but $M \ncong N$.

(19) Let M be a given structure. If $N \models \mathrm{ED}(M)$ then there exists an isomorphism of M onto a submodel of $N \restriction \mathrm{L}(M)$.

(20) Any elementary submodel of a simple group is simple.

(21) If $M \preceq N$ and $A \subseteq |M|$ then for every $n < \omega$ we have that $\mathrm{BA}^n_A(M) \cong \mathrm{BA}^n_A(N)$.

(22) Prove Theorem 3.20 using Proposition 3.9 directly without repeating the argument.

(23) Show that for every infinite cardinality λ there exists an algebraically closed field of cardinality λ.

(24) The purpose of this exercise is to show that for every field F there exists an algebraically closed field $E \supseteq F$ such that every element of E is algebraic over F (such a field is called *the algebraic closure of* F)[11].

  (a) Use to compactness theorem to show that for every field K there exists a field extension K^* such that for every polynomial from $K[x]$ has a root in K^*.

  (b) Let $\{K_n \mid n < \omega\}$ be an increasing chain of fields such that $K_0 = F$ and $K_{n+1} := (K_n)^*$ (i.e. every polynomial with coefficients from K_n has a root in K_{n+1}). Show that $K := \bigcup_{n<\omega} K_n$ is an algebraically closed field containing F.

  (c) Suppose K is an algebraically closed extension of F. Show that $E := \{a \in K \mid a \text{ is algebraic over } F\}$ is algebraically closed.

(25) Show that if E is an algebraic closure of the field F then $|E| \leq |F| + \aleph_0$.

(26) Suppose that E and F are algebraically closed fields of the same characteristic. Prove that there exists an elementary embedding from F into E

[11] The standard prof in algebra text-books uses Zorn's lemma to establish that every ideal can be extended to a maximal ideal for part (a), the advantage of the proof using the compactness theorem is that it is strictly weaker than Zorn's lemma.

or there is an elementary embedding from E into F. Hint: The case that both fields are countable requires understanding the proof of Steinitz's theorem **??**.

(27) Prove Theorem 3.33. Hint use the argument you have seen in the proof of Theorem 3.30.

(28) Let F and E be fields of characteristics zero. Is the following implication true? If F is a subfield of E then $F \preceq E$ when the language is the language of fields.

(29) The theory of atomless boolean algebras (see Exercise 2.42) is complete.

(30) Suppose that M is a model of the theory from Example 2.48(2) such that P^M is uncountable. Let $A \subseteq |M|$ and $B \subseteq |M| - P^M$ be infinite sets, prove that $\langle A \cup B, B\rangle$ is an elementary submodel of M.

(31) Let V and W be uncountable vector spaces over $\mathbb{Q}$.
Is necessarily $\langle V, +_V\rangle \equiv \langle W, +_W\rangle$?

(32) Let $M, N \models T_{ord}$. Prove that if $M \subseteq N$ then $M \preceq N$.

(33) Let $M, N \models T_{ind}$. Prove that if $M \subseteq N$ then $M \preceq N$.

(34) Let p be a prime or zero. Suppose that $M, N \models T_{ACF_p}$. Prove that if $M \subseteq N$ then $M \preceq N$.

(35) Suppose that $M \models T_{ord}$ is uncountable, show that there exists an elementary embedding from $\langle \mathbb{Q}, <\rangle$ into M.

(36) Let $E, F \models T_{ACF_0}$. If the cardinality of E is strictly less than the cardinality of the field F then there exists an elementary embedding from E into F.

(37) In the previous exercise the assumption that $|E| < |F|$ is not optimal, find a weaker condition on the fields, that is a necessary and sufficient condition for existence of an elementary embedding from E into F.

(38) (Hard) Define a finite language L and find an L-structure M of cardinality $\aleph_1$ that is a Jónsson algebra.

(39) Show that for every positive integer n there exists a Jónsson algebra of cardinality $\aleph_n$.

(40) Using basic properties of the constructible universe $\mathbf{L}$ one can show that:

$$ZF + \mathbf{V} = \mathbf{L} \vdash [\text{ for every cardinal } \lambda$$

$$\text{there exists a Jonsson algebra of cardinality } \lambda].$$

(41) Let T be a consistent theory, suppose that $\mathrm{L}(T)$ contains a unary predicate P and T has a model M such that P^M is an infinite set. Conclude that for every $\lambda \geq \aleph_0 + |\mathrm{L}(T)|$ there exists $N \models T$ such that the cardinality of P^N is exactly λ.

(42) Let T be a consistent theory, suppose that $\mathrm{L}(T)$ is a countable language containing a unary predicate P. If T has an uncountable model M such that P^M is countable then $\mathrm{I}(\aleph_1, T) \geq 2$.

(43) Let T be a theory and let $\lambda \geq \aleph_0 + |\mathrm{L}(T)|$. If T has an infinite model then there exists a model of cardinality λ such that every definable set is either finite or of cardinality λ.

(44) Suppose L contains a unary predicate P. If M is an L-structure such that $\|M\| > |P^M| \geq \aleph_0$ then there is no $N \equiv M$ such that $|N| - P^N$ is finite.

(45) Let L be a language and M and N both L-structures. Suppose that $M \precneqq N$. Let $L' := L \cup \{P\}$, when P is a new unary predicate. Consider the L'-structure $\mathfrak{A} := \langle |N|, |M|, F^N, R^N, c^N \rangle$. I.e. $P^{\mathfrak{A}} = |M|$. Show that if $\mathfrak{B} \equiv \mathfrak{A}$ then $P^{\mathfrak{B}}$ is a universe of an L-elementary submodel of $\mathfrak{B} \restriction L$.

I.e. If $N' := \langle |\mathfrak{A}|, F^{\mathfrak{B}}, R^{\mathfrak{B}}, c^{\mathfrak{B}} \rangle$ and $M' := \langle P^{\mathfrak{B}}, F^{\mathfrak{B}}, R^{\mathfrak{B}}, c^{\mathfrak{B}} \rangle$ then $M' \preceq N'$.

Hint: first verify that $M' \subseteq N'$ and consider L'-sentences of the form ψ^P for all $\psi \in \mathrm{Sent}(L)$ when

$$\psi^P := \begin{cases} \forall x[P(x) \to \varphi^P(x)] & \text{if } \psi \text{ is } \forall x \varphi(x)] \\ \exists x[P(x) \wedge \varphi^P(x)] & \text{if } \psi \text{ is } \exists x \varphi(x)] \\ [\varphi_1^P \wedge \varphi_2^P] & \text{if } \psi \text{ is } \varphi_1 \wedge \varphi_2 \\ \neg \varphi^P & \text{if } \psi \text{ is } \neg \varphi. \end{cases}$$

(46) Prove Theorem 3.31.

(47) (a) Every linearly ordered set is directed.
(b) Let S be a nonempty set, and let I be the power set of S. $\langle I, <_I \rangle$ is a directed set when $A <_I B$ is defined as $A \subseteq B$.

(48) For every uncountable directed set I there exists a countable subset I_0 which is also directed.

(49) Let M be an L-structure such that $\|M\| > |L| + \aleph_0$. Show that there exists an elementary chain $\{M_\alpha \mid \alpha < \|M\|\}$. Satisfying:
(a) $\alpha < \beta \implies M_\alpha \precneqq M_\beta$,
(b) $\|M_\alpha\| < \|M\|$ for all α and
(c) $M = \bigcup_{\alpha < \|M\|} M_\alpha$.
If you are using the coherence property (Lemma 3.10(2)) state where it is used.

(50) Let T be a first-order theory such that $\mathrm{L}(T)$ contains a unary predicate P and $\lambda \geq |\mathrm{L}(T)|$. Now suppose that there are $M, N \models T$ such that $M \precneqq N$, the set $P^M = P^N$ is infinite. Prove that if T is λ-categorical then there exists $M^* \models T$ of cardinality λ^+ such that $|P^{M^*}| = \lambda$.

Hint: Use Exercise 45 to show that we may assume that both models have cardinality λ.

(51) Show that the previous theorem is vacuously true when $\lambda > |\mathrm{L}(T)|$. I.e. prove that if T is categorical in some $\lambda > |\mathrm{L}(T)|$ and $P \in \mathrm{L}(T)$ then for any $M \preceq N$ both of cardinality λ we have for P^M infinite that $P^M = P^N$ implies $M = N$.

(52) Suppose $\{M_i \mid i < \alpha\}$ and $\{N_i \mid i < \alpha\}$ are elementary chains (for the same similarity type L). If $f_i : M_i \to N_i$ is an elementary embedding (for all $i < \alpha$) such that $i < j \implies f_i \subseteq f_j$ then $\bigcup_{i<\alpha} f_i$ is an elementary embedding from $\bigcup_{i<\alpha} M_i$ to $\bigcup_{i<\alpha} N_i$.

(53) Construct an example to show that the assumption "T does not have finite models" is necessary for Theorem 3.26.

The following group of exercises explore connections with closed unbounded sets. These are fundamental both in model theory and in set theory. As this is not used in elementary model theory, unless you are interested in connections with set theory, the reader to skip these exercise until the first mention of closed unbounded sets in the text.

(54) Suppose λ is a cardinal with uncountable cofinality and $f : \lambda \to \lambda$. Consider $M := \langle \lambda, f \rangle$, i.e. the language of M consists of a single unary function symbol interpreted in M by f. Show that there exists a limit ordinal $\delta < \lambda$ such that $\langle \delta, f \rangle$ is an elementary submodel of M.

(55) Let λ be an uncountable regular cardinal, $f : \lambda \to \lambda$ and $M := \langle \lambda, f \rangle$. Show that

$$\{\delta < \lambda \mid \langle \delta, f \rangle \prec M\} \text{ contains a closed unbounded set.}$$

(56) Let λ be an infinite cardinal. Show that

$$\{\delta < \lambda^+ \mid \lambda(1 + \delta) = \delta\} \text{ is a closed unbounded set.}$$

(57) Let M be an uncountable L-structure with $\|M\| > |L|$. If $f : M \cong N$ then there exists $M^* \prec M$ of smaller cardinality such that $f[|M^*|]$ is a universe of an elementary submodel of N.

(58) Let λ be an uncountable regular cardinal and $f : M \cong N$ when $|M| = |N| = \lambda > |\operatorname{L}(M)|$. Show that

$$\{\delta < \lambda \mid f : \langle \delta, F^M, R^M, c^M \rangle_{F,R,c \in \operatorname{L}(M)} \cong \langle \delta, F^N, R^N, c^N \rangle_{F,R,c \in \operatorname{L}(M)}\}$$

contains a closed unbounded subset of λ.

4. The Erdős-Rado Theorem

Starting at the mid-forties in a period of about 25 years Paul Erdős together with several collaborators (the most prominent among them were Richard Rado and András Hajnal) considered the possibility of doing combinatorics on uncountable sets. They developed a branch of set theory called "infinitary combinatorics" or "combinatorial set theory" (for an exposition see [**EHMR**]). The most important family of results of Erdős and his collaborators is called "partition calculus" or "Ramsey Theory". The starting points are Proposition 4.4 and Ramsey's Theorem. One of the more fundamental results of partition calculus is a theorem known as the Erdős-Rado theorem from 1952 (Erdős, Hajnal, and Rado in the introduction of their monumental paper [**EHR65**] state that essentially the same result (under GCH) was published earlier by Djuro Kurepa in 1935). Until the mid sixties combinatorial set theory was detached from the main subfields of mathematical logic. In the mid sixties Logicians discovered several connections: Michael Morley [**Mo4**] used the Erdős-Rado theorem in an omitting types theorem in model theory (Corollary 2.7.42 below), Robert Vaught in [**Va65**] (Theorem 2.12.7) used the same theorem to conclude a two cardinal theorem, Rowbottom discovered connections between infinite Ramsey theory and Chang's conjecture, Arhangel'skii ([**Ar69**]) in 1969 published his solution to Alexandroff's conjecture (it is the statement that every Hausdorff Lindelöf space that is first countable must be of cardinality no more than $2^{\aleph_0}$, his solution used the method of proof of the Erdős-Rado theorem). I view these applications to model theory, set theory and general topology to be a profound testimony to the insight Erdős and his friends had already in the forties and fifties. This section is dedicated to one of the fundamental theorems in infinitary combinatorics. While presenting it I am introducing the reader to several powerful ideas of model theory among them are: types and saturation.

One of the more fundamental principles of mathematics is the *pigeonhole principle* (also known as Dirichlet's principle). It has many formulations. The simplest infinite version is that "when you divide an infinite set into two pieces there is a piece containing infinitely many elements". A finite version of the pigeonhole principle can be stated as "for every $k < n < \omega$ and every $f : n \to k$ there are $j_1 < j_2 < n$ such that $f(j_1) = f(j_2)$". Recall that since every cardinal number is an ordinal and always we identify the ordinal α with the set of ordinals less than α i.e. when k is a natural number it stands for the set $\{0, \ldots, k-1\}$ the notation $f : n \to k$ makes sense.

The notion of regular uncountable cardinal is a generalization of the pigeonhole principle for uncountable sets.

DEFINITION 4.1. Let λ be an uncountable cardinal number. We say that λ *is a regular cardinal* iff for every $\kappa < \lambda$ every set S of cardinality λ and every $f : S \to \kappa$ there exists $H \subseteq S$ of cardinality λ such that the function f is constant on the set H.

NOTATION 4.2. Let μ be a cardinal, we denote by μ^+ the first cardinal that is strictly greater than μ, we also refer to μ^+ as the *successor cardinal* of μ. E.g. $\aleph_1$ is the successor of $\aleph_0$ and $\aleph_{\alpha+1}$ is the successor of $\aleph_\alpha$. An uncountable cardinal that is not regular is called *singular*. The cardinal $\aleph_\omega$ which is the least upper bound of $\{\aleph_n \ : \ n < \omega\}$, is the first singular cardinal (let $f : \aleph_\omega \to \omega$ defined by $f(\alpha) := Min\{n < \omega \mid \alpha < \aleph_n\}$, this function violates the condition from Definition 4.1).

PROPOSITION 4.3. *Suppose λ is an uncountable regular cardinal. For every family of sets $\{A_i \ : \ i < \lambda\}$ such that $i < j \implies A_i \subseteq A_j$ we have that for every $A \subseteq \bigcup_{i<\lambda} A_i$ of cardinality less than λ there exists $i_0 < \lambda$ such that $A \subseteq A_{i_0}$.*

PROOF. Let $\{A_i \ : \ i < \lambda\}$ such that $i < j \implies A_i \subseteq A_j$ be given. For the sake of contradiction suppose that there exists $A \subseteq \bigcup_{i<\lambda} A_i$ of cardinality $\kappa < \lambda$ and for every $i < \lambda$ there exists $a_i \in A$ such that $a_i \notin A_i$. Denote by $f : \lambda \to A$ the function $f(i) := a_i$. Since λ is regular and the range of f has cardinality $\kappa < \lambda$, there exists $S \subseteq \lambda$ of cardinality λ and there exists $a^* \in A$ such that for every $i \in S$ we have that $f(i) = a^*$. By the definition of f this imply that for every $i \in S$ the element $a^* \notin A_i$. But since the set S has cardinality λ, the set S is an unbounded subset of t λ. As $\{A_i\}$ is an increasing sequence of sets we have that for every $j < \lambda$ the element $a^* \notin A_j$ i.e. $a^* \notin \bigcup_{j<\lambda} A_j$ which is a contradiction to the hypothesis that $A \subseteq \bigcup_{i<\lambda} A_i$. □

Defining regular cardinals as the ones that satisfy the pigeonhole principle is not the usual way. The usual definition is closer to the conclusion of Proposition 4.3.

It turns out that in the presence of the Axiom of Choice there are plenty of regular cardinals:

PROPOSITION 4.4. *For every infinite cardinal μ the cardinal μ^+ is a regular cardinal.*

PROOF. Let $\kappa < \mu^+$ be given (notice that from the definition of successor cardinal it follows that $\kappa \leq \mu$) and suppose that $f : \mu^+ \to \kappa$. For the sake of contradiction suppose that μ^+ is not regular namely if $S_i := f^{-1}(i)$ for every $i < \kappa$ then we have that $i < \kappa \implies |S_i| \leq \mu$.
From the assumptions it follows that

$$\mu^+ = \mathrm{Dom}(f) = \bigcup_{i<\kappa} S_i.$$

Now estimate the cardinalities in the last formula:

$$\mu^+ = |\mu^+| = |\operatorname{Dom}(f)| = |\bigcup_{i<\kappa} S_i| \leq \sum_{i<\kappa} |S_i|$$
$$\leq \sum_{i<\kappa} \mu = \kappa \cdot \mu = \mu$$

which is a contradiction. □

Notice that in the very last step we have used the axiom of choice (to conclude that $\kappa \cdot \mu = \max\{\kappa, \mu\}$). There are models of set theory where even $\aleph_1$ is singular. In fact there are models of set theory where every uncountable cardinal is singular (see Gitik [**Gi**]). Since already the compactness and the Downward Löwenheim-Skolem theorems involve usage of the axiom of choice it seems that we must accept the axiom of choice when working in model theory.

NOTATION 4.5. Let n be a positive integer, and let S be a linearly ordered set. Denote the set $\{(j_1, \ldots, j_n) \ : \ j_1 < \cdots < j_n \in S\}$ by $[S]^n$ and for a subset H of S let

$$[H]^n := \{(j_1, \ldots, j_n) \in {}^nH \ : \ j_1 < \cdots < j_n \in H\}.$$

For example we may view $[\omega]^2$ as the complete undirected graph whose set of vertices is ω and any pair of distinct natural numbers is connected by an edge.

DEFINITION 4.6. Let $n < \omega$ and suppose that λ, κ, μ are cardinal numbers (not necessarily infinite). We denote by $\lambda \to (\mu)^n_\kappa$ the following statement:
For every set S of cardinality λ and every function[12] $F : [S]^n \to \kappa$ there exists $H \subseteq S$ of cardinality μ and there exists $i_0 < \kappa$ such that for every $\mathbf{j} \in [H]^n$ we have that $F(\mathbf{j}) = i_0$.

The negation of $\lambda \to (\mu)^n_\kappa$ is denoted by $\lambda \not\to (\mu)^n_\kappa$.

For reasons that will become clear shortly we call the function F *coloring with κ-colors*, and a set satisfying the requirement from H is called *monochromatic for F* also the word *homogeneous* is used in this context.

The above partition relation is called *ordinary partition symbol*.

To the reader who finds the above definition of the relation $\lambda \to (\mu)^n_\kappa$ very complicated I have a word of comfort: In 1965 Erdős, Hajnal and Rado published a 103-page long paper titled "Partition relations for cardinal numbers" (see [**EHR65**]), I have chosen in this book to mention only three of the 14 different partition relations from [**EHR65**] and $\lambda \to (\mu)^n_\kappa$ is not the most complicated or the most general. For some additional partition relations see Definitions II.13.7 and II.13.9

[12]Recall that κ is not only a cardinal but also stands for the set of ordinals less than κ, e.g. when $\kappa = 2$ then F is a function into the set $\{0, 1\}$.

A related partition relation with exponent $< \omega$ that connects model theory and set theory will be be used in later sections.

NOTATION 4.7. $[S]^{<\omega}$ stands for $\bigcup_{n<\omega}[S]^n$.

DEFINITION 4.8.

$\lambda \to (\mu)^{<\omega}_{\kappa} \stackrel{def}{\iff}$ For every $F : [\lambda]^{<\omega} \to \kappa$ there exists $S \subseteq \lambda$ of cardinality μ such that the image of $[S]^n$ under F is a singleton for all $n < \omega$.

Proposition 4.4 can be restated using our notation as "for every $\kappa \leq \lambda$ we have that $\lambda^+ \to (\lambda^+)^1_\kappa$". The celebrated theorem of combinatorics that is known as the infinite Ramsey's theorem in the above "arrow" notation is:

THEOREM 4.9 (infinite Ramsey). $\aleph_0 \to (\aleph_0)^2_2$

It turns out that Ramsey's theorem and some of its relatives are of fundamental importance in several different areas of mathematical logic. It is interesting to point out that Ramsey's original paper from 1930 [**Ra30**] is titled "On a problem of formal logic". There is also a finite version of Ramsey's theorem, first the most basic version:

THEOREM 4.10 (finite Ramsey). *For every positive integer m there exists a positive integer $l(m)$ such that $l(m) \to (m)^2_2$.*

The following more general version (can be derived directly from Theorem 4.10):

THEOREM 4.11 (finite Ramsey's theorem). *For every m, n, k positive integers there exists a positive integer $l(m, n, k)$ such that $l(m, n, k) \to (m)^n_k$.*

In the next section the compactness theorem is used to derive the finite version of Ramsey's theorem from the Erdős-Rado theorem. The fact that it is possible to use model theory to transfer results about uncountable structures to finitary mathematics is an example of a fundamental phenomenon: *Model theory is a bridge between the mathematics of uncountable sets and finite mathematics*. Another example is the application of Theorem 3.26 to the decidability of the theory of the field of complex numbers.

An inspection of Definition 4.6 show that the following monotonicity properties hold:

LEMMA 4.12. *Let n, λ, κ and μ be cardinalities. Suppose that $\lambda \to (\mu)^n_\kappa$ holds. Then the following are true*

(1) *if $\lambda_1 \geq \lambda$ then $\lambda_1 \to (\mu)^n_\kappa$*
(2) *if $\mu_1 \leq \mu$ the $\lambda \to (\mu_1)^n_\kappa$*
(3) *if $\kappa_1 \leq \kappa$ then $\lambda \to (\mu)^n_{\kappa_1}$*

(4) *if* $n_1 \leq n$ *then* $\lambda \to (\mu)^{n_1}_{\kappa}$.

PROOF. Trivial. □

In 1933 the great Polish mathematician W. Sierpiński managed to show that the infinitary version of Ramsey's theorem can not be generalized to $\aleph_1$. Namely, in the statement of Theorem 4.9 it is not possible to replace $\aleph_0$ by $\aleph_1$[13]. More formally his result is:

THEOREM 4.13 (Sierpiński 1933). $\aleph_1 \not\to (\aleph_1)^2_2$.

PROOF. By Lemma 4.12(1) and Cantor's theorem ($2^{\aleph_0} \geq \aleph_1$) it is enough to show that $2^{\aleph_0} \not\to (\aleph_1)^2_2$. Fix $<^*$ to be a well-ordering of the reals. Denote by $<_{\mathbb{R}}$ the usual ordering of the reals. For every $a <^* b \in \mathbb{R}$ define a coloring as follows:

$$F(a,b) := \begin{cases} 0 & \text{if } a <_{\mathbb{R}} b \\ \\ 1 & \text{otherwise.} \end{cases}$$

For the sake of contradiction suppose that $2^{\aleph_0} \to (\aleph_1)^2_2$ holds. Thus there exists $S \subseteq \mathbb{R}$ of cardinality $\aleph_1$ such that F is constant on $[S]^2$.

In case the constant value of F is 0 then by the definition of F we have that for every $a, b \in S$ we have that $a <^* b \iff a <_{\mathbb{R}} b$. Since S has cardinality $\aleph_1$ we can pick an $<^*$-increasing sequence $\{a_i \,:\, i < \omega_1\}$ of elements of S. Now for every $i < \omega$ define $I_i := (a_i, a_{i+1})$. Since the order $<^*$ coincide on S with the usual ordering of the reals and $\{a_i\}$ is increasing we have that $i < j < \omega_1$ implies that $I_i \cap I_j = \emptyset$. Since the rationals $\mathbb{Q}$ are dense in $\mathbb{R}$ for every $i < \omega_1$ we can pick $q_i \in I_i \cap \mathbb{Q}$. Since the intervals are pairwise disjoint we have that $i \neq j$ implies $q_i \neq q_j$ which contradicts the fact that $\mathbb{Q}$ is countable.

In case the constant value is 1 then by the definition of F we have that $a, b \in S$ we have that $a <^* b \iff b <_{\mathbb{R}} a$. This gives a decreasing (in the order $<_{\mathbb{R}}$) sequence of reals of order-type ω_1 and as before we get a contradiction by constructing $\aleph_1$ many rationals. □

Notice that the above proof gives $2^{\aleph_0} \not\to (\aleph_1)^2_2$. A natural question to ask is: Does there exists a cardinal number λ such that $\lambda \to (\aleph_1)^2_2$? What we know is that if such a λ exists it must be strictly greater than $2^{\aleph_0}$. It is a corollary of the Erdős-Rado Theorem that $(2^{\aleph_0})^+ \to (\aleph_1)^2_2$ which together with Sierpiński's result mean that the smallest λ such that $\lambda \to (\aleph_1)^2_2$ is the cardinality $(2^{\aleph_0})^+$. It is quite remarkable that for $\aleph_1$ we can show that the upper-bound and lower-bound

[13]The same result was discovered independently by Duro Kurepa.

coincide while for finite numbers closing the gap between the upper-bounds and lower-bounds is considered one of the central open problems of combinatorics.

In Exercise 10 the reader is expected to show that Sierpiński's result can be generalized from $\aleph_1$ to any successor cardinal. This mean that a priori there may be another way to generalize Ramsey's theorem to uncountable cardinals (which is not ruled out by Sierpiński). This brings us to

DEFINITION 4.14. An uncountable cardinal χ is called *weakly-compact* iff $\chi \to (\chi)^2_2$ holds.

A reader who may be puzzled by the usage of the word compact will probably be interested in the following explanation. William Hanf (see [**Han**]) in his Berkeley thesis in 1964 discovered a surprising connection between the above partition property and infinitary logics (to be defined later): An uncountable cardinal χ is weakly-compact iff the logic $L_{\chi,\chi}$ satisfies a form of the compactness theorem (see Theorem 8.**??**).

A natural question to ask is: Do weakly compact cardinals exist?

In 1943 Erdős and Tarski published a result that implies the following:

THEOREM 4.15. *(*ZFC*) Let χ be an uncountable cardinal, if χ is weakly compact then the first-order theory* ZFC *has a model (namely* ZFC *is consistent).*

However Gödel's incompleteness theorem imply:

THEOREM 4.16. *From* ZFC *alone it is impossible to prove the consistency of* ZFC.

Combining the last two facts we get the following immediate

COROLLARY 4.17. ZFC $\nvdash \exists\chi$ *weakly compact.*

About 35 years later in 1977 Jeff Paris and Leo Harrington have shown that there is a strengthening of the finite Ramsey's theorem that imply the consistency of Peano's arithmetic, namely there is a partition theorem (for integers) that with respect to PA plays an analog role to $\exists\chi[\chi > \aleph_0 \wedge \chi \to (\chi)^2_2]$ with respect to ZFC. See chapter **??**.

Let us return to the Erdős-Rado theorem. Instead of proving $(2^{\aleph_0})^+ \to (\aleph_1)^2_2$ directly we prove a generalization. For its statement we need another definition:

DEFINITION 4.18. Let α be an ordinal and let λ be a cardinal. By induction on ordinals we define the cardinal number $\beth_\alpha(\lambda)$[14]

[14] $\beth$ is the second letter of the Hebrew alphabet and is called beth .

(1) if $\alpha = 0$ then let $\beth_\alpha(\lambda)$ to be λ,
(2) if α is a limit ordinal then let $\beth_\alpha(\lambda) := \bigcup_{i<\alpha} \beth_i(\lambda)$,
(3) if $\alpha = \beta + 1$ then $\beth_\alpha(\lambda) := 2^{\beth_\beta(\lambda)}$.

When $\lambda = \aleph_0$ instead of $\beth_\alpha(\aleph_0)$ we write $\beth_\alpha$.

Thus $\beth_1(\lambda)$ is just another way to write 2^λ and $\beth_2 = 2^{2^{\aleph_0}}$. An alternative way to state the generalized continuum hypothesis is: For every ordinal α we have that the equality $\beth_\alpha = \aleph_\alpha$ holds.

THEOREM 4.19 (The Erdős-Rado theorem). *For every natural number n and every infinite cardinal λ we have that*

$$\beth_n(\lambda)^+ \to (\lambda^+)^{n+1}_\lambda.$$

Before starting the proof we need to introduce a central model-theoretic concept which is a generalization of a first-order formula. This is a very important concept on its own, the first sections of Chapter 4 are dedicated to the study of this concept.

DEFINITION 4.20. Let L be a similarity type, let M be an L-structure, $A \subseteq |M|$ and $\mathbf{a} \in |M|$. The *type of* $\mathbf{a}$ *over* A *in* M is denoted by $\mathrm{tp}(\mathbf{a}/A, M)$ and it is the set $\{\varphi(\mathbf{x}; \mathbf{b}) \ : \ \varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{Fml}(L), \mathbf{b} \in A, M \models \varphi[\mathbf{a}; \mathbf{b}]\}$.

EXAMPLE 4.21. Suppose L is the language of fields, and suppose that M is a field. Given $A \subseteq |M|$ and $\mathbf{a} \in |M|$ then $\mathrm{tp}(\mathbf{a}/A, M)$ contains all the equations with coefficients from A in unknown $\mathbf{x}$ that $\mathbf{a}$ is a solution for.

PROPOSITION 4.22. *Suppose $M \preceq N$, $A \subseteq |M|$ and $\mathbf{a} \in |M|$. Then*

$$\mathrm{tp}(\mathbf{a}/A, M) = \mathrm{tp}(\mathbf{a}/A, N).$$

PROOF. Exercise 13. □

DEFINITION 4.23. Let L be a similarity type, let M be an L-structure, $A \subseteq |M|$ and let $\mathbf{x}$ be a fixed finite sequence of variables. Suppose that p is a subset of $\{\varphi(\mathbf{x}; \mathbf{b}) \ : \ \varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{Fml}(L), \mathbf{b} \in A\}$. We say that *$p$ is realized in M* iff there exists $\mathbf{a} \in |M|$ such that $p \subseteq \mathrm{tp}(\mathbf{a}/A, M)$ for such an $\mathbf{a}$ we denote $\mathbf{a} \models p$ and this reads as *p is realized by $\mathbf{a}$* .

REMARK 4.24. Suppose that $A \subseteq |M|$ and $\mathbf{a}, \mathbf{a}' \in |M|$. If $\mathbf{a}'$ realizes $\mathrm{tp}(\mathbf{a}/A, M)$ then $\mathrm{tp}(\mathbf{a}'/A, M) = \mathrm{tp}(\mathbf{a}/A, M)$.

The following Lemma will be used to prove the Erdős-Rado Theorem:

LEMMA 4.25. *Let N be a given structure, and let A be a subset of $|N|$. If λ is a cardinal such that $\lambda \geq 2^{|\,\mathrm{L}(N)|+\aleph_0+|A|}$ then there exists $M \preceq N$ containing A such that $\|M\| \leq \lambda$ and for every $\mathbf{a} \in |N|$ the type $\mathrm{tp}(\mathbf{a}/A, N)$ is realized in M.*

PROOF. Denote by Ψ the set $\{\mathrm{tp}(\mathbf{a}/A,N) \ : \ \mathbf{a}\in |N|\}$. Let's estimate the cardinality of the set Ψ: Since $\mathrm{tp}(\mathbf{a}/A,N)$ is a subset of $\{\varphi(\mathbf{x};\mathbf{b}) \ : \ \mathbf{b}\in A, \varphi(\mathbf{x};\mathbf{y})\in \mathrm{Fml}(\mathrm{L}(N))\}$, the cardinality of Ψ is no more than the cardinality of the power set of the last set.

It is not difficult to see that it is $\leq 2^{|\mathrm{Fml}(\mathrm{L}(N))|\cdot\sum_{n<\omega}|A|^n}$ which is bounded by $2^{|\mathrm{L}(N)|+\aleph_0+|A|}$ which by the assumption on λ gives that $|\Psi|\leq\lambda$. There exists an $\alpha_0\leq\lambda$ such that $\{\mathbf{a}_i \ : \ i<\alpha_0\}\subseteq |N|$ satisfying the next requirement: For every $p\in\Psi$ there exists $i<\alpha_0$ such that $p=\mathrm{tp}(\mathbf{a}_i/A,N)$.

Let $C:=\{\mathbf{a}_i \ : \ i<\alpha_0\}\cup A$. Clearly C's cardinality is no more than λ. Apply the Downward Löwenheim-Skolem-Tarski theorem to get $M\preceq N$ containing C of cardinality $\leq\lambda$. It is left to the reader to verify that M is as required. □

Let λ be an infinite cardinal number and suppose that μ is a cardinal (can be finite). Recall that the cardinal λ^μ is defined as the cardinality of the set of functions from A to B when A is a set of cardinality μ and B is a set of cardinality λ. The cardinal λ^μ has a combinatorial significance it is also the cardinality of the set $\{C\subseteq B \ : \ |C|\leq\mu\}$.

The following corollary is a strengthening of the Downward Löwenheim-Skolem-Tarski theorem where we require the submodel not only to be closed with respect to

$$N\models\exists x\varphi(x;\mathbf{a}) \implies \text{there exists } b\in|M| \text{ such that } N\models\varphi[b;\mathbf{a}]$$

for all $\varphi\in\mathrm{Fml}(\mathrm{L}(N))$ and every $\mathbf{a}\in|M|$, but also with respect types over sets of cardinality $\leq\mu$ from a fixed subset A of M.

COROLLARY 4.26. *Let N be a given structure, and let B be a subset of $|N|$. If λ is a cardinal such that $\lambda\geq 2^{|\mathrm{L}(N)|+\aleph_0}+|B|$ and μ is a cardinal such that $\lambda^\mu=\lambda$ then there exists $M\preceq N$ containing B such that $\|M\|\leq\lambda$ and for every $A\subseteq B$ of cardinality less or equal to μ we have that for every $\mathbf{a}\in|N|$ the type $\mathrm{tp}(\mathbf{a}/A,N)$ is realized in M.*

PROOF. By the assumption that $\lambda^\mu=\lambda$ we can fix an enumeration $\{B_i \ : \ i<\lambda\}$ of all the subsets of B of cardinality $\leq\mu$. Define $\{M_i\preceq N \ : \ i<\lambda\}$ such that

(1) for every i the cardinality of M_i is λ,
(2) if $i<j$ then $M_i\preceq M_j$, and
(3) for every $\mathbf{a}\in|N|$ the type $\mathrm{tp}(\mathbf{a}/B_i,N)$ is realized in M_{i+1}.

The construction of the chain is by transfinite induction on $i<\lambda$:

For $i=0$ use the downward Löwenheim-Skolem-Tarski theorem to get $M_0\preceq N$ of cardinality λ.

For i limit define M_i as $\bigcup_{j<i} M_j$. It is an elementary submodel of N by Theorem 3.31 and its cardinality is bounded by $|i| \cdot \lambda = \lambda$.

For $i = j + 1$ get M' by applying Lemma 4.25 to N and the set B_j, now apply the Downward Löwenheim-Skolem-Tarski theorem to N and the set $A := |M'|$.

Using Zermelo-Konig it follows that $\mathrm{cf}\, \lambda > \mu$ use this to verify that $M := \bigcup_{i<\lambda} M_i$ is as required. □

It is natural to ask: Is it possible to strengthen the previous corollary and get a precise analogous to the Downward Löwenheim Skolem theorem with respect to all types over a subset of M of cardinality $\leq \mu$? The answer is affirmative but first we prove the Erdős-Rado theorem.

PROOF. (of Theorem 4.19): By induction of $n < \omega$ show that for every infinite λ, the partition relation $\beth_n(\lambda)^+ \to (\lambda^+)^{n+1}_\lambda$ holds.

For $n = 0$ we have to show that $\lambda^+ \to (\lambda^+)^1_\lambda$ This mean that for every $F : \lambda^+ \to \lambda$ there exists $S \subseteq \lambda^+$ of cardinality λ such that F is constant on S. But this is true by Proposition 4.4.

Suppose that $n \geq 0$ is such that for every infinite λ,

$$\beth_n(\lambda)^+ \to (\lambda^+)^{n+1}_\lambda \quad \text{holds.}$$

We have to show that for every infinite λ we that

$$\beth_{n+1}(\lambda)^+ \to (\lambda^+)^{n+2}_\lambda \quad \text{holds.}$$

Let λ be given. Denote by μ the cardinality $\beth_n(\lambda)$ we have to show that for every $F : [(2^\mu)^+]^{n+2} \to \lambda$ there is an F-monochromatic subset of $(2^\mu)^+$ of cardinality λ^+.

Let $N := \langle (2^\mu)^+, <, F, i\rangle_{i<\lambda}$ [15] thus the similarity type of N consists of a binary relation that is interpreted by the usual ordering on the ordinals, an $n + 2$-place function symbol, interpreted by F and a set $\{c_i \ : \ i < \lambda\}$ of constant symbols such that c_i is interpreted by the ordinal i.

Let $\{N_i \preceq N \ : \ i < \mu^+\}$ be such that

(1) for all i we have that $\|N_i\| \leq 2^\mu$,
(2) if $i < j$ then $N_i \preceq N_j$,
(3) for every $B \subseteq N_i$ of cardinality $\leq \mu$ and every $a \in |N|$ the type $\mathrm{tp}(a/B, N)$ is realized in N_{i+1}, and
(4) if i is limit then $N_i = \bigcup_{j<i} N_j$.

[15] Since $F(j_1 < \cdots < j_{n+2})$ is defined only for $j_1, \ldots, j_{n+2}$ we have to set a value for F also for non-increasing sequences. For $\langle \xi_1, \ldots, \xi_{n+2}\rangle$ that is not increasing
we set $F(\xi_1, \ldots, \xi_{n+2}) = 0$.

Construct the chain by induction on $i < \mu^+$: When $i = 0$ let N_0 be an arbitrary elementary submodel of cardinality 2^μ. When $i = j + 1$ apply Corollary 4.26 to get $A := N_j$ to get N_{j+1}. When i is limit just define $N_i := \bigcup_{j<i} N_j$ by the inductive hypothesis and Theorem 3.31 N_i is as required.

Let $M := \bigcup_{i<\mu^+} N_i$. Since $\|M\| \leq 2^\mu \cdot \mu^+ = 2^\mu$ and the fact that $|N| = (2^\mu)^+ > \|M\|$, by regularity of $(2^\mu)^+$ we have that
there exists $\alpha^* \in (2^\mu)^+ - Sup|M|$.

For $\{a_i : i < \mu^+\} \subseteq |N|$ denote by A_i the set $\{a_j : j < i\}$. By induction on $i < \mu^+$ define $a_i \in N_{i+1}$ as follows:

$$(*) \quad \mathrm{tp}(a_i/A_i, N) = \mathrm{tp}(\alpha^*/A_i, N).$$

This is possible by requirement (3).

Notice that by (*) we have that $i < j < \mu^+ \Longrightarrow a_i < a_j$. The reason is that since $\alpha^* > Sup|M|$ we have that $\alpha^* > \alpha_i$ so given $j > i$ since $a_i \in A_j$ we have that the formula $[x > a_i]$ is an element of $\mathrm{tp}(\alpha^*/A_j, N)$ and since a_j realizes the last type we have that $N \models [a_j > a_i]$.

Denote by A the set $\{a_i : i < \mu^+\}$.

LEMMA 4.27. *For every $i_1 < \cdots < i_{n+2} < \mu^+$ we have that*

$$F(a_{i_1}, \ldots, a_{i_{n+2}}) = F(a_{i_1}, \ldots, a_{i_{n+1}}, \alpha^*).$$

PROOF. Let $i_1 < \cdots < i_{n+2} < \mu^+$ be given. Let $j_0 < \lambda$ be such that $F(a_{i_1}, \ldots, a_{i_{n+1}}, \alpha^*) = j_0$. Since the language of N has constants for all the elements of λ by the definition of $\mathrm{tp}(\alpha^*/A_{i_{n+1}+1}, N)$ we have that the formula $[F(a_{i_1}, \ldots, a_{i_{n+1}}, x) = j_0]$ is
a member of $\mathrm{tp}(\alpha^*/A_{i_{n+1}+1}, N)$ since also $a_{i_{n+2}}$ realizes this type we have that $N \models [F(a_{i_1}, \ldots, a_{i_{n+2}}) = j_0]$. □

Now define $G : [A]^{n+1} \to \lambda$ as follows: For $i_1 < \cdots < i_{n+1} < \mu^+$ let $G(a_{i_1}, \ldots, a_{i_{n+1}}) := F(a_{i_1}, \ldots, a_{i_{n+1}}, \alpha^*)$. Since $|A| = \mu^+ = \beth_n(\lambda)^+$ by the induction hypothesis applied to the coloring G there exists $B \subseteq A$ of cardinality λ^+ and there exists $i_0 < \lambda$ such that for every $a_1 < \cdots < a_{n+1} \in B$ we have $G(a_1, \ldots, a_{n+1}) = i_0$.

We show now that the set B is monochromatic for the coloring F: It suffices to show that for every $a_1 < \cdots < a_{n+2} \in B$ we have that
$F(a_1, \ldots, a_{n+1}, a_{n+2}) = i_0$.

Let $a_1 < \cdots < a_{n+2} \in B$ be given. By Lemma 4.27 we have that
$F(a_1, \ldots, a_{n+1}, a_{n+2}) = F(a_1, \ldots, a_{n+1}, \alpha^*)$ by the definition of G we get
$F(a_1, \ldots, a_{n+1}, \alpha^*) = G(a_1, \ldots, a_{n+1})$.
By the assumption on B we have that $G(a_1, \ldots, a_{n+1}) = i_0$. □

There are two important ideas in the proof of the Erdős-Rado theorem. The first is the use of types and elementary submodels and the other is the notion of *end-homogeneous* set, it is a set satisfying the statement of Lemma 4.27, namely the coloring does not depend on the last coordinate. I will demonstrate this by using end-homogenuity to prove the infinite Ramsey's theorem.

PROOF OF THEOREM 4.9. We use induction on $n < \omega$ to prove that $\aleph_0 \to (\aleph_0)^n_k$ holds for all $k < \omega$. Suppose $F : [\omega]^{n+1} \to k$ is given.

By induction $i < \omega$ define $a_i < \omega$ and infinite sets $S_i \subseteq \omega$ such that

(1) $a_i < a_{i+1}$, and $S_{i+1} \subseteq S_i$,
(2) $a_{i+1} = \min S_i$,
(3) For every $j_1 < \cdots < j_n < i$ we have that
$$F(a_{j_1}, \ldots, a_{j_n}, a_{i+1}) = F(a_{j_1}, \ldots, a_{j_n}, j) \quad \text{for all } j \in S_{i+1}.$$

For $i = 0$ we take $S_0 := \omega$. Suppose S_i is defined. For every $j, j' \in S_i$ consider

$$j \equiv j' \iff F(a_{j_1}, \ldots, a_{j_n}, j) = F(a_{j_1}, \ldots, a_{j_n}, j') \text{ for every } j_1 < \cdots j_n < a_i.$$

As $\equiv$ is an equivalence relation with finitely many equivalence classes there exists an infinite equivalence class $S_{i+1} \subseteq S_i$.

Now let $G(j_1, \ldots, j_n) := F(a_{j_1}, \ldots, a_{j_n}, a_{i+1})$. By the induction hypothesis applied to G there exists $H \subseteq \omega$ homogeneous for G, verify that H is homogeneous also for F. □

This proof can be easily adapted to prove that every measurable cardinal is weakly compact. Also a careful computation of of the number of $\equiv$ classes in the previous proof can be used to give a proof of the finite Ramsey's theorem.

The following definition should be compared with Definition 3.8.

DEFINITION 4.28. Suppose $M \preceq N$, and let $\mu \leq \|M\|$. The model M *is a μ-relatively saturated submodel of N* iff for every $A \subseteq |M|$ of cardinality less than μ and every $\mathbf{a} \in N$ there exists $a' \in M$ realizing the type $\mathrm{tp}(\mathbf{a}/A, N)$.

The following is a generalization of the downward Löwenheim-Skolem-Tarski theorem since its proof is similar to that of the Erdős-Rado theorem we have included it:

THEOREM 4.29. *Let N be a given structure, let $A \subseteq N$ and let μ be an infinite cardinal number. If λ is a cardinality satisfying $\lambda = \lambda^\mu$ and $\lambda \geq |A|$ then there exists M a μ^+-relatively saturated submodel of N containing the set A of cardinality λ.*

PROOF. Construct a chain $\{N_i \preceq N \ : \ i < \mu^+\}$ such that

(1) $|N_0| \supseteq A$,
(2) if $i < j$ then $N_i \preceq N_j$,
(3) for every $i < \lambda^+$ we have that $\|N_i\| = \lambda$,
(4) if i is limit than $N_i = \bigcup_{j<i} N_j$,
(5) for every $i < \mu^+$ and every $B \subseteq |N_i|$ of cardinality $\leq \mu$ and every $\mathbf{a} \in |N|$ the type $\mathrm{tp}(\mathbf{a}/B, M)$ is realized in N_{i+1}.

Construct the chain by induction on $i < \mu^+$: For $i = 0$ let N_0 be an arbitrary elementary submodel of N of cardinality λ containing the set A. For i successor apply Corollary 4.26 to get N_i. When i is limit just define $N_i := \bigcup_{j<i} N_j$, by the inductive hypothesis and Theorem 3.31 N_i is as required. Define M to be $\bigcup_{i<\mu^+} N_i$. By Theorem 3.31 $M \preceq N$. To show that it is μ^+-relatively saturated let $B \subset |M|$ be of cardinality $\leq \mu$ and consider the type $\mathrm{tp}(\mathbf{a}/B, N)$ for some $\mathbf{a} \in |N|$. Since μ^+ is regular by Proposition 4.3 there exists $i < \mu^+$ such that $B \subseteq |N_i|$. Now by requirement (5) $\mathrm{tp}(\mathbf{a}/B, N)$ is realized in N_{i+1}. □

It is interesting to note that the idea of the proof of the last theorem can be used to get an alternative proof to Arhangelski's solution to Alexandroff's conjecture about the cardinality of compact Hausdorff spaces that satisfy the first countability axiom. See Exercise 22.

COROLLARY 4.30. *Let N be a given structure, let $A \subseteq N$ and let μ be an infinite cardinal such that $2^\mu \leq \|N\|$. Then there exists a μ^+-relatively saturated submodel M of N of cardinality 2^μ such that $|M| \supseteq A$.*

PROOF. Let $\lambda := 2^\mu$ and apply the last theorem. □

Exercises 1.4.

(1) Recall the relation $\equiv$ on finite sequences of rationals, introduced in Exercise 1.2.16 . Verify that

$$\mathbf{a} \equiv \mathbf{b} \iff \mathrm{tp}(\mathbf{a}/\emptyset, \mathbb{Q}) = \mathrm{tp}(\mathbf{b}/\emptyset, \mathbb{Q}) \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbb{Q}.$$

(2) Let M be an L-structure, $A \subseteq |M|$ and $\mathbf{c}, \mathbf{d} \in {}^{n}|M|$. Show that if $\mathrm{tp}(\mathbf{c}/A) \neq \mathrm{tp}(\mathbf{d}/A)$ then there exists a formula $\varphi(\mathbf{x}; \mathbf{y})$ with $\ell(\mathbf{x}) = n$ and $\mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A}$ such that $\varphi(\mathbf{x}; \mathbf{a}) \in \mathrm{tp}(\mathbf{c}/A)$ but $\neg\varphi(\mathbf{x}; \mathbf{a}) \in \mathrm{tp}(\mathbf{d}/A)$.

(3) Let M be an L-structure, $B \subseteq A \subseteq |M|$ and $\mathbf{c}, \mathbf{d} \in {}^{n}|M|$. Show that if $\mathbf{d} \models \mathrm{tp}(\mathbf{c}/B)$ then $\mathrm{tp}(\mathbf{c}/B) \subseteq \mathrm{tp}(\mathbf{d}/A)$.

(4) Prove Lemma 4.12.

(5) Often the infinite Ramsey theorem is stated as:

$$\forall n < \omega \; \forall k < \omega \quad \aleph_0 \to (\aleph_0)^n_k.$$

Derive the last statement from $\aleph_0 \to (\aleph_0)^2_2$.?

?!

(6) Let λ be an uncountable cardinal. If for every $\mu < \lambda$ we have that $\lambda \to (\lambda)^2_\mu$ holds then λ is a regular cardinal.

(7) Show that every infinite partially ordered set has an infinite chain or an infinite antichain.

(8) Let λ be a cardinal such that $\lambda \to (\lambda)^2_2$ show that this imply the following partition theorems:
   (a) For every $\kappa < \lambda$, we have that $\lambda \to (\lambda)^2_\kappa$ holds.
   (b) For every $n < \omega$, $\lambda \to (\lambda)^{n+1}_2$ holds.
   (c) For every $\kappa < \lambda$ and every $n < \omega$, $\lambda \to (\lambda)^{n+1}_\kappa$ holds.

(9) Generalize Sierpiński's result: For every infinite λ we have that $2^\lambda \not\to (\lambda^+)^2_2$.

(10) Prove that $\aleph_\omega \not\to (\aleph_\omega)^2_\omega$.

(11) Let λ be an uncountable cardinal such that $\lambda \to (\lambda)^2_2$, prove that λ must be regular.

(12) Use Exercise 9 to show that if $\lambda > \aleph_0$ satisfies $\lambda \to (\lambda)^2_2$ then λ is a strong limit (λ is *strong limit* iff $\forall \mu < \lambda [2^\mu < \lambda]$).

(13) Prove Proposition 4.22.

(14) Show that $\beth_\omega \to (\mu)^2_\omega$ holds for every $\mu < \beth_\omega$. Generalize this to show: If λ is strong limit then

$$\lambda \to (\mu^+)^n_\kappa \quad \text{for every } \mu, \kappa < \lambda \text{ and every } n < \omega.$$

(15) Let M be a structure, an equivalence relation R on the set $|M|$ is called *bounded* if there exists a cardinal number $\kappa < \|M\|$ such that the cardinality of $|M|/R$ is at most κ. Show that if $\{R_i \mid i < \alpha < \|M\|\}$ is a set of bounded equivalence relations then $R := \bigcap_{i<\alpha} R_i$ is a bounded equivalence relation.

Hint: Let $\kappa := |\alpha| + \sum_{i<\alpha} \kappa_i$ and use the Erdős-Rado theorem to show that R is bounded by $(2^\kappa)^+$.

(16) For an infinite cardinal μ let

$$\begin{aligned}\operatorname{Ded}\mu := \sup\{|I| \ : \ & \text{there exists a linearly ordered } I \\ & \text{with a dense subset } J \text{ of cardinality } \mu\}.\end{aligned}$$

Show that $\operatorname{Ded}\aleph_0 = 2^{\aleph_0}$ and that $\operatorname{Ded}\mu \leq 2^\mu$ for all μ.

(17) Let M be a structure and $\mathbf{a}$ a sequence of elements from $|M|$. Suppose that $\mathbf{c}$ is a sequence of constant symbols not in $\mathrm{L}(M)$. Show that

$$\{\varphi(\mathbf{c}) \mid \varphi(\mathbf{x}) \in \operatorname{tp}(\mathbf{a}/\emptyset, M)\}$$

is a complete theory in $L \cup \{\mathbf{c}\}$.

(18) Let T be a consistent theory that has an infinite model.
Let $\mathcal{K} := \{M \models T \ : \ \|M\| \geq 2^{\aleph_0}\}$ and set $\mathrm{LS}(\mathcal{K}) := 2^{\aleph_0}$, define $M \prec_{\mathcal{K}} N$ iff M is an $\aleph_0$-relatively saturated submodel of N. Show that $\langle K, \prec_{\mathcal{K}}\rangle$ is an abstract elementary class.

(19) Find an example of a theory T such that $\langle \mathcal{K}, \prec_{\mathcal{K}}\rangle$ is not an abstract elementary class. Where $\mathcal{K} := \{M \models T \ : \ \|M\| \geq 2^{\aleph_0}\}$ with $\mathrm{LS}(\mathcal{K}) = 2^{\aleph_0}$ and $M \prec_{\mathcal{K}} N$ iff M is an $\aleph_1$-relatively saturated submodel of N.

(20) Let M and $\mathbf{a} \in |M|$ be given then $\operatorname{tp}(\mathbf{a}/\emptyset, M)$ induces an ultrafilter in $\mathrm{BA}^n(M)$ when n is the length of the sequence $\mathbf{a}$.

(21) Let M be an L-structure. For a positive n and $A \subseteq |M|$ let L_A be the expansion of L with new the constants $\{c_a \ : \ a \in A\}$ and let M_A the expansion $\langle M, a\rangle_{a\in A}$ to L_A. Denote by $\mathrm{BA}^n_A(M)$ the Lindenbaum algebra $\mathrm{BA}^n(M_A)$. Using the notation of Exercise 2.3.83 show that for $\mathbf{a} \in {}^n|M|$ the function $f(\mathbf{a}) := \{[\varphi(\mathbf{x}; \mathbf{b})]/\approx : \ \varphi(\mathbf{x}; \mathbf{b}) \in \operatorname{tp}(\mathbf{a}/A, M)\}$ is a mapping from ${}^n|M|$ into the set of ultrafilters of the algebra $\mathrm{BA}^n_A(M)$.

(22) Prove the following strengthening of Theorem 4.29:

THEOREM 4.31. *Let N be a given structure, let $A \subseteq N$ and let λ be such that $\lambda \geq |\mathrm{L}(N)| + \aleph_0$ If μ is a cardinality satisfying $\mu = \mu^{<\lambda}$ then there exists M a λ-relatively saturated submodel of N containing the set A of cardinality μ.*

(23) *Alexandroff's conjecture*: Let X be a Hausdorff topological space satisfying the first countable axiom. If X is compact then $|X| \leq 2^{\aleph_0}$.

Hint: Suppose $|X| > 2^{\aleph_0}$. By first countability fix $\{U_n(x) \mid n < \omega\}$ a basic system of neighborhoods for every point $x \in X$.
Define $\{Y_\alpha \mid \alpha < \omega_1\}$ increasing and continuous chain with the following properties:
(a) $|Y_\alpha| = 2^{\aleph_0}$,
(b) for any countable $S \subseteq Y_\alpha$ if $\{U_n(x) \mid n < \omega, x \in S\}$ is not a cover of X then there exists $y \in Y_{\alpha+1} - \bigcup_{n<\omega, x\in S} U_n(x)$ and
(c) $\bar{Y}_\alpha \subseteq Y_{\alpha+1}$.
Let $Y := \bigcup_{\alpha<\omega_1} Y_\alpha$. Get a contradiction by showing that $Y = X$:

(a) Since Y is closed $\{X - Y\} \cup \{U_n(y) \mid n < \omega, y \in Y\}$ is an open cover of X since the space is Lindelöf there is a countable $S \subseteq Y$ such that $\{X - Y\} \cup \{U_n(y) \mid n < \omega, y \in S\}$ is a cover of X.

(b)

(24)

DEFINITION 4.32. Let $M \models T_{ord}$ be a model of cardinality $\aleph_\alpha$. We say that M is η_α iff for every $A \subseteq |M|$ of cardinality less than $\aleph_\alpha$ the following holds:

$$\exists b \forall a \in A[a \leq b].$$

Check that the real numbers is not an η_1 structure.

(25) If $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ then there exist an $\eta_{\alpha+1}$ model of T_{ord}.

(26) Suppose GCH (for simplicity). Show that there can not be an η_ω model.

(27) Is the converse to Exercise 20 true? Namely, is it true that for every ultrafilter F in $BA^n(M)$ there exists $\mathbf{a}_F \in |M|$ such that $F = \mathrm{tp}(\mathbf{a}_F/\emptyset, M)$?

5. Applications of the compactness theorem

This section contains several interesting models. Among them are: An ordered field satisfying all the first-order properties the reals have but non-Archimedean (hence not isomorphic to $\mathbb{R}$). Construction of $2^{\aleph_0}$ many countable and pairwise non-isomorphic models of true arithmetic. The purpose of this section is not only to present some interesting models for familiar theories but to emphasize the great importance of the compactness theorem that will be used repeatedly in future sections. It is possible to use the compactness theorem to get results that don't mention explicitly models or logic, e.g. James Ax's application to the complex numbers (see Theorem 5.12 below) and derive the finite Ramsey theorem from the Erdős-Rado Theorem.

Let me start with a property of ordered fields:

DEFINITION 5.1. Let T_{orf} be the following set of sentences in L_{orf}: It consists of the field axioms:

(1) $0 \neq 1$,

(2) $\forall x \forall y \forall z[x + (y + z) = (x + y) + z \wedge x \cdot (y \cdot z) = (x \cdot y) \cdot z]$,

(3) $\forall x[x + 0 = x]$,

(4) $\forall x \exists y[x + y = 0]$,

(5) $\forall x[x \cdot 1 = x]$,

(6) $\forall x[x \neq 0 \to \exists y[x \cdot y = 1]]$,

(7) $\forall x \forall y[x + y = y + x \wedge x \cdot y = y \cdot x]$,

(8) $\forall x \forall y \forall z[x \cdot (y + z) = x \cdot y + x \cdot z]$

with the following sentences

(9) $\forall x \forall y[x < y \vee y < x \vee x = y]$,

(10) $\forall x \forall y \forall z[x < y \wedge y < z \to x < z]$,

(11) $\forall x[\neg(x < x)]$,

(12) $\forall x \forall y \forall z[x < y \to x + z < y + z]$,

(13) $\forall x \forall y \forall z[x < y \wedge z > 0 \to x \cdot z < y \cdot z]$ and

(14) $\forall x \forall y \forall z[x < y \wedge z < 0 \to x \cdot z > y \cdot z]$.

T_{orf} is called the *theory of ordered fields*. When $F \models T_{orf}$ we say that F *is an ordered field*.

REMARK 5.2. Let $L_{orf'}$ be $\{+, -, \cdot, {}^{-1}, 0, 1\}$, define $T_{orf'}$ as T_{orf} when axiom (4) is replaced by
$\forall x[x - x = 0]$ and axiom (6) is replaced by $\forall x[x \neq 0 \to x \cdot x^{-1} = 1]$.

Notice that if $M \models T_{orf'}$ then $M \models T_{orf}$ and if $M \models T_{orf}$ then there are functions $f, g : |M| \to |M|$ such that $\langle |M|, +^M, G^M, \cdot^M, f, 0^M, 1^M \rangle \models T_{orf'}$.

So $T_{orf'}$ as another axiomatization of the theory of ordered fields. The advantage of this observation is that we avoid using the existential quantifier. I.e. this show that T_{orf} has a universal axiomatization. This will be used in Theorem 13.29

$\langle \mathbb{R}, +, \cdot, <, 0, 1 \rangle$ is an example for a model of the theory T_{orf}. Using axiom (12) one can show that when $F \models T_{orf}$ then F must have characteristic zero. It is an elementary exercise in complex numbers to show that the structure $\langle \mathbb{C}, +, \cdot, 0, 1 \rangle$ can not be expanded by a binary relation such that the expansion is a model of T_{orf}.

Often in analysis the following fundamental property of the real numbers is used: Let $a \in \mathbb{R}$ be a non-negative number. In order to show that $a = 0$ it is enough to prove that for every positive integer n we have that $a < 1/n$. This property of the real numbers is called the *Archimedean axiom* .

DEFINITION 5.3. Let $F \models T_{orf}$. We say that the field F has the *Archimedean property* if for every $a \in F$ if $F \models [a > 0]$ then there exists a positive integer n such that

$$F \models [\underbrace{a + a + \cdots + a}_{n\text{-times}} \geq 1].$$

An ordered field which fails to have the Archimedean property is called *non-Archimedean field*.

Notice that since there is no obvious way to define the positive integers in an ordered field, the Archimedean requirement as stated in the previous definition can not be translated directly into a first-order statement in the language of ordered fields. Is it possible that there is an indirect definition? Namely it is natural to ask the following

QUESTION 5.4. Does there exists a first-order theory $T' \supseteq T_{orf}$ such that if $M \models T'$ then M is an Archimedean field?

The following theorem provides a strong negative answer.

THEOREM 5.5. *There exists a field F which is an elementary extension of the field $N := \langle \mathbb{R}, +, \cdot, <, 0, 1 \rangle$ such that F is non-Archimedean.*

PROOF. Let $T := \mathrm{CD}(N)$. Let c be a new constant symbol not in the language of $\mathrm{CD}(N)$, let $L' := \mathrm{L}(T) \cup \{c\}$. Define

$$T' := T \cup \{1 < c, 1 + 1 < c, \ldots, \underbrace{1 + 1 + \cdots + 1}_{n\text{-times}} < c, \ldots\}.$$

By the compactness theorem T' has a model: Let T^* be a finite subset of T'. The set T^* contains finitely many sentences about the constant c. Suppose n is the maximal natural number such that $\underbrace{1 + 1 + \cdots + 1}_{n\text{-times}} < c$ appears in T^*. It is easy to

see that the expansion M^* of M to L' when $c^{M^*} := n+1$ is a model of T^*. Hence T' has a model M'. Let $a \in |M'|$ be the element interpreting the new constant c. Since for every positive n w have that $M' \models \underbrace{1+1+\ldots+1}_{n\text{-times}} < c$ the element a is greater then every natural number. Thus M' is a non-Archimedean field. Take F to be the reduct of M' to L_{orf}. □

Now let us consider number theory. Since the discovery by ancient Babylonians of the general solution to the quadratic equation, properties of the integers are considered one of the main areas of mathematics. In this century one may say that number theory is the gold standard of mathematics. Often mathematical progress is judged by its effect on Number theory.

The most popular axiomatization of elementary number theory is known as Peano's Arithmetic. The axioms can be formulated in predicate calculus (requiring infinitely many axioms).

DEFINITION 5.6 (Peano's Arithmetic). The similarity type of *Peano's arithmetic* is denoted by L_{PA} it is the set $\{+, \cdot, 0, 1\}$ and the set of axioms denoted by PA is:

$$\begin{aligned}
&0 \neq 1,\\
&\neg\exists x[x+1=0],\\
&\forall x[x+0=x],\\
&\forall x[x\cdot 0=0],\\
&\forall x\forall y[x+(y+1)=(x+y)+1,\\
&\forall x\forall y[x+1=y+1\to x=y],\\
&\forall x\forall y[(x+1)\cdot y=x\cdot y+y]
\end{aligned}$$

together with the set

$$\left\{\psi_\varphi \ : \ \varphi(x;y_0,\ldots,y_{n-1})\in \mathrm{Fml}(L_{PA})\right\}.$$

Where ψ_φ is the *induction axiom* for $\varphi(x;y_0,\ldots,y_{n-1})$, namely it is the following statement:
$\forall y_0\cdots\forall y_{n-1}\big[\varphi(0;y_0,\ldots,y_{n-1})$
$\wedge\,\forall x[\varphi(x;y_0,\ldots,y_{n-1})\to\varphi(x+1;y_0,\ldots,y_{n-1})]\to\forall x\varphi(x;y_0,\ldots,y_{n-1})\big]$.

One of Gödel's important discoveries is that the consistency of PA can be formulated using a single sentence in L_{PA} we write $Con(\mathrm{PA})$ for this statement. From this observation Gödel derived what is known as Gödel's third incompleteness theorem:

FACT 5.7 (Gödel's third incompleteness theorem). $\mathrm{PA} \nvdash Con(\mathrm{PA})$.

It is an easy exercise to show that ZF $\vdash Con(\mathrm{PA})$ just show that ω with the natural operations on it is a model of PA. That is ZF $\vdash [N \models \mathrm{PA}]$ where $N := \langle \omega, +, \cdot, 0, 1 \rangle$. The model N is called *The standard model of Peano's arithmetic* . Any model $M \models \mathrm{PA}$ that is not isomorphic to N is called a *non-standard model of Peano's arithmetic*.

For $x, y \in \omega$ recall that $x|y$ means that y is divisible by the number x. $x \nmid y$ stands for the negation of $x|y$. Notice that there is a first-order formula in L_{PA} which express this property, it is the formula: $\exists z[y = x \cdot z]$.

THEOREM 5.8. *Let P be the set of prime numbers. For every $S \subseteq P$ there exists $M_S \models \mathrm{PA}$ such that there is an element $a_S \in |M_S|$ with the following property:*

$$M_S \models p|a_S \iff p \in S \quad \textit{for every } p \in P.$$

Moreover, in addition we can require that $M_S \succ N$.

PROOF. Let c be a new constant. Let $N = \langle \omega, +, \cdot, 0, 1 \rangle$, define

$$T' = \mathrm{CD}(N) \cup \{p|c \ : \ p \in S\} \cup \{p \nmid c \ : \ p \in P - S\}.$$

When p stands for the term $1 + \cdots + 1$ (p many times),
$p|c$ is the sentence $\exists z[c = p \cdot z]$ and $p \nmid c$ is its negation. Clearly it is enough to find a model $M' \models T'$ since the interpretation of c is the required element in $M_S := M' \restriction L_{\mathrm{PA}}$. Apply the compactness theorem to T': Let T^* be a finite subset of T'. Suppose $p_0, \ldots, p_n \in S$ and $q_0, \ldots, q_k \in P - S$ are the elements of P such that

$$T^* \subseteq \mathrm{CD}(N) \cup \{p_0|c, \ldots, p_n|c\} \cup \{q_0 \nmid c, \ldots, q_k \nmid c\}.$$

Let $a := p_0 \cdot \cdots \cdot p_n$, by the fundamental theorem of arithmetic (=every positive integer has a unique factorization to powers of distinct primes) we have that

$$q_0 \nmid a, \ldots, q_k \nmid a \quad \text{(in the standard model) } N.$$

Hence M^* which is the expansion of N to $L_{\mathrm{CD}(N)} \cup \{c\}$ when the interpretation of c in M^* is a is a model of T^*. □

REMARK 5.9. Notice that the proof of the last theorem provides us with models of true arithmetic.

The following show existence of a countable non-standard model of PA.

COROLLARY 5.10. *There exists a countable non-standard elementary extension of the standard model of Peano's arithmetic.*

PROOF. Apply Theorem 5.8 to $S = P$ (the set of all prime numbers). Let M_P be an elementary extension of N such that there exists $a_P \in M_P$ with the property that $p \in P \iff M_P \models p|a_P$. By the Downward Löwenheim-Skolem-Tarski we may assume that M_P is countable (by taking an elementary submodel containing a_P). Show that $N \ncong M_P$: Suppose $f : M_P \to N$ is an isomorphism. Notice that

$f(1_{M_P}) = 1_N$, hence for every $p \in P$ we have the $f(p) = p$. Let $k \in \omega$ be such that $f(a_P) = k$. Since there are infinitely many primes and $k \in \omega$ there exists a prime number $p > k$. Certainly $N \models [p|k]$

(1) But the choice of a_P we have that $M_P \models p|a_P$.
(2) Since $p = f(p)$ and $k = f(a_P)$).

(1) and (2) contradict the statement of Claim2.8. □

COROLLARY 5.11 (The number of countable models of arithmetic). $\mathrm{I}(\aleph_0, \mathrm{PA}) = 2^{\aleph_0}$. *Moreover, true arithmetic has* $2^{\aleph_0}$*-many countable non-isomorphic models.*

PROOF. Let T be true arithmetic. Suppose for the sake of contradiction that there exists an infinite $\mu < 2^{\aleph_0}$ such that $\mathrm{I}(\aleph_0, T) \leq \mu$.
Let $\{M_i \ : \ i < \alpha \leq \mu\}$ be countable models of T such that for every countable $N \models T$ there exists $i < \alpha$ such that $N \cong M_i$.

Let $\{M_S \ : \ S \subseteq P\}$ be models of T from the proof of Theorem 5.8. By the choice of $\{M_i \ : \ i < \alpha \leq \mu\}$ for every S there exists $i < \alpha$ such that $M_S \restriction \mathrm{L(PA)} \cong M_i$. Namely for every S there exists $i < \alpha$ such that M_S is isomorphic to a structure from $S_i := \{\langle M_i, a\rangle \ : \ a \in |M_i|\}$. Since $|S_i| = \aleph_0$ we have

$$|\{M_S \ : \ S \subseteq P\}/\cong| \leq |\bigcup_{i<\alpha} S_i| \leq \sum_{i<\alpha} |S_i| = \sum_{i<\alpha} (\|M_i\| \cdot \aleph_0)$$
$$= \sum_{i<\alpha} \aleph_0 \leq |\alpha| \cdot \aleph_0 \leq \mu \cdot \aleph_0 = \mu.$$

□

I would like to bring to the attention of the reader that while there are $2^{\aleph_0}$-countable pairwise non-isomorphic models of PA (or even of true arithmetic) all the countable nonstandard models have the same order type. See Exercise 8.

The nature of non-standard models of Peano's arithmetic got much attention. Let $\varphi(x, y) := \exists z[x + z = y]$. Notice that

$$\begin{aligned}\mathrm{PA} \vdash & \forall x \forall y[\varphi(x,y) \vee \varphi(y,x)] \wedge \\ & \forall x_1 \forall x_2 \forall x_3[\varphi(x_1,x_2) \wedge \varphi(x_2,x_3) \to \varphi(x_1,x_3)] \wedge \\ & \forall x \forall y[\varphi(x,y) \wedge \varphi(y,x) \to x = y] \wedge \\ & \forall x[\varphi(0,x)].\end{aligned}$$

Namely, for any $M \models \mathrm{PA}$ the formula $\varphi(x, y)$ induces a linear order on $|M|$ with 0^M as its first element. When dealing with models of PA we use $x < y$

as a shorthand for $\varphi(x, y)$. It is not difficult to verify that

$$(*) \quad \begin{aligned} \mathrm{PA} \vdash & \forall x \forall y [x < y \to x + 1 < y + 1] \wedge \\ & \forall x \forall y \forall z [x < y \to x \cdot z \leq y \cdot z] \wedge \\ & \forall x [\neg \exists z [x < z < x + 1]] \wedge \\ & \forall x [x < x + 1]. \end{aligned}$$

By $(*)$ for every $M \models \mathrm{PA}$ there exists $f : \omega \to |M|$ injective such that $f(0) = 0^M$ and $f(n) = \underbrace{1^M + \cdots + 1^M}_{n\text{-times}}$. It is easy to show that f also preserves multiplication. $\mathrm{Range}\, f$ is often called the *standard integers of* M. $\mathrm{Range}\, f$ is usually identified with ω. An element $a \in |M|$ is *non-standard integer of* M iff $a \notin \omega$.

The common denominator to the applications of the compactness we have seen so far is that the compactness theorem was used to produce a new model with some surprising properties. The compactness theorem can be used also in quite different circumstances. The following is an example[16]:

THEOREM 5.12 (J. Ax 1966 and A. Grothendieck 1968). *Let $\mathbb{C}$ be the field of the complex numbers, and let n be a positive integer.*
For every polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ if f is injective then f is surjective.
We call f polynomial mapping if there are polynomials $p_1, \ldots, p_n \in \mathbb{C}[x_1, \ldots, x_n]$ such that for every n-tuple $\mathbf{a}$ of complex numbers we have that

$$f(\mathbf{a}) = (p_1(\mathbf{a}), \ldots, p_n(\mathbf{a})).$$

The above theorem is a non-trivial fact and its proof (in complex analysis) is quite involved. Below we will present a proof (due to James Ax) which uses the compactness theorem to get a simple proof. For a classical proof see S. Cynk and K. Rusek [**CyRu**]. A more recent simple proof using the Baire category theorem was found by Walter Rudin [**Ru**].

DEFINITION 5.13. Let T_f stand for the *theory of fields* namely it is the set axioms from $T_{or\,f}$ the language containing $\{+, \cdot, 0, 1\}$ only (no occurrence of the relation symbol $<$).

PROPOSITION 5.14 (A. Robinson). *Let φ be a sentence in the theory of fields. If φ is true in fields with arbitrarily large characteristic then φ is true in a field of characteristic zero.*

PROOF. Let

$$T := \{\varphi\} \cup T_f \cup \{1 + 1 \neq 0, 1 + 1 + 1 \neq 0, \ldots, 1 + \cdots + 1 \neq 0, \ldots\}.$$

Clearly any $F \models T$ is a field as required. By the compactness theorem in order to show that T has a model it is enough to show that every finite $T^* \subseteq T$ has a

[16] Another example of this kind is Theorem 5.21.

model. Let $T^* \subseteq T$ be a given finite subset, and let n be the largest integer such that $1 + 1 + \cdots + 1$ (n times) appears in T^*. By the hypothesis there exists a field $K \models \varphi$ of characteristic greater than n. I.e., $K \models T^*$. □

It is easy to see that the last Proposition can be generalized to

THEOREM 5.15. *Let $T \supseteq T_f$ in the language of fields, and suppose that Ψ is a set of sentences in the language of fields. If for every natural number n there exists $K_n \models T \cup \Psi$ of characteristic $\geq n$ then there exists $F \models T \cup \Psi$ of characteristic zero.*

PROOF. Exercise. □

In order to present our proof to Theorem 5.12 we need to recall an additional fact from algebra

FACT 5.16. *If F is a finite field of characteristics p then its algebraic closure (the minimal field containing F which is algebraically closed) is isomorphic to $\bigcup_{n<\omega} \mathbb{F}_{p^{n!}}$.*

Now we are ready to prove Theorem 5.12:

PROOF. Notice that the hypothesis of the Theorem can be formalized in the language of fields:
Define

$$\deg(f) := \max\{\deg(p_1), \ldots, \deg(p_n)\}.$$

For every positive integer d there is a first-order sentence ψ_d which say that: "for every polynomial mapping $f(x_1, \ldots, x_n)$ of degree $\leq d$ and we have if f is injective then f is surjective". Now let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a given polynomial mapping of degree d which is injective. Let $T := \{\psi_d\} \cup T_f$. Notice that if F is a finite field then $F \models T_f$ and $F \models \psi_d$ - since by the pigeonhole principle for finite sets we have that any injective mapping of F^n into F^n is surjective (remember that F^n is finite). Hence for every prime number p and every $n < \omega$ we have that $F_{p_n} \models T$. By Fact 5.11 also the algebraic closure of F_{p_n} satisfies T. In fact the algebraic closure of F_{p_n} also satisfies T_{ACF}. Notice that this is true for every prime p. By Proposition 5.9 there exists $F \models \{\psi_d\} \cup T_{ACF}$. In particular $\{\psi_d\} \cup T_{ACF_0}$ is a consistent theory. Recall that from Theorem 3.26 and Fact 2.38 it follows that T_{ACF_0} is a complete theory hence $T_{ACF_0} \models \psi_d$. Since $\mathbb{C} \models T_{ACF_0}$ we have that $\mathbb{C} \models \psi_d$. Since the polynomial mapping f appears in ψ_d it is surjective. □

The following application – a proof of Theorem 4.10 is included here in order to emphasize that model theory permits us not only to use facts about finite structures to prove results about infinite structures but it is possible to use it to derive results about finite structures from uncountable models.

PROOF OF THEOREM 4.11. Instead of proving the statement: For every $m < \omega$ there exists an integer $l(m) < \omega$ such that $l(m) \to (m)^2_2$ we prove the stronger statement (which is equivalent):

$$\forall m < \omega \, \forall n < \omega \, \forall k < \omega \, \exists l(m,n,k) < \omega \quad [l(m,n,k) \to (m)^n_k].$$

For the sake of contradiction suppose that there are $m, n, k < \omega$ such that for every $l < \omega$ have that $l \not\to (m)^n_k$. Namely for every $l < \omega$ there exists a coloring $F_l : [l]^n \to \{0,1\}$ such that for all $a_1 < \cdots < a_m < l$ there exist
$\mathbf{b} = \{b_1 < \cdots < b_n\} \subseteq \{a_1, \ldots, a_m\}$ and
$\mathbf{b}' = \{b'_1 < \cdots < b'_n\} \subseteq \{a_1, \ldots, a_m\}$ such that $F(\mathbf{b}) \neq F(\mathbf{b}')$.

Let $\lambda := \beth^+_{n-1}$. Let L be the similarity type $\{<, F, P\} \cup \{c_i \ : \ i < \lambda\}$ where c_i's are constant symbols, F is an m-place function symbol, P a unary predicate, and $<$ a binary predicate.

Let N be the structure $\langle \omega, <, n \rangle_{n<\omega}$ i.e. we have constant symbols for all natural numbers.

Let

$$\begin{aligned} T_F := &\operatorname{Th}(N) \cup \{c_i \neq c_j \ : \ i \neq j < \lambda\} \cup \{P(c_i) \ : \ i < \lambda\} \\ &\cup \{c_n = n \ : \ n < \omega\} \cup \{\psi_F\}. \end{aligned}$$

Where ψ_F is the following sentence:

$$\begin{aligned} F : [P]^n \to &\{0, 1, \ldots k-1\} \wedge \forall x_0 \cdots \forall x_{m-1} [\bigwedge_{k<m} P(x_k) \wedge \exists y_1 \cdots \exists y_n \exists y'_1 \cdots \exists y'_n \\ &\bigwedge_{i<j} [y_i < y_j \wedge y'_i < y'_j] \wedge \bigwedge_{1 \leq j \leq n} \bigvee_{\substack{k<m \\ k'<m}} [y_j = x_k] \wedge [y'_j = x_{k'}] \wedge \\ &\wedge F(y_1, \ldots, y_n) \neq F(y'_1, \ldots, y'_n)]. \end{aligned}$$

Where $F : [P]^n \to \{0, 1, \ldots k-1\}$ stands for "F is a function from the set $[P]^n$ into the set $\{0, 1, \ldots k-1\}$".

Use the hypothesis and the compactness theorem to show that T_F has a model (for a finite subset of T_F there is always an expansion of N that satisfy it).

Suppose $M \models T_F$. From T_F it follows that $F^M : [P^M]^n \to k$. By the λ many constants $|P^M| \geq \beth^+_{n-1}$. Since $M \models \psi_F$ have that F^M exemplify that $\beth^+_{n-1} \not\to (m)^n_k$ which is a contradiction to the Erdős-Rado theorem. □

The next application is in group theory.

DEFINITION 5.17. Let T_{gr} be the following set of sentences in L_{gr}:
$\{\forall x \forall y \forall z [x \cdot (y \cdot z) = (x \cdot y) \cdot z], \ \forall x [x \cdot e = x], \ \forall x \exists y [x \cdot y = e]\}$.

It should be clear that $\mathrm{Mod}(T_{gr})$ is the category of all groups.

DEFINITION 5.18. Let $G \models T_{gr}$. An element $a \in G$ is called of *finite order* if there exists a positive integer n such that $G \models a^n = e$. The group G is called *periodic* if every element of G is of finite order. A group is called *torsion-free* if the only element of finite order is the identity element.

In light of the above definition it is natural to ask: Does there exist a first-order theory $T \supseteq T_{gr}$ such that if $G \in \mathrm{Mod}(T)$ then G is periodic?

PROPOSITION 5.19. *The following periodic group*

$$G := \bigoplus_{1 \leq n < \omega} \mathbb{Z}/n\mathbb{Z}$$

has a non-periodic elementary extension.

PROOF. Let c be a new constant symbol, use the compactness theorem to show that the theory

$$T := \mathrm{CD}(G) \cup \{\underbrace{c + \cdots + c}_{n \text{ times}} \neq 0_G \ : \ 1 \leq n < \omega\}$$

is consistent. □

The next result is another example of applying the compactness theorem to a specific given structure (like Theorem 5.12). It concerns the concept of ordered group:

DEFINITION 5.20 (The theory of ordered groups). Let $L_{orgr} := \{\cdot, <, e\}$. The *theory of ordered groups* is denoted by T_{orgr} it consists of

- T_{gr},
- $<$ is a strict linear order and the following axiom

$$\forall x \forall y \forall z \big[x < y \wedge z > e \to [x \cdot z < y \cdot z] \wedge [z \cdot x < z \cdot y]\big],$$

THEOREM 5.21 (Robinson). *Let $G \models T_{gr}$. If every finitely generated subgroup H of G can be expanded by a binary relation $<^H$ such that*

$$\langle H, \cdot, <^H, e\rangle \models T_{orgr}$$

then there exists a binary relation $<^G$ on G such that

$$\langle G, \cdot, <^G, e\rangle \models T_{orgr}.$$

PROOF. Notice that it suffices to find $M \models T_{orgr}$ such that $G \subseteq M \restriction \{\cdot, e\}$. To accomplish this by Lemma 3.18(1) it is enough to show that

$$\mathrm{ED}(G) \cup T_{orgr} \quad \text{is a consistent theory.}$$

For this use the hypothesis on G and the compactness theorem. □

It turns out that the following innocent-looking application of the compactness theorem captures much of the power of the compactness theorem. More on this see Theorem II.7.38 and Corollary II.7.42.

Recall that by definition a linearly ordered set $\langle P, < \rangle$ is *well-ordered* iff every non-empty subset S of P has a first element which is equivalent to: There is no $\{a_n \ : \ n < \omega\} \subseteq P$ such that for all $n < \omega$ we have that $a_{n+1} < a_n$.

THEOREM 5.22. *For every infinite well-ordered set $\langle P, < \rangle$ there exists an elementary extension that is not well-ordered.*

PROOF. Let $\{c_n \ : \ n < \omega\}$ be new constant symbols. Define T to be the set $\mathrm{CD}(\langle P, < \rangle) \cup \{[c_{n+1} < c_n] \ : \ n < \omega\}$. Use the compactness theorem to show that T has a model. $\square$

Exercises 1.5.

(1) Prove Lemma
(2) Show that there is no expansion M of the field of complex numbers $\langle \mathbb{C}, +, \cdot, 0, 1\rangle$ to L_{orf} such that $M \models T_{orf}$.
(3) Show that $F \models T_{orf}$ is not Archimedean iff there exists $a \in F$ such that

$$F \models a > \underbrace{1 + 1 + \cdots + 1}_{n\text{-times}} \text{ for all positive } n.$$

(4) Let $R(x, y, z)$ be a three-place predicate symbol, in the language $L := \{R(x, y, z)\}$ define a set of sentences T that is an equivalent axiomatization to group theory. Namely for every $M \models T_g$ there is $M' \models T$ such that $M' = \langle |M|, R^{M'}\rangle$ and for every $N \models T$ there exists $N^* \models T_g$ such that $N^* = \langle |N|, \cdot^{N^*}\rangle$ such that $(M')^* = M$ and $(N^*)' = N$ for all M and N.
(5) Suppose that M is a proper elementary extension of $\langle \omega, +, \cdot, 0, 1\rangle$. Show that there exists $a \in |M|$ such that for every $k < \omega$ we have that $M \models \underbrace{1 + \cdots + 1}_{k\text{-times}} < a$ holds.
(6) Suppose that $M \models \text{PA}$ show that

$$M \models \forall x \exists y \big[x < y \wedge \forall z[z > x \to z \geq y]\big]$$

and

$$M \models \forall x \Big[x > 0 \to \exists y \big[y < x \wedge \forall z[z > y \to z \geq x]\big]\Big].$$

(7) Suppose that $M \models \text{PA}$. Show that there exists a formula $\varphi(x, y)$ in the language L_{PA} such that $\varphi(x, y)$ induces a linear order on the universe of M that extends the natural order of the natural numbers. Denote this ordering by $<^M$. Prove that there is an order-preserving mapping g from ω into $\langle |M|, <^M\rangle$ such that $g(0)$ is the first element of $\langle |M|, <^M\rangle$.
(8) (Hard) Suppose M and N are nonstandard models of PA and $<^M$, $<^N$ are the orderings induced by φ from the previous exercise. Let $M' := \langle |M|, <^M\rangle$ and $N' := \langle |N|, <^N\rangle$. Prove that if M and N are countable then
$M' \cong N'$.
(9) Prove that there exists a first-order formula $\varphi(x, y)$ in the language of fields such that for every $a, b \in \mathbb{R}$ we have that $\langle \mathbb{R}, +, \cdot\rangle \models \varphi[a, b] \leftrightarrow a < b$ when $<$ is the usual order on the reals.
(10) Suppose that if M is a proper elementary extension of $\langle \mathbb{R}, +, \cdot\rangle$ then M has an expansion which is a non-archimedean ordered field.
(11) Let ψ be a sentence in the language of fields. Suppose that: ψ is true is every field of characteristic zero. Show that there exists an integer n such that for every prime $p \geq n$ the sentence ψ is true in the finite field with p elements.

(12) Let M be an L-structure and $\varphi(x, y)$ an L-formula. Suppose that for every $n < \omega$ there exists $\{a_i \mid i < n\} \subseteq |M|$ such that

$$M \models \varphi[a_i, a_j] \iff i < j \text{ for all } i, j < n.$$

Show that there exists $N \equiv M$ and $\{a_n \mid n < \omega\} \subseteq |N|$ such that

$$N \models \varphi[a_i, a_j] \iff i < j \text{ for all } i, j < \omega.$$

Moreover, for any linearly order set I there exists $N_I \equiv M$ that contains elements$\{a_i \mid i \in I\}$ satifying

$$N_I \models \varphi[a_i, a_j] \iff i <_I j \text{ for all } i, j \in I.$$

(13) Let M be an L-structure and $\varphi(x, y)$ an L-formula. Suppose that for every $n < \omega$ there exists $\{a_i \mid i < n\} \subseteq |M|$ such that

$$M \models \exists y[\bigwedge_{i \in w} \varphi(a_i, y) \wedge \bigwedge_{j \in n - w} \neg\varphi(a_j, y)] \text{ for all } w \subseteq n.$$

Informally speaking the formula φ allows us to code the power set of n by elements of M.

Show that for any infinite set X there exists $N_X \equiv M$ and $\{a_i \mid i \in X\} \subseteq |N_X|$ such that there are $\{b_w \mid w \subseteq X\} \subseteq |N|$ such that

$$N \models \varphi[a_i, b_w] \iff i \in w \text{ for all } w \subseteq X.$$

(14) Let ψ be a sentence in the language of fields. If ψ holds in the field of complex numbers is the conclusion of the previous exercise still valid?

(15) Prove the following

Theorem For every $\langle P <\rangle$ be a partially ordered set there exists a relation $R \subseteq P \times P$ such that $\langle P, R\rangle$ is a linearly ordered set satisfying: $a < b \implies aRb$ for all $a, b \in P$.

(16) Let $G := \langle V, E\rangle$ be a graph (namely $E \subseteq [V]^2$). For a positive integer k we say that the graph G is *k-colorable* iff there exists $f : V \to \{1, \dots, k\}$ such that $\{a, b\} \in E \implies f(a) \neq f(b)$ for all $a, b \in V$. Prove:

Theorem A graph G is k-colorable iff every finite subgraph of G is k-colorable.

(17) Find a non-commutative locally finite group that has a non-periodic elementary extension. A group G is called *locally finite* iff for every finite $S \subseteq G$ the subgroup of G generated by S is finite.

(18) Show that $\mathrm{I}(2^{\aleph_0}, \mathrm{Th}(\langle \mathbb{R}, +, \cdot, 0, 1\rangle)) \geq 2$.

(19) Suppose that M is a proper elementary extension of $\langle \mathbb{R}, +, \cdot, 0, 1\rangle$, show that $M \ncong \langle \mathbb{R}, +, \cdot, 0, 1\rangle$.

(20) Let M be a non-standard model of PA. Prove that the standard model $\langle \omega, +, \cdot, 0, 1\rangle$ is isomorphic to a submodel of M.

(21) Let M be a proper elementary extension of $\langle \omega, +, \cdot, 0, 1\rangle$, show that $M \ncong \langle \omega, +, \cdot, 0, 1\rangle$.

(22) Let M be a proper elementary extension of $\langle \omega, +, \cdot, 0, 1\rangle$. Prove that the set $|M| - \omega$ does not have a minimal element with respect to the natural ordering induced by arithmetic (see Exercise 7).
(23) Suppose that M is a proper elementary extension of $\langle \mathbb{R}, +, \cdot\rangle$. Show that $\mathbb{Q}$ is not a dense subset of $|M|$.
(24) Let M be a countable non-standard model of PA. Denote by $<$ the linear ordering from Exercise 7. Show that there exists an order preserving mapping from $\langle \mathbb{Q}, <^{\mathbb{Q}}\rangle$ into $\langle |M|, <^M\rangle$.
(25) Let M be a non-standard model of PA. Show that the standard integers of M are not definable. I.e. there is no $\varphi(x) \in \mathrm{Fml}(L_{PA})$ such that $\omega = \varphi(M)$. What happens if we permit parameters from M to appear in φ?
(26) Let M be a countable non-standard model of PA. Let $a \in |M| - \omega$, show that a is not definable. I.e. there is no $\varphi(x) \in \mathrm{Fml}(L_{PA})$ such that $\{a\} = \varphi(M)$. What happens if we permit parameters from M to appear in φ?
(27) Let F be an algebraically closed field. Suppose that F is not the algebraic closure of its prime field. Show that the set $\{a \in F \mid a \text{ is transcendental}\}$ is not definable.
(28) Let T be a consistent theory. Show that for every $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ there exists $M_\lambda \models T$ such that every definable subset of M_λ is either finite or of cardinality λ.
(29) Let L be a similarity type, and suppose that L contains a unary predicate P. If M is an L-structure such that P^M is infinite then for every $\lambda \geq \|M\| + |L|$ there exists $N \succ M$ of cardinality λ such that the cardinality of the set P^N is λ.
(30) Suppose T is categorical in some $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ and $M \models T$ of cardinality λ. Prove that for every $\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))$ and every $\mathbf{a} \in {}^{\ell(\mathbf{y})}|M|$ if $\varphi(M; \mathbf{a})$ is infinite then $|\varphi(M; \mathbf{a})| = \lambda$.
(31) Let $L' = L_{\mathrm{PA}} \cup \{c\}$ when c is a new constant symbol. Let P be the set of prime numbers. Show that for every $S \subseteq P$ there exists a countable L'-structure M_S such that $M_S \restriction L_{\mathrm{PA}} \succ \langle \omega, +, \cdot, 0, 1\rangle$ and for every $S_1, S_2 \subseteq P$ we have that $S_1 \neq S_2$ implies that $M_{S_1} \ncong M_{S_2}$.
(32) Prove the following lemma:

LEMMA 5.23. *Let T be a first-order theory in the language L. Let μ be a cardinal number (maybe finite), let λ be an infinite cardinal $\geq |L|$ such that $\lambda^\mu = \lambda$. If C is a set of new constants (not in L) of cardinality μ and $T_C \supseteq T$ is any theory in $L \cup C$ then*

$$\mathrm{I}(\lambda, T_C) > \lambda \implies \mathrm{I}(\lambda, T_C) = \mathrm{I}(\lambda, T).$$

Hint: Use a generalization of the argument encountered in the proof of Corollary 5.11.
(33) Show that the converse of Theorem 5.12 is false. Namely there exists a surjective polynomial from $\mathbb{C}$ to $\mathbb{C}$ which is not injective.

Why the argument used to prove Theorem 5.12 can't be adapted to prove the converse?

(34) Prove Theorem 5.10 (repeat the proof of Proposition 5.8).

(35) Prove that every field F is a subfield of an algebraically closed field. (Hint: use the following algebraic fact (Kronecker's lemma): for every $g(x) \in F[x]$ there exists a field $K \geq F$ such that $g(x)$ factors to linear factors in $K[x]$).

(36) Find a countable model M for Presburger's arithmetic such that $M \ncong \langle \mathbb{Z}, +, 0 \rangle$.

(37) Prove Theorem 5.18 (Hint: Use Theorem 5.16).

(38) Let $\langle P, < \rangle$ be a partially ordered set. Prove that there exists a binary relation $<^*$ extending $<$ (i.e. for every $a, b \in P$ we have that $a < b$ implies that $a <^* b$) such that $\langle P, <^* \rangle$ is linearly ordered. Hint: For a finite P argue by induction on $|P|$. For P infinite use the compactness theorem.

(39)

DEFINITION 5.24. Let $\mathcal{F}$ be a family of sets.

A function $g : \mathcal{F} \to \bigcup \mathcal{F}$ is called a *transversal* iff g is an injective choice function i.e.

$$s \in \mathcal{F} \implies g(s) \in S \text{ and } s_1 \neq s_2 \in \mathcal{F} \implies g(s_1) \neq g(s_2).$$

Suppose that $\mathcal{F}$ is a family of finite sets. Show that: If every finite $\mathcal{F}' \subseteq \mathcal{F}$ has a transversal then there exists a transversal for $\mathcal{F}$.

(40) Let M be a structure with definable groups such that $H \leq G$. Namely there are formulas $\varphi(x),\ \psi(x),\ \varphi_1(u, x, y; \mathbf{z})$ in $\mathrm{L}(M)$, $\mathbf{c} \in |M|$ such that $H = \psi(M)$ and $G = \varphi(M)$ and $\{(f, g, h) \in G \mid M \models \varphi_1[f, g, h; \mathbf{c}]\}$ defines a group operation on the elements of G (via $fg = h$). Suppose that the index of H in G is infinite. Show that for every $\lambda \geq \aleph_0$ there exists $M' \equiv M$ such that $[G' : H'] \geq \lambda$ where $G' := \varphi(M')$ and $H' = \psi(M')$.

(41) Let M be a structure with a definable subgroups $H_n \leq G$. Namely there are formulas $\varphi(x; \mathbf{y}), \psi_n(x; \mathbf{z}_n)$ in $\mathrm{L}(M)$, $\mathbf{a} \in |M|$ and $\{\mathbf{b}_n \mid n < \omega\} \subseteq |M|$ such that $H_n = \psi_n(M; \mathbf{b}_n)$ for all $n < \omega$ and $G = \varphi(M; \mathbf{a})$. Suppose that $[H_n : N_{n+1}] \geq \aleph_0$ for all n. Show that for every $\lambda \geq \aleph_0$ there exists $M' \equiv M$ with definable subgroups $H'_{n+1} \leq H'_n \leq G'$ such that $[H'_n : H'_{n+1}] \geq \lambda$ for all n, where $G' := \varphi(M'; \mathbf{a}')$ and $H'_n = \psi_n(M'; \mathbf{b}'_n)$ for some $\mathbf{a}' \in M'$ and $\{\mathbf{b}'_n \mid n < \omega\} \subseteq |M'|$.

(42) Let $L = \{P\}$ when P is a unary predicate, for $n < \omega$ define

$$\varphi_n := \exists x_0 \cdots \exists x_{n-1} \big[\bigwedge_{i<n} \neg P(x_i) \wedge \bigwedge_{i<j<n} x_i \neq x_j \big].$$

and

$$\psi_n := \exists x_0 \cdots \exists x_{n-1} \big[\bigwedge_{i<n} P(x_i) \wedge \bigwedge_{i<j<n} x_i \neq x_j \big].$$

Show that

$$T := \{\varphi_n, \psi_n \mid n < \omega\} \cup \{\exists x P(x), \exists x \neg P(x)\}$$

is a complete theory.

(43) (Hard) Let T be a countable $\aleph_0$-categorical first-order theory. Show that for every countable $M \models T$ there is a family $\{M_n \mid n < \omega\}$ of countable models of T satisfying

$$M = \bigcap_{n<\omega} M_n \quad \text{and} \quad M \prec M_{n+1} \precneqq M_n \text{ for all } n < \omega.$$

There is no error in the previous line. Hint: This problem will be revisited in Proposition 2.8.7.

(44) Prove the following result:

PROPOSITION 5.25 (Lefschetz principle). *Let φ be a sentence in the language of fields L_f. The following are equivalent:*

(a) *φ holds in the field of complex numbers.*
(b) *$T_{ACF_0} \vdash \varphi$.*
(c) *The set of primes $\{p \mid T_{ACF_p} \vdash \varphi\}$ is cofinite.*
(d) *The set of primes such that there exists an algebraically closed field of given characteristics satisfying φ is co-finite.*
(e) *φ holds in every algebraically closed field of characteristics zero.*

(45) Use Lefschetz principle to derive an alternative proof of Theorem 5.12. In fact establish that the following stronger result holds:

THEOREM 5.26. *Let $V \subseteq \mathbb{C}^n$ be an algebraic set*[17]*. If $f : V \to V$ is an injective polynomial mapping then f is surjective.*

(46) The goal of this exercise[18] is to outline of an alternative proof of Corollary 5.11. Let $\mathcal{F} := \{M_S \mid S \text{ is a set of primes}\}$ be the countable models from Theorem 5.8.

(a) For sets of primes S_1, S_2, define $S_1 \sim S_2$ iff $M_{S_1} \cong M_{S_2}$. Show that this is an equivalence relation and each equivalence class has at most $\aleph_0$ members.
(b) Since there $2^{\aleph_0}$ many sets of primes, there are $2^{\aleph_0}$ pairwise non-isomorphic members of $\mathcal{F}$.

[17] See Definition 4.1

[18] This exercise was suggested to me by Peter LeFanu Lumsdaine.

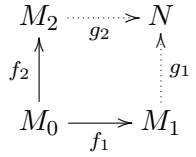
6. Joint embedding and the Amalgamation properties in first-order logic

The main result of this section is that when T is complete then $\mathrm{Mod}(T)$ has the amalgamation property with respect to elementary embeddings.

In the fifties Abraham Robinson identified two important properties of classes of models and embeddings and investigated them in the context of elementary classes. As in the seventies and eighties they found deep use on also other classes of structures, we present the definitions at the level of generality of Abstract Classes and the rest of this section will be dedicated to first–order logic.

DEFINITION 6.1. Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract class of similarity type L, and let $\lambda \geq \aleph_0 + |L|$.

(1) We say that $\mathcal{K}$ *has the* λ*-Amalgamation Property* if for every $M_\ell \in \mathcal{K}$ of cardinality λ (for $\ell = 0, 1, 2$) and every $f_\ell : M_0 \to M_\ell$ ($\ell = 1, 2$) $\mathcal{K}$-embeddings then there exists $N \in \mathcal{K}$ of cardinality λ, and $g_\ell : M_\ell \to N$ $\mathcal{K}$-embeddings such that the diagram



commutes. I.e. $g_1 \circ f_1 \restriction M_0 = g_2 \circ f_2 \restriction M_0$.

(2) We say that $\mathcal{K}$ *has the* λ*-Joint Embedding Property* if for every $M_\ell \in \mathcal{K}$ of cardinality λ (for $\ell = 1, 2$) there exists $N \in \mathcal{K}$ of cardinality λ, and $g_\ell : M_\ell \to N$ $\mathcal{K}$-embeddings. When the above requirement holds without any restriction on the cardinalities of the models we say that $\mathcal{K}$ *has the Joint Embedding Property* or $\mathcal{K}$ *has the JEP*.

(3) For a theory T we say that it has the JEP iff the class $\mathrm{Mod}(T)$ has it.

It is the contents of the following two results that when $\mathcal{K}$ satisfies: there is a complete first-order theory T such that $\mathcal{K} = \mathrm{Mod}(T)$ then $\mathcal{K}$ has the JEP (the requirement that T is complete is necessary) and the amalgamation property holds.

Both results were obtained by Abraham Robinson in 1956.

PROPOSITION 6.2 (Robinson). *Let T be a complete first-order theory. The class* $\mathrm{Mod}(T)$ *has the* JEP.

PROOF. For the next theorem we prove something little stronger than necessary. Suppose $T' \supseteq T$ (maybe $\mathrm{L}(T') \supseteq \mathrm{L}(T)$) Let $M_1 \models T'$ and $M_2 \models T$. Enough to show that $T^* := \mathrm{CD}(M_1) \cup \mathrm{CD}(M_2)$ is consistent. Without loss of generality we may assume that $|M_1| \cap |M_2| = \emptyset$.

Using the compactness theorem since $\mathrm{CD}(M_\ell)$ are closed under finite conjunctions it is enough to show that

$$\varphi(c_0,\dots,c_n)\wedge\psi(d_0,\dots,d_n)\quad\text{is consistent}$$

where $\varphi(c_0,\dots,c_n)\in\mathrm{CD}(M_1)$ and $\psi(d_0,\dots,d_n)\in\mathrm{CD}(M_2)$. Clearly it suffices to show that

$$M_1\models\exists x_0\cdots\exists x_n\varphi(x_0,\dots,x_n)\wedge\exists y_0\cdots\exists y_n\psi(y_0,\dots,y_n).$$

$M_1\models\exists x_0\cdots\exists x_n\varphi(x_0,\dots,x_n)$ is by the choice of φ. By the assumption on ψ we have that $M_2\models\exists y_0\cdots\exists y_n\psi(y_0,\dots,y_n)$, since T is complete we get the desired $M_1\models\exists y_0\cdots\exists y_n\psi(y_0,\dots,y_n)$. □

PROPOSITION 6.3 (Robinson). *Let T be a complete first-order theory. The class* $\mathrm{Mod}(T)$ *has the AP.*

PROOF. Suppose that $M_0\prec M_\ell\models T$ for $\ell=1,2$.

By renaming if necessary some of the elements of M_1-M_0 we may assume that $M_1\cap M_2=M_0$. Suppose $\psi(\mathbf{x};\mathbf{z}),\varphi(\mathbf{y};\mathbf{z})$ are formulas and $\mathbf{a}\in M_1-M_0,\mathbf{b}\in M_2-M_0,\mathbf{c}\in M_0$.

Using the compactness theorem as in the proof of the previous proposition it suffices to establish that

$$\psi(\mathbf{a};\mathbf{c})\wedge\varphi(\mathbf{b};\mathbf{c})\text{ has a model.}$$

Use the assumptions that $M_0\prec M_2$ and $M_1\succ M_0$ to verify that

$$M_1\models\psi(\mathbf{a};\mathbf{c})\wedge\exists\mathbf{y}\varphi(\mathbf{y};\mathbf{c}).$$

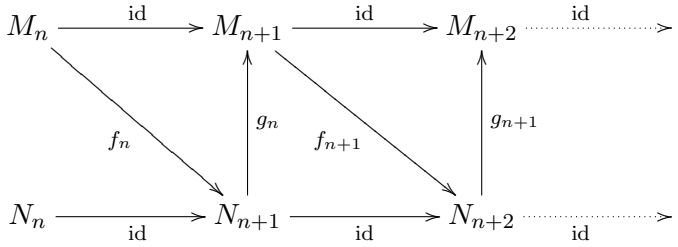
□

The amalgamation property can be derived also from the following more general theorem of Robinson (by taking $T=T_0=T_1=T_2$) note the use of the previous Proposition.

THEOREM 6.4 (Robinson's consistency property). *Let L_0,L_1,L_2 be similarity types such that $L_0=L_1\cap L_2$. Let T_ℓ be consistent theories in L_ℓ for $\ell=0,1,2$. If $T_0=T_1\cap T_2$ is complete theory in L_0 then $T_1\cup T_2$ has a model*

PROOF. Fix $M_0\models T_1$ and $N_0\models T_2$. By induction on $n<\omega$ define an increasing elementary chains $\{M_n\mid n<\omega\}$, $\{N_n\mid n<\omega\}$ and L_0-elementary embeddings $\{f_n:M_n\restriction L_0\to N_{n+1}\restriction L_0\ :\ n<\omega\}$
$\{g_n\ :\ N_{n+1}\restriction L_0\to M_{n+1}\restriction L_0\ :\ n<\omega\}$ such that for every $n<\omega$ the

following diagrams commute



Given M_n and N_n, since both $M_n \restriction L_0$ and $N_n \restriction L_0$ are models of T_0 and T_0 is complete by the proof of the JEP (Proposition 6.2) there exists N_{n+1} such that $N_{n+1} \succ N_n$ and an L_0-embedding $f_{n+1} : M_n \to N_{n+1}$. An application of the JEP to the pair M_n and N_{n+1} gives $M_{n+1} \succ M_n$ and an L_0-elementary embedding $g_n : N_{n+1} \to N_{n+1}$ such that that the above diagram commutes.

Let $M := \bigcup_{n<\omega} M_n \models T_1$, $N := \bigcup_{n<\omega} N_n \models T_2$ and $f := \bigcup_{n<\omega} f_n$. By commutativity of the above diagrams we get that $f : M \restriction L_0 \cong N \restriction L_0$.

Now we define an expansion of the model N to L_1.

Let $N^* := \langle N, \bar{F}, \bar{R}, \bar{c}\rangle_{F,R,c\in L_1 - L_0}$ where for $\mathbf{a} \in |N|$ the value of function $\bar{F}(\mathbf{a})$ is defined as $g(F^M(g^{-1}(\mathbf{a}))$ and $\bar{R}, \bar{c}$ are defined similarly.
Notice that since $f : M \restriction L_0 \cong N \restriction L_0$ the definition of N^* entails that $f : M \cong N^* \restriction L_1$, this together with the fact that $N \models T_1$ gives what we want, namely $N^* \models T_1 \cup T_2$. □

In Chapter 4 as an application of Theorem 4.2.46 we'll see a shorter proof of Robinson's consistency property.

Here is an application of Robinson's consistency property:

THEOREM 6.5 (Craig's interpolation theorem). *Let ψ_ℓ be sentences in a similarity type L_ℓ for $\ell = 1, 2$ and let $L_0 := L_1 \cap L_2$. If $\psi_1 \models \psi_2$ then there exists $\varphi \in \mathrm{Sent}(L_0)$ such that $\psi_1 \models \varphi$ and $\varphi \models \psi_2$.*

PROOF. Let $T_2 \subseteq \mathrm{Sent}(L_2)$ be a completion of

$$\{\neg\psi_2\} \cup \{\chi \in \mathrm{Sent}(L_0) \mid \psi_1 \models \chi\},$$

let $T_0 := T_2 \cap \mathrm{Sent}(L_0)$ and let $T_1 \subseteq \mathrm{Sent}(L_1)$ be a completion of $\{\psi_1\} \cup T_0$.

By Robinson's consistency property there exists $M \models T_1 \cup T_2$, this is a contradiction to $\psi_1 \models \psi_2$ as $M \models \psi_1 \wedge \neg\psi_2$. □

An important early result of Model theory is Beth's definability theorem.

DEFINITION 6.6. Let L be a similarity type and let P be a relation symbol not in L, denote by L' the similarity type $L \cup \{P\}$. For a consistent L'-theory T

we say that P *is implicitly definable by* T iff for every $M, N \models T$ we have that

$$M \restriction L = N \restriction L \implies P^M = P^N.$$

THEOREM 6.7 (Beth). *Let L, P and T be as above. If P is implicitly definable by T then there exists $\chi(\mathbf{x}) \in \mathrm{Fml}(L)$ such that*

$$T \vdash \forall \mathbf{x}[P(x) \leftrightarrow \chi(\mathbf{x})].$$

PROOF. Let Q be a new relation symbol with the same arity as P. Denote by L^* the similarity type $L \cup \{Q\}$ and by $T(Q)$ the L^*-theory obtained from T when every occurance of P is replaced by Q. The implicit definability assumption gives

$$(*) \quad T \cup T(Q) \vdash \forall \mathbf{x}[P(\mathbf{x}) \leftrightarrow Q(\mathbf{x})].$$

Let $\mathbf{c}$ be a sequence of new constants of length $\ell(\mathbf{x})$. By predicate calculus $(*)$ gives

$$T \cup T(Q) \vdash [P(\mathbf{c}) \to Q(\mathbf{c})].$$

There exists a finite $T_0 \subseteq T$ such that

$$(**) \quad T_0 \cup T_0(Q) \vdash [P(\mathbf{c}) \to Q(\mathbf{c})].$$

Let ψ be the sentence $\bigwedge T_0$. Now $(**)$ translates into

$$\psi \wedge \psi(Q) \vdash [P(\mathbf{c}) \to Q(\mathbf{c})].$$

The Deduction Theorem gives

$$\psi \wedge P(\mathbf{c}) \vdash [\psi(Q) \to Q(\mathbf{c})].$$

An application of Craig;'s interpolation theorem gives $\chi(\mathbf{c}) \in \mathrm{Sent}(L \cup \{\mathbf{c}\})$ such that

$$\psi \wedge P(\mathbf{c}) \vdash \chi(\mathbf{x}) \text{ and } \chi(\mathbf{c}) \vdash [\psi(Q) \to Q(\mathbf{c})].$$

Now we will establish that $\psi \cup \psi(P) \vdash [\forall \mathbf{x}[P(\mathbf{x}) \leftrightarrow \chi(\mathbf{x})]$:

An application of the Deduction Theorem to the first statement gives

$$(***) \quad \psi \vdash [P(\mathbf{c}) \to \chi(\mathbf{c})]$$

Since $\chi(\mathbf{c}) \vdash [\psi(Q) \to Q(\mathbf{c})]$ and Q does not appear in $\chi(\mathbf{c})$ we get

$$\chi(\mathbf{c}) \vdash [\psi(P) \to P(\mathbf{c})].$$

The Deduction Theorem gives

$$\psi(P) \vdash [\chi(\mathbf{c}) \to P(\mathbf{c})]$$

this together with $(***)$ establishes that P is definable by $\chi(\mathbf{x})$. □

Robinson's Consistency property is an easy corollary of Craig's interpolation theorem:

SECOND PROOF OF THEOREM 6.4. Suppose for the sake of contradiction that $T_1 \cup T_2$ is inconsistent. By the finite character of inconsistency there are $\{\varphi_i^\ell \mid i < n\} \subset T_\ell$ such that

$$\bigcup_{\ell=1,2} \{\varphi_\ell^i \mid i < n\} \text{ is inconsistent.}$$

Thus for every $L_1 \cup L_2$-structure M if $M \models \bigwedge_{i<n} \varphi_1^i$ then $M \models \neg[\bigwedge_{i<n} \varphi_2^i]$. By Craig's interpolation Theorem there exists $\chi \in \mathrm{Sent}(L_0)$ such that $\bigwedge_{i<n} \varphi_1^i \models \chi$ and $\chi \models \neg[\bigwedge_{i<n} \varphi_2^i]$. Since T_0 is complete and T_1 is consistent, necessarily $\chi \in T_0$ but as $T_2 \supseteq T_0$ is consistent it is impossible for a sentence from T_2 to imply a negation of a conjunction of sentences from T_2. □

The following is a fundamental property of elementary classes:

THEOREM 6.8. *Let T be a complete first-order theory. For every $\lambda \geq |\mathrm{L}(T)|$ the class $\mathrm{Mod}(T)$ has the λ-amalgamation property.*

PROOF. By renaming we may assume that $M_0 \preceq M_\ell$ for $\ell = 1, 2$. Let $T_\ell := \mathrm{CD}(M_\ell)$. Apply Robinson's consistency lemma to T_1, T_2 and T_0. □

It is possible also to give a more direct proof of Theorem 6.8. I suggest the to reader to do this as an exercise.

A direct application of the compactness theorem can also produce a proof for a stronger result. Recall Definition 1.30: An abstract class satisfy the *Disjoint Amalgamation Property* iff for every $M_\ell \in \mathcal{K}$ $\ell \in \{0, 1, 2\}$ such that $M_0 \prec_{\mathcal{K}} M_k$ for $k = 1, 2$ there are $N \succ_{\mathcal{K}} M_2$ and a $\mathcal{K}$-embedding $f : M_1 \to N$ such that $f \restriction M_0 = \mathrm{id}_{M_0}$ and $f[M_1] \cap M_2 = M_0$.

Now we introduce a fundamental technical device (in a later chapter it will be proved that this is equivalent to forking for many theories):

DEFINITION 6.9. Let N be a structure. Suppose A, B and C are subsets of N. We say that A *is free from C over B* iff for every $\mathbf{a} \in A$, every $\mathbf{c} \in C$ and every $\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{L}(N)$ we have

$$N \models \varphi[\mathbf{a}; \mathbf{c}] \implies \text{there exists } \mathbf{c}' \in B \text{ satisfying } N \models \varphi[\mathbf{a}; \mathbf{c}'].$$

We denote this by writing $A \overset{N}{\underset{B}{\downarrow}} C$. When the identity of N is clear we will omit the superscript and write $A \underset{B}{\downarrow} C$.

The following easy observation is a key to get *disjoint* amalgamation:

PROPOSITION 6.10. *Let N, A, B and C be as above. Denote by A' the set $A \cup B$ and let C' stand for $C \cup B$. We have that*

$$A \underset{M}{\downarrow} C \implies A' \cap C' = B.$$

PROOF. Otherwise, there is $a \in A \cap C - B$. Take $\varphi(x;y)$ to be $x = y$. Since $N \models \varphi[a;a]$ by $A \underset{M}{\downarrow} C$ there is $c' \in B$ such that $N \models a = c'$, contradiction to $a \notin B$. □

THEOREM 6.11 (Existence of free extensions over models). *For every M_0, M_1, M_2 L-structures such that $M_0 \preceq M_\ell$ for $\ell = 1, 2$ there are $N \succ M_2$ and an elementary embedding $f : M_1 \to N$ such that f is the identity over M_0 and $f[M_1] \underset{M_0}{\overset{N}{\downarrow}} M_2$.*

PROOF. By replacing M_1 (if necessary) with an isomorphic copy we may assume taht $M_1 \cap M_2 = M_0$.

$$\text{Consider } T := \mathrm{CD}(M_1) \cup \mathrm{CD}(M_2) \cup \{\neg\varphi \mid \ \varphi \in \Gamma\} \text{ where}$$
$$\Gamma := \{\varphi(\mathbf{b};\mathbf{c}) \mid \varphi(\mathbf{x};\mathbf{y}) \in L,\ \mathbf{b} \in |M_1|,\ \mathbf{c} \in |M_2| \text{ and}$$
$$M_1 \models \neg\varphi[\mathbf{b};\mathbf{c}'] \text{ for all } \mathbf{c}' \in |M_0|\}.$$

If T is consistent there is $N \models T$, by renaming some elements we may asume that $N \succ M_2$ and there is $f : M_1 \to N$ is an elementray embedding which is the identity over M_0. As $N \models \neg\varphi$ for every $\varphi \in \Gamma$ we have that $f[M_1] \underset{M_0}{\overset{N}{\downarrow}} M_2$ holds.

If T is inconsistent (by Robinson's consistency lemma (Theorem 6.4) $\mathrm{CD}(M_1) \cup \mathrm{CD}(M_2)$ is consistent) then there exists a finite $\Phi \subseteq \Gamma$ such that

$$\mathrm{CD}(M_1) \cup \mathrm{CD}(M_2) \vdash \neg \bigwedge_{\varphi \in \Phi} \neg\varphi.$$

Thus there is $\varphi[\mathbf{b};\mathbf{c}] \in \Gamma$ such that

$$\mathrm{CD}(M_1) \cup \mathrm{CD}(M_2) \vdash \varphi[\mathbf{b};\mathbf{c}].$$

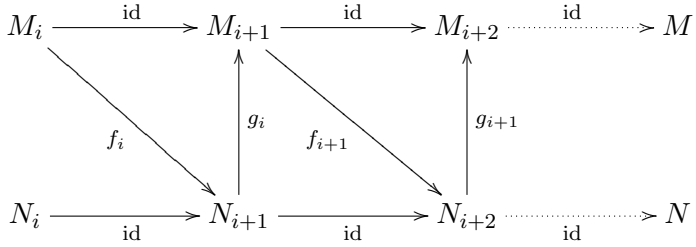
As $\mathrm{CD}(M_\ell)$ is complete we get that $M_1 \models \exists\mathbf{y}\varphi(\mathbf{b};\mathbf{y})$ and $M_2 \models \exists\mathbf{x}\varphi(\mathbf{x};\mathbf{c})$.

$$M_2 \models \exists\mathbf{y}\psi(\mathbf{a};\mathbf{y}) \text{ and } \mathbf{b} \in M_1, \mathrm{CD}(M_1) \cup \mathrm{CD}(M_2) \vdash \bigvee_{\varphi\in\Phi} \varphi.$$

Namely, there exists $N \models \mathrm{CD}(M_1) \cup \mathrm{CD}(M_2)$ such that $N \models \varphi[\mathbf{a};\mathbf{c}]$ for some $\varphi[\mathbf{a};\mathbf{c}] \in \Gamma$. Without loss of generality we may assume that $N \succ M_2$. □

Exercises 1.6.

(1) Establish that the DAP holds: I.e. for every $M_0 \prec M_\ell$ there exists $N \succ M_2$ and an elementary embedding $g : M_1 \to N$ such that $g \restriction M_0 = \mathrm{id}_{|M_0|}$ and $g[|M_1|] \cap |M_2| = |M_0|$.
(2) Verify that T_{ACF} does not have the JEP.
(3) If a first-order theory T is categorical in some $\lambda \geq |\mathrm{L}(T)|$ then T has the JEP.
(4) Show that T_{ACF} has the amalgamation property.
(5)
(6) Suppose that T is a first-order theory that has an infinite model and consider the AEC $\mathcal{K} := \langle \mathrm{Mod}(T), \preceq \rangle$, where $\prec$ is the usual elementary submodel relation, prove that the following are equivalent:
   (a) T is complete,
   (b) $\mathcal{K}$ has the λ-JEP for some $\lambda \geq |\mathrm{L}(T)|$
   (c) $\mathcal{K}$ has the λ-JEP for every $\lambda \geq |\mathrm{L}(T)|$.
(7) Derive Craig's interpolation theorem from the compactness theorem.
(8)
(9) Derive Craig's interpolation theorem from Robinson's consistency.
(10) Let T be a first-oder theory, and let K_T be the induced AEC (see Example 2.1.1.3(2). Prove that T is complete iff $\mathcal{K}_T$ has the JEP.
(11) Suppose that the abstract class $\mathcal{K}$ is elementary. Prove that for such $\mathcal{K}$ the Joint Embedding Property implies the Amalgamation Property.
(12)
(13)
(14) Let L_0, L_1, L_2 be similarity types such that $L_0 = L_1 \cap L_2$. Let T_ℓ be consistent theories in L_ℓ for $\ell = 1, 2$. If $T_1 \cup T_2$ is inconsistent then there exists $\chi \in \mathrm{Sent}(L_0)$ such that $T_1 \vdash \chi$ and $T_2 \vdash \neg\chi$.
(15) Suppose both $\{M_i \mid i < \omega\}$and $\{N_i \mid i < \omega\}$ are chains (i.e. $i < j \implies M_i \subseteq M_j$). Let $M := \bigcup_{i<\alpha} M_i$ and $N := \bigcup_{i<\omega} N_i$. If there are $\{f_i, g_i \mid i < \omega\}$ homomorphisms satisfying



for all $i < \omega$. Then $f := \bigcup_{i<\omega} f_i$ is an isomorphism from M onto N and $g := \bigcup_{i<\omega} g_i$ is an isomorphism from N onto M.
(16) State and prove a generalization of the previous exercise, where ω is replaced by any limit ordinal. Hint: for any large enough i there is f_i a homomorphism from M_i to some N_j with $j > i$ and there is g_j a homomorrphism of N_j to $M_{i(j)}$ for some some $i(j) > j$ such that $f_i^{-1} \subseteq g_j$...

7. Types and the diagram of T

Types are the direct generalization of formulas. A type relates to a single first-order formula similarly to the relationship between a (potentially infinite) system of equations and a single equation in algebra. As with formulas, the interesting object of study (for a model theorist) are not the syntactic objects but rather the sets of elements of a model which simultaneously satisfy all formulas in the type.

DEFINITION 7.1. Let L be a similarity type and let $\mathbf{x} = \langle x_i \mid i \in I \rangle$ be a (not necessarily finite) sequence of variables.

(1) A subset p of the set $\mathrm{Fml}(L)$ is called an $\mathbf{x}$-*type*. When $n = \ell(\mathbf{x})$ is finite we say that p is *n-type)* in L
if $\varphi \in p$ then $\mathrm{FV}(\varphi) \subseteq \{x_i \mid i \in I\}$. Following our convention of writing formulas we will occasionally write $p(\mathbf{x})$ to denote that p is an $\mathbf{x}$-type.
(2) An $\mathbf{x}$-type p is called a *complete type*, if for every $\varphi(\mathbf{x}) \in \mathrm{Fml}(L)$, we have either $\varphi \in p$ or $\neg\varphi \in p$.

EXAMPLES 7.2. The following sets of formulas are types:

(1) Let $L = L_{PA}$, and let
$$p(x) := \{0 \neq x, 1 \neq x, 1+1 \neq x, \ldots, 1+\cdots+1 \neq x, \ldots\}.$$
(2) Let $L = L_{orf}$ and let
$$q(x) := \{1 < x, 1+1 < x, \ldots, 1+\cdots+1 < x, \ldots\}.$$
(3) Let M be an L-structure, $a \in |M|$ the set $p := \mathrm{tp}(a/\emptyset, M)$ is a complete type. Notice that when p is a complete type then
$$\mathrm{Th}(M) \subseteq p.$$

DEFINITION 7.3. Let M be a structure, let p be an $\mathbf{x}$-type in $\mathrm{L}(M)$, and let $\mathbf{a} \in {}^{\ell(\mathbf{x})}|M|$.

(1) $\mathbf{a}$ *realizes* p if for every $\varphi \in p$ we have that $M \models \varphi[\mathbf{a}]$.
(2) $\mathbf{a}$ *omits* p if there exists $\varphi \in p$ such that $M \models \neg\varphi[\mathbf{a}]$.
(3) p *is realized in* M if there exists $\mathbf{a} \in {}^{\ell(\mathbf{x})}|M|$ such that $\mathbf{a}$ realizes p.
(4) p *is omitted in* M if for every $\mathbf{a} \in {}^{\ell(\mathbf{x})}|M|$ the sequence $\mathbf{a}$ omits p (I.e. p is not realized in M).

REMARK 7.4. Compare the last definition with Definition 1.4.23

EXAMPLE 7.5. Let p be the type from Example 7.2(1). It should be clear that p is omitted by the standard model of PA, and if $M \models PA$ is non-standard then p is realized in M.

EXAMPLE 7.6. Let q be the type from Example 7.2(2).
For $F \models T_{orf}$ the following equivalence holds: "F is archimedean" iff q is omitted in F.

EXAMPLE 7.7. Let $F \leq E$ be fields. Recall that an element $a \in E$ is called *algebraic over* F iff there exists a polynomial $g(x) \in F[x] \ \neq 0$ such that $g(a) = 0$. Otherwise we say that *the element* a *is transcendental over* F . Let F be a field of characteristics zero. Let

$$p(x) := \big\{[g(x) \neq 0] \ : \ g(x) \in F[x]\big\}.$$

Notice that $p(x)$ is a type and an element a of E realizes $p(x)$ iff a is a transcendental over F.

EXAMPLE 7.8 (unique non-algebraic type). Let $F \leq E$ be fields. Using Theorem 2.38 it follows that for all $a, b \in E$ if both elements are transcendental over F then

$$\mathrm{tp}(a/F, E) = \mathrm{tp}(b/F, E).$$

EXAMPLE 7.9. Let p be the type from Example 7.2(3) it is realized by a.

Examples 7.5 and 7.6 also show that by using types we have much more expressive power than using first-order formulas only.

LEMMA 7.10. *If $M \cong N$ then the models realize exactly the same types. Moreover, let M be an L-structure and p a set of formulas in L. If $g : M \cong N$ then $\mathbf{a} \in |M|$ realizes p iff $g(\mathbf{a})$ realizes p.*

PROOF. Exercise. □

DEFINITION 7.11. Let T be a theory, let $\mathbf{x}$ be a finite sequence of variables, and let $p(\mathbf{x})$ be an $\mathbf{x}$-type in $\mathrm{L}(T)$. We say that p *is a* T*-type* or p *is consistent with* T if for every finite subset q of p we have that the theory $\{\exists \mathbf{x} \bigwedge q\} \cup T$ is consistent.

PROPOSITION 7.12. *Let T be a theory, and let p be an $\mathbf{x}$-type in $\mathrm{L}(T)$. If p is a T-type then T has a model realizing p.*

PROOF. Supose $\mathbf{x} = \langle x_i \mid i \in I\rangle$ let $\{c_i \mid i \in I\}$ be new constants (not in $\mathrm{L}(T)$). Denote by $\mathbf{c}$ the sequence $\langle c_i \mid i \in I\rangle$. Define

$$T_p := T \cup \{\varphi(\mathbf{c}) \ : \ \varphi(\mathbf{x}) \in p\}.$$

Notice that since p is a T-type, every finite subset of T_p has a model. Thus by the compactness theorem there exists $M' \models T_p$. It is clear that $M \restriction \mathrm{L}(T)$ is a model of T and p is realized by the sequence $\mathbf{a}$ that interprets the constants $\mathbf{c}$ in the structure M. □

REMARK 7.13. While in our definition of type we do not require that all the free variables of the formulas are from a fixed finite set, types with infinitely many variables are quite useful they will start appearing in a substantial way about 400 pages from here.

Recall that for a structure M a finite sequence of elements $\mathbf{b}$ fro $|M|$ and any $A \subseteq |M|$ we already encountered

$$\mathrm{tp}(\mathbf{b}/A, M) := \{\varphi(\mathbf{x};\mathbf{a}) \mid \varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(M), \mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A}, M \models \varphi[\mathbf{b};\mathbf{a}]\}.$$

This can be used to define also a type of an infinite sequence $\mathbf{b}$ in an analogous way in algebra we define the ring of polynomial with infinitely many variables. Let

$$\mathrm{tp}_*(\mathbf{b}/A, M) := \bigcup_{\mathbf{b}^* \text{ finite subsequence of } \mathbf{b}} \{\varphi(\mathbf{x};\mathbf{a}) \mid \varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(M), \mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A}, M \models \varphi[\mathbf{b}^*;\mathbf{a}]\}.$$

LEMMA 7.14 (Every type can be completed). *Let L be a similarity type and let $\mathbf{x} = \langle x_i \mid i \in I\rangle$. If p is an $\mathbf{x}$-type in L then there exists a complete $\mathbf{x}$-type q (in L) such that $q \supseteq p$.*

PROOF. Use the previous proposition or alternatively Zorn's lemma. □

DEFINITION 7.15. Let M be a structure, let $\mathbf{x}$ be a sequence of variables.

(1) Let $p = p(\mathbf{x})$ be an $\mathbf{x}$-type in $\mathrm{L}(M)$. We say that p *is an M-type* iff p is $\mathrm{Th}(M)$-type.
(2) Let $A \subseteq |M|$. Let p be a type in $\mathrm{L}(\langle M, a\rangle_{a\in A})$.
We say that p *is M-type over A* iff p is a $\mathrm{Th}(\langle M, a\rangle_{a\in A})$-type.
(3) When the identity of the model is clear we say p *is type over A* instead of saying that p is an M-type over A.

PROPOSITION 7.16. *Let M be a structure, let $A \subseteq |M|$, and suppose $\mathbf{x}$ is a finite sequence of variables.*

(1) *If $A = \emptyset$ then p is M-type over A iff p is an M-type.*
(2) *Let $p(\mathbf{x})$ be a set of formulas in $\mathrm{L}(\langle M, a\rangle_{a\in A})$. p is an M-type over A iff every finite subset of p is realized in M.*

PROOF. (1) is obvious.

(2) Since $\mathrm{Th}(\langle M, a\rangle_{a\in A})$ is a complete theory in $\mathrm{L}(\langle M, a\rangle_{a\in A})$. □

Similarly to Lemma 7.14 we have:

LEMMA 7.17 (existence of completion of a type). *Let M be a structure, let $A \subseteq |M|$, and suppose $\mathbf{x}$ is a finite sequence of variables. If $p(\mathbf{x})$ is an M-type over A then there exists a complete type $\bar{p}(\mathbf{x}) \supseteq p$ which is an M-type over A.*

PROOF. Exercise 35. □

It is easy to realize a type:

THEOREM 7.18. *Let M be a structure, let $A \subseteq |M|$, and let p be an M-type over A. There exists an elementary extension N of M realizing p. Moreover, a model N with the above properties can be found such that also $\|N\| = \|M\| + |\operatorname{L}(M)| + \aleph_0$.*

PROOF. Apply Proposition 7.12 to the theory $\operatorname{CD}(M)$. □

A natural question to ask in this context is: Given a theory T and a T-type p, does there exists a model $M \models T$ omitting the type p? The following is a quote from Gerald Sacks' book [**Sa**] (which he attributes to a "not well-known model theorist"):

> Any fool can realize a type, but it takes a model theorist to omit one.

There is much truth in the above comment. As we will see again and again. The question is one of the most important technical questions in model theory. Namely, many interesting model theoretic problems can be reduced to omitting types.

There are several general methods to omit a type that we will encounter in later stages of this book:

(1) For a T in a countable language under a reasonable condition on a type p there exists a countable $M \models T$ omitting p. This is known as Henkin's omitting-types Theorem (see Theorem **??**).
(2) if we assume existence of large cardinals for any structure M that omits a type p there exists $N \succneqq M$ also omitting p. For this we will be using the ultrapower construction see **??**.
(3) using models introduced by Ehrenfeucht and Mostowski together withe the notion of indiscenible sequences we wil show that given a type p and exists of a certain small model generated by an infinite inndiscenible sequence p has arbitrarily large models omitting it.
(4) In a crucial step of the proof of Morely's categoricity theorem we will show that under the assumption that T is $\aleph_0$-stable certain types have models in all cardinals omitting them.

As was pointed out in section 3 of chapter 2 the understanding of the function $\lambda \mapsto \operatorname{I}(\lambda, T)$ is was one of the main forces in the development of the subject. So a natural question to ask at this stage is: Does there exists a connection between types and $\operatorname{I}(\lambda, T)$? It appears that there are several connections. Some are quite simple and some are very deep and difficult. The rest of this section is dedicated to a simple connection.

DEFINITION 7.19. Let T be a theory, let n be a positive integer, denote by $\mathbf{x}$ the sequence of variables $\langle x_1, \ldots, x_n \rangle$.

$$\mathrm{D}_n(T) := \{p(\mathbf{x}) \ : \ p \text{ is a complete } T\text{-type}\}, \quad \text{and let} \quad \mathrm{D}(T) := \bigcup_{n<\omega} \mathrm{D}_n(T).$$

$\mathrm{D}(T)$ is called *the diagram of* T .

REMARK 7.20. *It is easy to see that always we have that* $\aleph_0 \leq |\mathrm{D}(T)| \leq 2^{|T|}$ *holds. By Theorem 5.8.* $|\mathrm{D}(PA)| = 2^{\aleph_0}$. *It turns out that* $\mathrm{D}_n(T_{ord})$ *is finite. Later in Theorem 9.38 we will show that a theory* T *in a countable language is* $\aleph_0$*-categorical iff* $\mathrm{D}_n(T)$ *is finite for all* $n < \omega$.

The main theorem of this section is

THEOREM 7.21. *Let* T *be a countable theory.*
If $|\mathrm{D}(T)| > \aleph_0$ *then* $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$.

The proof is presented in several steps. The following lemma show that the above Theorem follows from Proposition 7.23 and Theorem 7.24.

LEMMA 7.22. *Let* T *be a countable theory. If* $|\mathrm{D}(T)| > \aleph_0$ *then there exists a positive integer* n *such that* $|\mathrm{D}_n(T)| > \aleph_0$.

PROOF. Let $D \subseteq \mathrm{D}(T)$ be a subset of cardinality $\aleph_1$. Since

$$D \subseteq \bigcup_{n<\omega} \mathrm{D}_n(T)$$

by the regularity of $\aleph_1$ there exists $n_0 < \omega$ such that $\mathrm{D}_{n_0}(T)$ has cardinality $\aleph_1$. □

The following relates the number of types to the number of pairwise non-isomorphic models.

PROPOSITION 7.23. *Let* T *be a countable theory.*
If there exists $1 \leq n < \omega$ *such that* $|\mathrm{D}_n(T)| = 2^{\aleph_0}$ *then* $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$.

PROOF. Suppose that $\mu = \mathrm{I}(\aleph_0, T) < 2^{\aleph_0}$. Hence there exists $S \subseteq \mathrm{Mod}(T)$ of cardinality μ such that $M \neq N \in S \Longrightarrow M \ncong N$, and for every countable $M \models T$ there exists (a unique) $N \in S$ such that $M \cong N$. By Proposition 7.12 and the DLS for every $p \in \mathrm{D}_n(T)$ there exists $M_p \in S$ such that p is realized in M_p. Thus we have that

$$\mathrm{D}_n(T) = \bigcup_{M \in S} \{p \in \mathrm{D}_n(T) \ : \ p \text{ is realized in } M\}.$$

This gives

$$|\mathrm{D}_n(T)| = \sum_{M \in S} |\{p \in \mathrm{D}_n(T) \ : \ p \text{ is realized in } M\}| \leq \mu \cdot \aleph_0 < 2^{\aleph_0}$$

which contradicts the hypothesis on the cardinality of $|\,\mathrm{D}_n(T)|$. □

In light of the last lemma and Proposition the next theorem is the main step needed to establish Theorem 7.21.

THEOREM 7.24 (Loś). *Let T be a countable theory, and let n be a positive integer. If $|\,\mathrm{D}_n(T)| > \aleph_0$ then $|\,\mathrm{D}_n(T)| = 2^{\aleph_0}$.*

PROOF. For every $\eta \in {}^{\omega>}2$ define a finite $p_\eta \subseteq \mathrm{Fml}(\mathrm{L}(T))$ with the following properties:

(1) if $\varphi \in p_\eta$ then $\mathrm{FV}(\varphi) \subseteq \{x_1, \ldots, x_n\}$
(2) p_η is a T-type
(3) for every $\eta \in {}^{\omega>}2$, if ν is an initial segment of η then $p_\nu \subseteq p_\eta$
(4) for every $\eta \in {}^{\omega>}2$, the set D_{p_η} is uncountable where for $\Phi \subseteq \mathrm{Fml}(\mathrm{L}(T))$ denote by D_Φ the set $\{p \in \mathrm{D}_n(T) \ : \ p \supseteq \Phi\}$.
(5) For every $\eta \in {}^{\omega>}2$ there exists $\psi_\eta \in \mathrm{Fml}(\mathrm{L}(T))$ such that $p_{\eta^\frown 0} = p_\eta \cup \{\psi_\eta\}$ and $p_{\eta^\frown 1} = p_\eta \cup \{\neg\psi_\eta\}$.

This is enough: For every $\eta \in {}^{\omega}2$ define $p_\eta = \bigcup_{n<\omega} p_{\eta\restriction n}$. By Lemma 7.14 there exists $q_\eta \in \mathrm{D}_n(T)$ containing p_η. Now to show that $|\,\mathrm{D}_n(T)| = 2^{\aleph_0}$ it is enough to show that $\eta \neq \nu \in {}^{\omega}2$ implies $q_\eta \neq q_\nu$. Let $\eta \neq \nu \in {}^{\omega}2$ be given. Let $\ell = \mathrm{Max}\{\ell < \omega \ : \ \eta \restriction \ell = \nu \restriction \ell\}$. Namely for every k such that $k < \ell$ we have $\eta[k] = \nu[k]$ and $\eta[\ell] \neq \nu[\ell]$. Without loss of generality we may assume that $\eta[\ell] = 0$ and $\nu[\ell] = 1$.
By (5) $\psi_{\eta\restriction\ell} \in p_{\eta\restriction\ell^\frown 0}$ which by the definition of q_η and (3) show that $\psi_{\eta\restriction l} \in q_\eta$.
Similarly by (5) and (3) we have that $\neg\psi_{\eta\restriction\ell} \in p_{\eta\restriction\ell^\frown 1} \subseteq q_\nu$. Thus the formula $\psi_{\eta\restriction\ell}$ is a witness to the fact that the types are distinct.

This is possible: We will construct $\{p_\eta \ : \ \eta \in {}^{\omega>}2\}$ satisfying (1) - (5) above by induction on $\ell(\eta)$: For $\ell(\eta) = 0$, let $p_{\langle\rangle} := \{x_1 = x_1, \ldots, x_n = x_n\}$. Suppose that p_η is defined and we need to define $p_{\eta^\frown 0}$ and $p_{\eta^\frown 1}$. This will be done by applying the following

CLAIM 7.25. *Let T be a countable theory. For every finite $\Phi \subseteq \mathrm{Fml}(\mathrm{L}(T))$ if $|\,\mathrm{D}_\Phi\,| > \aleph_0$ then there exists $\psi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T))$ such that both $\mathrm{D}_{\Phi\cup\{\psi\}}$ and $\mathrm{D}_{\Phi\cup\{\neg\psi\}}$ are uncountable.*

PROOF. We will use he following notation: $\varphi^0 := \varphi$ and $\varphi^1 := \neg\varphi$.

Suppose for the sake of contradiction that there exists a finite set of formulas Φ such that $\mathrm{D}_{\Phi\cup\{\psi\}}$ or $\mathrm{D}_{\Phi\cup\{\neg\psi\}}$ is countable for every formula ψ. Notice that since for every ψ we have that $\mathrm{D}_\Phi = D_{\Phi\cup\{\psi\}} \cup \mathrm{D}_{\Phi\cup\{\neg\psi\}}$. The uncountability of D_Φ and our hypothesis on Φ imply that exactly one of the sets $\mathrm{D}_{\Phi\cup\{\psi\}}$ or $\mathrm{D}_{\Phi\cup\{\neg\psi\}}$ is uncountable.

Since T is countable, we have a fixed enumeration $\{\varphi_k \ : \ k < \omega\}$ of all the formulas which are in the language $\mathrm{L}(T)$ and the free variables are among $\{x_1, \cdots, x_n\}$. For every $k < \omega$ let $t(k) \in \{0, 1\}$ be such that

$$(*) \quad \mathrm{D}_{\Phi \cup \{\varphi_0^{t(0)}, \ldots, \varphi_k^{t(k)}\}} \text{ is uncountable.}$$

By the first paragraph $t(k)$ is well defined for every $k < \omega$.

In particular $p := \{\varphi_k^{t(k)} \ : \ k < \omega\}$ is a T-type (by (*) every finite subset is included in uncountable many elements of $\mathrm{D}_n(T)$). Since $\{\varphi_k \ : \ k < \omega\}$ is an enumeration of all formulas in the variables $\mathbf{x}$ the set p is a complete type. Namely $p \in \mathrm{D}_n(T)$. Observe that

$$(**) \quad \mathrm{D}_\Phi = \{p\} \cup \bigcup_{k<\omega} \mathrm{D}_{\Phi \cup \{\varphi_k^{1-t(k)}\}}.$$

But by the assumption we made at the start of this proof and the definition of $t(k)$ we have that for every $k < \omega$ the set
$\mathrm{D}_{\Phi \cup \{\varphi_k^{1-t(k)}\}}$ is countable. Use (**) to find an upper bound to the cardinality of D_Φ:

$$\begin{aligned} |\,\mathrm{D}_\Phi\,| &= |\{p\} \cup \bigcup_{k<\omega} \mathrm{D}_{\Phi \cup \{\varphi_k^{1-t(k)}\}}\,| \\ &\leq 1 + \sum_{k<\omega} \aleph_0 = 1 + \aleph_0 \cdot \aleph_0 \\ &= 1 + \aleph_0 = \aleph_0. \end{aligned}$$

This is a contradiction to the assumption that D_Φ is uncountable. □

□

The argument employed in the proof of Theorem 7.24 is called "The perfect set argument". It is similar to Cantor's proof that every uncountable closed subset of the real numbers is of cardinality $2^{\aleph_0}$ (which lead him to believe in the Continuum Hypothesis). See Lemma I.4.2 in [**Je78**].

In light of Corollary 2.2.11, a natural question to ask is: What are the possible values at $\aleph_0$ of the spectrum functions for countable theories? Namely, let T be a countable complete first-order theory. What are the possible values of $\mathrm{I}(\aleph_0, T)$? Are there any impossible values? We have seen the following examples $\mathrm{I}(\aleph_0, T_{ord}) = 1$, $\mathrm{I}(\aleph_0, T_{ACF_p}) = \aleph_0$, $\mathrm{I}(\aleph_0, PA) = 2^{\aleph_0}$. In section 9 it is shown that $\mathrm{I}(\aleph_0, T)$ can not be 2! However, for any positive integer $n \geq 3$ there exists a theory T_n such that $\mathrm{I}(\aleph_0, T_n) = n$. So far we did not cover the range strictly between $\aleph_0$ and $2^{\aleph_0}$.

While it is the contents of the continuum Hypothesis that there are no cardinals as above, in models of set theory where the continuum hypothesis fails there are such cardinals. In the mid sixties Vaught made the following conjecture:

CONJECTURE 7.26 (Vaught's conjecture). *Let T be a countable first-order complete theory. If* $\mathrm{I}(\aleph_0, T) > \aleph_0$ *then* $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$.

Several people including the author think that the conjecture is probably false in its full generality, however it was verified for several classes of theories. More on this see in the historical comments.

The following is an easy observation: By Theorem 7.21 the problem is open for theories with countable diagram only. There are many partial results. Most of them require some strong hypothesis on the theory T (E.g. see Theorem XXX). We will mention at this stage only the following theorem that is proved using methods of descriptive set theory (which will not presented here).

THEOREM 7.27 (Morley 1968). *Let T be a countable first-order complete theory. If* $\mathrm{I}(\aleph_0, T) > \aleph_1$ *then* $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$.

In spring 2002 Robin Knight [**Kn**] announced a counterexample for Vaught's conjecture. So far (Spring 2015) the details of his counterexample were not confirmed.

The following is an important generalization of the set $\mathrm{D}_n(T)$.

DEFINITION 7.28. Let M be an L-structure, let $A \subseteq |M|$ and let n be a positive integer, suppose that $\mathbf{x} = \langle x_1, \dots, x_n \rangle$.
Denote by $\mathrm{S}^n(A, M)$ *the set of complete M-types over the set A in* $\mathbf{x}$. Namely $\mathrm{S}^n(A, M) := \{\mathrm{tp}(\mathbf{b}/A, N) \ : \ \mathbf{b} \in {}^n|N|, N \geq M\}$.
A relativized version: Suppose $\Delta \subseteq \mathrm{Fml}(\mathrm{L}(T))$ such that $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta \implies \ell(\mathbf{x}) = n$ then

$$\mathrm{S}^n_\Delta(A, M) := \{\mathrm{tp}_\Delta(\mathbf{b}/A, N) \ : \ \mathbf{b} \in {}^n|N|, N \geq M\}.$$

Where

$$\begin{aligned}\mathrm{tp}_\Delta(\mathbf{b}/A, N) :=& \{\varphi(\mathbf{x}; \mathbf{a}) \mid \mathbf{a} \in A, \varphi(\mathbf{x}; \mathbf{y}) \in \Delta, N \models \varphi[\mathbf{b}; \mathbf{a}]\} \cup \\ & \{\neg\varphi(\mathbf{x}; \mathbf{a}) \mid \mathbf{a} \in A, \varphi(\mathbf{x}; \mathbf{y}) \in \Delta, N \models \neg\varphi[\mathbf{b}; \mathbf{a}]\}.\end{aligned}$$

While Theorem 7.24 does not have a good extension for theories in uncountable languages (it depends on the value of the function $\lambda \mapsto 2^\lambda$). There is an extension that holds for also uncountable theories (it will be used in computing the first stability cardinal see XX¿YY).

THEOREM 7.29. *Let n be a positive integer. If there exists a set A in a model M such that* $|\mathrm{S}^n_\Delta(A, M)| > |A| + |T| + \aleph_0$ *then there are a countable $B \subseteq A$ and* $\Delta^* := \{\varphi_\eta(\mathbf{x}; \mathbf{y}_\eta) \mid \eta \in {}^{\omega >}2\} \subseteq \Delta$ *such that* $|\mathrm{S}^n_{\Delta^*}(B, M)| = 2^{\aleph_0}$.

PROOF. Exercise □

It is not difficult to see that for any theory T, and every $M \models T$ we have that $\mathrm{D}_n(T) \supseteq \mathrm{S}^n(\emptyset, M)$ holds for all n. Notice that for any structure M we have that $\mathrm{D}_n(\mathrm{Th}(M)) \supseteq \mathrm{S}^n(\emptyset, M)$. I.e. in Definition 7.28 we introduced a generalization of $\mathrm{D}_n(T)$. The letter S stands for "Stone space" the connection to set theoretic topology (via Stone's representation theorem of boolean algebras is not as central topic in model theory as it used to be thirty years ago, this line is developed in section 8).

The following is an obvious generalization of the notion of definable set (by a first-order formula):

DEFINITION 7.30. Let M be a structure, $A \subseteq |M|$ and α an ordinal. We say that the set $S \subseteq {}^{\alpha}|M|$ *is type-definable over* A iff there exists a set of formulas p in $\mathrm{L}(M)$ with the free variables from $\{x_{\mathbf{i}} \mid i < \alpha\}$ and parameters from A such that

$$S = \{\mathbf{b} \in {}^{\alpha}|M| \ : \ M \models \varphi[\mathbf{b}] \text{ for all } \varphi(\mathbf{x}) \in p\}.$$

We denote this by $S = p(M)$.

Clearly if $\varphi(\mathbf{x}) \in p \subseteq q$ are in $\mathrm{L}(M)$ then $\varphi(M) \supseteq p(M) \supseteq q(M)$. A set of formulas p in $\mathrm{L}(T)$ with variables from $\mathbf{x}$ is a T-type iff there exists $M \models T$ such that $p(M) \neq \emptyset$.

Notice that if M is a proper elementary extension of $\langle \mathbb{R}, +, \cdot, 0, 1 \rangle$ then the set of infinitesimals of M are type definable. I.e.

$$\left\{ a \in |M| \ : \ M \models \left[a \neq 0 \wedge [-\frac{1}{n} < a < \frac{1}{n}] \right], \text{ for every positive integre } n \right\}$$

is type definable, notice that this set is not definable by a single first-order formula.

DEFINITION 7.31. An M-type p is called *algebraic in* M iff the set of elements of M realizing p is finite.

A T-type p is called *algebraic* iff it is algebraic in M for every $M \models T$.

The idea behind the proof of Theorem 7.24 is very fundamental, in order to clarify it I present a theorem due to David Kueker [**Kue1**], while it is not directly related to the material of this section I hope that the reader will appreciate its beauty.

THEOREM 7.32 (Kueker). *Let M be a countable structure such that $\mathrm{L}(M)$ is at most countable. If $\mathrm{Aut}(M)$ is uncountable then $|\mathrm{Aut}(M)| = 2^{\aleph_0}$.*

PROOF. There are two steps, first we show that a certain condition on $\mathrm{Aut}(M)$ implies that there exists $\{f_\eta \in \mathrm{Aut}(M) \mid \eta \in {}^{\omega}2\}$ such that $\eta \neq \nu \in {}^{\omega}2 \implies$

$f_\eta \neq f_\nu$. In the second step we establish that $|\operatorname{Aut}(M)| > \aleph_0$ implies this condition.

Recall that for a structure N we say that N is *rigid* iff $\operatorname{Aut}(N) = \{\mathrm{id}_N\}$. For $\{a_k \mid n\} \subseteq |M|$ let $(M, a_k)_{k<n}$ stand for the structure N which is obtained by expanding the structure M by new n-many constants interpreted by the elements $a_0, \ldots, a_{n-1}$.

LEMMA 7.33 (Main Lemma). *Let M be a countable structure such that $|\operatorname{L}(M)| \leq \aleph_0$. If there exists $\{a_k \mid k < \omega\} \subseteq |M|$ such that $(M, a_k)_{k<n}$ is not rigid for every $n < \omega$ then $|\operatorname{Aut}(M)| = 2^{\aleph_0}$.*

PROOF. Use $\cong_p$ ...

Add Back and Forth construction subsection after Cantor's Theorem... □

LEMMA 7.34 (Second Step). *For M countable, if $\operatorname{Aut}(M)$ is uncountable then there exists $\{a_k \mid k < \omega\} \subseteq |M|$ such that $(M, a_k)_{k<n}$ is not rigid for every $n < \omega$.*

PROOF. □

□

Theorem 7.32 does not generalize to uncountable models: Thomas Jech proved

THEOREM 7.35 ([**Je72**]). *It is consistent with* ZFC *that $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} > \aleph_2$ and there exists a tree M of cardinality $\aleph_1$ with $|\operatorname{Aut}(M)| = \aleph_2$.*

Another related interesting result is

THEOREM 7.36 (Shelah). *Tree with λ branches gives an "abelian" structure M with $\operatorname{Aut}(M)$ of cardinality λ...*

Exercises 1.7.

(1) Let T be a consistent theory, let Γ
(2) Let T be a consistent theory, and let n be a positive integer. Prove that $p \in \mathrm{D}_n(T)$ iff there exists $M \models T$ and there exists $\mathbf{a} \in {}^n|M|$ such that $p = \mathrm{tp}(\mathbf{a}/\emptyset, M)$.
(3) Prove Lemma 7.14.
(4) Suppose $M \models T_{ACF}$ is uncountable and let p be a T_{ACF}-type. Show that if p is not algebraic then it is realized by $\|M\|$-many elements.
(5) Is it necessary in the previous exercise to work in T_{ACF}? Namely, given an uncountable M and an M-type p, does "p not algebraic" implies p is realized by $\|M\|$-many elements of M true?
(6) Use Zorn's lemma to prove Lemma 7.14, instead of using the completeness theorem.
(7) Find a shorter proof to Theorem **??**.
(8) Let M be a model of cardinality $\geq |\mathrm{L}(M)| + \aleph_0 + |D|$, and let $A \subseteq |M|$. Suppose D is a set of M-types over A. Show that there exists a model N of cardinality $\|M\|$ such that $N \succeq M$ and every type from D is realized in N.
(9) Let M be an infinite structure of cardinality $\geq 2^{|\mathrm{L}(M)|}$. Show that M has a proper extension N of the same cardinality as M such that every element of $\mathrm{D}(\mathrm{Th}(M))$ is realized in N.
(10) Let M and N be isomorphic models, and let p be a $\mathrm{Th}(M)$-type. Show that p is realized in M iff p is realized in N.
(11) Let $T = T_f$ (the theory of fields). Define a T-type $p(\mathbf{x})$ which is realized in a field iff the field has a transcendental element over its prime field.
(12) Let $T = \mathrm{CD}(\langle \mathbb{Q}, < \rangle)$. Show that $|\mathrm{D}(T)| = 2^{\aleph_0}$. However $\mathrm{D}(T_{\mathrm{ord}})$ is countable.
(13) Give an example of two fields F and E satisfying T_{ACF_0} and a type p in one variable without parameters such that $E \leq F$ and p is omitted by E but realized by F.
(14) Let M be a model, $A \subseteq |M|$, let p be an M-type over A, and let f be an automorphism of M. By $f(p)$ denote the set $\{\varphi(x; f(\mathbf{a})) \ : \ \varphi(x; \mathbf{a}) \in p\}$. Show the following:
   (a) $f(p)$ is an M-type over $\{f(a) \ : \ a \in A\}$.
   (b) For every $c \in |M|$ the type p is realized by c iff $f(p)$ is realized by $f(c)$.
(15) Let p be a T-type in a finite sequence of variables $\mathbf{x}$. Show that there exists a complete T- type q in $\mathbf{x}$ containing p.
(16) Let M be a structure $A \subseteq |M|$ and $\mathbf{a}, \mathbf{b}$ two sequences of elements of M of the same length. Show that for any $\mathbf{c} \in A$ we have

$$\mathrm{tp}(\mathbf{a}/A, M) = \mathrm{tp}(\mathbf{b}/A, M) \iff \mathrm{tp}(\mathbf{a}\hat{}\mathbf{c}/A, M) = \mathrm{tp}(\mathbf{b}\hat{}\mathbf{c}/A, M).$$

(17) Use Theorem 7.18 to prove Theorem 1.5.8.
(18) Assuming that ZF is consistent show that $|\mathrm{D}(\mathrm{ZF})| = 2^{\aleph_0}$.

(19) Using the notation and assumptions of Claim 7.25 show that if $\mathrm{D}(T)$ is uncountable then $|\mathrm{D}(T)| = 2^{\aleph_0}$.
(20) Without using Proposition 7.23 show that for a countable theory T we have that $\mathrm{I}(\aleph_0, T) = \aleph_0 \implies |\mathrm{D}(T)| = \aleph_0$.
(21) Suppose $M \precneqq N$, let $\mathbf{n}$ be an enumeration of $|N|$. Let $p(\mathbf{x})$ be $\mathrm{tp}(\mathbf{n}/|M|, N)$ and $N^* \succneqq M$. Show that N^* realizes p iff there exists an elementary embedding $f : N \to N^*$ such that $f \restriction |M| = \mathrm{id}_{|M|}$.
(22) What is the role of the assumption that T is countable in Theorem 7.21? Suppose $\lambda := |\mathrm{L}(T)|$ is uncountable. Is the implication $|\mathrm{D}(T)| > \lambda \implies \mathrm{I}(\lambda, T) = 2^\lambda$ true?
Does $|\mathrm{D}(T)| > \lambda \implies \mathrm{I}(\lambda, T) \geq \lambda$ hold?
(23) Let $\mu \geq |\mathrm{L}(T)|$, and let $\lambda > \mu$. If $|\mathrm{D}(T)| \geq \lambda$ then $\mathrm{I}(\mu, T) \geq \mu^+$.
(24) What is the role of μ^+ in Exercise 23?
Does $|\mathrm{D}(T)| \geq \lambda \implies \mathrm{I}(\mu, T) \geq \kappa$ for all $\mu < \kappa \leq \lambda$?
(25)

DEFINITION 7.37. Let $\lambda \geq |T|$ be a cardinal, we say $\mathrm{IE}(\lambda, T) \geq \mu$ iff there exists $\{M_i \mid i < \mu\}$ a family of models of T all of cardinality λ such that for every $i \neq j < \mu$, M_i is not elementarily embeddable into M_j. The equation $\mathrm{IE}(\lambda, T) = \mu$ stands for $\mathrm{IE}(\lambda, T) \geq \mu$ and $\mathrm{IE}(\lambda, T) \not\geq \mu^+$.

Verify that $1 \leq \mathrm{IE}(\lambda, T) \leq \mathrm{I}(\lambda, T)$ for all $\lambda \geq |T|$. Find an example of a theory T such that $\mathrm{IE}(\aleph_0, T) = 1$ but $\mathrm{I}(\aleph_0, T) = \aleph_0$.

(26) Is it possible to replace I by IE in Theorem 7.21?
(27) Let T be first order, suppose that $\mu := |\mathrm{D}(T)|$. Prove that if there exists λ satisfying $\mu > \lambda > |\mathrm{L}(T)| + \aleph_0$ then $\mathrm{IE}(\mu, T) \geq \mu$. Hint: Use Hajnal's free subset Theorem.
(28) Is it possible to replace I by IE in Exercise 23?
(29) Let M be a model, and let N be an elementary extension of M. Suppose $p \subseteq \mathrm{Fml}(\mathrm{L}(M))$ in the variable x. Show that if p is an M-type then p is an N-type.
(30) Verify that the set p defined in the proof of Claim 7.25 is a T-type.
(31) Give an example of two fields F and E satisfying T_{ACF_0} and a type p in one variable without parameters such that $E \leq F$ and p is omitted by E but realized by F.
(32) Let T be a complete theory. Prove that $p \in \mathrm{D}_n(T)$ iff $[p]$ is an ultrafilter in $\mathrm{BA}^n(T)$. Where $[p]$ is the set $\{[\varphi] \ : \ \varphi \in p\}$. See Exercise 2.3.83.
(33) Let T be a countable complete theory.

DEFINITION 7.38. T *has property* (A) iff for every $n < \omega$, $\mathbf{x}$ of length n and every formula $\chi(\mathbf{x})$ such that $T \vdash \exists \mathbf{x} \chi(\mathbf{x})$ there exists $p \in \mathrm{D}_n(T)$ containing $\chi(\mathbf{x})$ and there exists $\psi(\mathbf{x}) \in p$ such that

$$T \vdash \forall \mathbf{x}[\psi(\mathbf{x}) \to \varphi(\mathbf{x})] \text{ holds for every } \varphi(\mathbf{x}) \in p.$$

Prove

THEOREM 7.39. *If* $|\mathrm{D}(T)| < 2^{\aleph_0}$ *then* T *has property* (A).

Conclude that T has property (A) iff $|\mathrm{D}(T)| = \aleph_0$.

(34) Use the completeness theorem, without a direct application of Zorn's lemma (or the axiom of choice) to show that every T-type p can be extended to a complete T-type.

(35) Prove Lemma 7.17.

(36) Let M be a non-standard model of Peano's arithmetic, while the set of non-standard integers in M are not first-order definable (see Exercise 1.5.25) show that it is type-definable.

(37) For every algebraic type p there exists a formula $\varphi(\mathbf{x})$ such that for every $M \models T$ we have that

$$\mathbf{a} \in |M| \text{ is realizing } p \iff M \models \varphi[\mathbf{a}].$$

(38) The purpose of this exercise is to generalize Proposition 7.23. Suppose T is a complete theory and let $\lambda := |\mathrm{L}(T)|$. If $|\mathrm{D}_n(T)| > \lambda$ for some $n < \omega$ then $\mathrm{I}(\lambda, T) \geq |\mathrm{D}_n(T)|$.

(39) Let n be a positive integer. Prove that for any complete first-order T there exists $M \models T$ such that $\mathrm{D}_n(T) = \mathrm{S}^n(\emptyset, M)$, such M can be found of cardinality $\leq 2^{|\mathrm{L}(T)|}$.

(40) Given an infinite M there exists $N \succeq M$ of cardinality $\leq 2^{|\mathrm{L}(M)|+\|M\|}$ such that every M-type over A is realized in N for all $A \subseteq |M|$.

8. Some extensions of first-order logic

A common denominator to many of the applications of the compactness theorem in the previous sections is that first-order logic is too weak to express some interesting mathematical properties. Namely being an Archimedean field, being a periodic group (Proposition 5.19) or being a well-ordered set (Theorem 5.22), or being the standard model of arithmetic (Corollary 5.10) are all properties that can not be captured by (even infinite) sets of first-order sentences.

Already in 1931 this was viewed as a potential defect of first-order logic and the proposed remedy was supposed to be "replace first-order logic with a more expressible logic". The first paper on infinitary logics was published by Ernst Zermelo (see [**Ze**]).

In the fifties and sixties several logics were studied due to their greater expressive power than first order logic. A more detailed exposition can be found in books by Dickmann [**Di**], and Barwise & Feferman [**BaFe**]. It turns out that while it is fairly easy to introduce a stronger logic than first-order, it is difficult to prove interesting theorems about it. There are few logics that are simultaneously stronger than first-order and still they have a rich model theory.

The first half of this section is dedicated to the most popular and the best understood logic of this kind. It is $L_{\omega_1,\omega}$. It is a member of a family of infinitary logics. It is the simplest main-stream infinitary logic. Much work in the field was done in the period from the late fifties to the mid sixties, among the pioneers: Tarski, Hanf, Erdős, Henkin, Scott, Karp, Lopez-Escobar, Chang, Morley and Keisler.

DEFINITION 8.1. Let L be a similarity type. The set of formulas of $L_{\omega_1,\omega}$ is the smallest set of at most countable strings of symbols from $L \cup \{(,),[,],=,\wedge,\vee,\neg,\forall,\exists,\bigwedge,\bigvee\} \cup \{x_n \,:\, n<\omega\}$ such that it contains all first order formulas over L, each formula has only finitely many free variables and is closed under the following rules:

- if Φ is a countable set of formulas such that there exists $n<\omega$ and for every $\varphi\in\Phi$ we have that $\mathrm{FV}(\varphi)\subseteq\{x_0,\dots,x_{n-1}\}$ then both $\bigwedge\Phi$ and $\bigvee\Phi$ are formulas.
- if φ is formula and x a variable than also $\forall x\varphi,\ \exists x\varphi,\ \neg\varphi$ are all formulas.
- if φ and ψ are formulas than also $\varphi\vee\psi$ and $\varphi\wedge\psi$ are formulas.

The semantics of $L_{\omega_1,\omega}$ is defined in the obvious way:

DEFINITION 8.2. Let L be a similarity type, and suppose that M is an L-structure. For every $\mathbf{a}\in|M|$ by induction on the complexity of an $L_{\omega_1,\omega}$-formula ψ we define when $M\models_{\omega_1,\omega}\psi[\mathbf{a}]$ holds:

- if ψ is first order then $M\models_{\omega_1,\omega}\psi[\mathbf{a}]$ iff $M\models\varphi[\mathbf{a}]$.

- if ψ is $\bigwedge \Phi$ then $M \models_{\omega_1,\omega} \psi[\mathbf{a}]$ iff for every $\varphi(\mathbf{x}) \in \Phi$ we have that $M \models_{\omega_1,\omega} \varphi[\mathbf{a}]$ holds.
- if ψ is $\bigvee \Phi$ then $M \models_{\omega_1,\omega} \psi[\mathbf{a}]$ iff there exists $\varphi(\mathbf{x}) \in \Phi$ we have that $M \models_{\omega_1,\omega} \varphi[\mathbf{a}]$ holds.

Due to the fact that the relation $\models_{\omega_1,\omega}$ is a natural extension of $\models$ often the subscript is omitted and the same symbol as for first-order is used.

The following are simple observations (using the definition of $\models$ and De Morgan's laws):

- $M \models \neg \bigwedge \Phi[\mathbf{a}]$ holds iff there exists $\varphi \in \Phi$ such that $M \models \neg\varphi[\mathbf{a}]$ holds which is equivalent to $M \models \bigvee \Psi[\mathbf{a}]$ where $\Psi := \{\neg\varphi \ : \ \varphi \in \Phi\}$.
- $M \models \exists x \bigwedge \Phi[x, \mathbf{a}]$ holds iff there exists $a \in |M|$ such that for every $\varphi(x, \mathbf{y}) \in \Phi$ we have that $M \models \varphi[b, \mathbf{a}]$.

In the following examples we show that several important mathematical properties that can not be captured by first-order logic are easily definable in $L_{\omega_1,\omega}$.

REMARK 8.3. It follows immediately that the statement of compactness theorem fails, in fact even the upward Löwenheim-Skolem theorem fails for the following example.

EXAMPLE 8.4 (Archimedean axiom). : For an ordered field F

$$F \models \forall x \big[x > 0 \to [\bigvee_{n<\omega} \underbrace{x + \cdots + x}_{n\text{-times}} \geq 1]\big] \iff \text{The field } F \text{ is Archimedean.}$$

The fact that the archimedean axiom is $L_{\omega_1,\omega}$-definable makes it possible to have a natural framework for treating normed spaces over ultrametric fields and valuation fileds that occur naturally in number theory:

EXAMPLE 8.5. *Valuation fields*: .... *Normed fields*: *local fields* (see A. Frolich [**Fro**])

EXAMPLE 8.6 (Periodic groups). : Recall that a group G is *periodic* iff for every $a \in G$ there exists a positive integer $n(g)$ such that $a^{n(g)}$ is the identity element of the group.

$$\varphi_{pr} := \bigwedge T_g \wedge \forall x \big[\bigvee_{1\leq n<\omega} x^n = e\big].$$

Check that $M \models \varphi_{pr}$ iff M is a periodic group.

EXAMPLE 8.7 (Locally finite groups). A group G is *locally finite* iff for every finite $X \subseteq G$ the subgroup generated by X is finite:

$$\varphi_{lf} := \bigwedge T_g \wedge \bigwedge_{n<\omega} \forall x_0 \cdots \forall x_{n-1} \bigvee_{k<\omega} \exists y_0 \cdots \exists y_{k-1}$$
$$\bigwedge_{m<\omega} \Big[\bigwedge_{\substack{f:m\to n \\ e:m\to 2}} [\bigvee_{\ell<k} [\prod_{i<m} x_{f(i)}^{e(i)} = y_\ell]] \Big]$$

Check that G is a locally finite group iff $G \models \varphi_{lf}$.

EXAMPLE 8.8 (The standard model of PA).

$$\varphi := \bigwedge \mathrm{PA} \wedge \forall x \big[\bigvee_{n<\omega} x = \underbrace{1 + \cdots + 1}_{n\text{-times}} \big].$$

Notice that $M \models \varphi$ iff $M \cong \langle \omega, +, \cdot \rangle$.

EXAMPLE 8.9 (Free groups generated by a predicate). Let X be an infinite set, $F(X)$ the free group generated by X. In the similarity type $\{\cdot, P\}$ let T_{frg} be the $L_{\omega_1,\omega}$-theory consisting of the conjunction of

- The first-order sxioms of group theory,
- P is infinite,
- The elements of P are free generators,

$$\bigwedge_{\substack{n,m<\omega \\ e:n\to 2 \\ d:m\to 2}} \forall x_0 \ldots x_{n-1} \forall y_0 \ldots y_{m-1} \Big[\bigwedge_{\substack{i<n \\ j<m}} P(x_i) \wedge P(y_j) \to$$
$$[\prod_{k<n} x_i^{1-e(i)} = \prod_{k<m} y_i^{1-d(i)} \to \mathbf{x} = \mathbf{y} \wedge e = d] \Big].$$

- The structure is generated by P, i.e.

$$\forall z \bigvee_{\substack{n<\omega \\ e:n\to 2}} \exists x_0 \cdots x_{n-1} [\bigwedge_{i<n} P(x_i) \wedge z = \prod_{i<n} x_i^{1-d(i)}].$$

Notice that $M \models T_{frg}$ iff M is the free group generated by the set P^M.

Observe that T_{frg} has models in all infinte cardinals and also is totally categorical.

EXAMPLE 8.10 (Simple groups). Let $\mathcal{K} := \{G \mid G \text{ is infinite simple group}\}$, when $\mathrm{L}(\mathcal{K})$ is the language of groups L_{gr} (Example 1.1.2(3)). The class $\mathcal{K}$ is axiomatizable by an $L_{\omega_1,\omega}$ sentence: Recall that G is simple iff it does not have a proper normal subgroup. This is equivalent to

$(*)$ For every $x, y \in G,\ y$ is a finite product of conjugates of integer powers of$x$.

(*) can be axiomazied as:

$$\forall x \forall y [\bigvee_{n<\omega} \exists h_0 \cdots \exists h_{n-1} \bigvee_{e_k \in \mathbb{Z}} [y = \prod_{k<n} h_k g^{e_k} (h_k)^{-1}]].$$

EXAMPLE 8.11 (Algebraically closed fields with infinite transcendence degree). . Let T_{ACF_∞} be the $L_{\omega_1,\omega}$-theory in the similaritry type $\{+, \cdot, 0, 1\}$ consisting of the conjunction of

- The first-order axioms of algebraically closed fields and
- $$\bigwedge_{n<\omega} \exists x_0 \cdots x_{n-1} \psi_n(x_0, \ldots, x_{n-1}),$$
  where ψ_n has the following property: an algebraically closed field $F \models \psi_n[a_0, \ldots, a_{n-1}]$ iff $a_0, \ldots, a_{n-1}$ are algebraically independent over the prime field of F.

Notice that $M \models T_{ACF_\infty}$ iff M is an algebraically closed field of infinite transcendence degree.

Observe that T_{ACF_∞} has models in all infinite cardinals and while it is incomplete, adding the sentence that fixes the characteristics produces a totally categorical theory.

EXAMPLE 8.12 (Derived subgroups are definable). Let G be a group in the language of groups. Its commutator subgroup G' is definable by an $L_{\omega_1,\omega}$-formula: Recall that G' is the subgroup generated by the set $\{aba^{-1}b^{-1} \mid a, b \in G\}$. The formula

$$\psi_D(x) := \bigvee_{n<\omega} \bigvee_{\{e_k | k<n\} \subseteq \mathbb{Z}} [\exists a_0 \cdots \exists a_{n-1} \exists b_0 \cdots \exists b_{n-1} [x = \prod_{k<n} (a_k b_k a_k^{-1} b_k^{-1})^{e_k}]].$$

EXAMPLE 8.13 (Axiomatization of 1-Solvable groups).

$$\forall x [\psi_D(x) \to x = e]$$

EXAMPLE 8.14 (Axiomatization of 2-Solvable groups).

$$\forall x \bigvee_{n<\omega} \bigvee_{\{e_k | k<n\} \subseteq \mathbb{Z}} [\exists a_0 \cdots \exists a_{n-1} \exists b_0 \cdots \exists b_{n-1} \bigwedge_{k<n} [\psi_D(a_k) \wedge \psi_D(b_k)] \wedge [x = \prod_{k<n} (a_k b_k a_k^{-1} b_k^{-1})^{e_k}]] \to x = e].$$

An iteration of the previous construction gives:

EXAMPLE 8.15 (Axiomatization of m-Solvable groups). There is a sentence ψ_m such that $G \models \psi_m$ iff G an infinite group which is m-nilpotent.

Thus

EXAMPLE 8.16 (Axiomatization of Solvable groups). There is a sentence ψ_{solv} such that $G \models \psi_{solv}$ iff G is solvable. take

$$\psi_{solv} := \bigvee_{m<\omega} \psi_m.$$

Further evidence to the increased expressive power of $L_{\omega_1,\omega}$ over first-order logic can be found in the following result of Dana Scott. Its claim is that $\aleph_0$-categoricity can be axiomatized in $L_{\omega_1,\omega}$.

THEOREM 8.17 (Scott). *Given a countable structure M and suppose that $L := \mathrm{L}(M)$ is at most countable. There exists $\psi_M \in L_{\omega_1,\omega}$ such that*

$$N \models \psi_M \textit{ is countable } \implies N \cong M.$$

PROOF. For $\mathbf{a} \in {}^n|M|$ let

$$\varphi_{\mathbf{a}}(x_1, \ldots, x_n) := \bigwedge\{\psi(\mathbf{x}) \text{ is q.f. first order } : M \models \psi[\mathbf{a}]\}.$$

Given two L structures M and N, let

$$F_{M,N} := \{f : A \to |N| \ : \ A \subseteq |M| \text{ finite, such that } \mathbf{a} \in A \implies N \models \varphi_{\mathbf{a}}[f[\mathbf{a}]]\}.$$

To show that for countable M and N we have $M \cong N$ enough to show that $F_{M,N} \neq \emptyset$ and for every $f \in F_{M,N}$ $\forall a \in |M| \exists b \in |N| (\forall b \in |N| \exists a \in |M|)$ such that $f \cup \{\langle a, b\rangle\} \in F_{M,N}$.

Instead of working with $F_{M,N}$ we will work with an appropriately defined $F^*_{M,N} \subseteq F_{M,N}$ to be introduced below. First we define a countable family of $L_{\omega_1,\omega}$ formulas.

For every $\alpha < \omega$, every $n < \omega$ and every $\mathbf{a} \in {}^n|M|$ define (by induction on α) formulas $\{\varphi^\alpha_{\mathbf{a}}(x_1, \ldots, x_n) \in L_{\omega_1,\omega} \mid \alpha < \omega_1, n < \omega, \mathbf{a} \in {}^n|M|\}$ as follows:

For $\alpha = 0$:

$$\varphi^0_{\mathbf{a}}(x_1, \ldots, x_n) := \varphi_{\mathbf{a}}(\mathbf{x}).$$

For successor:

$$\varphi^{\alpha+1}_{\mathbf{a}}(x_1, \ldots, x_n) := \varphi^\alpha_{\mathbf{a}}(\mathbf{x}) \wedge \bigwedge_{a\in|M|} \exists y \varphi^\alpha_{\mathbf{a}^\frown a}(\mathbf{x}; y)) \wedge \forall y[\bigvee_{a\in|M|} \varphi^\alpha_{\mathbf{a}^\frown a}(\mathbf{x}; y))].$$

For limit α:

$$\varphi^\alpha_{\mathbf{a}}(\mathbf{x}) := \bigwedge_{\gamma<\alpha} \varphi^\gamma_{\mathbf{a}}(\mathbf{x}).$$

Notice that

$$\alpha < \beta < \omega_1 \implies M \models [\varphi^\beta_{\mathbf{a}}(\mathbf{x}) \to \varphi^\alpha_{\mathbf{a}}(\mathbf{x})] \text{ for all } \mathbf{a} \in |M|.$$

Since M is countable for every $\mathbf{a} \in |M|$ there exists $\alpha(\mathbf{a}) < \omega_1$ such that

$$\beta \geq \alpha(\mathbf{a}) \implies \varphi^{\beta}_{\mathbf{a}}(M) = \varphi^{\alpha(\mathbf{a})}_{\mathbf{a}}(M).$$

Using that M is countable once more, we get that

$$\alpha^* := \sup_{\mathbf{a} \in {}^{\omega >}|M|} \alpha(\mathbf{a}) \quad \text{is a countable ordinal.}$$

Finally define

$$\psi_M := \varphi^{\alpha^*}_{\langle\rangle} \wedge \bigwedge_{\mathbf{a} \in {}^{<\omega}|M|} \forall \mathbf{x}[\varphi^{\alpha^*}_{\mathbf{a}}(\mathbf{x}) \to \varphi^{\alpha^*+1}_{\mathbf{a}}(\mathbf{x})].$$

As $M \models \varphi^{\alpha^*}_{\mathbf{a}}[\mathbf{a}]$ for every $\mathbf{a} \in |M|$, clearly $M \models \psi_M$.

Given a countable $N \models \psi_M$, we establish that $M \cong N$ by showing that

$$F^*_{M,N} := \{f : A \to |N| \ : \ A \subseteq |M| \text{ finite, such that } \mathbf{a} \in A \implies N \models \varphi^{\alpha^*}_{\mathbf{a}}[f[\mathbf{a}]]\}$$

is not empty and has the back and forth property.

Since both M and N satisfy $\varphi_{\langle\rangle}$ we have that $\emptyset \in F^*_{M,N}$, thus we are left to verify the B & F property.

Suppose $f \in F^*_{M,N}$ and we are given $a \in |M|$, denote $\mathbf{a} := \operatorname{dom} f$, $\mathbf{b} := \operatorname{range} f$ satisfying $f(\mathbf{a}) = \mathbf{b}$ namely

$$N \models \varphi^{\alpha^*}_{\mathbf{a}}[\mathbf{b}]$$

we have to find $b \in |N|$ such that $N \models \varphi^{\alpha^*}_{\mathbf{a}^\frown a}[\mathbf{b}, b]$.

From $N \models \varphi^{\alpha^*}_{\mathbf{a}}[\mathbf{b}] \wedge \psi_M$ using $N \models \forall \mathbf{x}[\varphi^{\alpha^*}_{\mathbf{a}}(\mathbf{x}) \to \varphi^{\alpha^*+1}_{\mathbf{a}}(\mathbf{x})]$ we get that $N \models \varphi^{\alpha^*+1}_{\mathbf{a}}[\mathbf{b}]$, using the second part of the definition of the $\alpha + 1$th formulas we get $N \models \exists y \varphi^{\alpha^*}_{\mathbf{a},a}(\mathbf{b}, y)$. Let $b \in |N|$ such that

$$N \models \varphi^{\alpha^*}_{\mathbf{a}^\frown a}[\mathbf{b}, b].$$

This established that $f \cup \{\langle a, b\rangle\} \in F^*_{M,N}$.

The other case is similar. Given $b \in |N|$, from $N \models \varphi^{\alpha^*}_{\mathbf{a}}[\mathbf{b}]$ using the property of α^* and the third part of the definition of $\varphi^{\alpha+1}$ there exists $a \in |M|$ such that

$$N \models \varphi^{\alpha^*}_{\mathbf{a}^\frown a}[\mathbf{b}, b]$$

This means that $f \cup \{\langle a, b\rangle\} \in F^*_{M,N}$.

The verification of the claim is by following the definition of the formulas $\varphi^{\alpha}_{\mathbf{a}}$ and is left to the reader. □

DEFINITION 8.18. The sentence ψ_M in the previous theorem is called *Scott Sentence of* M.

The smallest $\alpha < \omega_1$ such that

$$M \models [\varphi^{\alpha}_{\mathbf{a}}(\mathbf{x}) \to \varphi^{\alpha+1}_{\mathbf{a}}(\mathbf{x})] \text{ for all } \mathbf{a} \in |M|$$

is called the *Scott height of* M.

Let us introduce a generalization of elementary equivalence:

DEFINITION 8.19. Let M and N be L structures. We say that M and N are $L_{\omega_1,\omega}$*-equivalent* iff

$$M \models \varphi \iff N \models \varphi \quad \text{for every } L_{\omega_1,\omega}\text{-sentence.}$$

We denote this by $M \equiv_{\omega_1,\omega} N$.

In light of Theorem 5.22 a natural question is whether well-ordering is definable in $L_{\omega_1,\omega}$?

EXAMPLE 8.20. Let α be a fixed countable ordinal.
Let $M = \langle |M|, P, <, \ldots, c_\beta\rangle_{\beta<\alpha}$ be a structure such that $P^M = \alpha$, $<^M$ is the $\in$ relation on α and $c^M_\beta = \beta$ for all $\beta < \alpha$. Let

$$\psi := \forall x\big[P(x) \to \bigvee_{\beta<\alpha} c_\beta = x\big].$$

Notice that $M \models \psi$ and if $N \equiv_{\omega_1,\omega} M$ then $\langle P^N, <^N\rangle$ is well ordered and isomorphic to the ordinal α

The previous example can be altered to capture a given countable ordinal without using any constants:

EXAMPLE 8.21 (pin down countable ordinals). Given $\alpha < \omega_1$, define

$$\varphi_\alpha(x) := \begin{cases} \neg\exists y[y < x] & \text{if } \alpha = 0 \\ \\ \forall y[y < x \leftrightarrow \bigvee_{\beta<\alpha} \varphi_\beta(y)] & \text{otherwise.} \end{cases}$$

Now let

$$\psi_\alpha := [\forall x \bigvee_{\beta<\alpha} \varphi_\beta(x)].$$

Verify that if $M := \langle |M|, <, \ldots\rangle$ is linearly ordered then

$$M \models \psi_\alpha \iff \langle |M|, <\rangle \cong \alpha.$$

So it remains to ask whether *uncountable* well-ordering is definable in $L_{\omega_1,\omega}$? Later (Theorem 7.38) we will show that the answer is no. It turns out that the indefinability of well-ordering in $L_{\omega_1,\omega}$ is of central importance as it can be used to get a replacement of the upward Löwenheim Skolem theorem and several other corollaries of the compactness theorem.

Below are several natural extensions of $L_{\omega_1,\omega}$:

DEFINITION 8.22. Let λ be an infinite cardinal number.

(1) *The logic* $L_{\lambda^+,\omega}$: Replace the first clause in Definition 8.1 with:
For a finite sequence of variables $\mathbf{x}$

$$\{\varphi_\alpha(\mathbf{x}) \mid \alpha < \lambda\} \subseteq L_{\lambda^+,\omega} \implies \bigvee_{\alpha<\lambda} \varphi_\alpha(\mathbf{x}) \in L_{\lambda^+,\omega}.$$

and for every variable y and any finite sequence of variables $\mathbf{x}$

$$\varphi(y,\mathbf{x}) \in L_{\omega_1,\omega} \implies \mathbf{Q}y\varphi(y,\mathbf{x}) \in L_{\omega_1,\omega}$$

for $\mathbf{Q} \in \{\forall, \exists\}$.

(2) Let μ be an infinite cardinal number. *The logic* $L_{\lambda,\mu}$: Replace the first clause in Definition 8.1 with:
If $\{\varphi_\alpha(\mathbf{x}) \mid \alpha < \lambda\} \subseteq L_{\lambda,\mu}$ with $\ell(\mathbf{x}) < \mu$ then

$$\bigvee_{\alpha<\lambda} \varphi_\alpha(\mathbf{x}) \in L_{\lambda,\mu}$$

and for every sequence $\mathbf{y} = \langle y_\beta \mid \beta < \alpha < \lambda\rangle$ and every sequence $\mathbf{x}$ of length $< \mu$:

$$\varphi(\mathbf{y};\mathbf{x}) \in L_{\lambda,\mu} \implies \mathbf{Q}_0 y_0 \mathbf{Q}_1 y_1 \cdots \mathbf{Q}_\beta y_{\beta\beta<\alpha} \varphi(\mathbf{y};\mathbf{x}) \in L_{\lambda,\mu}$$

there exists $\mathbf{Q} \in \{\forall, \exists\}$ such that $\mathbf{Q}_\beta = \mathbf{Q}$ for all $\beta < \alpha$.

(3) Let μ be an infinite cardinal number. *The logic* $L_{\infty,\mu}$:

$$L_{\infty,\mu} := \bigcup_{\lambda\in\mathrm{Card}} L_{\lambda,\mu}.$$

(4) *The logic* $L_{\infty,\infty}$:

$$L_{\infty,\infty} := \bigcup_{\mu\in\mathrm{Card}} L_{\infty,\mu}.$$

In an analogs way to Definition 8.2 we can define the relations $\models_{\lambda,\mu}$, $\models_{\infty,\mu}$ and $\models_{\infty,\infty}$. As before when $\mathbf{a} \in |M|$ and $\varphi(\mathbf{x}) \in L_{\lambda,\kappa}$ we will write $M \models \varphi[\mathbf{a}]$ for $M \models_{\lambda,\kappa} \varphi[\mathbf{a}]$.

Similarly to Definition 8.19 we can define $\equiv_{\lambda,\mu}$, $\equiv_{\infty,\mu}$ and $\equiv_{\infty,\infty}$.

Notice that the logic $L_{\omega,\omega}$ is first-order logic.

Like with first-order formulas, for infinitary logic there is a natural definition of types.

DEFINITION 8.23. Let Δ be a class of formulas from $L_{\infty,\infty}$ in a language L. If M is an L-structure, $A \subseteq |M|$ and $\mathbf{b}$ an arbitrary sequence of elements from A (in this definition sequences can be infinite).

$$\mathrm{tp}_\Delta(\mathbf{b}/A, M) := \{\varphi(\mathbf{x};\mathbf{a}) \mid M \models \varphi[\mathbf{a};\mathbf{b}], \varphi(\mathbf{x};\mathbf{y}) \in \Delta, \mathbf{a} \in A\}.$$

$\mathrm{tp}_{\omega,\omega}(\mathbf{b}/A, M)$ means that Δ is the set of first-order formulas in the similarity type of M.

$\mathrm{tp}_{\omega_1,\omega}(\mathbf{b}/A, M)$ means that Δ is all the formulas from $L_{\omega_1,\omega}$ etc.

The reader is invited to verify that the logics introduced in the previous definition have increasing expressive power. Namely:

$$L_{\lambda_1,\mu} \subsetneq L_{\lambda_2,\mu} \quad \text{for all } \lambda_1 < \lambda_2 \quad \text{and} \quad L_{\lambda,\mu_1} \subsetneq L_{\lambda,\mu_2} \quad \text{for all } \mu_1 < \mu_2.$$

Also we have

$$L_{\lambda,\mu} \subsetneq L_{\infty,\mu} \text{ and } L_{\lambda,\mu} \subsetneq L_{\lambda,\infty}.$$

There are properties that are expressible by an $L_{\aleph_1,\aleph_1}$ sentence and by not even by $L_{\infty,\omega}$. The following is a typical example:

REMARK 8.24 (Well ordering is definable in $L_{\aleph_1,\aleph_1}$). Let L be the similarity type consisting of the single binary relation $\{<\}$.

$$\psi := \bigwedge T_{lo} \wedge \neg\exists x_0 \exists x_1 \cdots x_n \exists x_{n+1} \cdots [\bigwedge_{i<j<\omega} x_i > x_j].$$

Clearly $\langle P, <\rangle \models \psi$ iff it is a well-ordered set. While later we will show that for every cardinal λ for arbitrary large ordinal α there exists $\langle P, <\rangle$ not well ordered such that $\langle P, <\rangle \equiv_{\lambda^+,\omega} \langle \alpha, \in\rangle$.

As in the case of first-order formulas we have the following connection between isomorphism and the relation $\equiv_{\infty,\infty}$

PROPOSITION 8.25. *If $g : M \cong N$ then*

$$M \models \varphi[\mathbf{a}] \iff N \models \varphi[g(\mathbf{a})] \quad \textit{for all } \varphi(\mathbf{x}) \in L_{\infty,\infty} \textit{ and every } \mathbf{a} \in |M|.$$

PROOF. Similar to the proof of Claim 2.17. □

Recall the relation $\cong^p$ introduced in Exercise 2.44. At first glance the following special case of a theorem of Carol Karp may look a bit surprising:

THEOREM 8.26 (Karp). *Let M and N be L-structures.*

$$M \cong^p N \iff M \equiv_{\infty,\omega} N.$$

Before proving the theorem let me introduce a concept that is natural for the proof and it will be used repeatedly in later chapters.

DEFINITION 8.27. Let L be a similarity type, and let Δ be a subclass of the $L_{\infty,\infty}$ formulas. For two L-structures M, N and $A \subseteq |M|$, $B \subseteq |N|$. A function $g : A \to B$ is *a Δ-(M, N)-mapping* iff

$$M \models \varphi[\mathbf{a}] \iff N \models \varphi[g(\mathbf{a})] \quad \text{for every } \varphi \in \Delta \text{ and every } \mathbf{a} \in \mathrm{dom}\, f.$$

Where $g(\mathbf{a}) = \langle g(a_0), \ldots, g(a_i)\rangle_{i<\alpha}$ for $\alpha = \ell(\mathbf{x})$.

If Δ is the set of first-order formulas over L we omit the parameter Δ and say that f *is an* (M, N)*-mapping*.

The following special case when Δ is the set of first-order quantifier free formulas appeared already in Exercise 44:

DEFINITION 8.28. Let M and N be both$L$-structures. A function f *is a partial isomorphism from* M *to* N provided there exists $A \subseteq |M|$ such that $f : A \to |N|$ satisfying:

$M \models \varphi[\mathbf{a}] \iff N \models \varphi[f(\mathbf{a})]$ for every $\mathbf{a} \in A$ and every quantifier free formula φ.

The following observations may clarify the concept:

PROPOSITION 8.29.

(1) *Let* M, N *be* L*-structures. If* g *is a partial isomorphism from* M *into* N *then* g *is* qf*-*(M, N)*-mapping, when* qf *is the class of quantifier free formulas in* $L_{\infty,\infty}$.
(2) *If the formula* $x = x$ *is an element of* Δ *then any* Δ*-*(M, N)*-mapping must be injective.*
(3)

PROOF. Exercise. □

THEOREM 8.30. *Let* M *and* N *be* L*-structures. If* $\mathcal{F} : M \cong^p N$ *then every* $g \in \mathcal{F}$ *is an* $L_{\infty,\omega}$*-*(M, N)*-mapping.*

PROOF. See Theorem 8.33. □

PROOF OF THEOREM 8.26. ($\Longrightarrow$): Suppose $\mathcal{F}$ is a non-empty family such that $\mathcal{F} : M \cong^p N$.

For every $g \in \mathcal{F}$ let $n(g) := |\operatorname{dom} g|$.

By induction on $\varphi(x_1, \ldots, x_n) \in L_{\infty,\omega}$ show that for all $g \in \mathcal{F}$ with $n(g) = n$ and every $a_1, \ldots, a_n \in \operatorname{dom} g$ we have

$$(*)_\varphi \quad M \models \varphi[a_1, \ldots, a_n] \iff N \models \varphi[g(a_1), \ldots, g(a_n)].$$

If φ is an atomic formula than $(*)_\varphi$ hold since g is a partial isomorphism. The case that φ is obtained from shorter formulas using propositional connectives follows from the inductive hypothesis.

If $\varphi(\mathbf{x}) = \exists y \psi(\mathbf{x}; y)$ use the back and forth property of $\mathcal{F}$ together with the hypothesis $(*)_\psi$.

($\Longleftarrow$): Notice that the assumption that $M \equiv_{\infty,\omega} N$ implies that

□

Is a similar characterization possible for $\equiv_{\infty,\mu}$? The answer is yes and requires a natural generalization of $\cong^p$:

DEFINITION 8.31. Let M and N be L-structures. Let $\mathcal{F}$ be a non-empty family of partial isomorphisms from M in to N[19].

$\mathcal{F} : M \cong^p_\mu N$ stands for the following requirements:

(1) For every $f \in \mathcal{F}$ and every $A \subseteq |M|$ of cardinality $< \mu$ there exists $g \in \mathcal{F}$ extending f such that $\mathrm{dom}\, g \supseteq A$ and
(2) For every $f \in \mathcal{F}$ and every $B \subseteq |N|$ of cardinality $< \mu$ there exists $g \in \mathcal{F}$ extending f such that $\mathrm{range}\, g \supseteq B$.

As before $M \cong^p_\mu N$ stands for "there exists a non-empty $\mathcal{F}$ such that $\mathcal{F} : M \cong^p_\mu N$".

EXAMPLE 8.32. If $g : M \cong N$ then $\mathcal{F} := \{g \restriction A \ : \ A \subseteq |M|,\ |A| < \mu\}$ satisfies $\mathcal{F} : M \cong^p_\mu N$.

Now we can state a pleasing theorem:

THEOREM 8.33 (Karp's test). *Let M and N be L-structures and suppose μ is an infinite cardinal number..*

$$M \equiv_{\infty,\mu} N \iff M \cong^p_\mu N.$$

PROOF. $\Longrightarrow$:

Suppose $M \equiv_{\infty,\mu} N$. Consider

$$\mathcal{F} := \Big\{ f : A \to |N| \ : \ A \subseteq |M|, |A| < \mu \text{ for every } \mathbf{a} \in A \text{ a sequence of length } < \mu$$
$$M \models \varphi[\mathbf{a}] \iff N \models \varphi[f[\mathbf{a}]] \text{ for every } \varphi(\mathbf{x}) \in \mathrm{Fml}(L_{\infty,\mu}) \Big\}.$$

Since $M \equiv_{\infty,\mu} N$ the empty function is an element of $\mathcal{F}$, thus it's not empty. We left the verification of the back and forth property. As the arguments are similar we provide details only for part (1) of the previous Definition.

Let $f \in \mathcal{F}$ and $A \subseteq |M|$ be of cardinality $< \mu$ be given. Denote by $\mathbf{a}$ an enumeration of $\mathrm{dom}\, f$ and by $\mathbf{a}_0$ an enumeration of the set A.

For any cardinal λ let $p_\lambda(\mathbf{x};\mathbf{y}) := \mathrm{tp}_{\lambda,\mu}(\mathbf{a};\mathbf{a}_0/\emptyset, M)$.

Let $\varphi_\lambda(\mathbf{x};\mathbf{y}) := \bigwedge p_\lambda(\mathbf{x};\mathbf{y})$.

[19]Partial isomorphisms are not required to have finite domains!

As $M \models \varphi_\lambda[\mathbf{a}; \mathbf{a}_0]$, we have

$$(*) \quad M \models \exists \mathbf{y} \varphi_\lambda(\mathbf{a}; \mathbf{y}).$$

Since $f \in \mathcal{F}$ an application of f to (*) gives

$$(**) \quad N \models \exists \mathbf{y} \varphi_\lambda(f(\mathbf{a}); \mathbf{y}).$$

Thus there exists $\mathbf{b}_\lambda \in |N|$ such that $N \models \varphi_\lambda[f(\mathbf{a}); \mathbf{b}_\lambda]$.

Since the cardinals is a proper class and $|N|$ is a set, an application of the pigeon hole principle to $\lambda \mapsto \mathbf{b}_\lambda$ gives us $\mathbf{b} \in |N|$ and a proper class of cardinals C such that $\lambda \in C \implies \mathbf{b}_\lambda = \mathbf{b}$.

Since C is a proper class of cardinals we have $L_{\infty,\mu} = \bigcup_{\lambda \in C} L_\lambda, \mu$, the function $g := f \cup \{\langle \mathbf{a}_0, \mathbf{b} \rangle\}$ is as required.

$\Longleftarrow$: Suppose that $\mathcal{F}$ is witnessing $M \cong^p_\mu N$. We will show by induction on $\varphi(\mathbf{x}) \in \mathrm{Fml}(L_{\infty,\mu})$ that

$$(*)_\varphi \quad M \models \varphi[\mathbf{a}], f \in \mathcal{F}, \mathbf{a} = \mathrm{dom}(f) \implies N \models \varphi[f(\mathbf{a})].$$

Note that as the set of formulas is closed under $\neg$ also the converse of the implication in $(*)_\varphi$ holds.

When φ is atomic $(*)_\varphi$ holds since f is a partial isomorphism. In case φ is built from shorter formulas using logical connectives, follows by induction hypothesis.

We left to deal with the case $\varphi(\mathbf{x})$ is $\exists \mathbf{y} \psi(\mathbf{x}; \mathbf{y})$. If $M \models \varphi[\mathbf{a}]$ then there exists $\mathbf{b} \in |M|$ such that

$$(**) \quad M \models \psi[\mathbf{a}; \mathbf{b}].$$

Using the back and forth property, there exists $g \in \mathcal{F} \supseteq f$ such that $\mathbf{b} \subseteq \mathrm{dom}\, f$. An application of g to $(**)$ gives $N \models \psi[g(\mathbf{a}); g(\mathbf{b})]$, thus $N \models \exists \mathbf{y} \psi(g(\mathbf{a}); \mathbf{y})$. Since $f(\mathbf{a}) = g(\mathbf{a})$ we get $N \models \varphi[f(\mathbf{a})]$. □

The following corollary of Karp's test will be used to provide an axiomatization for Abstract Elementary Classes (see Theorem **??**). One can prove a more general statement that is difficult to read, I will limit to the case when μ is a successor. Notice the similarity to the construction of Scott sentences.

THEOREM 8.34. *Let L be as similarity type and $\mu \geq |L| + \aleph_0$. For every L-structure M of cardinality $\leq 2^\mu$ there exists a sentence $\psi_M \in L_{(2^\mu)^+,\mu^+}$ such that if N in an L-structure satisfying ψ_M then $N \equiv_{\infty,\mu^+} M$.*

PROOF. For every $\mathbf{a} \in {}^{\leq \mu}|M|$ and $\alpha < (2^\mu)^+$ define $\varphi^\alpha_{\mathbf{a}}(\mathbf{x}) \in \mathrm{Fml}(L_{(2^\mu)^+,\mu^+})$. by induction on α;

For $\alpha = 0$:

$$\varphi^0_{\mathbf{a}}(\mathbf{x}) := \bigwedge \mathrm{tp}_{qf}(\mathbf{a}/\emptyset, M)^{20}.$$

For α limit; let $\varphi^\alpha_{\mathbf{a}}(\mathbf{x}) := \bigwedge_{\gamma<\alpha} \varphi^\gamma_{\mathbf{a}}(\mathbf{x})$.

For $\alpha = \gamma + 1$:

$$\varphi^\alpha_{\mathbf{a}}(\mathbf{x}) := \varphi^\gamma_{\mathbf{a}}(\mathbf{x}) \wedge \bigwedge_{\mathbf{a}_0 \in {}^{\leq\mu}|M|} \exists \mathbf{y} \varphi^\gamma_{\mathbf{a}^\frown\mathbf{a}_0}(\mathbf{x};\mathbf{y}) \wedge \forall \mathbf{y} \bigvee_{\mathbf{a}_0 \in {}^{\leq\mu}|M|} \varphi^\gamma_{\mathbf{a}^\frown\mathbf{a}_0}(\mathbf{x};\mathbf{y}).$$

Note that as the cardinality of ${}^{\leq\mu}|M|$ is at most $(2^\mu)^\mu = 2^\mu$. The formulas are from $L_{(2^\mu)^+,\mu^+}$. The fact that $M \models \varphi^\alpha_{\mathbf{a}}[\mathbf{a}]$ for every α can be verified by induction on α.

Clearly for any $\mathbf{a}$ we have that $\alpha < \beta \implies \varphi^\beta_{\mathbf{a}}(M) \subseteq \varphi^\alpha_{\mathbf{a}}(M)$.

For a fixed $\mathbf{a}$, as $\|M\| \leq 2^\mu$ there is $\alpha(\mathbf{a}) < 2^\mu)^+$ such that

$$\beta > \alpha(\mathbf{a}) \implies \varphi^\beta_{\mathbf{a}}(M) = \varphi^{\alpha(\mathbf{a})}_{\mathbf{a}}(M).$$

Consider

$$\alpha^* := \sup\{\alpha(\mathbf{a}) \mid \mathbf{a} \in {}^{\leq\mu}|M|\}.$$

Notice that $\alpha^* < (2^\mu)^+$.

Let ψ_M be the following sentence:

$$\varphi^{\alpha^*}_{\langle\rangle} \wedge \bigwedge_{\mathbf{a} \in {}^{\leq\mu}|M|} \forall \mathbf{x}[\varphi^{\alpha^*}_{\mathbf{a}}(\mathbf{x}) \to \varphi^{\alpha^*+1}_{\mathbf{a}}(\mathbf{x})].$$

Enough to show that if N is an L-structure such that $N \models \psi_M$ then $N \equiv_{\infty,\mu^+} M$.

Using Karp's theorem it is enough to show that

$$\mathcal{F} := \{f : A \to |N| \ : \ A \subseteq |M| \text{ of card } \leq \mu \text{ such that } \mathbf{a} \in A \implies N \models \varphi^{\alpha^*}_{\mathbf{a}}[f[\mathbf{a}]]\}$$

is not empty and has the back and forth property.

Since $N \models \varphi^{\alpha^*}_{\langle\rangle}$ we have that $\emptyset \in \mathcal{F}$.

Let $f \in \mathcal{F}$ and $\mathbf{a}_0$ a sequence (without repetitions) of elements of M of length $\leq \mu$ be given. Denote by $\mathbf{a}$ an enumeration of the domain of f.

Since $M \models \varphi^{\alpha^*}_{\mathbf{a}^\frown\mathbf{a}_0}[\mathbf{a};\mathbf{a}_0]$, by the definition of α^* we have also $M \models \varphi^{\alpha^*+1}_{\mathbf{a}^\frown\mathbf{a}_0}[\mathbf{a};\mathbf{a}_0]$. By the second conjunct in the definition of the successor case we get $M \models \exists \mathbf{y} \varphi^{\alpha^*}_{\mathbf{a}^\frown\mathbf{a}_0}[\mathbf{a};\mathbf{y}]$. Since $f \in \mathcal{F}$ we get $N \models \exists \mathbf{y} \varphi^{\alpha^*}_{\mathbf{a}^\frown\mathbf{a}_0}[f[\mathbf{a}];\mathbf{y}]$, let $\mathbf{b}_0 \in |N|$ be such that $N \models \varphi^{\alpha^*}_{\mathbf{a}^\frown\mathbf{a}_0}[f[\mathbf{a}];\mathbf{b}_0]$. Thus $g := f \cup \{\langle \mathbf{a}_0, \mathbf{b}_0\rangle\}$ is as required.

[20]Note that when $\mathbf{a} = \langle\rangle$ the empty sequence then $\varphi^0_{\langle\rangle}$ is the conjunction of all first-order quantifier-free sentences that hold in M.

Given $\mathbf{b}_0$ a sequence (without repetitions) of elements of $|N|$. Since $N \models \varphi^{\alpha^*}_{\mathbf{a}}[f[\mathbf{a}]]$, using that $N \models \psi_M$ we have that $N \models \varphi^{\alpha^*+1}_{\mathbf{a}}[f[\mathbf{a}]]$ holds. Using the third conjunct in the definition of the formulas we have that there exists $\mathbf{a}_0 \in |M|$ such that $N \models \varphi^{\alpha^*}_{\mathbf{a}\hat{}\mathbf{a}_0}[f[\mathbf{a}], \mathbf{b}_0]$. Now $g := f \cup \{\langle \mathbf{a}_0, \mathbf{b}_0\rangle\}$ is as required. □

For two L-structures M, N and $\Phi \subseteq L_{\infty,\infty}$ We can define $M \preceq_{TV,\Phi} N$ in a completely analogous way to Definition 3.8. When $\Phi = L_{\infty,\mu}$ we write $M \preceq_{\infty,\mu} N$ instead of $M \preceq_{TV,L_{\infty,\mu}} N$. The relation $M \preceq_{\lambda,\mu} N$ has a similar definition.

Similarly to Lemma 3.12. We want to show that for $\lambda \in \mathrm{Card} \cup \{\infty\}$ the relation $M \preceq_{\lambda,\mu} N$ is equivalent to $\langle M, \mathbf{a}\rangle_{a\in|M|} \equiv_{\lambda,\mu} \langle N, \mathbf{a}\rangle_{a\in|M|}$. Actually it is possible to prove little more, for the statement we need the technical device of the set of subformulas of φ: Extending $Sub : \mathrm{Fml}(L) \to \mathcal{P}(\mathrm{Fml}(L))$ from Definition 2.19 we can define

$$Sub : \mathrm{Fml}(L_{\infty,\infty}) \to \mathcal{P}(\mathrm{Fml}(L_{\infty,\infty}))$$

in an obvious way.

LEMMA 8.35 (Tarski-Vaught test). *Let M and N be L-structures and suppose μ is an infinite cardinal number, λ is an infinite cardinal number or $\lambda = \infty$. For $\Phi \subseteq L_{\lambda,\mu}$ such that $\varphi \in \Phi \implies Sub(\varphi) \subseteq \Phi$. The following are equivalent*

(1) $M \preceq_\Phi N$
(2) $M \subseteq N$ *and* $\langle M, a\rangle_{a\in|M|} \equiv_\Phi \langle N, a\rangle_{a\in|M|}$.

PROOF. Exercise. □

The following is a characterization of $\preceq_{\infty,\omega}$ using $\cong^p$:

PROPOSITION 8.36. *Let $M \subseteq N$ be L-structures.*

$$M \preceq_{\infty,\omega} N \iff \langle M, \mathbf{a}\rangle \cong^p \langle N, \mathbf{a}\rangle \quad \textit{for every finite } \mathbf{a} \in |M|.$$

PROOF. Exercise. □

The relation $\preceq_{\infty,\mu}$ and even $\preceq_{\omega_1,\omega}$ are sometimes difficult to manage since they involve too many formulas. E.g. $L_{\omega_1,\omega}$ has $2^{\aleph_0}$ many formulas even for a finite similarity type. This make the issue of finding elementary substructures of a given model problematic. There are two way to circumvent this difficulty. The first is to "slice" $L_{\infty,\omega}$ into layers according to the quantifier rank of formulas and the other is using the notion of fragment.

DEFINITION 8.37. For $\varphi(\mathbf{x}) \in L_{\infty,\omega}$ define an ordinal $qr(\varphi)$ called the *quantifier rank of* φ by induction on φ:

- If φ is quantifier free then
$$qr(\varphi) = 0.$$
- If $\varphi(\mathbf{x}) = \exists y \psi(y; \mathbf{x})$ then
$$qr(\varphi) := qr(\psi) + 1.$$
- If $\varphi = \bigwedge_{i<\gamma} \psi_i$ for some $\{\psi_i \mid i < \gamma\} \subseteq L_{\infty,\omega}$ then
$$qr(\varphi) := \sup_{i<\gamma} qr(\psi_i).$$

Now $\prec^{\alpha}_{\infty,\kappa}$ and $\prec^{\alpha}_{\lambda,\kappa}$ stand for $\preceq_{TV,\Phi}$ where $\Phi = \{\varphi \in L_{\infty,\kappa} \mid rk(\varphi) \leq \alpha\}$ and $\Phi = \{\varphi \in L_{\lambda,\kappa} \mid rk(\varphi) \leq \alpha\}$ respectively.

There are several generalizations of the Downward Löwenheim-Skolem-Tarski theorem:

THEOREM 8.38. *Let N be an L structure. For every $A \subseteq |N|$ and every infinite cardinal λ there exists $M \preceq_{\lambda^+,\omega} N$ of cardinality $\leq |L| + 2^{\lambda} + |A|$ containing the set A.*

PROOF. Let $\Phi := L_{\lambda^+,\omega}$.

Fix a well-ordering $<$ of the set $|N|$, denote by b_0 its first element. For every formula $\varphi(x; \mathbf{y}) \in \Phi$ and every $\mathbf{a} \in |N|$ define the following function:

$$G_{\varphi}(\mathbf{a}) := \begin{cases} Min\{b \in |N| \ : \ N \models \varphi[b, \mathbf{a}]\} & \text{if } N \models \exists x \varphi[x; \mathbf{a}] \\ \\ b_0 & \text{otherwise.} \end{cases}$$

Take $B \subseteq |N|$ a subset closed under the functions $\{G_{\varphi} \mid \varphi \in \Phi\}$. Now verify that B determines a universe of a $\preceq_{\lambda^+,\omega}$-submodel of N of the required cardinality. □

COROLLARY 8.39.

THEOREM 8.40. *Let N be an L structure. For every $A \subseteq |N|$ and every infinite cardinals λ, κ there exists $M \preceq_{\lambda^+,\kappa^+} N$ of cardinality $\leq |L| + 2^{\lambda} + |A|^{\kappa}$ containing the set A.*

PROOF. Fix a well-ordering $<$ of the set

$$\mathcal{P}_{\leq\kappa}(|N|) := \{S \subseteq |N| \ : \ |S| \leq \kappa\}.$$

Denote by $\mathbf{b}_0$ its first element. For every formula $\varphi(\mathbf{x}; \mathbf{y}) \in L_{\lambda^+,\kappa^+}$ and every $\mathbf{a} \in \mathcal{P}_{\leq\kappa}(|N|)$ define the following function:

$$G_{\varphi}(\mathbf{a}) := \begin{cases} Min\{\mathbf{b} \in \mathcal{P}_{\leq\kappa}(|N|) \ : \ N \models \varphi[\mathbf{b}, \mathbf{a}]\} & \text{if } N \models \exists \mathbf{x} \varphi[\mathbf{x}; \mathbf{a}] \\ \\ \mathbf{b}_0 & \text{otherwise.} \end{cases}$$

Clearly the closure of A under the functions $\{G_\varphi \mid \varphi \in L_{\lambda^+,\kappa^+}\}$ is a universe of a submodel as required. □

Notice that Theorem 8.40 implies the main model theoretic fact (Corollary 4.26) we used to prove the Erdős-Rado theorem.

DEFINITION 8.41. Let L be a countable similarity type. $\mathcal{L} \subseteq L_{\omega_1,\omega}$ is a *countable fragment* iff it is a countable set containing all the atomic formulas of L with the following closure properties: $\varphi \in \mathcal{L} \implies Sub(\varphi) \subseteq \mathcal{L}$ and $\mathcal{L}$ is closed permutations of variables in formulas, under the (first-order) propositional connectives as well as under $\forall$ and $\exists$.

PROPOSITION 8.42. *Let L be a countable similarity type. For every $\psi \in L_{\omega_1,\omega}$ there exists a countable fragment $\mathcal{L} \subseteq L_{\omega_1,\omega}$ such that $\psi \in \mathcal{L}$.*

PROOF. Define $\{\mathcal{L}_n \mid n < \omega\}$ such that

(1) $\psi \in \mathcal{L}_0$ and $\mathcal{L}_0 \supseteq \mathrm{AFml}(L)$,
(2) $\mathcal{L}_n \subseteq L_{\omega_1,\omega}$ is countable,
(3) $\mathcal{L}_n \subseteq \mathcal{L}_{n+1}$, for all $n < \omega$ and
(4) $\varphi, \psi \in \mathcal{L}_n \implies \{\neg\varphi,\ \varphi \wedge \psi,\ \varphi \vee \psi,\ \forall x\varphi,\ \exists x\varphi\} \bigcup Sub(\varphi) \subseteq \mathcal{L}_{n+1}$.

Check that $\mathcal{L} := \bigcup_{n<\omega} \mathcal{L}_n$ is a s required. □

Recall the notion of abstract elementary class. The next observation has great technical importance:

PROPOSITION 8.43. *Let L be a countable similarity type and $\psi \in L_{\omega_1,\omega}$. Suppose $\mathcal{L}$ is a countable fragment of $L_{\omega_1,\omega}$ containing ψ. Denote by $\mathcal{K}$ the class* $\mathrm{Mod}(\psi)$. *The class $\langle \mathcal{K}, \prec_{\mathcal{L}} \rangle$ is an abstract elementary class.*

PROOF. □

There are many more possible extensions of first-order logic. Either by adding more propositional connectives (infinite connectives) like in $L_{\omega_1,\omega}$ or adding new quantifiers (The simplest logic of this kind is $\mathrm{L}(\mathbf{Q})$. See Definition **??**.**??**).

examples and Stories....

********** Now couple of applications of the logic $L_{\aleph_{\omega+1},\omega}$. The first is to combinatorial set theory and the second to set-theoretic algebra.

THEOREM 8.44 (singular compactness for transversals). *Let λ be a singular cardinal of cofinality $\aleph_0$. Suppose $\mathcal{F}$ be a family of sets such that $|\mathcal{F}| = \lambda$ and $|s| < \lambda$ for all $s \in \mathcal{F}$. If $\mathcal{F}'$ has a transversal for every $\mathcal{F}' \subseteq \mathcal{F}$ then there exists a transversal for $\mathcal{F}$.*

PROOF. □

The next application is to group theory. The following property was introduced by László Fuchs in his 1960 book ([**Fu60**]).

DEFINITION 8.45. Let λ be an infinite cardinal. A group G is *λ-free* iff H is free for every subgroup H of G of cardinality $< \lambda$. A group G is *almost-free* iff it is $\|G\|$-free.

EXAMPLE 8.46. The group $\mathbb{Z}^\omega$ is $\aleph_1$-free but not free.

The following are natural questions:

QUESTION 8.47.

(1) When does λ-free imply λ^+-freeness?
(2) Are there groups such that almost free implies free?

It is reasonable to expect that an answer is easier to find for abelian groups. In his 1970 book ([**Fu1**]) László Fuchs suspected that this is a property of cardinal numbers and not groups. Specifically he asked:

QUESTION 8.48. Characterize the class

$$\{\lambda \in \mathrm{Card} \mid \text{for every every abelian group } G \text{ such that } |G| > \lambda \text{ we have}$$
$$\text{if } G \text{ is } \lambda\text{-free then } G \text{ is } \lambda^+\text{-free}\}.$$

The following is a partial answer:

THEOREM 8.49 (singular compactness for groups). *Let λ be a singular cardinal of cofinality $\aleph_0$ and G a group of cardinality λ. If G is almost-free then it is free.*

PROOF. □

THEOREM 8.50.

PROOF. □

As we have seen in Remark **??** the statement of the compactness theorem fails for $L_{\omega_1,\omega}$, it is natural to ask: Does it hold for $|L_\kappa, \omega$ when $\kappa > \aleph_0$?

DEFINITION 8.51. $L_{\kappa,\kappa}$

DEFINITION 8.52. Let $\kappa > \aleph_0$. κ is *strongly compact* iff the logic $L_{\kappa,\kappa}$ is compact. κ is *weakly compact* iff the logic $L_{\kappa,\kappa}$ is ....

The following is equivalent:

DEFINITION 8.53. Let $\kappa > \aleph_0$. κ is *strongly compact* iff every κ-complete filter can be extended to a κ-complete ultrafilter.

Since

DEFINITION 8.54. Let $\kappa > \aleph_0$. κ is *measurable* iff there exists a κ-complete ultrafilter.

every strongly compact cardinal is measurable.

THEOREM 8.55. *Let $\kappa > \aleph_0$. κ is weakly compact iff $\kappa \to (\kappa)^2_\kappa$.*

PROOF. □

THEOREM 8.56. *If κ is measurable then $\mu < \kappa \mid \mu$ is strongly inaccessible$\}$ is a stationary subset of κ.*

PROOF. □

The following (surprising) result show that in dealing with a large family of infinitary logics it suffices to deal with PC-clases.

THEOREM 8.57 (Chang's presentation). *Let λ be an infinite cardinal and L a similarity type of cardinality $\leq \lambda$. If $T \subseteq L_{\lambda^+,\omega}$ of cardinality $\leq \lambda$ such that $|L| \leq \lambda$ then $\mathrm{Mod}(T)$ is a PC_λ-class.*

Moreover, if $L^* \subseteq L_{\lambda^+,\omega}$ is a fragment of cardinality λ containing T then there are $L_1 \supseteq L$, a first-order T_1 in L_1 with built in Skolem Functions and there exists Γ a set of finitary T_1-types such that

$\mathrm{Mod}(T) = \mathrm{PC}(T_1, \Gamma, L)$ *and if $M, N \models T_1$ s.t. they omit all types from Γ then*

$$M \subseteq N \implies M \restriction L \prec_{L^*} N \restriction L.$$

PROOF. W.l.o.g. we may assume that L^* contains no infinite conjunction just infinite disjunction (if $\bigwedge_{\psi\in\Phi} \psi(\mathbf{x}) \in L^*$ replace it with $\neg \bigvee_{\psi\in\Phi} \neg\psi(\mathbf{x})$).

For every $\varphi(\mathbf{x}) \in L^*$ let $R_\varphi(\mathbf{x})$ be a new relation symbol. Let $L_1 := L \cup \{R_\varphi(\mathbf{x}) \mid \varphi(\mathbf{x}) \in L^*\}$. Notice that the variables of the relation symbol $R_\varphi(\mathbf{x})$ are exactly the free variables of $\varphi(\mathbf{x})$, so when φ is a sentence then R_φ is a propositional variable.

For every $\varphi \in L^*$ if $\varphi(\mathbf{x}) = \bigvee_{\psi\in\Phi} \psi(\mathbf{x})$ let

$$p_\varphi(\mathbf{x}) := \{R_{\varphi(\mathbf{x})}(\mathbf{x}) \wedge \neg R_{\psi(\mathbf{x})}(\mathbf{x}) \mid \psi(\mathbf{x}) \in \Phi\}.$$

$$\begin{aligned}T_1 :=&\big\{R_\psi \mid \psi \in T\big\}\\ &\cup \big\{\forall \mathbf{x}[R_{\neg\psi}(\mathbf{x}) \leftrightarrow \neg R_\psi(\mathbf{x})] \mid \psi \in L^*\big\}\\ &\cup \big\{\forall \mathbf{x}[R_{\psi_1\vee\psi_2}(\mathbf{x}) \leftrightarrow [R_{\psi_1}(\mathbf{x}) \vee R_{\psi_2}(\mathbf{x})]] \mid \psi_1,\psi_2 \in L^*\big\}\\ &\cup \big\{\forall \mathbf{x}[[\mathrm{R}_{\exists y\varphi(y;\mathbf{x})}(\mathbf{x}) \leftrightarrow \exists y R_{\varphi(y;\mathbf{x})}(y,\mathbf{x})] \mid \varphi(y;\mathbf{x}) \in L^*\big\}\\ &\cup \big\{\forall \mathbf{x}[R_\psi(\mathbf{x}) \to R_\varphi(\mathbf{x})] \mid \text{ if there exists } \{\varphi\}\cup\Phi \subseteq L^* \text{ such that } \varphi(\mathbf{x}) = \bigvee_{\psi\in\Phi}\psi(\mathbf{x})\big\}.\end{aligned}$$

Let $\Gamma := \{p_\varphi(\mathbf{x}) \mid$ if there exists $\{\varphi\}\cup\Phi \subseteq L^*$ such that $\varphi(\mathbf{x}) = \bigvee_{\psi\in\Phi}\psi(\mathbf{x})\}$. Check that $\mathrm{PC}(T_1,\Gamma,L)$ is as required.

For the moreover part. Expand further $\mathrm{L}(T_1)$ with Skolem Functions so that the new theory T_1 has built-in Skolem Functions, we are done by Remark 1.14.17. □

With little more coding it turns out that for any $T \subseteq L_{\lambda^+,\omega}$ such that $|T| + |\mathrm{L}(T)| \leq \lambda$ there exists a first-order T_1 in an expansion L_1 (of cardinality λ) and a T_1-type $q(y)$ such that

$$\mathrm{Mod}(T) = \mathrm{PC}(T_1, q, \mathrm{L}(T)).$$

This follows from Chang's representation theorem (Theorem 8.57) together with the following coding trick:

PROPOSITION 8.58 (Shelah's 1-type coding). *If $\mathcal{K}$ is $\mathrm{PC}_{\lambda,\lambda}$ then $\mathcal{K}_{\geq\lambda}$ has an axiomatization as a $\mathrm{PC}_{\lambda,1}$.*

PROOF. Suppose $K = \mathrm{PC}(T_1,\Gamma,L)$ when $|\mathrm{L}(T_1)| + |\Gamma| + \aleph_0 \leq \lambda$. We will introduce an expansion L^* of $\mathrm{L}(T_1)$ of cardinality λ, a theory T^* and a type q in L^* such that $\mathcal{K} = \mathrm{PC}(T^*, q, L)$.

Suppose

$$\Gamma = \{p_\alpha(\mathbf{x}_\alpha) \mid \alpha < \alpha_0 \leq \lambda\} \text{ and } p_\alpha(\mathbf{x}_\alpha) = \{\varphi^i_\alpha(\mathbf{x}_\alpha) \mid i < i_\alpha \leq \lambda\}.$$

Let $\{c^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\}$ be new constant symbols, Q and $\{P_\alpha \mid \alpha < \alpha_0\}$ new unary predicates and $\{F_\alpha(\mathbf{x}_\alpha) \mid \alpha < \alpha_0\}$ new function symbols. Let

$$L^* := \mathrm{L}(T_1)\cup\{c^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\}\cup\{Q\}\cup\{P_\alpha \mid \alpha < \alpha_0\}\cup\{F_\alpha(\mathbf{x}_\alpha) \mid \alpha < \alpha_0\}.$$

$$\begin{aligned}T^* := T_1 &\cup \{Q(c^i_\alpha), P_\alpha(c^i_\alpha) \mid \alpha < \alpha_0, i < i_\alpha\} \cup \{c^i_\alpha \neq c^j_\beta \mid (\alpha,i) \neq (\beta,j)\}\\ &\cup \{\forall x[P_\alpha(x) \to Q(x)] \mid \alpha < \alpha_0\}\\ &\cup \{\forall x[P_\alpha(x) \to \neg P_\beta(x)] \mid \alpha \neq \beta < \alpha_0\}\\ &\cup \{\forall \mathbf{x}_\alpha[P_\alpha(F_\alpha(\mathbf{x}_\alpha)] \mid \alpha < \alpha_0\}\\ &\cup \{\forall \mathbf{x}_\alpha[F_\alpha(\mathbf{x}_\alpha) = c^i_\alpha \to \neg\varphi^i_\alpha(\mathbf{x}_\alpha)] \mid \alpha < \alpha_0, i < i_\alpha\}.\end{aligned}$$

Let $q(y) := \{Q(y) \wedge y \neq c^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\}$.

CLAIM 8.59. $K_{\geq\lambda} = \mathrm{PC}(T^*, q, L)$.

PROOF. Suppose $M \in \mathcal{K}_{\geq\lambda}$. Fix a subset $\{a^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\}$ of M. Let $Q := \{a^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\}$, $P_\alpha := \{a^i_\alpha \mid i < i_\alpha\}$. Since M omits all the types from Γ. For every $\alpha < \alpha_0$ and every $\mathbf{a} \in {}^{\ell(\mathbf{x}_\alpha)}|M|$ there exists a first $i < i_\alpha$ such that $M \models \neg\varphi^i_\alpha[\mathbf{a}]$. Define $F_\alpha(\mathbf{a})$ as a^i_α. Clearly $\langle M, Q, P_\alpha, F_\alpha, a^i_\alpha\rangle_{\alpha<\alpha_0, i<i_0}$ is an element of $\mathrm{PC}(T^*, \{q\}, L)$.

Suppose $N \in \mathrm{EC}(T^*, q)$. It is enough to show that N omits all the types from Γ. Since N omits q we get that

$$Q^N = \{(c^i_\alpha)^N \mid \alpha < \alpha_0, i < i_0\}.$$

By the the first three groups of axioms we have that

$$Q^N = \bigcup_{\alpha<\alpha_0} P^N_\alpha,\ \alpha \neq \beta \implies P^N_\alpha \cap P^N_\beta = \emptyset$$

and $P^N_\alpha = \{(c^i_\alpha)^N \mid i < i_0\}$.

Now given $\alpha < \alpha_0$ let's show that N omits p_α: Suppose $\mathbf{a} \in {}^{\ell(\mathbf{x}_\alpha)}|N|$. By the forth group of axioms $F^N_\alpha(\mathbf{a}) \in P^N_\alpha$. Since $P^N_\alpha = \{(c^i_\alpha)^N \mid i < i_0\}$ there exists $i < i_0$ such that $F^N_\alpha(\mathbf{a}) = (c^i_\alpha)^N$. This by the fifth group of axioms imply that $N \models \neg\varphi^i_\alpha[\mathbf{a}]$. □

□

A combination of the last two results gives the following useful

COROLLARY 8.60. *Let λ be an infinite cardinal and L a similarity type of cardinality $\leq \lambda$. If $T \subseteq L_{\lambda^+,\omega}$ of cardinality $\leq \lambda$ such that $|L| \leq \lambda$ then there exists a first-order theory T_1 in an expansion of the L of cardinality λ and a T_1-type $q(y)$ such that*

$$\{M \models T\ :\ \|M\| \geq \lambda\} = \mathrm{PC}(T_1, q, L).$$

Note the remarkable feature of Theorem 7.6. Regardless of how complex the types from Γ our new type q is very simple and is not influenced at all by Γ. This is possible since T^* "codes" all the types from Γ with the help of q.

The following variation of Chang's presentation theorem is useful:

THEOREM 8.61. *Let M be an L-structure and $L^* \subseteq L_{\omega_1,\omega}$ a fragment of cardinality λ where $\lambda \geq |L| + \aleph_0$. ($L$ denotes the language of the fragment L^*). There exists an expansion L_1 of L of cardinality λ, a first-order T_2 in L_2 and a T_2-type q such that*

$$\{N\ :\ N \equiv_{L^*} M,\ \|N\| = \mu\} = \{N \in \mathrm{PC}(T_2, q, L)\ :\ \|N\| = \mu\} \text{ for all } \lambda \leq \mu \leq \|M\|.$$

Moreover, for any $N_1, N_2 \models T_1$ if $N_1 \prec N_2$ then $N_1 \prec_{L^*} N_2$.

PROOF. We use the same expansion as in the proof of Chang's theorem, also use the same set of types Γ. The definition of T_1 is as the definition of with only one change, the first set is replaced by

$$\{R_\psi \mid M \models \psi \text{ and } \psi \text{ is a sentence from } L^*\}.$$

Now apply Shelah's 1-type coding to obtain an expansion $L_2 \supseteq L_1$ and a first-order theory T_2 and a type q as required.

To obtain the moreover part, we expand further T_2 by Skolem functions. □

CONNECTION WITH SET THEORY

AND PROVE

In 1968 C. C. Chang proved the following very surprising theorem:

THEOREM 8.62 (Chang's presentation theorem). *If T is a theory in $L_{\lambda^+,\omega}$ of cardinality $\leq \lambda$ then* $\mathrm{Mod}(T)$ *is* PC_λ.

Exercises 1.8.

(1) Like in Exercise 2.10 show that for a countable similarity type L and for a countable $T \subseteq L_{\omega_1,\omega}$ there exists a sentence $\psi_T \in L_{\omega_1,\omega}$ such that $\mathrm{Mod}(T) = \mathrm{Mod}(\psi_T)$.
(2) In the similarity type $\{V, F, +_V, +_F, \cdot, N\}$ when V and F are unary predicates and N is a unary function symbol from V into F. Introduce a sentence $\psi \in L_{\omega_1,\omega}$ such that $\langle V, F, +_V, +_F, \cdot_F, N\rangle \models \psi$ iff V is a normed vector space over F, N is an Archimedean norm and F is isomorphic to a subfield of the reals.
(3) Let $\varphi(x) \in L_{\omega_1,\omega}$. Find a senttence $\psi_\varphi \in L_{\omega_1,\omega}$ such that

$$M \models \psi_\varphi \iff \{a \in M \mid M \models \varphi[a]\} \text{ is infinite.}$$

(4) Let L be a countable similarity type. Suppose $\psi \in L_{\omega_1,\omega}$ is a sentence that has an infinite model. Show that ψ has a countable model. Suppose now that Ψ is a countable set of $L_{\omega_1,\omega}$-sentences that has an infinite model, does the theory Ψ have a countable model?
(5) Prove that $\aleph_1$ is not a compact cardinal.
(6) Show that the class of all equivalence relations with finitely many equivalence classes (From Exercise 2.58) is axiomatizeable by an $L_{\omega_1,\omega}$ sentence.
(7) Show that the class of connected graphs (From Exercise 2.59) is axiomatizeable by an $L_{\omega_1,\omega}$ sentence.
(8) Can a compact cardinal be singular?
(9) The class of locally finite groups LF has an axiomatization by an $L_{\omega_1,\omega}$-sentence in the language of group theory. For definition of LF see II.1.3.
(10) The class ULF has an axiomatization by an $L_{\omega_1,\omega}$-sentence in the language of group theory. For definition of ULF see II.1.3.
(11)
(12)

DEFINITION 8.63. Let $\langle \mathcal{K}, \preceq_{\mathcal{K}}\rangle$ be an abstract class of similarity type L, and let $\lambda \geq \aleph_0 + |L|$. We say that $\mathcal{K}$ *has the* $\langle \kappa, \mu, \lambda\rangle$ *-amalgamation property* if for every $M_0 \in \mathcal{K}_\kappa$, $M_1 \in \mathcal{K}_\mu$ and every $M_2 \in \mathcal{K}$ of cardinality λ (for $\ell = 0, 1, 2$) and every $f_\ell : M_0 \to M_\ell$ ($\ell = 1, 2$) $\mathcal{K}$-embeddings then there exists $N \in \mathcal{K}$, and $g_\ell : M_\ell \to N$ $\mathcal{K}$-embeddings such that the diagram

$$\begin{array}{ccc} M_2 & \overset{g_2}{\dashrightarrow} & N \\ {\scriptstyle f_2}\uparrow & & \uparrow{\scriptstyle g_1} \\ M_0 & \xrightarrow[f_1]{} & M_1 \end{array}$$

commutes. I.e. $g_1 \circ f_1 \restriction M_0 = g_2 \circ f_2 \restriction M_0$.

Let $\langle \mathcal{K}, \preceq_{\mathcal{K}}\rangle$ be an AEC and suppose that $\mathrm{LS}(\mathcal{K}) \leq \lambda$. Prove that the λ-amalgamation property implies the $\langle \lambda, \lambda^+, \lambda\rangle$-amalgamation property.

(13) Prove that T_{frg} (from Example 8.9) is categorical in all infinte cardinals.
(14) Prove that T_{frg} can not be axiomatized by an $L_{\omega_1,\omega}$ theory. Hint: Use $\mathbb{Z}^\omega$.
(15) Describe a formula $\psi(x) \in L_{\omega_1,\omega}$ in the similarity type of fields such that for a field $F \models \exists x \psi(x)$ iff F has a transcendental element over its prime field.
(16) Describe the formula $\psi(x_0, \ldots, x_{n-1})$ necessary for Example 8.11.
(17) Why T_{ACF_∞} does not have a first-order axiomatization?
(18) Let L be a countable similarity type. For any $\psi \in L_{\omega_1,\omega}$ and α a countable limit ordinal the class $\langle \mathrm{Mod}(\psi), \prec^\alpha_{\omega_1,\omega} \rangle$ is an abstract elementary class.
(19) Prove Proposition 8.36. Hint: Use Theorem 8.26 together with Lemma 8.35.
(20) Let $\psi \in L_{\lambda^+,\omega}$ and suppose that the language has at most λ many symbols. Denote by Φ a set of $L_{\lambda^+,\omega}$ formulas of cardinality λ such that $\Phi \ni \psi$ and

$$\varphi \in \Phi \implies Sub(\varphi) \subseteq \Phi.$$

Show that $\mathcal{K} := \langle \mathrm{Mod}(\psi), \preceq_{TV,\Phi} \rangle$ is an abstract elementary class with $\mathrm{LS}(\mathcal{K}) = \lambda$.
(21) Prove Theorem 8.49.
(22) For any infinite countable ordinal α find a sentence $\varphi_\alpha \in L_{\omega_1,\omega}$ in the similarity type $\{<\} \cup \{c_i \mid i < \alpha\}$ such that

$$M \models \psi_\alpha \iff \langle |M|, <^M \rangle \cong \langle \alpha, \in \rangle.$$

(23) For every countable ordinal α define a sentence $\psi_\alpha \in L_{\omega_1,\omega}$ in the language $\{c_\beta \mid \beta < \alpha\} \cup \{\in\}$. With the following properties: ψ_α has a model of cardinality $\beth_\alpha$ but not larger.

Hint: Use the previous exercise together with Definition 10.2.

(24) For every countable M there exists a sentence $\varphi_M \in L_{\omega_1,\omega}$ such that

$$N \equiv_{\omega_1,\omega} M \iff N \models \varphi_M.$$

(25) Let $L_{\mathcal{A}}$ be a countable fragment. Show that in the previous exercise the assumption that M is countable is not necessary. Namely, prove that: For every M there exists a sentence $\varphi_M \in L_{\omega_1,\omega}$ such that

$$N \equiv_{L_{\mathcal{A}}} M \iff N \models \varphi_M.$$

(26) Let $L := \{<\}$ be a single binary relation. Show that for every $\beta < \omega_1$ there exists an L-structure M_β such that its Scott sentence has high $\geq \beta$.
(27) Prove the following generalization of Scott's theorem (due to C. C. Chang):

THEOREM 8.64. *Let λ be a strong limit of countable cofinality. For every L-structure M of cardinality λ there exists a sentence $\psi_M \in L_{\lambda^+,\omega}$ such that*

$$N \models \psi_M \implies N \cong M \text{ for all models } N \text{ of cardinality } \lambda.$$

(28) Let G be a free abelian group of cardinality $\aleph_1$ show that $G \equiv_{\infty,\omega} \mathbb{Z}^\omega$. Thus (since $\mathbb{Z}^\omega$ is not free) the logic $L_{\infty,\omega}$ is not sufficiently strong to capture the notion of free group for groups of cardinality $\aleph_1$.
(29) Construct an almost free but not free group of cardinality $\aleph_1$.
(30) Prove that an infinite group G is simple iff it satisfies $(*)$ from Example 8.10.
(31) Let T be the $L_{\omega_1,\omega}$ theory of the field of complex numbers. Find a first-order theory T^* in the language of rings such that $\mathrm{Mod}(T^*)_\lambda = \mathrm{Mod}(T)_\lambda$ for all $\lambda > \aleph_0$.
(32)

9. Countable models and Henkin's omitting types theorem

When $\mathrm{L}(T)$ is at most countable it make sense to deal with countable saturated models. They behave particularly nicely:

THEOREM 9.1 (Existence of countable saturated models). *Let M be a countable structure in a finite or countable similarity type. Denote by T its theory. $|\mathrm{D}(T)| = \aleph_0$ if and only if there exists $N \succeq M$ countable saturated.*

PROOF. Define N as a union of the following increasing elementary chain $\{N_n \mid n < \omega\}$ where:

(1) $M = N_0$,
(2) every N_n-type over A is realized in N_{n+1} for every finite $A \subseteq |N_n|$.

The construction can be carried out due to the following observation: If $A \subseteq |N_n|$ is of cardinality $k < \omega$ then $|\mathrm{S}(A)| = |\mathrm{D}_{k+1}(T)| \leq \aleph_0$.

The converse: Let $M \models T$ be countable saturated. Then since every type over the empty set is realized in M we have $|\mathrm{D}(T)| \leq \|M\|$. □

REMARK 9.2. A natural question is to ask whether the existence of uncountable saturated model (of cardinality $\lambda > \aleph_0$) has a similar characterization in terms of counting complete types. This requires much more machinery. Only in Chapter **??** we will be able to show exactly that.

THEOREM 9.3 (Uniqueness of countable saturated models). *Let M and N be countable saturated structures. If $M \equiv N$ then $M \cong N$.*

PROOF. This an application of Exercise 1.2.44: Notice that the family

$$\mathcal{F} := \{g \mid g \text{ is a partial isomorphism from } M \text{ into } N\}$$

is not empty (since the assumption $M \equiv N$ implies that $\langle\rangle \in \mathcal{F}$) and has the back and forth property. □

Countable saturated models have the nice property that a union of ω-chains of saturated models is countably saturated This property fails for uncountable saturated models. It is true only with additional assumptions on the theory T (this failure is very important as we will see in later chapters it is equivalent to super stability).

THEOREM 9.4 (preservation of countable saturation under union). *If $\{M_n \mid n < \omega\}$ is an increasing chain of countable saturated models then $\bigcup_{n<\omega} M_n$ is saturated*

PROOF. Exercise. □

As was shown in Proposition 7.12 it is easy to realize a type. A much harder problem is

QUESTION 9.5. Given a consistent theory T and a T- type p, does there exist a model $M \models T$ that omits the type p?

Let us assume that the theory T is complete (this is not a necessary assumption but it helps us to avoid inessential technical issues). There are several cases when finding a model that omits a given type is impossible:

EXAMPLE 9.6. Let c be a constant symbol in $\mathrm{L}(T)$ and let p be the type $\{x = c\}$. Certainly if $M \models T$ then c^M realizes p.

EXAMPLE 9.7. Let $\varphi(x) \in \mathrm{Fml}(\mathrm{L}(T))$, and let p be the type $\{\varphi(x)\}$. If p is a T- type then every model of T realizes the type p. Since when p is a T-type, by definition we have that $T \cup \{\exists x \varphi(x)\}$ is consistent; Since T is complete $T \vdash \exists x \varphi(x)$. By Exercise I.2.17 every model $M \models T$ satisfies $\exists x \varphi(x)$.

A natural question to ask is whether Example 9.7 is the most general one? Namely given a complete theory T and an infinite type p. Is p realized in every model of T? Certainly if the type p "depends on a single formula" then p will be realized. The following definition enables us to generalize Example 9.7 and formalize the above discussion.

DEFINITION 9.8. Let $\mathbf{x}$ be a finite sequence of variables, let T be a theory.

(1) Suppose that p and q are T- types in $\mathbf{x}$. Write $p \vdash q$ to denote the following: For every model $M \models T$ and every $\mathbf{a} \in {}^{\ell(\mathbf{x})}|M|$, if $\mathbf{a}$ realizes p then $\mathbf{a}$ realizes q.
(2) A T- type p in $\mathbf{x}$ is called *isolated type* if there exists a formula $\varphi(\mathbf{x})$ such that $\varphi(\mathbf{x}) \vdash p$ and $T \cup \{\exists \mathbf{x} \varphi(\mathbf{x})\}$ is consistent, we say that *the formula φ isolates p*.
(3) Let M be a structure and $A \subseteq |M|$. Suppose that p is an M-type in $\mathbf{x}$ over A. We say that *p is isolated over A* iff there are $\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(M))$ and $\mathbf{a} \in A$ satisfying $M \models \exists \mathbf{x} \varphi(\mathbf{x}; \mathbf{a})$ and $\varphi(\mathbf{x}; \mathbf{a}) \vdash p$.

REMARK 9.9. We will not elaborate at this stage on the source of the term "isolated". There is a natural topology on the set of complete types where the isolated points are the complete isolated types. The topological interpretation will be given in section 8 of Chapter 5. Often isolated types are called *principle*, or *atomic types* , or *types with support*.

PROPOSITION 9.10. *Let T be a complete theory, and let p be a T- type. If p is an isolated type then p is realized in every model of T.*

PROOF. Since p is isolated, there exists a formula $\varphi(\mathbf{x})$ such that $\varphi \vdash p$. Since $T \cup \{\exists \mathbf{x} \varphi(\mathbf{x})\}$ is consistent, since T is complete we have that $T \models \exists \mathbf{x} \varphi(\mathbf{x})$. Let $M \models T$, since $T \models \exists \mathbf{x} \varphi(\mathbf{x})$ there exists $\mathbf{a} \in {}^{\ell(\mathbf{x})}|M|$, such that $M \models \varphi[\mathbf{a}]$. Since

φ isolates the type p for every $\psi \in p$ we have that $M \models \psi[\mathbf{a}]$. Namely p is realized by $\mathbf{a}$. □

The following property of isolation will play a key role in the proof of Morley's theorem, however the reader may consider this result as an exercise concerning isolated types:

THEOREM 9.11 (Transitivity of isolation). *Suppose T is complete. Let $p := \mathrm{tp}(\mathbf{b}\hat{}\mathbf{a})$ Suppose $\psi(\mathbf{x}) \vdash \mathrm{tp}(\mathbf{a})$ and $\varphi(\mathbf{y}, \mathbf{a}) \vdash p$. If $\chi(\mathbf{y}, \mathbf{x}) := [\psi(\mathbf{x}) \wedge \varphi(\mathbf{y}, \mathbf{x})]$ then $\chi(\mathbf{y}, \mathbf{x}) \vdash p$.*

PROOF. Otherwise, there is $\rho(\mathbf{y}, \mathbf{x}) \in p$ and there are $\mathbf{b}'\hat{}\mathbf{a}'$ such that

$$(*) \quad \models \neg\rho[\mathbf{b}', \mathbf{a}'] \wedge \chi[\mathbf{b}', \mathbf{a}'].$$

Since $\models \psi[\mathbf{a}']$ by assumption of ψ we get that $\mathrm{tp}(\mathbf{a}/\emptyset) = \mathrm{tp}(\mathbf{a}'/\emptyset)$ thus there is $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(\mathbf{a}') = \mathbf{a}$. Apply f to $(*)$ to get

$$(**) \quad \models \neg\rho[f(\mathbf{b}'), \mathbf{a}] \wedge \chi[f(\mathbf{b}'), \mathbf{a}].$$

Since $\models \chi[f(\mathbf{b}'), \mathbf{a}]$ by assumption of φ we get that $\mathrm{tp}(f(\mathbf{b}')\hat{}\mathbf{a}) = \mathrm{tp}(\mathbf{b}\hat{}\mathbf{a})$ thus there is $g \in \mathrm{Aut}_{\mathbf{a}}$ such that $g(f(\mathbf{b}')) = \mathbf{b}$. Now an application of g in $(**)$ gives

$$\models \neg\rho[\mathbf{b}, \mathbf{a}] \wedge \chi[\mathbf{b}, \mathbf{a}].$$

This is a contradiction to the assumption that χ isolates p. □

COROLLARY 9.12.

Back to Problem 1.1. The only types we can hope to omit are the non-isolated types.

THEOREM 9.13 (Henkin's omitting types theorem). *Let T be a countable theory, and let p be a T- type in the variable x. If p is not isolated then there exists $M \models T$ that omits p.*

PROOF. Let $C := \{c_n \mid n < \omega\}$ a countable set of new constants (not in $\mathrm{L}(T)$). Let $L' = \mathrm{L}(T) \cup C$. Since T is countable there exists $\{\varphi_n : n < \omega\}$ an enumeration of $\mathrm{Sent}(L')$. Let $\{\psi_k(x) : k < \omega\} = p$. Define $\{T_m \subseteq \mathrm{Sent}(L') : m < \omega\}$ such that for every $m < \omega$ the following conditions hold:

(1) $T_0 = T$,
(2) T_m is a consistent theory in L',
(3) $T_m \subseteq T_{m+1}$ and $T_m - T$ is finite,
(4) $c_m = c_m \in T_{m+1}$, there exists $k(m) < \omega$ such that $\neg\psi_{k(m)}(c_m) \in T_{m+1}$. If $\varphi_m = \exists x\psi(x)$ and $\{\varphi_m\} \cup T_m$ is consistent then $\psi(c_n) \in T_{m+1}$ where $n := \min\{n < \omega \mid c_n \notin \mathrm{L}(\{\varphi_m\} \cup T_m)\}$ and

This is enough: Let $T_\omega := \bigcup_{m<\omega} T_m$. Since consistency depends only on finite subsets by (1) and (2) $T_\omega \supseteq T$ is a consistent theory.

Let $N \models T_\omega$. Note that by requirement (4) we have that $\mathrm{L}(T_\omega) \supseteq C$.

Define $A := \{c_n^N \ : \ n < \omega\}$. Let $M := \langle A, F^N, R^N, c^N \rangle_{F,R,c \in \mathrm{L}(T)}$.

CLAIM 9.14. *$M \preceq N \restriction \mathrm{L}(T)$ and M omits p.*

PROOF. It is enough to show that $A \subseteq_{TV} N \restriction \mathrm{L}(T)$. Let $\mathbf{a} \in A$ be given and suppose $N \models \exists x \varphi(x, \mathbf{a})$ for some $\varphi(x; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))$. Let $\mathbf{c}$ be the sequence of new constants such that $\mathbf{c}^N = \mathbf{a}$.

Let $m < \omega$ be such that $\varphi_m = \exists x \varphi(x; \mathbf{c})$.

Since $N \models \exists x \varphi(x, \mathbf{a})$ and T_ω is a complete theory the sentence $\exists x \varphi(x; \mathbf{c})$ belong to T_{m+1}, by (4) we have there exists a constant c_k such that that $\varphi(c_k; \mathbf{c}) \in T_{m+1}$ thus we found $b := c_k^M \in A$ such that $N \models \varphi[b, \mathbf{a}]$. By the Tarski-Vaught test A is a universe of an elementary submodel N with respect to $\mathrm{L}(T)$.

Since $T \subseteq T_\omega$ (by (1)) we have that $M \models T$.

We left to show that p is omitted by M.

Let $a \in |M|$ be given. By the definition of A there exist $m < \omega$ such that $a = c_m^N$. By property (4) $c_m \in \mathrm{L}(T_{m+1})$. Since $\neg \psi_{k(m)}(c_m) \in T_{m+1}$. I.e. $M \models \neg \psi_{k(m)}[a]$.

So for every element a of $|M|$ there exists a formula $\psi \in p$ such that $M \models \neg \psi[a]$. □

The construction is possible: Define T_m by induction on $m < \omega$: For $m = 0$ take T_0 to be the theory T. Suppose T_m has been defined.

If $\varphi_m = \exists x \psi(x)$ and $\{\varphi_m\} \cup T_m$ is consistent then

$$T_m^1 := \{\psi(c), c_m = c_m\} \cup T_m, \text{ otherwise } T_m^1 := \{c_m = c_m\} \cup T_m,$$

where $c \in C$ is the first not in $\mathrm{L}(T_m^1 \cup \{\psi\})$.

Let $\rho(\mathbf{y}; x) \in \mathrm{Fml}(\mathrm{L}(T))$ such that $\rho(\mathbf{d}; c_m) := \bigwedge(T_m^1 - T)$ for $\mathbf{d} \in C$. Notice that by (3) this is well defined, we may assume that the constant c does not occur in $\mathbf{d}$, thus x is not one of the variables in $\mathbf{y}$. Now let $\chi(x)$ be the formula $\exists \mathbf{y} \rho(\mathbf{y}; x)$. In case where T_m^1 does not contain any constants from C let $\chi(x)$ be any $\mathrm{L}(T)$-formula consistent with T.

As p is not isolated, by (2) there exists $k < \omega$ such that

$$(*) \quad T \cup \{\exists x [\chi(x) \wedge \neg \psi_k(x)]\} \text{ is consistent.}$$

Let $T_{m+1} := T_m^1 \cup \{\neg \psi_k(c_m)\}$. By $(*)$ the set T_{m+1} is consistent. □

The following is a most useful generalization of Henkin's omitting types theorem.

THEOREM 9.15 (Extended omitting types). *Let T be a countable theory. Let Γ be a countable set of T- types. If p is not isolated for every $p \in \Gamma$ then there exists $M \models T$ that omits all the elements of Γ.*

A further generalization dealing with uncountable languages and uncountable models is possible.

DEFINITION 9.16. Let λ be an infinite cardinal. We say that a T-type p is *λ-isolated* provided there exists a T-type q of cardinality $< \lambda$ such that for every $M \models T$ we have

$$\mathbf{a} \models q \implies \mathbf{a} \models p \text{ for all } \mathbf{a} \in |M|.$$

THEOREM 9.17 (Another Extended omitting types). *Let T be a first-order theory and λ an infinite cardinal such that $\lambda \geq |\mathrm{L}(T)|$. Let Γ be a set of T- types of cardinality $\leq \lambda$. If p is not λ-isolated for every $p \in \Gamma$ then there exists $M \models T$ that omits all the elements of Γ.*

It is possible to prove this theorem by revising the proof of Theorem 9.13 via a careful counting of all the tasks that needed to be accomplished. I suggest that the reader will try to do that. In section 10 of chapter 6 a general method to build models is presented, and a proof of Theorem 9.15 is one of its applications.

REMARK 9.18. It can be shown that the extended omitting types theorem for uncountable Γ of cardinality λ follows from MA_λ.

To satisfy the curiosity of the reader I will give now the following application of the extended omitting-types theorem, this theorem has many applications that play a central role in Section 12.

THEOREM 9.19 (Keisler and Morley). *For every countable $M \models \mathrm{ZF}$ there exists a countable $N \supsetneq M$ such that $\omega^N = \omega^M$.*

PROOF. Let $M \models \mathrm{ZF}$ be countable, for every infinite $a \in |M|$ consider

$$p_a(x) := \{x \in a \wedge x \neq b \ : \ b \in |M|, \ M \models [b \in a]\}.$$

Let c be a new constant and consider the following expansion of $\mathrm{CD}(M)$:

$$T^* := \mathrm{CD}(M) \cup \{\neg[c \in a] \ : \ a \in |M|\}.$$

Verify that $p_a(x)$ is a T^*-type for all infinite $a \in |M|$.

It suffices to find $N^* \models T^*$ omitting all the types in $\{p_a \mid a \in |M|\}$. By the extended omitting types theorem it is enough to verify that the types are not isolated.

CLAIM 9.20. *For every $\varphi(x; c) \in \mathrm{Fml}(\mathrm{L}(T^*))$ TFAE*

(1) $T^* \cup \{\exists x \varphi(x; c)\}$ *has a model.*
(2) $\mathrm{CD}(M) \models \forall y \exists z \exists x [\neg [z \in y] \wedge \varphi(x; z)]$.

PROOF. (1) $\implies$ (2): Clear.

(2) $\implies$ (1): By the compactness theorem, given a finite

$$\{\neg [c \in a_i] \mid a_i \in |M|,\ i < n\}$$

since $\mathrm{ZF} \subseteq T^*$ there is an element y of M satisfying $y = \bigcup_{i<n} a_i$. Thus by (2) the theory is finitely satisfiable. □

In order to complete the proof of the theorem it is enough to verify:

CLAIM 9.21. p_a *is not isolated for all infinite* $a \in |M|$.

PROOF. Let $\rho(x; c) \in \mathrm{Fml}(\mathrm{L}(T^*))$ be given and show that if $T^* \cup \{\exists x \rho(x; c)\}$ has a model then there exist $\psi(x; \mathbf{b}, a) \in p_a$ such that

$$T^* \cup \{\exists x [\rho(x; c) \wedge \neg \psi(x; \mathbf{b}, a)]\} \quad \text{is consistent.}$$

If

$$T^* \cup \{\exists x [\rho(x; c) \wedge \neg [x \in a]]\} \quad \text{is consistent}$$

then we are done. So suppose that

$$T^* \cup \{\exists x [\rho(x; c) \wedge [x \in a]]\} \quad \text{is consistent.}$$

Take

$$\varphi(x; c) := \rho(x; c) \wedge [x \in a].$$

An application of Claim 9.20 gives that

$$\mathrm{CD}(M) \models \forall y \exists z \exists x [\neg [z \in y] \wedge \rho(x; z) \wedge [x \in a]].$$

An application of the axiom of replacement gives

$$\mathrm{CD}(M) \models \exists x \forall y \exists z [\neg [z \in y] \wedge \rho(x; z) \wedge [x \in a]].$$

□

□

An immediate corollary is the following two-cardinal version of Skolem's paradox:

COROLLARY 9.22. *For every countable* $M \models \mathrm{ZF}$ *there exists a* $N \succ M$ *of cardinality* $\aleph_1$ *such that* $\omega^N = \omega^M$.

PROOF. Define an elementary chain $\{N_\alpha \mid \alpha < \omega_1\}$ such that

(1) $N_0 = M$,
(2) N_α is a countable model,

(3) $N_\alpha \precneqq N_{\alpha+1}$ and $\omega^{N_\alpha} = \omega^{N_\alpha+1}$ for all $\alpha < \omega_1$,
(4) for α limit $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$.

At successor stage of the construction use Theorem 9.19.

Check that $N := \bigcup_{\alpha<\omega_1} N_\alpha$ is as required. □

DEFINITION 9.23. A structure M is *atomic* iff for every $\mathbf{a} \in |M|$ the type $\mathrm{tp}(\mathbf{a}/\emptyset)$ is isolated.

Let M be a structure and $A \subseteq |M|$ we say that M is *atomic over* A iff $\mathrm{tp}(\mathbf{a}/A)$ is isolated over A for all $\mathbf{a} \in |M|$.

M is an *atomic model* iff M is atomic over $\emptyset$.

In the previous definition it is essential that we require that the condition holds for all *sequences* $\mathbf{a} \in |M|$ and not just elements of M.

One can easily show:

PROPOSITION 9.24. *Suppose M is a countable structure with $|\mathrm{L}(M)| \leq \aleph_0$. If M is atomic then for every $N \equiv M$ there is an elementary embedding of M into N.*

PROOF. Fix $\{a_n \mid n < \omega\}$ an enumeration of $|M|$. Construct an increasing sequence of (M, N)-mappings

$$g_n : \{a_k \mid k < n\} \to |N|$$

by induction on n. □

The previous Proposition (due to Vaught) inspired Morley for a model M and a subset A to introduce the notion *M is constructible over A* (see Definition 3.5.7) as a key in generalizing existence of prime models over arbitrary sets (see Lemma 3.5.8).

A similar argument can be used to show:

PROPOSITION 9.25 (Uniqueness of atomic models). *Let T be a countable and complete first-order theory. If $M, N \models T$ are atomic countable models then $M \cong N$.*

PROOF. This is a back and forth argument. □

Notice that the contents of Propositions 9.24 and 9.25 is that for countable T the notion of atomic model is dual to that of countable saturated model.

In Chapter 7 we'll study the notion of *prime model over a set A* (see Definition 3.5.1), here I introduce a special case, where $A = \emptyset$:

DEFINITION 9.26. An infinite model M is *prime model of* T iff for every infinite $N \models T$ there exists an elementary embedding $g : M \to N$.

Early on this notion was recognized as useful due to the following simple observation:

PROPOSITION 9.27. *Suppose T admits elimination of quantifiers. If there exists a prime model for T then T is complete.*

This allows us to conclude that T_{ACF_p} is complete, without using Steinitz's and Vaught's theorem using only the fact that the theory allows elimination of quantifiers and that the algebraic closure of the prime field is a prime model of this theory.

THEOREM 9.28 (Vaught). *Let T be a complete first-order theory in a countable language. For an infinite model $M \models T$, M is prime model of T iff M is atomic.*

PROOF. If M is atomic then by Proposition 9.24 M must be prime.

Suppose M is prime and let $\mathbf{a} \in |M|$ be given. If $\mathrm{tp}(\mathbf{a}/\emptyset, M)$ is not isolated then by Henkin's omitting-types theorem there exists $N \models T$ omitting it, this is a contradiction to the assumption that M is isomorphic to an elementary sub model of N. □

We left with the natural question of existence of atomic models. Clearly existence of an atomic model M of T implies that T is atomic, it turns out that when the language of T is at most countable then also the converse is true:

THEOREM 9.29. *Let T be a countable first-order theory. If $\mathrm{D}(T)$ is countable then T has an atomic model.*

PROOF. Let $\Gamma := \{p \in \mathrm{D}(T) \mid p \text{ is not isolated }\}$. By the extended omitting types theorem there exists a countable $M \models T$ omitting all the types from Γ. Check that M is atomic. □

COROLLARY 9.30. *Suppose T is a first-order theory in a countable language. If T is $\aleph_0$ categorical then every $M \models T$ is atomic (including the uncountable models).*

PROOF. Using Theorem **??**, $\aleph_0$-categoricity implies that $\mathrm{D}(T)$ is countable. Now an application of the previous theorem show that the countable model N_0 is atomic. Given any $M \models T$ and $\mathbf{a} \in |M|$ take a countable $M_0 \preceq M$ containing $\mathbf{a}$, by $\aleph_0$-categoricity there exists $f : M_0 \cong N_0$ and $\mathrm{tp}(\mathbf{a}/\emptyset, M_0)$ is isolated iff $\mathrm{tp}(f(\mathbf{a})/\emptyset, M_0)$. □

It turns out that uncountable atomic models are very important. Here is a sufficient condition for a model to be atomic model (which will be used in later chapters to build atomic models):

THEOREM 9.31 (Sufficient condition for being atomic model). *Suppose that M is a structure and $A \subseteq |M|$. If there exists an enumeration $\{a_i \mid i < \alpha\}$ of $|M| - A$, such that $\mathrm{tp}(a_i/A_i)$ is isolated for every $i < \alpha$ where $A_i := A \cup \{a_j \mid j < i\}$ then M is atomic over A*

PROOF. Exercise. □

A related property of a theory is the following:

DEFINITION 9.32. A complete theory T is called *atomic* iff for every formula $\varphi(\mathbf{x})$ there exists an isolated $p \in \mathrm{D}_{\ell(\mathbf{x})}(T)$ containing the formula $\varphi(\mathbf{x})$.

In topological language T is atomic iff the set $\Gamma \subseteq \mathrm{D}_n(T)$ of isolated types is dense in the topological space $\mathrm{D}_n(T)$.

There is a nice connection with the cardinality $\mathrm{D}(T)$:

THEOREM 9.33 (Density of isolated types). *Let T be a complete theory in a countable language. If $\mathrm{D}(T)$ is countable then T is atomic.*
Moreover already $|\mathrm{D}(T)| < 2^{\aleph_0}$ implies that the theory T is atomic.

PROOF. Suppose T is not atomic and let $\varphi(\mathbf{x})$ be a consistent formula that can not be extended to an isolated complete type.

For every $\eta \in {}^{\omega>}2$ define a formula $\psi_\eta(\mathbf{x})$ such that

$$p_\eta := \{\varphi(\mathbf{x})\} \cup \{\psi^{\eta[n]}_{\eta\restriction n}(\mathbf{x}) \ : \ n < \ell(\eta)\} \quad \text{is consistent and}$$

$$\bigwedge p_\eta \quad \text{does not belong to a complete isolated type.}$$

The definition of $\psi_\eta(\mathbf{x})$ is by induction on $\ell(\eta)$. Now for $\eta \in {}^{\omega}2$ let

$$p_\eta := \bigcup_{n<\omega} p_{\eta\restriction n}.$$

Clearly $\eta \neq \nu \in {}^{\omega}2 \implies p_\eta \neq p_\nu$.

Thus we have shown that the assumption that T is not atomic yields that $|\mathrm{D}(T)| = 2^{\aleph_0}$. □

Already from Theorem 7.21 we know that if $\mathrm{D}(T)$ is uncountable then $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$. Thus if $\mathrm{I}(\aleph_0, T)$ is less than the maximum then $\mathrm{D}(T)$ must be countable.

Apriory for theories with countable $\mathrm{D}(T)$ there are six possibilities:

(1) $\mathrm{I}(\aleph_0, T) = 1$
(2) $\mathrm{I}(\aleph_0, T) = 2$
(3) $\mathrm{I}(\aleph_0, T) \in \{n \mid 3 \leq n < \omega\}$
(4) $\mathrm{I}(\aleph_0, T) = \aleph_0$
(5) $\aleph_0 < \mathrm{I}(\aleph_0, T) < 2^{\aleph_0}$, and
(6) $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$.

As we know there are examples for $\aleph_0$-categorical theories (like T_{ord} and T_{ind}). First we present a characterization of $\aleph_0$-categoricity.

Next we present a theorem of Robert Vaught that rules out possibility 2. It is not difficult to construct examples for possibility 3.

Possibility 4 is exemplified by T_{ACF_0} and we will be able to say something about this later in Chapter x.y.

There are no known examples for theories satisfying 5 (obviously this is interesting only in models where the continuum hypothesis fails). This is the most famous conjecture of elementary model theory:

CONJECTURE 9.34 (Vaught's conjecture). *Let T be a first order theory in a countable language. If $\mathrm{I}(\aleph_0, T) > \aleph_0$ then $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$.*

There are several approximations to this conjecture, I will mention only three:

Using descriptive set theory Morley have shown:

FACT 9.35 (Morley 1969 [**Mo3**]). *Let T be a first order theory in a countable language. If $\mathrm{I}(\aleph_0, T) > \aleph_1$ then $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$.*

Baldwin and Lachlan have shown that Vaught' conjecture is true for $\aleph_1$-categorical theories.

FACT 9.36 (Baldwin and Lachlan []). *If T is $\aleph_1$-categorical then Vaught's conjecture holds for T.*

Using deep properties of *orthogonality calculus* in 1980 Shelah managed to prove the following generalization:

FACT 9.37 ([**ShHaMa**]). *Vaught's conjecture is true for $\aleph_0$-stable theories.*

A proof of Fact 9.36 appears in Chapter 7. Proofs for the other results are not presented in this book.

There are partial results for some superstable theories due to Buechler and Newelski (see [**Bu1**]). Several people believe that the conjecture in its full generality is false. In spring 2002 Robin Knight [**Kn**] announced a counterexample, as of today a paper containing a proof was not published in a refereed journal.

There are examples for theories satisfying $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$ (even with $\mathrm{D}(T) = \aleph_0$).

In Theorem 7.21 we have seen that for a countable first-order theory T if $|\mathrm{D}(T)|$ is uncountable then $\mathrm{I}(\aleph_0, T) = 2^{\aleph_0}$. Around 1959 another fundamental fact connecting the number of types over the empty set and the number of countable models was discovered independently by Erwin Engeler, Ceslaw Ryll-Nardzewski, and Lars Svenonius:

THEOREM 9.38 (Engler, Svenonious, Ryll-Nardzweski's). *Let T be a countable complete first-order theory, the following conditions are equivalent:*

(1) *For every $n < \omega$ we have that $|\mathrm{D}_n(T)| < \aleph_0$.*
(2) *For every $n < \omega$ every $p \in \mathrm{D}_n(T)$ is an isolated type.*
(3) *If $M \models T$ then M is atomic.*
(4) *T is $\aleph_0$-categorical.*
(5) *if $M \models T$ is countable then M is saturated.*
(6) *for every $n < \omega$ and $\mathbf{x} = \langle x_0, \ldots, x_{n-1} \rangle$ there exists $k_n < \omega$ and there exists $\{\varphi_i(\mathbf{x}) \ : \ i < k_n\}$ such that for every formula $\psi(\mathbf{x})$ there exists $i < k_n$ such that $T \models \forall \mathbf{x}[\psi(\mathbf{x}) \leftrightarrow \varphi_i(\mathbf{x})]$.*

PROOF. We will show that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (2) $\Longrightarrow$ (5) $\Longrightarrow$ (4), (2) $\Longrightarrow$ (1), (2) $\Longrightarrow$ (1) $\Longrightarrow$ (6), and (6) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2): Let $p \in \mathrm{D}_n(T)$ be given. For every $q \in \mathrm{D}_n(T) - \{p\}$ there is a formula $\varphi_{p,q}(\mathbf{x}) \in p$ such that $\neg\varphi_{p,q}(\mathbf{x}) \in q$. Now since $\mathrm{D}_n(T)$ is finite

$$\psi(\mathbf{x}) := \bigwedge_{q \in \mathrm{D}_n(T) - \{p\}} \varphi_{p,q}$$

is a first-order formula isolating p.

(2) $\Longrightarrow$ (3): Let $M \models T$ and suppose $\mathbf{a} \in |M|$. the type $\mathrm{tp}(\mathbf{a}/\emptyset) \in \mathrm{D}_n(T)$ must be isolated.

(3) $\Longrightarrow$ (4): Since atomic models are unique (Proposition 9.25).

(4) $\Longrightarrow$ (2): Suppose that (2) fails. Let $n < \omega$ be such that there exists $p \in \mathrm{D}_n(T)$ is not isolated. By Henkin's omitting types theorem (Theorem 9.13) there exists a countable $M \models T$ omitting p. Thus $\mathrm{I}(\aleph_0, T) \geq 2$.

(2) $\Longrightarrow$ (5): Immediate.

(5) $\Longrightarrow$ (4): By uniqueness of Saturated models (Theorem 9.3).

(2) $\Longrightarrow$ (1): Let $n < \omega$ be given. Consider

$$\Gamma := \{\neg\varphi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T)) \mid \exists p \in \mathrm{D}_n(T) \text{ such that } \varphi(\mathbf{x}) \vdash p\}.$$

Since every $p \in \mathrm{D}_n(T)$ is isolated there can not be a type $q \in \mathrm{D}_n(T) \supseteq \Gamma$. Thus there is a finite inconsistent $\Gamma_0 \subseteq \Gamma$. Since T is consistent this translates into

$$T \vdash \neg\exists\mathbf{x}[\bigwedge \Gamma_0].$$

Which means that

$$T \vdash \forall\mathbf{x}[\bigvee_{\neg\varphi\in\Gamma_0} \varphi(\mathbf{x})].$$

Thus for any $\mathbf{a} \in \mathfrak{C}$ there exists a formula $\neg\varphi \in \Gamma_0$ such that $\mathfrak{C} \models \varphi[\mathbf{a}]$.

For any $\neg\varphi \in \Gamma_0$ let $p_\varphi := \{\psi(\mathbf{x}) \mid T \vdash \forall\mathbf{x}[\varphi(\mathbf{x}) \to \psi(\mathbf{x})]\}$. Since $\neg\varphi(\mathbf{x}) \in \Gamma$ it isolates a complete type, thus p_φ is that complete type.

So we found a finite set of complete types $\Phi := \{p_\varphi \mid \neg\varphi \in \Gamma_0\} \subseteq \mathrm{D}_n(T)$ such that for every $\mathbf{a} \in \mathfrak{C}$ there is $p \in \Phi$ such that $\mathbf{a} \models p$. So we have shown that $|D_n(T)| \leq |\Phi| < \aleph_0$.

(6) $\Longrightarrow$ (1): Let $\{\varphi_i(\mathbf{x}) \ : \ i < k_n\}$ be as in the statement. for $i < k_n$. Clearly $|\,\mathrm{D}_n(T)| \leq 2^{k_n} < \aleph_0$.

(1) $\Longrightarrow$ (6): For every formula $\varphi(\mathbf{x})$ in n-variables let

$$[\varphi] := \{p \in \mathrm{D}_n(T) \mid \varphi(\mathbf{x}) \in p\}.$$

Clearly $T \vdash \forall\mathbf{x}[\varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x})]$ iff $[\varphi] = [\psi]$. Thus the equivalence class of the formula $\varphi(\mathbf{x})$ is determined by the set $[\varphi] \subseteq \mathrm{D}_n(T)$. Let $\{\varphi_i(\mathbf{x}) \ : \ i < \alpha\}$ be a complete set of representatives of the above classes. Clearly $|\alpha| \leq 2^{|\,\mathrm{D}_n(T)|}$. $\square$

A natural question to ask immediately after introduction of the spectrum function is the following implication true?

$$T' \text{ is an expansion of } T \implies \mathrm{I}(\lambda, T) = \mathrm{I}(\lambda, T').$$

Notice that always

$$T' \text{ is an expansion of } T \implies \mathrm{I}(\lambda, T) \leq \mathrm{I}(\lambda, T').$$

The following corollary show that in case of categoricity in $\aleph_0$ the above is true for expansion by constants and in Corollary 3.**??** we will derive an analog property also for theories categorical in an uncountable cardinal.

COROLLARY 9.39 (Countable categoricity is preserved by expansions). *Let T be a countable complete theory. If T is categorical in $\aleph_0$ then for every finite A also T_A is categorical in $\aleph_0$.*

PROOF. Let $\mathbf{a}$ be an enumeration of A. If T_A is not $\aleph_0$-categorical then by Theorem 9.38 there exists $n < \omega$ such that there exists an infinite set $\Phi \subseteq \mathrm{Fml}(\mathrm{L}(T_A))$ such that $\varphi \in \Phi \implies \varphi = \varphi(\mathbf{a}; \mathbf{x})$ where $\ell(\mathbf{x}) = n$ and

$$\varphi_1 \neq \varphi_2 \in \Phi \implies T \vdash \exists \mathbf{x} \neg [\varphi_1(\mathbf{a}; \mathbf{x}) \leftrightarrow \varphi_2(\mathbf{a}; \mathbf{x})].$$

Thus $\Psi := \{\varphi(\mathbf{y}; \mathbf{x}) \mid \varphi(\mathbf{a}; \mathbf{x}) \in \Phi\}$ is an infinite set of pairwise non-equivalent formulas in $|A| + n$ many variables in contradiction to part (6) of Theorem 9.38. □

ADD LATER

(1) If T is countable and complete then
(2) examples
(3) Structure of $\mathrm{Aut}(M)$ - transitivity
(4) $\aleph_0$ categorical projective planes ?

THEOREM 9.40 (Vaught 1962). *If T be a complete countable first-order theory then* $\mathrm{I}(\aleph_0, T) \neq 2$.

PROOF. For the sake of contradiction suppose that $\mathrm{I}(\aleph_0, T) = 2$.
Since $\mathrm{I}(\aleph_0, T) < 2^{\aleph_0}$ by Theorem 7.21 the diagram of T is countable. By Theorems 9.29 and 9.1 the theory T has an atomic model N and a countable saturated model M. Since $\mathrm{I}(\aleph_0, T) \neq 1$, by Theorem 9.38 $M \ncong N$. Thus there is $\mathbf{a} \in M$ such that $p := \mathrm{tp}(\mathbf{a}/\emptyset)$ is not isolated. Consider the theory $T_{\mathbf{a}}$ defined as $\mathrm{Th}(\langle M, \mathbf{a} \rangle)$. Namely the expansion of T by constants for $\mathbf{a}$.

Let $n := \ell(\mathbf{a})$. Since T is not $\aleph_0$-categorical by Theorem 9.38 there are $\{\varphi(x_0, \ldots, x_n) \mid n < \omega\}$ such that

$$i \neq j < \omega \implies T \vdash \neg \forall \mathbf{x} [\varphi_i(\mathbf{x}) \leftrightarrow \varphi_j(\mathbf{x})].$$

Notice that we have now that

$$(*) \quad i \neq j < \omega \implies T_{\mathbf{a}} \vdash \neg \forall x_0 [\varphi_i(x_0, \mathbf{a}) \leftrightarrow \varphi_j(x_0, \mathbf{a})].$$

Since also $\mathrm{D}(T_{\mathbf{a}})$ is countable an application of Theorem 9.29 gives a countable atomic $N_{\mathbf{a}} \models T_{\mathbf{a}}$. I claim that $N_{\mathbf{a}} \restriction \mathrm{L}(T)$ is a third countable model. Clearly the above reduct is not an atomic model of T. It suffices to show that it not saturated. Notice that the model $M_{\mathbf{a}} := \langle M, \mathbf{a} \rangle$ (the expansion of M by constants from $\mathbf{a}$) is a countable saturated model of $T_{\mathbf{a}}$. Had $N_{\mathbf{a}} \cong M_{\mathbf{a}}$ then by universality of saturated models every countable model of $T_{\mathbf{a}}$ is atomic thus by Theorem 9.38 the theory $T_{\mathbf{a}}$ is categorical in $\aleph_0$. But this contradicts $(*)$.

□

EXAMPLE 9.41. For every $n < \omega$ such that $n > 2$ there exists a complete countable theory T_n such that $\mathrm{I}(\aleph_0, T_n) = n$.

How about stable example (still open, Pillay's favorite question)?

Peter Cameron's book.

DEFINITION 9.42. Let M be an L-structure. We say that $G \leq \mathrm{Aut}(M)$ is *oligomorphic* iff the action of G on ${}^{<\omega}|M|$ has finitely many orbits for every $n < \omega$.

Using the correspondence between complete types and orbits of groups with the above action. Ryll-Nardzweski's theorem can be restated as: For a complete f.o. theory T, the theory T is $\aleph_0$-categorical iff $\mathrm{Aut}(M)$ is oligomorphic for all infinite $M \models T$.

Exercises 1.9.

(1) Use Corollary 2.20 to obtain an alternative proof for of Theorem 9.1.
(2) Let T be a consistent theory and p a T-type. If $p(x)$ is not isolated then for every $\psi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T))$ (the variable x occurs in the sequence $\mathbf{x}$) such that $T \cup \{\exists \mathbf{x}\psi(\mathbf{x})\}$ is consistent there $\varphi(x) \in p$ such that

$$T \cup \{\exists \mathbf{x}[\psi(\mathbf{x}) \wedge \neg\varphi(x)]\} \text{ is consistent.}$$

(3) The goal of this exercise is to show that in Henkin's omitting types theorem we can replace the single variable x by a finite sequence $\mathbf{x}$. Namely, prove the following generalization of Theorem 9.13:

THEOREM 9.43. *Let T be a countable theory, let $\mathbf{x}$ be a finite sequence of variables, and let p be a T- type in $\mathbf{x}$. If p is not isolated then there exists $M \models T$ that omits p.*

(4) Let T be a theory. If every $p \in \mathrm{D}(T)$ is isolated then $|\mathrm{D}(T)| \leq |\mathrm{L}(T)| + \aleph_0$.
(5) Let T be a countable theory. Suppose that there exists a consistent (with respect to T) formula $\varphi(x)$ such that for revery $p \in \mathrm{D}(T)$ containing φ the type p is not isolated. Show that $|\mathrm{D}(T)| = 2^{\aleph_0}$.
(6) Let T be a theory in a countable language. Suppose $\Gamma \subseteq \mathrm{D}(T)$ such that

$$p \in \Gamma \implies p \text{ is not isolated.}$$

If $|\Gamma| < 2^{\aleph_0}$ then there exists a countable $M \models T$ omitting all the types in Γ.
(7) Let n be a positive integer, and let p, q be n-types, let T be a theory. Suppose that q is a finite set and that $p \vdash q$. Conclude that there exists a finite $p_0 \subseteq p$ such that $p_0 \vdash q$. Hint: For $c = \langle c_0, \ldots, c_n \rangle$ a sequence of new constants (not in $\mathrm{L}(T)$) let $p(c)$ be the obvious thing - the set $\{\varphi(c) \mid \varphi(x) \in p\}$. Prove that $p \models q$ iff $T \cup p(c) \vdash \psi$ for every $\psi \in q(c)$.
(8) Let $\mathbb{C}$ the field of complex numbers. Show that for every transcendental $t \in \mathbb{C}$ the type $\mathrm{tp}(t/\mathbb{Q})$ is not isolated.
(9) Let $p := \mathrm{tp}(\mathbf{a}/A, M)$. Show that if p is isolated over A then there exists $\varphi(\mathbf{x}; \mathbf{b}) \in p$ such that $\varphi(\mathbf{x}; \mathbf{b}) \vdash p$.
(10) In this exercise you will establish *transitivity of isolation*. Let T be a complete theory, $A \subseteq |\mathfrak{C}|$ and $\mathbf{a}, \mathbf{b} \in |\mathfrak{C}|$ are given. If $\mathrm{tp}(\mathbf{a}/A)$ is isolated over A and $\mathrm{tp}(\mathbf{b}/A \cup \{\mathbf{a}\})$ is isolated over $A \cup \{\mathbf{a}\}$ then $\mathrm{tp}(\mathbf{a}\hat{}\mathbf{b}/A)$ is isolated over A.
(11) Prove Theorem 9.31. Hint: by induction using Theorem 9.11.
(12) The goal of this exercise is to provide a crucial fact used in the proof of Morley's theorem to show that certain model is not saturated. Suppose that T is a complete first-order theory, let $A_0 \subseteq A \subseteq |\mathfrak{C}|$ be such that $|A_0| + \aleph_0 < |A|$ and let M be atomic over A.

If there exists $p \in \mathrm{S}(A_0)$ not isolated over A_0 then the model M is not $|A|^+$-saturated.

(13) For every countable $M \models \mathrm{ZF}$ there exists a $N \succeq M$ such that ω_1^N is uncountable and $\omega^N = \omega^M$.
(14) Let χ be regular uncounable. Show that every countable $M \preceq \langle H(\chi), \in\rangle$ has an uncountable elementary extension N such that $\omega^N = \omega^M$.
(15) Let T be a first-order theory in a countable language. Let $\varphi(x) \in \mathrm{Fml}(\mathrm{L}(T))$ be given. If T has a model M such that

$$\|M\| > |\varphi(M)| \geq \aleph_0$$

then there exists an uncountable $N \models T$ such that $|\varphi(N)| = \aleph_0$.
(16)
(17) Show that T_{ACF} has $\aleph_0$ many pairwise non-siomorphic prime models.
(18) Permutation groups ....
(19) Suppose M is a countable atomic model, prove that for any $a, b \in |M|$ if $\mathrm{tp}(a/\emptyset, M) = \mathrm{tp}(b/\emptyset, M)$ then there exists $f \in \mathrm{Aut}(M)$ such that $f(a) = b$.
(20) Suppose M is a countable atomic model and let $A \subseteq |M|$ be a finite set. Prove that for any $a, b \in |M|$ if $\mathrm{tp}(a/A, M) = \mathrm{tp}(b/A, M)$ then there exists $f \in \mathrm{Aut}_A(M)$ such that $f(a) = b$.
(21) Suppose M is a countable atomic model, prove that for any $\mathbf{a}, \mathbf{b} \in |M|$ if $\mathrm{tp}(\mathbf{a}/\emptyset, M) = \mathrm{tp}(\mathbf{b}/\emptyset, M)$ then there exists $f \in \mathrm{Aut}(M)$ such that $f(\mathbf{a}) = \mathbf{b}$.
(22) Let T be a countable theory. If T is $\aleph_0$-categorical then for every $M \models T$ and a finite $A \subseteq |M|$ also $\mathrm{Th}(\langle M, a\rangle_{a\in A})$ is $\aleph_0$-categorical.
(23) Why did we require in the previous exercise the set A to be finite? Is $\mathrm{Th}(\langle M, a\rangle_{a\in A})$ $\aleph_0$-categorical also when A is infinite?
(24) Describe the countable atomic models of T_{ord}, T_{ACF_0} and Peano's arithmetic.
(25) Is $\langle \mathbb{R}, <\rangle$ an atomic model of T_{ord}?
(26) Every atomic model is homogenous.
(27) Let T be a countable first-order theory that does not have finite models. If $M \models T$ is a countable atomic model then M is elementarily embeddable into every model of T.
(28) Suppose that M and N are countable structures such that $M \equiv N$ and let $A \subseteq |M|$, $B \subseteq |N|$ be finite sets. If M is atomic over A and N is atomic over B then for any (M, N) embedding f satisfying $\mathrm{dom}\, f = A$ and $\mathrm{range}\, f = B$ there exists $f^* : M \cong N$ extending f.
(29) Can a countable complete theory have an uncountable atomic model?
(30) Is ZFC an atomic theory?
(31) The following notion is of central importance for the theory developed in Chapter **??** of Volume III. Suppose T is a countable complete first-order theory.

DEFINITION 9.44. A set $A \subseteq \mathfrak{C}$ is called *good* iff for every $\varphi(\mathbf{x}; \mathbf{a})$ with parameters from A such that $\models \exists \mathbf{x}\varphi[\mathbf{x}; \mathbf{a}]$ there exists $p \in \mathrm{S}(A)$ isolated over A containing $\varphi(\mathbf{x}; \mathbf{a})$.

Denote by D_A the set $\{\mathrm{tp}(\mathbf{a}/A, \mathfrak{C}) \mid A \cup \{\mathbf{a}\}$ is an atomic set$\}$.

Prove the following:

PROPOSITION 9.45. *If* $|\,\mathrm{D}_A\,| < 2^{\aleph_0}$ *then* A *is good.*

(32) For each of the theories from Examples I.2.48, I.2.53 and I.2.59 decide which theory is atomic and present at least one atomic mode (if there are any).
(33) Let T be a theory with recursive axiomatization. Suppose that T has elimination of quantifiers. Prove that if T has an atomic model then it is decidable.
(34) Suppose that every subset of $\mathfrak{C}$ is good. Prove that $\mathrm{Th}(\mathfrak{C})$ has arbitrarily large atomic models.

10. Models of weak set theory

In several proofs we started with a given L-structure M, took an expansion N of M and applied some model-theoretic manipulation to N. E.g. see the proof of the Erdős-Rado theorem (Theorem 4.19). A natural question to ask is. How do we know how to select the expansion N? In this section I describe how to take a massive expansion that always will do the job.

It turns out that there is a "generic" way to choose expansions. What we do is to take for N a model that contains much of set theory and also has all the relevant information about structure M.

We introduce in this section some simple but powerful facts about models of set theory or fragments of set theory.

For this we need to learn few basic facts about models of set theory.

DEFINITION 10.1. *Zermelo Fraenkel set theory* is denoted by ZF, it is the following infinite set of sentences in the language consisting of a single binary relation $\{\in\}$:

(1) *Axiom of extensionality*: $\forall a \forall b[a = b \leftrightarrow \forall x[x \in a \leftrightarrow x \in b]]$.
(2) *Axiom of pairing*: $\forall a \forall b \exists c \forall x[x \in c \leftrightarrow x = a \vee x = b]$.
(3) *Axiom of foundation*:

$$\forall a \big[\exists x[x \in a] \rightarrow \exists y[y \in a \wedge \forall z[z \in a \rightarrow [z = y \vee y \in z]]]\big].$$

(4) *Axiom of union*:

$$\forall a \exists b \forall x[x \in b \leftrightarrow \exists y[y \in a \wedge x \in y]].$$

(5) *Power set axiom*:

$$\forall a \exists b \forall x[x \in b \leftrightarrow \forall y[y \in x \rightarrow y \in a]].$$

.

(6) *Axiom schema of comprehension*: For every first-order formula $\varphi(x; \mathbf{y})$ (in the language $\{\in\}$):

$$\forall \mathbf{y} \forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \varphi(x; \mathbf{y})].$$

(7) *Axiom schema of replacement*: For every formula $\varphi(x, y; \mathbf{z})$

$$\forall \mathbf{z} \forall a \Big[\big[\forall x \forall y_1 \forall y_2[x \in a \wedge \varphi(x, y_1; \mathbf{z}) \wedge \varphi(x, y_2; \mathbf{z})] \rightarrow y_1 = y_2\big] \rightarrow$$

$$\exists b \forall y[y \in b \rightarrow \exists x[x \in a \wedge \varphi(x, y; \mathbf{z})]\Big].$$

(8) *Axiom of infinity:*

$$\exists x[x \neq \emptyset \wedge \forall y[y \in x \rightarrow y \cup \{y\} \in x]].$$

One should check that all the standard notions of mathematics can be formulated in the language $\{\in\}$. See [**HJ**] or [**Ku**] for a detailed formal development. For any family of sets S we say that $f : S \to \bigcup S$ is a *choice function for* S iff

$$f(x) \in x \quad \text{for every } x \in S.$$

The *Axiom of Choice* is denoted by AC and is the statement:

$$\forall a \exists f [f \text{ is a choice function for } a].$$

By ZFC *Zermelo Fraenkel with Choice* denote the theory ZF plus the axiom of choice.

By ZFC^- and ZF^- (respectively) we denote ZFC and ZF without the power set axiom.

DEFINITION 10.2 (The cumulative hierarchy). By induction on the ordinal α define a set $\mathbf{V}_\alpha$:

$$\mathbf{V}_\alpha := \begin{cases} \emptyset & \text{if } \alpha = 0 \\ \bigcup_{\beta<\alpha} \mathbf{V}_\beta & \text{if } \alpha \text{ is limit} \\ \mathcal{P}(\mathbf{V}_\beta) & \text{if } \alpha = \beta + 1. \end{cases}$$

Let $\mathbf{V} := \bigcup_{\alpha \in \mathrm{On}} \mathbf{V}_\alpha$.

For $x \in \mathbf{V}$ let $\mathrm{rk}(x) := \min\{\alpha \in \mathrm{On} \mid x \in \mathbf{V}_\alpha\}$.

It is easy to see that

PROPOSITION 10.3. *If $\beta > \omega$ is a limit ordinal then*

$$\langle \mathbf{V}_\beta, \in \rangle \models \Big[(1) \wedge (2) \wedge (3) \wedge (4) \wedge (5) \wedge (6) \wedge (8) \wedge \forall x \forall f [f \textit{ is a function } \wedge \operatorname{dom} f = x \to \exists z [z = \operatorname{range} f]] \Big].$$

PROOF. Exercise. □

The following is an important basic fact:

THEOREM 10.4 (Levy's reflection principle). *For any sentence φ (in the language of set theory) and every ordinal α if $\langle \mathbf{V}, \in \rangle \models \varphi$ then there exists an ordinal $\beta > \alpha$ such that $\langle \mathbf{V}_\beta, \in \rangle \models \varphi$.*

PROOF. The proof is similar to the argument used for Downward Löenhiem-Skolem Theorem: □

By Gödel's third incompleteness theorem (i.e. $T \nvdash Con(T)$ for any T extending PA) together with the easy direction of the completeness theorem (if there is a *set* M such that $M \models T$ then T is consistent) it follows that

$$(*) \quad \text{ZFC} \nvdash \text{“}\langle \mathbf{V}_\alpha \in \rangle \models \text{ZFC”} \quad \text{for every ordinal} \quad \alpha.$$

Thus using the reflection principle we get the following interesting corollary:

COROLLARY 10.5. ZFC *does not have a finite axiomatization.*

PROOF. Otherwise, let Φ be a finite axiomatization of ZFC an application of the reflection theorem will produce α such that $\langle \mathbf{V}_\alpha \in \rangle \models \Phi$ thus $\langle \mathbf{V}_\alpha \in \rangle \models \text{ZFC}$ which contradicts $(*)$. □

It is interesting to contrast $(*)$ with the following theorem:

THEOREM 10.6. *(*ZFC*) If κ is an uncountable strongly inaccessible cardinal then $\langle \mathbf{V}_\kappa \in \rangle \models \text{ZFC}$. More precisely*

$$\text{ZFC} \vdash \forall \kappa > \aleph_0 [\kappa \textit{ is strongly inaccessible } \rightarrow \langle \mathbf{V}_\kappa \in \rangle \models \text{ZFC}].$$

PROOF. □

A standard corollary of the last theorem is

$$\text{ZFC} \nvdash \exists \kappa > \aleph_0 [\kappa \text{ is strongly inaccessible }].$$

DEFINITION 10.7. Let χ be an infinite cardinal number. The set $H(\chi)$ is called the *set of elements of hereditary cardinality less than* χ

$$H(\chi) := \{S \ : \ |\operatorname{tr.cl}(S)| < \chi\}.$$

Where $\operatorname{tr.cl}(S)$ is the *transitive closure of* S namely it is defined as

$$\operatorname{tr.cl}(S) := \bigcup_{n<\omega} \bigcup^n S,$$

where $\bigcup^n S$ is defined by induction on $n < \omega$:

$$\bigcup^n S := \begin{cases} S & \text{if } n = 0 \\ \\ S \cup \bigcup S \cdots \cup \bigcup^{n-1} S & \text{if } n \text{ is positive integer.} \end{cases}$$

REMARK 10.8. $H(\chi)$ is a set and not a proper class. This can be seen by proving that

$$H(\chi) \subseteq \mathbf{V}_\chi.$$

Show that by induction on $rk(x)$ for all $x \in H(\chi)$.

DEFINITION 10.9. *weak model of* ZFC

THEOREM 10.10 (Reflection for $H(\chi)$). *If χ is regular uncountable then $\langle H(\chi), \in\rangle$ is a model of* ZFC^-. *Moreover, for any first order sentence φ (in the language $\{\in\}$) such that* $\mathrm{ZFC} \vdash \varphi$ *there exists χ regular such that $\langle H(\chi), \in\rangle \models \mathrm{ZFC}^- \cup \{\varphi\}$.*

PROOF. LATER. □

PROPOSITION 10.11. *Let $M := \langle P, <_P\rangle$ be a poset of cardinality λ. If $\chi > 2^\lambda$ then there exists $M' \in H(\chi)$ isomorphic to M.*

PROOF. □

THEOREM 10.12. *Suppose $\chi > 2^\lambda$ is regular, then for every similarity type L of cardinality $\leq \lambda$ and any L-structure of cardinality λ there exists $M' \in H(\chi)$ isomorphic to M.*

PROOF. □

REMARK 10.13. Notice that since for an uncountable regular χ the structure $\langle H(\chi), \in\rangle$ is a model of ZFC^-, for every $M \preceq \langle H(\chi), \in\rangle$ the following holds:

(1) $\omega \subseteq |M|$.
(2) If $\alpha \in |M|$ then also $\alpha + 1 \in |M|$. If α is a successor ordinal then its predessor is also an element of $|M|$.
(3) If $\alpha < \chi$ then $\alpha \in H(\chi)$, thus $\chi \subseteq H(\chi)$.
(4) If $f \in |M|$ is a set of ordered pairs then also $\mathrm{Dom}\, f \in |M|$ and $\mathrm{Range}\, f \in |M|$; In case f is a function we have $f(a) \in |M|$ for all $a \in \mathrm{Dom}\, f$.
(5) While usually M is not transitive we have that: If $A \in |M|$ is a finite set then also $A \subseteq |M|$.
(6) When $\lambda \in H(\chi)$ is a cardinal number then $\lambda \cap |M|$ does not have a last element (with respect to $\in$).

DEFINITION 10.14 (definable subsets). For a set S by $\mathrm{Def}(S)$ denote *the collection of definable subsets of S with parameters from S*, namely

$$\mathrm{Def}(S) := \{\varphi(S; \mathbf{a}) \mid \mathbf{a} \in {}^{\omega>}\mathrm{S},\ \varphi(x; \mathbf{y}) \in \mathrm{Fml}(\{\in\})\},$$
$$\text{where } \varphi(S; \mathbf{a}) := \{b \in S \mid \langle \mathbf{V}, \in\rangle \models \varphi[b; \mathbf{a}]\}.$$

DEFINITION 10.15 (The constructible hierarchy). By induction on the ordinal α define a set $\mathbf{L}_\alpha$:

$$\mathbf{L}_\alpha := \begin{cases} \emptyset & \text{if } \alpha = 0 \\ \bigcup_{\beta<\alpha} \mathbf{L}_\beta & \text{if } \alpha \text{ is limit} \\ \mathrm{Def}(\mathbf{L}_\beta) & \text{if } \alpha = \beta + 1. \end{cases}$$

Let $\mathbf{L} := \bigcup_{\alpha\in\mathrm{On}} \mathbf{L}_\alpha$.

The axiom of constructibility is the statement $\mathbf{V} = \mathbf{L}$.

The axiom $\mathbf{V} = \mathbf{L}$ was introduced by Kurt Gödel in 1938 as a tool to show the relative consistency of the axiom of choice and the continuum hypothesis.

The next major contribution to the study of the constructible universe was Ronald Jensen's 1972 paper ([**Je72**]), where a theory named *fine structure theory* was introduced. A standard technology developed by Jensen was to choose an increasing and continuous elementary chain $\{N_\alpha \preceq \langle \mathbf{L}_{\omega_2}, \in\rangle \mid \alpha < \omega_1\}$ satisfying

(1) $\|N_\alpha\| = \aleph_0$,
(2) $\alpha + 1 \subseteq |N_\alpha|$ and
(3) $\langle N_\beta \mid \beta \leq \alpha\rangle \in N_{\alpha+1}$.

An important object to study is $\{\omega_1 \cap N_\alpha \mid \alpha < \omega_1\}$. This can be used to construct various different mathematical objects.

About the same time Saharon Shelah (see [**Sh 11**]) used similar chains of elementary submodels of $\langle V_\alpha, \in\rangle$ to obtain results (in ZFC) about saturated models and to prove non-structure theorems in model theory (see [**Sh e**] and section VII.1 of [**Sh a**]). During the late seventies and early eighties this technique was applied to submodels of $\langle H(\chi), \in\rangle$ instead of $\langle \mathbf{L}_\chi, \in\rangle$. Much of the development of the theory of *proper forcing* (see [**Sh f**]) and *pcf*-theory (see [**Sh g**]) would be impossible without studying such chains.

One of the most powerful tools in using models of weak set theory to construct specific mathematical objects. The following is a generic construction (or "the magic"):

DEFINITION 10.16. Let λ be an uncountable cardinal number, M a structure of cardinality λ (for the language $\mathrm{L}(M)$ of cardinality less than λ), χ a regular cardinal large enough such that $M \in H(\chi)$ and the model $\langle H(\chi), \in\rangle$ reflects all the properties of M and λ (usually it suffices to require that $\chi \geq (2^\lambda)^+$). $\mathfrak{B} := \langle H(\chi), \in, \lambda, M, \ldots\rangle$ where λ is a unary predicate interpreted by the set of ordinals less than λ and there is a unary predicate for $|M|$, function and relations and constants from $\mathrm{L}(M)$.
A sequence $\{\mathfrak{B}_i \preceq \mathfrak{B} \ : \ i < \lambda\}$ is called an *internal chain of models* iff it is an increasing continuous chain such that:

(1) $\|\mathfrak{B}_i\| < \lambda$ for every $i < \lambda$,
(2) $i + 1 \subseteq \mathfrak{B}_i$ for all i and
(3) $\langle \mathfrak{B}_j \ : \ j \leq i\rangle \in \mathfrak{B}_{i+1}$ for every $i < \lambda$.

Notice that by the Downward Löwenheim-Skolem theorem such a chain always exists. Often when constructing internal chains we will have some additional condition. For example the induction step in the proof of the Erdős-Rado

Theorem

$$\beth_n(\mu)^+ \to (\mu^+)^{n+1}_\mu \implies \beth_{n+1}(\mu)^+ \to (\mu^+)^{n+2}_\mu.$$

can be obtained from taking an internal chain of length λ inside the structure $\langle H(\chi), \in, \lambda, \lambda_1, F, i\rangle_{i<\mu}$ where $\lambda_1 := \beth_{n+1}(\mu)^+$, $\lambda := \beth_{n+1}(\mu)$ and F is the coloring with the additional condition

(4) $\mathfrak{B}_{i+1}$ is a $\beth_n(\mu)^+$-saturated submodel of $\langle H(\chi), \in, \lambda, \lambda_1, F, j\rangle_{j<\mu}$.

REMARK 10.17. Often it is desirable to have a model of weak set theory with definable Skolem functions. In light of Theorem 14.18 if we pick $<^*$ a well-ordering of the set $H(\chi)$ then the structure $\langle H(\chi), \in, <^*, \ldots\rangle$ has definable Skolem functions.

The following is a generalization of the main step used in the proof of cardinal transfer of model-completeness (see Theorem 13.11).

PROPOSITION 10.18 (Coding first-order properties). *Suppose $N \prec M$ and let $L := \mathrm{L}(M)$ and χ regular large enough so that $M, N, L, \models_L \in H(\chi)$ and $\langle H(\chi), \in\rangle \models [N \prec M]$. Consider $\mathfrak{B} := \langle H(\chi), \in, |M|, |N|, L, R^M, F^M, c^M\rangle_{R,F,c\in L}$ when $\mathrm{L}(\mathfrak{B})$ contains two new unary predicates interpreted by the sets $|M|$ and $|N|$, F^M is defined for all the elements of $|M|$ as usual and for $a \in H(\chi) - |M|$ as some fixed element of $|M|$.*

If $\mathfrak{B}' \equiv \mathfrak{B}$ then $M' \equiv M$ and $N' \prec M'$.

PROOF. Exercise. □

PROPOSITION 10.19 (Coding omitting a type). *Let T be a first-order theory, $p(x)$ a T-type and $M \models T$. let $L := \mathrm{L}(M)$, $\lambda := |L| + \aleph_0$, fix a surjective function $f : \lambda \to p$, a bijection $g : \lambda \to \mathrm{Fml}(L)$ and χ regular large enough so that $M, p, f, g, \lambda, L, \models_L \in H(\chi)$ and*

$\langle H(\chi), \in\rangle \models [M \models_L T \wedge M$ *omits the type* $p \wedge f$ *is surjective* $\wedge\, g$ *is a bijection*$]$.

Consider $\mathfrak{B} := \langle H(\chi), \in, |M|, \lambda, f, g, p, L, R^M, F^M, c^M, i\rangle_{R,F,c\in L, i\in\lambda}$ when $\mathrm{L}(\mathfrak{B})$ contains three new unary predicates interpreted by the sets $|M|$, λ, p and two unary function symbols interpreted by f and g.

If $\mathfrak{B}^* \equiv \mathfrak{B}$ such that $\lambda^* = \lambda$ then $\mathrm{Fml}(L)^* = \mathrm{Fml}(L)$, $p^* = p$ and

$$M^* \text{ omits } p \iff M \text{ omits } p.$$

PROOF. Notice that $\mathfrak{B} \models$ □

Suppose that L is a similarity type and $\chi > \aleph_0$ is regular such that $L \in H(\chi)$. Observe that $\mathrm{Fml}(L) \subseteq H(\chi)$ and if in addition we suppose that also the cardinality of L and the enumeration of L are elements of $H(\chi)$ then $\mathrm{Fml}(L) \in H(\chi)$.

Sometime we will construct internal chains for λ uncountable regular and pay special attention to the interpretation of the unary predicate that stands for λ in the structure $\mathfrak{B}_i$:

PROPOSITION 10.20. *Let λ be an uncountable regular cardinal. If $\{\mathfrak{B}_i \preceq \mathfrak{B} \ : \ i < \lambda\}$ is an internal chain and let*

$$C := \{\delta < \lambda \mid \delta = \lambda^{\mathfrak{B}_\delta}\}.$$

Then the set C contains a closed unbounded subset of λ.

PROOF. This is a special case of the "moreover" part in Theorem 1.9. □

A reader interested to see more applications of weak models of set theory is advised to jump ahead and read section 13 of Chapter 4. It is independent of the rest of Chapter 4.

Many more applications are in section **??**.**??**, Chapters **??** and **??**.

Exercises 1.10.

(1) Let δ be an infinite ordinal and let β be an ordinal large enough so that $\delta \in \mathbf{V}_\beta$. Consider $\mathfrak{B} := \langle \mathbf{V}_\beta, \in, \delta\rangle$ when δ is a unary predicate interpreted by the set δ. Show that if $\mathfrak{B}^* \equiv \mathfrak{B}$ then $\langle \delta^{\mathfrak{B}^*}, \in^{\mathfrak{B}^*}\rangle$ starts with order-type ω. If $\langle \delta^{\mathfrak{B}^*}, \in^{\mathfrak{B}^*}\rangle$ is not well-ordered then there exists $\{a_n \mid n < \omega\} \subseteq \delta^{\mathfrak{B}^*}$ such that $\mathfrak{B}^* \models [a_{n+1} \in a_n]$ for every $n < \omega$.

Let $I := \{a \in \delta^{\mathfrak{B}^*} \mid \mathfrak{B}^* \models [a \in a_n] \text{ for every } n < \omega\}$. Conclude that I is infinite.

(2) Let δ be an infinite ordinal and let β be an ordinal large enough so that $\delta \in \mathbf{V}_\beta$ Consider $\mathfrak{B} := \langle \mathbf{V}_\beta, \in, \delta\rangle$ when δ is a unary predicate interpreted by δ. If $\mathfrak{B}^* \equiv \mathfrak{B}$ such that $\langle \delta^{\mathfrak{B}^*}, \in^{\mathfrak{B}^*}\rangle$ is not well-ordered then there exists and order preserving mapping from the rational numbers into $\langle \delta^{\mathfrak{B}^*}, \in^{\mathfrak{B}^*}\rangle$.

(3) Let T be a first-order theory that has an infinite model. Prove that for every $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ there exists $\{M_n \models T \mid n < \omega\}$ all of cardinality λ such that

$$n < \omega \implies M_{n+1} \precneqq M_n \text{ and } \bigcap_{n<\omega} M_n \prec M_0 \text{ of cardinality } \lambda.$$

While almost always we are interested in increasing chains and unions, the peculiar properties of the chain of models as above are actually useful (see the proof of Thoerem 8.5 and Proposition 8.7). Using indiscernible sequence and SkolemHulls, it is possible to obtain a a stronger result (see Exercise 4.36, however the methods of this section ca be extended also to non first-order context.

(4) Verify the following strong form of Keisler's theorem:

THEOREM 10.21 (Strong Keisler's two cardinal theorem). *Let L be a countable similarity type containing a unary predicate P. Suppose that M^* is an L-structure such that*

$$\|M^*\| > |P^{M^*}| \geq \aleph_0$$

If χ is regular large enough such that $\{M^*, L\} \in H(\chi)$ then for every countable $\mathfrak{B}_0 \prec \langle H(\chi), \in, P^{M^*}, M^*, \ldots\rangle$ there exists $\mathfrak{B}_1 \succ \mathfrak{B}_0$ such that $P^{\mathfrak{B}_1} = P^{\mathfrak{B}_0}$ and $M^{*\mathfrak{B}_1}$ uncountable.

(5) Corollary 12.22 implies the strong version of Keisler's two cardinal theorem.

(6)

11. Absoluteness

We need few set-theoretic facts:

DEFINITION 11.1. Σ_0 formula in $\mathrm{L(ZF)}$.

FACT 11.2. *Let $H(\omega_1)$ be the set of hereditary at most countable sets i.e. it consists of all sets whose transitive closure (with respect to the relation $\in$) is at most a countable set.*

$$ZFC \vdash \text{“}\langle H(\omega_1), \in\rangle \models ZFC - P\text{”}$$

where P is the power-set axiom

For this and other basic results see any book on set theory (e.g. section 7 of Chapter IV in Kunen's book [**Ku**]). Note that among other things ω is a definable element of $H(\omega_1)$.

REMARK 11.3 (Absoluteness of first-order statements). Suppose that L is a similarity type and $\chi > \aleph_0$ is regular such that $L \in H(\chi)$. Let T be a set of L-sentences. Consider $\mathfrak{B} := \langle H(\chi), \in, T, L, \mathrm{Fml}(L), c_\varphi\rangle_{\varphi(\mathbf{x})\in \mathrm{Fml}(L)}$ be such that T and L are interpretations of unary predicates and c_φ is a constant symbol interpreted by φ.

(1) If $\mathfrak{B}' \equiv \mathfrak{B}$ such that $T^{\mathfrak{B}'} = T$ then T is consistent iff $\mathfrak{B}' \models T$ is consistent.
(2) If $\mathfrak{B}' \equiv \mathfrak{B}$ such that $T^{\mathfrak{B}'} = T$ then T is complete iff $\mathfrak{B}' \models T$ is complete.
(3) Suppose p is a set of L-formulas in a finite sequence of variables $\mathbf{x}$. If $\mathfrak{B}' \equiv \langle \mathfrak{B}, p\rangle$ where p is a unary predicate interpreted by the set of formulas in p and p is the same element of both structures (this holds provided $\mathrm{Fml}(L)^{\mathfrak{B}} = \mathrm{Fml}(L)^{\mathfrak{B}'}$ then p is a T-type iff $\mathfrak{B}' \models p$ is a T-type.
(4) Moreover, p is a complete T-type iff $\mathfrak{B}' \models p$ is a T-complete type.
(5) Let $p(\mathbf{x})$ be a set of L-formulas, $M \models T$, $\lambda := |L| + \aleph_0$ and fix $f : \lambda \to p$ be a surjection. Suppose that M omits the type p. Consider $\langle \mathfrak{B}, \lambda, M, f\rangle$ to be an expansion of $\mathfrak{B}$ by a unary predicate and function symbol.
(6) Let $\lambda := |L| + \aleph_0$ and let q be a T-type fix injective functions g_F, g_T and g_q from λ into $\mathrm{Fml}(L)$, T and q. Suppose M is an L-structure. Consider $\mathfrak{B} := \langle H(\chi), \in, \lambda, M, g_F, g_T, g_q\rangle$ if $\mathfrak{B}' \equiv \langle \mathfrak{B}, p\rangle$ such that $\lambda^{\mathfrak{B}'} = \lambda^{\mathfrak{B}}$ then $T^{\mathfrak{B}'} = T$, $\mathrm{Fml}^{\mathfrak{B}'} = \mathrm{Fml}(L)$ and $q^{\mathfrak{B}'} = q$. Also we have

$$M^{\mathfrak{B}'} \in \mathrm{EC}(T, q) \iff M \in \mathrm{EC}(T, q).$$

FACT 11.4 (Levy-Shoenfield absoluteness lemma). *Let P be a poset and let V^P be the generic extension of V. Let σ be a sentence of the form $\exists x \rho(x)$ where $\rho(x)$ is a Σ_0-formula in the language of ZF.*

$$\textit{If } \langle H(\omega_1), \in\rangle^{V^P} \models \sigma \textit{ then } \langle H(\omega_1), \in\rangle^{V} \models \sigma.$$

Reference: Exercise 14.18 in Jech's book [**Je78**].

Using ideas of Gödel involving numbering any finite signature τ can be identified with an element of $H(\omega)$ as $H(\omega)$ is closed under finitary set-theoretic operations including exponentiation and induction on elements of ω all the first-order formulas over τ are in $H(\omega)$. Similarly we can make observations about $L_{\omega_1,\omega}(\tau)$ and $H(\omega_1)$ where τ is an at most countable signature (it can be identified with an element of $H(\omega_1)$).

FACT 11.5. *If τ is an at most countable signature and $\varphi(\mathbf{x};\mathbf{y})$ is an $L_{\omega_1,\omega}(\tau)$ formula then $\varphi \in H(\omega_1)$.*

FACT 11.6. *If τ_1 as an at most countable signature, T a first-order theory in τ_1 and Γ is a set of T-types such that $|\Gamma| \leq \aleph_0$ then $\{T, \Gamma\} \subseteq H(\omega_1)$.*

The formula of $\rho(M, T, \Gamma)$ in the language of ZF stating that

$$M \models T \text{ and } M \text{ omits every } p \in \Gamma$$

is Σ_0.

FACT 11.7. *The following sentence (in the language of ZF):*

$$\exists M \models T \exists \{\mathbf{a}_n \mid n < \omega\} \subseteq |M| \forall p \in \Gamma [M \text{ omits } p] \wedge [\{\mathbf{a}_n\} \text{ is an indic seq}]$$

Is Σ_1 over $H(\omega_1)$ with the parameters T and Γ

A related fact that seems to be quite interesting is

FACT 11.8. *If ψ is an $L_{\omega_1,\omega}(\mathbf{Q})(\tau)$-sentence where τ is as above and $\mathbf{Q}$ stands for $\exists^{\geq \aleph_1}$ then*

$$\exists M \models \psi$$

is also a Σ_1-sentence

12. Two cardinal theorems, by Vaught, Chang, Keisler and Morley

In this section we introduce the notion of *a theory admitting a two cardinal model*, this property introduced in order to consider a generalization of the downward and upward Löwenhiem Skolem theorems. The main results of this section are Vaught's and Keisler's two cardinal theorems which are generalizations of the downward Löwenhiem Skolem theorem for countable theories.

It turns out that there are fundamental and interesting connections between existence of certain two-cardinal models and omitting types. In this section we will focus on models of cardinality $\aleph_0$ and $\aleph_1$. In sections 7 and 12 of Chapter 6 we will exploit similar connections for larger models.

For many years the value and importance of Keisler's two cardinal theorem was overlooked. We will show later that this theorem (or its equivalent Corollary 12.22) are the key for certain constructions of models in $\aleph_1$ and in some cases also in $\aleph_2$ for countable $L_{\omega_1,\omega}$-theories.

DEFINITION 12.1. Let T be a first-order theory with a unary predicate P in $\mathrm{L}(T)$.

(1) For cardinals $\lambda \geq \mu$ we say that T *admits a model of type* $\langle \lambda, \mu \rangle$ iff there exists a model $M = \langle |M|, P^M, \ldots \rangle \models T$ such that $\|M\| = \lambda$ and $|P^M| = \mu$ and we say that the model M is of *type* $\langle \lambda, \mu \rangle$.
(2) For cardinals $\lambda_\ell \geq \mu_\ell$ (for $\ell = 1, 2$) we say that $\langle \lambda_1, \mu_1 \rangle$ *implies* $\langle \lambda_2, \mu_2 \rangle$ iff the existence of a model (for T) of type $\langle \lambda_1, \mu_1 \rangle$ implies that T has a model of type $\langle \lambda_2, \mu_2 \rangle$.
    In symbols we write $\langle \lambda_1, \mu_1 \rangle \to \langle \lambda_2, \mu_2 \rangle$.
(3) T is said *have a two-cardinal model* iff there are $\lambda > \mu$ such that T admits a $\langle \lambda, \mu \rangle$-model.

The following is are easy consequence of the DLS and the compactness theorems:

PROPOSITION 12.2 (Easy implications).

(1) *For cardinals* $\lambda \geq \mu \geq \aleph_0$ *we have* $\langle \lambda, \mu \rangle \to \langle \chi, \chi \rangle$ *for every* $\chi \geq |T|$, *where* $|T|$ *is* $|\mathrm{L}(T)| + \aleph_0$.
(2) $\langle \lambda_1, \mu_1 \rangle \to \langle \lambda_2, \mu_1 \rangle$ *holds for every* λ_2 *satisfying* $\mu_1 \leq \lambda_2 \leq \lambda_1$.
(3) *if* T *has a two-cardinal model then* T *admits gap*-1.

PROOF. The statements are left as an exercise. □

Notice that by Proposition 12.2

DEFINITION 12.3.

(1) Let T be a first-order theory with a unary predicate P in $\mathrm{L}(T)$. The theory T *admits a gap*-1 iff there exists $\lambda \geq |T|$ such that T admits a model of type $\langle \lambda^+, \lambda \rangle$.
(2) For $n < \omega$ and a cardinal number λ let

$$\lambda^{+n+1} := \begin{cases} (\lambda^n)^+ & \text{if } n \geq 1 \\ \\ \lambda & \text{if } n = 0. \end{cases}$$

The theory T *admits a gap-*n iff there exists $\lambda \geq |T|$ such that T admits a model of type $\langle \lambda^{+n}, \lambda \rangle$.
(3) The theory T *admits a gap-*$\beth_\omega$ iff there exists $\lambda \geq |T|$ such that T admits a model of type $\langle \beth_\omega(\lambda), \lambda \rangle$.
(4) The theory T *admits a gap-*$\aleph_\omega$ iff there exists $\lambda \geq |T|$ such that T admits a model of type $\langle \aleph_\omega(\lambda), \lambda \rangle$.

The following examples describe some easy facts and eventually help us to focus on the main theorems.

EXAMPLES 12.4. (1) let $L := \{P\}$ be a single unary predicate. The theory of $\langle \mathbb{Q}, \mathbb{Z} \rangle$ is an L-theory that admits a $\langle \lambda, \mu \rangle$-model for every $\lambda \geq \mu \geq \aleph_0$.

(2) let : $L = \{P, F\}$, with P unary predicate and F a unary function symbol. Suppose $f : \mathbb{Q} \to \mathbb{Z}$ is a bijection, show that $\mathrm{Th}(\langle \mathbb{Q}, \mathbb{Z}, f \rangle)$ does not have a two cardinal model.

(3) Take $T := \mathrm{Th}(\langle \mathcal{P}(\omega), \omega, \in \rangle)$ when the set ω is interpreted by a unary predicate. Show that for every $\mu \geq \aleph_0$ T has a $\langle 2^\mu, \mu \rangle$-model and observe that if T has a $\langle \lambda, \mu \rangle$-model then $\lambda \leq 2^\mu$.

Here is another example of a theory T_1 that has a $\langle 2^\mu, \mu \rangle$-model, but if $M \models T_1$ then $\|M_1\| \leq |P^M|$: Let $\mathrm{L}(T_1)$ consists of a unary predicate P and a binary predicate E. The idea is that we consider sets of elements of P where E is the membership relation. More formally T_1 consists of the following groups of axioms:

- there are infinitely many elements satisfying P and there are infinitely many elements satisfying $\neg P$,
- $\forall x \forall y [xEy \to P(x) \wedge \neg P(y)]$,
- (Extensionality)

$$\forall x \forall y [\neg P(x) \wedge \neg P(y) \to [x = y \leftrightarrow \forall z [P(z) \to [zEx \leftrightarrow zEy]]].$$

(4) For every $\alpha \in \omega_1 - \omega$ fix a bijection $f_\alpha : \alpha \to \omega$. Let

$$g(\alpha, \beta) := \begin{cases} f_\alpha(\beta) & \text{if } \beta < \alpha \\ \\ 0 & \text{if } \beta \geq \alpha. \end{cases}$$

Consider $M := \langle \omega_1, \omega, g \rangle$ for the language $\{P, G\}$, when P^M is ω_1 and g is G^M. Show that $\mathrm{Th}(M)$ admits a $\langle \mu^+, \mu \rangle$-model for every $\mu \geq \aleph_0$ and if the theory admits a $\langle \lambda, \mu \rangle$-model then $\lambda \leq \mu^+$.

An iteration of the second example from 3 gives

PROPOSITION 12.5 (Gap $\beth_n$). *For every $n < \omega$ there exists a finite set of axioms T_n in a finite language such that if T_n has a $\langle \beth_n(\mu), \mu \rangle$-model then T_n does not admit a $\langle \beth_n(\lambda)^+, \lambda \rangle$-model for any infinite λ.*

This leads to the following question: Suppose T is a countable first-order theory. Suppose for every $n < \omega$ there exists $M_n \models T$ such that $\|M_n\| \geq \beth_n(|P^{M_n}|)$ and P^{M_n} is infinite does this imply that T admits a $\lambda, \mu\rangle$ model for every $\lambda > \mu \geq \aleph_0$. It turns out that the answer to the question is affirmative and this is known as Vaught's gap-$\beth_\omega$ presented in the next section together with a powerful technique to construct models.

An iteration of Example 4 gives

PROPOSITION 12.6 (Gap n). *For every $n < \omega$ there exists a finite set of axioms T_n in a finite language such that if T_n has a $\langle \mu^{+n}, \mu \rangle$-model then T_n does not admit a $\langle \lambda^{n+1}, \lambda \rangle$-model for any infinite λ.*

While the focus of this section are first-order theories, for future reference I want to mention two facts concerning the model theory for $L_{\omega_1,\omega}$ that can be proved by extending the previous examples and using Example 1.8.21.

PROPOSITION 12.7. *For every $\alpha < \omega_1$ there is a countable similarity type L and a sentence $\psi_\alpha \in L_{\omega_1,\omega}$ such that ψ_α has a model of cardinality $\beth_\alpha$ but does not have a model of greater cardinality.*

PROPOSITION 12.8. *For every $\alpha < \omega_1$ there is a countable similarity type L and a sentence $\psi_\alpha \in L_{\omega_1,\omega}$ such that ψ_α has a $\langle \aleph_\alpha, \aleph_0 \rangle$-model but not a $\langle \aleph_{\alpha+1}, \aleph_0 \rangle$-model.*

The first non-trivial theorem of this section is the following celebrated 1961 theorem of Vaught. Vaught's original proof (presented in 4.2) made use of countable homogenous models (as opposed to countable saturated models), below we derive it using a later and different method due to Keisler from a stronger theorem.

THEOREM 12.9 (Vaught's two cardinal theorem). *Suppose T is a first-order theory in a countable language. If T admits $\langle \lambda, \mu \rangle$ for some $\lambda > \mu \geq \aleph_0$ then T admits $\langle \aleph_1, \aleph_0 \rangle$.*

Thus Vaught's theorem can be restated as: If a countable T admits a gap-1 model than it admits a $\langle \aleph_1, \aleph_0 \rangle$-model.

A simple argument can give a generalization of Vaught's two cardinal theorem (and all other two cardinal theorems).

THEOREM 12.10 (Generalized Vaught's two cardinal theorem). *Suppose T is a first order theory in a countable language and $\varphi(\mathbf{x})$ is a formula in the language of T. If there exists $M \models T$ such that $\|M\| > |\varphi(M)| \geq \aleph_0$ then there exists $N \equiv M$ with $\|N\| = \aleph_1$ and $|\varphi(N)| = \aleph_0$.*

PROOF. Let P be a new unary predicate and F a new unary function symbol. Pick a subset A of M of cardinality $|\varphi(M)|$ and a bijection f from $\varphi(M)$ onto A. Let M^* be the expansion of M to $L^* := \mathrm{L}(T) \cup \{P, F\}$ where F is interpreted by f and P by A.

Using Theorem 12.9 there is $N^* \equiv M^*$ of cardinality $\aleph_1$ with P^{N^*} countable. Check that $N := N^* \restriction \mathrm{L}(T)$ is as required.

□

Shortly after the proof of Vaught's two cardinal theorem, C. C. Chang Conjectured that the following strengthening is true: For every $\langle \lambda^+, \lambda \rangle$-structure M (in a countable language) there exists $N \prec M$ of cardinality $\aleph_1$ such that the predicate is countable. This is known by the name of *Chang's Conjecture*. According to a legend for several years Chang suggested this conjecture as a thesis project for some of his graduate students (eventually it was proved to be independent, in fact now it is a viewed as a large cardinal axiom)[21].

In 1966 Keisler published a strengthening of Vaught's theorem that brings us closer (in ZFC) to Chang's Conjecture.

Since Vaught's theorem follows instantly from Keisler's stronger result and its proof uses facts about homogenous models we postpone Vaught's original proof to after Corollary 4.2.20.

THEOREM 12.11 (Keisler's two cardinal theorem). *Let L be a countable similarity type containing a unary predicate P. If there is an L-structure M^* such that*

$$\|M^*\| > |P^{M^*}| \geq \aleph_0$$

then for every countable $A \subseteq |M^*|$ there are $M_0 \preceq M_{\omega_1}$ such that

$M_0 \preceq M^*$, $|M_0| \supseteq A$, *is countable and M_{ω_1} uncountable such that $P^{M_0} = P^{M_{\omega_1}}$.*

Instead of proving the two cardinal theorem directly I will present a more abstract result that imply it as well as several important results in the context of infinitary logics, that previously were considered to be unrelated theorems but really follow quite simply from what we will do here.

[21] It turns out that under stability-like assumptions, not only Chang's conjecture is true but even stronger statements, see **??**.

DEFINITION 12.12. Suppose that L is a similarity type containing a unary predicate P and a binary relation symbol $<$.

For a $\varphi(x, y; \mathbf{z}) \in \mathrm{Fml}(L)$ the *Keisler's sentence for* φ is the following sentence

$$\psi_\varphi := \forall \mathbf{z}\Big[\forall x \exists x' \exists y[x' > x \wedge [P(y) \wedge \varphi(x', y; \mathbf{z})]] \rightarrow \\ \exists y \forall x \exists x'[x' > x \wedge [P(y) \wedge \varphi(x', y; \mathbf{z})]]\Big]$$

Let

$$T_{Keisler} := \{\psi_\varphi \mid \varphi(x, y; \mathbf{z}) \in \mathrm{Fml}(L)\} \cup \\ \{< \text{ "is a linear order without a last element on the universe"}\}$$

Notice that when M is an infinite structure and $<$ is a binary relation in $\mathrm{L}(M)$ such that

$$M \models \forall x \forall y \forall z[x < y \wedge y < z \rightarrow x < z] \wedge \\ \forall x \forall y[x < y \vee x = y \vee y < x] \wedge \forall x \exists y[x < y].$$

I.e. $<$ is a linear order of the universe of M without endpoints then for any formula $\varphi(x)$ we have

$$M \models \forall x \exists x'[x' > x \wedge \varphi(x)] \iff \{a \in |M| \ : \ M \models \varphi[a]\} \text{ is unbounded.}$$

The intuitive meaning of Keisler's sentence is that we have a first-order way to formalize that a universe of a structure is of regular cardinality and is greater than the cardinality of P. More precisely we have:

CLAIM 12.13 (Alternation of Quantifiers). *Suppose M is an L-structure such that L is containing a unary predicate P and a binary relation symbol $<$. Further we assume that $<^M$ linearly orders $|M|$ without a last element.*

If $\|M\| = \lambda$ is regular and $\lambda > |P^M|$ then $M \models T_{Keisler}$.

PROOF. Given $\varphi(x, y; \mathbf{z})$ and $\mathbf{a} \in {}^{\ell(\mathbf{z})}|M|$. Suppose that

$$M \models \forall x \exists x' \exists y\Big[x' > x \wedge [P(y) \wedge \varphi(x', y; \mathbf{a})]\Big]$$

consider $G_{\varphi,\mathbf{a}} : |M| \rightarrow P^M$ chosen so that for every $c \in |M|$

$$M \models \Big[\exists x'[x' > c \wedge [P(G_{\varphi,\mathbf{a}}(c)) \wedge \varphi(x', G_{\varphi,\mathbf{a}}(c); \mathbf{a})]]\Big].$$

Using the pigeon hole principle there are $S \subseteq \lambda$ of cardinality λ and $b \in P^M$ such that $G_{\varphi,\mathbf{a}}(c) = b$ for all $c \in S$. Thus we found an element b such that

$$M \models \forall x \exists x'[x' > x \rightarrow [P(b) \wedge \varphi(x', b; \mathbf{a})]].$$

□

Our theorem that will be used to derive Keisler's two cardinal theorem and many other results is the following.

THEOREM 12.14 (Extension of countable structures). *Suppose M is a countable L-structure such that L is at most countable, containing a unary predicate P and a binary relation symbol $<$.*

If $M \models T_{Keisler}$ then there exists a countable $N \supsetneqq M$ such that $P^N = P^M$.

PROOF. Consider $\mathrm{CD}(M)$ and let c be a new constant. Let

$$T := \mathrm{CD}(M) \cup \{a < c \mid a \in |M|\}.$$

As M is infinite, the set T is consistent and if $N' \models T$ then $N' \restriction \mathrm{L}(M)$ is isomorphic to some $N \supsetneqq M$. Let

$$q(y) := \{P(y)\} \cup \{y \neq a \mid a \in |M|\}.$$

Notice that if $N \in \mathrm{EC}(T, q)$ and $M \subseteq N$ then $P^N = P^M$.

Using Henkin's omitting-types theorem it is sufficient to show that q is not isolated.

CLAIM 12.15 (Characterization of consistent formulas). *$T \cup \{\chi(c)\}$ is consistent iff $M \models \forall x \exists x'[x' > x \wedge \chi(x')]$ for all $\chi(y) \in \mathrm{L}(\mathrm{CD}(M))$.*

PROOF. Clear. □

Now we show that q is not isolated. Suppose $\chi(y, c)$ is such that

$$(*) \quad T \cup \{\exists y[\chi(y, c)]\} \text{ is consistent,}$$

we need to find $\rho(y) \in q$ such that

$$T \cup \{\exists y[\chi(y, c) \wedge \neg\rho(y)]\} \text{ is consistent.}$$

By $(*)$ there two possibilities: Either

$$\text{Case 1: } T \cup \{\exists y[P(y) \wedge \chi(y, c)]\} \text{ is consistent}$$

or

$$\text{Case 2: } T \cup \{\exists y[\neg P(y) \wedge \chi(y, c)]\} \text{ is consistent.}$$

If it is Case 2 then we found $\rho \in p$ as required (take $P(y)$). So suppose it is Case 1. By the characterization of consistent formulas and Case 1, we get that

$$M \models \forall x \exists x' \exists y[x' > x \wedge P(x') \wedge \chi(y, x')].$$

As $M \models \psi_\varphi$ where, $\varphi(x, y, \mathbf{z})$ is $\chi(x, y)$ we get that there exists $a \in P^M$ such that

$$M \models \forall x \exists x'[x' > x \wedge \chi(a, x')].$$

Namely

$$M \models \forall x \exists x'[x' > x \wedge \exists y[\chi(y, x') \wedge \neg(y \neq a)]].$$

Now by the Characterization of consistent formulas from the previous statement we get that $T \cup \{\exists y[\chi(y,c) \wedge \neg\rho(y)]\}$ is consistent when $\rho(y)$ is the formula $y \neq a$. □

PROOF OF KEISLER'S TWO CARDINAL THEOREM. Using DLST we may assume without loss of generality that $\|M^*\| = \mu^+$ where $\mu := |P^{M^*}|$. Pick a well-ordering $<$ of $|M^*|$ such that $\langle P^{M^*}, <\rangle \cong \mu$ and $\langle |M^*|, <\rangle \cong \mu^+$. Let M be the expansion of M^* to $\mathrm{L}(M^*) \cup \{<\}$ I.e. $M := \langle M^*, <\rangle$. By Claim 12.13 $M \models T_{Keisler}$.

Let A be a given countable subset of $|M^*|$. Pick $M_0 \prec M$ be countable such that $|M_0| \supseteq A$. By induction on $\alpha < \omega_1$ an increasing and continuous elementary chain $\{M_\alpha \mid \alpha < \omega_1\}$ of countable models such that

$$\alpha < \beta \implies M_\alpha \precneqq M_\beta \text{ and } P^{M_\alpha} = P^{M_\beta}.$$

As M_0 was already chosen, by continuity it is enough to define the chain at the successor stage. Suppose M_α is given. As $M_\alpha \models T_{Keisler}$ by The extension theorem (Theorem 12.14) there exists $M_{\alpha+1} \succneqq M_\alpha$ countable such that $P^{M_{\alpha+1}} = P^{M_\alpha}$.

Clearly $M_{\omega_1} := \bigcup_{\alpha<\omega_1} M_\alpha \restriction \mathrm{L}(M^*)$ is as required. □

In section 6.12 I present a couple of applications of Keisler's theorem to obtain "easy proofs" of facts relevant to Morley's categoricity theorem.

Observe that the argument we presented for Keisler's two-cardinal theorem gives:

COROLLARY 12.16. *Let L be at most countable containing a unary predicate P and μ an infinite cardinal. Suppose M is an L-structure such that $|M\| > \mu \geq |P^M|$ and $<$ is a well-order of $|M|$ such that $\langle P^M, <\rangle$ has order-type $\leq \mu$. Consider the following expansion of M:*

$$\mathfrak{B}^* := \langle M, <\rangle.$$

If $\mathfrak{B}_0 \prec \mathfrak{B}^*$ is countable then there exists $\mathfrak{B}_{\omega_1} \succ \mathfrak{B}_0$ of cardinality $\aleph_1$ such that $P^{M_{\omega_1}} = P^{M_0}$, where $M_\alpha := \mathfrak{B}_\alpha \restriction L$.

It is reasonable to ask, is the expansion by $<$ necessary? Namely, could we amalgamate the statements of Keisler's theorem with the corollary: If M is a two-cardinal model for a countable language then every countable elementary submodel of M has an uncountable elementary extension with the same predicate? It turns out that this is impossible. An example is presented in [**CK**] pages 158-160.

Keisler's two-cardinal theorem can be extended also t o $L_{\omega_1,\omega}$.

THEOREM 12.17. *Let M be uncountable in a countable language that contains a unary predicate P such that $\|M\| > |P^M| \geq \aleph_0$, suppose that $M \models \psi$*

for some $\psi \in L_{\omega_1,\omega}$. For any countable $A \subseteq |M|$ there are $N_0 M$ countable countable containing A and an uncountable $N_1 \succ N_0$ such that $P^{N_0} = P^{N_1}$.

Moreover, if T_1 is a countable first-order theory, Γ a countable set of T_1-types and $L \subseteq \mathrm{L}(T_1)$ containing a unary predicate P then if there exists $M \in \mathrm{PC}(T_1, \Gamma, L)$ such that $\|M\| > |P^M| \geq \aleph_0$ then there exists a countable $T_2 \supseteq T_1$

PROOF. By Theorem 8.61 we may assume that $M \models T_2$ omitting a type q. Suppose $\{\varphi_n(\mathbf{x}) \mid n < \omega\} = q$.

Let $<$ be a linear order of $|M|$ without a last element, let $\{c_n \mid n < \omega\}$ be new constants, $\{S\}$ a binary predicate (that will code satisfaction). Pick $A \subseteq |M|$ countable and replace M by its expansion to the new symbols as follows:

$$\{c_n^M \mid n < \omega\} = A,\ P^M := A \text{ and } S \subseteq A \times {}^{\omega >}|M| \text{ such that}$$
$$(*) \quad M \models \forall \mathbf{x}[S(c_n, \mathbf{a}) \leftrightarrow \varphi_n(\mathbf{x})] \text{ for all } n < \omega.$$

By our definitions we have also

$$(**) \quad M \models \{P(c_n) \mid n < \omega\}.$$

As M omits q we have also

$$(***) \quad M \models \neg \exists \mathbf{x} \forall y [P(y) \to S(y, \mathbf{x})].$$

By the definition of P^M we have that M omits the following type

$$p(y) := \{P(y) \wedge y \neq c_n \mid n < \omega\}.$$

By Claim 12.13 since $\|M\| > |P^M| = \aleph_0$ we have that $M \models T_{Keisler}$.

If $N_0 \prec_{L^*} M$ is countable omitting p by $(**)$ we have that $P^{N_0} = \{c_n^M \mid n < \omega\}$. Now by Theorem 12.14 iterated ω_1-many times there exists $N_1 \succ N_0$ of cardinality $\aleph_1$ such that $P^{N_1} = P^{N_0}$.

Enough to establish that N_1 omits q. Note that since $P^{N_1} = P^{N_0}$ the type p is omitted by M. By $(***)$ if $\mathbf{a} \in |N_1|$ then there exists $b \in P^{N_1}$ such that $M \models \neg S(b, \mathbf{a})$. As $P^{N_1} = \{c_n^{N_1} \mid n < \omega\}$ there exists $n < \omega$ such that $M \models b = c_n$. Thus by $(*)$ we get $M \models \neg \varphi_n[\mathbf{a}]$. □

COROLLARY 12.18 (Vaught's two-cadinal theorem for $L_{\omega_1,\omega}$). *Let ψ be an $L_{\omega_1,\omega}$-sentence in a countable similarity type that contains a unary predicate P. If there exists $M \models \psi$ such that $\|M\| > |P^M| \geq \aleph_0$ then there exists $N \models \psi$ uncountable with P^N countable.*

It turns out that the method of proof of Keisler's two-cardinal theorem can be used to prove a few stronger (and useful) theorems.

THEOREM 12.19. *Let T be first-order in a countable language and q a T-type. There exists $T_1 \supseteq T$ in $\mathrm{L}(T) \cup \{P, S, <\} \cup \{c_n \mid n < \omega\}$ such that when*

$$p(x) := \{P(x) \wedge x \neq c_n \mid n < \omega\},$$

the following are equivalent

(1) *There exists an uncountable member of* $\mathrm{EC}(T, q)$.
(2) *There exists a countable member of* $\mathrm{EC}(T_1, p)$.

PROOF. If q is finite then assertion is trivially true, so we may assume that q is infinite. Fix $\{\varphi_n(x) \mid n < \omega\} = q$.

For every $\varphi(x) \in \mathrm{Fml}(\mathrm{L}(T))$ let

$$\psi_\varphi(y) := \forall x[R(y, x) \leftrightarrow \varphi(x)].$$

For $n < \omega$ denote by χ_n the following sentence

$$\forall y_0 \cdots \forall y_{n-1}[\bigwedge_{i<n} P(y_i) \rightarrow \exists x[\bigwedge_{i<n} R(y_i, x)]].$$

****

$$\begin{aligned} T_1 :=& T \cup T_{Keisler} \cup \{\chi_n \mid n < \omega\} \cup \\ & \{\neg\exists x \forall y[P(y) \rightarrow R(y, x)]\} \cup \\ & \{\psi_{\varphi_n}(c_n) \mid n < \omega\} \cup \\ & \{P(c_n) \mid n < \omega\}. \end{aligned}$$

Now observe that If $N \in \mathrm{EC}(T_1, p)$ then we have the following:

$$\begin{aligned} & N \models T \cup T_{Keisler}, \\ & P^N = \{c_n^N \mid n < \omega\}, \\ & q \text{ is an } N\text{-type and} \\ & q \text{ is omitted by } N. \end{aligned}$$

Clearly any uncountable element of $\mathrm{EC}(T, q)$ has an expansion to a member of $\mathrm{EC}(T_1, p)$.

Suppose $M \in \mathrm{EC}(T_1, p)$ is countable. As $M \models T_{Keisler}$ by iterating Theorem 12.14 ω_1-many times we get $N \succ M$ of cardinality $\aleph_1$ such that $P^M = P^N$, thus N omits p.

We are done by the previous observation. □

Notice that if T is a first-order theory in a countable language, then due to the compactness theorem, existence of uncountable models for T is an absolute property (i.e. does not depend on the model of ZFC we work in). On the other hand it is far from being the case that "ψ has an uncountable model" is absolute for $\psi \in L_{\omega_1,\omega}$.

COROLLARY 12.20 (Existence of uncountable models is absolute). *Let L be at most countable and ψ an $L_{\omega_1,\omega}$-sentence. The statement "ψ has an uncountable model" is absolute.*

PROOF. By Chang's presentation theorem and Shelah's 1-type coding, there are a countable similarity type L, a first-order theory T in L and a T-type q such that

$$\mathrm{Mod}(\psi) = \mathrm{PC}(T, q, L).$$

Let T_1 and p be as in Theorem 12.19.

Clearly if $M \models \psi$ is uncountable then it has an expansion to an element of $\mathrm{EC}(T_1, p)$.

As T_1 is a consistent theory in a countable language and p is a non-isolated T-type, existence of a countable model of T_1 omitting p is absolute. □

Using a generalization of Theorem 12.19 we can obtain a much stronger version of the last Corollary.

COROLLARY 12.21 (Existence of uncountable models for $L_{\omega_1,\omega}(\mathbf{Q})$ is absolute). *Let L be at most countable and ψ an $L_{\omega_1,\omega}(\mathbf{Q})$-sentence. The statement "ψ has an uncountable model" is absolute.*

PROOF. Let F_φ and G_ψ be new function symbols such that ******

□

An immediate corollary is the following two-cardinal version of Skolem's paradox:

COROLLARY 12.22 (Keisler and Morley 1966). *For every countable $M \models$* ZF *there exists a $N \succneqq M$ of cardinality $\aleph_1$ such that $\omega^N = \omega^M$.*

PROOF. Define an elementary chain $\{N_\alpha \mid \alpha < \omega_1\}$ such that

(1) $N_0 = M$,
(2) N_α is a countable model,
(3) $N_\alpha \precneqq N_{\alpha+1}$ and $\omega^{N_\alpha} = \omega^{N_\alpha+1}$ for all $\alpha < \omega_1$,
(4) for α limit $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$.

At successor stage of the construction use Theorem 9.19.

Check that $N := \bigcup_{\alpha<\omega_1} N_\alpha$ is as required. □

DEFINITION 12.23. Let T be a theory and $\varphi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T))$ be given. We say that M, N *is Vaughtian pair for* φ iff

$$M \precneqq N \models T \quad \text{and} \quad \varphi(M) = \varphi(N).$$

We say that T *admits a Vaughtian pair* iff there are infinite models M, N and a formula φ as above.

PROPOSITION 12.24. *If M, N is Vaughtian pair of T for φ then for every $\lambda \geq |\mathrm{L}(T)|$ the theory T has a Vaughtian pair M_λ, N_λ both of cardinality λ.*

PROOF. Exercise. □

The following is a converse of Vaught's theorem (it is a two-cardinal version of an "upward Löwenhiem Skolem"):

THEOREM 12.25 (Chang's two cardinal theorem). *Suppose T is a first-order theory in a countable language and $\lambda > |T|$ is satisfying $\lambda = \lambda^{<\lambda}$. If T admits a gap-1 model then T admits $\langle \lambda^+, \lambda \rangle$.*

Chang's trick.

As the assumption $\lambda = \lambda^{<\lambda} > |T|$ implies that λ must be a regular cardinal for several years it was unknown whether the statement of Chang's theorem holds also for singular λ. This was settled after the introduction of a new set-theoretic principle:

THEOREM 12.26 (Jensen 1972). *Suppose T is a first-order theory in a countable language and $\lambda > |T|$ is singular satisfying $\square_\lambda$ and there exists an unbounded family of cardinals $S \subseteq \lambda$ such that $\mu \in S \implies 2^\mu = \mu^+$. If T admits a gap-1 model then T admits $\langle \lambda^+, \lambda \rangle$.*

As the proofs of the last two theorems involves using saturated and special models we postpone this to a later chapter. See section 4.2, immediately after the proof of Corollary 4.2.20.

Exercises 1.12.

(1) For each of the theories from Example 12.4 write down a finite axiomatization.
(2) Show that $\mathrm{Th}(\langle \mathcal{P}(\mathcal{P}(\omega)), \omega, \in\rangle)$ admits a $\langle 2^{2^\mu}, \mu\rangle$-model for all $\mu \geq \aleph_0$ but not a $\langle (2^{2^\mu})^+, \mu\rangle$-model.
(3) Prove Proposition 12.6.
(4) For very $n < \omega$ find a finite T_n in the similarity type $\{P, \in\}$ such that T_n admits $\langle \beth_n(\mu), \mu\rangle$-models for every $\mu \geq \aleph_0$ and if the theory admits a $\langle \lambda, \mu\rangle$-model then $\lambda \leq \beth_n(\mu)$.
(5) There is a first order theory (in a finite language) admitting a $\langle (2^{\aleph_0})^+, \aleph_1\rangle$-model.
(6) Prove Proposition 12.7.
(7) Prove Proposition 12.8.
(8) Let $M \prec N$, $\mathbf{a} \in |M|$ we say that (M, N) is *Vaughtian Pair* if there are $\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(M))$, $\mathbf{a} \in {}^{\ell(\mathbf{y})}|M|$ such that

$$M \precneqq N,\ \varphi(M; \mathbf{a}) = \varphi(N; \mathbf{a}) \text{ and is infinite.}$$

Prove that if T is in a countable language and has a Vaughtian Pair (M, N) such that $\|M\| < \|N\|$ then T has Vaghtian Pair (M_0, N_0) with $\|M_0\| = \aleph_0$ and N_0 is uncountable.
(9) Suppose a theory T has a Vaughtian Pair then for every $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ the theory has Vaughtian Pair with models of cardinality λ.
(10) Suppose that $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ and T is λ-categorical. If T has a Vaghtian Pair then there exists a Vaghtian Pair (M, N) such that $\|M\| = \lambda$ and $\|N\| = \lambda^+$.
(11) Let T be a first-order theory in a countable language which is categorical in some $\lambda > \aleph_0$. If T has a Vaghtian pair then T can't be categorical in $\aleph_1$.
(12) Verify the following strong form of Keisler's theorem:

THEOREM 12.27 (Strong Keisler's two cardinal theorem). *Let L be a countable similarity type containing a unary predicate P. Suppose that M^* is an L-structure such that*

$$\|M^*\| > |P^{M^*}| \geq \aleph_0$$

If χ is regular large enough such that $\{M^*, L\} \in H(\chi)$ then for every countable $\mathfrak{B}_0 \prec \langle H(\chi), \in, P^{M^*}, M^*, \ldots\rangle$ there exists $\mathfrak{B}_1 \succ \mathfrak{B}_0$ such that $P^{\mathfrak{B}_1} = P^{\mathfrak{B}_0}$ and $M^{*\mathfrak{B}_1}$ uncountable.

(13) Find a direct proof to Vaught's two cardinal theorem from the omitting types theorem.
(14) Use Corollary 12.22 to derive Keisler's two cardinal theorem.
(15) Prove Proposition 12.24.
(16) Let $\langle T, \leq\rangle$ be a tree, $I \subseteq T$ is called an *antichain* iff

$$\langle T, \leq\rangle \models \forall x \forall y [x \in I \wedge y \in I \wedge x \neq y \to [x \not\leq y \wedge y \not\leq x].$$

Show that every special $\aleph_1$-Aronszjan tree has an uncountable antichain.

(17) Let T be a λ^+-Aronszjan tree. Prove that T is special iff T is a union of λ-many antichains.

(18) Use Vaught's and Chang's two-cardinal theorems to conclude that for every uncountable λ such that $\lambda = \lambda^{<\lambda}$there exist a λ^+-special Aronszjan tree.

(19)

(20)

(21)

13. Model complete-theories

In this section certain concepts that play a central role in applications to algebra are presented. I suggest that even a reader who is not interested in applications will read this section due to the importance of the idea used in the proof of Theorem 13.11. Also the proof of Robinson's test (Theorem 13.19) provides us with early and relatively simple example of two dimensional chain construction that are essential in classification theory for non-elementary classes (Chapter **??**) as well as in some set-theoretic constructions e.g. Theorem 2.12.5.

The property of being model-complete was identified by Abraham Robinson and studied extensively by him and his students due to the connection between model-completeness and decidability.

DEFINITION 13.1. Let T be a first-order theory. T *is model-complete* if for every infinite $M, N \models T$ we have that $M \subseteq N \implies M \preceq N$.

Later we will see several examples of model-complete theories. At this point I would like to make the obvious observation that whenever T has elimination of quantifiers then it is model-complete (use Theorem 2.6). In particular, a theory with built-in Skolem functions is model complete.

Note that we must require in the definition of model-complete theory to be infinite (see Exercise 1).

The following concept (of algebraically prime model) was introduced in this context by Robinson as a generalization of several natural examples from algebra.

DEFINITION 13.2. Let T be a first-order theory, and suppose that $M \models T$. We say that M *is algebraically-prime* iff every $N \models T$ has a submodel N^* isomorphic to M.

Few years after Robinson's work (mainly in the fifties and early sixties) Michael Morely for his categoricity theorem (Theorem 2.2) introduced a different kind of a model he named *prime-model* (see Definition 3.5.1). Later Shelah while developing general classification theory studied several families of models that are generalizations of Morely's definition. The current usage of the term "prime model" is not the concept from Definition 13.2. For the connection between the two see the comment after Definition 7.5.1.

Instead of discussing examples now, let me state a simple connection between model completeness and complete theories.

THEOREM 13.3. *Let T be a first-order theory. Suppose T is model-complete. If T has an algebraically-prime model then T is a complete theory.*

PROOF. By Proposition 3.2 it is enough to show that any pair of models $M_l \models T$ (for $l = 1, 2$) are elementary equivalent. Suppose N is an algebraically-prime model of T Let M_1 and M_2 be given models of T. By Definition 13.2 there are $M_l' \subseteq M_l$ such that $N \cong M_l'$. Since T is model-complete we have that $M"_l \preceq M_l$. Thus we have $M_1 \equiv M_1' \equiv N \equiv M_2' \equiv M_2$. □

EXAMPLE 13.4. The standard model of Peano's arithmetic $\langle \omega, +, \cdot, 0, 1 \rangle$ is algebraically-prime model, but PA is not model complete (actually no completion of PA is model complete).

EXAMPLE 13.5. Let p be a prime number or zero, and let $T_{f,p}$ be the theory of fields of characteristic p. The prime field of characteristic p is algebraically-prime model** of $T_{f,p}$.

EXAMPLE 13.6. Let p be a prime number or zero. The algebraic closure of prime field of characteristic p is algebraically-prime model of T_{ACF_p}, it will be shown below that T_{ACF_p} is model complete.

EXAMPLE 13.7. $\langle \mathbb{Q}, < \rangle$ is a prime model of T_{ord} .

EXAMPLE 13.8. Since PA is incomplete (by Gödel's incompleteness theorem), it is a corollary to Theorem 9.2 and example (1) that PA is not model complete.

EXAMPLE 13.9. Any countable model of T_{ind} is algebraically prime.

It turns out that the theories T_{ord}, T_{ind}, and T_{ACF_p} all have elimination of quantifiers which together with the last result yields alternative proofs to the fact that they are complete, thus decidable (compare with Corollary 3.28).

The following definition is a localization (via cardinality) of the property of being model-complete and it will provide us a handy tool to show that certain theories are model-complete.

DEFINITION 13.10. Let T be a first-order theory, and let λ be a cardinal number greater or equal to $|T| + \aleph_0$. T is *λ-model-complete* if for every $M, N \models T$ of cardinality λ we have that $N \subseteq M \Longrightarrow N \preceq M$.

The next Theorem is the our first example for a "transfer theorem" which carries a property from one cardinality to another (we will see later few additional examples; Morley's theorem is the most famous example).

THEOREM 13.11. *Let T be a first-order theory. The following conditions are equivalent*

(1) *T is model complete*
(2) *T is λ-model-complete for every $\lambda \geq |\mathrm{L}(T)| + \aleph_0$.*
(3) *There exists $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ such that T is λ-model-complete.*

PROOF. The non-trivial direction is to show that $(3) \implies (1)$. Suppose $\lambda \geq |\mathrm{L}(T)|$ is infinite and is such that T is λ-model complete. For the sake of contradiction assume that $M, N \models T$ and $N \subseteq M$, but N is not an elementary submodel of M. By the definition of being an elementary submodel (using the TV-property) there are a formula $\exists x \varphi(x; \mathbf{y})$ in the language of T, and a sequence $\mathbf{a} \in N$ such that
$M \models \exists x \varphi(x; \mathbf{a})$ but for no $b \in N$ we have that $M \models \varphi[b; \mathbf{a}]$.
Let $L' := \mathrm{L}(T) \cup \{P, \mathbf{c}\}$ when P is a new unary predicate, and $\mathbf{c}$ are $l(\mathbf{a})$ many new constants. Let M' be the structure $\langle M, |N|, \mathbf{a} \rangle$ - it is the expansion of M to L' when $P^{M'} = |N|$, and $\mathbf{c}^{M'} = \mathbf{a}$. Let $T' := \mathrm{Th}(M')$.

From the choice of φ and $\mathbf{a}$ it is clear that

$$(*) \qquad \exists x \varphi(x; \mathbf{a}) \wedge \forall y [P(y) \to \neg \varphi(y; \mathbf{a})] \in T'.$$

Since $|N|$ is infinite by applying the compactness theorem and than the DLS there exists $M^* \models T'$ of cardinality λ such that also P^{M^*} has cardinality λ. Add λ many new constants $\{c_i : i < \lambda\}$ and the sentences $P(c_i)$ to T', apply the compactness theorem, and than take an elementary submodel of cardinality λ.
Notice the following fundamental fact[22]: P^{M^*} is a universe of a model of T (since $N \models T$, and the definition of T').
Since M^* satisfies the sentence from $(*)$, P^{M^*} is a universe of a submodel of $M^* \restriction L$ which is not an elementary submodel of $M^* \restriction L$ which is a contradiction to assumption (3) (since both are models of T of cardinality λ). □

In the following theorem we present two classical examples for model complete theories:

COROLLARY 13.12.

(1) *T_{ord} is model-complete.*
(2) *T_{ACF_p} is model-complete (similar argument to 1)*

PROOF. By Theorem 13.11, it is enough to show that for every pair M and N of countable models of T_{ord} if $M \subseteq N$ then $M \preceq N$. STRONG ISOMORPHISM □

Theorem 13.12(2) is also known as *Hilbert's weak Nullstelensatz theorem*.

THEOREM 13.13. *STATE A GENERALIZATION OF 9.6 (exercise)*

ROBINSON TEST (?)

THEOREM 13.14. *(Lindstrom)*

[22]See Exercise 3.45 for a generalization.

COROLLARY 13.15. *(Tarski) T_{ACF_p} is model-complete. (prove also directly using Steinitz and Th9.4 or definition of $\preceq$) T_{ord} and T_{ind} are model complete*

PROPOSITION 13.16. *If T admits elimination of quantifiers then T is model complete.*

PROOF. Follows immediately from Theorem **??** □

Later in Theorem 13.28 we will present a method due to Robinson to prove elimination of quantifiers via a partial converse to Proposition 13.16

Examples: definable well ordering (V=L), T_{ACF_p} Lindstrom's characterization, cardinal transfer of MC

DEFINITION 13.17. Let T be a first-order theory, denote by $T_\forall := \{\varphi \in \mathrm{Sent}(\mathrm{L}(T)) \ : \ T \vdash \varphi$, and φ is a universal sentence $\}$. The theory $T_\forall$ is called *model companion* of T iff $T_\forall$ is model complete.

BASIC FACTS

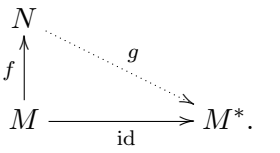
THEOREM 13.18. *Existence/Uniqueness.*

The following category-theoretic test of model completion (due to Abraham Robinson) will be used in section 1 of Chapter 4

THEOREM 13.19 (Robinson's test). *Let T be a first-order theory. T is model complete iff*

for every $M, N \models T$ and a monomorphism $f : M \to N$
there are $M^* \succ M$ and a monomorphism $g : N \to M^*$
such that the following diagram commutes:

(*)



The idea of the proof is similar to the argument in the proof of Theorem 6.4:

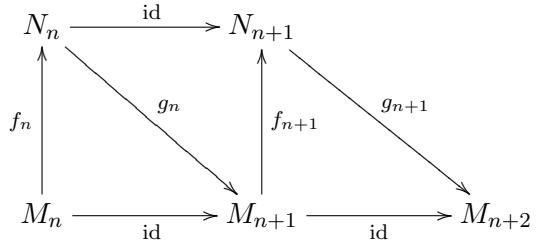
PROOF. Let $M \subseteq N$ be two models of T. Take $f_0 := \mathrm{id}_{|M|}$.

By induction on $n < \omega$ define $\{M_n, N_n, g_n, f_n \mid n < \omega\}$ such that

(1) $f_0 = f$, $g_0 = g$, $M_0 = M$, $M_1 = M^*$ and $N_0 = N$.
(2) $M_{n+1} \succ M_n$ and $N_{n+1} \succ N_n$

(3) $g_n : N_n \to M_{n+1}$ and also $f_n : M_n \to N_n$ are monomorphisms
(4) M_{n+1} is the model obtained when the assumption is applied to f_n.
(5) N_{n+1} is the model obtained when the hypothesis is applied to g_n

Thus the following diagram commutes:



for all $n < \omega$.

Now define $M^+ := \bigcup_{n<\omega} M_n$ and $N^+ := \bigcup_{n<\omega} N_n$.
Clearly $F^+; + \bigcup_{n<\omega} f_n$ is an isomorphism from M^+ onto N^+. By the construction $N^+ \succ N$ and f is an elementary embedding of M into N^+.
Since $F^+ \restriction |M| = \mathrm{id}_{|M|}$ we have that $M \preceq N^+$ and $M \subseteq N \preceq N^+$ thus by Lemma 3.10(2) we get that $M \preceq N$. □

There is another test for model completion of a theory that was discovered by Robinson, it is also called "Robinson's test". It is rather syntactic while the previous one is more "category theoretic".

THEOREM 13.20 (Robinson's test). *Let T be a first-order theory, the following are equivalent:*

(1) *T is model complete.*
(2) *For all $M, N \models T$ if $M \subseteq N$ then $M \subseteq_{TV,\mathrm{qf}} N$.*
(3) *For every $\varphi(\mathbf{x})$ there exists an existential $\psi(\mathbf{x})$ such that*
$$T \vdash \forall \mathbf{x}[\varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x})].$$
(4) *For every $\varphi(\mathbf{x})$ there exists a universal $\psi(\mathbf{x})$ such that*
$$T \vdash \forall \mathbf{x}[\varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x})].$$

PROOF. LATER. □

For certain (incomplete) theories T one can find a theory $T^* \supseteq T$ in the same similarity type such that T^* is model complete.

EXAMPLES 13.21. The following are examples to the above:

(1) $T = T_f$ and $T^* = T_{ACF}$.
(2) T the theory of undirected graphs and $T^* = T_{ind}$.
(3) $T = T_{or}$ (is the theory of linear orders) and $T^* = T_{ord}$.

Robinson have identified a canonical operation on theories that produces (whenever possible) a unique model complete extension:

DEFINITION 13.22. Let L be a similarity type. T and T^* theories in L. T^* is a *model completion* of T iff

(1) $M \models T^* \implies M \models T$,
(2) for every $M \models T$ there exists $N \supseteq M$ such that $N \models T^*$, and
(3) for all M, N_ℓ for $\ell = 1, 2$ L-structures satisfying
   (a) $M \models T$, $N_\ell \models T^*$ and
   (b) $M \subseteq N_\ell$
   we have that $\langle N_1, a\rangle_{a\in|M|} \equiv \langle N_2, a\rangle_{a\in|M|}$ holds.

It is not difficult to show that T_{ACF} is a model completion of the theory of fields (see Theorem 13.25 below).

The following observation justify the name:

PROPOSITION 13.23. *If T^* is a model completion of a theory than T^* is model complete.*

PROOF. Exercise. □

The following shows that a model completion is canonical:

THEOREM 13.24 (Uniqueness of model completion). *Let T be a theory. If T_1^* and T_2^* are both model completions of T then $T_1^* = T_2^*$.*

PROOF. It suffices to show that for every $\mathrm{L}(T)$ structure M we have

$$M \models T_1^* \iff M \models T_2^*.$$

Since the statement is symmetric it is enough to show that $M \models T_1^* \implies M \models T_2^*$ for all M.
By induction on $n < \omega$ we will define $\{M_n \mid n < \omega\}$ such that

(1) $M_0 = M$,
(2) $M_n \subseteq M_{n+1}$,
(3) $M_{2n} \models T_1^*$ and $M_{2n+1} \models T_2^*$.

This is enough: Since T_1^* is model complete the chain $\{M_{2n} \mid n < \omega\}$ is elementary. Since also T_2^* is model complete the chain $\{M_{2n+1} \mid n < \omega\}$ is elementary. Thus $N := \bigcup_{n<\omega} M_{2n}$ is an elementary extension of M, but since $N = \bigcup_{n<\omega} M_{2n+1}$ the structure N is also a model of T_2^* we get that $M \models T_2^*$.

This is possible: For $n = 0$ let $M_0 := M$.

For $n = 2k + 1$; since $M_{2k} \models T_1^*$ and T_1^* is a model completion of T we get that $M_{2k} \models T$. Using the hypothesis that T_2^* is also a model completion of T

and the second clause in the definition of model completion we get that there exists $N \supseteq M_{2k}$ which is a model of T_2^*. Let $M_n := N$.

For $n = 2k$; it is analogues to the case when n is odd. □

The following three theorems provides us with examples of model completions as well as yields alternative proofs to the fact that the theories T_{ACF}, T_{ord} and T_{ind} are model complete.

THEOREM 13.25 (Robinson). *T_{ACF} is the model completion of the theory of fields T_f.*

PROOF. Requirement (1) in the definition of model completion holds trivially.

Since every field can be extended to an algebraically closed field (see Exercise 5.35) requirement (2) holds.

We left to verify requirement (3): Let F be a field and $E_\ell \supseteq F$ algebraically closed fields. By the Upward Löwenheim-Skolem theorem there are $\bar{E}_\ell \succ E_\ell$ of cardinality λ where $\lambda := \max\{|E_1|, |E_2|\}+(|F|+\aleph_0)^+$. Let $\mathbf{I}_\ell \subseteq |\bar{E}_\ell|$ transcendence bases over F. By the choice of λ we have that $\lambda = |\mathbf{I}_1| = |\mathbf{I}_2|$. Let f be a bijection from $\mathbf{I}_1$ onto $\mathbf{I}_2$. Denote by f' the field isomorphism from $F(\mathbf{I}_1)$ onto $F(\mathbf{I}_2)$ such that $f' \supseteq f$ and $f' \restriction F = \mathrm{id}_F$. Let $\bar{f} : \bar{E}_1 \cong \bar{E}_2$ such that $\bar{f} \supseteq f'$. Since $\bar{f} \restriction F = \mathrm{id}_F$ we have (by Claim 2.17) that

$$(*) \qquad \langle \bar{E}_1, a\rangle_{a\in F} \equiv \langle \bar{E}_2, a\rangle_{a\in F}.$$

Thus we have

$$\begin{aligned} \langle E_1, a\rangle_{a\in F} &\equiv \langle \bar{E}_1, a\rangle_{a\in F} && \text{since } \bar{E}_1 \succ E_1 \\ &\equiv \langle \bar{E}_2, a\rangle_{a\in F} && \text{by } (*) \\ &\equiv \langle E_2, a\rangle_{a\in F} && \text{since } E_2 \preceq \bar{E}_2. \end{aligned}$$

□

THEOREM 13.26 (Robinson). *T_{ord} is the model completion of the theory of linear order T_{lo}.*

PROOF. Requirement (1) holds trivially.

Since every linear order can be extended to a dense linear order without endpoints also (2) holds.

If requirement (3) fails then there are $M \subseteq N_\ell$ such that $M \models T_{lo}$ and $N_\ell \models T_{ord}$ such that $\langle N_1, a\rangle_{a\in|M|} \not\equiv \langle N_2, a\rangle_{a\in|M|}$. Let $\varphi(\mathbf{x})$ be a formula in L_{lo} and $\mathbf{a} \in |M|$ a finite sequence such that

$$(*) \qquad N_1 \models \varphi[\mathbf{a}] \quad \text{and} \quad N_2 \models \neg\varphi[\mathbf{a}].$$

By the Downward Löwenheim-Skolem-Tarski there are countable $\bar{N}_\ell \preceq N_\ell$ such that $|N_\ell| \supseteq \mathbf{a}$. By a variation on Cantor's theorem (see Exercise 2.44) there exists

$$(**) \qquad f : \bar{N}_1 \cong_{\mathbf{a}} \bar{N}_2.$$

Now Claim 2.17 applied to $(**)$ contradicts $(*)$. □

THEOREM 13.27 (Shelah). *T_{ind} is the model completion of the theory of undirected graphs.*

PROOF. Essentially the same as the proof of Theorem 13.26. □

The following connects elimination of quantifiers and model completions in a non-trivial way:

THEOREM 13.28 (Robinson). *Suppose T has an axiomatization involving universal sentences. If T^* is the model completion of T then T^* admits elimination of quantifiers.*

PROOF. LATER. □

A nice corollary which was previously proved by C. Chevalley in 193x and Alfred Tarski in 194Y:

COROLLARY 13.29 (Elimination of quantifiers in T_{ACF}). *The theory T_{ACF} admits elimination of quantifiers.*

PROOF. Notice (by Remark 5.2) that T_f has a universal axiomatization. An application of the previous theorem and Theorem 13.25 gives what we want. □

A much simpler proof that the theories T_{ord}, T_{ind} and T_{ACF} admits elimination of quantifiers without using the notion of "model completion" will be presented in Theorem 2.1.26.

Exercises 1.13.

(1) If $M \preceq N$ and one of the structures is finite then $M = N$. Thus, in Definition 13.1 the restriction to all *infinite* models instead of all finite models in only technical as it streamline some proofs.
(2) Show that if T is model complete than $(*)$ from Theorem 13.19 holds.
(3) Give an alternative proof of Theorem 13.11 without using the Tarski-Vaught test.
(4) Show that T_{ACF} is model complete.
(5) This is a generalization of Exercise 8.4:
Prove that if T is model complete then $\langle \mathrm{Mod}(T) \subseteq \rangle$ has the amalgamation property.
(6)
(7)
(8)

14. Skolemization

Many interesting problems are questions concerning the group $\mathrm{Aut}(M)$ for M a model of T. The following is an easy observation:

LEMMA 14.1. *For any infinite $M \models T$ we have that $1 \leq |\mathrm{Aut}(M)| \leq 2^{\|M\|}$ holds.*

PROOF. Exercise 2. □

EXAMPLE 14.2. When M is one of the following structures: $\langle \omega, +, \cdot, 0, 1\rangle$, $\langle \mathbb{Q}, +, \cdot, 0, 1\rangle$, $\langle \mathbb{R}, +, \cdot, 0, 1\rangle$ (with the usual operations) or $M = \langle \alpha, \in\rangle$ where α is an ordinal then necessarily $|\mathrm{Aut}(M)| = 1$ holds.

EXAMPLE 14.3. When M has an infinite universe with the empty language (no symbols at all) then $|\mathrm{Aut}(M)| = 2^{\|M\|}$.

EXAMPLE 14.4. Suppose $M \models T_{V,F}$ (see Example 2.45) If B is a basis then every permutation of the set B iduces an automorphism of V and this number has to be multiplied by the number of bases (which is the cardinality of the group $GL(V)$) add to this the number of bijections between anytwo different bases.

If $\dim_F V$ is infinite and $|V| \geq |F|$ then $|GL(V)| = 2^{\dim_F V}$.

EXAMPLE 14.5. Using the notion of transcendence basis instead of linear basis as in the previous example we get that the group of automorphisms of $\langle \mathbb{C}, +, \cdot, 0, 1\rangle$ has cardinality $2^{2^{\aleph_0}}$.

EXAMPLE 14.6. The group of automorphisms of $\langle \mathbb{Q}, <\rangle$ has cardinality $2^{\aleph_0}$.

Structures as in Example 14.2 have a special name.

DEFINITION 14.7. A structure M is called *rigid* iff $|\mathrm{Aut}(M)| = 1$, namely if $f : M \cong M$ then $f = \mathrm{id}_{|M|}$.

By inspecting Example 14.3 one realizes that not every model of every theory can be rigid. In light of the above, the following question is natural:

QUESTION 14.8. Given a theory T and a cardinal $\lambda \geq |T| + \aleph_0$. Does there exists $M \models T$ of cardinality λ such that $|\mathrm{Aut}(M)| = 2^\lambda$?

Ehrenfeucht and Mostowski in 1956 published a paper that answered the above question affirmatively. It turns out that in their solution to the question they discovered two fundamental ideas that have found many applications both in model theory and set theory.

Let us return to a more careful inspection of Example 14.4. Recall the following fact from linear algebra:

FACT 14.9. *Let F be a field, suppose that V is a vector-space over F, let $B \subseteq V$ be a basis for V. Every permutation f of B can be extended to a (unique) automorphism of V.*

COROLLARY 14.10. $|\operatorname{Aut}(V)| \geq 2^{\dim_F V}$.

By Exercise 1 when $|V| > |F| + \aleph_0$ we have that $\dim_F V = |V|$ we automatically have that for $\lambda > |F| + \aleph_0$ for every $M \models T_{V,F}$ of cardinality λ, $|\operatorname{Aut}(M)| = 2^\lambda$. That is to say that we have an affirmative solution to this question in a much more general formulation than it was asked.

The basic idea is to try to imitate Fact 14.9 for an arbitrary theory. The fact can be broken down into two different ingredients:

(1) There exists a set $B \subseteq |M|$ that generates M, and
(2) Every permutation of B has an extension to an automorphism.

Since in a language consisting of relation symbols only we can not expect to have a good notion of generation (1) fails badly for arbitrary models.

Also (2) seems to be close to impossible, in Example 14.2 it is impossible to find any automorphism except the identity mapping.

In this section we deal with problem (1) and delay the solution of problem (2) for the next chapter.

The approach to solve (1) is by adding function symbols to the language (and later taking reducts).

The key property is a generalization of the idea of adding Henkin's witnesses (in the proof of the completeness theorem):

DEFINITION 14.11. Let T be a first-order theory.

(1) *T has built-in Skolem functions* iff
for every $\varphi = \varphi(x; \mathbf{y}) \in \operatorname{Fml}(\mathrm{L}(T))$ there exists a function symbol $F_\varphi(\mathbf{y})$ such that
$$T \vdash \forall \mathbf{y}[\exists x \varphi((x; \mathbf{y}) \to \varphi((F_\varphi(\mathbf{y}); \mathbf{y})].$$
$F_\varphi(\mathbf{y})$ is called *Skolem function for* $\varphi((x; \mathbf{y})$.
(2) *T has definable Skolem functions* iff
for every $\varphi = \varphi(x; \mathbf{y}) \in \operatorname{Fml}(\mathrm{L}(T))$ there exists a formula $\psi_\varphi(\mathbf{y}; z) \in \operatorname{Fml}(\mathrm{L}(T))$ such that
$$T \vdash \forall \mathbf{y}[\exists x \varphi(x; \mathbf{y}) \to \exists z[\psi_\varphi(\mathbf{y}; z) \wedge \varphi(z; \mathbf{y})]]$$
and
$$T \vdash \forall \mathbf{y}[\forall z_1 \forall z_2 \psi_\varphi(\mathbf{y}; z_1) \wedge \psi_\varphi(\mathbf{y}; z_2) \to z_1 = z_2].$$

A natural question at this stage is: Are there any examples for these properties? The answer at first may look little disappointing. There are no natural theories in mathematics that have built-in Skolem functions. However we will see in chapters 3, 4 and 11 that the concept is of great importance (it will be used in the proof of Morley's theorem as well as in constructing many non-isomorphic models). There are some important theories with definable Skolem functions (as we will show below).

For practical reasons it is useful to consider a relativized notion for structures:

DEFINITION 14.12. Let M be an L-structure.

(1) The structure M *has built-in Skolem functions* iff $\mathrm{Th}(M)$ has built-in Skolem functions.
(2) The structure M *has definable Skolem functions* iff $\mathrm{Th}(M)$ has definable Skolem functions.

In the following theorem we show that every theory can be expanded to a theory that have built-in Skolem functions. The idea of the proof is identical to what we did in order to prove the Downward Löwenheim-Skolem theorem.

THEOREM 14.13 (Skolemization of a theory).

(1) *For every consistent T has there exists a similarity type $L' \supseteq \mathrm{L}(T)$ of cardinality $\leq |\mathrm{L}(T)| + \aleph_0$ and a theory $T^{sk} \supseteq T$ in L' such that T^{sk} has built-in Skolem functions.*
(2) *For every model M there is an extension $L' \supseteq \mathrm{L}(M)$ and an expansion M^{sk} of M to L' such that M^{sk} has built-in Skolem functions.*

PROOF. It suffices to prove (2): Let T be given. By consistency of T there exists $M \models T$. By (2) let M^{sk} be an expansion of M with built-in Skolem functions, let $T^{sk} := \mathrm{Th}(M^{sk})$. Since $M^{sk} \models T$ we have that $T^{sk} \supseteq T$.

(2): Fix $\{a_i \ : \ i < \|M\|\} = |M|$. For revery $\varphi(x; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(M))$ and every $\mathbf{b} \in {}^{\ell(\mathbf{y})}|M|$ let

$$G_\varphi(\mathbf{b}) := \begin{cases} a_{Min\{i<\|M\| \, : \, M \models \varphi[a_i, \mathbf{b}]\}} & \text{if } M \models \exists x \varphi[x, \mathbf{b}] \\ a_0 & \text{otherwise.} \end{cases}$$

Denote by M^* the expansion of M to $\{G_\varphi \ : \ \varphi(x; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(M))\}$. Now by induction on $n < \omega$ define $\{M_n \ : \ n < \omega\}$ such that $M_0 = M$ and $M_{n+1} = (M_n)^*$.

For $n < \omega$ let $T_n := \mathrm{Th}(M_n)$ and $T_\omega := \bigcup_{n<\omega} T_n$.

Let M^{sk} is the expansion of M to T_ω given by

$$\langle |M|, R, F, c\rangle_{F,R,c\in\bigcup_{n<\omega}\mathrm{L}(M_n)}.$$

□

CLAIM 14.14. *M^{sk} has built-in Skolem functions*

PROOF. Let $\varphi(x;\mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(M^{sk}))$ be given.

By the definition of T_ω and the fact that $T_{n+1} \supseteq T_n$ there exists $n < \omega$ such that $\varphi(x;\mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T_n)$. By the definition of G_φ we have that

$$M_{n+1} \models \forall \mathbf{y}[\exists x \varphi((x;\mathbf{y}) \to \varphi((G_\varphi(\mathbf{y});\mathbf{y})].$$

Namely we have that

$$\forall \mathbf{y}[\exists x \varphi((x;\mathbf{y}) \to \varphi((G_\varphi(\mathbf{y});\mathbf{y})] \in T_{n+1} \subseteq \mathrm{Th}(M^{sk}).$$

□

DEFINITION 14.15. A theory T *admits elimination of quantifiers* iff for every $\varphi(\mathbf{x})$ in $\mathrm{L}(T)$ there exists a quantifier free $\psi(\mathbf{x}) \in \mathrm{L}(T)$ such that

$$T \vdash \forall \mathbf{x}[\varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x})].$$

Identifying theories that admits elimination of quantifiers was an important sport of the first generation of model theorists and still is important for several applications. One can show that the theories T_{ord}, T_{ind} and T_{ACF_0} all admit elimination of quantifiers while Peano's arithmetic does not. While it is not difficult to verify that PA does not admit elimination of quantifiers, I will delay positive results about elimination of quantifiers to section **??** of Chapter 2 where a uniform method for proving elimination of quantifiers will be presented. Impatient readers are advised to work on Exercises 4 and 5 in this section.

There is an easy connection between Skolem functions and elimination of quantifiers:

LEMMA 14.16. *If T has built-in Skolem functions then T has elimination of quantifiers.*

PROOF. We have to show that for every $\varphi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T))$ there exists a quantifier-free formula ψ_φ such that $T \vdash \forall \mathbf{x}[\varphi(\mathbf{x}) \leftrightarrow \psi_\varphi(\mathbf{x})]$. By induction on the complexity of $\varphi(\mathbf{x})$: For atomic formulas there is nothing to prove, all cases except quantification follow from the induction hypothesis. The exception is when $\varphi(\mathbf{x})$ is $\exists y \psi(y;\mathbf{x})$: By the assumption on T we have that there exists a function symbol F_χ such that $T \vdash \forall \mathbf{x}[\exists y \chi(y;\mathbf{x}) \to \chi(F_\chi(\mathbf{x});\mathbf{x})]$. Notice that by elementary logic we have $T \vdash \forall \mathbf{x}[\chi(F_\chi(\mathbf{x});\mathbf{x}) \to \exists y \chi(y;\mathbf{x})]$.
Thus we have $T \vdash \forall \mathbf{x}[\exists y \chi(y;\mathbf{x}) \leftrightarrow \chi(F_\chi(\mathbf{x});\mathbf{x})]$. □

REMARK 14.17. Notice that by Theorem 2.6 if T admits elimination of quantifiers than it is model-complete. So by the last lemma if T has built-in Skolem functions then it is model complete. However using the Tarski-Vaught test it is not difficult to see that even if T has definable Skolem functions then it is already model complete.

The following is an easy but important observation:

THEOREM 14.18. *Let M be an L-structure*[23]*. If there exists an L-formula $\varphi(x, y; \mathbf{a})$ with parameters from M such that the binary relation induced by φ on $|M|$ is a well-ordering then M has definable Skolem functions.*

PROOF. Given a formula $\psi(x; \mathbf{y})$, let $\chi_\psi(\mathbf{y}; x)$ stand for
$[\exists z \psi(z; \mathbf{y}) \to [\psi(x; \mathbf{y}) \wedge \forall x'[\psi(x'; \mathbf{y}) \to \neg\varphi(x', x; \mathbf{a})]]$.
Thus $\{\langle \mathbf{a}, b\rangle \ : \ M \models \chi_\psi[\mathbf{a}; b]\}$ is a Skolem function for ψ. □

Let $\mathbf{L}$ be the class of constructible sets. It follows from Gödel's work on the constructible universe (see Kunen [**Ku**] or Jech [**Je78**]) that:

COROLLARY 14.19. *The theory of $\langle \mathbf{L}, \in\rangle$ has definable Skolem functions*

PROOF. By Theorem 33 in [**Je78**] (or Lemma VI 4.4 of [**Ku**]) there is a formula $\varphi(x, y)$ in the language $\{\in\}$ (without parameters) such that φ defines a well-ordering on $\mathbf{L}$. Apply the previous theorem to $\varphi(x, y)$. □

DEFINITION 14.20. Let M be a structure with built-in Skolem functions. A set $I \subseteq |M|$ is called *a set of generators of M* iff for every $b \in |M|$ there exist a term $\tau_b(x_1, \ldots, x_{n_b})$ and a sequence $a_1, \ldots, a_{n_b} \in I$ such that

$$b = \tau_b^M[a_1, \ldots, a_{n_b}].$$

The following models will play an important role in many investigations:

DEFINITION 14.21. Suppose that T has built-in Skolem functions and let $M \models T$. For every $A \subseteq |M|$ denote by $\mathrm{EM}(A)$ the *Skolem Hull of the set A* which is the set

$$\{\tau^M[a_0, \ldots, a_{n-1}] \ : \ \mathbf{a} \in A, \tau(\mathbf{x}) \text{ is a term from } \mathrm{L}(T)\}.$$

Notice that by Remark 14.17 we have that $\mathrm{EM}(A)$ is a universe of an elementary submodel of M.

In section 5 of Chapter II Skolem hulls are used to obtain several surprising and fundamental results. See e.g. Corollary II.3.28

[23]Here M may be also a proper class we do not require that its universe is a set

Exercises 1.14.

(1) Let V be a vector space over a field F.
If $|V| > |F| + \aleph_0$ then $\dim_F V = |V|$.
(2) Prove Lemma 14.1.
(3) Let T be a theory with definable Skolem functions, suppose that $M_\ell \models T$ for $\ell = 1, 2$ and $\mathbf{I}_\ell \subseteq M_\ell$ are sets of generators. If $g : \mathbf{I}_1 \to \mathbf{I}_2$ is an (M_1, M_2)-mapping [24] then there exists a unique elementary embedding $\bar{g} : M_1 \to M_2$ extending g.
(4) Use Exercise 1.2.17 to show that T_{ord} has elimination of quantifiers.
(5) Formulate analogs of Exercise 1.2.16 for T_{ind} and T_{ACF_p} and use them to show that the theories admits elimination of quantifiers.
(6) Show that ZFC does not admit elimination of quantifiers.
(7) Suppose that T has built-it Skolem functions. Let $\{c_n \mid n < \omega\}$ be new constants, $L' := \mathrm{L}(T) \cup \{c_n \mid n < \omega\}$ and T' an L'-theory such that $T' \supseteq T \cup \{c_n \neq c_k \mid n \neq k < \omega\}$. If T' is consistent then T' has built-in Skolem functions.
(8) Suppose T has built in Skolem functions, $M \models T$ and $A \subseteq |M|$. Prove that

$$\mathrm{EM}(A) = \{F^M(\mathbf{a}) \mid \mathbf{a} \in A, F(\mathbf{y}) \text{ is a function symbol of } \mathrm{L}(T)\}.$$

(9) Suppose T is a consistent theory, show that there exists a complete $T^* \supseteq T$ with built in Skolem functions.
(10) Suppose T is a consistent theory with built in Skolem functions and T^c is a completion of T. Does T^c has built in Skolem functions?
(11)
(12)
(13)
(14)

[24] A function $g : \mathbf{I}_1 \to \mathbf{I}_2$ is (M_1, M_2)-mapping iff for any $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1} \in \mathbf{I}_1$ and every formula $\varphi(\mathbf{x}_0, \ldots, x_{n-1})$ we have

$$M_1 \models \varphi[\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}] \iff M_2 \models \varphi[g(\mathbf{a}_0), \ldots, g(\mathbf{a}_{n-1})].$$

CHAPTER 2

Abstract Elementary Classes

Introduction

A better understanding of the notion of elementary sub model can be achieved by considering a generalization.

In this chapter we will present an axiomatization of the notion of elementary substructure.

DEFINITION 0.1. Let L be a similarity type. A class $\mathcal{K}$ of L-structures is called *elementary class* provided there exists a first-order theory T in the language L such that $\mathcal{K} = \mathrm{Mod}(T)$. A class $\mathcal{K}$ is called *non-elementary class* provided it is not elementary.

Notice that by Theorem 5.22 the class of well-ordered sets (when $L = \{<\}$) in non-elementary. There are many examples of mathematically interesting class that are not elementary among them is the classes of periodic groups and the Archimedean ordered fields.

This weakness of first-order logic brought Ernst Zermelo already in 1921 to propose considering infinitray logics that are much stronger to formalize logically many properties, in the early thirties he proposed infinitary logic again as a tool that among other things provides a solution to Gödel's incompleteness theorem (see Ebbinghaus [**Ebb**]). Some of the basics of infinitary logic will be presented in section 8.

First, we will deal with a concept named *Abstract Elementary Class* that turns out to be more fundamental and is also a key to understand the deeper new developments in model theory of infinitary logic.

Volume III of this book series is dedicated entirely to the study of Abstract Elementary Classes.

In the last four decades some of the deepest work in pure model theory was carried out in an area called *classification-theory for Abstract Elementary Classes*. This field is an amalgam of a theory that was called in the early seventies *model theory for infinitary languages* together with *classification theory* which is also known as *stability theory*. There are several possible approaches. The usual one is

through a stronger logic than first-order and the other is a more category-theoretic via a generalization of the notion of "elementary embedding". We present here some of the very basic aspects of the theory mainly in order to clarify the notion of elementary substructure (and elementary embedding) that are the main goals of this book.

DEFINITION 0.2. Let L be a language and $\mathbf{x}$ a finite sequence of variables. Let p be a subset of $\{\varphi(\mathbf{x}) \mid \varphi(\mathbf{x}) \in \mathrm{Fml}(L)\}$ and M be an L-structure. We say that p is *realized in* M iff there exists $\mathbf{a} \in {}^{\ell(\mathbf{a})}|M|$ such that

$$M \models \varphi[\mathbf{a}] \quad \text{for all } \mathbf{a} \in {}^{\ell(\mathbf{a})}|M|.$$

We say that M *omits* p iff M does not realize p. Any set $p \subseteq \{\varphi(x) \mid \varphi(\mathbf{x}) \in \mathrm{Fml}(L)\}$ is called *a finitary L-type*.

DEFINITION 0.3. Let $L \subseteq L_1$ be languages, let T_1 be an L_1-theory and let Γ be a set of finitary types in L_1.

$$\mathrm{PC}(T_1, \Gamma, L) := \{M \restriction L \ : \ M \models T_1, M \text{ omits } p, \text{ for every } p \in \Gamma\}.$$

A class $\mathcal{K}$ of L structures such that there are an expansion L_1 of L and T_1, Γ as above with $|L_1| \leq \lambda$ and $|\Gamma| \leq \mu$ is called $\mathrm{PC}_{\lambda,\mu}$ provided

$$\mathcal{K} = \mathrm{PC}(T_1, \Gamma, L).$$

We will write $\mathcal{K}$ is PC_λ as a short hand for "there exists $\mu \leq \lambda$ such that $\mathcal{K}$ is $\mathrm{PC}_{\lambda,\mu}$.

LEMMA 0.4. *Suppose $\mathcal{K} = \mathrm{PC}(T_1, \Gamma, L)$. If $M \in \mathcal{K}$ and $N \cong M$ then $N \in \mathcal{K}$.*

PROOF. Use Theorem 1.2.16 , Lemma 1.3.5 and Lemma 1.7.10.

□

Overview of the structure of this chapter:

Section 1: While the main object of study in this chapter is that of an Abstract Elementary Class, in this section we introduce a more general object, satisfying fewer requirements.

Section 2: In this section Abstract Elementary Classes are introduced. As can be seen from the table of contents we have in a later part of the book an entire chapter dedicated to these objects. The purpose of including this section in the first chapter is to clarify certain issues concerning sub models and the notion of elementary sub models as well as extensions of first-order logic.

Section 4: I discuss several very hard problems and conjectures concerning model theory of Abstract Elementary Classes that not only where the main motivation for the development for almost all the current theory but still are the major open questions in the field.

Section 5: In this section I will discuss Shelah's presentation Theorem.

We establish that every AEC can be axiomatized as a $\mathrm{PC}_{\lambda,\mu}$-class for some λ and μ that depends on $\mathrm{LS}(\mathcal{K})$ only.

Section 7: Contains some of the most difficult material of this book. We use models of weak set theory (as developed in Section 1.10) to establish consistency of certain first-order theories. This important technology is used to investigate which ordinals can be described as a $\mathrm{PC}_{\lambda,\kappa}$. As an application we will conclude a form of upward Lönehiem-Skolem theorem: If an AEC $\mathcal{K}$ has a model of cardinality $\beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$ then $\mathcal{K}$ has a model of cardinality μ for every $\mu \geq \mathrm{LS}(\mathcal{K})$. The theorems of this section are also necessary for first-order model theory to the development of *forking* and the study of *two-cardinal models*.

Section 8: Is dedicated for the following ZFC theorem: If $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an AEC such that it is $\mathrm{PC}_{\aleph_0}$ and also $\prec_{\mathcal{K}}$ can be viewed as a $\mathrm{PC}_{\aleph_0}$-class then $\mathrm{I}(\aleph_0, \mathcal{K}) = \mathrm{I}(\aleph_1, \mathcal{K}) = 1 \implies \mathcal{K}_{\aleph_2} \neq \emptyset$.

Section 8: Several extensions of first-order logic are presented among them: Infinitary logics and cardinality quantifiers. Due to the increased expressive power of these logics the compactness theorem fails. This presents several very interesting challenges when trying to develop model theory for those logics. Chapter **??** is dedicated to a certain aspect of this.

1. Abstract Classes

DEFINITION 1.1. Let L be a similarity type. An ordered pair $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is called an *Abstract Class in* L iff

(1) $\mathcal{K}$ is a class of L-structures.
(2) $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is partially ordered and for $M_1, M_2 \in \mathcal{K}$ we have that $M_1 \leq_{\mathcal{K}} M_2$ implies that $M_1 \subseteq M_2$.
(3) *First isomorphism axiom*: $\mathcal{K}$ is closed under isomorphism i.e. for every L-structure N and every $M \in \mathcal{K}$ if $N \cong M$ then $N \in \mathcal{K}$ and for every $M \in \mathcal{K}$ and every L-structure N if $M \cong N$ then $N \in \mathcal{K}$.
(4) *Second isomorphism axiom*: Let $N_1, N_2 \in \mathcal{K}$ and let $M_1, M_2 \in \mathcal{K}$ such that there exist $f_\ell : N_\ell \cong M_\ell$ (for $\ell = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \leq_{\mathcal{K}} N_2$ implies that $M_1 \leq_K M_2$. We have the following commutative diagram:

$$\begin{array}{ccc} M_1 & \xrightarrow[\mathrm{id}]{} & M_2 \\ {\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle f_2} \\ N_1 & \xrightarrow[\mathrm{id}]{} & N_2 \end{array}$$

Denote by $\mathrm{L}(\mathcal{K})$ the similarity type L.

An abstract class is called *coherent* provided for every $M_1, M_2, M_3 \in \mathcal{K}$ if $M_1 \leq M_3,\ M_2 \leq_{\mathcal{K}} M_3$ and $M_1 \subseteq M_2$ then $M_1 \leq_{\mathcal{K}} M_2$.

Let λ be a cardinal number, by $\mathcal{K}_\lambda$ denote the subclass of $\mathcal{K}$ consisting of structures of cardinality λ. Namely $\mathcal{K}_\lambda := \{M \in \mathcal{K} : \|M\| = \lambda\}$.

By $\mathcal{K}_{<\lambda}$ denote the class $\{M \in \mathcal{K} : \|M\| < \lambda\}$. $\mathcal{K}_{\leq\lambda}$ denote the class $\{M \in \mathcal{K} : \|M\| \leq \lambda\}$. $\mathcal{K}_{\geq\lambda}$ denote the class $\{M \in \mathcal{K} : \|M\| \geq \lambda\}$.

DEFINITION 1.2 ($\mathrm{LS}(\mathcal{K})$). Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an Abstract Class. The *Löwenheim-Skolem number of* $\mathcal{K}$ is denoted by $\mathrm{LS}(\mathcal{K})$. If there exists a cardinal λ such that for every $M \in \mathcal{K}$ and every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $A \subseteq |N|$, $N \leq_{\mathcal{K}} M$ and $\|N\| \leq \lambda + |A|$, the smallest infinite such cardinal which is $\geq |\mathrm{L}(\mathcal{K})|$ (if exists) is denoted by $\mathrm{LS}(\mathcal{K})$ if there is no such λ we write $\mathrm{LS}(\mathcal{K}) = \infty$.

Later we will see several interesting examples, below are few trivial examples:

EXAMPLE 1.3.

(1) Let T be a first-order theory that has an infinite model, let $\mathcal{K}_T := \mathrm{Mod}(T)$ and define $M \leq_{\mathcal{K}_T} N$ iff M is an elementary submodel of N. The only nontrivial part of the definition to verify is (4) but this is the statement of Lemma 3.10. The class $\langle \mathcal{K}_T, \leq_{\mathcal{K}} \rangle$ is called *the Abstract Elementary Class induced by* T. Noticed that by the Downward Löwenheim-Skolem-Tarski theorem we have that $\mathrm{LS}(\mathcal{K}_T)$ is $|\mathrm{L}(T)|$.
(2) Let T and $\mathcal{K}_T$ be as above, it is even easier to show that $\langle \mathcal{K}_T, \subseteq \rangle$ is an abstract class with $\mathrm{LS}(\mathcal{K}_T) = |\mathrm{L}(T)|$.
(3) Let $\mathcal{K}$ be the class of all groups and $\leq_{\mathcal{K}}$ stands for the subgroup relation. This is an abstract class with $\mathrm{LS}(\mathcal{K}) = \aleph_0$.
(4) Let $\mathcal{K}$ be the class of all groups and $\leq_{\mathcal{K}}$ stands for being a *pure subgroup*, namely $G \leq_{\mathcal{K}} H$ iff for every $a \in H$ and every positive integer n from $H \models \exists x[a = n \cdot x]$ follows that there exists $b \in G$ such that $H \models [a = n \cdot b]$. This is an abstract class with $\mathrm{LS}(\mathcal{K}) = \aleph_0$.
(5) If $\mathcal{K}$ is $\mathrm{PC}_{\lambda,\mu}$ where $\lambda \geq |\mathrm{L}(\mathcal{K})| + \aleph_0$ then $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is AEC with $\mathrm{LS}(\mathcal{K}) = \lambda$ where $\leq_{\mathcal{K}}$ is the first-order relation of being elementary sub model in the language $\mathrm{L}(K)$.
(6) Let $\mathcal{K}$ be the class of all infinite sets, $M \leq_{\mathcal{K}} N$ stands for M is subset of N. In this trivial case $\mathrm{LS}(\mathcal{K}) = 0$.
(7) Let $L = \{E\}$ when E is a binary relation, Let $\mathcal{K}$ be the class of all graphs with $G_1 \leq_{\mathcal{K}} G_2$ defined as G_1 is a subgraph of G_2. Also in this case $\mathrm{LS}(\mathcal{K}) = 0$.
(8) Let $L = \{P\}$ be a unary predicate and $\mathcal{K}$ be the class of all L structures with the property that $M \in \mathcal{K} \iff |P^M| \geq \aleph_0 \wedge ||M| - P^M| \geq \aleph_0$ and let $M \leq_{\mathcal{K}} N$ be defined as $P^M \subseteq P^N \wedge |M| - P^M \subseteq |N| - P^N$,

another interesting relation we could define on $\mathcal{K}$ is

$$M \leq_{\mathcal{K}'} N \iff P^M = P^N \wedge |M| - P^M \subseteq |N| - P^N.$$

Another related class is $\langle \mathcal{K}_0, \leq_{\mathcal{K}'} \rangle$ where $\mathcal{K}_0 := \{M \in \mathcal{K} \mid P^M \text{ is countable.}\}$. In the last example $\mathrm{LS}(\mathcal{K}') = \aleph_0$.

(9) Let $\mathcal{K}$ be the class $\{\langle \alpha, \in \rangle \mid \alpha \text{ is an ordinal}\}$ and let $\langle \alpha, \in \rangle \leq_{\mathcal{K}} \langle \beta \in \rangle$ be defined by $\alpha \leq \beta$. This class is not an abstract class, however $\mathcal{K}_{wo}$ consisting of all well-ordered sets when $\langle P, \leq^P \rangle \leq_{wo} \langle Q, \leq^Q \rangle$ stands for P is an initial segment of Q is an abstract class with $\mathrm{LS}(\langle \mathcal{K}_{wo}, \leq_{wo} \rangle) = \infty$. However $\mathrm{LS}(\langle \mathcal{K}_{wo}, \subseteq \rangle) = 0$.

(10) The class of Noetherian rings with the subring relation is an Abstract Class with $\mathrm{LS}(\mathcal{K}) = \aleph_0$.

(11) Let $\mathcal{K}$ be a class of infinite fields, for $F, E \in \mathcal{K}$ define $E \leq_{\mathcal{K}} F$ iff "the field E is relatively algebraically closed in the field F" (namely for every polynomial p with coefficients from E if p has a root in F then it has a root in E).

(12) Let $\mathcal{K}$ be all fields of characteristics zero of infinite transcendence degree over $\mathbb{Q}$ with $\leq_{\mathcal{K}}$ the subfield relation..

(13) Let LF be the class of *locally finite groups* ($G \in \mathrm{LF}$ iff every finite subset of G generates a finite subgroup of G), the relation $\leq_{\mathrm{LF}}$ is the subgroup relation.

(14) Let ULF be the class of *Universally Locally Finite Groups* with the relation $\leq_{\mathrm{ULF}}$ stands for the subgroup relation. A group G is *universally locally finite group* $G \in \mathrm{LF}$, for every finite group H there exists $H^* \leq G$ such that $H \cong H^*$ and in addition for all finite $K_1, K_2 \leq G$ if there exists $\varphi : K_1 \cong K_2$ then there exists $h \in H$ such that $x \mapsto h \cdot x \cdot h^{-1}$ extends φ to G.

(15) Let K be the class of all Banach spaces over the complex numbers, and for $B_1, B_2 \in K$ define $B_1 \leq_K B_2$ iff B_1 is a subspace of B_2.

In analogous way to the definition of elementary embedding using the notion of elementary sub model, there is a natural way to associate a notion of embedding to the abstract class $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$:

DEFINITION 1.4 ($\mathcal{K}$-embedding). Let $M, N \in \mathcal{K}$, we say that a $f : |M| \to |N|$ is a *$\mathcal{K}$-embedding* iff there exists $N^* \leq_{\mathcal{K}} N$ (in $\mathcal{K}$) such that $f : M \cong N^*$.

EXAMPLE 1.5.

(1) When $\mathcal{K}$ is the class of all groups and $\leq_{\mathcal{K}}$ is the subgroup relation then the $\mathcal{K}$-embeddings are exactly the injective group homomorphisms.

(2) When $\mathcal{K}$ is the class of all fields and $\leq_{\mathcal{K}}$ is the subfield relation then the $\mathcal{K}$-embeddings are exactly the injective field-homomorphisms.

(3) When $\mathcal{K}$ is an abstract class and $\leq_{\mathcal{K}}$ is the elementary sub model relation then the $\mathcal{K}$-embeddings are the elementary embeddings.

REMARK 1.6. A typical application of the second isomorphism axiom is as follows: Suppose $M \in \mathcal{K}$, M' is an $\mathrm{L}(\mathcal{K})$-structure and there exists $f : M' \cong M$. If $N \in \mathcal{K}$ and $N \prec_{\mathcal{K}} M$ then $f^{-1}[N]$ determines a universe of a substructure N' of M' such that $N' \prec_{\mathcal{K}} M'$.

Similarly to Lemma 1.2.12 we have

LEMMA 1.7 (Fundamental extension Lemma). *Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract class. Suppose $M, N \in \mathcal{K}$. If there exists a $\mathcal{K}$-embedding f from M to N then there exists $M^* \in \mathcal{K}$ such that $M \leq_{\mathcal{K}} M^*$ and there exists $f^* : M^* \cong N$ satisfying $f^* \supseteq f$.*

PROOF. Exercise, notice the usage of the isomorphism axioms.. □

NOTATION 1.8. For an abstract class $\mathcal{K}$ and $M, N \in \mathcal{K}$ we write $g : M \to N$ to denote g is a $\mathcal{K}$-embedding from M to N.

When we draw a diagram, this will be written as $M \underset{g}{\to} N$, thus $M \underset{\mathrm{id}}{\to} N$ stands for $M \leq_{\mathcal{K}} N$.

When $M_0 \leq_{\mathcal{K}} M$ then $g : M \underset{M_0}{\to} N$ stands for g is a $\mathcal{K}$-embedding from M to N such that $g \restriction M_0 = \mathrm{id}_{M_0}$.

DEFINITION 1.9. Let L be a similarity type. A class $\mathcal{K}$ of L-structures is called an *elementary class* iff there exists a first-order theory T in L such that $\mathcal{K} = \mathrm{Mod}(T)$.

The class of algebraically closed fields is elementary, however the class of algebraically closed fields of infinite transcendence degree is not elementary (see Exercise **??**).

DEFINITION 1.10. Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract class and $\lambda \geq |\mathrm{L}(\mathcal{K})|$. We say that $\mathcal{K}$ is λ-*categorical* iff all elements of $\mathcal{K}_\lambda$ are isomorphic.

It is easy to show that $(\mathcal{K}, Embedd_{\mathcal{K}})$ is a category when $Embedd_{\mathcal{K}}$ is the class of $\mathcal{K}$-embeddings.

The following is very useful:

LEMMA 1.11. *Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract class. If $M, N \in \mathcal{K}$ and f is a $\mathcal{K}$-embedding of M to N then there are $N' \in \mathcal{K}$ and $g : N' \cong N$ such the $N' \succeq_{\mathcal{K}} M$ and $g \supseteq f$.*

PROOF. Use the construction from Proposition 1.3.18(1) and the isomorphism axioms. □

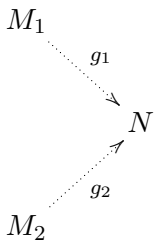
Later we will deal with derived class of an abstract class. We introduce this here.

DEFINITION 1.12 (Derived class). Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an abstract class let $\mathcal{K}^{\leq} := \{\langle |M|, |N| \rangle \ ; \ M \prec_{\mathcal{K}} N, M, N \in \mathcal{K}\}$. I.e. $\mathrm{L}(\mathcal{K}^{\leq})$ consists of s ingle unary predicate. For $\langle |M|, |N| \rangle, \langle |M'|, |N'| \rangle \in \mathcal{K}^{\leq}$ let $\langle |M|, |N| \rangle \prec^*_{\mathcal{K}} \langle |M'|, |N'| \rangle$ iff $M \prec_{\mathcal{K}} M' \wedge N \subseteq N'$.

REMARK 1.13. If $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an abstract class than also $\mathcal{K}^{\leq}$ is abstract class, $\mathrm{LS}(\mathcal{K}) = \mathrm{LS}(\mathcal{K}^{\leq})$ and if $\mathcal{K}$ is coherent then so is also $\mathcal{K}^{\leq}$.

DEFINITION 1.14. Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an Abstract Class. We say that $\mathcal{K}$ has the *Joint Mapping Property* iff for every $M_1, M_2 \in \mathcal{K}$ there exists N and $\mathcal{K}$-embeddings $g_\ell : M_\ell \to N$.

I.e.



Some authors call JMP also *Joint Embedding Property* denoted by *JEP*.

For some $\lambda \geq \mathrm{LS}(\mathcal{K})$ we say that $\mathcal{K}$ has the λ-*JEP* iff $\mathcal{K}_\lambda$ has the Joint Embedding Property.

The following is a reformulation of JEP.

LEMMA 1.15. *Let $\langle \mathcal{K}, \leq \rangle$ be an abstract class. The class has the JEP iff For every $M_\ell \in \mathcal{K}$ ($\ell = 1, 2$) there are $N \in \mathcal{K}$ and $g : M_1 \to N$ such that $N \succeq_{\mathcal{K}} M_2$*

PROOF. Exercise, using the isomorphism axioms. □

EXAMPLES 1.16.

(1) When $\mathcal{K}$ is the class of all groups and $g \leq_{\mathcal{K}} H$ stands for G is a subgroup of H has the JMP (since it is closed under direct product).
(2) By Robinson (Proposition 1.6.2), $\mathcal{K} := \langle \mathrm{Mod}(T), \leq_{\mathcal{K}} \rangle$ when T is a complete first-order theory and $\leq_{\mathcal{K}}$ stands for elementary sub model has the JMP.
(3) Both classes $\mathcal{K}_\ell := \langle \mathrm{Mod}(T_{ACF}), \leq_{\mathcal{K}_\ell} \rangle$, when $\leq_{\mathcal{K}_1}$ stands for elementary sub model and $\leq_{\mathcal{K}_2}$ stands for the subfield relation don't have the JMP as fields of different characteristic can not be embedded into a third field.

REMARK 1.17. Similarly to the proof of Lemma 1.29 one can show that without loss of generality in the definition of JMP we may assume that one of the

mappings is the identity (both mappings can't be the at the same time identity, for trivial reasons)

DEFINITION 1.18. Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract class. $\mathcal{K}$ has the *amalgamation property* iff for every $M_\ell \in \mathcal{K}$ (for $\ell = 0, 1, 2$) and every $f_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) there exists $N \in \mathcal{K}$ and there are $\mathcal{K}$-embeddings $g_\ell : M_\ell \to N$ such that $g_1((f_1(a)) = g_2(f_2(a))$ for every $a \in |M_0|$, we denote this by writing: the following diagram

$$\begin{array}{ccc} M_1 & \overset{g_1}{\dashrightarrow} & N \\ {\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle g_2} \\ M_0 & \underset{f_2}{\longrightarrow} & M_2 \end{array}$$

commutes. The model N is called an *amalgam of the models M_1 and M_2 over M_0*.

An Amalgam is a weakening of what in category theory is called pushout.

A model $M_0 \in \mathcal{K}$ is called an *amalgamation base* iff every $M_\ell \in \mathcal{K}$ for $\ell = 1, 2$ can be amalgamated over M_0.

We say that a class $\mathcal{K}$ has the *λ-amalgamation property* iff $\mathcal{K}_\lambda$ has the amalgamation property. We will write *$\mathcal{K}$ has the λ-AP* instead of writing $\mathcal{K}$ has the λ-Amalgamation Property.

More generally we can consider

DEFINITION 1.19. Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract class and let λ, μ and κ be cardinal numbers. $\mathcal{K}$ has the *(λ, μ, κ)-amalgamation property* iff for every $M_0 \in \mathcal{K}_\lambda, M_1 \in \mathcal{K}_\mu$ and $M_2 \in \mathcal{K}_\kappa$ (for $\ell = 0, 1, 2$) and every $f_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) there exists $N \in \mathcal{K}$ an amalgam of the triple

The following is a useful reformulation of AP.

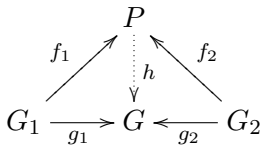
LEMMA 1.20. *Let $\mathcal{K}, \leq_{\mathcal{K}}$ be an abstract class. $\mathcal{K}$ has the Amalgamation Property iff for every $M_\ell \in \mathcal{K}$ ($\ell = 0, 1, 2$) such that $M_0 \leq_{\mathcal{K}} M_\ell$ for ($\ell = 1, 2$) there are $N \in \mathcal{K}$ and $g : M_1 \to N$ such that $N \geq_{\mathcal{K}} M_2$ and $g(a) = a$ for every $a \in |M_0|$.*

PROOF. Exercise, using the isomorphism axioms. □

EXAMPLES 1.21.

(1) Let $\mathcal{K}$ be the class of all sets and $\leq_{\mathcal{K}}$ the subset relation.
Given $M_0 \subseteq M_\ell$, clearly $|M_1| \cup |M_2|$ is an amalgam.
(2) $\mathrm{Mod}(T_{ACF})$ when $M \leq_{\mathcal{K}} N$ stands for M is a subfield of N has the amalgamtion property,

(3) let $\mathcal{K}$ be the class of free groups when $\leq_{\mathcal{K}}$ is the subgroup relation, it has the amalgamation property. We need to recall some basic facts about group theory as can be found in Chapter 11 of Rotman's [**Rot**] book. For groups G_1, G_2 the *free product* of G_1 and G_2 is a group P such that there are group hommorphisms $f_\ell : G_\ell \to P$ such that for group G and hommorphisms $g_\ell : G_\ell \to G$ there exists a unique hommorphism $h : P \to G$ such that the following diagram commutes:



It turns out that always a free product exists and it is unique. The free product P of G_1 and G_2 is denoted by $G_1 * G_2$. A concrete construction: We may assume that $G_1 \cap G_2$ is only the identity element $\{1\}$. $G_1 * G_2$ is the set of reduced words in the alphabeth $G_1 \cup (G_2 - \{1\})$. One can prove a *normal form* theorem: Let $g \in G_1 * G_2$ if $g \neq 1_{G_1 * G_2}$ then there are (unique) $a_\ell \in G_\ell - \{1_{G_\ell}\}$ such that $g = a_1 a_2$. It is not difficult to show that if both groups are free generated by I_1 and I_2 then $G_1 * G_2$ is the free group generated by $I_1 \cup I_2$. This observation can be used to prove: If G_ℓ is given by the presentation $(I_\ell \mid \Delta_\ell)$ and $I_1 \cap I_2 = \emptyset$ then

$$G_1 * G_2 = (I_1 \cup I_2 \mid \Delta_1 \cup \Delta_2).$$

Suppose $G_0 \leq G_1$ and $f : G_0 \to G_2$ is a group monomorphism and $G_1 \cap G_2 = \{1\}$. denote by N the normal subgroup of $G_1 * G_2$ generated by the set $\{f(a)a^{-1} \mid a \in G_0\}$. Denote by $G_1 *_{G_0} G_2$ the group $G_1 * G_2/N$. The group $G_1 *_{G_0} G_2$ is called *free amalgamated product*. It is easy to see that when G_ℓ are free groups then $G_1 *_{G_0} G_2$ is free and is an amalgam of the triple.

(4) The class of all groups has the AP. Given $\langle G_0, G_1, G_2 \rangle$ the previous argument establishes that $G_1 *_{G_0} G_2$ is an amalgam of the triple.

(5)

A related notion to Abstract Class was introduced by Ronald Fraisse in the mid fifties, he also was the first to be abstractly study the Joint Mapping and the Amalgamation Properties.

PROPOSITION 1.22 (Fraisse). *Let M be an infinite structure in a finite language. Let $\mathcal{K} = \mathcal{K}_f(M)$ the collection of all finitely-generated L-structures that are isomorphic to substructures of M and let $N_1 \prec_{\mathcal{K}} N_2 \iff N_1 \subseteq N_2$. Then $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an Abstract Class that has both the Joint Mapping and Amalgamation properties.*

PROOF. Trivial. □

DEFINITION 1.23 (Fraisse limit). Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an abstract class. Given $\bar{M} = \{M_n \mid n < \omega\} \subseteq \mathcal{K}$ such that $n < \omega \implies M_n \prec_{\mathcal{K}} M_{n+1}$. An $\mathrm{L}(\mathcal{K})$-structure M^* is called a *Fraisse limit of* $\bar{M}$ exactly when $M^* = \bigcup_{n<\omega} M_n$.

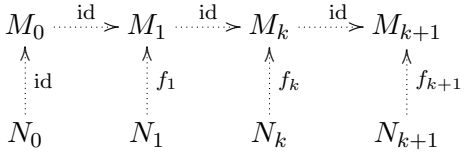
We say that an infinite $\mathrm{L}(\mathcal{K})$-structure M^* is a *Fraisse limit of* $\mathcal{K}$ iff there exists $\bar{M}$ as above.

Notice that since $M_n \prec_{\mathcal{K}} M_{n+1} \implies M_n \subseteq M_{n+1}$ the structure M^* is an $\mathrm{L}(\mathcal{K})$-structure but we don't require that necessarily $M^* \in \mathcal{K}$.

Another simple proposition is

PROPOSITION 1.24. *If an abstract class $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ has the JEP and $\{M/\cong \mid M \in \mathcal{K}\}$ is infinite then there exists an infinite Fraisse limit of $\mathcal{K}$.*

PROOF. Let $\{N_k \mid k < \omega\} \subseteq \mathcal{K}$ be such that $i \neq j \implies N_i \ncong N_j$. Using Lemma1.15 construct $\{M_n \mid n < \omega\} \subseteq \mathcal{K}$ to be $\prec_{\mathcal{K}}$-increasing and $f_k : N_k \to M_k$, such that $M_0 := N_0$ and apply JEP to N_k and M_{k-1} to get M_k.



Finally let $M^* := \bigcup_{k<\omega} M_k$. □

PROPOSITION 1.25 (Fraisse).

INCOMPLETE ADD MORE

EXAMPLE 1.26. It turns out that one can use Fraisse limits to describe some interesting countable structures for the theories T_{ord}, T_{ind} and T_{ACF_p} as well as the countable universal locally-finite group.

(1) When $\mathcal{K}$ is the class of finite linearly-ordered sets then the Fraisse limit of $\mathcal{K}$ is isomorphic the rational numbers.
(2) When $\mathcal{K}$ is the class of finite graphs then the Fraisse limit is the countable random graph, namely a countable model of T_{ind}.
(3) When $\mathcal{K}$ is the class of finite groups then the Fraisse limit is the countable universal locally-finite group. While it is easy to see that $\mathcal{K}$ has the JEP as direct product of two finite groups is a finite group establishing the amalgamation property is not easy. When $G_0 \leq G_\ell, \ (\ell = 1, 2)$ are finite groups the free amalgamated product $G_1 *_{G_0} G_2$ is usually not a finite group. The proof of existence of an amalgam uses the notion of *constricted symmetric group* (for details see Chapter 6 of [**KW**]).

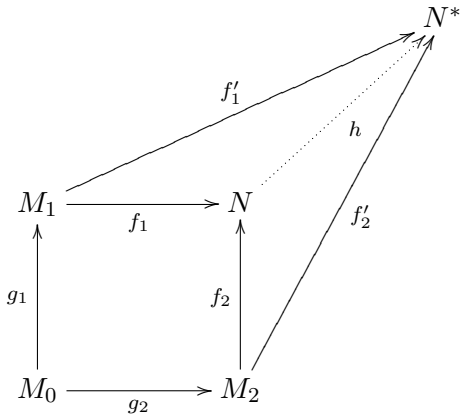
(4) Let F be a fixed finite field and $\mathcal{K}$ the class of finite-dimensional vector spaces over F. The Fraisse limit of $\mathcal{K}$ is the F-vector space of dimension $\aleph_0$.
(5) Let p be a prime number and $\mathcal{K}$ the class of finite fields of characteristics p. The Fraisse limit of $\mathcal{K}$ is the algebraically closed field of transcendence degree $\aleph_0$.

HRUSHOVSKI's CONSTRUCTIoNS?

There is category-theoretic notion called *pushout* that is stronger than amalgamation and also plays a central role in model theory.

DEFINITION 1.27. A category $(\mathcal{K}, Embedd_{\mathcal{K}})$ is *closed under pushouts* iff for every $M_\ell \in \mathcal{K}$ ($\ell \in \{0,1,2\}$) and $g_\ell : M_0 \to M_\ell$ (for $\ell = 1,2$) there are $N \in \mathcal{K}$ and embeddings $f_\ell : M_\ell \to N$ such that $f_1 \circ g_1(a) = f_2 \circ g_2(a)$ for all $a \in |M_0|$ and for every $N^* \in \mathcal{K}$ and embeddings $f'_\ell : M_\ell \to N'$ there exists a unique $h : N \to N^*$ such that $h \circ f_\ell = f'_\ell$ for $\ell = 1,2$.

Here is diagram version of pushout:



REMARK 1.28 (Amalgamation is not pushout). Suppose $\langle K, \leq_{\mathcal{K}} \rangle$ is an abstract class. If the category $(\mathcal{K}, Embedd_{\mathcal{K}})$ is closed under pushouts then certainly the class $\langle K, \leq_{\mathcal{K}} \rangle$ has the amalgamation property, but not conversly.

? EXAMPLE: Let T be complete first-order with DOP. ?!

LEMMA 1.29. *Let $\mathcal{K}$ be an abstract class. The following are equivalent:*

(1) *$\mathcal{K}$ has the λ-amalgamation property.*
(2) *for every $M_\ell \in \mathcal{K}_\lambda$ (for $\ell = 0,1,2$) such that $M_0 \leq_{\mathcal{K}} M_\ell$ (for $\ell = 1,2$) there exists $N \in \mathcal{K}_\lambda$ such that $N \succ_{\mathcal{K}} M_2$ and there exists $\mathcal{K}$-embeddings*

$f : M_1 \to N$ such that the following diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & N \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_0 & \xrightarrow[\mathrm{id}]{} & M_2 \end{array}$$

commutes.

(3) *for every $M_\ell \in \mathcal{K}_\lambda$ (for $\ell = 0, 1, 2$) such that is $g_\ell : M_0 :\to M_\ell$ (for $\ell = 1, 2$) and $\mathcal{K}$-embeddings there exists $N \in \mathcal{K}_\lambda$ there are $\mathcal{K}$-embeddings $f_\ell : M_\ell \to N$ such that the following diagram*

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & N \\ {\scriptstyle g_1}\uparrow & & \uparrow{\scriptstyle f_2} \\ M_0 & \xrightarrow[g_2]{} & M_2 \end{array}$$

commutes.

PROOF. Exercise. □

A stronger version of the amalgamation property is

DEFINITION 1.30. Let $\mathcal{K}$ be an abstract class. $\mathcal{K}$ has the *Disjoint Amalgamation Property* iff for every $M_\ell \in \mathcal{K}$ (for $\ell = 0, 1, 2$) a $\mathcal{K}$-embedding $g : M_0 \to M_1$ such that $M_0 \leq_{\mathcal{K}} M_2$ there are $N \in \mathcal{K}$ which is a $\mathcal{K}$-extension of M_2 and a $\mathcal{K}$-embedding $f : M_1 \to N$ such that

$$f(g(a)) = a \quad \text{for all} \quad a \in M_0.$$

As usual, for $\lambda \geq \mathrm{LS}(\mathcal{K})$ we say that a class has the *λ-Disjoint Amalgamation Property* iff $\mathcal{K}_\lambda$ has the Disjoint Amalgamation Property.

The following follows from the axioms of Abstract Classes:

PROPOSITION 1.31. *Let $\mathcal{K}$ be an abstract class and let λ be a cardinal number. The following are equivalent:*

(1) *$\mathcal{K}$ has the λ-DAP.*
(2) *For every $M_\ell \in \mathcal{K}_\lambda$ (for $\ell = 0, 1, 2$) such that $M_0 \leq_{\mathcal{K}} M_\ell$ (for $\ell = 1, 2$) and $M_1 \cap M_2 = M_0$ there are $N \in \mathcal{K}_\lambda$ and a $\mathcal{K}$-embedding $f : M_1 \to N$, $g \restriction M_0 = \mathrm{id}_{M_0}$ such that $N \geq_{\mathcal{K}} M_2$ and $f[M_1] \cap M_2 = M_0$.*

PROOF. Exercise. □

In the presence of the disjoint amalgamation property the following is a natural property to consider:

DEFINITION 1.32. Let $M_0, M_1, M_2 \in \mathcal{K}$ be given such that $M_0 \leq_{\mathcal{K}} M_\ell$ for $\ell = 1, 2$ suppose that $N \geq_{\mathcal{K}} M_2$ and $f : M_1 \to N$ is a $\mathcal{K}$-embedding such that the following commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & N \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_0 & \xrightarrow[\mathrm{id}]{} & M_2 \end{array}$$

The amalgam N is *minimal* iff there is no $N' \precneqq N$ such that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & N' \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_0 & \xrightarrow[\mathrm{id}]{} & M_2 \end{array}$$

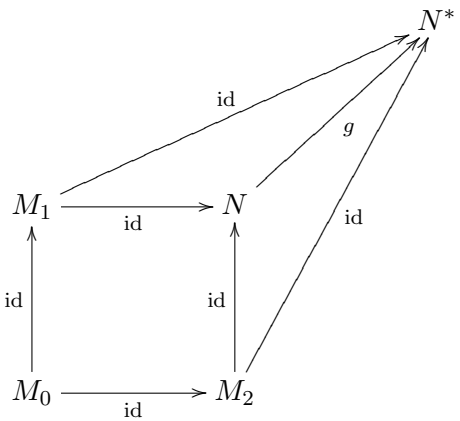
commutes.

This notion of minimal amalgam is of interest in particular also when $f = \mathrm{id}_{M_1}$.

The amalgam N is *prime over* $|M_1| \cup |M_2|$ iff for every N^* such that

$$\begin{array}{ccc} M_1 & \xrightarrow{\mathrm{id}} & N^* \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_0 & \xrightarrow[\mathrm{id}]{} & M_2 \end{array}$$

commutes, there exists a $\mathcal{K}$-embedding $g : N \to N^*$ such that $g \restriction (|M_1| \cup |M_2|) = \mathrm{id}_{|M_1| \cup |M_2|}$, i.e. the following diagram commutes:



EXAMPLES 1.33. In the case when $\mathcal{K}$ is the class of all sets and $\leq_{\mathcal{K}}$ is the subset relation then the union of two sets is a prime and minimal amalgam.

For the class of free groups $G_1 *_{G_0} G_2$ is also prime and minimal amalgam (among free groups). However in the class of all groups for the free amalgamated product is not always prime or minimal amalgam HNN THEOREM ? 1/6 ?

PROPOSITION 1.34. *Let $\mathcal{K}$ be an abstract class.*

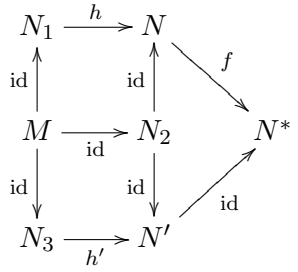
PROOF. Exercise. □

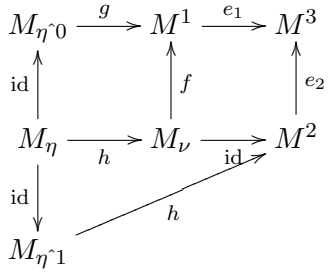
DEFINITION 1.35. Let $\mathcal{K}$ be an abstract class. $\mathcal{K}$ hs the (λ, μ, κ)-AP iff in Definition 1.18 we require the cardinalities of the models M_0, M_1 and M_3 to be λ, μ and κ respectively.

We say that $\mathcal{K}$ has the *Amalgamation Property* or *AP* for short, iff it has the (λ, μ, κ)-AP for all $\kappa, \mu \geq \lambda \geq \mathrm{LS}(\mathcal{K})$.

Notice that a class has the λ-AP iff it has the $(\lambda, \lambda, \lambda)$-AP.

In the study of Abstract Classes, much more complicated amalgamation properties show up and also quite complex diagrams play a role. For example the following diagrams occur in proofs:



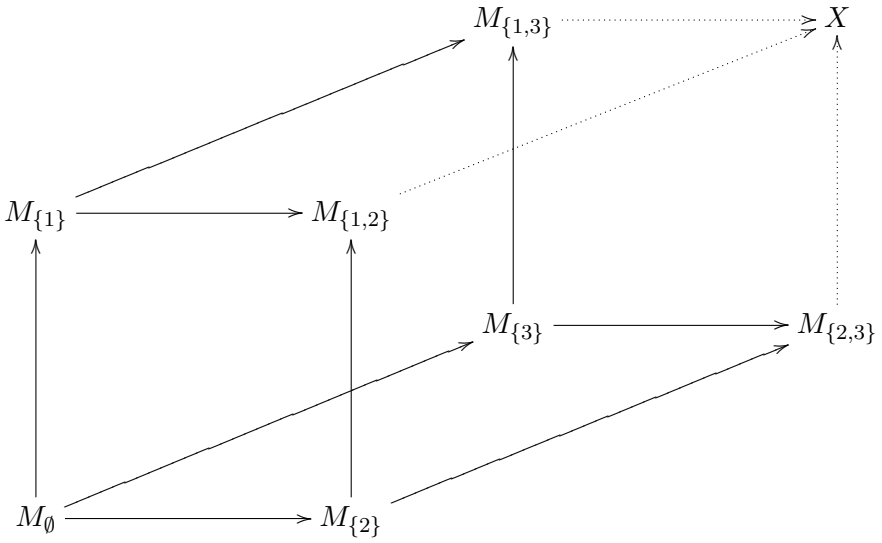


While this kind of diagrams appear also in arguments in other fields of mathematics the following are few pictures that are not so common:

It turns out that n-dimensional cubes occur naturally (for all positive n), here is what they look like:

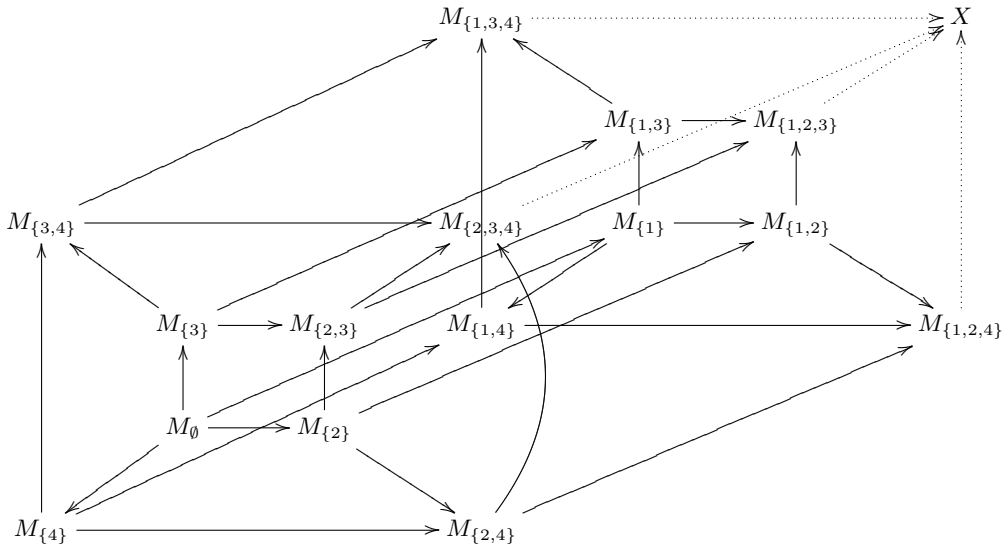
For $n = 3$:

For every $M_\emptyset, M_{\{1\}}, M_{\{2\}}, M_{\{3\}}, M_{\{1,2\}}, M_{\{1,3\}}, M_{\{2,3\}} \in \mathcal{K}_{\aleph_0}$ the following diagram can be amalgamated



For $n = 4$:

Given 15 countable models we need to find a solution for X in the following diagram:



DEFINITION 1.36 (n-dimensional cube). Let n be a positive integer, recall that $n = \{0, \ldots, n-1\}$. By $\mathcal{P}(n)$ denote the power set of n, it is a partially ordered set by $\subseteq$. Suppose $I \subseteq \mathcal{P}(n)$. When $I = \mathcal{P}(n) - n$ denote I by $\mathcal{P}^-(n)$.

DEFINITION 1.37 (n-dimensional system). Let n be a positive integer, and $\lambda \geq \mathrm{LS}(\mathcal{K})$.

$$\{M_s \mid s \in \mathcal{P}^-(n)\} \cup \{f_{s,t} : M_s \to M_t \mid s,t \in \mathcal{P}^-(n)\}$$

is a (λ, n)-*system* provided $M_s \in \mathcal{K}_\lambda$ for all s and

$$f_{s_1,s_3} = f_{s_2,s_3} \circ f_{s_1,s_2} \quad \text{for every } s_1 \subseteq s_2 \subseteq s_3.$$

.

DEFINITION 1.38 ((λ, n)-amalgamation property). For every $\{M_s \mid s \in \mathcal{P}^-(n)\} \cup \{f_{s,t}M_s \to M_t \mid s,t \in \mathcal{P}^-(n)\}$ (λ, n)-system there are $N \in \mathcal{K}_\lambda$ and $g_s M_s \to N$ such that

*****

Let n be a positive integer, and $\lambda \geq \mathrm{LS}(\mathcal{K})$.

$$\{M_s \mid s \in \mathcal{P}^-(n)\} \cup \{f_{s,t} : M_s \to M_t \mid s,t \in \mathcal{P}^-(n)\}$$

is a (λ, n)-*system* provided $M_s \in \mathcal{K}_\lambda$ for all s and

$$f_{s_1,s_3} = f_{s_2,s_3} \circ f_{s_1,s_2} \quad \text{for every } s_1 \subseteq s_2 \subseteq s_3.$$

.

DEFINITION 1.39 ((λ, n)-uniqueness property). For every $\{M_s \mid s \in \mathcal{P}^-(n)\} \cup \{f_{s,t} : M_s \to M_t \mid s,t \in \mathcal{P}^-(n)\}$ (λ, n)-system, $N^\ell \in \mathcal{K}_\lambda$ and $g^\ell_s M_s \to N^\ell$ there are $N^* \in \mathcal{K}_\lambda$ and $h^\ell : N_\ell \to N^*$ such that

*****

Let n be a positive integer, and $\lambda \geq \mathrm{LS}(\mathcal{K})$.

$$\{M_s \mid s \in \mathcal{P}^-(n)\} \cup \{f_{s,t}M_s \to M_t \mid s,t \in \mathcal{P}^-(n)\}$$

is a (λ, n)-*system* provided $M_s \in \mathcal{K}_\lambda$ for all s and

$$f_{s_1,s_3} = f_{s_2,s_3} \circ f_{s_1,s_2} \quad \text{for every } s_1 \subseteq s_2 \subseteq s_3.$$

.

DEFINITION 1.40 (NMM). Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract class. We say that $\mathcal{K}$ has the *no maximal model property* NMM for short, if for every $M \in \mathcal{K}$ there exists $N \in \mathcal{K}$ such that $M <_{\mathcal{K}} N$.

A related property is

DEFINITION 1.41 (AL). Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract class. We say that $\mathcal{K}$ has the *arbitrary large models* AL for short, if for every μ there exists a cardinal $\lambda > \mu$ such that $\mathcal{K}_\lambda \neq \emptyset$.

When $\mathcal{K}$ is an AEC then NMM implies AL, other connections between the various properties are in the exercises in several sections to come.

Generalizing the definition from Exercise 55 from section 3 of chapter I, we have:

DEFINITION 1.42. Let $\mathcal{K}$ be an abstract class. $\lambda \geq |\mathrm{L}(\mathcal{K})|$ be a cardinal, we say $\mathrm{IE}(\lambda, \mathcal{K}) \geq \mu$ iff there exists $\{M_i \mid i < \mu\} \subseteq \mathcal{K}$ all of cardinality λ such that for every $i \neq j < \mu$, here is no $\mathcal{K}$-embedding from M_i into M_j. The equation $\mathrm{IE}(\lambda, \mathcal{K}) = \mu$ stands for $\mathrm{IE}(\lambda, \mathcal{K}) \geq \mu$ and $\mathrm{IE}(\lambda, \mathcal{K}) \ngeq \mu^+$.

Exercises 2.1.

(1) Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract class show that for every $\lambda \geq |\operatorname{L}(\mathcal{K})|$ we have that $\operatorname{I}(\lambda, \mathcal{K}) \leq 2^{\lambda}$.

(2) Let M be a structure of cardinality $\lambda > |\operatorname{L}(M)| + \aleph_0$ and let

$$\mathcal{K} := \big\{ N \text{ an } \operatorname{L}(M)\text{-structure } \mid \text{ there exist } N^* \prec M \text{ such that } N \cong N^* \text{ and } |\operatorname{L}(M)| + \aleph_0 \leq \|N\| < \lambda \big\}$$

with $\prec$ being elementary sub model. Verify that $\mathcal{K}_{<\lambda}$ has the JEP, AP and NMM properties.

If in addition we assume that M is model-homogenous (or saturated) then $\mathcal{K}$ has also the AP.

(3) Consider

$$\mathcal{K} := \big\{ N \mid \text{ there exist } N^* \prec \langle \mathbf{R}, +, \cdot, 0, 1 \rangle \text{ such that } N \cong N^* \big\}$$

with $\prec$ being elementary sub model. Verify that $\mathcal{K}$ has the JEP but not the NMM.

(4) Suppose $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an abstract class of fields. Show that if $\mathcal{K}$ has the JEP then $F, E \in \mathcal{K} \implies \operatorname{Char} F = \operatorname{Char} E$.

(5) Let T be a first-order theory, $\mathcal{K} := \operatorname{Mod}(T)$ and $\prec_{\mathcal{K}}$ the usual notion of elementary sub model (with respect to $\operatorname{L}(T)$). Show that T is complete iff $\mathcal{K}$ has the JEP.

(6) Prove Lemma 1.7.

(7) Suppose that $\langle \mathcal{K}_\ell, \leq_\ell \rangle$ (for $\ell = 1, 2$) are both Abstract Classes such that $\operatorname{L}(K_1) = \operatorname{L}(\mathcal{K}_2)$. Let $\mathcal{K}_3 := \mathcal{K}_1 \cap \mathcal{K}_2$ show that $\langle \mathcal{K}_3, \leq_3 \rangle$ is an Abstract Class, where $\leq_3 := \leq_1 \cap \leq_2$.

(8) Prove Lemma 1.20.

(9) Prove that the amalgamation property is equivalent to: For every $M_0, M_1, M_2 \in \mathcal{K}$ such that $M_0 \leq_{\mathcal{K}} M_\ell$ (for $\ell = 1, 2$) and $M_1 \cap M_2 = M_0$ there exists $N \in \mathcal{K}$ and there exists $f : |M_1| \cup |M_2| \to |N|$ such that $f \restriction M_\ell$ is a $\mathcal{K}$-embedding.

(10) Let $\langle \mathcal{K} \leq \rangle$ be an abstract class and λ an infinite cardinal number. Suppose $\mathcal{K}$ has the λ-AP. If $\{M_k \mid k < n\} \subseteq \mathcal{K}_\lambda$ such that $M_0 \leq M_k$ for all $k < n$ then there are $N \in \mathcal{K}$ and $f_k : M_k \to N$ embeddings such that $f_k \restriction M_0 = \operatorname{id}_{|M_0|}$ for all $k < n$.

(11) Let ψ be the following $L_{\omega_1,\omega}$ sentence in the language of ordered fields:

$$\big[\bigwedge \operatorname{Th}(\langle \mathbf{R}, +, \cdot, <, 0, 1 \rangle) \wedge \forall x [x > 0 \to \bigvee_{n<\omega} nx \geq 1] \big].$$

Let $\mathcal{K}$ be $\operatorname{Mod}(\psi)$ and $\leq_{\mathcal{K}}$ be elementary submodel. Does $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ has the NMM property?

Does $\mathcal{K}^*$ has the NMM property when $\mathcal{K}^* := \operatorname{Mod}(\psi^*)$? Where

$$\psi^* := \big[\bigwedge \operatorname{Th}(\langle \mathbf{R}, +, \cdot, <, 0, 1 \rangle) \big].$$

(12)
(13) Construct an Abstract Class with AL but not the NMM property.
(14)
(15)

2. Abstract Elementary Classes

There are no significant theorems that hold at the level of generality of Abstract Classes. The following additional requirement appears to be very useful and covers a large number of interesting cases.

DEFINITION 2.1. Let L be a similarity type. An ordered pair $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is called an *Abstract Elementary Class* iff

(1) $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is a coherent abstract class in L and
(2) (a) *DLST property* $\mathrm{LS}(\mathcal{K}) < \infty$.
    (b) *First chain axiom* for every ordinal α and every $\{M_i : i < \alpha\} \subseteq \mathcal{K}$ such that
    $i < j < \alpha \implies M_i \leq_{\mathcal{K}} M_j$ we have that $M := \bigcup_{i<\alpha} M_i$ is a member of $\mathcal{K}$ and for every $i < \alpha$ we have that $M_i \leq_{\mathcal{K}} M$ holds.
    (c) *Second chain axiom* for every $N \in \mathcal{K}$ and every $\{M_i \leq_{\mathcal{K}} N : i < \alpha\} \subseteq \mathcal{K}$ such that
    $i < j < \alpha \implies M_i \leq_{\mathcal{K}} M_j$ we have that $\bigcup_{i<\alpha} M_i \leq_{\mathcal{K}} N$.

We denote by $\mathrm{L}(\mathcal{K})$ the similarity type L. Recall that for an AEC, always $\mathrm{LS}(\mathcal{K}) \geq |\mathrm{L}(\mathcal{K})| + \aleph_0$.

An Abstract class $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is called *weak Abstract Elementary Class* if it satisfies requirements (a) and (b).

Let $\lambda \geq |\mathrm{L}(\mathcal{K})|$. An abstract class $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ in L is an *Abstract Elementary Class in* λ iff all structures from $\mathcal{K}$ are of cardinality λ and axioms $(2)(b)$ and $(2)(c)$ holds for all $\alpha < \lambda^+$.

As at present there are no interesting theorems about models in an AEC of cardinality $< \mathrm{LS}(\mathcal{K})$. In order to simplify statements of many results and definitions of important notions we make the following additional assumption:

HYPOTHESIS 2.2. If $\mathcal{K}$ is an AEC then $\mathcal{K} = \mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$. Thus $M \in \mathcal{K} \implies \|M\| \geq \mathrm{LS}(\mathcal{K})$.

EXAMPLES 2.3.

(1) Notice that all examples from 1.3 with the exception of the last are all AECs.
(2) All the class of solvable groups with $\leq$ being the subgroup relation is an AEC.
(3) The class of free groups with the subgroup relation is an AEC.
(4) Wilfrid Hodges (see Corollary 11.5.5 in [**Ho2**]) using results of Gulliksen [**Gu**] and Lopez-Escobar (Theorem 7.22) established that the class of Abelian Noetherian rings (with the subring relation) is not axiomatizeable in $L_{\omega_1,\omega}$, since it is not even $\mathrm{PC}_{\aleph_0}$.

It is unclear to me if the class is AEC, probably it is not. However the class $\mathcal{K}_n$ of rings whose ideals have at most n-generators is an AEC.

(5) The class of Noetherian rings in the language of rings with $R \leq_{\mathcal{K}} S$ iff R is an elementary sub model of S and R is relatively $\aleph_0$-saturated sub model of S, i.e., R realizes every f.o. type over a finite subset of R which is realized in S.

(6) The class of all well-ordered sets of cardinality λ (when $\leq_{\mathcal{K}}$ is the initial segment relation) is an example of an AEC in λ which is not an AEC.

Here are two interesting non-trivial examples concerning first-order logic, that are included for the more advanced readers.

EXAMPLES 2.4. The verification of the following two assertions let to Chapters **??** and **??**.

Let T be a complete countable superstable and not $\aleph_0$-stable, $\prec_{\mathcal{K}}$ be the elementary submodel relation and $\mathrm{LS}(\mathcal{K}) = 2^{\aleph_0}$. $\mathcal{K} := \{M \models T \ : \ M \text{ is } \aleph_1\text{-saturated}\}$. By Theorem III.3.12 of [**Sh c**] $\mathcal{K}$ is closed under unions (recall that superstability implies $\kappa(T) = \aleph_0$).

The above class, when T is $\aleph_0$-stable is also AEC with $\mathrm{LS}(\mathcal{K}) = \aleph_0$.

EXAMPLES 2.5 (From first-order logic). (1) Given T first-order let $\mathcal{K} := \mathrm{Mod}(T)$ and $\leq_{\mathcal{K}}$ be the relation of elementary submodel. It is the contents of the DLST and the Tarski-Vaught chain theorems that $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is an AEC with $\mathrm{LS}(\mathcal{K}) = \aleph_0 + |\mathrm{L}(T)|$.

(2) Let M be an L-structure. Let $\mathcal{K}_M := \{N \ : \ N \equiv M\}$ and $\leq_{\mathcal{K}}$ the elementary sub model relation. Clearly $\langle \mathcal{K}_M, \leq_{\mathcal{K}} \rangle$ is an AEC with $\mathrm{LS}(\mathcal{K}_M) = \aleph_0 + |L|$.

Chapter **??** is dedicated to the basics of the model-theory of abstract elementary classes.

Condition (2)(a) is just a statement of the Downward Löwenheim-Skolem-Tarski Theorem, and condition (2)(b) is the statement of Theorem 3.30. It is easy to check that (1)-(4) in Example 1.3 are all abstract elementary classes while (5) fail to satisfy any of the parts of requirement (2). I hope that the reader can without much difficulties imagine several generalizations of the above definition (by weakening each of the requirements (2)(a) and (2)(b) or eliminating requirement 2(c). It appears that there are several weakening of the above properties that still permit developing an interesting theory. Since not all weakening are interesting, and the model theory of these classe is beyond the limited scope of this book I have decided to limit the level of generality to the one in Definition 2.1.

It is interesting that already the "modest" concept in Definition 2.1 is very powerful

DEFINITION 2.6. Let $\mathbf{I} := \langle I, \leq_I \rangle$ be a partially ordered set. $\mathbf{I}$ is called *directed set* iff for every $s, t \in I$ there exists $r \in I$ such that $s \leq_I r$ and $t \leq_I r$.

Suppose $\langle I, \leq_I \rangle$ is a directed set. $\{M_s \mid s \in I\}$ is called a *directed system* iff

$$s \leq_I t \implies M_s \leq_{\mathcal{K}} M_t \text{ for all } s, t \in I.$$

For examples see Exercise 47.

The contents of the next result is that if $\mathcal{K}$ is an AEC then the induced category $(\mathcal{K}, Embedd_{\mathcal{K}})$ is closed under directed limits, namely in the definition of AEC the ordinal α that serves as an index set in $(2)(b)$ and $(2)(c)$ can be replaced by a directed set.

THEOREM 2.7 (directed union exists). *Suppose that $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is an abstract elementary class. For every directed set $\langle I, \leq_I \rangle$ and every directed system $\{M_i : i \in I\} \subseteq \mathcal{K}$*

(1) $\bigcup_{i \in I} M_i \in \mathcal{K}$ *and*

$$M_j \leq_{\mathcal{K}} \bigcup_{i \in I} M_i \text{ for every } j \in I.$$

(2) *If in addition $M_i \leq_{\mathcal{K}} N$ for all $i \in I$ then we have also $\bigcup_{i \in I} M_i \leq_{\mathcal{K}} N$.*

PROOF. By induction on the cardinality of I:

If I is finite then since I is directed there exists $i_0 \in I$ such that $j \leq_I i_o$ for every $j \in I$. Since in this case $\bigcup_{i \in I} M_i = M_{i_0}$ we are done.

Suppose $|I| = \aleph_0$. Fix $\{a_n \mid n < \omega\} = I$. By induction on $n < \omega$ define $\{b_n \in I : n < \omega\}$ such that $b_n \geq a_n$ and $b_{n+1} \geq b_n$.

Since $\bigcup_{i \in I} M_i = \bigcup_{n < \omega} M_{b_n}$ an application of the chain axioms for $\alpha = \omega$ gives what we want.

If $\lambda := |I|$ is uncountable let $\{I_\alpha : \alpha < \lambda\}$ be such that I_α is a directed subset of I, $|I_\alpha| < \lambda$ for all α and $I = \bigcup_{\alpha < \lambda} I_\alpha$. For existence of $\{I_\alpha \mid \alpha < \lambda\}$ see Exercise 16.

Now let $N_\alpha := \bigcup_{j \in I_\alpha} M_j$ for all α. Verify using $(2)(b)$ and $(2)(c)$ that $\{N_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{K}$ and it is an $\leq_{\mathcal{K}}$-increasing chain satisfying $M_i \leq_{\mathcal{K}} N_\alpha$ for all $i \in I_\alpha$. Another application of $(2)(b)$ gives us that $\bigcup_{\alpha < \lambda} N_\alpha \in \mathcal{K}$. But $\bigcup_{\alpha < \lambda} N_\alpha = \bigcup_{i \in I} M_i$.

For (2): Notice that as $|I_\alpha| < \lambda$ by the inductive assumption as $M_j \leq_{\mathcal{K}} N$ for all $j \in I_\alpha$ we get that $N_\alpha = \bigcup_{j \in I_\alpha} M_j \leq_{\mathcal{K}} N$ for all $\alpha < \lambda$ so by $(2)(c)$ we get $\bigcup_{\alpha < \lambda} N_\alpha \leq_{\mathcal{K}} N$. □

COROLLARY 2.8. *Let $\langle I, \leq_I \rangle$ be a directed set and $\{M_i \mid i \in I\} \subseteq \mathcal{K}$ a directed system. If J is a directed subset of I i.e. $J \subseteq I$ and $\langle J, \leq_I \rangle$ is directed then $\bigcup_{j \in J} M_j \leq_{\mathcal{K}} \bigcup_{i \in I} M_i$.*

Similarly to Theorem 2.7 one can show that in Definition 2.1 in 2(c) it is possible to replace chains of models indexed by an ordinal, by chains indexed by an arbitrary directed set.

Related material to the latter part of this section can be found in section 8.

THEOREM 2.9 (Existence of resolution). *Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an abstract elementary class. Suppose $\lambda > \mathrm{LS}(\mathcal{K}) + \aleph_0$. Show that for every $M \in \mathcal{K}$ of cardinality λ there exists $\{M_i \leq_{\mathcal{K}} M \mid i < \lambda\}$ such that*

(1) $i < j \implies M_i \leq_{\mathcal{K}} M_j$,
(2) *i limit implies that* $M_i = \bigcup_{j<i} M_j$,
(3) $i < \lambda \implies \|M_i\| < \lambda$ *and*
(4) $M = \bigcup_{i<\lambda} M_i$.

PROOF. Similar to the proof of the parallel statement for first-order theories, note the use of the coherence axiom. □

The following is a fundamental property of AECs!

THEOREM 2.10 (Decomposition of models into smaller models). *Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be a coherent Abstract Class with $\mathrm{LS}(\mathcal{K}) < \infty$. For every $\lambda > \mathrm{LS}(\mathcal{K})$ and every $M \in \mathcal{K}_\lambda$ there exists a directed set I of cardinality λ and there is $\{M_i \mid i \in I\} \subseteq \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ such that $M = \bigcup_{i \in I} M_i$.*

PROOF. Given M let I be the set of all finite subsets of $|M|$ ordered by inclusion. We define a directed system $\{M_A \mid A \in I\}$ by induction on $n < \omega$ for every $A \in I$ of cardinality n.

$n = 0$: Using the DLST property pick $M_\emptyset \leq_{\mathcal{K}} M$ of cardinality $\mathrm{LS}(\mathcal{K})$.

Given A of cardinality > 0: By induction assumption $\{M_B \mid B \subsetneq A\} \subseteq \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$. Apply the DLST property to get $M_A \leq_{\mathcal{K}} M$ of cardinality $\mathrm{LS}(\mathcal{K})$ containing the set $\bigcup_{B \subsetneq A} M_B \cup A$. From the coherence axiom it follows that $M_B \leq_{\mathcal{K}} M_A$ for every $B \subsetneq A$. As $\bigcup_{A \in I} M_A = M$ and $\{M_A \mid A \in I\}$ is directed we are done. □

Theorem 2.10 is interesting even when $\mathcal{K} = \mathrm{Mod}(T)$ and T is a first-order theory in a countable language. In this case the theorem provides a decomposition of any uncountable model into a directed system of countable models.

More formally we have

COROLLARY 2.11 (Countable models are the building blocks of all models). *Let T be a first-order theory such that $\mathrm{L}(T)$ is at most countable. For every infinite $M \models T$ there are a directed set $\langle I_M, \leq_M \rangle$ and a directed system $S_M := \{M_i \mid i \in I_M\}$ of countable models of T such that $\bigcup S_M = M$.*

This property of first-order theories was probably known already in the late fifties and I suspect is the reason for important work in the early sixties on countable models (see section 9 of Chapter 4 and the discussion of Vaught's conjecture in section 7 of Chapter 4).

Shelah's work in classification theory (the proof of the Main Gap Theorem) indicate that under reasonable (and necessary) assumptions on T it is possible to understand the structure of S_M namely with additional information on T there is a strong refinement of Corollary 2.11.

Theorem 2.10 and its proof have several applications that are interesting for their own, the most important is Shelah's presentation theorem presented in the next section but here are few simpler examples:

THEOREM 2.12. *Let $\mathcal{K}$ be an abstract coherent class. Suppose it satisfies both chain axioms. If there exists a cardinal $\lambda \geq |\mathrm{L}(\mathcal{K})| + \aleph_0$ such that for every $M \in \mathcal{K}_{>\lambda}$ and every $B \subseteq |M|$ of cardinality $\leq \lambda$ there exists $N \leq_{\mathcal{K}} M$ of cardinality λ containing the set B, then $\mathcal{K}$ is an AEC with $\mathrm{LS}(\mathcal{K}) = \lambda$ when λ is the minimal with the above property.*

PROOF. Given $M \in \mathcal{K}_{>\lambda}$ and $A \subseteq |M|$. Let I_A be the set of all finite subsets of A. By induction on $n < \omega$ define $M_X \in \mathcal{K}_\lambda$ such that $M_X \leq_{\mathcal{K}} M$ and $|M_X| \supseteq \bigcup_{Y \subsetneq X} M_Y \cup X$ for all $X \in I_A$ with $|X| = n$.

Since $\{M_X \mid X \in I_A\} \subseteq \mathcal{K}_\lambda$ is directed,

$$N := \bigcup_{X \in I_A} M_X \in K_{\lambda + |A|} \text{ containing the set } A.$$

□

Similarly to Theorem 2.12 one can show any AEC $\mathcal{K}$ is induced by $\mathcal{K}_{\mathrm{LS}(\mathcal{K})}$:

THEOREM 2.13. *Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. For every $M \in K_\lambda$ there exists a directed set I_M and a directed system $\{N_i \leq_{\mathcal{K}} M \mid i \in I_M\} \subseteq \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ such that*

$$M = \bigcup_{i \in I_M} N_i.$$

PROOF. Exercise. □

Here is a variation on Theorem 2.7 that will be used in the next section.

THEOREM 2.14 (directed system theorem). *Suppose $\mathcal{K}$ is an AEC and N is an $\mathrm{L}(\mathcal{K})$-structure. If there exists $I \subseteq \mathcal{K}$ such that $\bigcup I = N$ and $\langle I, \leq_{\mathcal{K}} \rangle$ is directed then $N \in \mathcal{K}$ and $N_0 \in I \implies N_0 \leq_{\mathcal{K}} N$. Moreover, if there exists $N^* \in \mathcal{K}$ such that $M \in I \implies N_0 \leq_{|} K N^*$ for every $N_0 \in I$ then $M \leq_K N^*$.*

PROOF. By induction on $|I|$: If $|I| = \aleph_0$ pick $\{N_k \mid k < \omega\} \subseteq I$ cofinal in $\langle I, \leq_{\mathcal{K}} \rangle$. As clearly $N = \bigcup I = \bigcup_{k<\omega} N_k$ we have that $N \in \mathcal{K}$ by the first chain axion and $M \leq_K N^*$ by the second chain axiom.

Suppose $\chi := |I| > \aleph_0$. Using the first-order downward Lowenheim-Skolem-Tarski there exists $\{\langle I_\alpha, \leq_{\mathcal{K}} \rangle \prec \langle I, \leq_{\mathcal{K}} \rangle \mid \alpha < \chi\}$ an increasing continuous elementary chain such that $I = \bigcup_{\alpha<\chi} I_\alpha$ and $\alpha < \chi \implies |I_\alpha| < \chi$.

For $\alpha < \chi$ let $N_\alpha := \bigcup I_\alpha$. As I_α is directed using the inductive hypothesis $N_\alpha \in \mathcal{K}$ and that $M_0 \in I_\alpha \implies M_0 \leq_{\mathcal{K}} N_\alpha$. Since $\{I_\alpha \mid \alpha < \chi\}$ is increasing and the moreover part we have for $\alpha < \beta < \chi$ that $M_0 \in I_\alpha \implies M_0 \leq_{\mathcal{K}} N_\beta$ using the moreover part of the inductive hypothesis we get $N_\alpha \leq_{\mathcal{K}} N_\beta$.

Taking care of the "moreover" part is similar. □

LEMMA 2.15 (union of chain of $\mathcal{K}$-embeddings is an embedding). *Let $\mathcal{K}$ be an AEC. Suppose $\{M_i \mid i < \alpha\}$ is an $\leq_{\mathcal{K}}$-increasing chain and $N \in \mathcal{K}$. If there exists a family of $\mathcal{K}$-embeddings $\{f_i \mid i < \alpha\}$ such that $f_i : M_i \to N$ satisfying also $f_i \subseteq f_j$ for all $i < j < \alpha$ then $\bigcup_{i<\alpha} f_i$ is a $\mathcal{K}$-embedding from $\bigcup_{i<\alpha} M_i$ into N.*

PROOF. Exercise. □

THEOREM 2.16. *Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an AEC. For every $\lambda > \mathrm{LS}(\mathcal{K})$, every $N^1 \leq_{\mathcal{K}} N^2 \in \mathcal{K}_{\geq\lambda}$ there are directed sets $I^1 \subseteq I^2$ and there are $\{M_i^\ell \mid i \in I^\ell\} \subseteq \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ directed systems of models such that $N^\ell = \bigcup_{i \in I} M_i^\ell$ and $M_i^1 = M_i^2$ for all $i \in I^1$.*

PROOF. Exercise. □

THEOREM 2.17. *Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an AEC. Suppose $\langle I_1, \leq_1 \rangle$ and $\langle I_2, \leq_2 \rangle$ are directed sets such $I_1 \subseteq I_2$ and $a \leq_1 b \implies a \leq_2 b$ for all $a, b \in I_1$.*

If $\{M_i^\ell \mid i \in I_\ell\} \subseteq \mathcal{K}$ are directed sets such that $M_i^1 \leq_{\mathcal{K}} M_i^2$ for all $i \in I_1$ then

$$N^1 \leq_{\mathcal{K}} N^2 \text{ where } N^\ell := \bigcup_{i \in I^\ell} M_i^\ell.$$

PROOF. Exercise. □

THEOREM 2.18. *Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an AEC. For every $\lambda \geq \mathrm{LS}(\mathcal{K})$, every $M \in \mathcal{K}_{\geq\lambda}$ and every $A \subseteq |M|$ there exists a $N \leq_{\mathcal{K}} M$ containing the set A such that $\|N\| = \lambda + |A|$.*

PROOF. Exercise. □

In what follows we present another generalization of the notion of elementary submodel. For every formula in a given language we associate its set of subformulas.

DEFINITION 2.19. Let L be a similarity type.
$Sub : \mathrm{Fml}(L) \to \mathcal{P}(\mathrm{Fml}(L))$ is the following function defined by induction on the complexity of φ:

(1) if φ is atomic then $Sub(\varphi) = \{\varphi\}$,
(2) if φ is $\neg\psi$ then $Sub(\varphi) = \{\varphi\} \cup Sub(\psi)$,
(3) if φ is $\psi \wedge \chi$ then $Sub(\varphi) = \{\varphi\} \cup Sub(\psi) \cup Sub(\chi)$,
(4) φ is $\forall x\psi$ then $Sub(\varphi) = \{\varphi\} \cup Sub(\psi)$.

Now it is possible to generalize Theorems 3.30 and 3.31 to the notion $\leq_\Phi$ of Definition 3.8:

THEOREM 2.20. *Let L be a similarity type, and suppose that $\Phi \subseteq \mathrm{Fml}(L)$ such that $\varphi \in \Phi \implies Sub(\varphi) \subseteq \Phi$. Let $\mathcal{K}$ be the class of all L structures of cardinality $\geq |L| + \aleph_0$ then $(\mathcal{K}, \leq_\Phi)$ is an abstract elementary class.*

PROOF. Exercise 39. □

One of the interesting features of AECs in comparison to Abstract Classes is that we can derive the full amalgamation property from a localized version. Namely we have the following theorem:

THEOREM 2.21. *Let $\langle \mathcal{K}, \leq_\mathcal{K} \rangle$ be an AEC. If $\mathcal{K}$ has the μ-AP for all $\mu \geq \mathrm{LS}(\mathcal{K})$ then $\mathcal{K}$ has the AP.*

To get a better understating of this theorem and its proof I suggest to the reader either before reading the proof of while gong over it to attempt doing Exercise 35, this may clarify what is going on.

It is enough to prove by induction on $\zeta := \mathrm{Max}\{i, j\}$ that the $(\aleph_\alpha, \aleph_{\alpha+i}, \aleph_{\alpha+j})$-AP holds for all α.

The base for the induction (when $\zeta = 0$) is the assumption of the theorem. For $\zeta > 0$ we treat the cases when $i < j$ and $i = j$ separately.

LEMMA 2.22. *Let $\langle \mathcal{K}, \leq_\mathcal{K} \rangle$ be an AEC, $\mu > \kappa \geq \lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ has the (λ, μ', κ')-AP for all λ, μ' and κ' satisfying $\mathrm{LS}(\mathcal{K}) \leq \lambda \leq \kappa' \leq \kappa + \mu'$, $\lambda \leq \mu' < \mu$ then $\mathcal{K}$ has the (λ, μ, κ)-AP.*

PROOF. Let $M_0 \in \mathcal{K}_\lambda$, $M_1 \in \mathcal{K}_\kappa$ and $M_2 \in \mathcal{K}_\mu$ be given such that $M_0 \preceq_\mathcal{K} M_\ell$ for $\ell = 1, 2$.

Fix $\{M_2^\beta \in \mathcal{K}_{<\mu} \mid \beta < \mu\}$ to be $\leq_{\mathcal{K}}$ increasing and continuous such that $M_2 = \bigcup_{\beta<\mu} M_2^\beta$ and $M_2^0 \geq M_0$. By the induction assumption there is $N_0 \in \mathcal{K}_\mu$ an amalgam of M_1, M_0 and M_2^0. I.e. $N_0 \geq M_1$ and there exists a $\mathcal{K}$ embedding $f_0 : M_2^0 \to N_0$ such that

$$\begin{array}{ccc} M_2^0 & \xrightarrow{f_0} & N_0 \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_0 & \xrightarrow[\mathrm{id}]{} & M_1 \end{array}$$

commutes.

By induction on β define $\{N_\beta \in \mathcal{K}_{<\mu} \mid \beta < \mu\}$ to be $\leq_{\mathcal{K}}$ increasing and continuous and a family of $\mathcal{K}$-embeddings $\{f_\alpha \mid \beta < \mu\}$ such that

$$\begin{array}{ccc} M_2^{\beta+1} & \xrightarrow{f_{\beta+1}} & N_{\beta+1} \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_2^\beta & \xrightarrow[f_\beta]{} & N_\beta \end{array}$$

commutes. For β limit let $f_\beta := \bigcup_{\gamma<\beta} f_i$, and $N_\beta := \bigcup_{\gamma<\beta} N_i$.

By Lemma 2.15 for $\beta < \mu$ limit and every $\gamma < \beta$ we have that

$$\begin{array}{ccc} M_2^\beta & \xrightarrow{f_\beta} & N_\beta \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_\beta & \xrightarrow[f_\gamma]{} & N_\gamma \end{array}$$

commutes. Thus $N := \bigcup_{\beta<\mu} N_\beta$ is an amalgam as required. □

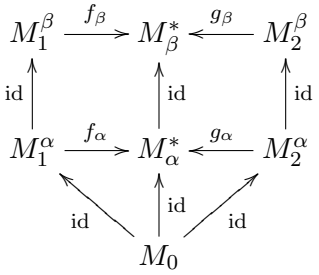
The case when $i = j$:

LEMMA 2.23. *Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an AEC and $\mu > \lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is satisfying the $(\lambda', \mu', \kappa')$-AP for all $\kappa', \mu' < \mu$ and every λ' satisfying $\lambda \leq \lambda' \leq \min\{\mu', \kappa'\}$ then $\mathcal{K}$ has the (λ, μ, μ)-AP.*

PROOF. Let $M_0 \in \mathcal{K}_\lambda$ and $M_1, M_2 \in \mathcal{K}_\mu$ be given such that M_0 is a $\leq_{\mathcal{K}}$-substructure of both M_1 and M_2.

Fix continuous resolutions $\{M_\ell^\alpha \mid \alpha < \mu\} \subseteq \mathcal{K}_{<\mu}$ of M_ℓ such that M_ℓ^0 are $\geq_{\mathcal{K}}$-extensions of M_0.

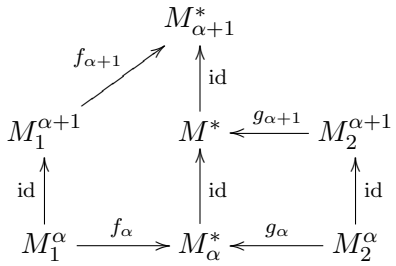
It is enough to define increasing and continuous chains $\{M^*_\alpha, f_\alpha, g_\alpha \mid \alpha < \mu\}$ such that for every $\alpha < \beta$ the diagram



commutes. For $\alpha = 0$, using the induction hypothesis pick M^*_0, f_0 and g_0 an amalgam of M^0_1 and M^0_2 over M_0.

Given $M^*_\alpha, f_\alpha, g_\alpha$, by continuity it is enough to define $M^*_{\alpha+1}, f_{\alpha+1}, g_{\alpha+1}$.

First, select M^* and $g_{\alpha+1}$ such that M^* is an amalgam of $M_2^{\alpha+1}$ and M^*_α over M_2^α, now let $M^*_{\alpha+1} \geq_{\mathcal{K}} M^*$ and $f_{\alpha+1}$ be chosen such that it is an amalgam of $M_1^{\alpha+1}$ and M^* over M_1^α. Since the diagram



commutes we are done. □

DEFINITION 2.24. Let $\mathcal{K}$ be an abstract class and $\lambda > \mathrm{LS}(\mathcal{K})$. We say that $\mathcal{K}_{<\lambda}$ *has the amalgamation property* iff any triple of models from $\mathcal{K}_{<\lambda}$ has an amalgam.

THEOREM 2.25. *Let $\mathcal{K}$ be an abstract class and $\lambda > \kappa \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}_{<\lambda}$ has the amalgamation property then $\mathcal{K}$ has the $(\kappa, \kappa, \lambda)$-AP.*

PROOF. Exercise. □

DEFINITION 2.26. Let $\mathcal{K}$ be an abstract class. $\mathcal{K}$ has the *Joint Mapping Property* often it is also called *Joint Embedding Property* iff for every $M_\ell \in \mathcal{K}$ (for $\ell = 1, 2$) there are $N \in \mathcal{K}$ and $\mathcal{K}$-embeddings $f_\ell : M_\ell \to N$.

We say that $\mathcal{K}$ has the λ-JMP (or JEP) when all structures have cardinality λ.

EXAMPLES 2.27.

(1) $\mathrm{Mod}(T_{ACF_p})$ has the JMP,
(2) $\mathrm{Mod}(T_{ACF})$ does not have the λ-JMP for any λ, however it has the AP.
(3) The class of all groups has the JMP since it is closed under direct product. The JMP property can also be deduced from the fact that the class of groups is closed under free product.
(4) Taking direct products also show that the class of locally finite groups has the JMP.
(5) There is $\mathcal{K}$ which is λ-categorical (hence has the λ-JEP), however does not have the λ-AP. (see Theorem 9.16?)
(6) ?!

For AECs in the presence of AP it is usually easy to derive JEP for all models assuming only $\mathrm{LS}(\mathcal{K})$-JEP:

PROPOSITION 2.28. *Suppose $\mathcal{K}$ is an AEC with AP. If the $\mathrm{LS}(K)$-JEP holds then $\mathcal{K}$ has the JEP.*

PROOF. Exercise. □

As trivially we have that λ-categoricity implies λ-JEP (even for Abstract Classes) we have following useful

COROLLARY 2.29. *Let $\mathcal{K}$ be an AEC satisfying AP and $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is λ-categorical then $\mathcal{K}_{\geq\lambda}$ has the JEP.*

PROOF. Exercise. □

Exercises 2.2.

(1) Two abstract classes $\mathcal{K}_1$ and $\mathcal{K}_2$ are *equal* iff $\mathrm{L}(K_1) = \mathrm{L}(\mathcal{K}_2)$ and $M \in \mathcal{K}_1 \iff M \in \mathcal{K}_2$ for every $\mathrm{L}(K_1)$-structure M.

Use Theorem 2.13 to prove that there are at most $2^{\mathrm{LS}(\mathcal{K})}$ abstract elementary classes.

(2) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an abstract class, $M, N \in \mathcal{K}$. Suppose that $g : |M| \to |N|$ is function. Prove that g is a $\mathcal{K}$-embedding iff g is a monomorphism of $\mathrm{L}(\mathcal{K})$ structures and $g[|M|]$ determines a universe of a $\mathcal{K}$-substructure of N.

(3) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an AEC. If $\mathcal{K}$ has the amalgamation property and the $\mathrm{LS}(\mathcal{K})$-JEP then $\mathcal{K}$ has the JEP.

(4) Let $\langle \mathcal{K}, \leq \rangle$ be an AEC. Suppose that $\mathcal{K}_{<\lambda}$ has NMM. If $\mathcal{K}$ is categorical in λ then $\mathcal{K}_{<\lambda}$ has the JEP.

(5) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an AEC. If $\mathcal{K}$ has the NMM and the $\mathrm{LS}(\mathcal{K})$-JEP then $\mathcal{K}$ has the JEP.

(6) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an abstract elementary class. Suppose that $\{M_i \mid i < \alpha\}$ and $\{N_i \mid i < \alpha\}$ are both $\preceq_{\mathcal{K}}$-increasing chains of structures from $\mathcal{K}$ such that $M_i \preceq_{\mathcal{K}} N_i$ for all $i < \alpha$, let $M := \bigcup_{i<\alpha} M_i$ and $N := \bigcup_{i<\alpha} N_i$. Prove that $M \preceq_K N$.

(7) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an abstract elementary class. Suppose $N \in \mathcal{K}$, $\{M_i \mid i < \alpha\} \subseteq \mathcal{K}$ such that $i < j < \alpha \implies M_i \preceq_{\mathcal{K}} M_j$ and $\{f_i : M_i \to N \mid i < \alpha\}$ are $\mathcal{K}$-embeddings satisfying $i < j < \alpha \implies f_i \subseteq f_j$. Prove that

$$\bigcup_{i<\alpha} f_i : \bigcup_{i<\alpha} M_i \to N \text{ is a } \mathcal{K}\text{-embedding.}$$

(8) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an abstract elementary class and let $\lambda \geq |\mathrm{L}(\mathcal{K})|$. If $\mathcal{K}$ has the λ-AP then for any $\{M_\beta \mid \beta < \lambda\} \subseteq \mathcal{K}_\lambda$ and every $M \in \mathcal{K}_\lambda$ such that $M \preceq_{\mathcal{K}} M_\beta$ for all $\beta < \lambda$ there are $N \in \mathcal{K}_\lambda$ and $\{f_\beta : M_\beta \xrightarrow[M]{} N \mid \beta < \lambda\}$.

(9) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an abstract elementary class. Suppose that $\lambda \geq \mathrm{LS}(\mathcal{K})$.

If $\mathrm{I}(\lambda, \mathcal{K}) = 1$ and $\mathrm{I}(\lambda^+, \mathcal{K}) \geq 1$ then for every $M \in \mathcal{K}$ of cardinality λ there exists $N \in \mathcal{K}$ such that $N \neq M$ and $M \preceq_{\mathcal{K}} N$.

(10) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an abstract elementary class. Suppose that $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathrm{I}(\lambda, \mathcal{K}) = 1$ and the λ-AP fails then $\mathcal{K}_{\lambda^+} \neq \emptyset$.

(11) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an AEC. Suppose that $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathrm{I}(\lambda^+, \mathcal{K}) = \mathrm{I}(\lambda, \mathcal{K}) = 1$ then $\mathcal{K}_\lambda$ has NMM.

(12) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an abstract elementary class. Suppose that $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathrm{I}(\lambda, \mathcal{K}) = 1$ and the λ-AP fails then there is a counterexample for the amalgamation property $\{M_\ell \in \mathcal{K}_\lambda \mid \ell \in \{0, 1, 2\}\}$ such that $|M_\ell| - |M_0| = \lambda$ for $\ell = 1, 2$

(13) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an abstract elementary class such that there exists a unary predicate $P \in \mathrm{L}(K)$. Suppose that $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathrm{I}(\lambda, \mathcal{K}) = 1$ and there are $M, N \in \mathcal{K}_\lambda$ such that $M \preceq_{\mathcal{K}} N$ and $P^M = P^N$ then there exists $N^* \in \mathcal{K}_{\lambda^+}$ with P^{N^*} of cardinality $\leq \lambda$.

(14) Suppose that $\langle \mathcal{K}_\ell, \preceq_\ell \rangle$ (for $\ell = 1, 2$) are both AECs such that $\mathrm{L}(K_1) = \mathrm{L}(\mathcal{K}_2)$. Let $\mathcal{K}_3 := \mathcal{K}_1 \cap \mathcal{K}_2$ show that $\langle \mathcal{K}_3, \preceq_3 \rangle$ is an AEC, where $\preceq_3 := \preceq_1 \cap \preceq_2$ and $\mathrm{LS}(K_3) = \max\{\mathrm{LS}(K_1), \mathrm{LS}(K_2)\}$.

(15) Suppose that $\langle \mathcal{K}^\ell, \preceq_\ell \rangle$ (for $\ell = 1, 2$) are both AECs such that $\lambda := \mathrm{LS}(\mathcal{K}^1) = \mathrm{LS}(\mathcal{K}^2)$. Show that $\mathcal{K}^1_\lambda = \mathcal{K}^2_\lambda$ implies that $\mathcal{K}^1 = \mathcal{K}^2$.

(16) For every infinite directed set I of cardinality λ there exists $\{I_\alpha : \alpha < \lambda\}$ such that
   (a) For every $\alpha < \lambda$ the set I_α is directed and has cardinality less than λ,
   (b) $\alpha < \beta < \lambda$ implies that $I_\alpha \subseteq I_\beta$ and
   (c) $I = \bigcup_{\alpha<\lambda} I_\alpha$.

(17) The purpose of this exercise is to show that in the definition of AEC (Definition 2.1) we could weaken the LS-axiom. Let

   weak Löwnheim Skolem axiom be the following:

   There exists an infinite cardinal $\kappa \geq |\mathrm{L}(\mathcal{K})|$ such that for every $M \in \mathcal{K}$ and every finite $A \subseteq |M|$ there exists $N \preceq_{\mathcal{K}} M$ containing the set A such that $\|N\| \leq \kappa$.

(18) Prove the assertion made in Remark 2.13.

(19) In the definition of of AEC the chain axioms can be weaken to require instead "for all chain indexed by an ordinal α..." to require only "for all chain indexed by a regular cardinal α".

(20) Let $\mathcal{K}$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$. Show that for every μ satisfying $\mathrm{LS}(\mathcal{K}) \leq \mu \leq \lambda$, $M \in \mathcal{K}_\lambda$ and $A \subseteq |M|$ of cardinality μ there exists $N \in \mathcal{K}_\mu$ such that $A \subseteq |N|$ and $N \leq_{\mathcal{K}} M$.

(21) Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. Show that for every μ satisfying $\mathrm{LS}(\mathcal{K}) < \mu \leq \lambda$ and every $M \in \mathcal{K}_\lambda$ there exists a directed set I such that there exists a directed system $\{M_i \leq_{\mathcal{K}} M \mid i \in I\} \subseteq \mathcal{K}_\mu$ satisfying $M = \bigcup_{i \in I} M_i$.

(22) Let $\mathcal{K}$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$. For any $M \in \mathcal{K}_\lambda$ there exists an *increasing and continuous* $\{M_i \preceq_{\mathcal{K}} M \mid i < \lambda\}$ such that $M = \bigcup_{i<\lambda} M_i$ and $\|M_i\| = \mathrm{LS}(\mathcal{K}) + |i|$. I.e $i < j < \lambda \implies M_i \precneqq_{\mathcal{K}} M_j$, if i is limit then $M_i = \bigcup_{j<i} M_j$. A chain as above is called a *resolution* of M. resolution of M

(23) Let $\mathcal{K}$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$ a cardinal. Suppose that $M \in \mathcal{K}_\lambda$. Show that for every μ satisfying $\mathrm{LS}(\mathcal{K}) \leq \mu < \lambda$ and every $A \subseteq |M|$ of cardinality μ there exists $N \leq_K M$ of cardinality μ containing A such that N is also an elementary submodel of M.

(24) Let $\mathcal{K}$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$ a regular cardinal. Suppose that $\{M_i \mid i < \lambda\} \subseteq \mathcal{K}_{<\lambda}$ is a resolution of M then for every increasing and continuos $\leq_{\mathcal{K}}$-chain $\{N_i \mid i < \lambda\} \subseteq \mathcal{K}_{<\lambda}$ if $M = \bigcup_{i<\lambda} N_i$ then for every $j < \lambda$ there exists $i < \lambda$ greater than j such that $N_i = M_i$.

(25) Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. For any $M \in \mathcal{K}_{\lambda^+}$ there exists an increasing and continuous $\{M_i \preceq_{\mathcal{K}} M \mid i < \lambda^+\} \subseteq \mathcal{K}_\lambda$ such that $M = \bigcup_{i<\lambda^+} M_i$.

(26) Let $\mathcal{K}$ be an AEC and suppose that $\lambda > \mathrm{LS}(\mathcal{K})$ is a regular cardinal. For any $M \in \mathcal{K}_\lambda$ if both $\{M_i \mid i < \lambda\}$ and $\{N_i \mid i < \lambda\}$ are resolutions of M then the set $\{i < \lambda \mid M_i = N_i\}$ contains a club.

(27) Let $\mathcal{K}$ be an AEC and suppose that $\lambda > \mathrm{LS}(\mathcal{K})$ is a regular cardinal. For any $M, N \in \mathcal{K}_\lambda$ and $f : M \cong N$. If $\{M_i \mid i < \lambda\}$ is a resolution of M and $\{N_i \mid i < \lambda\}$ a resolution of N then the set $\{i < \lambda \mid f : M_i \cong N_i\}$ contains a club.

(28) Let $\mathcal{K}$ be an AEC and suppose that $\lambda > \mathrm{LS}(\mathcal{K})$ is a regular cardinal. Suppose $M \in \mathcal{K}_\lambda$ such that $|M| = \lambda$. For any resolution $\{M_i \mid i < \lambda\}$ of M there exists a limit ordinal $\delta < \lambda$ such that $\delta = |M_\delta|$.

(29) Let $\mathcal{K}$ be an AEC, P a unary predicate in $\mathrm{L}(\mathcal{K})$ and suppose that $\lambda \geq \mathrm{LS}(\mathcal{K})$ is a cardinal. Suppose $M \in \mathcal{K}_{\lambda^+}$ such that $P^M = \lambda^+$. For any resolution $\{M_i \mid i < \lambda^+\}$ of M there exists a limit ordinal $\delta < \lambda^+$ such that $\lambda(1 + \delta) = \delta = P^{M_\delta}$.

(30) Let $\mathcal{K}$ be an AEC, P a unary predicate in $\mathrm{L}(\mathcal{K})$ and suppose that $\lambda \geq \mathrm{LS}(\mathcal{K})$ is a cardinal. Suppose $M \in \mathcal{K}_{\lambda^+}$ such that $P^M = \lambda^+$. For any resolution $\{M_i \mid i < \lambda^+\}$ of M

$$\{\delta < \lambda^+ \mid \lambda(1 + \delta) = \delta = P^{M_\delta}\} \quad \text{contains a club of } \lambda^+.$$

(31) Show that in Definition 2.1 it suffices to require that (b) and (c) hold for all α satisfying $\alpha = \mathrm{cf}\,\alpha$.

(32) Prove Theorem 2.10.

(33) Show that in Theorem 2.9 we could replace requirement (3) by the stronger one:

$(3)'$: $i < \lambda \implies \|M_i\| \leq |i| + \mathrm{LS}(\mathcal{K})$.

(34) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ fails to have the λ-AP then there are $M_\ell \in \mathcal{K}_\lambda$ for $\ell = 0, 1, 2$ such that M_1 and M_2 can not be amalgamated over M_0 in $\mathcal{K}_{\lambda^+}$.

(35) In purpose of this exercise to improve your understanding of the proof of Theorem 2.21. Try without using anything proved in this section to establish

(a) If $\mathcal{K}$ has the λ-AP then it has the $(\lambda, \lambda, \lambda^+)$-AP.

(b) If $\mathcal{K}$ has the λ-AP then it has the $(\lambda, \lambda^+, \lambda^+)$-AP.

(36) Let $\mathcal{K}$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$ be a cardinal. Derive that $\mathcal{K}_{<\lambda}$ has the AP from the assumption that $\mathcal{K}_\mu$ has the AP for every μ satisfying $\mathrm{LS}(\mathcal{K}) \leq \mu < \lambda$.

(37) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an AEC.
Suppose that $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathrm{I}(\lambda, \mathcal{K}) = 1$ and there are $M, N \in \mathcal{K}_\lambda$ such that $M \precneqq_{\mathcal{K}} N$ then $\mathrm{I}(\lambda^+, \mathcal{K}) \geq 1$.

(38) Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. Suppose $\mathcal{K}$ is categrocial in λ^+. If $\mathcal{K}_{\lambda^{++}} \neq \emptyset$ then for every $\{N_i \mid i < \lambda^+\} \subseteq \mathcal{K}_\lambda$ increasing and continuous chain there exists an increasing continuous chain $\{N^1_{\mathbf{i}} \mid i < \lambda^+\} \subseteq \mathcal{K}_\lambda$ such that $N_i \precneqq_{\mathcal{K}} N^1_i$ for all $i < \lambda^+$.

(39) Prove Theorem 2.20.

(40) Let T be a theory, denote by $\mathrm{Mod}(T) = \{M \ : \ M \models T\}$ and by $Embedd_{\mathrm{L}(T)} := \{f \ : \$ there are $M, N \in \mathrm{Mod}(T)$ such that f is an elementary embedding of M into $N\}$. Show that $(\mathrm{Mod}(T), Embedd_{\mathrm{L}(T)})$ is a category. It is called the *induced category by* T.

(41) Let $\mathcal{K}$ be the class of all Banach spaces over the reals, define $B_1 \preceq_{\mathcal{K}} B_2$ to hold iff B_1 is a Banach subspace of B_2 (i.e. in addition of being a subspace as a normed space it is also required to be complete). Why $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ is not an abstract elementary class?

(42) Let $\mathcal{K}$ be the class of well-ordered sets with $\mathrm{L}(\mathcal{K}) = \{<\}$ and $\preceq_{\mathcal{K}}$ the initial segment relation. Is $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ an AEC? What is $\mathrm{LS}(\mathcal{K})$?

(43) Let $\mathcal{K}$ be the class of well-ordered sets with $\mathrm{L}(\mathcal{K}) = \{<\}$ and $\preceq_{\mathcal{K}}$ the subset relation. Is $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ is an AEC. What is $\mathrm{LS}(\mathcal{K})$? What is $\lambda \mapsto \mathrm{I}(\lambda, \mathcal{K})$?

(44) Let $\mathcal{K}_{\mathrm{On}}$ be the class of ordinals with $\mathrm{L}(\mathcal{K}) = \{\in\}$ and $\langle \alpha, \in \rangle \preceq_{\mathcal{K}_{\mathrm{On}}} \langle \beta, \in \rangle$ defined as $\alpha \subseteq \beta$. Is $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ is an AEC?

(45) Let $\lambda = \aleph_\alpha$ and $\mathcal{K}$ the class of well-ordered sets of order type $\leq \lambda$ with $\mathrm{L}(\mathcal{K})$ containing a single binary relation $<$, i.e. $M \in \mathcal{K}$ iff $\langle |M|, <^M \rangle$ is well-ordered of order type $\leq \lambda$. Compute $\mathrm{I}(\aleph_\beta, \mathcal{K})$ for every $\beta \leq \alpha$.

(46) Let $\mathcal{K}$ be an abstract class. Suppose $M, N \in \mathcal{K}$. If g is a $\mathcal{K}$ embedding from M into N then there exists $N' \in \mathcal{K}$ such that $M \preceq_{\mathcal{K}} N'$ and there exists $\overline{g} : N' \cong N$ extending g.

(47) Let $\mathcal{K}$ be an abstract class. Suppose $M, N \in \mathcal{K}$ such that $f : M \cong N$. For every $M' \in \mathcal{K}$ if $M \prec M'$ then there are $N' \in \mathcal{K}$ and $g : M' \cong N'$ such that $g \supseteq f$ and $N' \succ_{\mathcal{K}} N$.

(48) Let $\mathcal{K}$ be an abstract class.
Prove Lemma **??**.

(49) Let T be a first-order theory and suppose that $\mathcal{K}_T := \langle \mathrm{Mod}(T), \leq \rangle$ is the induced AEC by T, i.e. $\leq$ stands for sub model. Show that if $\mathcal{K}_T$ has the JEP then $\mathcal{K}_T$ has the AP.

(50) Let T be a first-order theory and suppose that $\mathcal{K}_T := \langle \mathrm{Mod}(T), \leq \rangle$ is the induced AEC by T, i.e. $\leq$ stands for elementary sub model. Show that if $\mathcal{K}_T$ is categorical in some $\mu \geq \mathrm{LS}(\mathcal{K})$ then $\mathcal{K}_T$ has the AP.

(51) Let LF be the class of *locally finite groups*. I.e. a group G is an element of LF iff for every finite subset X of G the subgroup generated by X is finite. Prove that LF is an AEC with $\mathrm{LS}(LF) = \aleph_0$.

(52) For p a prime or zero, let $\mathcal{K}_p$ be the class of all algebraically closed fields of characteristics p with infinite transcendence degree. Verify that $\mathcal{K}_p$ is an AEC with $\mathrm{LS}(\mathcal{K}_p) = \aleph_0$ and is categorical in every $\lambda \geq \aleph_0$.

(53) Prove that the class $\mathcal{K}_p$ from the previous exercise is not elementary (i.e. there is no first-order T such that $\mathcal{K}_p = \mathrm{Mod}(T)$).

(54) Prove Lemma 2.15.

(55) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an AEC. Suppose $\lambda > \mathrm{LS}(\mathcal{K})$. For any $M \preceq_{\mathcal{K}} N$ both from $\mathcal{K}_\lambda$ there are $\{M_\alpha \mid \alpha < \lambda\}$, and $\{N_\alpha \mid \alpha < \lambda\}$ both $\preceq_{\mathcal{K}}$-increasing

and continuous chains of elements of $\mathcal{K}_{<\lambda}$ such that

$$M = \bigcup_{\alpha<\lambda} M_\alpha, \quad N = \bigcup_{\alpha<\lambda} N_\alpha \quad \text{and } M_\alpha \preceq_{\mathcal{K}} N_\alpha \text{ for all } \alpha < \lambda.$$

(56) Prove Theorem 2.25.

(57) Later (see Theorems 4.2.6 and 4.2.8) we will show that for AECs μ-JMP for some $\mu \geq \mathrm{LS}(\mathcal{K})$ is an assumption that is similar to completeness of a theory. The purpose of this exercise is to derive an analog for AECs of Vaught's Theorem 1.3.26.

Let $\mathcal{K}$ be an AEC. If $\mathcal{K}$ is categorical in some $\mu \geq \mathrm{LS}(\mathcal{K})$ then $\mathcal{K}$ has the λ-JMP for all λ satisfying $\mathrm{LS}(\mathcal{K}) \leq \lambda \leq \mu$.

(58) Let $L := \{E\}$ when E is a binary relation symbol. Let $\mathcal{K}$ be the class of all L-structures such that $M \in \mathcal{K} \iff |M|/E$ is finite. Show that $\mathcal{K}$ is not elementary.

(59) Let $L := \{E\}$ when E is a binary relation. Let $\mathcal{K}$ be the class of all L-structures such that $M \in \mathcal{K} \iff \langle |M|, E^M \rangle$ is directed connected graph. Show that $\mathcal{K}$ is not elementary.

3. Axiomatization of AECs and Kueker's theorem

A natural question is, can an AEC be axiomatized in a reasonable logic?

THEOREM 3.1 (Kueker). *Suppose $\mathcal{K}$ is an AEC with $\mu := \mathrm{LS}(\mathcal{K})$ and $M \in \mathcal{K}$. If N is an $\mathrm{L}(\mathcal{K})$-structure satisfying $N \equiv_{\infty,\mu^+} M$ then $N \in \mathcal{K}$.*

PROOF. Using the directed system theorem Theorem 2.14 it is sufficient to find $I \subseteq \mathcal{K}_\mu$ such that $N = \bigcup I$ and $\langle I, \leq_{\mathcal{K}} \rangle$ is directed.

Using Karp's Theorem 1.8.33 there $\mathcal{F}$ exists a family of partial isomorphism from M to N witnessing that $M \equiv_{\infty,\mu^+} N$. Let

$$I := \{f_0[M_0] \mid f_0 \in \mathcal{F}, M_0 \in \mathcal{K}_\mu, M_0 \leq_{\mathcal{K}} M\}.$$

Clearly it's enough to verify the following claims with similar arguments:

CLAIM 3.2. *For every $a \in |N|$ there exists $N_0 \in I$ such that $a \in |N_0|$.*

CLAIM 3.3. *Given $N_1, N_2 \in I$ there exists $N^* \in I$ such that $N_\ell \leq_{\mathcal{K}} N^*$ for $\ell = 1, 2$.*

PROOF OF CLAIM 3.2. Let $a \in |N|$ be given. Using the backward property of $\mathcal{F}$ applied to $f = \emptyset$ and a, there exists $g \in \mathcal{F}$ such that $a \in \mathrm{Range}(g)$. Apply the LS axiom to find $M_0 \leq_{\mathcal{K}} M$ of cardinality μ containing $\mathrm{dom}\, g$. By the forward property of $\mathcal{F}$ there is $f^* \in \mathcal{F} \supseteq g$ such that $\mathrm{dom}\, f^* = |M_0|$. The model $N_0 := f^*[M_0]$ is as required. □

PROOF OF CLAIM 3.3. By induction on $k < \omega$ define two $\leq_{\mathcal{K}}$-increasing chains $\{N_\ell^k \mid k < \omega\} \in \mathcal{K}_\mu\}$ such that $N_\ell^0 = N_\ell$ for $\ell = 1, 2$ and for all $k < \omega$ we have that the universe of N_1^{k+1} contains $|N_2^k|$ as a subset and $|N_2^{k+1}| \supseteq |N_1^{k+1}|$.

Namely we have or every $k < \omega$ towers that mutually include each other

$$N_1^k \xrightarrow{\text{id}} N_1^{k+1} \xrightarrow{\text{id}} N_1^{k+2} \xrightarrow{\text{id}} \cdots$$

$$N_2^k \xrightarrow[\text{id}]{} N_2^{k+1} \xrightarrow[\text{id}]{} N_2^{k+2} \xrightarrow[\text{id}]{} \cdots$$

This is enough: Let $N_\ell^* := \bigcup_{k<\omega} N_\ell^k$, for $\ell = 1, 2$. Notice that by the last two requirements on the chains $N^* := N_1^* = N_2^*$. Thus N^* is as required.

For the inductive step, it suffices to show that given $N_0 \in I$ of cardinality μ and a subset A of N of cardinality μ there exists $N_1 \in I$ such that $|N_1| \supseteq A$ and $N_1 \geq_{\mathcal{K}} N_0$.

□

□

?!

Probably ? we can prove

THEOREM 3.4. *Suppose $\mathcal{K}$ is an AEC with $\mu := \mathrm{LS}(\mathcal{K})$ and $M \in \mathcal{K}$. If N is an $\mathrm{L}(\mathcal{K})$-structure satisfying $N \leq_{\infty,\mu^+} M$ then $N \leq_{\mathcal{K}} M$.*

Shelah and Villaveces in a recent paper **??** claim to have proven that

THEOREM 3.5. *Suppose $\mathcal{K}$ is an AEC with $\mu := \mathrm{LS}(\mathcal{K})$. There exists a sentence $\psi_K \in L_{,\mu^+}$ such that $\mathcal{K} = \mathrm{Mod}(\psi_{\mathcal{K}})$, where $\lambda = \beth_2(\mu)^{+++}$.*

4. The major open questions concerning AEC

Categoricity, eventual categoricity and computing the Hanf number for categoricity.

They belong into two families: Existence and categoricity.

Due to the LS axiom for any $\lambda \geq \mu \geq \mathrm{LS}(\mathcal{K})$, trivially we have that $\mathcal{K}_\lambda \neq \emptyset \implies K_\mu \neq \emptyset$. The implication $\mathcal{K}_\mu \neq \emptyset \implies K_\lambda \neq \emptyset$ is the interesting one. Due to its difficulty we focus in:

Does $\mathcal{K}_\lambda \neq \emptyset \implies \mathcal{K}_{\lambda^+} \neq \emptyset$ for $\lambda \geq \mathrm{LS}(\mathcal{K})$?

It is easy to see that in general the answer is no. It turns out that for some λ and certain families of AECs we have the following implication: $\mathrm{I}(\lambda, \mathcal{K}) = \mathrm{I}(\lambda^+, \mathcal{K}) = 1 \implies \mathcal{K}_{\lambda^{++}} \neq \emptyset$.

This brings us to:

CONJECTURE 4.1. *Let $\mathcal{K}$ be AEC with $\lambda \geq \mathrm{LS}(\mathcal{K})$.*
If $\mathrm{I}(\lambda, \mathcal{K}) = \mathrm{I}(\lambda^+, \mathcal{K}) = 1$ then $\mathcal{K}_{\lambda^{++}} \neq \emptyset$.

The case when $\lambda = \aleph_0$ and $\mathcal{K}$ has an axiomatization by an $L_{\omega_1,\omega}(\mathbf{Q})$-sentence in a countable language is a theorem of Shelah (see Theorem 8.1), however for $\lambda > \aleph_0$ it is a very hard problem. Attempts on this problem are responsible to most of the important developments in the subject in the last 20+ years.

While the conjecture is stated as an assertion in ZFC, taking into consideration the present state of our knowledge concerning it, it would be a fantastic result to have a proof assuming $\mathrm{ZFC} + \mathbf{V} = \mathbf{L}$.

Perhaps even the stronger statement is true:

PROBLEM 4.2. *Let $\mathcal{K}$ be AEC with $\lambda \geq \mathrm{LS}(\mathcal{K})$.*
If $\mathrm{I}(\lambda, \mathcal{K}) = 1$ and $1 \leq \mathrm{I}(\lambda^+, \mathcal{K}) < 2^{\lambda^+}$ then $\mathcal{K}_{\lambda^{++}} \neq \emptyset$.

Also in this problem the case when $\lambda = \aleph_0$ and $\mathcal{K}$ has an axiomatization by an $L_{\omega_1,\omega}(\mathbf{Q})$-sentence in a countable language is a theorem of Shelah. However the only known proof is in a slight extension of ZFC, the theory $\mathrm{ZFC} + 2^{\aleph_0} < 2^{\aleph_1}$. A ZFC proof is not known. A proof for λ uncountable even from $\mathrm{ZFC} + \mathbf{V} = \mathbf{L}$ is science fiction.

Later we will prove

THEOREM 4.3 (Morley, Shelah). *Let T be a first-order theory and $\mu := |\mathrm{L}(T)| + \aleph_0$. If T is categorical in some $\lambda_0 > \mu$ then T is categorical in every $\lambda > \mu$.*

CONJECTURE 4.4 (Shelah's categoricity conjecture). *Let $\mathcal{K}$ be an AEC and $\mu_0 := \beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$. If there exists $\lambda_0 \geq \mu_0$ such that $\mathcal{K}$ is categorical in λ_0 then $\mathcal{K}$ is categorical in every $\lambda \geq \mu_0$.*

Shelah's categoricity conjecture for AECs and the more specialized version for $L_{\omega_1,\omega}$ are among the most studied conjectures in model theory, with thousands of pages of partial solutions.

The following much easier conjecture is also open:

CONJECTURE 4.5 (The eventual categoricity conjecture). *For every μ there exists a cardinal μ_0 such that for every AEC $\mathcal{K}$ with $\mathrm{LS}(\mathcal{K}) \leq \mu$ if there exists $\lambda_0 \geq \mu_0$ such that $\mathcal{K}$ is categorical in λ_0 then $\mathcal{K}$ is categorical in every $\lambda \geq \mu_0$.*

When $\mathcal{K}$ has an axiomatization by a sentence in $L_{\omega_1,\omega}$ in a countable language we get a stronger bound on μ_0:

CONJECTURE 4.6 (Shelah's categoricity conjecture for $L_{\omega_1,\omega}$). *Let ψ be a sentence in $L_{\omega_1,\omega}$ in at most countable language. If there exists $\lambda_0 \geq \beth_{\omega_1}$ such that $\mathcal{K}$ is categorical in λ_0 then $\mathcal{K}$ is categorical in every $\lambda \geq \beth_{\omega_1}$.*

This conjecture is the oldest in the field, first stated by Shelah in the summer of 1976. At present we seems to be far from being able to verify this conjecture.

While I believe in the eventual categoricity conjecture, in my opinion it is quit plausible that Conjectures 4.4 and 4.6 are not correct and the threshold for categoricity is perhaps a much larger cardinal. The first $\mu_0 > \mathrm{LS}(\mathcal{K})$ such that $\mu_0 = \beth_{\mu_0}$ seems to be a more natural candidate for the threshold than the Hanf number of $\mathcal{K}$.

Probably if there is a counterexample to categoricity, it should be relatively easy to construct it, but it is very hard to find it.

The effect of the AP, DAP and related properties on the existence problem. Suppose for simplicity that $\lambda > \mathrm{LS}(\mathcal{K})$ is a successor cardinal, namely there exists $\mu \geq \mathrm{LS}(\mathcal{K})$ such that $\lambda = \mu^+$. Below I describe properties of $\mathcal{K}_\lambda$ sufficient to show that $\mathcal{K}_{\lambda^+} \neq \emptyset$.

Using the first chain axiom we can establish

PROPOSITION 4.7. *Let $\mathcal{K}$ be a AEC, $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}_\lambda$ has* No Maximal model*, namely for every $M \in \mathcal{K}_\lambda$ there exists $N \in K_\lambda$ satisfying $N \geq_{\mathcal{K}} M$ and $N \neq M$ then $\mathcal{K}_{\lambda^+} \neq \emptyset$.*

PROOF. Exercise. □

DEFINITION 4.8 (The (λ^+, λ)-existence property). Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. We say that $\mathcal{K}$ has the (λ^+, λ)-*existence property* provided for every $M_0, N \in \mathcal{K}_\lambda$ such that $M_0 \leq_{\mathcal{K}} N$ and every $M \in \mathcal{K}_{\lambda^+}$ satisfying $M \succ_{\mathcal{K}} M_0$ there are $N_0 \in \mathcal{K}$ and $f : N \cong N_0$ such that $N_0 \cap M = M_0$ and $f \restriction M_0 = \mathrm{id}_{M_0}$.

It turns out that this existence property is strong enough to settle Conjecture XX (under the λ-AP).? ?!

λ-DAP ? ?!

THEOREM 4.9. *Suppose an AEC $\mathcal{K}$ has the (λ^+, λ)-existence property and has the λ-AP. If $\mathcal{K}_{\lambda^+} \neq \emptyset$ then $\mathcal{K}_{\lambda^{++}} \neq \emptyset$.*

LEMMA 4.10. *Let $\mathcal{K}$ be an Abstract Class. Suppose $M, N, M_0, N_0 \in \mathcal{K}$ such that $M_0 \leq_{\mathcal{K}} M$, $N_0 \leq_{\mathcal{K}} N$ and suppose $f : M \to N$ is a $\mathcal{K}$-embedding. If $M \cap N_0 = \emptyset$ then Without loss of generality we may assume that $f = \mathrm{id}_M$. Namely if the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_0 & \xrightarrow[\mathrm{id}]{} & N_0 \end{array}$$

commutes then there exists $N' \cong N$ such that

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{id}} & N' \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_0 & \xrightarrow[\mathrm{id}]{} & N_0 \end{array}$$

commutes.

PROOF. Exercise. □

PROOF. By the first chain axiom it is enough to define a $\leq_{\mathcal{K}}$-increasing chain $\{M^\alpha \mid \alpha < \lambda^{++}\} \subseteq \mathcal{K}_{\lambda^+}$. We carry out the construction by induction on $\alpha < \lambda^{++}$:

For $\alpha = 0$; Let $M^0 := M$.

If α is limit let $M^\alpha := \bigcup_{\beta<\alpha} M^\beta$.

If $\alpha = \beta + 1$; Fix $\{M_i^\beta \mid i < \lambda^+\}$ be a continuous resolution of M^β. Now by induction on $i < \lambda^+$ define a $\prec_{|K}$-increasing $\{N_i \mid i < \lambda^+\} \subseteq \mathcal{K}_\lambda$ satisfying for all $i < \lambda^+$

(1) $M_i^\beta \leq_{\mathcal{K}} N_i$,
(2) $M^\beta \cap N_i = M_i^\beta$,

Clearly this is enough. Take $N := \bigcup_{i<\lambda^+} N_i$. As $M_0^\beta \subsetneq N_0$ we get $M^\beta \leq_{\mathcal{K}} N$.

For $i = 0$ let $N' \leq_{\mathcal{K}}$ of cardinality $\mathcal{K}_\lambda$ such that $M_0^\beta \leq_{\mathcal{K}} N$, by the (λ^+, λ)-existence property there exists $N_0 \in \mathcal{K}_\lambda$ such that $N_0 \geq_{\mathcal{K}} M_0^\beta$ and $M \cap N_0 = \emptyset$.

For i limit, take $N_i := \bigcup_{j<i} N_j$.

If $i = j + 1$, apply the λ-AP to get $\square$

. . . .

3 & 4 DIMENSIONAL EXISTENCE PROPERTIES

$\mathcal{K}_{\lambda^{+++}} \neq \emptyset$.

Exercises 2.3.

(1)

(2) The purpose of this exercise is to connect properties of models of cardinality λ with the issue of existence of model of cardinality λ^+.

Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. Suppose that $\mathcal{K}$ has the λ-DAP. Prove: If there are $M \precneqq_{\mathcal{K}} N$ both elements of $\mathcal{K}_\lambda$ then $\mathcal{K}_{\lambda^+} \neq \emptyset$.

(3) The purpose of this exercise is to connect properties of models of cardinality λ with the issue of existence of model of cardinality λ^+.

?Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. Suppose that $\mathcal{K}$ has the λ-DAP. Prove: If there are $M \precneqq_{\mathcal{K}} N$ both elements of $\mathcal{K}_\lambda$ then $\mathcal{K}_{\lambda^+} \neq \emptyset$. ?!

5. Shelah's presentation Theorem and 2-categories

We will establish here an extension of Chang's presentation Theorem **??** to AECs.

THEOREM 5.1 (Shelah's presentation theorem). *Every AEC $\mathcal{K}$ with $\chi = \mathrm{LS}(\mathcal{K})$ is a $\mathrm{PC}_{\chi,2^\chi}$-class. Moreover there exists L_1 an expansion of $\mathrm{L}(\mathcal{K})$ of cardinality χ, a first-order theory T_1 in L_1 and a set of L_1-types Γ of cardinality $\leq 2^\chi$ such that $\mathcal{K} = \mathrm{PC}(T_1, \Gamma, \mathrm{L}(\mathcal{K}))$ and for any $M_1 \in \mathrm{EC}(T_1, \Gamma)$, if $N_1 \subseteq M_1$ then $N_1 \restriction \mathrm{L}(\mathcal{K}) \preceq_{\mathcal{K}} M_1 \restriction \mathrm{L}(\mathcal{K})$.*

Moreover, for an AEC $\mathcal{K}$ let T_1 and Γ be as above then the relation $\preceq_{\mathcal{K}}$ is also $\mathrm{PC}_{\chi,2^\chi}$. Every $M \in \mathcal{K}$ has an expansion $M_1 \models T_1$ such that If N_1 is an $\mathrm{L}(T_1)$-structure and N_1 is a submodel of M_1 then $N \preceq_{\mathcal{K}} M$ where $N := N_1 \restriction \mathrm{L}(\mathcal{K})$.

Here are two unexpected applications that demonstrate the power of Shelah's presentation theorem.

COROLLARY 5.2. *For any infinite cardinal χ the class*

$$\mathcal{F} := \{\mathcal{K} \mid \mathcal{K} \text{ is an AEC with } \mathrm{LS}(\mathcal{K}) \leq \chi\}$$

is a set. Moreover its cardinality is $\leq 2^{(2^\chi)}$

PROOF. Given an AEC $\mathcal{K}$ with $\mathrm{LS}(\mathcal{K}) = \chi$, it is $\mathrm{PC}_{\chi,2^\chi}$. As $|\mathrm{L}(\mathcal{K})| \leq \chi$ there at most χ-many language of cardinality $\leq \chi$, there are at most 2^χ-many theories in a language L where $|L| \leq \chi$. Now given a first-order theory T in L there at most $2^{(2^\chi)}$ possible sets of L-formulas Γ. Thus $|\mathcal{F}| \leq \chi \cdot 2^\chi \cdot 2^{(2^\chi)} = 2^{(2^\chi)}$. □

COROLLARY 5.3 (Hanf). *For every χ there exists a cardinal $\mu(\chi)$ such that if $\mathcal{K}$ is an AEC with $\mathrm{LS}(\mathcal{K}) \leq \chi$ and $\mathcal{K}_{\mu(\chi)} \neq \emptyset$ then $\mathcal{K}_\lambda \neq \emptyset$ for every $\lambda \geq \mathrm{LS}(\mathcal{K})$. We have that $\mu(\chi) \leq \beth_{(2^\chi)^+}$.*

PROOF. Use Shelah's presentation Theorem together with Theorems 7.35 and 7.31, notice that we are using here that $\delta(\chi, 2^\chi) \leq (2^\chi)^+$. □

NOTATION 5.4. Given a set S we use the following notation: For $\mathbf{a} \in {}^{\omega>}\mathrm{S}$ denote by $\cup\mathbf{a}$ the *set* of elements in the sequence $\mathbf{a}$. Let $\leq$ be the partial oder on ${}^{\omega>}\mathrm{S}$ defined as follows: Suppose $\mathbf{a} = \langle a_0, \ldots, a_{n-1}\rangle$ and $\mathbf{b} = \langle b_0, \ldots, b_{k-1}\rangle$. We write $\mathbf{a} \leq \mathbf{b}$ iff there exists a sequence $\langle i_0, \ldots, i_{n-1}\rangle$ of integers less than k such that $a_\ell = b_{i_\ell}$ for all $\ell < n$. Thus $\mathbf{a} \leq \mathbf{b}$ means that $\mathbf{a}$ is a subsequence of $\mathbf{b}$ (but not necessarily an initial segment).

Note that $\mathbf{a} \leq \mathbf{b}$ implies $\cup\mathbf{a} \subseteq \cup\mathbf{b}$.

The proof involves a generalization of the notion of L-structures. Given an L-structure M and a subset of the universe A. We consider $M_A := \langle A, R^M, F^M, c^M\rangle_{R,F,c\in L}$

while M_A may be not even closed under the functions it is an interesting object. In particular, given M_1 and M_2 both L-structures and $A_\ell \subseteq |M_\ell|$ the notion of an isomorphism from $M^1_{A_1}$ onto $M^2_{A_2}$ makes sense (= a bijection that presevers, constants, functions and relations).

PROOF OF THEOREM 5.1. Let

$$L_1 := \mathrm{L}(\mathcal{K}) \cup \{F^n_i(\mathbf{x}) \mid i < \chi, n < \omega, \ell(\mathbf{x}) = n\}.$$

The set $\{F^0_i \mid i < \chi\}$ is a set of individual constants.

Now we define a first-order theory T_1 in L_1 and a set of types Γ such that

$$\mathcal{K} = \mathrm{PC}(T_1, \Gamma, \mathrm{L}(\mathcal{K})).$$

Let

$$T_1 := \{\exists x[x = x]\} \cup \{\forall \mathbf{x} F^n_k(x_0, \dots, x_{n-1}) = x_k \mid k < n < \omega\}.$$

For $M_1 \models T_1$ and any $\mathbf{a} \in {}^{\omega>}|M_1|$ consider

$$A^{M_1}_{\mathbf{a}} := \{F^{M_1}(\mathbf{a}) \mid F \text{ is a function or constant symbol of } L_1\} \cup \{c^{M^1} \mid c \in \mathrm{L}(\mathcal{K})\}.$$

Let $N^{M_1}_{\mathbf{a}}$ be

$$\langle A^{M_1}_{\mathbf{a}}, F^n_i, R^{M_1}, G^{M_1}, c^{M_1}\rangle_{i<\chi, n<\omega, R, G, c \in \mathrm{L}(\mathcal{K})}$$

Notice that since the set $A^{M_1}_{\mathbf{a}}$ is perhaps not closed under the functions I don't claim that $N^{M_1}_{\mathbf{a}} \restriction \mathrm{L}(\mathcal{K}) \in \mathcal{K}$ or even that it is an L_1-structure. However this can happen.

When the identity of the ambient structure M_1 is clear we omit it and write $N_{\mathbf{a}}$ for $N^{M_1}_{\mathbf{a}} \restriction \mathrm{L}(\mathcal{K})$.

For $\mathbf{a} \in {}^{\omega>}|M_1|$ let $p^{M_1}_{\mathbf{a}}$ be the quantifier-free L_1-type of $\mathbf{a}$, namely

$$p^{M_1}_{\mathbf{a}} := \{\varphi(\mathbf{x}) \mid \varphi(\mathbf{x}) \text{ is a quantifier free formula in } L_1, M_1 \models \varphi[\mathbf{a}]\}$$

Clearly

CLAIM 5.5. *Let $M_\ell \models T_1$ for $\ell = 1, 2$ and $n < \omega$. For any $\mathbf{a} \in {}^n|M_1|$ and any $\mathbf{b} \in {}^n|M_2|$ the following are equivalent*

(1) $p^{M_1}_{\mathbf{a}} = p^{M_2}_{\mathbf{b}}$
(2) *there exists $f : N^{M_1}_{\mathbf{a}} \cong N^{M_2}_{\mathbf{b}}$ (it is bijection preserving L_1) such that $f(\mathbf{a}) = \mathbf{b}$.*

PROOF. (2) $\implies$ (1) is immediate.

For (1) $\implies$ (2): Given $a \in N^{M_1}_{\mathbf{a}}$ let $\mathbf{x}$ a sequence of variables of length $\ell(\mathbf{a})$ and an L_1-term $\tau(\mathbf{x})$ such that $a = \tau^{M_1}[\mathbf{a}]$. Define $f(a) := \tau^{M_2}[\mathbf{b}]$. Using $p^{M_1}_{\mathbf{a}} = p^{M_2}_{\mathbf{b}}$ show that f is a well defined isomorphism. $\square$

Let $N^1_{\mathbf{a}} := N^{M_1}_{\mathbf{a}} \restriction \mathrm{L}(\mathcal{K})$. We associate to M_1 a set of types

$$\Gamma^{M_1} := \bigcup_{\mathbf{a} \in {}^{\omega >}|M_1|} \{p^{M_1}_{\mathbf{a}} \ : \ N^1_{\mathbf{a}} \notin \mathcal{K} \text{ or } \exists \mathbf{b} < \mathbf{a} [N^1_{\mathbf{b}} \not\preceq_{\mathcal{K}} N^1_{\mathbf{a}}]\}.$$

The idea is as we will see below is that M_1 omits all the types from Γ^{M_1} iff $M_1 \restriction \mathrm{L}(\mathcal{K})$ is the union of the directed system $\{N^1_{\mathbf{a}} \mid \mathbf{a} \in {}^{\omega >}|M_1|\} \subseteq \mathcal{K}$.

Now let

$$\Gamma := \bigcup_{M_1 \models T_1} \Gamma^{M_1}.$$

Notice that in the definition even without a restriction on the cardinality of M_1 we know that of Γ is not a proper class. In fact since Γ is a set of quantifier-free types in the language L_1 we have that $|\Gamma| \leq 2^{|L_1|} = 2^{\chi}$. We left to show that $\mathcal{K} = \mathrm{PC}(T_1, \Gamma, \mathrm{L}(\mathcal{K}))$:

Suppose $M_1 \in \mathrm{EC}(T_1, \Gamma)$. For $\mathbf{a} \in {}^{\omega >}|M_1|$ as $p^{M_1}_{\mathbf{a}}$ is realized in M_1 (by $\mathbf{a}$) the type $p^{M_1}_{\mathbf{a}}$ can't be a member of Γ thus $N_{\mathbf{a}} \in \mathcal{K}$ and for every $\mathbf{b}$ such that $\mathbf{b} \leq \mathbf{a}$ and $N_{\mathbf{b}} \preceq_{\mathcal{K}} N_{\mathbf{a}}$. By the requirement on T_1 we have that $|N_{\mathbf{a}}| \supseteq \cup \mathbf{a}$, thus $M_1 \restriction \mathrm{L}(\mathcal{K})$ is the directed union of

$$\{N_{\mathbf{a}} \mid \mathbf{a} \in {}^{\omega >}|M_1|\}$$

we are done by Theorem 2.7.

Given $M \in \mathcal{K}$ we will define M_1 an expansion of M to L_1 by providing interpretations of the function symbols $\{F^n_i \mid i < \chi, n < \omega\}$.

We define a directed system $\{M_{\mathbf{a}} \ : \ \mathbf{a} \in {}^{\omega >}|M|\} \subseteq \mathcal{K}_{\chi}$ such that $|M_{\mathbf{a}}| \supseteq \cup \mathbf{a}$ for every $\mathbf{a} \in {}^{\omega >}|M|$ and $M = \bigcup M_{\mathbf{a}}$.

By induction on n: For every $\mathbf{a} \in {}^{n}|M|$ pick $M_{\mathbf{a}} \preceq_{\mathcal{K}} M$ of cardinality χ satisfying $\mathbf{b} \leq \mathbf{a} \implies M_{\mathbf{b}} \preceq_{\mathcal{K}} M_{\mathbf{a}}$.

$n = 0$: Fix $M_{\langle\rangle} \preceq_{\mathcal{K}} M$ be of cardinality χ and let $\{b_i \mid i < \chi\}$ be an enumeration of the universe of $M_{\langle\rangle}$. Let $F^0_i := b_i$ for all $i < \chi$.

$n \geq 1$: Denote by A the set

$$\cup \mathbf{a} \cup \bigcup_{\mathbf{b} \leq \mathbf{a}} M_{\mathbf{b}}.$$

Take $M_{\mathbf{a}} \preceq_{\mathcal{K}} M$ be of cardinality χ containing the set A. For $\mathbf{b} \leq \mathbf{a}$ as both $M_{\mathbf{b}}$ and $M_{\mathbf{a}}$ are $\preceq_{\mathcal{K}}$-substructures of M and $M_{\mathbf{b}} \subseteq M_{\mathbf{a}}$ by the coherence axiom we get that $M_{\mathbf{b}} \preceq_{\mathcal{K}} M_{\mathbf{a}}$.

Fix an enumeration $\langle b_i \mid i < \chi \rangle$ of $|M_{\mathbf{a}}|$ such that $\langle b_0, \ldots, b_{n-1} \rangle = \mathbf{a}$. Now for every $i < \chi$ let $F^n_i(\mathbf{a}) := b_i$. Let

$$M_1 := \langle M, F^n_i \rangle_{i < \chi, n < \omega}$$

Clearly by the choice of enumerations $M_1 \models T_1$. We left to show that M_1 omits all the types from Γ. Consider $\mathbf{a} \in |M_1|$ if there exists $M_2 \models T_1$ and $\mathbf{b} \in |M_2|$ such that $\mathbf{a} \models p^{M_2}_{\mathbf{b}}$ then by the definition of M_1 we have that $N^{M_1}_{\mathbf{a}} \restriction \mathrm{L}(\mathcal{K}) = M_{\mathbf{a}}$ and the Claim gives us that $N^{M_1}_{\mathbf{a}} \cong N^{M_2}_{\mathbf{b}}$. Since $N^{M_2}_{\mathbf{b}} \restriction \mathrm{L}(\mathcal{K}) \in \mathcal{K}$ we have to show that for every $\mathbf{c} < \mathbf{b}$ we have that $N^{M_2}_{\mathbf{c}} \prec_{\mathcal{K}} N^{M_2}_{\mathbf{b}}$: As $\mathbf{c} < \mathbf{b}$ and $p^{M_1}_{\mathbf{a}} = p^{M_2}_{\mathbf{b}}$ there exists $\mathbf{d} < \mathbf{a}$ such that $p^{M_1}_{\mathbf{d}} = p^{M_2}_{\mathbf{c}}$ another application of the Claim gives us that $M_{\mathbf{d}}$ and $N^{M_2}_{\mathbf{c}} \restriction \mathrm{L}(\mathcal{K})$ are isomorphic, now an application of the second isomorphism axiom to $M_{\mathbf{d}} \preceq_{\mathcal{K}} M_{\mathbf{a}}$ gives us that $N^{M_2}_{\mathbf{c}} \preceq_{\mathcal{K}} N^{M_2}_{\mathbf{b}}$.

Now to the moreover part.

Given L_1-structure N and M such that $N, M \in \mathrm{EC}(T_1, \Gamma)$ we establish that

$$N \subseteq M \implies N \restriction \mathrm{L}(\mathcal{K}) \leq_{\mathcal{K}} M \restriction \mathrm{L}(\mathcal{K}) :$$

Since M omits all types from Γ we have that $\{N^M_{\mathbf{a}} \mid \mathbf{a} \in I_M\} \subseteq \mathcal{K}$ and it is a directed system of models where $I_M := \{\cup \mathbf{a} \mid \mathbf{a} \in {}^{\omega >}|M|\}$, similarly let I_N be the collection of all finite subsets of $|N|$. Notice that $\langle I_N, \subseteq \rangle$ and $\langle I_M, \subseteq \rangle$ are directed sets satisfying $I_N \subseteq I_M$. As N is a sub model of M and both satisfying T_1 we get

$$N^M_{\mathbf{a}} = N^N_{\mathbf{a}} \in \mathcal{K} \text{ and } \mathbf{a} \subseteq N^M_{\mathbf{a}} \text{ for every } \mathbf{a} \in I_N.$$

By two applications of Theorem 2.7 we have that $\bigcup_{\mathbf{a} \in I_N} N^N_{\mathbf{a}} \in \mathcal{K}, \bigcup_{\mathbf{a} \in I_N} N^N_{\mathbf{a}} \leq_{\mathcal{K}} N$ and $\bigcup_{\mathbf{a} \in I_M} N^M_{\mathbf{a}} \in \mathcal{K}, \bigcup_{\mathbf{a} \in I_M} N^M_{\mathbf{a}} \leq_{\mathcal{K}} M$. We get $\mathbf{a} \in I_N \implies |N^M_{\mathbf{a}}| \subseteq |N|$, thus an application of Theorem 2.17 gives

$$N \restriction \mathrm{L}(\mathcal{K}) = \bigcup_{\mathbf{a} \in I_N} N^M_{\mathbf{a}} \leq_{\mathcal{K}} \bigcup_{\mathbf{a} \in I_M} N^M_{\mathbf{a}} = M \restriction \mathrm{L}(\mathcal{K}).$$

□

The following notion relates to Abstract Classes in an analogous way to 2-categories relates to categories. I will start with a brief informal discussion of 2-categories. Suppose $C := \langle \mathbf{Ob}, \mathbf{Mor} \rangle$ is a category. Already in the fifties it was observed that it is very interesting to consider also the class

$$C^2 := \{\langle \langle (O_1, O_2), \mathbf{Mor}(O_1, O_2) \rangle, \mathbf{Mor}^2 \rangle \mid O_\ell \in C\}.$$

C^2 is also a category with $\mathbf{Mor}^2$, the morphisms taking elements of $\mathbf{Mor}(O_1, O_2)$ to elements of $\mathbf{Mor}(O'_1, O'_2)$ where O'_ℓ are the images of O_ℓ with the additional natural requirements that everything is associaltiv and existence of identity mappings. More precisely, $f \in \mathbf{Mor}^2$ take a diagram $g : O_1 \to O_2$ to a diagram $f(g) : f(O_1) \to f(O_2)$.

While one would expect the following notion to be considered by Shelah, it was not and as far as I know it is new here.

DEFINITION 5.6 (The 2-AC of $\mathcal{K}$). Let $\mathcal{K}$ be an abstract class.

$$\mathcal{K}^2 := \{\langle |M|, |N|, F^M, R^M, c^M\rangle_{F,R,c\in \mathrm{L}(\mathcal{K})} \ : \ M, N \in \mathcal{K}, N \leq_{\mathcal{K}} M\}$$

while the language of $\mathcal{K}^2$ is $\mathrm{L}(\mathcal{K}) \cup \{P\}$ where P is a unary predicate. Where

$$\langle |M_1|, |N_1|, \ldots\rangle \preceq_2 \langle |M_2|, |N_2|, \ldots\rangle$$

iff $M_1 \preceq_{\mathcal{K}} M_2$ and $N_1 \preceq_{\mathcal{K}} N_2$.

The following basic observation is fundamental.

PROPOSITION 5.7. *If $\mathcal{K}$ is an AEC then $\langle \mathcal{K}^2, \leq_2\rangle$ is also an AEC with* $\mathrm{LS}(\mathcal{K}^2) = \mathrm{LS}(\mathcal{K})$.

PROOF. Exercise. □

Some of the deeper theorems of AECs depend on analyzing the $\mathcal{K}^2$ of a given AEC $\mathcal{K}$ and several higher-order categories induced by $\mathcal{K}$. First we establish a simple definability result extending Shelah's presentation theorem:

THEOREM 5.8 (Second presentation Theorem). *Let $\mathcal{K}$ be an AEC. There exists T_1 first-order in a language of cardinality* $\mathrm{LS}(\mathcal{K})$ *and a set of parameter-free of finitary T_1-types Γ such that* $|\Gamma| \leq 2^{\mathrm{LS}(\mathcal{K})}$ *and* $\mathcal{K}^2 = \mathrm{PC}(T_1, \Gamma, \mathrm{L}(\mathcal{K}^2))$.

Moreover for every $M, N \in \mathcal{K}$ such that $N \leq_{\mathcal{K}} M$ the element $(M, N) \in \mathcal{K}^2$ has an expansion (M^*, N^*) to $\mathrm{L}(T_1)$ *such that $(M^*, N^*) \models T_1$ and $N' \subseteq M'$ where N' (and M') are reducts of N^* and M^* to* $\mathrm{L}(T_1) - \{P\}$.

PROOF. Instead of using Theorem 5.1 we repeat the construction in its proof with little more care.

*** THE ARGUMENT IS REALLY AN EXERCISE, I PROBABLY WILL WRITE A COMPLETE PROOF. WHAT IS WRITTEN IS INCOMPLETE DETAILS WILL BE ADDED SOON.

Let

$$L_1 := \mathrm{L}(\mathcal{K}) \cup \{F^n_i(\mathbf{x}) \mid i < \chi, n < \omega, \ell(\mathbf{x}) = n\} \cup \{P\}.$$

The set $\{F^0_i \mid i < \chi\}$ is a set of individual constants and P is a unary predicate not in $\mathrm{L}(\mathcal{K})$.

Now we define a first-order theory T_1 in L_1 and a set of types Γ such that

$$\mathcal{K} = \mathrm{PC}(T_1, \Gamma, \mathrm{L}(\mathcal{K})).$$

Let

$$T_1 := \{\exists x[x = x]\} \cup \big\{\forall \mathbf{x} F^n_k(x_0, \ldots, x_{n-1}) = x_k \mid k < n < \omega\big\} \cup$$

$$\cup\big\{\forall \mathbf{x}[\bigwedge_{j<\ell(\mathbf{x})} P(x_j) \to P(F(\mathbf{x}))] \ : \ F(\mathbf{x}) \text{ is a function symbol of } L_1, j < \ell(\mathbf{x})\big\}.$$

For $M_1 \models T_1$ and any $\mathbf{a} \in {}^{\omega>}|M_1|$ let $A^{M_1}_{\mathbf{a}}$ and $A^{M_1}_{\mathbf{a}}$ be like in the previous proof.

Given $(M, N) \in \mathcal{K}^2$ we will define M_1 an expansion of M to L_1, such that $P^{M_1} = |N|$ by providing interpretations of the function symbols $\{F^n_i \mid i < \chi, n < \omega\}$.

We define a directed system $\{M_{\mathbf{a}} \leq_{\mathcal{K}} M \ : \ \mathbf{a} \in {}^{\omega>}|M|\} \subseteq \mathcal{K}_\chi$ such that $|M_{\mathbf{a}}| \supseteq \cup\mathbf{a}$ for every $\mathbf{a} \in {}^{\omega>}|M|$, with the additional requirement that $\mathbf{a} \in |N|$ implies $M_{\mathbf{a}} \leq_{\mathcal{K}} N$ this definition automatically gives us that $M = \bigcup_{\mathbf{a} \in {}^{\omega>}|M|} M_{\mathbf{a}}$ and $N = \bigcup_{\mathbf{a} \in {}^{\omega>}|N|} M_{\mathbf{a}}$ hold.

By induction on n: For every $\mathbf{a} \in {}^{n}|M|$ pick $M_{\mathbf{a}} \preceq_{\mathcal{K}} M$ of cardinality χ satisfying $\mathbf{b} \leq \mathbf{a} \implies M_{\mathbf{b}} \preceq_{\mathcal{K}} M_{\mathbf{a}}$.

$n = 0$: Fix $M_{\langle\rangle} \preceq_{\mathcal{K}} M$ be of cardinality χ and let $\{b_i \mid i < \chi\}$ be an enumeration of the universe of $M_{\langle\rangle}$. Let $F^0_i := b_i$ for all $i < \chi$.

$n \geq 1$: Denote by A the set

$$\cup\mathbf{a} \cup \bigcup_{\mathbf{b} \trianglelefteq \mathbf{a}} M_{\mathbf{b}}.$$

Take $M_{\mathbf{a}} \preceq_{\mathcal{K}} M$ be of cardinality χ containing the set A. For $\mathbf{b} \leq \mathbf{a}$ as both $M_{\mathbf{b}}$ and $M_{\mathbf{a}}$ are $\preceq_{\mathcal{K}}$-substructures of M and $M_{\mathbf{b}} \subseteq M_{\mathbf{a}}$ by the coherence axiom we get that $M_{\mathbf{b}} \preceq_{\mathcal{K}} M_{\mathbf{a}}$.

Fix an enumeration $\langle b_i \mid i < \chi\rangle$ of $|M_{\mathbf{a}}|$ such that $\langle b_0, \ldots, b_{n-1}\rangle = \mathbf{a}$. Now for every $i < \chi$ let $F^n_i(\mathbf{a}) := b_i$. Let

$$M_1 := \langle M, F^n_i\rangle_{i<\chi, n<\omega}$$

Clearly by the choice of enumerations $M_1 \models T_1$. We left to show that M_1 omits all the types from Γ. Consider $\mathbf{a} \in |M_1|$ if there exists $M_2 \models T_1$ and $\mathbf{b} \in |M_2|$ such that $\mathbf{a} \models p^{M_2}_{\mathbf{b}}$ then by the definition of M_1 we have that $N^{M_1}_{\mathbf{a}} \restriction \mathrm{L}(\mathcal{K}) = M_{\mathbf{a}}$ and the Claim gives us that $N^{M_1}_{\mathbf{a}} \cong N^{M_2}_{\mathbf{b}}$. Since $N^{M_2}_{\mathbf{b}} \restriction \mathrm{L}(\mathcal{K}) \in \mathcal{K}$ we have to show that for every $\mathbf{c} < \mathbf{b}$ we have that $N^{M_2}_{\mathbf{c}} \prec_{\mathcal{K}} N^{M_2}_{\mathbf{b}}$: As $\mathbf{c} < \mathbf{b}$ and $p^{M_1}_{\mathbf{a}} = p^{M_2}_{\mathbf{b}}$ there exists $\mathbf{d} < \mathbf{a}$ such that $p^{M_1}_{\mathbf{d}} = p^{M_2}_{\mathbf{c}}$ another application of the Claim gives us that $M_{\mathbf{a}}$ and $N^{M_2}_{\mathbf{c}} \restriction \mathrm{L}(\mathcal{K})$ are isomorphic, now an application of the second isomorphism theorem to $M_{\mathbf{d}} \preceq_{\mathcal{K}} M_{\mathbf{a}}$ gives us that $N^{M_2}_{\mathbf{c}} \preceq_{\mathcal{K}} N^{M_2}_{\mathbf{b}}$.

Now to the moreover part.

Given an L_1-structure N_1 we establish that

$$N_1 \subseteq M_1 \implies N_1 \restriction \mathrm{L}(\mathcal{K}) \preceq_{\mathcal{K}} M :$$

Since M_1 omits all types from Γ we have that $\{N^{M_1}_{\mathbf{a}} \mid \mathbf{a} \in I_M\} \subseteq \mathcal{K}$ and is a directed system of models where $I_M := \{\cup\mathbf{a} \mid \mathbf{a} \in {}^{\omega>}|M_1|\}$. Notice that

$\langle I_M, \subseteq \rangle$ is a directed set. As N_1 is a sub model of M_1 we get

$$N_{\mathbf{a}}^{M_1} = N_{\mathbf{a}}^{N_1} \in \mathcal{K} \text{ for every } \mathbf{a} \in I_N.$$

By Theorem 2.7 we have that $\bigcup_{\mathbf{a} \in I_N} N_{\mathbf{a}}^{M_1} \in \mathcal{K}$ and $\bigcup_{\mathbf{a} \in I_N} N_{\mathbf{a}}^{M_1} \preceq M$, we get $\mathbf{a} \in I_N \implies |N_{\mathbf{a}}^{M_1}| \subseteq |N_1|$, thus $N_1 \restriction \mathrm{L}(\mathcal{K}) = \bigcup_{\mathbf{a} \in I_N} N_{\mathbf{a}}^{M_1}$.

□

REMARK 5.9 (Second Shelah's presentation theorem). It is possible to derive a little weaker version of the moreover part directly from the first part of presentation theorem.

Every $\preceq_{\mathcal{K}}$ AEC with $\chi = \mathrm{LS}(\mathcal{K})$ is a $\mathrm{PC}_{\chi, 2^\chi}$-class.

PROOF. Let $L := \{P\}$, when P is a unary predicate, now consider a class of L-structures defined as

$$\mathcal{K}^* := \{\langle |M|, |N| \rangle \ : \ M, N \in \mathcal{K}, N \preceq_{\mathcal{K}} M\}.$$

For $\langle |M_1|, |N_1| \rangle, \ \langle |M_2|, |N_2| \rangle \in \mathcal{K}^*$ we define $\langle |M_1|, |N_1| \rangle \preceq_{\mathcal{K}^*} \langle |M_2|, |N_2| \rangle$ iff $M_1 \preceq_{\mathcal{K}} M_2$ and $N_1 \preceq_{\mathcal{K}} N_2$. Verify that $\langle \mathcal{K}^*, \preceq_{\mathcal{K}^*} \rangle$ is an AEC and $\mathrm{LS}(\mathcal{K}^*) = \mathrm{LS}(\mathcal{K})$. An application of the first presentation theorem gives the result. □

REMARK 5.10. In the special case where the AEC is axiomatized in infinitrary logic then Chang's presentation theorem gives more than Shelah's. Namely, when $T \subseteq L_{\lambda^+, \omega}$, its language of cardinality $\leq \lambda$ and L^* is a fragment of cardinality λ then the AEC $\langle \mathrm{Mod}(T), \preceq_{L^*} \rangle$ is by Chang's presentation theorem is $\mathrm{PC}_{\lambda, \lambda}$ and in fact by Theorem **??** is actually $\mathrm{PC}_{\lambda, 1}$, Shelah's presentation theorem gives us the weaker result that it is $\mathrm{PC}_{\aleph_0, 2^\lambda}$.

While every AEC is $\mathrm{PC}_{\lambda, \kappa}$ for some λ, κ, not every PC is AEC.

EXAMPLE 5.11. There is $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ which is $\mathrm{PC}_{\aleph_0, 0}$ but not AEC.

Let $L_1 := \{P, F\}$ where P is a unary predicate and F is a unary function symbol. Let $L := \{P\}$. The theory T_1 consists only of the two axiom: $\forall x [P(F(x)]$ and $\forall x \forall y [F(x) = F(y) \to x = y]$.

For $\alpha < \omega_1$ but $\alpha \geq \omega$ let $M_\alpha^1 := \langle \alpha, \omega, f_\alpha \rangle$ where f_α is an injective function from α onto ω. Let $M_\alpha := M_\alpha^1 \restriction L$. Define $M \preceq_{\mathcal{K}} N$ as M is a submodel of N.

Clearly $\{M_\alpha \mid \omega \leq \alpha < \omega_1\}$ is an increasing $\preceq_{\mathcal{K}}$-chain such that $\bigcup_{\alpha < \omega_1} M_\alpha \notin \mathcal{K}$.

While every AEC is a PC-class the converse is not true.

EXAMPLE 5.12. Let $L := \{P, <\}$ when P is unary and $<$ is a binary predicate, take $L_1 := \{P, <, F\}$ when F is a unary function symbol and $\Gamma = \emptyset$.

Let T_1 be the first-order theory that asserts that f is an injective function from the universe into P and $<$ is a linear order.

For any infinite cardinal λ consider $M^* := \langle \lambda^+, \lambda, \in \rangle$ as an L-structure what the set of ordinals λ is the interpretation of P and $\in$-the natural ordering of the ordinals interprets $<$.

Pick $\{M_\alpha \mid \alpha < \lambda^+\}$ an increasing and continuous elementary chain of submodels of M^* such that $\|M_\alpha\| = \lambda$ and $|M_\alpha| \supseteq \lambda \cup \alpha$ for all $\alpha < \lambda^+$.

For every $\alpha < \lambda$ pick a bijection $f_\alpha : |M_\alpha| \to \lambda$. Now define $N^*_\alpha := \langle M_\alpha, f_\alpha \rangle$ the expansion of M_α to L_1, i.e. $M_\alpha = N_\alpha \restriction \alpha$.

Clearly $\{M_\alpha \mid \alpha < \lambda^+\} \subseteq \mathrm{PC}(T_1, \emptyset, L)$ however M^* the union of this chain is not a member of $\mathrm{PC}(T_1, \emptyset, L)$.

I conclude this section with another remarkable result of Shelah that provides further definability information on $\leq_{\mathcal{K}}$ in the case $\mathcal{K}$ is categorical in some "nice" cardinal.

THEOREM 5.13. *Suppose an AEC $\mathcal{K}$ is λ-categorical in some λ satisfying $\lambda = \lambda^{\mathrm{LS}(\mathcal{K})}$ then $M \leq_{\mathcal{K}} N \implies M \prec_{\infty,\mathrm{LS}(\mathcal{K})^+} N$ for every $M, N \in \mathcal{K}_\lambda$.*

PROOF. We argue by induction on the quantifier depth of $\varphi(\mathbf{x}) \in L_{\infty,\mathrm{LS}(\mathcal{K})^+}$ that

$$(*) \quad M \models \varphi[\mathbf{a}] \iff N \models \varphi[\mathbf{a}] \text{ for all } \mathbf{a} \in |M|^{\leq \mathrm{LS}(\mathcal{K})}.$$

As $M \leq_{\mathcal{K}} N \implies M \subseteq N$ the statement $(*)$ holds for all quantifier free φ.

By the induction hypothesis it is enough to verify $(*)$ for $\varphi(\mathbf{x})$ of the form $\exists \mathbf{y} \psi(\mathbf{y}; \mathbf{x})$. As the direction $(\implies)$ follows also form the inductive hypothesis it is sufficient to establish the direction $(\impliedby)$ in $(*)$.

Suppose it fails, namely there are $\mathbf{a} \in |M|^{\leq \mathrm{LS}(\mathcal{K})}$ and $\mathbf{b} \in N$ such that

$$N \models \psi[\mathbf{b}; \mathbf{a}] \text{ and } M \models \neg \exists \mathbf{y} \psi(\mathbf{y}; \mathbf{a}).$$

Using λ-categoricity, by induction on $i < \lambda^+$ define an increasing continuous $\leq_{\mathcal{K}}$-chain $\{M_i \in \mathcal{K}_\lambda \mid i < \lambda^+\}$ and $f_i : M \cong M_i$, $f^*_i : N \cong M_{i+1}$ such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{f^*_i} & M_{i+1} \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M & \xrightarrow[f_i]{} & M_i \end{array}$$

commutes.

Consider $S = \{i < \lambda^+ \mid \operatorname{cf} i = \operatorname{LS}(\mathcal{K})^+\}$.

As $\lambda = \lambda^{\operatorname{LS}(\mathcal{K})}$ necessarily we have $\lambda > \operatorname{LS}(\mathcal{K})$ the set S is stationary.

$$F(i) := \min\{j < \lambda^+ \mid f_i(\mathbf{a}) \in M_j\}.$$

For $i \in S$ as $M_i = \bigcup_{j<i} M_j$, $\operatorname{cf} i = \operatorname{LS}(\mathcal{K})^+$ and $\ell(\mathbf{a}) \leq \operatorname{LS}(\mathcal{K})$ we get that F is regressive, namely $i \in S \implies F(i) < i$. By Fodor's lemma there exists a stationary $S^* \subseteq S$ and $j_0 < \lambda^+$ such that

$$i \in S^* \implies F(i) = j_0.$$

Since $|S^*| = \lambda^+$ and the cardinality of $|M_{j_0}|^{\leq \operatorname{LS}(\mathcal{K})}$ is $\lambda^{\operatorname{LS}(\mathcal{K})} = \lambda$ an application of the pigeon hole principle to the mapping $i \mapsto f_i(\mathbf{a})$ for $i \in S^*$ give $S^{**} \subseteq S^*$ of cardinality λ^+ and $\mathbf{a}^* \in M_{j_0}$ such that $i \in S^{**} \implies f_i(\mathbf{a}) = \mathbf{a}^*$.

Fix $i_1 < i_2 \in S^{**}$ such that $i_1 > j_0$.

By the definitions of M_{i_1} and $f^*_{i_1}$ and the assumptions on $\mathbf{a}$ and $\mathbf{b}$ we get

$$M_{i_1+1} \models \psi[f^*_{i_1}(\mathbf{b}); f^*_{i_1}(\mathbf{a})].$$

As $f^*_{i_1} \restriction M = f_{i_1} \restriction M$ and $\mathbf{a} \in M$ we get

$$M_{i_1+1} \models \psi[f^*_{i_1}(\mathbf{b}); f_{i_1}(\mathbf{a})].$$

Since $i_1 \in S^{**}$

$$(A) \quad M_{i_1+1} \models \psi[f^*_{i_1}(\mathbf{b}); \mathbf{a}^*].$$

Since $\mathbf{a}^*, f^*_{i_1}(\mathbf{b}) \in M_{i_1+1}$ and $M_{i_2} \geq_{\mathcal{K}} M_{i_1+1}$ an application of the inductive hypothesis $(*)$ to (A) gives

$$M_{i_2} \models \psi[f^*_{i_1}(\mathbf{b}); \mathbf{a}^*].$$

As $f^*_{i_1}(\mathbf{b}) \in M_{i_1+1} \subseteq M_{i_2}$ we get

$$(B) \quad M_{i_2} \models \exists \mathbf{y} \psi[\mathbf{y}; \mathbf{a}^*].$$

Using that $f_{i_2}(\mathbf{a}) = \mathbf{a}^*$ and $f_{i_2} : M \cong M_{i_2}$ an application of $f^{-1}_{i_2}$ to (B) gives $M \models \exists \mathbf{y} \psi(\mathbf{y}; \mathbf{a})$ which is a contradiction to our assumption on ψ and $\mathbf{a}$. □

The idea used in the proof of Theorem 5.13 can be adapted to establish:

THEOREM 5.14. *Suppose an AEC $\mathcal{K}$ is λ-categorical in some λ satisfying $\lambda > \operatorname{LS}(\mathcal{K})$ then $M \leq_{\mathcal{K}} N \implies M \prec_{\infty,\omega} N$ for every $M, N \in \mathcal{K}_\lambda$.*

PROOF. Exercise. □

Exercises 2.5.

(1) If $\mathcal{K}$ is an AEC then also $\mathcal{K}^* := \{\langle |M|, |N| \rangle \ : \ M, N \in \mathcal{K}, N \preceq_{\mathcal{K}} M\}$ is also an AEC where $\mathrm{L}(\mathcal{K}^*)$ consists of a single unary predicate and $\preceq_{\mathcal{K}^*}$ is defined by $\langle |M_1|, |N_1| \rangle \preceq_{\mathcal{K}^*} \langle |M_2|, |N_2| \rangle$ iff $M_1 \preceq_{\mathcal{K}} M_2$ and $N_1 \preceq_{\mathcal{K}} N_2$.

(2) Let $\mathcal{K}$ be an abstract class. Let

$$\mathcal{K}^2 := \{\langle |M|, |N|, F^M, R^M, c^M \rangle_{F,R,c \in \mathrm{L}(\mathcal{K})} \ : \ M, N \in \mathcal{K}, N \leq_{\mathcal{K}} M\}$$

while the language of $\mathcal{K}^{\leq}$ is $\mathrm{L}(\mathcal{K}) \cup \{P\}$ where P is unary predicate. Show that if $\mathcal{K}$ is an AEC then also $\mathcal{K}^2$ is an AEC, where

$$\langle |M_1|, |N_1|, \cdots \rangle \preceq_{\mathcal{K}^2} \langle |M_2|, |N_2|, \cdots \rangle$$

iff $M_1 \preceq_{\mathcal{K}} M_2$ and $N_1 \preceq_{\mathcal{K}} N_2$.

(3) Prove Theorem 5.14.

(4) Using the notation in the proof of Theorem 5.1 show that if $p^{M_1}_{\mathbf{a}} = p^{M_2}_{\mathbf{b}}$ then there exists $f : N^{M_1}_{\mathbf{a}} \cong N^{M_2}_{\mathbf{b}}$ such that $f(\mathbf{a}) = \mathbf{b}$.

(5)

(6) Two AECs $\mathcal{K}_1$ and $\mathcal{K}_2$ are *equal* iff $\mathrm{L}(K_1) = \mathrm{L}(\mathcal{K}_2)$ and $M \in \mathcal{K}_1 \iff M \in \mathcal{K}_2$ for every $\mathrm{L}(K_1)$-structure M.

Use Shelah's presentation theorem to prove that there are exactly $2^{\mathrm{LS}(\mathcal{K})}$ abstract elementary classes.

(7)

6. Basic Examples

EXAMPLE 6.1. $\mathcal{K}$ AEC such that $\mathrm{LS}(\mathcal{K}) = \aleph_0, \mathrm{I}(\aleph_0, \mathcal{K}) = \aleph_1, \mathrm{I}(\aleph_1, \mathcal{K}) = \aleph_2$ and $\mathcal{K}_{\aleph_2} = \emptyset$.

Let $L(\mathcal{K}) = \{<\}$ when $<$ is a binary relation and $\mathcal{K}$ is the class of well-ordered sets of cardinality $\leq \aleph_1$ with $\preceq_{\mathcal{K}}$ the sub model relation.

EXAMPLE 6.2. $\mathcal{K}$ AEC such that $\mathrm{LS}(\mathcal{K}) = \aleph_0, \mathrm{I}(\aleph_0, \mathcal{K}) = \aleph_1, \mathrm{I}(\aleph_1, \mathcal{K}) = 1$ and $\mathcal{K}_{\aleph_2} = \emptyset$.

Let $L(\mathcal{K}) = \{<\}$ when $<$ is a binary relation and $\mathcal{K}$ is the class of well-ordered sets of order-type $\leq \omega_1$ with $\preceq_{\mathcal{K}}$ the sub model relation.

EXAMPLE 6.3. There is a $\mathrm{PC}_{\aleph_0,0}$ which is not an AEC.

Let $L_1 := \{P, F\}$ where P is a unary predicate and F is a unary function symbol. Let T_1 consists of the following two axioms

(1) $\forall x[P(F(x))]$ and
(2) $\forall x \forall y[F(x) = F(y) \to x = y]$.

Let $\mathrm{L}(\mathcal{K}) := \{P\}$ and $\preceq_{\mathcal{K}}$ be $\subseteq$. For every countable ordinal α fix a bijection $f : \alpha \to \omega$. For every such α let $M^1_\alpha := \langle \alpha, f_\alpha, \omega \rangle$ Clearly $M^1_\alpha \models T_1$. However $\{M^1_\alpha \restriction L ;: \ \omega \leq \alpha < \omega_1\}$ is a strictley increasing $\preceq_{\mathcal{K}}$-chain, its union is not an element of $\mathcal{K}$, this violates the first chain axiom in the definition of AEC.

EXAMPLE 6.4 (Coding the successor of a cardinal). Let $L_1 := \{P, <, F\} \cup \{c_i \mid i < \lambda\}$ where P is a unary predicate, $<$ a binary predicate, F a binary function symbol and c_i constants. Let T_1 be the theory

$$\{P(c_i) \mid i < \lambda\} \cup \{c_i \neq c_j \mid i \neq j < \lambda\}$$
$$< \text{ is a linear order,}$$
$$\exists x[\neg P(x)] \wedge \forall x \exists y[y > x]$$
$$\forall x \forall y[y < x \to P(F(x, y))]$$
$$\forall x \forall y_1 \forall y_2[y_1 < x \wedge y_2 < x \wedge F(x, y_1) = F(x, y_2) \to y_1 = y_2].$$

Let $q(x) := \{P(x) \wedge x \neq c_i \mid i < \lambda\}$. Let $L := \{P\}$ and $\mathcal{K} := \mathrm{PC}(T_1, \{q\}, L)$.

Notice that if $M \models T_1$ omitting the type q then $P^M = \{c_i^M \mid i < \lambda\}$. The standard model of T_1 has universe λ^+ the order is $\in$ and P is interpreted by λ.

PROPOSITION 6.5. *If $M \models T_1$ then $\|M\| \leq |P^M|^+$.*

PROOF. Exercise. □

Notice that this allows us to have a sentence $\psi \in L_{\lambda^+,\omega}$ in L_1 with the property $M \models \psi \implies \lambda \leq \|M\| \leq \lambda^+$:

$$\psi := \bigwedge T_1 \wedge \neg\exists x \bigwedge q.$$

PROPOSITION 6.6. (1) *Suppose* $\lambda = \aleph_0$. *We have* $I(\aleph_0, \mathcal{K}) = I(\aleph_1, \mathcal{K}) = 1$ *and* $I(\aleph_2, \mathcal{K}) = 0$.
(2) $I(\mu, \psi) = 2^\mu$ *for all* $\mu > \lambda$.
(3)

PROOF. The first part, easy exercise. For the second part use Proposition 1.2.62. □

Check that we have that $\mathrm{I}(\lambda, \mathcal{K}) = \mathrm{I}(\lambda^+, \mathcal{K}) = 1$ but $\mathcal{K}_{\lambda^{++}} = \emptyset$.

Compare this with conjecture **??**.

The idea in the previous example can be iterated to get

THEOREM 6.7. *For any* $\alpha < \lambda^+$ *there exists a class* $\mathcal{K}^\alpha$ *which is* $\mathrm{PC}_{\lambda,1}$ *such that* $\mathcal{K}^\alpha_\mu \neq \emptyset$ *for every* $\mu \leq \lambda^{+\alpha}$ *but* $K^\alpha_{\lambda^{+\alpha+1}} = \emptyset$.

PROOF. Exercise. □

EXAMPLE 6.8 (Coding the power set). Let P and Q be unary predicates, E a binary predicate, and λ a fixed infinite cardinal. Let

$$L_1 := \{P, Q, E\} \cup \{c_i \mid \ i < \lambda\}.$$

Let T be a first-order theory in L_1 consisting of the following axioms:

(1) $\forall x[P(x) \vee Q(x)]$,
(2) $\neg\exists x[P(x) \wedge Q(x)]$,
(3) $\forall x \forall y[xEy \to P(x) \wedge Q(y)]$,
(4) $\forall x[P(x) \to \exists y[Q(y) \wedge xEy \wedge \forall z[P(z) \wedge zEy \to z = x]]$,
(5) $\forall y_1 \forall y_2[Q(y_1) \wedge Q(y_2) \to [y_1 = y_2 \leftrightarrow \forall x[P(x) \to [xEy_1 \leftrightarrow xEy_2]]]$,
(6) $\{P(c_i) \mid i < \lambda\} \cup \{c_i \neq c_j \mid i \neq j < \lambda\}$.

The idea is that if $M \models T$ then its universe consists of two disjoint sets P^M the objects and its complement Q^M a collection of subsets of P^M where E is the membership relation. Notice that the fifth axiom is an extensionallity axiom for E.

Observe that

FACT 6.9. *If* $M \models T$ *then* $\|M\| \leq 2^{|P^M|}$.

and

FACT 6.10. *Let $L := \{P\}$. If $M, N \models T$ then*

$$M \restriction L \cong N \restriction L \iff |P^M| = |P^N| \text{ and } ||M| - P^M| = ||N| - P^N|.$$

Let $p(x)$ be the type $\{P(x) \wedge x \neq c_i \mid i < \lambda\}$. If $M \in \mathcal{K}(\lambda) := \mathrm{EC}(T, \{p\})$ then $P^M = \{c_i^M \mid i < \lambda\}$. Thus $\|M\| \leq 2^\lambda$.

Thus $\mathrm{I}(\mu, \mathcal{K}(\lambda)) = 1$ for every μ satisfying $\lambda \leq \mu \leq 2^\lambda$.

If T' and $\mathcal{K}'$ are exactly like in the above example, just axiom (4) is deleted (i.e. we don't require that all singletons are sets) then $\mathrm{I}(\mu, \mathcal{K}'(\lambda)) = 1$ for every μ satisfying $\lambda < \mu \leq 2^\lambda$. But when $\aleph_\alpha = \lambda$ then $\mathrm{I}(\lambda, \mathcal{K}'(\lambda)) = \aleph_0 + |\alpha|$.

EXAMPLE 6.11 (Silver 1969?). Let $L_1 := \{P, Q, E\}$ when P, Q are unary predicates and E is a binary predicate.

Consider T_1 to be the axioms (1) to (5) from Example 6.8.

The intuitive idea is that P is a set of urelements (atoms) and the universe of a structure consists of two kinds of elements, atoms and sets of atoms. Let $K_1 := \mathrm{Mod}(T)$.

Let $L := \{P\}$, i.e. L_1 is the expansion of L by the relations Q and E and denote by $\mathcal{K}$ the class $\mathrm{PC}(T_1, \emptyset, L)$.

For $M \in \mathcal{K}$, its isomorphism class is determined by the cardinalities of the sets P^M and $|M| - P^M$. By axioms (4) and (5) we have that the cardinality of $|M| - P^M$ is at least that of P^M. It is easy to show:

A corollary of Facts 6.9 and 6.10 is the following:

COROLLARY 6.12. *Let $M \in \mathcal{K}$ be infinite. $\|M\|$ is a strong limit cardinal iff $\mathcal{K}_\lambda$ is categorical. Thus the class $\mathcal{K}$ is categorical in λ iff λ is a strong limit cardinal.*

PROOF. Let λ be a strong limit. If $M \in \mathcal{K}_\lambda$ and $|P^M| < \lambda$ then by the extensionallity axiom $\|M\| \leq 2^{|P^M|}$, as $|P^M| < \lambda$ and λ strong limit we get that $\|M\| < \lambda$, thus whenever $M \in \mathcal{K}_\lambda$ necessarily also P^M has cardinality λ. Using axiom (4) we get that $|M| - P^M$ has cardinality λ. □

Thus Silver's example is a $\mathrm{PC}_{\aleph_0,0}$ that contradicts Shelah's eventual categoricity conjecture, namely that the conjecture is false for a very simple PC .

Example 6.8 can be iterated to get

THEOREM 6.13. *For every $\alpha < \lambda^+$ there exists a language of cardinality λ, a first-order theory T in L and a T-type p such that the class $\mathrm{EC}(T, p)$ has models in all $\mu \leq \beth_\alpha(\lambda)$ but no model of cardinality $> \beth_\alpha(\lambda)$.*

PROOF. Exercise. □

Examples 6.8 and 6.14 can be combined to produce a $\mathrm{PC}_{\lambda,1}$ class $\mathcal{K}$ such that $\mathcal{K}_{(2^\lambda)^+} \neq \emptyset$ but $\mathcal{K}_{(2^\lambda)^{++}} = \emptyset$.

EXAMPLE 6.14. There exists a class $\mathcal{K}$ which is $\mathrm{PC}_{\aleph_0,1}$ such that $\mathrm{I}(\aleph_0, \mathcal{K}) = \mathrm{I}(\aleph_1, \mathcal{K}) = 1$ and $\mathcal{K}_{\aleph_2} = \emptyset$.

Let $L_1 := \{P, N, F, G\} \cup \{c_n \mid n < \omega\}$ where P and N are unary predicates and F, G are unary function symbols. Let T_1 consists of the following axioms

(1) $\forall x[P(x) \to [N(G(x))]$,
(2)
(3) $\forall x[P(F(x))]$
(4) $\forall x[P(F(x))] \wedge \forall y[P(y) \to \exists x[F(x) = y]$and
(5) $\forall x \forall y[F(x) = F(y) \to x = y]$.

Let $\mathrm{L}(\mathcal{K}) := \{P\}$ and $\preceq_{\mathcal{K}}$ be $\subseteq$.

Is this class an AEC?

EXAMPLE 6.15.

EXAMPLE 6.16.

EXAMPLE 6.17.

EXAMPLE 6.18.

Exercises 2.6.

(1) Find an AEC $\mathcal{K}$ with $\mathrm{LS}(\mathcal{K}) = \aleph_0$ and $\mathrm{I}(\aleph_0, \mathcal{K}) = \aleph_0$ but $\mathcal{K}_{\aleph_1} = \emptyset$.
(2) For every $\alpha < \omega_1$ find $\mathcal{K}_\alpha$ which is a $\mathrm{PC}_{\aleph_0,0}$ such that there exists $\{Mi \mid i < \aleph_{\alpha+1}\} \subseteq \mathcal{K}$ which is $\preceq_{\mathcal{K}}$-increasing such that $\bigcup_{i<\aleph_{\alpha+1}} M_i \notin \mathcal{K}$.
(3) For every $\alpha < \omega_1$ there exist $\mathcal{K}_\alpha$ which is $\mathrm{PC}_{\aleph_0,1}$ such that $\mathrm{L}(\mathcal{K}) = \{P\}$ when P is a unary relation symbol, $\mathcal{K}_{\aleph_\alpha} \neq \emptyset$ but $\mathcal{K}_{\aleph_{\alpha+1}} = \emptyset$.
(4) For $\alpha < \omega_1$ there exist $\mathcal{K}^\alpha$ which is $\mathrm{PC}_{\aleph_0,1}$ such that $\mathrm{L}(\mathcal{K}^\alpha) = \{P\}$ when$P$ is a unary relation symbol, $\mathcal{K}^\alpha$ has a model of cardinality $\beth_\alpha$ but no models of cardinality $> \beth_\alpha$.
(5) Why the class $\mathcal{K}$ from Silver's example is not an AEC?
(6)
(7) For every $\lambda \geq \aleph_0$ define an AEC $\mathcal{K}$ such that $\mathrm{L}(\mathcal{K}) = \{P\}$, where P is a unary predicate, $\mathcal{K}$ is $\mathrm{PC}_{\lambda,1}$ and $M \models \mathcal{K} \implies \lambda \leq \|M\| \leq 2^\lambda$.
(8) For every $\alpha, \beta < \lambda$ construct a class $\mathcal{K}$ which is $\mathrm{PC}_{\lambda,1}$ such that for every $\lambda \leq \mu \leq (\beth_\alpha)^{+\beta}$ there exists $M \in \mathcal{K}_\mu$ but $\mathcal{K}$ has no larger models.

7. PC-classes and omitting types

In Theorem 1.5.22 we have seen that the notion of well-ordered set is not elementary (can not be captured by a first-order axiomatization). It turns out that even $L_{\lambda^+,\omega}$ is not strong enough to define well ordering. This fact is an important weak replacement for the role of the compactness theorem in first-order logic. In this section it will used to prove replacements of the upward Löwenheim-Skolem theorem, getting new results also for first-order theories.

Given a first order complete theory T and a set Γ of (not necessarily complete) *finitary types* in $\mathrm{L}(T)$ over the empty set, we let

$$\mathrm{EC}(T,\Gamma) = \{\, M \models T \mid M \text{ omits every type in } \Gamma \,\}.$$

We say that an $\mathrm{L}(T)$-type p is *finitary* iff there exists a finite sequence of variables $\mathbf{x}$ such that $\varphi \in p \implies \mathrm{FV}(\varphi) = \mathbf{x}$.

DISCUSS EXAMPLES, CONNECTIONS WITH $L_{\omega_1,\omega}$,

PROPOSITION 7.1. *Suppose $\mathcal{K}$ is a class such that $\mathcal{K} = \mathrm{EC}(T,\Gamma)$. If $\lambda \geq |L(\mathcal{K})| + \aleph_0$ then $\langle \mathcal{K}, \preceq \rangle$ is an AEC, where $\preceq$ is the usual elementary sub model relation.*

PROOF. Use the DLS and the TV chain theorems. Observe that in order to show that for any elementary chain $\{M_i \mid i < \alpha\} \subseteq \mathcal{K}$ also $\bigcup_{i<\alpha} M_i \in \mathcal{K}$ we use the assumption that Γ is a set of *finitary* types.

□

PROPOSITION 7.2. *Suppose $\mathcal{K}$ that $\mathcal{K} = \mathrm{EC}(T,\Gamma)$. If $\lambda \geq |\mathrm{L}(\mathcal{K})| + \aleph_0$ then there exists a sentence $\psi \in L_{\lambda^+,\omega}$ such that $\mathcal{K} = \mathrm{Mod}(\psi)$..*

PROOF. Take ψ to be the sentence

$$\bigwedge T \wedge \bigwedge_{p\in\Gamma} [\neg\exists \mathbf{x} \bigwedge_{\varphi(\mathbf{x})\in p} \varphi(\mathbf{x})].$$

□

DEFINITION 7.3. An abstract class $\mathcal{K}$ is called a PC-*class* iff there is a language $L_1 \supseteq \mathrm{L}(\mathcal{K})$, a theory T_1 and a set of finitary types Γ_1 in L_1 such that

$$\mathcal{K} := \{M \restriction \mathrm{L}(\mathcal{K}) \mid M \in \mathrm{EC}(T_1,\Gamma_1)\}.$$

$\mathcal{K}$ is a PC_λ if it is a PC-class and we have $|L_1| + |T_1| + |\Gamma| \leq \lambda$.

$\mathcal{K}$ is a $\mathrm{PC}_{\lambda,\mu}$ if there are $L_1 \supseteq \mathrm{L}(\mathcal{K})$ and T_1, Γ such that

$$\mathcal{K} := \{M \restriction \mathrm{L}(\mathcal{K}) \mid M \in \mathrm{EC}(T_1,\Gamma_1)\} \text{ and } \lambda \geq |L_1| + |T_1| + \aleph_0, \mu \geq |\Gamma|.$$

REMARK 7.4. Keisler's notation in his book [**Ke71**] for $\mathrm{PC}_{\aleph_0,\aleph_0}$-class is PC_δ.

The following (surprising) result show that in dealing with a large family of infinitary logics it suffices to deal with PC-clases.

THEOREM 7.5 (Chang's representation). *Let λ be an infinite cardinal and L a similarity type of cardinality $\leq \lambda$. If $T \subseteq L_{\lambda^+,\omega}$ of cardinality $\leq \lambda$ such that the similarity type has at most λ-many symbols then $\mathrm{Mod}(T)$ is a PC_λ-class.*

Moreover, if $L^* \subseteq L_{\lambda^+,\omega}$ is a fragment of cardinality λ containing T then there are $L_1 \supseteq L$, a first-order T_1 in L_1 with built in Skolem Functions and there exists Γ a set of finitary T_1-types such that

$\mathrm{Mod}(T) = \mathrm{PC}(T_1, \Gamma, L)$ *and if $M, N \models T_1$ s.t. they omit all types from Γ then*

$$M \subseteq N \implies M \restriction L \prec_{L^*} N \restriction L.$$

PROOF. W.l.o.g. we may assume that L^* contains no infinite conjunction just infinite disjunction (if $\bigwedge_{\psi\in\Phi} \psi(\mathbf{x}) \in L^*$ replace it with $\neg \bigvee_{\psi\in\Phi} \neg\psi(\mathbf{x})$).

For every $\varphi(\mathbf{x}) \in L^*$ let $R_\varphi(\mathbf{x})$ be a new relation symbol. Let $L_1 := L \cup \{R_\varphi(\mathbf{x}) \mid \varphi(\mathbf{x}) \in L^*\}$. Notice that the variables of the relation symbol $R_\varphi(\mathbf{x})$ are exactly the free variables of $\varphi(\mathbf{x})$, so when φ is a sentence then R_φ is a propositional variable.

For every $\varphi \in L^*$ if $\varphi(\mathbf{x}) = \bigvee_{\psi\in\Phi} \psi(\mathbf{x})$ let

$$p_\varphi(\mathbf{x}) := \{R_{\varphi(\mathbf{x})}(\mathbf{x}) \wedge \neg R_{\psi(\mathbf{x})}(\mathbf{x}) \mid \psi(\mathbf{x}) \in \Phi\}.$$

$$\begin{aligned}
T_1 :=& \{R_\psi \leftrightarrow \psi \mid \psi \in T\} \\
&\cup \{\forall \mathbf{x}[R_{\neg\psi}(\mathbf{x}) \leftrightarrow \neg R_\psi(\mathbf{x})] \mid \psi \in L^*\} \\
&\cup \{\forall \mathbf{x}[R_{\psi_1\vee\psi_2}(\mathbf{x}) \leftrightarrow [R_{\psi_1}(\mathbf{x}) \vee R_{\psi_2}(\mathbf{x})]] \mid \psi_1, \psi_2 \in L^*\} \\
&\cup \{\forall \mathbf{x}[[\mathrm{R}_{\exists y\varphi(y;\mathbf{x})}(\mathbf{x}) \leftrightarrow \exists y R_{\varphi(y;\mathbf{x})}(y, \mathbf{x})] \mid \varphi(y;\mathbf{x}) \in L^*\} \\
&\cup \{\forall \mathbf{x}[\psi(\mathbf{x}) \to \varphi(\mathbf{x})] \mid \text{ if there exists } \{\varphi\} \cup \Phi \subseteq L^* \text{ such that } \varphi(\mathbf{x}) = \bigvee_{\psi\in\Phi} \psi(\mathbf{x})\}.
\end{aligned}$$

Let $\Gamma := \{p_\varphi(\mathbf{x}) \mid$ if there exists $\{\varphi\} \cup \Phi \subseteq L^*$ such that $\varphi(\mathbf{x}) = \bigvee_{\psi\in\Phi} \psi(\mathbf{x})\}$. Check that $\mathrm{PC}(T_1, \Gamma, L)$ is as required.

For the moreover part. Expand further $\mathrm{L}(T_1)$ with Skolem Functions so that the new theory T_1 has built-in Skolem Functions, we are done by Remark 1.14.17. □

With little more coding it turns out that for any $T \subseteq L_{\lambda^+,\omega}$ such that $|T| + |\mathrm{L}(T)| \leq \lambda$ there exists a first-order T_1 in an expansion L_1 (of cardinality λ) and a T_1-type $q(y)$ such that

$$\mathrm{Mod}(T) = \mathrm{PC}(T_1, \{q\}, \mathrm{L}(T)).$$

This follows from Chang's representation theorem together with the following coding trick:

THEOREM 7.6 (Shelah's 1-type coding). *If $\mathcal{K}$ is $\mathrm{PC}_{\lambda,\lambda}$ then $\mathcal{K}_{\geq\lambda}$ has an axiomatization as a $\mathrm{PC}_{\lambda,1}$.*

PROOF. Suppose $K = \mathrm{PC}(T_1, \Gamma, L)$ when $|\mathrm{L}(T_1)| + |\Gamma| + \aleph_0 \leq \lambda$. We will introduce an expansion L^* of $\mathrm{L}(T_1)$ of cardinality λ, a theory T^* and a type q in L^* such that $\mathcal{K} = \mathrm{PC}(T^*, \{q\}, L)$.

Suppose

$$\Gamma = \{p_\alpha(\mathbf{x}_\alpha) \mid \alpha < \alpha_0 \leq \lambda\} \text{ and } p_\alpha(\mathbf{x}_\alpha) = \{\varphi^i_\alpha(\mathbf{x}_\alpha) \mid i < i_\alpha \leq \lambda\}.$$

Let $\{c^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\}$ be new constant symbols, Q and $\{P_\alpha \mid \alpha < \alpha_0\}$ new unary predicates and $\{F_\alpha(\mathbf{x}_\alpha) \mid \alpha < \alpha_0\}$ new function symbols. Let

$$L^* := \mathrm{L}(T_1) \cup \{c^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\} \cup \{Q\} \cup \{P_\alpha \mid \alpha < \alpha_0\} \cup \{F_\alpha(\mathbf{x}_\alpha) \mid \alpha < \alpha_0\}.$$

$$\begin{aligned} T^* := T \cup & \{Q(c^i_\alpha), P_\alpha(c^i_\alpha) \mid \alpha < \alpha_0, i < i_\alpha\} \\ & \cup \{\forall x[P_\alpha(x) \to Q(x)] \mid \alpha < \alpha_0\} \\ & \cup \{\forall x[P_\alpha(x) \to \neg P_\beta(x)] \mid \alpha \neq \beta < \alpha_0\} \\ & \cup \{\forall \mathbf{x}_\alpha[P_\alpha(F_\alpha(\mathbf{x}_\alpha)) \mid \alpha < \alpha_0\} \\ & \cup \{\forall \mathbf{x}_\alpha[F_\alpha(\mathbf{x}_\alpha) = c^i_\alpha \to \neg\varphi^i_\alpha(\mathbf{x}_\alpha)] \mid \alpha < \alpha_0, i < i_\alpha\}. \end{aligned}$$

Let $q(y) := \{Q(y) \wedge y \neq c^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\}$.

CLAIM 7.7. $K_{\geq\lambda} = \mathrm{PC}(T^*, \{q\}, L)$.

PROOF. Suppose $M \in \mathcal{K}_\lambda$. Fix a subset $\{a^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\}$ of M. Let $Q := \{a^i_\alpha \mid \alpha < \alpha_0, i < i_\alpha\}$, $P_\alpha := \{a^i_\alpha \mid i < i_\alpha\}$. Since M omits all the types from Γ. For every $\alpha < \alpha_0$ and every $\mathbf{a} \in {}^{\ell(\mathbf{x}_\alpha)}|M|$ there exists a minimal $i < i_\alpha$ such that $M \models \neg\varphi^i_\alpha[\mathbf{a}]$. Define $F_\alpha(\mathbf{a})$ as a^i_α. Clearly $\langle M, Q, P_\alpha, F_\alpha, a^i_\alpha\rangle_{\alpha<\alpha_0, i<i_0}$ is an element of $\mathrm{PC}(T^*, \{q\}, L)$.

Suppose $N \models T^*$ omitting the type q and we will show that $N \restriction \mathrm{L}(T_1)$ omits all the types from Γ namely $N \restriction L \in \mathcal{K}$.

Since N omits q we get that

$$Q^N = \{(c^i_\alpha)^N \mid \alpha < \alpha_0, i < i_0\}.$$

By the the first three groups of axioms we have that

$$Q^N = \bigcup_{\alpha<\alpha_0} P^N_\alpha,\ \alpha \neq \beta \implies P^N_\alpha \cap P^N_\beta = \emptyset$$

and $P^N_\alpha = \{(c^i_\alpha)^N \mid i < i_0\}$.

Now given $\alpha < \alpha_0$ let's show that N omits p_α: Suppose $\mathbf{a} \in {}^{\ell(\mathbf{x}_\alpha)}|N|$. By the forth group of axioms $F^N_\alpha(\mathbf{a}) \in P^N_\alpha$. Thus there exists $i < i_0$ such that $F^N_\alpha(\mathbf{a}) = (c^i_\alpha)^N$ and this by the fifth group of axioms imply that $N \models \neg\varphi^i_\alpha[\mathbf{a}]$ $\square$

$\square$

Note that the proof gives a stronger result:

THEOREM 7.8. *Suppose* $\mathcal{K} = \mathrm{PC}(T_1, \Gamma, L)$. *If* $\mu \geq |\mathrm{L}(T_1)| + \Gamma + \aleph_0$ *then there exits* $T^* \supseteq T_1$ *in an expansion of* $\mathrm{L}(T_1)$ *of cardinality* μ *and a type* q *such that* $\mathcal{K}$ *and* $\mathrm{PC}(T^*, \{q\}, L)$ *have the same structures in all cardinals* $\geq \mu$.

Note the remarkable feature of Theorem 7.6. Regardless how complex the types from Γ our new type q is very simple and is not influenced at all by Γ.

CONNECTION WITH SET THEORY

AND PROVE

In 1968 C. C. Chang proved the following very surprising theorem:

THEOREM 7.9 (Chang's presentation theorem). *If* T *is a theory in* $L_{\lambda^+,\omega}$ *of cardinality* $\leq \lambda$ *then* $\mathrm{Mod}(T)$ *is* PC_λ.

Later we will prove a much stronger result due to Shelah that asserts that any AEC is a PC-class. However we recommend to the reader to construct a direct proof for Chang's presentation theorem.

THEOREM 7.10 (Shelah's presentation theorem). *Let* $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *be an AEC and let* $\mu := \mathrm{LS}(\mathcal{K})$. *Then* $\mathcal{K}$ *is* $\mathrm{PC}_{\mu,2^\mu}$.

COROLLARY 7.11. *Let* $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *be an AEC. If* $\mathcal{K}_\lambda \neq \emptyset$ *for* $\lambda = \beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$ *then for all* χ, $\mathcal{K}_\chi \neq \emptyset$.

THEOREM 7.12. *Let* $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *be an AEC.*
$\mathcal{K}^* := \{(M, N) \mid N \prec_{\mathcal{K}} M; M, N \in \mathcal{K}\}$ *is a* 2^μ-PC *class.*

We will start assuming that our classes are not only AECs but also are $\aleph_0$-PC classes (both $\mathcal{K}$ and $\mathcal{K}^*$) and that $\mathrm{LS}(\mathcal{K}) = \aleph_0$.

Notice that for L countable, $T \subseteq L_{\omega_1,\omega}(\mathbf{Q})$ countable and $\mathcal{L}^*$ a countable fragment of $L_{\omega_1,\omega}(\mathbf{Q})$ containing T, $\langle \mathrm{Mod}(T), \prec_{\mathcal{L}^*} \rangle$ is an $\aleph_0$-PC class.

THEOREM 7.13. *If* $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *is an AEC, then there exists* $\mu \leq 2^{\mathrm{LS}(\mathcal{K})}$ *such that* $\mathcal{K}$ *is a* PC_μ *class.*

COROLLARY 7.14. *If* $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ *is an AEC, then* $\mathcal{K}$ *is a* $\mathrm{PC}_{2^{\mathrm{LS}(\mathcal{K})}}$ *class.*

7.1. Undefinability of well-order. Recall

DEFINITION 7.15. A linearly-ordered set $\langle P, <\rangle$ is *well-ordered* iff $\neg\exists\{a_n \mid n < \omega\} \subseteq P$ such that $a_{n+1} < a_n$ for all $n < \omega$.

One of the basic conesquences of the compactness theorem is that well-ordering can not be axiomatized in first-order logic. Theorem 1.5.22 implies:

FACT 7.16. *If α is an infinite ordinal and $M = \langle |M|, <, P, \ldots\rangle$ is some expansion of α (i.e. $P^M = \alpha$ and $<$ is the natural order on α) then there exists $N \equiv M$ such that $\langle P^N, <^N\rangle$ is not well-ordered.*

It is natural to ask:

QUESTION 7.17. Is well-ordering definable in $L_{\omega_1,\omega}$ or perhaps in $L_{\lambda^+,\omega}$? I.e. let α be an ordinal, M a structure such that $\mathrm{L}(M)$ contains a unary predicate P and a binary relation $<$ such that $\alpha \subseteq |M|$. Does the following implication hold:

$$\langle |M|, \alpha, \in, \ldots\rangle \equiv_{\lambda^+,\omega} N \implies \langle P^N, <^N\rangle \text{ is well-ordered.}$$

Notice that well-ordering is already definable in L_{ω_1,ω_1}. It is also easy to see that for any $\alpha < \lambda^+$ the order-type of α is definable using a sentence of $L_{\lambda^+,\omega}$ using α-many constants.

The first result of this nature:

THEOREM 7.18 (Lopez-Escobar 1965). *For an ordinal $\alpha \geq \omega_1$ there exists $N \equiv_{\omega_1,\omega} \langle |M|, \alpha, \in, \ldots\rangle$ such that $\langle \alpha^N, \in^N\rangle$ is not well-ordered.*

Theorem 7.22 is a generalization of this result. In order to introduce the generalized setting, we will consider several functions.

DEFINITION 7.19. Let κ and λ be cardinals, κ can be finite (even zero). Let

$$\delta(\lambda,\kappa) := \min\Big\{\delta \in \mathrm{On} \mid \text{ for every language } L \text{ of cardinality } \leq \lambda,$$
$$\text{containing the binary relation } < \text{ and a unary predicate } P,$$
$$T \text{ a first-order theory in } L,\ \Gamma \text{ a set of } T\text{-types of cardinality } \leq \kappa$$
$$\text{if there exists } M \in \mathrm{EC}(T,\Gamma) \text{ s.t. } \mathrm{otp}(P^M, <^M) \geq \delta \text{ then}$$
$$\text{there exists } N \in \mathrm{EC}(T,\Gamma) \text{ s.t. } \langle P^N, <^N\rangle \text{ is not well-ordered.}\Big\}$$

Denote by $\delta(\lambda)$ the ordinal $\delta(\lambda, 1)$.

A related cardinal-valued function is

DEFINITION 7.20. Let κ and λ be cardinals, κ can be finite (even zero). Let

$$\mu(\lambda,\kappa) := \min\Big\{\mu \in \mathrm{Card} \mid \text{ for every language } L \text{ of cardinality } \leq \lambda, \text{ any first-order theory } T \text{ in } L \text{ and for every set of types } \Gamma \text{ of cardinality } \leq \kappa \text{ if for every } \mu' < \mu \text{ there exists } M \in \mathrm{EC}(T,\Gamma) \text{ of cardinality } \geq \mu' \text{ then for every } \chi \geq \lambda + \aleph_0 \text{ there exists } N \in \mathrm{EC}(T,\Gamma) \text{ of cardinality } \geq \chi.\Big\}$$

Denote by $\mu(\lambda)$ the cardinal $\mu(\lambda,1)$ and let $\delta(\lambda)$ stand for $\delta(\lambda,1)$.

REMARK 7.21. By Theorem 7.6 we have that $1 \leq \kappa \leq \lambda \implies \delta(\lambda) = \delta(\lambda,\kappa)$ and $\mu(\lambda) = \mu(\lambda,\kappa)$.

Naturally for the above to make sense we need to argue that the definitions makes sense. We will prove the following fundamental result:

THEOREM 7.22 (Morley and Lopez-Escobar). $\delta(\aleph_0) = \omega_1$.

PROOF. . Let κ be such that $1 \leq \kappa\aleph_0$. For $\delta(\aleph_0,\kappa) \geq \alpha$ for every $\alpha < \omega_1$ see **??**.

Let $\mathcal{K}$ be a $\mathrm{PC}_{\aleph_0,\kappa}$. By Theorem 7.6 we may assume that $\mathcal{K}$ given by $\mathrm{PC}(T_1,\{q(y)\},L)$ and suppose $M_1 \models T_1$ omitting q such that there exists $\{a_\alpha \mid \alpha < \omega_1\} \subseteq P^{M_1}$ such that $\alpha < \beta < \omega_1 \implies M_1 \models a_\alpha < a_\beta$.

As every model can be expanded with built in Skolem Functions (see Theorem 1.14.13(2)) we may assume without of generality that T_1 has built in Skolem functions.

Suppose that $q(y) = \{\varphi_i(y) \mid i < i_0 \leq \omega\}$ and let $\{\tau^n_k(x_0,\dots,x_{n-1}) \mid k < \omega\}$ be enumeration of all the terms from $\mathrm{L}(T_1)$ in the variables $x_0,\dots,x_{n-1}$ such that always $n < k$. Let $\{c_n \mid n < \omega\}$ be new constants not in $\mathrm{L}(T_1)$. We will construct a function $i : \omega \times \omega \to i_0$ such that the theory. T^* defined as

$$T_1 \cup \{c_{n+1} < c_n \mid n < \omega\} \cup \{\neg\varphi_{i(n,k)}(\tau^n_k(c_0,\dots,c_{n-1})) \mid n,k < \omega\} \cup \{P(c_n) \mid n < \omega\}$$

is consistent.

This is enough: Let $N^* \models T^*$ and $N := \mathrm{EM}(\{c^{N^*}_n \mid N < \omega\}) \restriction L$. Check that N omits q and P^N not well-ordered.

The consistency: For every $\mathbf{a} \in |M_1|$ and $\tau(\mathbf{x})$ let

$$\xi(\mathbf{a},\tau) := \min\{\xi < i_0 \mid M_1 \models \neg\varphi_{\xi(\mathbf{a},\tau)}[\tau(\mathbf{a})]\}.$$

By induction on $n < \omega$ we define $S_n \subseteq \omega_1$ of cardinality $\aleph_1$ and $i < \alpha_i(n, n-1) < \cdots \alpha_i(n, 0) < \omega_1$ such that $S_{n+1} \subseteq S_n$ for all $n < \omega$ and

$$n < m,\ i \in S_{n+1} \text{ and } j \in S_m \implies$$
$$(*) \quad \left\langle \langle \ell, \tau^\ell_k, \xi(\mathbf{a}_n(i), \tau^\ell_k)\rangle \mid \ell, k \leq n \right\rangle = \left\langle \langle \ell, \tau^\ell_k, \xi(\mathbf{a}_m(j), \tau^\ell_k)\rangle \mid \ell, k \leq n \right\rangle$$

Where $\mathbf{a}_n(i) := \langle a_{\alpha_i(n,n-1)}, \ldots, a_{\alpha_i(n,0)}\rangle$ for $i \in S_n$.

For $n = 0$ let $S_0 := \omega_1$ and $\alpha_i(0) \in \omega_1$ such that $\alpha_i(0) > i$. Given S_n and $\{\langle \alpha_i(n, n-1), \ldots, \alpha_i(n, 0)\rangle \mid i \in S_n\}$ as above.

For every $i < \omega_1$ as S_n is unbounded let $j = j(i) \in S_n$ such that $j > i$. Define $\alpha_i(n+1, k) := \alpha_j(n, k)$ for all $k < n$ and $\alpha_i(n+1, n) := j$. For every $i \in S_n$ denote by $f_n(i)$ the left hand side of (*). As f_n is a function from S_n which is a set of cardinality $\aleph_1$ into a countable set (the set of all possible finite sequences of triples) there exists an uncountable $S_{n+1} \subseteq S_n$ such that $i, j \in S_{n+1} \implies f_n(i) = f_n(j)$.

For every $n < \omega$ pick $j_n \in S_n$. Let $i(n, k) = \xi(\mathbf{a}_n(j_n), \tau^n_k)$. Note that by (*) the definition of $i(n, k)$ does not depend on the particular choice of j_n.

Let T_0 be a finite subset of T^*, it is enough to show that there is an expansion of M_1 which is a model of T_0. Let $n < \omega$ be large enough so all the indices of c's and τ's in T_0 are $\leq n$. Verify that $\langle M_1, a_{\alpha_{j_n}(n,n-1)}, \ldots, a_{\alpha_{j_n}(n,0)}\rangle \models T_0$. □

THEOREM 7.23 (C. C. Chang). *If λ is a strong limit cardinal with cofinality $\aleph_0$ then $\delta(\lambda) = \lambda^+$.*

PROOF. Fix $\{\lambda_n \mid n < \omega\}$ be a sequence of cardinals such that $\lambda = \sum_{n<\omega} \lambda_n$ and $2^{\lambda_n} < \lambda_{n+1}$ for all $n < \omega$.

Alter the proof of the previous theorem to incorporate into the definition of $f_n(i)$ also the first n such that $\xi(\mathbf{a}, \tau) \in \lambda_n$. □

It turns out that the last result does not generalize to uncountable cofinality. The proof is the first instance when we use models of set theory to obtain result that is in model theory. For basic background consult section **??**.

THEOREM 7.24 (Shelah?). *If λ is of uncountable cofinality then $\delta(\lambda) > \lambda^+$.*

PROOF. Let β be a large enough limit so that $\lambda^+ \in \mathbf{V}_\beta$ and

$$\langle \mathbf{V}_\beta, \in\rangle \models \Big[(1) \quad \forall x[x \geq \lambda \wedge x < \lambda^+ \to \exists f : x \overset{1-1}{\to} \lambda] \wedge$$
$$(2) \quad \forall f[[f : \omega \to \lambda] \to \exists y[y < \lambda \wedge \forall n[n < \omega \to f(n) \in y]] \wedge$$
$$(3) \quad \forall x \forall f[f \text{ is a function } \wedge \operatorname{dom} f = x \to \exists z[z = \operatorname{range} f]] \wedge$$
$$(4) \quad [S \subseteq \lambda^+ \wedge [|S| < \lambda \to \exists g : S \overset{\text{order-preserving}}{\longrightarrow} \lambda]\Big].$$

Let $\mathfrak{B}_\lambda := \langle \mathbf{V}_\beta, \in, \lambda^+, \lambda, i\rangle_{i<\lambda}$, where λ and λ^+ are unary predicates and i individual constants.

Let $p := \{\lambda(x) \wedge x \neq i \mid i < \lambda\}$ and let $T := \operatorname{Th}(\mathfrak{B}_\lambda)$. Notice that $\mathfrak{B}_\lambda \in \operatorname{EC}(T, \{p\})$. It is enough to prove

CLAIM 7.25. $\mathfrak{B} \in \operatorname{EC}(T, \{p\}) \implies ((\lambda^+)^{\mathfrak{B}}, \in^{\mathfrak{B}}\rangle$ *is well-ordered.*

PROOF. Notice that since $\mathfrak{B}$ omits p and $\mathfrak{B} \models T$ we have automatically that $\langle \lambda^{\mathfrak{B}}, \in^{\mathfrak{B}}\rangle \cong \langle \lambda, \in\rangle$.

For sake of contradiction suppose that there exists $\{a_n \mid n < \omega\} \subseteq (\lambda^+)^{\mathfrak{B}}$ such that $\mathfrak{B} \models [a_{n+1} \in a_n]$ for every $n < \omega$. As $otp(\lambda^{\mathfrak{B}}, \in^{\mathfrak{B}}) = \lambda$ we have that $a_n \in (\lambda^+)^{\mathfrak{B}} - \lambda^{\mathfrak{B}}$ for all $n < \omega$.

By (1) there exists a function $f \in \mathfrak{B}$ such that

$$\mathfrak{B} \models [f : a_0 \overset{1-1}{\to} \lambda].$$

By (2) there exists $a^* \in \lambda^{\mathfrak{B}}$ such that

$$\mathfrak{B} \models [f(a_n) \in a^*] \text{ for every } n < \omega.$$

Let $S := \{a \in (\lambda^+)^{\mathfrak{B}} \mid \mathfrak{B} \models [a \in a_0 \wedge f(a) \in a^*]\}$. By Proposition 1.10.3 requirement (3) holds, thus $S \in \mathfrak{B}$. Since f is injective we have that $|S| < \lambda$. By (4) there exists $g \in \mathfrak{B}$ such that

$$g : S \overset{\text{order-preserving}}{\longrightarrow} a^*.$$

Since $\{a_n \mid n < \omega\} \subseteq S$ we get that for every $n < \omega$ we have

$$\mathfrak{B} \models [g(a_{n+1}) \in g(a_n)].$$

This is a contradiction to $\lambda^{\mathfrak{B}}$ being well-ordered. □

□

QUESTION 7.26.

THEOREM 7.27 (Shelah?). *Let $\mathcal{K} = \operatorname{EC}(T, \Gamma)$. where $|T| + |\operatorname{L}(T)| \leq \lambda$ and $|\Gamma| \leq \kappa$ such that the predicates P and $<$ are from $\operatorname{L}(T)$. Suppose for every $\alpha < \delta(\lambda, \kappa)$ there exists $M_\alpha \in \mathcal{K}$ such that $\alpha \leq otp(P^{M_\alpha}, <^{M_\alpha})$ there exists $N \in \mathcal{K}$ such that $(P^N, <^N)$ is not well-ordered.*

The main idea in the proof of Theorem 7.27 is to use once more models of set theory to code classes of models. This idea will also be used in the proof of the much more important Theorem **??**.

LEMMA 7.28 (Using models of set theory to code EC). *Let $\mathcal{K} = \mathrm{EC}(T, \Gamma)$ and $\{M_\alpha \in \mathcal{K} \mid \alpha < \delta\}$ be given. Denote by G the function from δ into $\{M_\alpha \in \mathcal{K} \mid \alpha < \delta\}$ given by $G(\alpha) = M_\alpha$ and let $F := \mathrm{Fml}(L)$.. Suppose β is an ordinal big enough so that*

$$\{\delta, F, T, L, \models_L\ G\} \subseteq \mathbf{V}_\beta$$

and

$$\langle \mathbf{V}_\beta, \in\rangle \models \forall x[x \in \delta \to G(x) \models_L T].$$

Let $\mathfrak{B} := \langle \mathbf{V}_\beta, \in, \delta, F, T, L, \models_L, G, \ldots, \varphi\rangle_{\varphi\in\mathrm{Fml}(L)}$, where δ, F, T and L are unary predicates, $\models_L$ is a binary relation, F a unary function symbol and φ's are individual constants.

For every $\varphi(\mathbf{x}) \in \mathrm{Fml}(L)$ we define a formula $\varphi^*(y; \mathbf{x})$ in $\mathrm{Fml}(\mathfrak{B})$ as follows $\varphi^*(y, \mathbf{x}) := [y \in \delta \wedge \mathbf{x} \in G(y) \wedge G(y) \models \varphi(\mathbf{x})]$. For every $p \in \Gamma$ let p^* be $\{\varphi^* \mid \varphi \in p\}$. Now let

$$\Gamma^* := \{p^* \mid p \in \Gamma\} \cup \{z \in F \wedge z \neq \varphi \mid \varphi \in \mathrm{Fml}(L)\}.$$

Let $T^* := \mathrm{Th}(\mathfrak{B})$. Notice that $\mathfrak{B} \in \mathrm{EC}(T^*, \Gamma^*)$. If $\mathfrak{B}^* \in \mathrm{EC}(T^*, \Gamma^*)$ then $N_a \in \mathrm{EC}(T, \Gamma)$ where $N_a := G^{\mathfrak{B}^*}(a)$ for all $a \in \delta^{\mathfrak{B}^*}$.

PROOF. Since $\mathfrak{B}^*$ omits the type $\{z \in F \wedge z \neq \varphi \mid \varphi \in \mathrm{Fml}(L)\}$ we get that $F^{\mathfrak{B}^*} = \mathrm{Fml}(L)$ and in particular $T^{\mathfrak{B}^*} = T$. Thus $N_a \models T$. For the sake of contradiction suppose that there are $p \in \Gamma$ and $\mathbf{a} \in |N|$ such that $\mathbf{a} \models p$. This gives $N_a \models \varphi[\mathbf{a}]$ for all $\varphi \in p$, thus $a\hat{}\mathbf{a} \models_{\mathrm{L}(\mathfrak{B})} p^*$ which is a contradiction to the assumption that $\mathfrak{B}^*$ omits p^*. □

PROOF OF THEOREM 7.27. Denote by δ the ordinal $\delta(\lambda, \kappa)$ and let $L := \mathrm{L}(T)$. Consider the function $G(\alpha) := M_\alpha$ for all $\alpha < \delta$ and for every $\alpha < \delta$ fix an order-preserving $G_1(\alpha, \cdot) : \langle \alpha, \in\rangle \to \langle P^{M_\alpha}, <^{M_\alpha}\rangle$.

Let β big enough so that

$$\{G, G_1, \delta, \Gamma, T_1, L, \models_L, \mathrm{Fml}(L)\} \in \mathbf{V}_\beta$$

and $\langle \mathbf{V}_\beta, \in\rangle$ "knows" all the relevant information about the above sets,

In particular we have

$$\langle \mathbf{V}_\beta, \in\rangle \models \Big[\forall x[x \in \delta \to [G(x) \models T]]] \wedge$$
$$\forall x[x < \delta \to [G_1(x, \cdot) \text{ is order preserving from } \langle x, \in\rangle \text{ into } \langle P^{M_x}, <^{M_x}\rangle]\Big].$$

Let

$$\mathfrak{B} := \langle \mathbf{V}_\beta, \in, \delta, F, T, L, \models_L, G, G_1, \varphi \rangle_{\varphi \in \mathrm{Fml}(L)},$$

where $\in, \delta, T, \Gamma, F, \models_L$ are predicates interpreted by the approperiat sets, G, G_1 functions and φ individual constants.

Let $T^* := \mathrm{Th}(\mathfrak{B})$ and Γ^* from Lemma 7.28. As $|\mathrm{L}(T^*)| + |T^*| \leq \lambda$, $|\Gamma^*| \leq \kappa + 1$, using Theorem 7.6 we get that $\delta(\lambda, \kappa) = \delta(\lambda, \kappa + 1)$. Since $\langle \delta^{\mathfrak{B}}, \in^{\mathfrak{B}} \rangle$ has order type δ and $\mathfrak{B} \in \mathrm{EC}(T^*, \Gamma^*)$ by the definition of δ there exists $\mathfrak{B}^* \equiv \mathfrak{B}$ such that $\langle \delta^{\mathfrak{B}^*}, \in^{\mathfrak{B}^*} \rangle$ is not well-founded and $\mathfrak{B}^*$ omits all the types in Γ^*.

Let $\{a_n \mid n < \omega\} \subseteq \delta^{\mathfrak{B}^*}$ such that $\mathfrak{B}^* \models [a_{n+1} \in a_n]$ for all $n < \omega$.

Let $N := G_1^{\mathfrak{B}^*}(a_0)$. By Lemma 7.28 we get that $N \in \mathrm{EC}(T, \Gamma)$. Now let $b_n := G_1^{\mathfrak{B}^*}(a_{n+1})$ for all $n < \omega$. Since $G_1^{\mathfrak{B}^*}(a_{n+1}) \in P^N$ and G_1 is order preserving we get that $\mathfrak{B}^* \models [a_{n+1} \in a_n \to b_{n+1} < b_n]$ for all $n < \omega$. Namely $\langle P^N, <^N \rangle$ is not well-founded. □

LEMMA 7.29 (Morley's method). *Let T be a theory with built in Skolem Functions and Γ is a set of T-types. If $\{p_n \in \mathrm{D}_n(T) \mid n < \omega\}$ and $\{c_n \mid n < \omega\}$ are new constants such that*

(1) $T^* \supseteq T \cup \{p_n(c_0, \ldots, c_{n-1}) \mid n < \omega\}$ *is consistent*

and

(2) *for every $n < \omega$ there are $M \in \mathrm{EC}(T, \Gamma)$ and $\mathbf{b} \in {}^n|M|$ realizing p_n*

then there exists $N \in \mathrm{EC}(T, \Gamma)$.

PROOF. Let $N^* \models T^*$, let $N := \mathrm{EM}(\{c_n^{N^*} \mid n < \omega\})$. As $N \models T^*$ it is enough to show that N omits q for every $q \in \Gamma$. Suppose that $\mathbf{a} \in |N|$ and $q \in \Gamma$ be given such that $m := \ell(\mathbf{a})$ is also the number of free variables in q.

Let $\langle a_1, \ldots, a_m \rangle = \mathbf{a}$ for every $1 \leq i \leq m$ there are $\tau_i(\mathbf{x}) \in \mathrm{Term}(\mathrm{L}(M))$ and $\mathbf{a}_i \in \{c_n^N \mid n < \omega\}$ such that $a_i = \tau_i[\mathbf{a}_i]$ by renaming the variables in the terms there exists $n_{\mathbf{a}} < \omega$ such that $a_i = \tau^N[c_0, \ldots, c_{n(\mathbf{a})-1}]$.

By (2) let $\mathbf{b} \in |M|$ realizing $p_{n(\mathbf{a})}$, since M omits q there exists $\varphi(x_0, \ldots, x_{n(\mathbf{a})-1}) \in q$ such that $M \models \neg\varphi[\tau_1(\mathbf{b}), \ldots \tau_m(\mathbf{b})]$, since p_n is a complete type and (1) we have that $N \models \neg\varphi[\tau_1(\mathbf{c}), \ldots, \tau_m(\mathbf{c})]$ when $\mathbf{c} = \langle c_0, \ldots, c_{n-1} \rangle$, thus $N \models \neg\varphi[\mathbf{a}]$. □

THEOREM 7.30. $\delta(\lambda) \leq (2^\lambda)^+$.

THEOREM 7.31 (Morley, Schmerl and Shelah). *For infinite λ we have that $\delta(\lambda, \kappa) \leq (2^\lambda)^+$ for every $\kappa \leq 2^\lambda$.*

PROOF. Fix $\mathcal{K} = \mathrm{PC}(T, \Gamma, L)$ with $|\mathrm{L}(T)| \leq \lambda$ and $|\Gamma| \leq \kappa$. We may assume that T has built-in Skolem functions. Suppose $M \in \mathrm{PC}(T, \Gamma, L)$ such that there

exists $\{a_\alpha \mid \alpha < (2^\lambda)^+\} \subseteq Q^M$ satisfying

$$\alpha < \beta < (2^\lambda)^+ \implies M \models [a_\alpha < a_\beta].$$

Using Morley's method it is enough to show that there are $\{p_n \in \mathrm{D}_n(T) \mid n < \omega\}$ such that

$$(1) \quad \begin{aligned} T^* :=& T \cup \{p_n(c_0, \dots, c_{n-1}) \mid n < \omega\} \cup \\ & \{Q(c_n) \mid n < \omega\} \cup \{c_{n+1} < c_n \mid n < \omega\}. \end{aligned}$$

is consistent and

$$(2) \quad \text{for every } n < \omega \text{ there exists } \mathbf{b} \in {}^n|Q^M| \text{ realizing } p_n$$

We construct a sequence of sets

$$\langle S_n \subseteq (2^\lambda)^+ \mid n < \omega \rangle,$$

of types

$$p_n(x_0, \dots, x_{n-1}) \in \mathrm{D}_n(T) \quad \text{for } n > 0,$$

and ordinals

$$\langle \alpha_i(n,0), \dots, \alpha_i(n,n-1) \rangle, \quad \text{for } i \in S_n$$

such that

(1) $S_0 = (2^\lambda)^+$ and $S_{n+1} \subseteq S_n$;
(2) $|S_n| = (2^\lambda)^+$;
(3) $(2^\lambda)^+ > \alpha_i(n,0) > \cdots > \alpha_i(n,n-1) > i$ for each $i \in S_n$;
(4) $p_n \subseteq p_{n+1}$;
(5) For every $n < m$, $i \in S_n$ and $j \in S_m$ we have

$$p_n = \mathrm{tp}(a_{\alpha_i(n,0)}, \dots, a_{\alpha_i(n,n-1)}/\emptyset, M) = \mathrm{tp}(a_{\alpha_j(m,0)}, \dots, a_{\alpha_j(m,n-1)}/\emptyset, M).$$

This is by induction on $n < \omega$. For $n = 0$, we only need to define S_0 and we let $S_0 = (2^\lambda)^+$.

Having constructed S_n, for each $i \in S_n$ pick $j := j(i) \in S_n$ such that $j > i$, this is possible since S_n is unbounded in $(2^\lambda)^+$. Define

$$\alpha_i(n+1,0) := \alpha_j(n,0), \dots, \alpha_i(n+1,n-1) := \alpha_j(n,n-1)$$

and

$$\alpha_i(n+1,n) := j.$$

We also let

$$p^i_{n+1} := \mathrm{tp}(a_{\alpha_i(n+1,0)}, \dots, a_{\alpha_i(n+1,n)}/\emptyset, M).$$

Then $p^i_{n+1} \in \mathrm{D}_{n+1}(T)$. By the pigeonhole principle, since $|S_n| = (2^\lambda)^+ > |\mathrm{D}_{n+1}(T)|$, we can find $S_{n+1} \subseteq S_n$ of cardinality $(2^\lambda)^+$ and $p_{n+1} \in \mathrm{D}_{n+1}(T)$ such that $p^i_{n+1} = p_{n+1}$ for all $i \in S_{n+1}$. It is easy to see that this satisfies all the requirements.

Now let $\{ c_n \mid n < \omega \}$ be new constants and let T_1 be the union of the following set of sentences

- T^*;
- $\{Q(c_n) \mid n < \omega\}$;
- $c_n > c_{n+1}$, for every $n < \omega$;
- $p_n(c_0, \dots, c_{n-1})$, for every $n < \omega$.

Then T_1 is consistent: for a finite subset T_0 choose n maximal appearing in T_0. Then $T_0 \subseteq T^* \cup \{ c_k > c_{k+1} \mid k < n \} \cup p_n(c_0, \dots, c_{n-1})$.

Pick $i \in S_n$, verify that $\langle M, a_{\alpha_i(n,k)} \rangle_{k<n} \models T_0$.

Let $N^* \models T_1$ and $a_n := c_n^{N^*}$. Consider

$$N := \mathrm{EM}(\{ a_n \mid n < \omega \}) \restriction L.$$

To verify that $N \in \mathrm{EC}(T, \Gamma)$it is enough to show that N omits every $q \in \Gamma$. For the sake of contradiction suppose there are $q \in \Gamma$ and $\mathbf{d} \in |N|$ such that $\mathbf{d} \models q$.

As N is a Skolem Hull of $\{a_n \mid n < \omega\}$ there exists a finite sequence of terms $\overline{\tau}(x_0, \dots, x_{n-1})$ such that

$$\mathbf{d} = \overline{\tau}(a_0, \dots, a_{n-1}).$$

Since $\mathbf{d} \models q$ we have that

$$N \models \varphi[\overline{\tau}(a_0, \dots, a_{n-1})] \text{ for every } \varphi \in q.$$

Namely,

$$N^* \models \varphi[\overline{\tau}(c_0, \dots, c_{n-1})] \text{ for every } \varphi \in q.$$

As $N^* \models T_1$ we get

$$\varphi(\overline{\tau}(x_0, \dots, x_{n-1})) \in p_n.$$

Using requirement (2) in the p_ns there exists $\mathbf{b} \in Q^M$ realizing p_n, thus

$$\overline{\tau}(\mathbf{b}) \models q$$

which is a contradiction to the assumption that M omits q. □

One reason why results about undefinability of well-ordering in PC-classes is important because of the following theorem. Notice that this is *not* the reason why we study them here.

One of the most important theorems of model theory from the sixties is the following result of Morley:

THEOREM 7.32 (Morley's Hanf number first result). *Let T be an $L_{\omega_1,\omega}$-theory in a countable language. If there exists $M \models T$ of cardinality $\beth_{\omega_1}$ then T has models in all infinite cardinals.*

It turns out that one can weaken the hypothesis of the previous result:

THEOREM 7.33 (Morley's Hanf number 2nd result). *Let T be an $L_{\omega_1,\omega}$-theory in a countable language. If for every $\alpha < \omega_1$ there exists $M_\alpha \models T$ of cardinality $\geq \beth_\alpha$ then T has models in all infinite cardinals.*

Another extension also due to Morley is

THEOREM 7.34 (Morley's two cardinal theorem). *Let T be an $L_{\omega_1,\omega}$-theory in a countable language and suppose that the similarity type contains a unary predicate P. If for every $\alpha < \omega_1$ there exists $M_\alpha \models T$ of cardinality $\geq \beth_\alpha(|P^{M_\alpha}|)$ then for every μ there exists $N \models T$ of cardinality μ such that $|P^N| = \aleph_0$.*

In his 1968 paper C. C. Chang extended these three theorems to PC-classes covering also the case when the base similarity type is uncountable:

THEOREM 7.35 (Morley and C.C. Chang). $\mu(\lambda,\kappa) = \beth_{\delta(\lambda,\kappa)}$. *Moreover, suppose that T first-order such that $|\mathrm{L}(T)| + |T| \leq \lambda$, Γ a set of $\leq \kappa$-many types and P is a unary predicate in $\mathrm{L}(T)$. If for every $\alpha < \delta(\lambda,\kappa)$ there exists $M_\alpha \in \mathrm{EC}(T,\Gamma)$ such that $\|M_\alpha\| \geq \beth_\alpha(|P^{M_\alpha}| + \lambda)$ then for every $\mu \geq \lambda$ there exists $N_\mu \in \mathrm{EC}(T,\Gamma)$ such that P^N has cardinality λ and $\|N_\mu\| = \mu$.*

I will present an interesting argument of Shelah using non well-founded (=non standard) models of set theory. A technology that is useful for many other results.

Before the proof of Theorem 7.35 we will introduce that technique and obtain additional information on $\delta(\lambda,\kappa)$:

THEOREM 7.36 (C. C. Chang ?). *Let T be first-order such that $|\mathrm{L}(T)| + |T| \leq \lambda$, Γ a set of $\leq \kappa$-many types and Q is a unary predicate in $\mathrm{L}(T)$ and $<$ is a binary predicate in $\mathrm{L}(T)$. If for every $\alpha < \delta(\lambda,\kappa)$ there exists $M_\alpha \in \mathrm{EC}(T,\Gamma)$ such that $\langle\alpha,\in\rangle$ can be embedded into $\langle Q^{M_\alpha},<\rangle$ then there exists $N \in \mathrm{EC}(T,\Gamma)$ such that $\langle Q^N,<^N\rangle$ is not well-ordered.*

PROOF. Let $\delta := \delta(\lambda,\kappa)$. Let

$$F(\alpha) := M_\alpha \text{ for all } \alpha < \delta.$$

For every $\alpha < \delta$ fix an order-preserving

$$F^*(\alpha,\cdot) : \langle\alpha,\in\rangle \to \langle Q^{M_\alpha},<^{M_\alpha}\rangle.$$

Using the reflection principle find a limit ordinal $\beta > \alpha$ such that

$$\delta, \{M_\alpha \mid \alpha < \delta\}, F, F^*, T, \Gamma, \mathrm{Fml}(\mathrm{L}(T)), \models_{\mathrm{L}(T)}, \ldots \in \mathbf{V}_\beta$$

The relation $\models_{\mathrm{L}(T)}$ stands for the three place relation $F(\alpha) \models \varphi[\mathbf{a}]$ when $\alpha < \delta$ and $\mathbf{a} \in F(\alpha)$. Note that some of the requirements above are redundant as for

example from the assumption that δ and F are elements of $\mathbf{V}_\beta$ and β is limit ordinal it follows that in $\mathbf{V}_\beta$ we can compute the set $\{M_\alpha \mid \alpha < \delta\}$.

Using the reflection principle we can require that the structure satisfies $\langle \mathbf{V}_\beta, \in \rangle$ all the "relevant information". In particular we have:

$$\begin{aligned}\langle \mathbf{V}_\beta, \in \rangle \models & M_\alpha \in \mathrm{EC}(T, \Gamma) \text{ for all } \alpha \in \delta \\ & F^*(\alpha, \cdot) : \langle \alpha, \in \rangle \to \langle Q^{M_\alpha}, <^{M_\alpha} \rangle \text{ is increasing} \\ & \ldots\end{aligned}$$

Let

$$\mathfrak{B} := \langle \mathbf{V}_\beta, \in T, F^*, \delta, T \models_{\mathrm{L}(T)}, \mathrm{Fml}(\mathrm{L}(T)) \rangle_{\varphi \in \mathrm{L}(T)}.$$

□

Let $\mathfrak{B}_1$ be an elementary submodel of $\mathfrak{B}$ of cardinality $|\delta|$ such that $\mathfrak{B}_1 \supseteq \delta$. Notice that automatically we have that $\delta^{\mathfrak{B}_1} = \delta$.

Fix a bijection $G : \delta \to |\mathfrak{B}_1|$, let $<^*$ be the induced well ordering of $|\mathfrak{B}_1|$ i.e. we have that

$$G : \langle \delta, \in \rangle \cong \langle |\mathfrak{B}_1|, <^* \rangle.$$

Now let

$$\mathfrak{B}_2 := \langle \mathfrak{B}_1, <^*, G \rangle \text{ and } T_1 := \mathrm{Th}(\mathfrak{B}_2).$$

For every $\varphi \in \mathrm{Fml}(\mathrm{L}(T))$ let

$$\varphi^1(y; \mathbf{x}) := [\mathbf{x} \in F(y) \wedge F(y) \models_{\mathrm{L}(T)} \varphi(\mathbf{x})].$$

Note that

$$\mathfrak{B}_2 \models \varphi^1[\alpha, \mathbf{a}] \iff F(\alpha) \models_{\mathrm{L}(T)} \varphi[\mathbf{a}].$$

Let q be the type $\{x \in \mathrm{Fml}(\mathrm{L}(T)) \wedge x \neq \varphi \mid \varphi \in \mathrm{Fml}(\mathrm{L}(T))\}$.

Note that if $\mathfrak{B}^* \models T_1$ omitting the type q is that $\mathrm{Fml}(\mathrm{L}(T))^{\mathfrak{B}^*}$ is the original set $\mathrm{Fml}(\mathrm{L}(T))$.

For $p \in \Gamma$ let $p^1 := \{\varphi^1(y; \mathbf{x}) \mid \varphi(\mathbf{x}) \in p\}$ and $\Gamma^1 := \{p^1 \mid p \in \Gamma\} \cup \{q\}$. Clearly $\mathfrak{B}_2 \in \mathrm{EC}(T_1, \Gamma^1)$. Let $\lambda_1 := |T_1|$ and $\kappa_1 := |\Gamma_1|$. Notice that $\lambda \leq \lambda_1$ and $\kappa_1 \leq \kappa + 1$.

****

In the case of $\kappa \leq \lambda$ by Theorem **??** we have that $\delta(\lambda_1, \kappa_1) \leq \delta$. In the case delta rightarrow

PROOF. First we prove the harder claim: $\mu(\lambda,\kappa) \leq \beth_{\delta(\lambda,\kappa)}$:

We may assume that T has built-in Skolem Functions. Using Morley's method it is enough to show that there are complete T-types $\{p_n \mid n<\omega\}$ satisfying $p_n \subseteq p_{n+1}$ for all $n<\omega$ such that for every $\mu \geq \lambda$

$$\begin{aligned}(1)\quad T^* :=& T \cup \{p_n(c_0,\dots,c_{n-1}) \mid n<\omega\} \cup \{c_i \neq c_j \mid i \neq j < \chi\} \cup \\ & \{\psi(c_0,\dots,c_{n-1}) \leftrightarrow \psi(c_{i_0},\dots,c_{i_{n-1}}) \mid i_0 < \cdots i_{n-1} < \mu, \\ & \psi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T)),\ n<\omega\}.\end{aligned}$$

is consistent and

(2) There exists $M \in \mathrm{EC}(T,\Gamma)$ and for every $n<\omega$ there exists $\mathbf{b} \in {}^n|M|$ realizing p_n

Let T,Γ be given such that $|\mathrm{L}(T)| + \aleph_0 \leq \lambda$ and $|\Gamma| \leq \kappa$. Suppose $M \in \mathrm{EC}(T,\Gamma)$ of cardinality $\beth_{\delta(\lambda,\kappa)}$. We may assume that T has built-in Skolem functions. Using the Erdős-Rado theorem there exists a finite $\Psi \subseteq \mathrm{ZFC}$ such that

$$\Psi \vdash \forall \alpha < \delta(\lambda,\kappa) \forall n < \omega [(\beth_n(\beth_\alpha)^+ \to (\beth_\alpha)^{n+1}_{\beth_\alpha}].$$

For every $\beta < \delta(\lambda,\kappa)$ pick $M_\beta \in \mathrm{EC}(T,\Gamma)$ be of cardinality $\beth_\beta$. Let F be a function from $\delta(\lambda,\kappa)$ into $\mathrm{EC}(T,\Gamma)$ defined as $F(\beta) := M_\beta$.

Using Theorem 10.4 there exists a limit ordinal α such that

$$\{T, \mathrm{L}(T), \Gamma, \delta(\lambda,\kappa), \beth_{\delta(\lambda,\kappa)}, F\} \in \mathbf{V}_\alpha$$

and $\langle \mathbf{V}_\alpha, \in\rangle$ reflects all the information about the above sets. Namely,

$\langle \mathbf{V}_\alpha, \in\rangle \models \Psi$,

$\langle \mathbf{V}_\alpha, \in\rangle \models \forall \beta < \delta(\lambda,\kappa) F(\beta) \in \mathrm{EC}(T,\Gamma)$,

$\langle \mathbf{V}_\alpha, \in\rangle \models \forall \beta < \delta(\lambda,\kappa)[\|F(\beta)\| = \beth_\beta]$ and

$\langle \mathbf{V}_\alpha, \in\rangle \models$"the function $\gamma \mapsto \beth_\gamma$ is increasing and converging to $\beth_{\delta(\lambda,\kappa)}$ for $\gamma < \delta(\lambda,\kappa)$".

Now define

$$\mathfrak{B} := \langle \mathbf{V}_\alpha, \in, T, \models_T, \mathrm{Fml}\,\mathrm{L}(T), \delta(\lambda,\kappa), \beth_{\delta(\lambda,\kappa)}, F, \varphi\rangle_{\varphi \in \mathrm{L}(T)}.$$

When the sets and ordinals are coded by unary relation symbols, let Δ be a relation symbol for the set $\delta(\lambda,\kappa)$, let T be a unary predicate for the set T and suppose that F is a unary function symbol and φ individual constants.

For $p(\mathbf{x}) \in \Gamma$ let

$$p^*(y;\mathbf{x}) := \{\Delta(y) \wedge [\mathbf{x} \in F(y)] \wedge [F(y) \models \varphi(\mathbf{x})] \mid \varphi \in p\}.$$

Let

$$p_T := \{T(x) \wedge x \neq \varphi \mid \varphi \in T\}$$

and

$$p_{Fml} := \{Fml(x) \wedge x \neq \varphi \mid \varphi \in \mathrm{Fml}(\mathrm{L}(T))\}.$$

Now let

$$\Gamma^* := \{p^* \mid p \in \Gamma\} \cup \{p_T\} \cup \{p_{Fml}\}.$$

Let $T^* := \mathrm{Th}(\mathfrak{B})$ and $L_{\mathfrak{B}} := \mathrm{L}(\mathfrak{B})$. Notice that $\mathfrak{B}$ omits all the types from Γ^* and if $\mathfrak{B}^* \in \mathrm{EC}(T^*, \Gamma^*)$ then

$$\alpha \in \Delta^{\mathfrak{B}^*} \implies F^{\mathfrak{B}^*}(\alpha) \in \mathrm{EC}(T, \Gamma)$$

and

$$Fml^{\mathfrak{B}^*} = \mathrm{Fml}(\mathrm{L}(T)),\ T^{\mathfrak{B}^*} = T.$$

Consider now the following $L_{\mathfrak{B}}$-type

$$p(x) := \{x \in \mathrm{Fml}\,\mathrm{L}(T) \wedge x \neq \varphi \mid \varphi \in \mathrm{L}(T)\}.$$

Notice that $\mathrm{Fml}\,\mathrm{L}(T)$ is a unary predicate in $L_{\mathfrak{B}}$.

Clearly $\mathfrak{B} \in \mathrm{EC}(T^{\mathfrak{B}}, \{p\})$. Since $\langle \Delta^{\mathfrak{B}}, \in^{\mathfrak{B}} \rangle$ has order-type $\delta(\lambda, \kappa)$ there exists $\mathfrak{B}' \in \mathrm{EC}(T^{\mathfrak{B}}, \{p\})$ such that there is $\{a_n \mid n < \omega\} \subseteq \Delta^{\mathfrak{B}'}$ satisfying

$$\mathfrak{B}' \models \forall n < \omega[a_{n+1} < a_n].$$

By renaming the elements of $\{a_n \mid n < \omega\}$ we may assume that

$$\mathfrak{B}' \models \forall n < \omega[a_{n+1} + n + 1 < a_n].$$

Fix $N := F^{\mathfrak{B}'}(a_0)$. By induction on $n < \omega$ define $\{X_n \subseteq |N| \ : \ n < \omega\}$ such that

(1) $X_n \in |\mathfrak{B}'|$ and $\mathfrak{B}' \models [|X_n| = \beth^+_{a_n}]$,
(2) $\mathfrak{B}' \models [X_n$ is a sequence of n-indiscernibles$]$ and
(3) $X_{n+1} \subseteq X_n$ for all $n < \omega$.

The construction of X_{n+1} is possible by applying

$$\mathfrak{B}' \models [\beth^+_{a_n} \to (\beth^+_{a_{n+1}})^{n+1}_{2^\lambda}]$$

to the coloring

$$G(a_1, \ldots, a_{n+1}) := \mathrm{tp}(a_1, \ldots, a_{n+1}/\emptyset, M) \text{ for every } a_1, \ldots, a_{n+1} \in [X_n]^{n+1}.$$

Let $p_n := \mathrm{tp}(a_1, \ldots, a_n/\emptyset, N)$. While the definition of p_n is carried out in $\mathfrak{B}'$ as the theory T and it's set of formulas are the same as in $\mathfrak{B}$ and complete types in $\mathfrak{B}'$ are also complete types in $\mathfrak{B}$ we get that p_n are complete types in $\mathbf{V}$.

Using M and $\{p_n \mid n < \omega\}$ establish that the theory T_1 is consistent where $\mathrm{L}(T_1) = \mathrm{L}(T) \cup \{c_n \mid n < \omega\}$ and T_1 is the union of the following set of sentences

- T;
- $\varphi(c_0, \ldots, c_{n-1}) \leftrightarrow \varphi(c_{i_0}, \ldots, c_{i_{n-1}})$, whenever $\varphi \in \mathrm{L}(T)$, $i_0 < \ldots i_{n-1}$, and $n < \omega$.
- $p_n(c_0, \ldots, c_{n-1})$, for every $n < \omega$.

As before the reduction to $\mathrm{L}(T)$ of skolem hull of the interpretations of $\{c_n \mid n < \omega\}$ omits all the types from Γ and by stretching theorem has arbitrarily large elementary extensions in $\mathrm{EC}(T, \Gamma)$.

THEOREM 7.37.

For a cardinal λ, the ordinal $\delta(\lambda)$ is defined as the least ordinal δ such that for any T and Γ and M, if

$|T| \leq \lambda$,
$T \vdash (P, <)$ is a linear order,
$M \in \mathrm{EC}(T, \Gamma)$, and
$(P^M, <^M)$ has order type at least δ,

then there exists $N \in \mathrm{EC}(T, \Gamma)$, such that $(P^N, <^N)$ is not well-ordered.

The following lemma is a generalization of the indefinability of well ordering in an elementary class (Theorem 1.5.22). It was proved independently by Lopez-Escobar and Morley for the context of $L_{\omega_1,\omega}$:

THEOREM 7.38 (Undefinability of well order). *Let P be a unary predicate and $<$ a binary predicate in $\mathrm{L}(T)$ such that*

$$T \vdash \text{“}< \textit{ is a linear order on } P\text{”}.$$

Suppose $M \in \mathrm{EC}(T, \Gamma)$ with $P^M = \{ a_i : i < (2^{|T|})^+ \}$ is such that

$$M \models a_i < a_j \quad \textit{if and only if} \quad i < j < (2^{|T|})^+.$$

Then there exists $N \in \mathrm{EC}(T, \Gamma)$ such that P^N is not well-ordered by $<^N$.

PROOF. □

COROLLARY 7.39. $\delta(|T|) \leq (2^{|T|})^+$.

We could not find a proof of the following theorem in the literature, so we have included a complete proof here.

THEOREM 7.40. *Let T be any theory. For every $\langle a_i \mid i < \beth_{\delta(|T|)} \rangle$ there exists a sequence of indiscernibles $\langle b_n \mid n < \omega \rangle$ with the following property: for every $n < \omega$ there are $i_0 < \cdots < i_{n-1}$ satisfying*

$$\mathrm{tp}(b_0, \ldots, b_{n-1}/\emptyset) = \mathrm{tp}(a_{i_0}, \ldots, a_{i_{n-1}}/\emptyset).$$

PROOF. Let $I = \langle a_i \mid i < \beth_{\delta(|T|)} \rangle$. We define the following functions:

- For every $n < \omega$, functions $f_n \colon [\beth_{\delta(|T|)}]^n \to \mathrm{D}(T)$ given by $f_n(i_0, \ldots, i_{n-1}) = \mathrm{tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}/\emptyset)$;
- A bijection $g \colon \beth_{\delta(|T|)} \to I$, defined by $g(i) = \bar{a}_i$;
- A bijection $h \colon \mathrm{D}(T) \to \kappa$, where $\kappa = |\mathrm{D}(T)|$;

Let χ be a regular cardinal large enough so that $H(\chi)$ contains $\mathrm{L}(T)$, $\mathrm{D}(T)$, I, $\beth_{\delta(|T|)}$ and the functions f_n, g and h as subsets. Assume in addition that $H(\chi)$ "knows" the Erdős-Rado Theorem; more precisely:

$$H(\chi) \models \beth_n(\beth_\alpha) \to (\beth_\alpha^+)^{n+1}_{\beth_\alpha} \quad \forall \alpha \in \delta(|T|)\, \forall n \in \omega.$$

Next, choose new predicates J, D, λ, μ; new constants κ and φ for every $\varphi \in \mathrm{L}(T)$; and new function symbols f_n for every $n \in \omega$, g,h, and b. Now form the following expansion of $H(\chi)$.

$$V = \langle\, H(\chi), \in, J, D, , \lambda, \mu, \kappa, f_n, g, h, b, \varphi \,\rangle_{\varphi \in L(T), n \in \omega},$$

where $J^V = I$, $D^V = D(T)$, $\lambda^V = \beth_{\delta(|T|)}$, $\mu^V = (2^{|T|})^+$, $f_n^V = f_n$, $g^V = g$, $h^V = h$, $b(\alpha)^V = \beth_\alpha$, $\kappa^V = \kappa$, and $\varphi^V = \varphi$ for $\varphi \in \mathrm{L}(T)$.

Let $T^* = \mathrm{Th}(V)$. Then T^* contains the following sentences (written informally for readability):

- $|J| = \lambda$;
- $|D| = \kappa < \mu$;
- $\forall \alpha \in \mu\, b(\alpha + n + 1) \to (b(\alpha))^{n+1}_\kappa$, for every $n \in \omega$;

We define a set of types Γ in the language of T^* as follows:

$$\Gamma := \{ p \in S(\emptyset) \mid \text{ no } \bar{a} \in I \text{ realizes } p \,\}. \tag{*}$$

Then $V \in \mathrm{EC}(T^*, \Gamma)$ and $\mu^V = \delta(|T|)$, so by Corollary7.39, there is $V' \in \mathrm{EC}(T^*, \Gamma)$ such that $\mu^{V'}$ is not well-ordered. Let $\{\, \alpha_n \mid n < \omega \,\} \subseteq \mu^{V'}$ witness this. We may assume that $V' \models \alpha_n > \alpha_{n+1} + n + 1$.

We now construct sets $X_n \subseteq \lambda^{V'}$ for $n < \omega$ such that

(1)
(2) $V' \models |X_n| \geq b(\alpha_n)$ for every $n < \omega$;
(3) $V' \models f_n(g(i_0), \ldots, g(i_{n-1})) = f_n(g(j_0), \ldots, g(j_{n-1}))$, for $i_0 < \cdots < i_{n-1} \in X_n$ and $j_0 < \cdots < j_{n-1} \in X_n$.

The construction is by induction on n. Let $X_0 = \lambda^{V'}$ and clearly $V' \models |X_0| \geq b(\alpha_0)$ since $V' \models b(\alpha_0) \in \lambda$.

Having constructed X_n, notice the following.

(1) $V' \models |X_n| \geq b(\alpha_n)$;
(2) $V' \models c : [X_n]^{n+1} \to D$, with $c(i_0, \ldots, i_n) := f_{n+1}(g(i_0), \ldots, g(i_n))$;
(3) $V' \models |D| = \kappa$;
(4) $V' \models b(\alpha_n) \to (b(\alpha_{n+1}))^{n+1}_\kappa$.

Then (4) applied to (1) and (2) implies that there is $X_{n+1} \subseteq X_n$ such that $V' \models |X_{n+1}| \geq b(\alpha_{n+1})$ monochromatic with respect to c, so X_{n+1} is as required.

Since every $\varphi \in L$ has a name in $\mathrm{L}(T^*)$, there are $p_n \in \mathrm{D}_n(T)$ for $n \in \omega$ such that $\mathrm{tp}(g(i_0), \ldots, g(i_n)/\emptyset) = p_n$ for $\bar{a}_{i_0} = g(i_0), \ldots, \bar{a}_{i_{n-1}} = g(i_{n-1}) \in J^{V'}$, and $i_0 < \cdots < i_{n-1} \in X_n$.

Now let $\{ c_n \mid n < \omega \}$ constants not in $\mathrm{L}(T^*)$ and let T_1 be the union of the following set of sentences:

- T^*;
- $p_n(c_0, \ldots, c_{n-1})$, for every $n < m$;
- $g(c_n) < g(c_m)$, for every $n < m$;
- $\varphi(c_0, \ldots, c_{n-1}) \leftrightarrow \varphi(c_{i_0}, \ldots, c_{i_{n-1}})$, whenever $\varphi \in \mathrm{L}(T)$, $i_0 < \ldots i_{n-1}$, and $n < \omega$.

Then T_1 is consistent: for every finite subset of T_1 use $g(X_n)$ to realize the c_k's.

Let $N_1 \models T_1$ and $b_n = c_n^{N_1}$ for every $n \in \omega$. Certainly $\{ b_n \mid n < \omega \}$ is indiscernible (in $\mathrm{L}(T)$). Now we show that for every $n < \omega$ there exist $i_0 < \cdots < i_{n-1} < \beth_{(2^{|T|})^+}$ such that

$$\mathrm{tp}(a_{i_0}, \ldots, a_{i_{n-1}}/\emptyset) = p_n.$$

But by construction, there are $j_0 < \cdots < j_{n-1}$ such that

$$q_n = \mathrm{tp}_{\mathrm{L}(T^*)}(g(j_0), \ldots, g(j_{n-1})/\emptyset) = p_n.$$

Hence, $\{ x_i \in J \} \in q_n$ for every $i < n$, and since $\{ J, g, \in \} \subseteq \mathrm{L}(T^*)$, we have

$$\{ g(x_0) < g(x_1), \ldots, g(x_{n-2}) < g(x_{n-1}) \} \subseteq q_n$$

But since $V' \in \mathrm{EC}(T^*, \Gamma)$, we must have $q_n \notin \Gamma$. Thus, by (*), there are $a_{i_0}, \ldots a_{i_{n-1}}$ in I realizing q_n. □

Now to the proof of Theorem A.1:

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PROOF. Let T^* be an expansion of T of the same cardinality with Skolem functions. We construct sequences of sets $\langle S_n \mid n < \omega \rangle$, types

$$p_n(x_0, \ldots, x_{n-1}) \in S^n_{\mathrm{L}(T^*)}(\emptyset) \quad \text{for } n > 0,$$

and ordinals

$$\alpha_i(n, 0), \ldots, \alpha_i(n, n-1) \in S_n, \quad \text{for } i \in S_n$$

such that

(1) $S_0 = (2^{|T|})^+$ and $S_{n+1} \subseteq S_n$;
(2) $|S_n| = (2^{|T|})^+$;
(3) $(2^{|T|})^+ > \alpha_i(n, 0) > \cdots > \alpha_i(n, n-1) > i$ for every $i \in S_n$;
(4) $p_n \subseteq p_{n+1}$;

(5) For every $n < m$, $i \in S_n$, and $j \in S_m$,

$$p_n = \mathrm{tp}(a_{\alpha_i(n,0)}, \ldots, a_{\alpha_i(n,n-1)}/\emptyset, M) = \mathrm{tp}(a_{\alpha_j(m,0)}, \ldots, a_{\alpha_j(m,n-1)}/\emptyset, M).$$

The construction is by induction on n. For $n = 0$, we only need to define S_0 and we let $S_0 = (2^{|T|})^+$.

Having constructed S_n for every $i \in S_n$, fix $j \in S_n$ with $j > i$ (which exists by (2)). Define

$$\begin{aligned} \alpha_i(n+1,0) &:= \alpha_j(n,0) \\ &\vdots \\ \alpha_i(n+1,n-1) &:= \alpha_j(n,n-1) \\ \alpha_i(n+1,n) &:= j. \end{aligned}$$

We also let

$$p^i_{n+1} := \mathrm{tp}(a_{\alpha_i(n+1,0)}, \ldots, a_{\alpha_i(n+1,n)}/\emptyset, M).$$

Then $p^i_{n+1} \in S^{n+1}(\emptyset)$. Since $|S_n| = (2^{|T|})^+ > |S^{n+1}_{\mathrm{L}(T^*)}(\emptyset)|$,
there exist $S_{n+1} \subseteq S_n$ of cardinality $(2^{|T|})^+$ and $p_{n+1} \in S^{n+1}(\emptyset)$ such that $p^i_{n+1} = p_{n+1}$ for every $i \in S_{n+1}$. It is easy to see that all the requirements are satisfied.

Now let $\{ c_n \mid n < \omega \}$ be new constants and let T_1 be the union of the following set of sentences:

- T^*;
- $c_n > c_{n+1}$, for every $n < \omega$;
- $p_n(c_0, \ldots, c_{n-1})$, for every $n < \omega$.

Then T_1 is consistent: If Γ is a finite subset of T and n maximal appearing in Γ, then $\Gamma \subseteq T^* \cup \{ c_i > c_{i+1} \mid i < n \} \cup p_{n+1}(c_0, \ldots, c_n)$ is satisfied by interpreting c_i as $\alpha_{(n,i)}$.

Let $N_1 \models T_1$ and $b_n = c^{N_1}_n$.
Let N be the closure of $\{ b_n \mid n < \omega \}) \restriction L(T)$. under the Skolem functions. Then $N \models T$ and P^N is not well-ordered. We claim that $N \in \mathrm{EC}(T, \Gamma)$. In fact very type p (over the empty set) that N realizes is realized in M: Let p be a given type and suppose $\bar{d} \in N$ its realization. Since N is a Skolem Hull there exists a $\bar{\tau} \in L(T^*)$ such that $\bar{d} = \bar{\tau}(b_0, \ldots, b_{n-1})$. Hence,

$$p = \mathrm{tp}(\bar{d}/\emptyset) = \mathrm{tp}(\bar{\tau}(b_0, \ldots, b_{n-1})/\emptyset) \subseteq p_n.$$

By construction of p_n and S_n, there are $\langle a_{i_0}, \ldots, a_{i_{n-1}} \rangle \in P^M$ realizing p_n, so $\bar{\tau}(a_{i_0}, \ldots, a_{i_{n-1}})$ realizes p, which is a contradiction, since
$\bar{\tau}(a_{i_0}, \ldots, a_{i_{n-1}}) \in M \in \mathrm{EC}(T, \Gamma)$.

□

COROLLARY 7.41. *Let $\langle I, <_I \rangle$ be an infinite linearly ordered set. For every sequence $\langle \mathbf{a}_i \mid i < \beth_{(2^{|T|})^+} \rangle$ there exists a sequence of indiscernibles $\langle \mathbf{b}_i \mid i \in I \rangle$ with the following property: for every $n < \omega$ and any $j_0 <_I \cdots <_I j_{n-1} \in I$ there exist $i_0 < \cdots < i_{n-1}$ such that*

$$\mathrm{tp}(\mathbf{b}_{j_0}, \ldots, \mathbf{b}_{j_{n-1}}/\emptyset) = \mathrm{tp}(\mathbf{a}_{i_0}, \ldots, \mathbf{a}_{i_{n-1}}/\emptyset).$$

COROLLARY 7.42 (Morley's omitting types theorem). *Let T be a first-order theory and Γ be a set of types over the empty set such that $|\Gamma| \leq |T|$. If there exists $M \in \mathrm{EC}(T, \Gamma)$ of cardinality $\beth_{\delta(|T|)}$ then $\mathrm{EC}(T, \Gamma)$ contains models of arbitrary large cardinality.*

ADD CONNECTIONS WITH THE GALVIN-HAJNAL RANK.

Exercises 2.7.

(1) Show that not every PC-class is an abstract elementary class. Define a PC-class which is not closed under unions.
(2) Show that every EC-class is AEC.
(3) Give an example of an AEC with $\mathrm{LS}(\mathcal{K}) = \aleph_0$ which is not an EC-class.
(4) Prove that the class LF of locally finite groups is an EC-class.
(5) Is LF an EC-class?
(6) Prove that $\mathcal{K}_p$ from Exercise 1.3.52 is a PC-class.
(7)
(8) Suppose $\mathcal{K}$ is a $\mathrm{PC}_{\aleph_0,1}$ class such that $<, P \in \mathrm{L}(\mathcal{K})$ show that if there exists $M \in \mathcal{K}$ such that there exists an order preserving $f : \omega_1 \to (P^M, <^M)$ then there exist $N \in \mathcal{K}$ containing $\{a_r \mid r \in \mathbb{Q}\} \subseteq P^N$ satisfying $\mathbb{Q} \models r_1 < r_2 \implies N \models [a_{r_1} < a_{r_2}]$.
Hint: Use the proof of Theorem 7.22 together with Exercise 1.5.24.
(9)
(10) Suppose that T and Γ are at most countable and $\mathrm{L}(T)$ contains two unary predicates P and Q. Suppose that for every $\alpha < \omega_1$ there exists $M_\alpha \in \mathrm{EC}(T, \Gamma)$ such that $\|M_\alpha\| \geq \beth_\alpha(|Q^{M_\alpha}|)$ and $|Q^{M_\alpha}| \geq \beth_\alpha(|P^{M_\alpha}|)$. Conclude that for any pair of infinite cardinals χ_1, χ_2 there exists $N \in \mathrm{EC}(T, \Gamma)$ such that $|P^N| = \chi_1$ and $|Q^N| = \chi_2$.
(11)
(12)

8. $\mathrm{I}(\aleph_0, \mathcal{K}) = \mathrm{I}(\aleph_1, \mathcal{K}) = 1 \implies \mathcal{K}_{\aleph_2} \neq \emptyset$.

This section is dedicated to the proof of a special case of Conjecture 4.1.

THEOREM 8.1 (Shelah). *If $\mathcal{K}$ is an AEC such that $\mathcal{K}$ is $\mathrm{PC}_{\aleph_0}$ and $\mathcal{K}^{\prec} := \{\langle |M|, |N| \rangle \mid N \prec_{\mathcal{K}} M \in \mathcal{K}\}$ is also $\mathrm{PC}_{\aleph_0}$ in a language consisting of single unary predicate then $\mathrm{I}(\aleph_0, \mathcal{K}) = \mathrm{I}(\aleph_1, \mathcal{K}) = 1 \implies \mathcal{K}_{\aleph_2} \neq \emptyset$.*

Notice that this theorem has the following

COROLLARY 8.2. *Let $\psi \in L_{\omega_1,\omega}$ be in a countable language. If $\mathrm{I}(\aleph_1, \psi) = 1$ then ψ has a model of cardinality $\aleph_2$.*

PROOF. Let $M \models \psi$ be a model of cardinality $\aleph_1$. Pick L^* a countable fragment containing ψ, we may assume that M has built-in Skolem functions for every formula from L^*. Let $M_0 \prec_{L^*} M$ be countable. Let ψ_0 be a Scott sentence of M_0. Now by Chang's presentation theorem the AEC $\mathcal{K} := \langle \mathrm{Mod}(\psi_0), \prec_{L^*} \rangle$ is $\mathrm{PC}_{\aleph_0}$. Why $\mathcal{K}^{\prec}$ is $\mathrm{PC}_{\aleph_0}$?

Suppose T and Γ are both countable such that $\mathcal{K} = \mathrm{PC}(T, \Gamma, L)$.

Let $L' = L(T) \cup \{P\} \cup \{R_\varphi(\mathbf{x}) \mid \varphi(\mathbf{x}) \in L^*\}$ when P is a new unary predicate and R_φ is a new $\ell(\mathbf{x})$-place relation symbol.

Like in the proof of Theorem 7.5 Let

$$T' := T \cup \{\forall \mathbf{y}[\exists x[P(x) \wedge \varphi(x; \mathbf{y})] \to \exists x'[P(x) \wedge \varphi(x'; \mathbf{y})]] \mid \varphi(x; \mathbf{y}) \in L^*\}.$$

categorical both in $\aleph_0$ and $\aleph_1$ and $N \models \psi_0 \implies N \models \psi$.

□

REMARK 8.3. Let $\psi \in L_{\omega_1,\omega}$ be in a countable language. It is natural to ask whether existence of an uncountable model of ψ is an absolute property? It turns out that it depends if we are interested in models of cardinality $\aleph_1$ or $> \aleph_1$.

(1) The statement "there exists an uncountable model for ψ" is absolute.
(2) For any $\lambda \geq \aleph_2$ and every $1 < n < \omega$ there exists $\psi_n \in L_{\lambda^+,\omega}$ such that ψ has a model of cardinality λ^{+n} iff there exists a Kurepa tree of height λ^n. It is known that existence of Kurepa trees is independent even of ZFC+GCH.

Does this contradict Example 6.14?

DEFINITION 8.4 (pre-types).

$$\mathcal{K}^3_\lambda := \{(a, M, N) \mid M, N \in \mathcal{K}_\lambda, M \leq_{\mathcal{K}} N, a \in |N| - |M|\}.$$

While at first the reader might find it difficult to believe the following generalization of clarifies the role of $\aleph_0$ in the Theorem **??**.

THEOREM 8.5. *Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in both λ and λ^+ then there exists $M \in \mathcal{K}_{\lambda^{++}}$, provided that $\delta(\lambda) = \lambda^+$ and both $\mathcal{K}$ and $\mathcal{K}^{\leq}$ are PC_λ-class.*

Recall

FACT 8.6.

(1) *(Lopez-Escobar) $\delta(\aleph_0) = \omega_1$.*
(2) *(C.C. Chang) If α is an ordinal of cofinality $\aleph_0$ and $\lambda = \beth_\alpha$ then $\delta(\lambda) = \lambda^+$.*

We will prove Theorem 8.5 in a sequence of claims. For the rest of this section suppose that $\mathcal{K}$ is an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$.

PROPOSITION 8.7 (The magic). *Suppose $\mathcal{K}$ is an AEC, $\lambda \geq \mathrm{LS}(\mathcal{K})$, the classes $\mathcal{K}$ and $\{(M, N) \mid M, N \in \mathcal{K}, M \prec_{\mathcal{K}} N\}$ are both PC_λ-classes and $\delta(\lambda) = \lambda^+$. If $\mathcal{K}_{\lambda^+} \neq \emptyset$ and $\mathcal{K}$ is λ-categorical. then there exists $\{M_n \mid n < \omega\} \subseteq \mathcal{K}_\lambda$ such that*

(1) $\bigcap_{n<\omega} M_n \in \mathcal{K}_\lambda$,
(2) $\bigcap_{n<\omega} M_n \precneqq_{\mathcal{K}} M_k$ *for all* $k < \omega$, *and*
(3) $M_{n+1} \precneqq_{\mathcal{K}} M_n$ *for all* $n < \omega$.

We denote the conclusion of the last Proposition by $(*)_\lambda$.

PROPOSITION 8.8. *If $\mathcal{K}$ is categorical in λ^+ and there are $M \precneqq_{\mathcal{K}} N$ both of cardinality λ^+ then $\mathcal{K}_{\lambda^{++}} \neq \emptyset$.*

PROOF. Define by induction on $\alpha < \lambda^{++}$ an increasing and continuous chain $\{M_\alpha \mid \alpha < \lambda^{++}\} \subseteq \mathcal{K}_{\lambda^+}$. For $\alpha = 0$ take an arbitary model $M_0 \in \mathcal{K}_{\lambda^+}$, at limit ordinals take unions and given $M_\alpha \in \mathcal{K}_{\lambda^+}$ by λ^+-categoricity fix $f : M \cong M_\alpha$ since $N \precneqq_{\mathcal{K}} M$ there are $M_{\alpha+1} \precneqq_{\mathcal{K}} M_\alpha$ and $\bar{f} \supseteq f$ an isomorphism from N onto $M_{\alpha+1}$. The model $\bigcup_{\alpha<\lambda^{++}} M_\alpha$ is as required. □

DEFINITION 8.9. For $(a, M, N),\ (a', M', N') \in \mathcal{K}^3_\lambda$ define

$$(a, M, N) <_{\mathcal{K}} (a', M', N') \iff a = a',\ M \precneqq_{\mathcal{K}} M' \text{ and } N \preceq_{\mathcal{K}} N'.$$

What we mean here by the name of this section? What *extension property* we refer to? Recall that for first-order theories we have the following fundamental fact:

FACT 8.10 (Lindenbaum before 1930). *Let T be a set of first-order sentences. If T is consistent then there exists a complete $\mathrm{L}(T)$-theory $T' \supseteq T$.*

When working with a fixed complete first-order theory T and inside its monster model $\mathfrak{C}$ this translates to:

Extension property for every type p in $\mathbf{x}$ and every $B \supseteq \operatorname{dom} p$ there exists a complete $\mathbf{x}$-type $q \supseteq p$.

DEFINITION 8.11 (Weak extension property for pre-types). Suppose $\lambda \geq \mathrm{LS}(\mathcal{K})$. Let $(a, M, N) \in \mathcal{K}^3_\lambda$ be given. *The pre-type* (a, M, N) *has the weak extension property* iff there are $M' \supsetneq_{\mathcal{K}} M$ and $N' \succeq_{\mathcal{K}} N$ such that $(a, M', N') > (a, M, N)$. $\mathcal{K}^3_\lambda$ *has the weak extension property* iff every element of $\mathcal{K}^3_\lambda$ has the weak extension property.

We say that $\mathcal{K}_\lambda$*-has the weak extension property* iff for every $p \in \mathrm{gS}(M_0)$ for all $M_0 \in \mathcal{K}_\lambda$ all the pretypes of p have the weak extension property.

REMARK 8.12. Recall that in Proposition **??**.3.12 we proved that under the λ-AP for every $M_0 \prec_{\mathcal{K}} M_1$ both of cardinality λ every $p_0 \in \mathrm{gS}(M_0)$ has an extension to $\mathrm{gS}(M_1)$.

The weak extension property is much stronger requirement as it requires the realization of p_0 be fixed.

DEFINITION 8.13 (Extension property for pre-types). Suppose $\lambda \geq \mathrm{LS}(\mathcal{K})$. ? Let $(a, M, N) \in \mathcal{K}^3_\lambda$ be given. *The pre-type* (a, M_0, N_0) *has the extension property* iff for every $N_1 \in \mathcal{K}_\lambda$ and every $f : M_0 \to M_1$ there are $N_1 \succ_{\mathcal{K}} N_0$ and $g : M_1 \to N_1$ extending f such that $(g(a), N_0, N_1) \in \mathcal{K}^3_\lambda$. ?!

Notice the following connection between the order on $\mathcal{K}^3_\lambda$ from Definition 8.9 and the extension property for Galois types.

PROPOSITION 8.14. *Suppose $\mathcal{K}$ has the λ-AP. $\mathcal{K}^3_\lambda$ has no maximal element (in the ordering $<_{\mathcal{K}}$) if and only if $\mathcal{K}_\lambda$ has the weak extension property.*

PROOF. Exercise. □

REMARK 8.15. Notice that if $\mathcal{K}^3_\lambda \neq \emptyset$ then the failure of the extension property for $\mathcal{K}_\lambda$ is equivalent to "there exists a maximal $(a, M, N) \in \mathcal{K}^3_\lambda$".

Let $(a, M, N) \in \mathcal{K}^3_\lambda$ be given. Suppose $N^* \in \mathcal{K}_\lambda$ and that $f : N \cong N^*$, denote by M^* the image of M under f.

$$(a, M, N) \text{ is maximal} \iff (f(a), M^*, N^*) \text{ is maximal.}$$

PROPOSITION 8.16. *Suppose $\mathcal{K}$ is λ^+-categorical. If $\mathcal{K}^3_\lambda$ has no maximal element then $\mathcal{K}_{\lambda^{++}} \neq \emptyset$.*

PROOF. It is enough to show that there are $M \precneqq_{\mathcal{K}} N$ both of cardinality λ^+. By induction on $\alpha < \lambda^+$ construct an $<_{\mathcal{K}}$-increasing $\{(a, M_\alpha, N_\alpha) \mid \alpha < \lambda^+\} \subseteq \mathcal{K}^3_\lambda$.

Let $M := \bigcup_{\alpha<\lambda^+} M_\alpha$ and $N := \bigcup_{\alpha<\lambda^+} N_\alpha$. Since $M_\alpha \prec_{\mathcal{K}} N_\alpha$ for all α, by Exercise 2.6 we have that $M \prec_{\mathcal{K}} N$. Since $a \in |N| - |M|$ we are done. $\square$

The proof of Theorem 8.5 will be complete after proving:

THEOREM 8.17. *Suppose $\mathcal{K}$ satisfies $(*)_\lambda$ and is λ-categorical. If $\mathcal{K}^3_\lambda$ has a maximal element then $I(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$.*

Recall

FACT 8.18 (Ulam). *There are $\{S_\alpha \mid \alpha < \lambda^+\}$ pairwise disjoint stationary subsets of λ^+. Thus there exists a family $\{S_X \mid X \subseteq \lambda^+\}$ of stationary subsets of λ^+ such that $X \neq Y \subseteq \lambda^+ \implies S_X \not\equiv S_Y \mod D_{\lambda^+}$.*

PROOF OF THEOREM 8.17. Using the last Fact, there is a family $\mathcal{F}$ of 2^{λ^+}-many stationary subsets of λ^+ such that

$$S_1 \neq S_2 \in \mathcal{F} \implies S_1 - S_2 \text{ is stationary or } S_2 - S_1 \text{ is stationary.}$$

We will construct $M^S \in \mathcal{K}_{\lambda^+}$ for all $S \subseteq \lambda^+$ such that

$$S_1 - S_2 \text{ is stationary} \implies M^{S_1} \not\cong M^{S_2}.$$

Fix $(a, M, N) \in \mathcal{K}^3_\lambda$ to be a maximal triple.

For any $S \subseteq \lambda^+$ define an increasing and continuous chain $\{M^S_\alpha \mid \alpha < \lambda^+\} \subseteq \mathcal{K}_\lambda$ as follows: Let M^S_0 be an model of universe λ, at limit stage take unions. The remaining case is when $\alpha = \beta + 1$: There are two possibilities:

If $\beta \in S$, by λ-categoricity fix $f : M \cong M^S_\beta$. by the isomorphism axioms take $M^S_{\beta+1} \succneqq_{\mathcal{K}} M^S_\beta$ and $\bar{f} \supseteq f$ such that $\bar{f} : N \cong M^S_{\beta+1}$.

If $\beta \notin S$, let $\{M_n \mid n < \omega\} \subseteq \mathcal{K}_\lambda$ from $(*)_\lambda$ for the model M^S_β. I.e. $M^S_\beta = \bigcap_{n<\omega} M_n$ and $M_{n+1} \precneqq_{\mathcal{K}} M_n$ for all $n < \omega$. Now define $M^S_{\beta+1}$ as M_0.

Finally define $M^S := \bigcup_{\alpha<\lambda^+} M^S_\alpha$. Clearly the models are all of cardinality λ^+.

It is enough to prove:

CLAIM 8.19. $S_1 \not\equiv S_2 \mod D_{\lambda^+} \implies M^{S_1} \not\cong M^{S_2}$

PROOF. For the sake of contradiction suppose that $S_1 \not\equiv S_2 \mod D_{\lambda^+}$ and $f : M^{S_1} \cong M^{S_2}$. Without loss of generality we may assume that $S_1 - S_2$ is stationary (intersects every club).

Now consider

$$C := \{\delta < \lambda^+ \mid (f \restriction M^{S_1}_\delta) : M^{S_1}_\delta \cong M^{S_2}_\delta\}.$$

Since C is a club, fix $\delta \in (S_1 - S_2) \cap C$.

Since $\delta \in S_1$ there exists $g : N \cong M^{S_1}_{\delta+1}$ such that $g[M] = M^{S_1}_\delta$. Let $a^1 := g(a)$. Since $a^1 \notin M^{S_1}_\delta$, using $\delta \in C$ we get that also $a^2 := f(a^1) \notin M^{S_2}_\delta$.

Since $\delta \notin S_2$ we have that $M^{S_2}_{\delta+1} \cong M_0$, we may assume that

$$M^{S_2}_{\delta+1} = M_0 \text{ and } M^{S_2}_\delta = \bigcup_{n<\omega} M_n.$$

Since $a^2 \notin \bigcap_{n<\omega} M_n$ there exists $n < \omega$ such that $a^2 \notin M_n$. As $f[M^{S_1}_{\delta+1}] \cup M_0 \cup \{a^2\}$ is a subset of M^{S_2} of cardinality λ there exists $\gamma < \lambda^+$ such that $M^{S_2}_\gamma \supseteq f[M^{S_1}_{\delta+1}] \cup M_0 \cup \{a^2\}$. Now let $M^* := f^{-1}[M_n]$ and $N^* := f^{-1}[M^{S_2}_\gamma]$. Notice that $(a^1, M^*, N^*) >_{\mathcal{K}} (a^{S_1}_\delta, M^{S_1}_\delta, M^{S_1}_{\delta+1})$ is a contradiction to the assumption that (a, M, N) is maximal. □

□

To complete the proof of Theorem 8.5 we need to prove Proposition 8.7.

PROOF OF PROPOSITION 8.7. Notice that by λ-categoricity and the assumption that $M \precneqq_{\mathcal{K}} N$ are both elements of $\mathcal{K}_\lambda$ there exists an increasing continuous $\{M_\alpha \mid \alpha < \lambda^+\} \subseteq \mathcal{K}_\lambda$. Let $F : \lambda^+ \to \mathcal{K}_\lambda$ defined by $F(\alpha) = M_\alpha$. For every $x \in \bigcup_{\alpha<\lambda^+} M_\alpha$ let

$$i(x) := \min\{\alpha < \lambda^+ \mid x \in M_\alpha\}.$$

Let T, q, L and T_0, q_0, L_0 be such that $\mathcal{K} = \mathrm{PC}(T_0, q_0, L_0)$ and $\{(N, M) \mid M, N \in \mathcal{K}, M \precneqq_{\mathcal{K}}\} = \mathrm{PC}(T, q, L)$.

Fix $g : \lambda^+ \to \lambda^+$ be the function defined by: $g(0) := 0$, $g(\alpha + 1) := g(\alpha) + \omega + 1$ and when α is limit then $g(\alpha) := \sup\{g(\beta) \mid \beta < \alpha\}$.

Let $\mathfrak{B}_\chi$ be an expansion by λ^+, i, g and F of the structure from Remark 11.3. I.e.

$$\mathfrak{B}_\chi := \langle H(\chi), \in, F, i, \lambda^+, g, \lambda, i, T, T_0, q, q_0, L, L_0, \beta, \varphi_0, \varphi\rangle_{\beta<\lambda, \varphi\in \mathrm{Fml}(L), \varphi_0 \in \mathrm{Fml}(L_0)}$$

We consider F and i as unary function symbols, $\lambda, \lambda^+, T, T_0, L, L_0, q$ and q_0 are all unary predicates, while β, φ and φ_0 are all individual constants.

χ is chosen to be regular big enough so that all the above mentioned sets are elements of $H(\chi)$ and

$$\begin{aligned}\mathfrak{B}_\chi \models &\forall x[\exists \alpha < \lambda^+ x \in F(\alpha) \to [x \in F(i(x))]] \wedge \forall x \in F(\alpha)[i(x) \leq \alpha] \wedge \\ &\forall \alpha < i(x) \exists y \in F(i(x))[y \notin F(\alpha)] \wedge \\ &\forall \alpha < \beta < \lambda^+ [F(\alpha) \in \mathrm{PC}(T_0, q_0, L_0) \wedge (F(\beta), F(\alpha)) \in \mathrm{PC}(T, q, L)] \wedge \\ &\forall \alpha < \lambda^+ |F(\alpha)| = \lambda.\end{aligned}$$

Let T^1 be the first-order theory of the structure $\mathfrak{B}_\chi$. Similarly to what we have in the proof of Lemma 7.28 let Γ be a finite set of T^1-types such that if $\mathfrak{B} \in \mathrm{EC}(T^1, \Gamma)$ then we have

$$(*) \quad \lambda^{\mathfrak{B}} = \lambda, L^{\mathfrak{B}} = L, q^{\mathfrak{B}} = q, T^{\mathfrak{B}} = T, L_0^{\mathfrak{B}} = L_0, T_0^{\mathfrak{B}} = T_0, q_0^{\mathfrak{B}} = q_0.$$

(*) Notice that since the theorem we proved immediately after the definition of AECs (i.e. they are closed under unions of directed systems) this also holds in $\mathfrak{B}$.

By the assumption that $\delta(\lambda) = \lambda^+$ there is a model $\mathfrak{B} \equiv \mathfrak{B}_\chi$ such that all the sets in $(*)$ are preserved and there is $\{b_n \mid n < \omega\} \subseteq (\lambda^+)^{\mathfrak{B}}$ satisfying:

$$\mathfrak{B} \models b_n \ni b_{n+1} \text{ for all } n < \omega.$$

Let $a_n := g^{\mathfrak{B}}(b_n)$ for all $n < \omega$.

Since $\alpha < \beta < \lambda \implies \mathfrak{B}_\chi \models [g(\beta) > g(\alpha) + \omega]$, we have that

$$\mathfrak{B} \models a_n \ni a_{n+1} + \omega \text{ for all } n < \omega.$$

Since

$$\mathfrak{B} \models [\alpha \in \lambda^+ \implies F(\alpha) \in \mathrm{PC}(T_0, q_0, L)]$$

we have that $M_a := F^{\mathfrak{B}}(a) \in \mathcal{K}_\lambda$ for all $a \in (\lambda^+)^{\mathfrak{B}}$.

Let

$$I := \{a \in (\lambda^+)^{\mathfrak{B}} \mid \mathfrak{B} \models a \in a_n \text{ for all } n < \omega\}.$$

Notice that I is directed. I is not empty, moreover, $\lambda \subseteq I$ and

$$\alpha \in I \implies \alpha + 1 \in I.$$

Let $M := \bigcup_{a \in I} F^{\mathfrak{B}}(a)$, by (*) it is in $\mathcal{K}_\lambda$ and let $M_{a_n} := F^{\mathfrak{B}}(a_n)$.

It is enough to verify that $\{M_{a_n} \mid n < \omega\}$ is as required, namely

CLAIM 8.20. *$M_{a_{n+1}} \leq_{\mathcal{K}} M_{a_n} \in \mathcal{K}_\lambda$ for all $< \omega$ and $M = \bigcap_{n<\omega} M_n$.*

PROOF. The first part follows since $\{a_n \mid n < \omega\}$ is decreasing and the function F is monotonic from $\langle (\lambda^+)^{\mathfrak{B}}, \in^{\mathfrak{B}} \rangle$ into $\langle \mathcal{K}_\lambda, \leq_{\mathcal{K}} \rangle$.

The inclusion $\subseteq$ is clear as $\{F^{\mathfrak{B}}(a) \mid a \in {\lambda^+}^{\mathfrak{B}}\}$ is $\precneqq_{\mathcal{K}}$-increasing.

Now given $x \in \bigcap_{n<\omega} F^{\mathfrak{B}}(a_n)$ by the definition of $i(x)$ we get that $i(x) \leq a_n$ for every n. As $a_{n+1} \in a_n$ we get that $i(x) < a_n$ for all n. I.e. $i(x) \in I$ thus $x \in F^{\mathfrak{B}^*}(i(x)) \subseteq M$. □

□

It is a major open question if the assumption $\delta(\lambda) = \lambda^+$ can be removed from Theorem 8.5.

The following is a plausible conjecture:

CONJECTURE 8.21. *Let $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in λ and $\mathrm{I}(\lambda^+, \mathcal{K}) < 2^{\lambda^+}$ then $\mathcal{K}_{\lambda^{++}} \neq \emptyset$.*

Perhaps a much easier is the following:

CONJECTURE 8.22. *Let $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical both in λ and λ^+ then $\mathcal{K}_{\lambda^{++}} \neq \emptyset$.*

At present for $\lambda = \aleph_0$ under a weak set-theoretic axiom we have a positive solution for $\mathcal{K}$ which are also $\mathrm{PC}_{\aleph_0}$ (see [**Sh88**]). There is no known instance of the conjectures that is proved in ZFC.

When λ is uncountable the conjectures are much more difficult. Even under the additional assumption that $\mathcal{K}$ is a PC_λ-class the conjecture seems to be very difficult. A proof to the following approximation of Conjecture 8.22 took Shelah 99 pages:

THEOREM 8.23 (Shelah [**Sh 576**]). *Suppose that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. Let $\mathcal{K}$ be an AEC with $\mathrm{LS}(\mathcal{K}) \leq \lambda$ which is PC_λ.*

If $\mathrm{I}(\lambda, \mathcal{K}) = \mathrm{I}(\lambda^+, \mathcal{K}) = 1 \leq \mathrm{I}(\lambda^{++}, \mathcal{K}) < 2^{\lambda^{++}}$ then $\mathcal{K}_{\lambda^{+++}} \neq \emptyset$.

Shelah developed a highly complicated and important technology for his proof (around the notion of *λ-good frame*). Perhaps the most important step in his proof of Theorem 8.23 is concluding from all the assumptions that a λ^+-good frame exists. The machinery needed for this theorem of Shelah will be not developed in this book, I plan to present it in Volume III.

9. Categoricity in $\aleph_1$ for AECs is not absolute

We will show in Theorem 6.11.4 that

THEOREM 9.1 (Shelah). *Let T be a first-order theory in a countable language. The statement "T is $\aleph_1$-categorical" is absolute (i.e. if this is true in a model of ZFC then it is true in all models of ZFC).*

However this is not true for AECs.

In this section we introduce an AEC $\mathcal{K}$ which is $\mathrm{PC}_{\aleph_0}$ with $\mathrm{LS}(\mathcal{K}) = \aleph_0$, fails to have the $\aleph_0$-amalgamation property and is $\aleph_0$-categorical. We get two very different behaviors of this class:

THEOREM 9.2 (WCH). $\mathrm{ZFC} + 2^{\aleph_0} < 2^{\aleph_1} \vdash \mathrm{I}(\aleph_1, \mathcal{K}) = 2^{\aleph_1}$.

In the next section we will establish that the statement of this theorem holds for a fairly large family of AECs, see Theorem 13.1.

On the other hand we will show that

THEOREM 9.3 (MA). $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1} + \neg\mathrm{CH} \vdash \mathrm{I}(\aleph_1, \mathcal{K}) = 1$.

DEFINITION 9.4 (Martin's Axiom). Let $\langle P, <_P \rangle$ be a partially-ordered set. A subset D of P is called *dense* provided for every $p \in P$ there exists $q \in D$ such that $p \leq_P q$.

A partially-ordered set $\langle P, <_P \rangle$ satisfies the *countable chain condition*, *ccc* for short if for every $\{p_i \mid i < \omega_1\} \subseteq P$ there are $i \neq j < \omega_1$ such that $p_i \leq p_j$ or $p_j \leq p_i$.

Martin's Axiom for λ is denoted by MA_λ is the statement: For every partially-ordered set $\langle P, <_P \rangle$ and any family of dense sets $\{D_i \subseteq P \mid i < \lambda\}$ there exists a directed $G \subseteq P$ such that $G \cap D_i \neq \emptyset$ for all $i < \lambda$.

FACT 9.5 (Solovay and Tennebaum 1969?).

$\mathrm{Con}(\mathrm{ZF}) \implies \mathrm{Con}(\mathrm{ZFC} + \mathrm{MA}_\lambda + \neg\mathrm{CH})$ *for every* $\lambda < 2^{\aleph_0}$.

NOTATION 9.6. Let μ be a cardinal and P a set, by $S_{<\mu}(P)$ denote the set

$$\{S \subseteq P \ : \ |S| < \mu\}.$$

Recall that if $|P| = \lambda$ then $|S_{<\mu}(P)| = \lambda^{<\mu}$.

First I will give an informal description of $\mathcal{K}$. Fix a cardinal number λ satisfying $\lambda = \lambda^{<\lambda}$.

The idea behind the definition of $\mathcal{K}$ is to consider a class of bipartite graphs, such that the left hand side (denoted by P) is always λ and the right-hand side

(denoted by Q) is a family of subsets of λ and the binary relation $\in$ (subset of $P \times Q$) satisfies the axioms of random bipartite graphs.

Recall the first-order theory T_{ind}: Its language consists of a single binary relation E and the axioms are $\{\forall x \forall y[xEy \to yEx] \wedge \forall x[\neg(xEx)]\} \cup \{\varphi_{n,k} : n, k < \omega\}$ where $\varphi_{n,k}$ stands for the sentence

$$\forall x_0 \dots \forall x_{n-1} \forall y_0 \dots \forall y_{k-1} \Big[[\bigwedge_{i<j<n} x_i \neq x_j \wedge \bigwedge_{i<j<k} y_i \neq y_j \wedge \bigwedge_{\substack{i<n \\ j<k}} x_i \neq y_j] \to \exists z \bigwedge_{\substack{i<n \\ j<k}} zEx_i \wedge \neg(zEy_j) \Big].$$

The countable model of T_{ind} is unique up to isomorphism (see previous exercise) and is known by the name of *the countable random graph* and also as *The Erdős-Rényi graph*, see [**ErRe 1960**].

Before going into the technical definition of the class $\mathcal{K}$ I will explain how we can make two structures from $\mathcal{K}_{\aleph_1}$ to be isomorphic and will see that very few homogeneity properties of $\mathcal{K}$ will give this.

The following is important.

THEOREM 9.7 (Failure of amalgamation). *For every $\lambda \geq \aleph_0$ There exists an Abstract Class $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ such that $\mathcal{K}$ fails to have the λ-amalgamation property.*

DEFINITION 9.8 ((λ, μ)-independent family). Let $\lambda \geq \mu \geq \aleph_0$, P a set of cardinality λ, $\mathcal{F} := \{S_i \mid i \in I\}$ a family of subsets of P. We say that $\mathcal{F}$ is (λ, μ)- *independent* (ind. for short) iff for every pairwise disjoint $J_1, J_2 \in S_{<\mu}(I)$ we have that

$$|\bigcap_{i \in J_1} S_i \cap \bigcap_{j \in J_2} (P - S_j)| = \lambda.$$

REMARK 9.9. If $\mathcal{F}$ is a (λ, μ)-independent family of subsets of P then also
$\mathcal{F} - \{S\} \cup \{P - S\}$ is (λ, μ)-independent for every $S \in \mathcal{F}$.

REMARK 9.10. If $\mathcal{F}$ is a (λ, μ)-independent family of subsets of P and $S \in \mathcal{F}$ also
$\mathcal{F} - \{S\} \cup \{S \cup B - A\}$ is (λ, μ)-ind. for every $A, B \in S_{<\mu}(P)$.

DEFINITION 9.11 (μ-independent family). $\mathcal{F} := \{S_i \mid i \in I\}$ a family of subsets of P. We say that $\mathcal{F}$ is μ- *independent* iff for every pairwise disjoint $J_1, J_2 \in S_{<\mu}(P)$ we have that

$$\bigcap_{i \in J_1} S_i \cap \bigcap_{j \in J_2} (P - S_j) \neq \emptyset.$$

When $\mathcal{F}$ is a family of subsets of a set of cardinality λ then clearly μ-independence follows from (λ, μ)-independence. However if we assume in addition a separation property (not unlike like T_1 for topological spaces) then μ-independence is equivalent to (λ, μ)-indepependence We are interested in this as it is easier for axiomatize in a formal logic the μ-independence requirement.

DEFINITION 9.12 (Separation property). Let $\mathcal{F}$ be a family of subsets of P. We say that $\mathcal{F}$ has the *separation property* provided for every $a_1 \neq a_2 \in P$ there exists $S \in \mathcal{F}$ such that $a_1 \in S$ and $a_2 \notin S$.

We say that $\mathcal{F}$ has the *μ-separation property* provided for every disjoint $A, B \in S_{<\mu}(P)$ there exists $S \in \mathcal{F}$ such that $A \subseteq S$ and $B \cap S = \emptyset$.

PROPOSITION 9.13. *Let $\mu \leq \lambda$ be infnite cardinal numbers. Suppose P is a set of cardinality λ and $\mathcal{F}$ is a family of subsets of P.*

If $\mathcal{F}$ is μ-independent and satisfy the separation property then $\mathcal{F}$ is (λ, μ)-independent

PROOF. Exercise. □

PROPOSITION 9.14 (Extension of independent families). *Suppose λ is regular and satisfies $\lambda = \lambda^{<\mu}$. Let P be a set of cardinality λ. For every family $\mathcal{F} := \{S_i \mid i < \alpha \leq \lambda\}$ of (λ, μ)-independent subsets of P there exists $X \subseteq P$ of cardinality λ such that $X \notin \mathcal{F}$ and $\mathcal{F} \cup \{X\}$ is (λ, μ)-independent.*

Moreover, ? fix for $\lambda > \aleph_0$? ?!

if $\mathcal{F}$ is a (λ, μ)-independent family of subset of P of cardinality $\leq \lambda$ then there exists $\{S_i \mid i < \lambda\} \subseteq \mathcal{P}(P)$ such that $\mathcal{F} \cup \{S_i \mid i < \lambda\}$ is (λ, μ)-ind and has the μ-separation property.

PROOF. Without loss of generality we may assume that $P = \lambda$. By induction on α : For $\alpha = 0$ (i.e. $\mathcal{F} = \emptyset$). Let $X := \{2i + 1 \mid i < \lambda\}$.

For $\alpha > 0$: Let

$$\mathcal{F}^* := \left\{ \bigcap_{i \in J_1} S_i \cap \bigcap_{j \in J_2} (P - S_j) \ : \ J_1, J_2 \subseteq \alpha, J_1 \cap J_2 = \emptyset, |J_1| + |J_2| < \mu \right\}.$$

Notice that if $X \subseteq P$ is such that both $X \cap S$ and $(P - X) \cap S$ have cardinality λ for all $S \in \mathcal{F}^*$ then $\mathcal{F} \cup \{X\}$ is (λ, μ)-independent.

Since $\lambda = \lambda^{<\mu}$ fix $\langle B_i \mid i < \lambda \rangle = \mathcal{F}^*$ such that every element of $\mathcal{F}^*$ appears λ-many times.

We will define functions $f, g : \lambda \to \lambda$ both increasing such that the set $X := \{f(i) \mid i < \lambda\}$ is as required.

The requirements on the functions are:

(1) $0 < i < j \implies [g(i) < f(i) < f(j) \wedge f(i) < g(j)]$
(2) $f(i) \in B_i$ and $g(i) \in P - B_i$ for all $i > 0$

The definitions of the functions are by induction on $i < \lambda$:

For $i = 0$ let $g(0) = f(0) := 0$.

For $i > 0$: Notice that $|P - B_i| = \lambda$. Using $\lambda = \lambda^{<\lambda}$ we have that $\sup_{\alpha<i} f(\alpha) < \lambda$ let

$$g(i) := \min\{j \in P - B_i \mid j > \sup_{\alpha<i} f(\alpha)\}.$$

Similarly as $|B_i| = \lambda$ there exists $j \in B_i$ such that $j > g(i)$, let

$$f(i) := \min\{j \in B_i \mid j > g(i)\}.$$

Given $S \in \mathcal{F}^*$ let $\{i_\alpha \mid \alpha < \lambda\}$ be increasing such that $S = B_{i_\alpha}$ for every $\alpha < \lambda$. By the above we get $X \cap S \supseteq \{f(i_\alpha) \mid \alpha < \lambda\}$, so $|X \cap S| = \lambda$. The fact that $|(P - X) \cap S| = \lambda$ holds follows from $(P - X) \cap S \supseteq \{g(i_\alpha) \mid \alpha < \lambda\}$. $\square$

The description of $\mathcal{K}$. First, $\mathrm{L}(\mathcal{K}) = \{P, Q, \in, E\}$, where P and Q are unary predicates and $\in$ is a binary predicate. Then $M \in \mathcal{K}$ if $M = \langle |M|, P^M, Q^M, \in^M, E^M\rangle$ and

(1) P^M has cardinality λ;
(2) $|M| = P^M \cup Q^M$, $\in^M \subseteq P^M \times Q^M$ and $P^M \cap Q^M = \emptyset$;
(3) E^M is an equivalence relation on Q^M;
(4) If we let $A^M_y := \{x \in P^M \mid x \in^M y\} \subseteq P^M$, we make the requirement: There exists $S \subseteq Q^M$ of cardinality $\|M\|$ such that $b_1 \neq b_2 \in S \implies \neg[b_1 E^M b_2]$ and the family $\mathcal{F} := \{A^M_b \mid b \in S\}$ is μ- independent.
(5) Given sets $A, B \subseteq P^M$, we denote

$$\Gamma^M_{A,B} := \{y \in Q^M \mid A \subseteq A^M_y \wedge B \cap A^M_y = \emptyset\}$$

We require that for any disjoint A, B subsets of P^M of cardinality $< \mu$,

$$|\Gamma^M_{A,B}| = \|M\| \text{ and } \forall y \in Q^M \exists y' \in \Gamma^M_{A,B}[y \neq y' \wedge yEy'].$$

For models $M, N \in \mathcal{K}$, we define $M \preceq N$ iff

$$M \subseteq N \text{ and } P^M = P^N.$$

On $\mathcal{K}$ we define a second relation $\preceq^*$ as follows:

$M \prec^* N \iff M \preceq N$ and $|\Gamma^N_{A,B} - \Gamma^M_{A,B}| \geq \lambda$ for all disjoint $A, B \subseteq P^M$ s.t. $|A| + |B| < \mu$.

First, $\mathrm{L}(\mathcal{K}^2) = \{P, Q, \in, E\}$, where P and Q are unary predicates and $\in, E$ are binary predicates. Then $M \in \mathcal{K}^2$ if $M = \langle |M|, P^M, Q^M, \in^M, E^M\rangle$ and

(1) P^M has cardinality λ;

(2) $|M| = P^M \cup Q^M$, $\in^M \subseteq P^M \times Q^M$ and $P^M \cap Q^M = \emptyset$;

Note that the definition of the class $\mathcal{K}$ and $\preceq^*$ depends on the cardinals μ and λ. When the identity of the cardinals is clear we write $\mathcal{K}$ instead of $\mathcal{K}(\lambda, \mu)$.

REMARK 9.15. Notice that any $M \in \mathcal{K}$ satisfies the extensionality axiom: Given $b_1 \neq b_2 \in Q^M$ then by independence we get that $S^M_{b_1} \cap (P - S^M_{b_2})$ has cardinality λ so in particular there exists $a \in P^M$ such that $M \models [a \in b_1] \wedge \neg[a \in b_2]$.

Thus if $M \in \mathcal{K}$ then $\lambda \leq \|M\| \leq 2^\lambda$

THEOREM 9.16. *Let $\lambda \geq \mu \geq \aleph_0$ satisfying $\lambda = \lambda^{<\mu}$. The class $\langle \mathcal{K}, \preceq \rangle$ is a λ-AEC, it fails the first union axiom (for chains of length λ^+). The class $\langle \mathcal{K}, \preceq^* \rangle$ is an abstract class satisfying the first union axiom for chains of lenght $\leq \lambda^+$ and the DSL axiom; the coherence axiom fails. In addition we have the following properties:*

(1) *$\mathcal{K}_\lambda \neq \emptyset$ and $\mathcal{K}$ has no maximal elements of cardinality λ, namely if $M \in \mathcal{K}$ is of cardinality λ then there exists $N \in \mathcal{K}$ such that $M \prec^* N$. Thus there exists $N \in \mathcal{K}$ of cardinality λ^+;*
(2) *$\mathrm{LS}(\mathcal{K}) = \lambda$;*
(3) *The λ-AP fails for $\prec$ and also for $\prec^*$. Moreover for any $M_0 \in \mathcal{K}_\lambda$ and every $M_1 \in \mathcal{K}_\lambda$ such that $M_0 \prec M_1$ (or $M_0 \prec^* M_1$) there exists $M_2 \in \mathcal{K}_\lambda$ satisfying $M_0 \prec M_2$ (or $M_0 \prec^* M_2$) such that M_1 and M_2 can't be amalgamated over M_0;*
(4) *If $\mu = \lambda$ then $\mathrm{I}(\lambda, \mathcal{K}) = 1$;*
(5) *If $\mu \leq \lambda$ and $2^\lambda < 2^{\lambda^+}$ then $\mathrm{I}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$. If in addition $(2^\lambda)^+ < 2^{\lambda^+}$ then $\mathrm{IE}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$.*
(6) *If $\mu \leq \lambda$ and $2^\lambda < 2^{\lambda^+}$ then $\mathcal{K}_{\lambda^+}$ has no universal model.*
(7) *If $\mu = \lambda$ then $\mathcal{K}$ has the λ-JMP.*
(8) *If $\lambda = \aleph_0$ then $\mathrm{MA} + \neg\mathrm{CH}$ implies that $\mathrm{I}(\aleph_1, \mathcal{K}) = 1$;*
(9) *If $\lambda = \mu = \aleph_1$ there is a model of set theory where $2^{\aleph_0} = \aleph_1,\ 2^{\aleph_1} = 2^{\aleph_2}$ and $\mathrm{I}(\aleph_2, \mathcal{K}) = 1$.*

PROOF OF (1). Construction of $M \in \mathcal{K}_\lambda$:

Let $P^M := \lambda$. Take S to be the set of even ordinals $< \lambda$ as $\{S\}$ is a stochastically independent family of subsets of P^M an application of Proposition 9.14 λ-many times produces a stochastically independent family $\mathcal{F}$ of subsets of P^M of cardinality λ such that $\mathcal{F} \supset \{S\}$. Another application of Proposition 9.14 (λ-many times) gives $\{S_i \subseteq P^M \mid i < \lambda\}$ such that $\mathcal{F} \cup \{S_i \mid i < \lambda\}$ is stochastically independent. Since $\lambda = \lambda^{<\mu}$ there exists $\langle (A_i, B_i) \mid i < \lambda \}$ enumeration of $\{(A, B) \mid A, B \in \mathcal{P}_{<\mu}(P^M), A \cap B = \emptyset\}$ such that for every (A, B) as above the set $\{i < \lambda \mid (A_i, B_i) = (A, B)\}$ has cardinality λ.

Let $\mathcal{F}_1 := \mathcal{F} \cup \{S_i \cup A_i - B_i \mid i < \lambda\}$. By Remark 9.10 it is also s.i. Now let $M := \langle \lambda \cup \mathcal{F}_1, \lambda, \mathcal{F}_1, \in \rangle$. Verify that $M \in K_\lambda$.

Suppose $M \in \mathcal{K}_\lambda$ is given. Let $\mathcal{F} := \{S^M_b \mid b \in Q^M\}$. Since $M \in \mathcal{K}$ the family $\mathcal{F}$ is s.i. By Proposition 9.14 there exists $\{S_i \subseteq P^M \mid i < \lambda\}$ such that $\mathcal{F} \cup \{S_i \mid i < \lambda\}$ is stochastically independent. Using the above notation, let $N := \langle \lambda \cup \mathcal{F}_1, \lambda, \mathcal{F}_1, \in \rangle$. Verify that $N \in \mathcal{K}_\lambda$ and that $M \prec^* N$.

As $\mathcal{K}$ satisfies the first union axiom for $\prec^*$-chains of length $\leq \lambda^+$ using the previous part by induction on $\alpha \leq \lambda^+$ define an increasing and continuous $\prec^*$-chain $\{M_\alpha \mid \alpha < \lambda^+\} \subseteq \mathcal{K}_\lambda$, and let $M := \bigcup_{\alpha < \lambda^+} M_\alpha$. Clearly $M \in \mathcal{K}_{\lambda^+}$. $\square$

PROOF OF (2). Trivial. $\square$

PROOF OF (3). Let $N_0 \prec^* N_1$ both of cardinality λ, fix $b \in Q^{N_1} - Q^{N_0}$. We define N_2 as follows, its universe is $|N_1|$, namely $P^{N_2} = P^{N_0}$ and $Q^{N_2} = Q^{N_1}$. We define $\in^{N_2}$ as follows:

$$a \in^{N_2} b := \begin{cases} a \in^{N_1} c & \text{if } c \neq b \\ \neg[a \in^{N_1} b] & \text{otherwise.} \end{cases}$$

Suppose N is an amalgam of N_1 and N_2 over N_0, we may assume $N \succ N_2$ and there exists an embedding $f : N_1 \underset{N_0}{\to} N$. Suppose $a \in A^{N_1}_b$. By the choice we have $N_1 \models [a \in b]$. An application of f gives $N \models [f(a) \in f(b)]$. As $f \restriction P = \mathrm{id}_P$ we get $N \models a \in f(b)$. Thus $a \in S^N_{f(b)}$, By the definition of $\in^{N_2}$ we get $a \notin S^N_b$. By the extensionality axiom we get in particular that $f(b) \neq b$. By the independence of $\{S^N_c \mid c \in Q^N\}$ the set $S^N_b \cap S^B_{f(b)}$ has cardinality λ, but the above argument established that they are disjoint! $\square$

PROOF OF (4). Let $M, N \in \mathcal{K}_\lambda$, fix $\{c_i \mid i < \lambda\} = |M|$ and $\{d_i \mid i < \lambda\} = |N|$. Define $\{f_i \mid i < \lambda\}$ partial isomorphisms from M into N such that $|f_i| < \lambda$ for all $i < \lambda$ and for every $i < \lambda$ there exists $j(i) < \lambda$ such that $c_i \in \mathrm{dom}\, f_{j(i)}$ and $d_i \in \mathrm{range}\, f_{j(i)}$.

The definition is by induction on i: For $i = 0$ let $f_0 := \emptyset$. For $i > 0$, there two possibilities:

i is even. Let $f := \bigcup_{j<i} f_j$ and $\alpha := \min\{\alpha < \lambda \mid c_\alpha \notin \mathrm{dom}\, f\}$.

Case A $c_\alpha \in P^M$. Let

$$\{b_\gamma \mid \gamma < \beta < \lambda\} = \mathrm{dom}\, f \cap Q^M \text{ and } \gamma_1 \leq \beta$$

such that

$$M \models c_\alpha \in b_\zeta \text{ for all } \zeta < \gamma \text{ and } M \models \neg[c_\alpha \in b_\zeta] \text{ for every } \zeta \geq \gamma_1.$$

As f is injective $\zeta_1 \neq \zeta_2 \implies f(b_{\zeta_1}) \neq f(b_{\zeta_2})$. By the independence assumption we get

$$(*) \quad \bigcap_{\zeta<\gamma_1} S^N_{f(b_\zeta)} \cap \bigcap_{\gamma_1 \leq \zeta < \beta} (P^N - S^N_{f(b_\zeta)}) \text{ is a subset of } P^N \text{ of cardinality } \lambda.$$

As $\operatorname{range} f$ has cardinality $< \lambda$ there exist a element of $(*)$ such that $a \notin \operatorname{range} f$. Let $f_i := f \cup \{\langle c_\alpha, a\rangle\}$.

Case B $c_\alpha \in Q^M$. Let

$$\{a_\gamma \mid \gamma < \beta < \lambda\} = \operatorname{dom} f \cap P^M \text{ and } \gamma_1 \leq \beta$$

such that

$$M \models a_\zeta \in c_\alpha \text{ for all } \zeta < \gamma \text{ and } M \models \neg[a_\zeta \in c_\alpha] \text{ for every } \zeta \geq \gamma_1.$$

As f is injective the sets

$$A := \{f(a_\zeta) \mid \zeta < \gamma_1\} \text{ and } B := \{f(a_\zeta) \mid \zeta \geq \gamma_1\} \text{ are pairwise disjoint.}$$

Since $|\Gamma^N_{A,B}| = \lambda$ there exists $b \in \Gamma^N_{A,B} - \operatorname{range} f$. Define f_i as $f \cup \{\langle a_\alpha, b\rangle\}$.

i is odd. Let $f := \bigcup_{j<i} f_j$ and $\alpha := \min\{d_\alpha \notin \operatorname{dom} f\}$. Now repeat the previous argument for f^{-1} to find an extension f_i of f such that $d_\alpha \in \operatorname{range} f_i$.

Verify that $\bigcup_{i<\lambda} f_i : M \cong N$. □

PROOF OF (5). For every $\eta \in {}^{\lambda^+ >}2$ define $M_\eta \in \mathcal{K}_\lambda$ as follows

(1) $\eta \lhd \nu \implies M_\eta \prec^* M_\nu$,
(2) if $\ell(\eta)$ is limit then $M_\eta; = \bigcup_{i<\ell(\eta)} M_{\eta\restriction i}$,
(3) for every $\rho \in {}^{\lambda^+ >}2$ the models $M_{\rho^\frown 1}$ and $M_{\rho^\frown 0}$ can't be amalgamated over $M_{\langle\rangle}$.

The construction of $\{M_\eta \mid \eta \in {}^{\lambda^+ >}2\} \subseteq \mathcal{K}_\lambda$ is by induction o $\ell(\eta)$:

For $\ell(\eta) = 0$; pick $M_{\langle\rangle} \in \mathcal{K}_\lambda$.

For $\ell(\eta)$ limit let $M_\eta := \bigcup_{\alpha<\ell(\eta)} M_{\eta\restriction\alpha}$. Notice that since $\langle K, \prec^*\rangle$ satisfies the first union axiom for chains of length $\leq \lambda^+$ we have that $M_\eta \in \mathcal{K}_\lambda$ and for all $\nu \lhd \eta$ we have that $M_\nu \prec^* M_\eta$.

Given $\eta \in {}^{\lambda^+ >}2$, using (1) and the moreover part of (3) select $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ both from $\mathcal{K}_\lambda$ such that $M_\eta \prec^* M_{\eta^\frown \ell}$ and the models $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ can't be amalgamated over $M_{\langle\rangle}$.

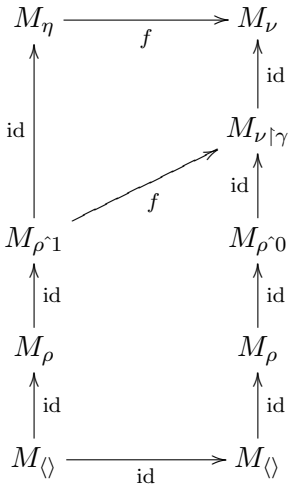
For $\eta \in {}^{\lambda^+}2$ let $M_\eta := \bigcup_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$. Clearly $M_\eta \in K_{\lambda^+}$.

CLAIM 9.17. $\eta \neq \nu \in 2^{\lambda^+} \implies \langle M_\eta, a\rangle_{a\in M_{\langle\rangle}} \ncong \langle M_\nu, a\rangle_{a\in M_{\langle\rangle}}$.

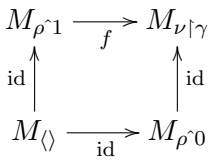
PROOF. For the sake of contradiction, suppose $\eta \neq \nu \in {}^{\lambda^+}2$ and there exists $f : \langle M_\eta, a\rangle_{a\in M_{\langle\rangle}} \to \langle M_\nu, a\rangle_{a\in M_{\langle\rangle}}$. Namely, $f : M_\eta \to M_\nu$ and $f \restriction M_{\langle\rangle} = \mathrm{id}_{M_{\langle\rangle}}$.

Let $\alpha := \min\{\alpha < \lambda \mid \eta[\alpha] \neq \nu[\alpha]\}$. Denote by ρ the sequence $\eta \restriction \alpha$ (notice that it is equal also to $\nu \restriction \alpha$).

Without loss of generality we may assume that $\eta[\alpha] = 1$ and $\nu[\alpha] = 0$. Thus $M_{\rho\hat{}0} \prec^* M_\nu$. Since $M_{\rho\hat{}1} \prec^* M_\eta$ and has cardinality λ also the set $|M_{\rho\hat{}0}| \cup f[|M_{\rho\hat{}1}|]$ is a subset of M_ν of cardinality λ there exists $\gamma < \lambda^+$ such that $|M_{\rho\hat{}0}| \cup f[|M_{\rho\hat{}1}|] \subseteq M_{\nu\restriction\gamma}$. Namely, the following diagram commutes



So we get that next diagram commutes:



This is a contradiction to requirement (3) in the construction.

□

We finish the proof of (5) by the following:

PROPOSITION 9.18. *If* $2^\lambda < 2^{\lambda^+}$ *then* $\mathrm{I}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$ *and*

$$(2^\lambda)^+ < 2^{\lambda^+} \implies \mathrm{IE}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}.$$

PROOF. For the sake of contradiction suppose

$$\mu := \mathrm{I}(\lambda^+, \mathcal{K}) + \aleph_0 < 2^{\lambda^+}$$

Let $\langle N_i \in \mathcal{K}_\lambda \mid i < \mu\rangle$ such that $M \in \mathcal{K}_\lambda \implies \exists i < \mu[M \cong M_i]$.

Observe the following

$$\{\langle M_\eta, a\rangle_{a\in M_{\langle\rangle}}/\cong :\ \eta \in {}^{\lambda^+}2\} \subseteq$$
$$\{\langle N_i, a\rangle_{a\in A} \mid i<\mu, A\subseteq |N_i|, \text{ and } A \text{ is of cardinality } \lambda\}.$$

Now estimate the cardinality of the second set: Its cardinality is

$$\leq \mu^+ \cdot (\lambda^+)^\lambda = \mu\cdot\lambda^+\cdot 2^\lambda \text{ by our assumptions } < 2^{\lambda^+}.$$

This is a contradiction to Claim 9.17.

Now suppose that $(2^\lambda)^+ < 2^{\lambda^+}$. Consider

$$f(\eta) := \{\nu \in {}^{\lambda^+}2 \mid \exists g : M_\nu \to M_\eta\} \text{ for every } \eta \in {}^{\lambda^+}2\,.$$

CLAIM 9.19. *$|f(\eta)| \leq 2^\lambda$ for every $\eta \in {}^{\lambda^+}2$.*

PROOF. Otherwise there are $\eta \in {}^{\lambda^+}2$ and a set $S \subseteq {}^{\lambda^+}2$ of cardinality $(2^\lambda)^+$ such that $\nu\in S \implies \exists g_\nu : M_\nu \to M_\eta$. As $g_\eta[M_{\langle\rangle}]$ is a subset of M_η of cardinality λ and M_η has at most $(\lambda^+)^\lambda (= 2^\lambda)$ such sets there exists $S^* \subseteq S$ of cardinality $(2^\lambda)^+$ such that $\nu_1,\nu_2\in S^* \implies g_{\nu_1}[M_{\langle\rangle}] = g_{\nu_2}[M_{\langle\rangle}]$. Let $\nu_1\neq\nu_2\in S^*$ and $g := g_{\nu_2}^{-1}\circ g_{\nu_1}$, g is an isomorphism from M_{ν_1} onto M_{ν_2} such that $g\restriction M_{\langle\rangle} = \mathrm{id}$. This leads to a contradiction of requirement (3) in the construction of the tree of models exactly like in the argument in the proof of Claim 9.17. □

Since f is a set valued function such that $|\operatorname{dom} f| = 2^{\lambda^+} > (2^\lambda)^+$ and $|f(\eta)|\leq 2^\lambda$ by Hajnal's free subset theorem **??** there exists $S\subseteq {}^{\lambda^+}2$ of cardinality 2^{λ^+} such that $\eta\neq\nu\in S \implies \nu\notin f(\eta)$. □

PROOF OF (6). Similar to the argument used in (6). □

PROOF OF (7). Use (1) and (4). □

PROOF OF (8). Suppose $M,N\in K_{\aleph_1}$. By the isomorphism axiom we can rename the elements of the universe of N and assume that $|M|\cap|N| = \emptyset$. Fix $\{M_i \mid i<\omega_1\}$ and $\{N_i \mid i<\omega_1\}$ both $\prec^*$-increasing and continuous resolutions of M and N into countable $\prec^*$-substructures such that

$$i<\omega_1 \implies |M_{i+1}-M_i| = |N_{i+1}-N_i| = \aleph_0,\ P^{M_0} = P^M \text{ and } P^{N_0} = P^N.$$

They exists since $\langle \mathcal{K}, \prec^*\rangle$ satisfies the DLS axiom and the first union axiom for chains of length $\leq\lambda^+$. Consider

$$P := \{f \text{ a finite partial isomorphism from } M \text{ into } N\ :$$
$$\forall i<\omega_1 \forall a\in \operatorname{dom} f[a\in M_i \leftrightarrow f(a)\in N_i]\}.$$

P is a partially ordered set with the relation $\subseteq$.

Let

$$D_a := \{f \in P \mid a \in \operatorname{dom} f\}, \text{ for } a \in M,$$
$$D^*_a := \{f \in P \mid a \in \operatorname{range} f\}, \text{ for } a \in N.$$

Notice that if $G \subseteq P$ is generic for $\{D_a \mid a \in M\} \cup \{D^*_a \mid a \in N\}$ then $\cup G : M \cong N$.

CLAIM 9.20. *D_a is a dense subset of P for every $a \in M$.*

PROOF. Let $f \in P$ and $a \in M$ be given. Consider $\alpha := \min\{\alpha < \omega_1 \mid a \in M_\alpha\}$, by continuity of the chain we have that $\alpha = 0$ of $\alpha = \beta + 1$ for some ordinal β.

Let B and C be subsets of $\operatorname{dom} f$ such that

$$B \cup C = \operatorname{dom} f \text{ and } B \subseteq P^M, C \subseteq Q^M.$$

If $a \in P^M$. Let

$$p_a := \{x \in c \mid c \in C \wedge M \models [a \in c]\} \cup \{\neg x \in c \mid c \in C \wedge M \models \neg[a \in c]\}.$$

Since $\{A^N_{f(c)} \mid c \in C\}$ are independent there exists $a' \in P^N$ satisfying $f(p_a)$ and $a' \notin \operatorname{range} f$. Check that $h := f \cup \langle a, a' \rangle \in P$.

Suppose $a \in Q^M$. Consider

$$q_a := \{b \in Rx \mid b \in B \wedge M \models b \in a\} \cup \{\neg b \in x \mid b \in B \wedge M \models \neg[b \in a]\}.$$

Let $A_1 := \{f(b) \mid M \models b \in a\}$ and $A_2 := \{f(b) \mid M \models \neg[b \in a\}$. A_1 and A_2 are disjoint finite subsets of Q^{N_α}.

In case $\alpha = 0$ by the definition of $\mathcal{K}$ the set $\Gamma^{N_0}_{A_1,A_2}$ is infinite and thus there exists $a' \in \Gamma^{N_0}_{A_1,A_2}$ satisfying $f(q_a)$ such that $a' \notin \operatorname{range} f$. Clearly $h := f \cup \{\langle a, a' \rangle\} \in P$.

In case $\alpha = \beta + 1$ by the definition of $\prec^*$ the set $\Gamma^{N_\alpha}_{A_1,A_2} - \Gamma^{N_\beta}_{A_1,A_2}$ is infinite and taking a' from this set satisfying $f(q_a)$ such that $a' \notin \operatorname{range} f$. gives us that $f \cup \{\langle a, a' \rangle\} \in P$. □

CLAIM 9.21. *P satisfies the countable chain condition.*

PROOF. Let $\{f_\alpha \mid \alpha < \omega_1\} \subseteq P$ be given. By applying the pigeonhole principle to $i \mapsto P^M \cap \operatorname{dom} f_i$ we may assume that there are

$$\{a_k \mid k < n\} \subseteq P^M \text{ and } \{b_\ell(i) \mid \ell < m\} \subseteq Q^M$$

satisfying

$$i < \omega_1 \implies \operatorname{dom} f_i = \{a_k \mid k < n\} \cup \{b_\ell(i) \mid \ell < m\}.$$

For every $i < \omega_1$ consider consider

$$\operatorname{tp}(i) := \{[x_k \in y_\ell]^{t_i(k,\ell)} \mid k < n, \ell < m\}$$

where

$$t_i(k,\ell) := \begin{cases} True & \text{if } a_k \in A^M_{b_\ell(i)} \\ \\ False & \text{otherwise.} \end{cases}$$

As the set of possible types has cardinality 2^{nm}, using the pigeonhole principle once more we may assume that $i,j \in \omega_1 \implies \mathrm{tp}(i) = \mathrm{tp}(j)$.

Denote by A_α the set $\mathrm{dom}\, f_\alpha \cup \mathrm{range}\, f_\alpha$. Consider $\mathcal{F} := \{A_\alpha \mid \alpha < \omega_1\}$. By the Δ-system lemma there are an uncountable $S \subseteq \omega_1$ and a finite set K such that $\alpha \neq \beta \in S \implies A_\alpha \cap A_\beta = K$.

Let X and Y be subsets of K such that $X \cup Y = K$, $X \subseteq |M|$ and $Y \subseteq |N|$. If $K = \emptyset$ then $f_\alpha \cup f_\beta \in P$ for any $\alpha \neq \beta \in S$. Otherwise we will find $\alpha \neq \beta \in S$ such that $f_\alpha \cup f_\beta \in P$:

Without loss of generality we may assume that $X \neq \emptyset$.

LEMMA 9.22. *The set $S_x := \{\alpha \in S \mid f_\alpha(x) \notin Y\}$ is at most countable for every $x \in X$.*

PROOF. Suppose for the sake of contradiction that S_x is uncountable for some $x \in X$. Let $i(x) := \min\{i \in \omega_1 \mid x \in M_i\}$. Since the elements of P are level preserving we get that $f_\alpha(x) \in N_{i(x)}$ for every $\alpha \in S_x$. By the pigeon hole principle there are uncountable $S^* \subseteq S_x$ and $y \in N_{i(x)} - Y$ such that $\alpha \in S^* \implies f_\alpha(x) = y$. Thus we get that $\alpha \in S^* \implies \mathrm{range}\, f_\alpha \supseteq K \cup \{y\}$, by the disjointness of M and N we get that $\alpha \neq \beta \in S^* \implies S_\alpha \cap S_\beta \supsetneq K$ which is a contradiction to the choice of K. □

By applying the Lemma $|X|$-many times there exists $S' \subseteq S$ uncountable such that $\alpha \in S' \implies f_\alpha \restriction X : X \to Y$. As the number of functions from X to Y is finite there are $\alpha \neq \beta \in S'$ such that $f_\alpha \restriction X = f_\beta \restriction X$.

Since $\mathrm{dom}\, f_\alpha \cap \mathrm{dom}\, f_\beta = X$ and $\mathrm{tp}(\alpha) = \mathrm{tp}(\beta)$ we get that $f_\alpha \cup f_\beta \in P$. □

□

PROOF OF (9). □

Notice that $\mathcal{K}(\lambda,\lambda)$ has three homogeneity property, the first two homogeneity with respect to P (via s.i.) and homogeneity in Q are used to get categoricity in λ and the third provides us with supply of many elements of $Q^N - Q^M$ when $M \prec^* N$ that is used to prove the density of the domain and range sets in the application of $\mathrm{MA}_{\aleph_1}$.

9.1. Getting an AEC with properties (1)-(9). As noted the class $\langle \mathcal{K}, \prec \rangle$ fails to satisfy the coherence axiom as well as the second union axioms. We will show that a small change in the definition of the previous class $\langle \mathcal{K}, \prec \rangle$ produces an AEC satisfying the above properties. The idea is to add an equivalence relation E on Q^M that has infinitely many equivalence classes but each equivalence class has cardinality at most λ and require that a little stronger independence requirement and that taking extensions of models will not add new elements to old equivalence classes. More formally:

DEFINITION 9.23. The description of $\mathcal{K}^2$: First, $\mathrm{L}(\mathcal{K}^2) = \{P, Q, \in, E\}$, where P and Q are unary predicates and $\in, E$ are binary predicates. Then $M \in \mathcal{K}^2$ if $M = \langle |M|, P^M, Q^M, \in^M, E^M \rangle$ and

(1) P^M has cardinality λ;
(2) $|M| = P^M \cup Q^M$, $\in^M \subseteq P^M \times Q^M$ and $P^M \cap Q^M = \emptyset$;
(3) E^M is an equivalence relation on Q^M;
(4) If we let $A^M_y := \{x \in P^M \mid x \in^M y\} \subseteq P^M$, we make the requirement: There exists $S \subseteq Q^M$ of cardinality $\|M\|$ such that $b_1 \neq b_2 \in S \implies \neg[b_1 E^M b_2]$ and the family $\mathcal{F} := \{A^M_b \mid b \in S\}$ is independent.
(5) Given disjoint sets $A, B \subseteq P^M$, we denote
$$\Gamma^M_{A,B} := \{y \in Q^M \mid A \subseteq A^M_y \wedge B \cap A^M_y = \emptyset\}$$
We require that for any disjoint A, B subsets of P^M of cardinality $< \mu$,
$$|\Gamma^M_{A,B}| = \|M\| \text{ and } \forall y \in Q^M \exists y' \in \Gamma^M_{A,B}[y \neq y' \wedge yEy'].$$

For models $M, N \in \mathcal{K}^2$, we define $M \preceq N$ iff $M \subseteq N$ and $P^M = P^N$. On $\mathcal{K}^2$ we define a second relation $\prec^*_2$ as follows: $M \precneqq^*_2 N \iff M \preceq N$,
$$|\Gamma^N_{A,B} - \Gamma^M_{A,B}| \geq \lambda \text{ for all disjoint } A, B \subseteq P^M \text{ s.t. } |A| + |B| < \mu$$
and
$$b \in Q^M \implies b/E^M = b/E^N.$$

The reason for $\langle \mathcal{K}, \preceq^* \rangle$ failing to satisfy the coherence and the second union axioms is due to the fact given $M \precneqq^* N$ for any $b \in Q^N - Q^M$ also $N' := \langle P^N \cup P^N, P^N, Q^N - \{b\}, \in^N \rangle$ is a $\precneqq^*$ extension of M but N' and N fail to satisfy the requirement on $\Gamma^{N'} - \Gamma^M$.

This can't happen in $\langle \mathcal{K}^2, \prec^*_2 \rangle$:

REMARK 9.24 (Strong model completeness). Suppose $M, N \in \mathcal{K}_2$. If $M \subsetneqq N$ such that $P^M = P^N$ and $b/E^M = b/E^N$ for every $b \in Q^M$ then $M \precneqq^*_2 N$.

PROPOSITION 9.25. *Suppose $\lambda = \mu$ and $\lambda = \lambda^{<\lambda}$. Then $\langle \mathcal{K}^2, \precneqq^*_2 \rangle$ is an AEC which is PC_λ and satisfying the following properties: $\mathrm{LS}(\mathcal{K}) = \lambda$, for every $M \in \mathcal{K}^2_\lambda$ there exists $N \in \mathcal{K}^2_\lambda$ such that $M \precneqq^*_2 N$, the λ-weak amalgamation property fails and $\mathcal{K}^2_\lambda$ is categorical.*

PROOF. Exercise.

□

PROPOSITION 9.26. *Suppose* $\lambda = \mu = \aleph_0$. *The class* $\mathcal{K}^2$ *is* $\mathrm{PC}_{\aleph_0}$.

PROOF. □

Combining the results of Theorem 9.16, Proposition **??** we get

COROLLARY 9.27 (The main result of this section). *There exists an AEC* $\mathcal{K}$ *with* $\mathrm{LS}(\mathcal{K}) = \aleph_0$, *which has* $L(\mathbf{Q})$ *axiomatization, is categorical in* $\aleph_0$ *and failing the* $\aleph_0$*-weak amalgamation property such that*

$$2^{\aleph_0} < 2^{\aleph_1} \implies \mathrm{I}(\aleph_1, \mathcal{K}) = 2^{\aleph_1}, \text{ and}$$

$$\mathrm{MA}_{\aleph_1} + 2^{\aleph_0} > \aleph_1 \implies \mathrm{I}(\aleph_1, \mathcal{K}) = 1.$$

REMARK 9.28. It is unknown if there exists an AEC with above properties (like $\mathcal{K}^2$) which has an $L_{\omega_1,\omega}$ axiomatization. Several times Shelah claimed that he found such a class but his argument turned to be false.

It is interesting to point out that for elementary classes we can't get such independence. From our second proof of Morley's categoricity theorem (see section 11) we will establish:

THEOREM 9.29. *Let* T *be a first-order theory in a countable language.* $\mathrm{I}(\aleph_1, \mathrm{T}) = 1$ *is absolute, namely if it holds in a model of* ZFC *then it holds in all of them.*

Exercises 2.9.

(1) Let $\mathcal{K} = \mathcal{K}(\lambda, \mu)$ be the class of bi-partite graphs. If $\mu \leq \lambda$ and $2^\lambda < 2^{\lambda^+}$ then $\mathcal{K}_{\lambda^+}$ has no universal model.
(2) $\mathcal{K}$ is PC_λ.

10. Random power set in higher order

DEFINITION 10.1 (Shelah's Generalized Martin's Axiom($\mathrm{Ax}_0\lambda$)). Suppose λ is an uncountable regular cardinal. Let P be a poset which is λ-complete (i.e. every increasing chain of length $< \lambda$ has a least upper bound) and the following strong λ^+-chain condition: For every $\{p_\alpha \mid \alpha < \lambda^+\} \subseteq P$ there exists $\{A_\alpha \mid \alpha < \lambda^+\}$ increasing sequence of sets such that $|A_\alpha| \leq \lambda$ for every α and if $\operatorname{cf} \alpha = \lambda$ then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and there exists a function $h : \lambda^+ \to \bigcup_{\alpha<\lambda^+} A_\alpha$ such that $h(\alpha) \in A_\alpha$ for every α and there exists C a club in λ^+ such that for every

$$\alpha, \beta \in C \cap S^{\lambda^+}_\lambda \text{ if } h(\alpha) = h(\beta) \text{ then the conditions } p_\alpha, p_\beta \text{ are compatible.}$$

For every $\{D_i \mid i < \beta < 2^\lambda\}$ dense subsets of P there exists a directed $G \subseteq P$ such that $G \cap D_i \neq \emptyset$ for every $i < \beta$.

FACT 10.2 (Shelah [**Sh 80**], a more general result is announced in [**BKSh**]). *For every uncountable regular λ the following is consistent:*

$$\mathrm{ZFC} + \lambda = \lambda^{<\lambda} > \aleph_0 + (Ax_0\lambda) + 2^\lambda > \lambda^+.$$

.

THEOREM 10.3 (The main result of this section). *For any λ satisfying $\lambda = \lambda^{<\lambda}$ there exists an AEC $\mathcal{K}$ with $\mathrm{LS}(\mathcal{K}) = \lambda$, which is PC_λ, is categorical in λ and failing the λ-weak amalgamation property such that*

$$\mathrm{ZFC} + (\mathrm{Ax}_0\lambda) + 2^\lambda > \lambda^+ \vdash \mathrm{I}(\lambda^+, \mathcal{K}) = 1.$$

PROOF. Let λ be uncountable satisfying $\lambda = \lambda^{<\lambda}$. Let $\mu = \lambda$ by Proposition **??** $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be the AEC introduced in Definition 9.23 and denoted there by $\mathcal{K}^2$.

□

11. $\mathrm{Ext}^1_{\mathbf{Z}}(\mathrm{G}, \mathbf{Z})$

DEFINITION 11.1. Let A, B, C be R-modules.

$$0 \to A \to B \to C \to 0$$

is a *short exact sequence*

equivalent extensions

12. Weak amalgamation

From the point of view Shelah's categoricity conjecture it is natural to wonder how much amalgamation is supplied us by the categoricity assumption?

Theorem **??** has the following immediate consecuance:

COROLLARY 12.1. *Let λ be such that Θ'_{λ^+} holds. Suppose $\mathcal{K}$ is an AEC such that $\lambda \geq \mathrm{LS}(\mathcal{K})$, $\mathrm{I}(\lambda, \mathcal{K}) = 1$ and $\mathcal{K}_{\lambda^+} \neq \emptyset$. If $\mathcal{K}$ is categorical in λ^+ then $\mathcal{K}_\lambda$ has the amalgamation property.*

The goal of this section is dual: Get λ-AP or perhaps something weaker from less assumption.

The following is an important notion that appeared implicitly in previous work of Shelah:

DEFINITION 12.2 (weak amalgamation property). Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. We say that $\mathcal{K}$ has the *λ-weak amalgamation property* iff for every $N \in \mathcal{K}_\lambda$ there exists $M \succ_{\mathcal{K}} N$ of cardinality λ such that for any $M_\ell \in \mathcal{K}_\lambda$ ($\ell = 0, 1$) and any $f_\ell : M \to M_\ell$ there exists $M^* \in \mathcal{K}_\lambda$ and mappings $g_\ell : M_\ell \underset{N}{\to} M^*$.

REMARK 12.3. As before we may in this definition assume $M_\ell \succ_{\mathcal{K}} M$ and $g_1 = \mathrm{id}_{M_1}$. I.e. $\mathcal{K}$ has the λ-WAP iff for every $N \in \mathcal{K}_\lambda$ there exists $M \succ_{\mathcal{K}} N$ of cardinality λ such that for every $M_\ell \in \mathcal{K}_\lambda$ both $\prec_{\mathcal{K}}$-extensions of M there are $M^* \succ_{\mathcal{K}} M_1$ and $g : M_0 \underset{N}{\to} M^*$.

Notice that unlike the λ-AP the negation of λ-WAP in addition of being a statement on models that can be amalgamaed implies existence of a model of cardinality λ^+.

A tree-construction similar to the one used in Theorem 3.**??** can be used to get:

THEOREM 12.4. *Let λ be such that $2^\lambda < 2^{\lambda^+}$ holds. Suppose $\mathcal{K}$ is an AEC such that $\lambda \geq \mathrm{LS}(\mathcal{K})$ and every model of cardinlity λ has a proper $\prec_{\mathcal{K}}$-extension. If $\mathcal{K}$ fails to have the λ-weak amalgamation property then $\mathrm{I}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$. Moreover, if $(2^\lambda)^+ < 2^{\lambda^+}$ then $\mathrm{IE}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$.*

PROOF. Fix $N \in \mathcal{K}_\lambda$ witnessing the failure of the λ-WAP. Pick $M_\ell \succ_{\mathcal{K}} M$ of cardinality λ such that M_0 and M_1 can not be amalgamated over M and no $N_0, N_1 \succ_{\mathcal{K}} M_\ell$ can be amalgamated over N.

By induction on $\alpha < \lambda^+$ define $\{M_\eta \in \mathcal{K}_\lambda \mid \eta \in {}^\alpha 2\}$ such that the following conditions are satisfied:

(1) $\eta \lhd \rho$ implies $N \prec_{\mathcal{K}} M_\eta \precneqq_{\mathcal{K}} M_\rho$,

(2) If $\ell(\rho)$ is a limit, then $M_\rho = \bigcup_{\beta<\ell(\rho)} M_{\rho\restriction\beta}$,
(3) $M_{\rho^\frown 0}$ and $M_{\rho^\frown 1}$ cannot be amalgamated over N.

The construction is possible by taking $M_{\langle\rangle} := N$ and $M_{\langle\ell\rangle} := M_\ell$. By the continuity requirement it suffices to deal with α successor. Use the failure of the λ-WAP over N and the choice of M_ℓ to get $M_{\eta^\frown\ell} \succ_{\mathcal{K}} M_\eta$ such that they can not be amalgamated over N and there are λ-many elements in $M_{\eta^\frown\ell}$ which are not in M_η.

For $\eta \in {}^{\lambda^+}2$, let $M_\eta := \bigcup_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$. By requirement (2) we have that $M_\eta \in \mathcal{K}_{\lambda^+}$.

CLAIM 12.5. $\eta \neq \nu \implies \langle M_\eta, a\rangle_{a\in|N|} \ncong \langle M_\nu, a\rangle_{a\in|N|}$

PROOF. Otherwise, there are

$$\eta \neq \nu \in {}^{\lambda^+}2 \text{ and } f : \langle M_\eta, a\rangle_{a\in|N|} \cong \langle M_\nu, a\rangle_{a\in|N|}.$$

Let $\alpha < \lambda^+$ be such that $\eta \restriction \alpha = \nu \restriction \alpha$, $\eta[\alpha] = 0$ and $\nu[\alpha] = 1$.

Denote by ρ the sequence $\eta \restriction \alpha$. Since $M^* := f[M_{\rho^\frown 0}] \prec_{\mathcal{K}} M_\nu$, $M^* \succ_{\mathcal{K}} M_\rho$ and $f \restriction N = \mathrm{id}_N$ the model M_ν is an amalgam of $M_{\rho^\frown 0}$ with $M_{\rho^\frown 1}$ over N, a contradiction to requirement (4). □

Suppose for the sake of contradiction that $\mu := \mathrm{I}(\lambda^+, \mathcal{K}) < 2^{\lambda^+}$. Take $\{M_i \mid i < \mu\}$ a complete set of representatives for isomorphism types from $\mathcal{K}_{\lambda^+}$.

Clearly,

$$|\{\langle M_\eta, a\rangle_{a\in|N|}/\cong \ : \ \eta \in {}^{\lambda^+}2\}| \leq |\{\langle M_i, a\rangle_{a\in A}/\cong : \ i < \mu, A \subseteq |M_i|, \ |A| = \lambda\}|$$
$$\leq \mu\|M_i\|^\lambda = \mu(\lambda^+)^\lambda = \mu(\lambda^+ 2^\lambda) = \mu 2^\lambda < 2^{\lambda^+}.$$

Notice that the assumption $2^\lambda < 2^{\lambda^+}$ was used only in the last inequality.

But this is a contradiction to the fact we have many pairwise non-embeddable models. Notice how we used the assumption $2^\lambda < 2^{\lambda^+}$ in the final step.

Now to the moreover part, suppose that $(2^\lambda)^+ < 2^{\lambda^+}$. Consider

$$f(\eta) := \{\nu \in {}^{\lambda^+}2 \mid \exists g : M_\nu \to M_\eta\} \text{ for every } \eta \in {}^{\lambda^+}2\,.$$

CLAIM 12.6. $|f(\eta)| \leq 2^\lambda$ *for every* $\eta \in {}^{\lambda^+}2$.

PROOF. Otherwise there are $\eta \in {}^{\lambda^+}2$ and a set $S \subseteq {}^{\lambda^+}2$ of cardinality $(2^\lambda)^+$ such that $\nu \in S \implies \exists g_\nu : M_\nu \to M_\eta$. As $g_\nu[M_{\langle\rangle}]$ is a subset of M_η of cardinality λ and M_η has at most $(\lambda^+)^\lambda (= 2^\lambda)$ such sets there are $B \subseteq M_\eta$ and $S^* \subseteq S$ of cardinality $(2^\lambda)^+$ such that $\nu \in S^* \implies g_\nu[M_{\langle\rangle}] = B$. As the number

of functions from $M_{\langle\rangle}$ to B is $\lambda^\lambda = 2^\lambda$ a second application of the pigeon hole principle gives $S^{**} \subseteq S^*$ of cardinality $(2^\lambda)^+$ and $g : M_{\langle\rangle} \to B$ such that

$$\nu \in S^{**} \implies g_\nu \restriction M_{\langle\rangle} = g.$$

Let $\nu_0 \neq \nu_1 \in S^{**}$ and ρ the largest common initial segment of ν_0 and ν_1. We may assume that $\rho^\frown \ell \lessdot \nu_\ell$ for $\ell = 0, 1$.

Thus we have that the diagram

$$\begin{array}{ccc} M_{\rho^\frown 0} & \xrightarrow{g_{\nu_0}} & M_\eta \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle g_{\nu_1}} \\ M_{\langle\rangle} & \xrightarrow[\mathrm{id}]{} & M_{\rho^\frown 1} \end{array}$$

commutes.

Since $N \leq_{\mathcal{K}} M_{\langle\rangle}$, by the above diagram $M_{\rho^\frown 0}$ and $M_{\rho^\frown 1}$ are amalgamated over N in contradiction to requirement (3). $\square$

Since f is a set valued function such that $|\operatorname{dom} f| = 2^{\lambda^+} > (2^\lambda)^+$ and $|f(\eta)| \leq 2^\lambda$ for every η, by Hajnal's free subset Theorem 15.9.3 there exists $S \subseteq {}^{\lambda^+}2$ of cardinality 2^{λ^+} such that $\eta \neq \nu \in S \implies \nu \notin f(\eta)$. But by the definition of the function f, this means that $\{M_\eta \mid \eta \in S\}$ is a family of pairwise un embeddable structures. $\square$

Using the same construction as in the last proof we can get an unrelated result that is interesting for its own:

THEOREM 12.7. *Let λ be such that $2^\lambda < 2^{\lambda^+}$ holds. Suppose $\mathcal{K}$ is an AEC such that $\lambda \geq \mathrm{LS}(\mathcal{K})$ and is categorical in λ. If $\mathcal{K}$ fails to have the λ-amalgamation property then there is no universal model in $\mathcal{K}_{\lambda^+}$.*

PROOF. Suppose $N_0 <_{\mathcal{K}} N_\ell$ for $\ell = 1, 2$ witness the failure of the λ-AP.

Define $\{M_\eta \in \mathcal{K}_\lambda \mid \eta \in {}^{\lambda^+ >}2\}$ satisfying:

(1) $|M_\eta| = \lambda(1 + \ell(\eta))$,
(2) $\eta \lessdot \rho$ implies $M_\eta <_{\mathcal{K}} M_\rho$,
(3) If $\ell(\rho)$ is a limit, then $M_\rho = \bigcup_{\beta < \ell(\rho)} M_{\rho \restriction \beta}$,
(4) $M_{\rho^\frown 0}$ and $M_{\rho^\frown 1}$ cannot be amalgamated over M_ρ.

First observe that by λ-categoricity and the failure of the λ-AP we may assume that $|N_\ell| - |N_0|$ has cardinality λ.

The construction is carried out by induction on $\ell(\eta)$, by λ-categoricity there is $f : M_\eta \cong N_0$ and using Lemma 1.7 we find $M_{\eta^\frown \ell} > M_\eta$ (for $\ell = 0, 1$) and $f_\ell : M_{\eta^\frown \ell} \cong N_{\ell+1}$ both extending the function f.

Now for $\eta \in {}^{\lambda^+}2$ let $M_\eta := \bigcup_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$. For the sake of contradiction suppose that $N \in \mathcal{K}_{\lambda^+}$ is universal. WLOG we may assume that $|N| = \lambda^+$. Thus there are $\mathcal{K}$-embeddings $f_\eta : M_\eta \to N$.

Let $C := \{\delta < \lambda^+ \mid \delta = \lambda(1 + \delta)\}$. Note that it is a club of λ^+.

By Θ_{λ^+} there are $\delta \in C$ and $\eta \neq \nu \in {}^{\lambda^+}2$ such that $\eta \restriction \delta = \nu \restriction \delta$, $\eta[\delta] \neq \nu[\delta]$ and $f_\eta \restriction \delta = f_\nu \restriction \delta$. We may assume that $\eta[\delta] = 0$. Let $\rho := \eta \restriction \delta$.

Let $N^* < N$ be a model of cardinality λ such that $|N^*| \supseteq f_\eta[M_{\rho^\frown 0}] \cup f_\nu[M_{\rho^\frown 1}]$

As $f_\eta \restriction |M_\rho| = f_\nu \restriction |M_\rho|$ we have that the diagram

$$\begin{array}{ccc} M_{\rho^\frown 0} & \xrightarrow{f_\eta} & N^* \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle f_\nu} \\ M_\rho & \xrightarrow[\mathrm{id}]{} & M_{\rho^\frown 1} \end{array}$$

commutes. This is a contradiction to (4). □

An answer to the following is unknown:

PROBLEM 12.8. *Does* $\mathrm{IE}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$ *follow from only* $2^\lambda < 2^{\lambda^+}$?

Later we will prove

THEOREM 12.9 (Shelah). *Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{Hanf}(\mathcal{K})$. If $\mathcal{K}_\lambda$ is categorical then $\mathcal{K}$ is μ-Galois stable for every* $\mathrm{LS}(K) \leq \mu < \lambda$.

13. Few models imply the amalgamation property

A fundamental question is whether there is a "reasonable" model theoretic property on $\mathcal{K}$ that implies that the class $\mathcal{K}$ has the amalgamation property?

In Theorem **??** we will show that if $\lambda \geq \mathrm{LS}(\mathcal{K})$ and $2^\lambda = \lambda^+$ then for classes which are categorical in λ, the λ-AP is equivalent to existence of a universal model in $\mathcal{K}_{\lambda^+}$.

The purpose of this section is to prove a special instance of the statement in the name of this section.

THEOREM 13.1. *Under the assumption* $2^\lambda < 2^{\lambda^+}$*, for an AEC* $\mathcal{K}$*, with* $\lambda \geq \mathrm{LS}(\mathcal{K})$*,* $\mathrm{I}(\lambda, \mathcal{K}) = 1$*, and* $\mathcal{K}$ *fail to have the* λ*-amalgamation property, we have that* $\mathrm{I}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$.

As the main step of the argument will use what is called the Devlin-Shelah's weak diamond, see section 4 of chapter 15. I suggest that the reader will first read the proof of Theorem 15.7.10 which is a simpler application of the weak diamond.

The assumption $2^\lambda < 2^{\lambda^+}$ will be used used twice, once as a cardinal arithmetic assumption and at the second time in a more substantial in the following form:

FACT 13.2. *Suppose* $\Phi^2_{\lambda^+}$ *holds, then there are stationary sets* $\{S_\alpha \subseteq \lambda^+ \mid i < \lambda^+\}$ *so that* $\alpha \neq \beta \to S_\alpha \cap S_beta = \emptyset$*, and*
$\Phi^2_{\lambda^+}(S_\alpha)$ *holds for all* $\alpha < \lambda^+$.

OUTLINE OF A PROOF. This is a combination of two results established in the Appendix. The first is that the set

$$J = \{X \subseteq \lambda^+ \mid \neg\Phi^2_{\lambda^+}(X)\}$$

is a normal ideal of λ^+. This is by Theorem 8.3 of the Appendix.

Now $\Phi^2_{\lambda^+}$ implies existence of a stationary set $S \subseteq \lambda^+$, as $\Phi^2_{\lambda^+}$ implies that $^+ \notin J$, an application of of Ulam's theorem (Theorem 9.2 of the appendix) when taking $I := J$ and $S = \lambda^+$ we get pairwise disjoint subsets of λ^+ all satisfying the weak diamond.

□

REMARK 13.3. Observe the following: If $\mathcal{K}$ is λ-categorical and $N_0 \precneqq_{\mathcal{K}} N_\ell$ all from $\mathcal{K}_\lambda$ for $\ell = 1, 2$ are such that N_1 and N_2 can't amalgamated over N_0 then for any $N'_\ell \succ_{\mathcal{K}} N_\ell$ also N'_1 and N'_2 can't be amalgamated over N_0.

Notice that the assumptions $\mathrm{I}(\lambda, \mathcal{K}) = 1$ and $\mathcal{K}_{\lambda^+} \neq \emptyset$ imply that $\mathcal{K}_\lambda$ has no maximal element.

PROOF. For the sake of contradiction suppose that $\mathrm{I}(\lambda^+, \mathcal{K}) < 2^{\lambda^+}$. By Theorem 12.4 we may assume that $\mathcal{K}$ has the λ-WAP.

Recall that $\mathrm{I}(\lambda, \mathcal{K}) = 1$ and $\mathcal{K}$ fail to have the λ-amalgamation property implies that $\mathcal{K}_{\lambda^+}$ is nonempty.

By assumptions, we may take $N_0, N_1, N_2 \in \mathcal{K}_\lambda$ such that N_1 and N_2 that can not be amalgamated over N_0 and $N_\ell - N_0$ has cardinality λ for $\ell = 1, 2$ (see Exercise 2.12). Using the λ-WAP we may assume that any pair (from $\mathcal{K}_\lambda$) extending N_ℓ (for $\ell = 1, 2$) can be amalgamated over N_0.

For $\eta \in {}^{\lambda^+}2$ we will define a family of $M_\eta \in \mathcal{K}_\lambda$ so that the following hold:

(1) $|M_\eta| = \lambda(1 + \ell(\eta))$,
(2) $\nu \lhd \eta \to M_\nu \prec_{\mathcal{K}} M_\eta$,
(3) when $\ell(\eta)$ is a limit ordinal, $M_\eta = \bigcup_{\alpha<\ell(\eta)} M_{\eta\restriction\alpha}$,
(4) $M_{\eta^\frown 0}$ and $M_{\eta^\frown 1}$ cannot be amalgamated over M_η.
(5) for $\ell \in \{0, 1\}$ if M^0 and M^1 are extensions of $M_{\eta^\frown \ell}$, then M^0 and M^1 can be amalgamated over M_η.

The construction is by induction on $\ell(\eta)$:

For $\ell(\eta) = 0$: fix $M_{\langle\rangle}$ to be any member of $\mathcal{K}_\lambda$. For $\ell(\eta)$ limit follow requirement (3).

For the successor case: given η, by λ-categoricity, fix $f : N_0 \cong M_\eta$. Using the triple N_0, N_1, N_2 find $M'_{\eta^\frown \ell} \geq M_\eta$ and $f_\ell \supseteq f$ such that $f_\ell : N_\ell \cong M'_{\eta^\frown \ell}$. Now by the weak λ-AP let $M_{\eta^\frown \ell} \geq M'_{\eta^\frown \ell}$ such that requirement (5) holds.

Let $C := \{\delta < \lambda^+ : \delta = \lambda(1 + \delta)\}$, notice that it is a club. By Fact 13.2 there are $\{S_\gamma \subseteq C : \gamma < \lambda^+\}$ stationary sets such that
$\gamma_1 \neq \gamma_2 \implies S_{\gamma_1} \cap S_{\gamma_2} = \emptyset$ and for all $\gamma < \lambda^+$ we have that $\Phi_{\lambda^+}(S_\gamma)$ holds.

For every $\delta < \lambda^+$ such that $\delta = \lambda(1 + \delta)$, $h : \delta \to \delta$ and $\eta, \nu \in {}^\delta 2$ let

$$F(\eta, \nu, h) := \begin{cases} 1 & \text{if } h : M_\eta \to M_\nu \quad \text{and} \\ & M_{\eta^\frown 0} \\ & \uparrow \text{id} \\ & M_\eta \xrightarrow{h} M_{\nu^\frown 0} \\ & \text{can be amalgamated.} \\ 0 & \text{Otherwise.} \end{cases} \tag{15}$$

By $\Phi^2_{\lambda^+}(S_\gamma)$ and Lemma 7.9 pick $g_\gamma : \lambda^+ \to 2$ such that
for all $\eta, \nu \in {}^{\lambda^+}2$ and every $h : \lambda^+ \to \lambda^+$ we have that

$$S'_\gamma := \{\delta \in S_\gamma : F(\eta \restriction \delta, \nu \restriction \delta, h \restriction \delta) = g_\gamma(\delta)\} \quad \text{is stationary.}$$

For $X \subseteq \lambda^+$ and every $\delta < \lambda^+$ let

$$\eta_X[\delta] := \begin{cases} g_\gamma(\delta) & \text{if } \delta \in S_\gamma \text{ and } \gamma \in X \\ 0 & \text{Otherwise.} \end{cases}$$

Notice that since $\{S_\gamma : \gamma < \lambda^+\}$ are pairwise disjoint for any $\delta \in X$ there is at most one γ such that $\delta \in S_\gamma$ (maybe none), so η_X is well defined. We finish by showing:

CLAIM 13.4. *For every $X \neq Y \subseteq \lambda^+$ we have that $M_{\eta_X} \ncong M_{\eta_Y}$*

PROOF. We can prove little more: If $X \not\subseteq Y \subseteq \lambda^+$ then there is no $\mathcal{K}$-embedding from M_{η_X} to M_{η_Y}.

For the sake of contradiction suppose $X \not\subseteq Y \subseteq \lambda^+$ and $h : M_{\eta_X} \hookrightarrow M_{\eta_Y}$. Pick $\gamma \in X - Y$.

An application of $\Phi_{\lambda^+}(S_\gamma)$ to η_X, η_Y and h yields S'_γ as above.
Let $D := \{\delta < \lambda^+ \mid (h \restriction \delta) : \delta \to \delta\}$, it is a club and let $S''_\gamma := S'_\gamma \cap D$.

Let $\delta \in S''_\gamma$. Denote by η the sequence $\eta_X \restriction \delta$ and by ν the sequence $\eta_Y \restriction \delta$. Since $\gamma \notin Y$ by the definition of the sequence η_Y we have that $\eta_Y[\delta] = 0$, namely $\nu \lessdot \nu^\frown 0 \lessdot \eta_Y$.

Now consider $\eta_X[\delta]$. There are two possibilities (according to the value of $\eta_X[\delta]$):

(1) If $\eta_X[\delta] = 1$, by the definition of $\eta_X[\delta]$ necessarily $\eta_X[\delta] = g_\gamma(\delta)$, since $\delta \in S'_\gamma$ we have that $F(\eta, \nu, h) = 1$. From the definition of F we get that h is a $\mathcal{K}$-embedding from M_η to M_ν and there exists $M^1 \succ M_{\nu^\frown 0}$ and g be such that the following diagram commutes:

$(*)_1$
$$\begin{array}{ccc} M_{\eta^\frown 0} & \xrightarrow{g} & M^1 \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_\eta & \xrightarrow[h]{} & M_{\nu^\frown 0} \end{array}$$

Since $\eta^\frown 1 \lessdot \eta_X$, $\nu^\frown 0 \lessdot \eta_Y$ and the assumption that $h : M_{\eta_X} \hookrightarrow M_{\eta_Y}$ the following diagram

$$\begin{array}{ccc} M_{\eta^\frown 1} & \xrightarrow{h} & M_{\eta_Y} \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_\eta & \xrightarrow[h]{} & M_{\nu^\frown 0} \end{array}$$

must commute. By axiom $A4$ there exists $M^2 \prec_{\mathcal{K}} M_{\eta_Y}$ of cardinality λ such that $h(M_{\eta^\frown 1}) \prec_{\mathcal{K}} M^2$ and $M_{\nu^\frown 0} \prec_{\mathcal{K}} M^2$.

Namely the diagram

$$(*)_2 \qquad \begin{array}{ccc} M_{\eta^\frown 1} & \xrightarrow{h} & M^2 \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_\eta & \xrightarrow[h]{} & M_{\nu^\frown 0} \end{array}$$

commutes.

Consider

$$\begin{array}{ccc} M^1 & & \\ {\scriptstyle \mathrm{id}}\uparrow & & \\ M_{\nu^\frown 0} & \xrightarrow[\mathrm{id}]{} & M^2 \end{array}$$

by requirement (5) there are $M^3 \in K_\lambda$ such that $M^3 \geq M^2$ and a $\mathcal{K}$-mapping $e : M^1 \hookrightarrow M^3$ such that

$$(*)_3 \qquad \begin{array}{ccc} M^1 & \xrightarrow{e} & M^3 \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_\nu & \xrightarrow[\mathrm{id}]{} & M^2 \end{array}$$

is commutative. Combining $(*)_1$, $(*)_2$ and $(*)_3$ together we get

$$(*)_4 \qquad \begin{array}{ccccc} M_{\eta^\frown 0} & \xrightarrow{g} & M^1 & & \\ {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & \searrow^{e} & \\ M_\eta & \xrightarrow[h]{} & M_\nu & & M^3 \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & \nearrow_{\mathrm{id}} & \\ M_{\eta^\frown 1} & \xrightarrow[h]{} & M^2 & & \end{array}$$

While we don't know that $(*)_4$ is commutative as e may not fix $M_{\nu^\frown 0}$, by $(*)_3$, we have that the mappings $g \circ e$ and h agree on M_η thus we have that

$$\begin{array}{ccc} M_{\eta^\frown 1} & \xrightarrow{h} & M^3 \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle g\circ e} \\ M_\eta & \xrightarrow[\mathrm{id}]{} & M_{\eta^\frown 0} \end{array}$$

is commutative, which is a contradiction to requirement (4).

(2) If $\eta_X[\delta] = 0$ then we have that $\eta^\frown 0 \lessdot \eta_X$. Since $h : M_{\eta_X} \hookrightarrow M_{\eta_Y}$ we get that M_{η_Y} is an amalgam of $M_{\eta^\frown 0}$ and $M_{\nu^\frown 0}$ over M_η so by the definition of F we get that $F(\eta, \nu, h \restriction \delta) = 1$. Since $\delta \in S''_\gamma$ we have

that $g_\gamma(\delta) = 1$ which by the definition of η_X gives $\eta_X[\delta] = 1$ and this contradicts the assumption of this case.

□

□

By Corollary 9.27 and Theorem **??** for any regular λ the previous theorem can't be a proven in ZFC alone. However, we don't know an example axiomatizable in $L_{\omega_1,\omega}$. It is plausible that further development in the model theory of $L_{\omega_1,\omega}$ (perhaps a new omitting types theorem) will allow answering the following question in ZFC:

QUESTION 13.5. Let $\psi \in L_{\omega_1,\omega}$ be in a countable language and L^* be a countable fragment containing ψ. Suppose ψ is $\aleph_0$-categorical and fail the $\aleph_0$-AP (with respect to L^*-embeddings). Is necessarily $\mathrm{I}(\aleph_1, \psi) \geq 2$?

By the moreover part of Theorem 12.4 we know that under a certain set-theoretic situation from the failure of the λ-WAP we can get that $\mathrm{I}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$. The analogue for Theorem 13.1 is completely open. The following is the simplest open problem of this kind, even when we assume in addition that $\mathcal{K}$ is axiomatizeable in $L_{\omega_1,\omega}$:

QUESTION 13.6. Let $\psi \in L_{\omega_1,\omega}$ be in a countable language and L^* be a countable fragment containing ψ. Suppose ψ is $\aleph_0$-categorical and fail the $\aleph_0$-AP (with respect to L^*-embeddings). Is there a model of ZFC where $\mathrm{IE}(\aleph_1, \psi) \geq 3$?

Why 3 and not 2?

A simple but important corollary of Theorem 13.1 is that in the context of Shelah's categoricity conjecture (at least under the Generalized Weak Continuum Hypothesis) the AP for all models above a certain cardinal is a natural property. Namely:

THEOREM 13.7. *Suppose $2^\lambda < 2^{\lambda^+}$ for all λ. If there exists $\mu_0 \geq \mathrm{LS}(\mathcal{K})$ such that $\mathcal{K}$ is λ-categorical for every $\lambda \geq \mu_0$ then $\mathcal{K}_{\geq \mu_0}$ has the AP.*

PROOF. Exercise. □

Exercises 2.13.

(1) Prove Theorem 13.7.
(2) Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. Suppose $2^\lambda < 2^{\lambda^+}$. If M is not an amalgamation base for every $M \in \mathcal{K}_\lambda$ then $\mathrm{I}(\lambda^+, \mathcal{K}) = 2^{\lambda^+}$.

CHAPTER 3

More Fundamentals

Introduction

We discuss some additional basics:

Overview of the structure of this chapter:

Section 1: Given an uncountable model M of cardinality λ by renaming we may assume that its universe is the set $\{\alpha \in \mathrm{On} \mid \alpha < \lambda\}$. In this section we develop an analog to the Lebesgue measure on the subsets of $[0,1]$ interval of the reals. The analog to *subsets of measure* 1 *of* $[0,1]$ is the *set of closed unbounded subsets of* λ. It turns out that a consequence of the downward Löwenheim-Skolem-Tarski is that the latter has close interaction with model theoretic properties of M. The results of this section are fundamental for the interaction between model theory and set theory, benefiting both fields.

Section 2 : The ultraproduct construction is one of the most fundamental techniques used to produce new models from given models. We present a proof of the compactness theorem that is completely free from dependence on the notion of provability (thus this proof of the compactness theorem does not require the completeness theorem).

Section 3 : Ehrenfeucht-Fraisse games.

Section 4: Two applications to algebra.

Section 5: Non-standard analysis.

Section 2: This is mathematical introduction to the book. It also contain a compact overview of the material in the book. Perhaps the reader will benefit from attempting to read it already after completing the first five sections of this chapter. This section was designed so that even a reader who completed chapter 3 may benefit from rereading.

Section 6 : The section is dedicated to a precise mathematical formulation of what is *structure theory* for a class of models and its potential consequences.

1. The filter of closed unbounded sets

It turns out that for an uncountable regular cardinal λ the order topology on the set $\{\alpha \mid \alpha < \lambda\}$ has a surprising connection with a natural measure on the same set and this measure has deep connections with the structure of models of cardinality λ whose similarity type is of cardinality less than λ.

This section is dedicated to explore this important connection that provides a powerful tool for a model theorist to study models M where $\|M\|$ is regular and $|\operatorname{L}(M)| < \|M\|$.

On the microscopic level this section serves a dual purpose: The reader can view it as an application of model-theory to (elementary) set-theory. The secondary goal (and the more important one) is to establish certain set-theoretic facts that will be used later in Section **??** of Chapter **??** and the proof of Theorem **??** we use some of the results presented here to get some fundamental results of model theory.

Let λ be an uncountable cardinal. On the subsets of λ we introduce below a concept which is a refinement of the concept of cardinality of a set. In many respects it is a parallel to Lebesgue measure on the subsets of $[0, 1]$ interval of the reals.

DEFINITION 1.1. Let α be an ordinal. A set $S \subseteq \alpha$ is *unbounded* if for every $\beta < \alpha$ there exists $\gamma \in S$ such that $\gamma > \beta$.

Recall that when S is a set of ordinals then there exists a minimal ordinal α such that $\alpha \geq \beta$ for every $\beta \in S$ (namely α is the supremum of S in the class of the ordinals). It is an elementary fact that this ordinal α is equal to $\bigcup S$.

EXAMPLE 1.2. $S := \{2n \ : \ n \in \omega\}$ is unbounded in ω.

EXAMPLE 1.3. Suppose that λ is an uncountable regular cardinal.
The set $S := \{\delta < \lambda \ : \ \delta \text{ is a limit ordinal } \}$ is unbounded in λ: Let $\gamma < \lambda$ be given. Notice that $\gamma + \omega$ (which is the ordinal $\bigcup_{n<\omega}(\gamma + n)$) is an ordinal less than λ (argument: $|\gamma + \omega| \leq |\gamma| + |\omega| = |\gamma| + \aleph_0 < \lambda$).

DEFINITION 1.4. Let α be an ordinal. A set $S \subseteq \alpha$ is *closed unbounded* (club) iff

(1) S is unbounded in α , and
(2) S is *closed* which means that for every bounded $S' \subseteq S$ we have that $\bigcup S' \in S$.

Given a linearly-ordered set $\langle P, < \rangle$ there is a natural way to define a topology on P, it turns out that for a set of ordinals $S \subseteq \alpha$ the set is closed in the order topology iff it is closed in the sense of the last definition.

EXAMPLE 1.5. Let λ be a regular uncountable cardinal.
The set $\{\delta < \lambda \ : \ \delta \text{ is a limit ordinal } \}$ is a club.

EXAMPLE 1.6. $D := \{\alpha + 1 \ : \ \alpha < \lambda\}$ is not club since there exists $S = \{\alpha_n \in D \ : \ n < \omega\}$ such that $\bigcup S \notin D$ Take $\alpha_n := \alpha + n$ then $\bigcup S = \omega + \omega$ which is not a successor ordinal.

It appears that the closed unbounded subsets of a regular (uncountable) cardinal have much in common with subsets of the interval $[0, 1]$ of Lebesgue measure 1. Namely:

LEMMA 1.7. *Let $\lambda > \aleph_0$ be regular.*

(1) *If $C_1, C_2 \subseteq \lambda$ are closed unbounded then $C_1 \cap C_2$ is closed unbounded.*
(2) *Moreover, for every cardinal $\mu < \lambda$ and every $\{C_i \ : \ i < \mu\}$ clubs of λ also $\bigcap_{i<\mu} C_i$ is a club.*

A direct proof of the Lemma is fairly easy, we prefer to postpone it for awhile and derive the Lemma from a more general model-theoretic principle to be presented below. From the definition of a closed unbounded subset it is clear that the following is a true

PROPOSITION 1.8. *Let λ be a regular uncountable cardinal; For every closed unbounded subset C of λ, there exists a bijection $F_C : \lambda \to C$ which has the following properties*

(1) *F_C is* order preserving *(i.e. $[\alpha < \gamma \Longrightarrow F_C(\alpha) < F_C(\gamma)]$), and*
(2) *F_C is* continuous *if $\delta < \lambda$ is a limit ordinal then $F_C(\delta) = \bigcup_{\alpha<\delta} F_C(\alpha)$.*

A function like F_C is called normal enumeration *of the set C.*

PROOF. Define F_C by transfinite induction:

- $F_C(0) := \min C$,
- $F_C(\alpha + 1) := \min\{\gamma \in C \ : \ \gamma > F(\alpha)\}$, and
- for a limit ordinal $\delta < \lambda$ let $F_C(\delta) := \bigcup_{\alpha<\delta} F_C(\alpha)$.

Notice that the definition of $F_C(\alpha + 1)$ is possible since C is unbounded the set $\{\gamma \in C \ : \ \gamma > F_C(\alpha)\}$ is not empty. The definition for a limit $\delta < \lambda$ is possible because C is closed. The function defined above is onto C since if it were not than we get an injection from λ onto an initial segment of C which is impossible. □

The next fact is one of the most fundamental connections between model theory and set theory.

THEOREM 1.9. *Let λ be an uncountable regular cardinal, and let L be a similarity type of cardinality less than λ. Suppose that M is an L- structure such that $M = \langle |M|, P, \in, \ldots\rangle$ where P^M is a unary predicate interpreted by the set λ. If $\{M_\alpha \preceq M \ : \ \alpha < \lambda\}$ with the following properties:*

(1) $\alpha < \lambda \Longrightarrow \|M_\alpha\| < \lambda$,
(2) $\alpha < \gamma < \lambda \Longrightarrow M_\alpha \preceq M_\gamma$,
(3) $M = \bigcup_{\alpha<\lambda} M_\alpha$,
(4) *the chain is continuous; Namely for every* $\delta < \lambda$ *limit* $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$.

Then the set $\{\delta < \lambda \ : \ |P^{M_\delta}| = \delta\}$ *is a club.*

Moreover for every chain $\{N_\alpha \preceq M \ : \ \alpha < \lambda\}$ *satisfying (1)-(4) we have that the set* $\{\alpha < \lambda \ : \ P^{N_\alpha} = P^{M_\alpha}\}$ *is a club.*

PROOF. *Existence:* Construct $\{M_\alpha \preceq M \ : \ \alpha < \lambda\}$ by induction on $\alpha < \lambda$:

(1) for $\alpha = 0$; Use the downward Löwenheim-Skolem theorem to get $M_0 \preceq M$ of cardinality μ when μ satisfies $|L| \leq \mu < \lambda$.
(2) for $\alpha + 1$; Apply the downward Löwenheim-Skolem theorem to the set $|M_\alpha| \cup \alpha$ to get $M_{\alpha+1} \preceq M$ of cardinality less than λ which contains the set $|M_\alpha| \cup \alpha$.
(3) for α limit; Define $M_\alpha := \bigcup_{\gamma<\alpha} M_\gamma$.

By Theorem 3.30 we have that for all $\gamma < \alpha$ the structure M_γ is an elementary submodel of M_α, and by Theorem 3.31 we have that M_α is an elementary submodel of M. From the definition of $M_{\alpha+1}$ it is clear that $M = \bigcup_{\alpha<\lambda} M_\alpha$. We need only to verify that (5) holds. Let $C := \{\delta < \lambda \ : \ |M_\delta| = \delta\}$. Let $\alpha < \lambda$ be given. By induction on $n < \omega$ define $\{\gamma_n \ : \ n < \omega\}$ with the following properties:

(1) $\gamma_0 = \alpha$, and
(2) $\gamma_{n+1} = \sup |M_{\gamma_n}| + \gamma_n + 1$.

From (1) it follows that $\|M_{\gamma_n}\| < \lambda$, hence $Sup|M_{\gamma_n}| + 1 < \lambda$ is an ordinal less than λ. Now let $\delta := \bigcup_{n<\omega} \gamma_n$. Clearly $\delta > \alpha$.
Show that $\delta \in C$:

$$\beta \in \delta \Longrightarrow (\exists n < \omega)\beta \in \gamma_n \Longrightarrow (\exists n < \omega)\beta \in |M_{\gamma_n}|$$
$$\Longrightarrow \beta \in \bigcup_{n<\omega} M_{\gamma_n} \Longrightarrow \beta \in |M_\delta|.$$

The fact that C is closed follows from the continuity of the chain $\{M_\alpha \preceq M \ : \ \alpha < \lambda\}$.

Uniqueness: Let $\{N_\alpha \preceq M \ : \ \alpha < \lambda\}$ be satisfying (1)-(4). Define $C := \{\delta < \lambda \ : \ M_\delta = N_\delta\}$. Show that C is a club. By continuity of the chain C is closed. Hence it suffices to show that C is unbounded: Let $\alpha < \lambda$ be given. Define $\{\gamma_k < \lambda \ : \ k < \omega\}$: Let $\gamma_0 = \alpha$, choose $\gamma_{2n+1} > \gamma_{2n} > \gamma_{2n-1}$ such that $N_{\gamma_{2n}} \subseteq M_{\gamma_{2n+1}}$ and $M_{\gamma_{2n+1}} \subseteq N_{\gamma_{2n+2}}$. Note that since λ is regular and greater than $\|M_\alpha\| + \|N_\alpha\|$ the definition of γ_k is possible. By continuity of both chains we have that $\delta := \bigcup_{k<\omega} \gamma_k \in C$.
Since $N_\delta = \bigcup_{n<\omega} N_{\gamma_{2n}} = \bigcup_{n<\omega} M_{\gamma_{2n+1}} = M_\delta$. □

Now to the proof of Lemma 1.7:

PROOF. Let $\{C_i \subseteq \lambda \ : \ i < \mu\}$ be closed unbounded subsets of λ. By Proposition 1.8 let $F_{C_i} :\to C_i$ be continuous increasing surjective mappings. Let $M := \langle \lambda, \in, i\rangle_{i<\mu}$. Since $\lambda > \mu$ by the first half of Theorem 1.9 the set

$$D := \{\delta < \lambda \ : \ M_\delta \preceq M\}$$

is a club (when $M_\delta = \langle \delta, \in, i\rangle_{i<\mu}$). Given $\delta \in D - \mu$ for every $i < \mu$ we have that

$$M \models \forall x \exists y[y > x \wedge y \in C_i].$$

Hence

$$M_\delta \models \forall x \exists y[y > x \wedge y \in C_i].$$

Namely $\delta \cap C_i$ is unbounded in M_δ (which is δ). Thus

$$\delta = \sup \delta \cap C_i \in C_i \text{ for all } i < \mu.$$

□

The following corollary is the form the previous theorem is used in many applications:

COROLLARY 1.10. *Let λ be an uncountable regular cardinal. Suppose $f : \lambda \to \lambda$. Then the set*

$$\{\delta < \lambda \ : \ (\forall i < \lambda)\ [i < \delta \to f(i) < \delta]\}$$

is a closed unbounded subset of λ.

PROOF. Apply Theorem 1.9 to $M := \langle \lambda, \in, f\rangle$. □

DEFINITION 1.11. Let $\lambda > \aleph_0$ be regular, the set

$$D_\lambda := \{C \subseteq \lambda \ : \ C \text{ contains a club subset of } \lambda\}$$

is called the *club filter of* λ.

The contents of Lemma 1.7 is that D_λ is a μ-complete filter for every $\mu < \lambda$. In the analogy to Lebesgue measure on subsets of $I = [0, 1]$ we can define the analog notion to $X \subseteq I$ "has positive measure"

DEFINITION 1.12. Let D_λ be as in Definition 1.11. A set $S \subseteq \lambda$ is called *stationary* iff $\forall C \in D$ we have that $C \cap S \neq \emptyset$ holds.

EXAMPLE 1.13. Let λ be a regular uncountable cardinal. The set $S^\lambda_{\aleph_0} := \{\delta < \lambda \ : \ cf(\delta) = \aleph_0\}$ is stationary.

A more general fact is the following

PROPOSITION 1.14. *Suppose λ is an uncountable regular cardinal. For every infinite regular $\mu < \lambda$ the set $S^\lambda_\mu := \{\delta < \lambda \ : \ cf(\delta) = \mu\}$ is stationary subset of λ.*

PROOF. Exercise. □

EXAMPLE 1.15. Let $\lambda = \aleph_2$, the set $S^{\aleph_2}_{\aleph_0}$ is stationary which is not a club (since it is disjoint to the set $S^{\aleph_2}_{\aleph_1}$).

DEFINITION 1.16. Let λ be an uncountable regular cardinal.

(1) Let $S_1, S_2 \subseteq \lambda$ we write $S_1 \equiv S_2 \mod D_\lambda$ iff $C \cap S_1 = C \cap S_2$ holds for every $C \in D_\lambda$.
(2) $S_1 \subseteq S_2 \mod D_\lambda$ iff for every $C \in D_\lambda$ we have that $C \cap S_1 \subseteq C \cap S_2$.

PROPOSITION 1.17. *Let λ be an uncountable regular cardinal.*

(1) *D_λ is a λ-complete filter.*
(2) *$S_1 \equiv S_2 \mod D_\lambda$ if and only if $S_1 \Delta S_2$ is not stationary.*
(3) *$S_1 \equiv S_2 \mod D_\lambda$ is an equivalence relation.*

PROOF. Easy. □

The following is a generalization of the pigeonhole principle for regular cardinals.

THEOREM 1.18 (Geza Fodor 1956). *Let $\lambda > \aleph_0$ be regular; Suppose that $S \subseteq \lambda$ is stationary. If $f : S \to \lambda$ is a regressive function (i.e for every $i \in S - \{0\}$ we have that $f(i) < i$) then there are $S' \subseteq S$ stationary and $i_0 < \lambda$ such that $i \in S' \Longrightarrow f(i) = i_0$.*

PROOF. Suppose for the sake of contradiction that

$$S_{i_0} := f^{-1}(i_0) \text{ is not stationary for all } i_0 < \lambda.$$

Namely for every $\alpha < \lambda$ there exists a club $C_\alpha \subseteq \lambda$ such that

$$(*) \quad \gamma \in C_\alpha \cap S \Longrightarrow f(\gamma) \neq \alpha.$$

Define $M = \langle \lambda, \in, R \rangle$ where

$$R := \{\langle \gamma, \alpha \rangle \ : \ \gamma \in C_\alpha\}.$$

By Theorem 1.9 there exists an elementary chain $\{M_\alpha \preceq M \ : \ \alpha < \lambda\}$ such that $M = \bigcup_{\alpha<\lambda} M_\alpha$ and $C := \{\delta < \lambda \ : \ |M_\delta| = \delta\}$ contains a club. Since S is stationary there exists $\delta \in C \cap S$.

$$M \models \forall x \exists y [y > x \to R(y, \gamma)] \text{ for all } \gamma < \delta.$$

Since $M_\delta \preceq M$ and $\gamma \in M_\delta$ we have also that

$$M_\delta \models \forall x \exists y [y > x \to R(y, \gamma)].$$

Hence for every $\alpha < \delta$ the set $C_\alpha \cap \delta$ is unbounded in δ (remember that $\delta = |M_\delta|$). Thus (since C_α is closed) $\delta = Sup C_\alpha \cap \delta \in C_\alpha$. We have shown that

$$\alpha < \delta \Longrightarrow \delta \in C_\alpha.$$

Now combining this with $(*)$ gives:

$$\alpha < \delta \Longrightarrow f(\delta) \neq \alpha.$$

Namely $f(\delta) \geq \delta$. This is a contradiction to the assumption that f is a regressive function on S. □

It appears that stationary sets are the building blocks of many uncountable structures. In a later section in the construction of Jónsson algebras (Theorem 2.13.13) and in when we will deal with constructing many non-isomorphic models (E.g. Theorem 4.**??**) the existence of many disjoint stationary subsets of λ will be used. The existence is the contents of the next theorem which is a particular case of a theorem of Solovay. For $\lambda = \aleph_1$ it is known as Ulam's theorem.

THEOREM 1.19 (Ulam). *Let $\lambda > \aleph_0$ be regular, and let $\mu < \lambda$. If $S \subseteq \lambda$ is a stationary set such that $\alpha \in S \Longrightarrow cf\alpha \leq \mu$ then there are $\{S_i \ : \ i < \lambda\}$ pairwise disjoint stationary subsets of S.*

PROOF. For every $\delta \in S$ (since $\operatorname{cf} \delta \leq \mu$) there exists $\eta_\delta : \mu \to \delta$ unbounded and non-decreasing. For $i < \mu$, and $\gamma < \lambda$ let

$$S(\gamma, i) := \{\delta \in S \ : \ \eta_\delta(i) = \gamma\}.$$

Notice that for

$$i < \mu \wedge \gamma_1 \neq \gamma_2 \Longrightarrow S(\gamma_1, i) \cap S(\gamma_2, i) = \emptyset.$$

It suffices to show that there exists $i < \mu$ such that

$$|\{\gamma < \lambda \ : \ S(\gamma, i) \text{ is stationary}\}| = \lambda.$$

Suppose for the sake of contradiction that

$$|\{\gamma < \lambda \ : \ S(\gamma, i) \text{ is stationary } \}| < \lambda \text{ for all } i < \mu.$$

Since λ is regular for every $i < \mu$ there exists $\gamma(i) < \lambda$ such that

$$\gamma \geq \gamma(i) \Longrightarrow S(\gamma, i) \text{ is not stationary.}$$

Pick $C(\gamma, i) \subseteq \lambda$ club witnessing that, i.e.

$$\gamma \geq \gamma(i) \Longrightarrow S(\gamma, i) \cap C(\gamma, i) = \emptyset.$$

Consider now $M := \langle \lambda, \in, R_0, R_1, \gamma^* \rangle$. Where

$$\begin{aligned} R_1 :=& \{\langle \beta, \gamma, i \rangle \ : \ \beta \in C(\gamma, i)\} \\ R_0 :=& \{\langle \delta, i, \gamma \rangle \ : \ \eta_\delta(i) = \gamma\}, \text{ and} \\ \gamma^* :=& \sup_{i<\mu} \gamma(i) \cup \mu. \end{aligned}$$

By Theorem 1.9 there exists an elementary chain $\{M_\alpha \preceq M \ : \ \alpha < \lambda\}$ such that $M = \bigcup_{\alpha<\lambda} M_\alpha$ and $C := \{\delta < \lambda \ : \ |M_\delta| = \delta\}$ contains a club. Since

$$C_0 := \{\delta < \lambda \ : \ \delta \geq \gamma^*\}$$

is also a club and S is stationary there exists $\delta \in S \cap C \cap C_0$.
For every $i < \mu$ and every $\gamma < \delta$ such $\gamma \geq \gamma^*$ we have that

$$M \models \forall x \exists y [y > x \wedge y \in C(\gamma, i)].$$

Since $M_\delta \preceq M$ we have that

$$M_\delta \models \forall x \exists y [y > x \wedge y \in C(\gamma, i)] \text{ for every } i < \mu \text{ and every } \gamma \in [\gamma^*, \delta).$$

Namely $C(\gamma, i)$ is unbounded in δ.
Since $C(\gamma, i)$ is closed

$$\delta \in C(\gamma, i), \ \ \forall i < \mu \forall \gamma \in [\gamma^*, \delta).$$

In particular (since $C(\gamma, i)$ is disjoint to $S(\gamma, i)$),

$$\forall i < \mu \forall \gamma \in [\gamma^*, \delta) \ \delta \notin S(\gamma, i).$$

Now by the definition of $S(\gamma, i)$ we have that

$$\forall i < \mu \forall \gamma \in [\gamma^*, \delta) \ \eta_\delta(i) \neq \gamma.$$

Namely $Sup_{i<\mu} \eta_\delta(i) \leq \gamma^* < \delta$ which is a contradiction to the assumption that η_δ is unbounded in δ. □

We will use the following corollary as a quarry for building stones of structures M such that $\|M\|$ is uncountable regular. For the first application see Theorem 2.8.17.

COROLLARY 1.20. *Let $\lambda > \aleph_0$ be regular, and let*

$$S^\lambda_{\aleph_0} := \{\delta \in S \ : \ \mathrm{cf}\, \delta = \aleph_0\}.$$

There exists $\{S_i \subseteq S^\lambda_{\aleph_0} \ : \ i < 2^\lambda\}$ such that for every $i < j < 2^\lambda$ we have that $S_i \not\equiv S_j \mod D_\lambda$. Moreover for every $i \neq j$ we have that $S_i \not\subseteq S_j \mod D_\lambda$.

PROOF. By the previous theorem there exists

$$\{X_i \subseteq S^\lambda_{\aleph_0} \ : \ i < \lambda\} \text{ such that } i \neq j \implies X_i \cap X_j = \emptyset.$$

Now for every $A \subseteq \lambda$ let $S_A := \bigcup_{i \in A} X_i$. Check that if $A \neq B \subseteq \lambda$ then $S_A \not\equiv S_B \mod D_\lambda$. Since there are 2^λ subsets of λ such that no one is a subset of the other we are done.

Since $|\lambda \times \lambda| = \lambda$ fix $\langle \alpha_i \ : \ i < \lambda \rangle$ such that for every $\beta < \lambda$ the cardinality of $\{i < \lambda \ : \ \alpha_i = \beta\}$ is λ.
Given $S \subseteq \lambda$ let $Y_S := \{i \mid \alpha_i \in S\}$. Clearly $S_1 \not\subseteq S_2$ implies that $Y_{S_1} - Y_{S_2}$ is infinite. □

DEFINITION 1.21. Let $\mathcal{F}$ be a family of sets. $\mathcal{F}$ is called a Δ-*system* provided there exists a set K called the *kernel of* $\mathcal{F}$ satisfying

$$A \neq B \in \mathcal{F} \implies A \cap B = K.$$

THEOREM 1.22 (Δ-system Lemma). *If $\mathcal{F}$ is an uncountable family of finite sets then there exists an uncountable Δ-system $\mathcal{F}' \subseteq \mathcal{F}$.*

FIRST PROOF. Pick $\{A_i \mid i < \omega_1\} \subseteq \mathcal{F}$. Since the mapping $i \mapsto |A_i|$ is a function from ω_1 into ω there are an uncountable $S \subseteq \omega_1$ and an integer n such that $i \in S \implies |A_i| = n$.

Argue by induction on n: If $n = 1$ then $\{A_i \mid i \in S\}$ is a Δ-system with kernel $\emptyset$..

If there exists a such that $S_a := \{i \in S \mid a \in A_i\}$ is uncountable then apply the inductive hypothesis to $\mathcal{F} := \{A_i - \{a\} \mid i \in S_a\}$ and if K is the kernel of $\mathcal{F}$ then K is also the kernel of $\{A_i \mid \in S_a\}$.

Suppose that S_a is countable for every a. By induction on $i < \omega_1$ define $B_i \in \mathcal{F}$ such that $\bigcup_{j<i} B_j \cap B_i = \emptyset$. Now $\{B_i \mid i < \omega_1\}$ is a Δ-system with kernel $\emptyset$. □

Our next will be a generalization for families that are not necessarily finite, to understand its proof I will illustrate it first by presenting a second proof of the Δ-system lemma. Notice the similarity to the argument used in the proof of Theorem 7.**??**.

SECOND PROOF. Let $\{A_i \mid i < \omega_1\} \subseteq \mathcal{F}$. As $i < \omega_1 \implies A_i$ is finite, $|\bigcup_{i<\omega_1} A_i| \leq \aleph_1$. By fixing a bijection $g : \bigcup_{i<\omega_1} A_i \to \omega_1$ we may assume without loss of generality that $A_i \subseteq \omega_1$ for all $i < \omega_1$. Let $M := \langle \omega_1, \in, g\rangle$. Pick an increasing and continuous resolution $\{M_i \mid i < \omega_1\}$ of M into countable elementary sub models.

Let $C := \{\delta < \omega_1 \ : \ |M_\delta| = \delta\}$, recall (by Theorem 3.1.9) it is a club of ω_1. Notice that $i \in C \implies A_i \subseteq i$.

Let $S \subseteq \omega_1$ be the set of all ordinals of cofinality ω. Now consider

$$f(i) := \min\{j < \omega_1 \mid A_i \cap i \subseteq j\} \text{ for all } i \in S.$$

As $A_i \cap i$ is a finite subset of i and i is limit there exists an ordinal $j < i$ such that $A_i \cap i \subseteq j$. Thus we established that f is pressing down, namely

$$i \in S \implies F(i) < i \text{ for every } i \in S.$$

As $S \cap C$ is stationary an application of Fodor's Lemma 3.1.18 gives a stationary $S' \subseteq S \cap C$ and an ordinal $j_0 < \omega_1$ such that

$$f(i) = j_0 \text{ for all } i \in S'.$$

Now consider the mapping

$$i \mapsto A_i \cap i \text{ from } S' \text{ in to the family of finite subsets of } j_0.$$

Since there are only countably many finite subsets of j_0 and the domain of the above mapping is S' which is a set of cardinality $\aleph_1$, by the pigeon hole principle there exists $S^* \subseteq S'$ uncountable and $K \subseteq j_0$ such that

$$i \in S^* \implies A_i \cap i = K.$$

Let $\mathcal{F}' := \{A_i \mid j_0 \leq i \in S^*\}$. Now verify that $\mathcal{F}'$ is a Δ-system with kernel K: Given $i_1 < i_2$ elements of S^* such that $i_1 \geq j_0$.

$$\begin{aligned} A_{i_1} \bigcap A_{i_2} \subseteq & \text{ (as } i_\ell \in C \text{ we have } A_{i_\ell} \subseteq i_\ell) \\ & A_{i_1} \cap i_i \bigcap A_{i_2} \cap i_2 \subseteq \text{ (since } f(i_\ell) = j_0) \\ & A_{i_1} \cap j_0 \bigcap A_{i_2} \cap j_0 = \text{ (since } i_\ell \in S^*) \\ & K \bigcap K = K. \end{aligned}$$

On the other hand since $K \subseteq A_{i_\ell}$ for both $\ell = 1, 2$ we have that

$$i_1 \neq i_2 \in S^* \text{ and } \geq j_0 \implies A_{i_1} \cap A_{i_2} = K.$$

□

A generalization:

THEOREM 1.23. *Suppose λ is regular and $\mu < \lambda$ such that $\kappa^{<\mu} < \lambda$ for all $\kappa < \lambda$.*

If $\mathcal{F}$ is a family of λ-many sets all of cardinality $< \mu$ then there exists $\mathcal{F}' \subseteq \mathcal{F}$ a Δ-system of cardinality λ.

PROOF. Without loss of generality we may assume that $A \in \mathcal{F} \implies A \subseteq \lambda$. Fix $\{A_i \mid i < \lambda\}| = \mathcal{F}$. Let

$$\kappa := \begin{cases} \mu & \text{if } \mu \text{ is regular} \\ \\ \mu^+ & \text{Otherwise.} \end{cases}$$

Let $S^\lambda_\kappa := \{\delta < \lambda \mid \operatorname{cf} \delta = \kappa\}$, it is a stationary subset of λ. Consider

$$f(i) := \min\{j < \lambda \mid A_i \cap i \subseteq j\}.$$

Clearly $f(i) < i$ for every $i \in S^\lambda_\kappa$. By Fodor's Lemma there are a stationary $S' \subseteq S^\lambda_\kappa$ and $j_0 < \lambda$ such that

$$i \in S' \implies f(i) = j_o.$$

Consider now the mapping $i \mapsto A_i \cap i$ as

$$i \in S' \implies A_i \cap i \in \mathcal{S}_{<\mu}(j_0).$$

The range of the above mapping is of cardinality $|j_0|^{<\mu}$ which by assumption on λ and μ is less than λ. As the domain of above mapping is a set of cardinality λ by regularity of λ we get that there exists $S^* \subseteq S'$ of cardinality λ and there is $B \in \mathcal{S}_{<\mu}(j_0)$ such that

$$i \in S^* \implies A_i \cap i = B.$$

Let $\mathcal{F}' := \{A_i \mid i \in S^*, i \geq j_0\}$. Now check that $\mathcal{F}'$ is as required. □

DEFINITION 1.24. Weak Δ-system.

Exercises 3.1.

(1) Prove Lemma 1.7 directly without using the concept of elementary submodel.

(2) Let $\lambda > \aleph_0$ be regular and M is an L-structure such that $|M| = \lambda > |\mathrm{L}(M)|$. Let $f : \lambda \to \lambda$ and $\{M_\alpha \mid \alpha < \lambda\}$ an increasing and continuous chain of elementary submodels of M such that $\alpha < \lambda \implies \|M_\alpha\| < \lambda$ and $M = \bigcup_{\alpha<\lambda} M_\alpha$. Prove that

$$\{\alpha < \lambda \mid \alpha = |M_\alpha| \wedge f \restriction \alpha : \alpha \to \alpha\} \in D_\lambda.$$

(3) Let λ be an infinite cardinal. Show that

$$\{\delta < \lambda \mid \lambda\delta = \delta\}$$

is a club of λ^+, where $\lambda\delta$ stands for ordinal multiplication.

(4) Suppose that M is a structure such that $\mu := |M|$ is uncountable regular and $\mu > |\mathrm{L}(M)|$. If $f \in \mathrm{Aut}(M)$ then for any increasing continuous resolution $\{M_\alpha \prec M \mid \alpha < \mu\}$

$$\{\delta < \mu \ : \ |M_\delta| = \delta, f \in \mathrm{Aut}(M_\delta)\}$$

is a club of μ.

Suppose that G is an uncountable group such that $\mu := |G|$ is a regular cardinal and $f \in \mathrm{Aut}(G)$ an outer automorphism. If $\{G_\alpha \mid \alpha < \mu\}$ is an increasing continuous chain of subgroups which is resolution of G then

$$\{\delta < \mu \mid f \notin \mathrm{Inn}(G_\delta)\}$$

is a club of μ.

(5) Let $\lambda > \aleph_0$ be regular and n is a positive integer.
If $H : \underbrace{\lambda \times \cdots \times \lambda}_{n\text{-times}} \to \lambda$ then

$$C = \{\delta < \lambda^+ \mid \forall\beta_1 \cdots \forall\beta_n [\bigwedge_{1\leq k\leq n} \beta_k < \delta \to H(\beta_1, \ldots, \beta_n) < \delta]\} \in D_\lambda.$$

(6) Prove Corollary 1.10 directly without using Theorem 1.9.

(7) Let L be a countable similarity type, and let λ be an uncountable regular cardinal number. Suppose M and N are L-structures of cardinality λ and $F : M \cong N$. If $\{M_\alpha \preceq M \ : \ \alpha < \lambda\}$ and $\{N_\alpha \preceq N \ : \ \alpha < \lambda\}$ are increasing continuous elementary chains such that $M = \bigcup_{\alpha<\lambda} M_\alpha$, $N = \bigcup_{\alpha<\lambda} N_\alpha$, and for every $\alpha < \lambda \ \|M_\alpha\| \cdot \|N_\alpha\| < \lambda$ then
$\{\delta < \lambda \ : \ F \restriction M_\delta : M_\delta \cong N_\delta\}$ contains a club.

(8) Let λ be an uncountable regular. Given a club $C \subseteq \lambda$ show that for any similarity type L of cardinality less than λ and an L-structure M such that its universe is the set λ for any resolution $\{M_\alpha \mid \alpha < \lambda\}$ of M we have that $C \equiv \{\alpha \ : \ |M_\alpha| = \alpha\} \mod D_\lambda$.

(9) Let $\lambda > \aleph_0$ be regular and M is an L-structure such that $|M| = \lambda > |\mathrm{L}(M)|$. For any increasing and continuous $\{A_\alpha \mid \alpha < \lambda\}$ family of subsets of $|M|$ such that $\alpha < \lambda$ implies that $|A_\alpha| < \lambda$ and $|M| = \bigcup_{\alpha<\lambda} A_\alpha$

the set $\{\alpha < \lambda \mid A_\alpha$ is a universe of an elementary submodel of $M\}$ contains a club.

(10) Let λ be an uncountable regular cardinal. Show that there exists a family $\{S_X \subseteq \lambda \ : \ X \subseteq \lambda\}$ of stationary sets with the following property: if $X \neq Y$ then for every club $C \subseteq \lambda$ we have that $S_X \cap C \not\subseteq S_Y \cap C$.

(11) Let λ be an uncountable regular cardinal. Let A be a set of cardinality λ, and suppose $\{A_i \subseteq A \ : \ i < \lambda\}$ is increasing and continuous (with respect to i) such that if $i < \lambda$ then $|A_i| < \lambda$ and $A = \bigcup_{i<\lambda} A_i$. If $F : \lambda \to \mathcal{P}(A)$ satisfies $\alpha < \lambda \implies |F(\alpha)| < \aleph_0$ then there exists $S \subseteq \lambda$ of cardinality λ and a finite subset X of A such that $\alpha \in S \implies F(\alpha) = X$.

(12) Let λ be a regular cardinal and $\mu \leq \lambda$ suppose that $\forall \chi < \lambda [\chi^{<\mu} = \chi]$. If $\mathcal{F}$ is a family of sets of cardinality λ such that $A \in \mathcal{F} \implies |A| < \mu$ then for every $g : \lambda \to \mathcal{F}$ there are $S \subseteq \lambda$ of cardinality λ and $A_0 \in \mathcal{F}$ such that $i \in S \implies g(i) = A_0$.

(13) Let L be a countable similarity type, let λ be an uncountable regular cardinal. Suppose that M is an L-structure such that the universe of M is the set of ordinals λ. Show that for every $\alpha < \lambda$ there exists $N \preceq M$ such that $|N|$ is an ordinal greater than α.

(14) Theorem 1.9 can be strengthened by replacing requirement (5) with:
(5') the set $\{\delta < \lambda \ : \ |M_\delta| = \delta\}$ is a club. Moreover for every chain of sets $\{A_\alpha \subseteq |M| \ : \ \alpha < \lambda\}$ such that $\{A_\alpha \ : \ \alpha < \lambda\}$ is continuous. If for every α we have that $|A_\alpha| < \lambda$ and $|M| = \bigcup_{\alpha<\lambda} A_\alpha$ then the set $\{\alpha < \lambda \ : \ A_\alpha = |M_\alpha|\}$ is a club.

(15) Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an AEC. Suppose $\lambda > \mathrm{LS}(\mathcal{K})$ is regular and suppose $M \in \mathcal{K}_\lambda$. For every $\prec_{\mathcal{K}}$-increasing and continnuous (see Definition **??**) $\{M_i \prec_{\mathcal{K}} M \mid i < \lambda\}, \{N_i \prec_{\mathcal{K}} M \mid i < \lambda\} \subseteq \mathcal{K}_{<\lambda}$ such that $M = \bigcup_{i<\lambda} M_i = \bigcup_{i<\lambda} N_i$ the set $\{i < \lambda \mid M_i = N_i\}$ cointains a club.

(16)

(17)

(18) MOVE TO ANOTHER SECTION??

(19) Prove the claims of Remark 10.13.

(20) $|\mathbf{V}_\omega| = \aleph_0$ and $|\mathbf{V}_{\omega+\alpha}| = \beth_\alpha$.

(21) $\alpha < \beta \implies \mathbf{V}_\alpha \subseteq \mathbf{V}_\beta$ and $\mathbf{L}_\alpha \subseteq \mathbf{L}_\beta$.

(22) $|\mathbf{L}_\omega| = |\mathbf{L}_{\omega+1}| = |\mathbf{L}_{\omega+\alpha}| = \aleph_0$ for every $\alpha < \omega_1$.

(23) $|H(\omega)| = \aleph_0$.

(24) $H(\aleph_1) \neq \mathbf{V}_{\omega_1}$.

(25) $|H(\aleph_1)| = 2^{\aleph_0}$.

(26) $\mathrm{ZFC} \vdash |H(\chi)| = 2^{<\chi}$.

(27) $\mathrm{ZFC} \vdash \forall \chi > \aleph_0 [H(\chi) = \mathbf{V}_\chi \iff \chi = \beth_\chi \iff \chi$ is strongly inaccesible].

(28) Let $M \models \mathrm{ZF}$. Show that On^M is first-order definable in M without parameters.

(29) It is a basic result of ZF that the class On is well ordered by the relation $\in$ namely, $\langle \mathrm{On}, \in \rangle$ is a linear order and for every non-empty subclass S of On there exists $a \in S$ minimal with respect to $\in$. Explain why the previous exercise does not contradict Theorem 5.22.

(30) Let $M \models \mathrm{ZF}$. Suppose that $\mathrm{On}^M \supseteq \omega$. Show that unlike the situation in Peano's arithmetic (Exercise 5.25) ω is first-order definable in M without parameters.

(31) Let χ be uncountable regular. Let $N \preceq \langle H(\chi), \in, \ldots \rangle$.
   (a) If $S \in |N|$ is a countable set then $S \subseteq |N|$.
   (b) For every $\alpha \in \omega + 1$ we have that the set $\alpha \cap |N|$ is an ordinal.

(32) Let L be an uncountable similarity type and T a complete first-order theory in L. Suppose χ is a regular cardinal large enough so that

$$L \cup T \cup \{L, T\} \subseteq H(\chi)$$

and $\langle H(\chi), \in \rangle$ reflects the fact that T is a complete L-theory.

Let $\mathfrak{B} := \langle H(\chi), \in, L, T, \vdash \rangle$, where L, T are unary predicates and $\vdash$ is the binary relation of provability. If $M \preceq \mathfrak{B}$ is a countable then T^M is a countable complete theory in L^M.

(33) Show that requirement (2) in Definition 10.16 is redundant. It follows from requirement (3).

(34) Let L be a countable similarity type and $\psi \in L_{\omega_1,\omega}$. Suppose χ is a regular cardinal large enough so that

$$L_{\omega_1,\omega} \cup \{L_{\omega_1,\omega}\} \subseteq H(\chi).$$

Let $\mathfrak{B} := \langle H(\chi), \in, L_{\omega_1,\omega}, \varphi \rangle_{\varphi \in L}$, where $L_{\omega_1,\omega}$ is a unary predicate and φ individual constants. If $\langle U, \in \mathcal{L}, \varphi \rangle_{\varphi \in L}$ is a countable elementary submodel of $\mathfrak{B}$ containing ψ then $\mathcal{L}$ is a countable fragment of $L_{\omega_1,\omega}$ containing ψ.

(35) Let $<^*$ be a well-ordering of the set $H(\omega_2)$, let $\mathfrak{B} := \langle H(\omega_2), \in, \omega_1, <^* \rangle$. When ω_1 is a unary predicate with the usual interpretation. Prove that if $M \preceq \mathfrak{B}$ is of cardinality $\aleph_1$ such that $|M| \supseteq \omega_1$ then M is a Jónsson model.

(36) Let $\{F_n \mid n < \omega\}$ be list of all definable Skolem functions in $\mathfrak{B}$ from the previous exercise. Let $\mathfrak{B}^1 := \langle \mathfrak{B}, F_n \rangle_{n<\omega}$.
Show that if $\mathfrak{B}^2 \preceq \mathfrak{B}^1 \restriction \{F_n \mid n < \omega\}$ is of cardinality $\aleph_1$ then $\mathfrak{B}^2$ is a Jónsson algebra.

(37)

(38)

(39)

(40)

2. Ultraproducts

2.1. Introduction. A particular case of the ultraproduct construction was introduced in 1934 by Skolem ([**?**]) in order to get non-standard models of true arithmetic. In 1948 E. Hewitt ([**Hew**] used the same construction to study real-closed fields. Only in 1955 in a paper of J. Los ([**Lo55**]) this construction was applied to an arbitrary model with tremendous effect both on model theory and set theory.

A standard construction in mathematics is to consider products of structures. It is natural to consider products of L-structures. However, even for two models there are difficulties in preserving the theory: If $M, N \models T$ then $M \times N$ may not be a model of T:

EXAMPLE 2.1. Suppose that F_ℓ for $\ell = 1, 2$ are fields. Let $E := F_1 \times F_2$. The operations of E are defined coordinatewise, namely

$$\langle a, b\rangle +_E \langle a', b'\rangle := \langle a +_{F_1} a', b +_{F_2} b'\rangle,$$
$$\langle a, b\rangle \cdot_E \langle a', b'\rangle := \langle a \cdot_{F_1} a', b \cdot_{F_2} b'\rangle \quad \text{for } \langle a, b\rangle, \langle a', b'\rangle \in E.$$

Clearly E is a commutative ring, but is not a field since it has zero divisors: $\langle 0_{F_1}, 1_{F_2}\rangle \cdot_E \langle 1_{F_1}, 0_{F_2}\rangle = \langle 0_{F_1}, 0_{F_2}\rangle (= 0_E)$.

2.2. The basics. The most basic measure of a size of a set is its cardinality. In measure theory there is a finer and a more interesting concept. Consider I the interval $[0, 1]$ of real numbers. For $S \subseteq I$...

First let's look at another notion of largeness

DEFINITION 2.2. Let I be a non-empty set. A non-empty $D \subseteq \mathcal{P}(I)$ is a *filter on* I iff

(1) $S_1, S_2 \in D \implies S_1 \cap S_2 \in D$ and
(2) $S' \supseteq S$ and $S \in D \implies S' \in D$, for every $S, S' \subseteq I$.

A filter D on I is called a *proper filter* iff $\emptyset \notin D$.
A filter D on I is called a *principle filter* iff there exists $a \in I$ such that $a \in S$ for every $S \in D$.

REMARK 2.3. Notice that if D is a filter on I which is not proper (i.e. $\emptyset \in D$) than by the second requirement in the definition it follows that D is the powerset of I. Since we already have an efficient notation for the power set of I from now on we always assume that all *filters are proper*.

EXAMPLES 2.4. The following are all filters:

(1) Let $I = \mathcal{P}(\omega)$ and

$$D^1_\omega := \{S \subseteq \omega \ : \ |\omega - S| < \aleph_0\}.$$

(2) Let λ be an infinite cardinal number, $I := \mathbf{P}(\lambda)$ and
$$D^1_\lambda := \{S \subseteq \lambda \ : \ |\lambda - S| < \lambda\}.$$
(3) Let λ be an uncountable regular cardinal number, $I := \mathbf{P}(\lambda)$ and
$$D^2_\lambda := \{S \subseteq \lambda \ : \ \exists S' \in D_\lambda [S \supseteq S']\}.$$
It is the contents of Lemma 1.7 that D^2_λ is a filter.
(4) Let I be the powerset of the interval of real numbers.
(5)

How to construct filters?

DEFINITION 2.5. Let I be a non-empty set. A non-empty family D of subsets of I has the *finite intersection property* iff for every $n < \omega$ and every $\{S_k \mid k < n\} \subseteq D$ we have that $\bigcap_{k<n} S_k \neq \emptyset$.

The next is the main device to construct filters.

LEMMA 2.6. *Let I be a non-empty set and D a non-empty family of subsets of I. If D has the finite intersection property than there exists a filter over $\bar{D}$ a filter over I such that $\bar{D} \supseteq D$.*

PROOF. Define
$$D^* := \left\{ \bigcap_{k<n} S_k \;\middle|\; n < \omega, \ \{S_k \mid k < n\} \subseteq D \right\}.$$
Now define
$$\bar{D} := \{S \subseteq I \mid \text{ there exists } S' \in D^* \text{ such that } S \supseteq S'\}.$$
Verify that $\bar{D}$ is a filter containing the family D. □

It is easy to see that $\bar{D}$ is the minimal (with respect to $\subseteq$) filter containing the family D. Thus it is called the *filter generated by* D.

PROOF. □

DEFINITION 2.7. Let D be a filter on I and λ be an infinite cardinal. The filter D is λ-*complete* iff for every $\mu < \lambda$ and every $\{S_i \mid i < \mu\} \subseteq D$ we have that $\bigcap_{i<\mu} S_i \in D$.

In measure theory often $\aleph_1$-complete filters are called σ-*complete*.

DEFINITION 2.8. A (proper) filter D on I is an *ultrafilter* iff for every
$$S \in D \text{ or } I - S \in D \quad \text{for every } S \subseteq I.$$

The next is a useful technical characterization of ultrafilters.

PROPOSITION 2.9 (characterization of ultra filters). *Let D be a filter on I. The following are equivalent:*

(1) *D is an ultrafilter.*
(2) *D is a maximal filter. I.e. $D' \supsetneq D$ implies that D' is not a filter, for every $D' \subseteq \mathcal{P}(I)$.*

PROOF. $(1) \implies (2)$: Let D be an ultrafilter on I and for the sake of contradiction suppose that there exists D' which is a proper extension of D to a filter on I. Since $D' \supsetneq D$ there exists $S \in D' - D$. Since $S \notin D$ and D is an ultrafilter necessarily $I - S \in D$ thus by the first requirement in the definition of a filter we get that $S \cap (I - S) \in D$, namely $\emptyset \in D$ this contradicts the assumption that D is a proper filter.

$(2) \implies (1)$: Let D' be a maximal filter on I. Given $S \subseteq I$ such that $S \notin D'$. For the sake of contradiction suppose that $I - S \notin D'$. By Lemma 2.6 it suffices we contradict the maximality of D' if

$$D' \cup \{S\} \quad \text{or} \quad D' \cup \{I - S\}$$

have the finite intersection property.
Thus there are $\{S_k \mid k < n\} \subseteq D'$ and $\{S'_k \mid k < n'\} \subseteq D'$ such that

$$S \cap \bigcap_{k<n} S_k = \emptyset \text{ and } (I - S) \cap \bigcap_{k<n'} S'_k = \emptyset.$$

Namely we have that

$$\bigcap_{k<n} S_k \cap \bigcap_{k<n'} S'_k = \emptyset.$$

This is a contradiction to clause (1) in the definition of filter. □

LEMMA 2.10 (ultrafilter extension). *Let I be a non-empty set. For every filter D over I there exists an ultrafilter D^* on I such that $D^* \supseteq D$.*

PROOF. Let

$$P_D := \{D' \subseteq \mathcal{P}(I) \mid D' \supseteq D \text{ is a filter}\}.$$

Apply Zorn's lemma to the poset $\langle P_D, \subseteq \rangle$. By Proposition 2.9 a maximal element D^* of P is an ultrafilter as required.

□

It is interesting to point out that there is a huge number of ultrafilters. In the next subsection it will be shown that the number of ultrafilters over I (when I is infinite) is $2^{2^{|I|}}$.

DEFINITION 2.11. Let λ be an uncountable cardinal number. The cardinal λ is *measurable* iff there exists a λ-complete non-principle ultrafiler on $I := \mathcal{P}(\lambda)$.

DEFINITION 2.12. Let L be a similarity type, I a nonempty set, D a filter on I and $\{M_i \mid i \in I\}$ a family of L-structures. For $f, g \in \prod_{i\in I} |M_i|$ let

$$f \sim_D g \iff \{i \in I \mid f(i) = g(i)\} \in D.$$

For $f \in \prod_{i\in I} |M_i|$ let $[f] := \{g \in \prod_{i\in I} |M_i| \ : \ f \sim_D g\}$.

PROPOSITION 2.13. *The relation $\sim_D$ is an equivalence relation.*

PROOF. Exercise. □

DEFINITION 2.14 (Reduced product). Let L be a similarity type, I a nonempty set, D a filter on I and $\{M_i \mid i \in I\}$ a family of L-structures.
The *reduced product of* $\{M_i \mid i \in I\}$ *with respect to* D is an L-structure N whose universe is

$$\prod_{i\in I} |M_i|/\sim_D .$$

On the set $\prod_{i\in I} |M_i|/\sim_D$ we define interpretation for the symbols from L in a natural way:

(1) For individual constant $c \in L$ let define $f \in \prod_{i\in I} |M_i|$ by $f(i) := c^{M_i}$ for all $i \in I$. Now let
$$c^M := [f].$$
(2) For a function symbol $F(x_0, \dots, x_{n-1}) \in L$ and $f_0, \dots, f_{n-1} \in \prod_{i\in I} |M_i|$, let $g(i) := F^{M_i}(f_0(i), \dots, f_{n-1}(i))$ for all $i \in I$. Define
$$F^M(f_0, \dots, f_{n-1}) := [g].$$
(3) For a relation symbol $F(x_0, \dots, x_{n-1}) \in L$ and $f_0, \dots, f_{n-1} \in \prod_{i\in I} |M_i|$ let

$$\langle f_0, \dots, f_{n-1}\rangle \in R^M \iff \{i \in I \mid M_i \models R(f_0(i), \dots, f_{n-1}(i))\} \in D.$$

The structure N is denoted by $\prod^I M_i/D$.

It is necessary to show that the above is independent of choice of representatives:

PROPOSITION 2.15. *Let L be a similarity type, I a nonempty set, D a filter on I and $\{M_i \mid i \in I\}$ a family of L-structures.*

(1) *Suppose $F(x_0, \dots, x_{n-1}) \in L$ is a function symbol and for every*
$$\langle f_0, \dots, f_{n-1}\rangle, \langle f'_0, \dots, f'_{n-1}\rangle \in \prod_{i\in I} |M_i|$$
if $g(i) := F^{M_i}(f_0(i), \dots, f_{n-1}(i))$ and $g'(i) := F^{M_i}(f'_0(i), \dots, f'_{n-1}(i))$ for all $i \in I$ then $g \sim_D g'$.

(2) *Suppose* $R(x_0, \ldots, x_{n-1}) \in L$ *is a relation symbol and for every*

$$\langle f_0, \ldots, f_{n-1}\rangle, \langle f'_0, \ldots, f'_{n-1}\rangle \in \prod_{i\in I} |M_i|$$

we have

$$\{i \in I \mid M_i \models R(f_0(i), \ldots, f_{n-1}(i))\} \in D \quad \textit{iff}$$
$$\{i \in I \mid M_i \models R(f'_0(i), \ldots, f'_{n-1}(i))\} \in D.$$

PROOF. □

DEFINITION 2.16. Let I, D and $\{M_i \mid i \in I\}$ be as in Definition 2.14. When D is an ultrafilter then $\prod^I M_i/D$ is the *ultra product of* $\{M_i \mid i \in I\}$ *with respect to* D. When $M = M_j$ for all $i \in I$ then $\prod^I M/D$ is the *reduced power of* M *with respect to* D. When $M = M_i$ for all $i \in I$ and D is an ultrafilter then $\prod^I M/D$ is the *ultrapower of* M *with respect to* D.

It turns out that $\prod^I M_i/D$ when D is a principle ultrafilter is not interesting (see Exercise 11).

When dealing with reduced powers (and ultrapowers) we will consider the mapping

$$(*) \quad a \mapsto [f_a] \quad \text{where } f_a(i) := a \text{ for all } i \in I.$$

It is called *the natural embedding of* M *into* $\prod^I M/D$.

As was shown above, while the cartesian product of of models of T may not be a model of T the next theorem assures that ultraproducts do preserve the first-order theory.

THEOREM 2.17 (Loś's theorem). *Let* I, D, $\{M_i \mid i \in I\}$ *and* N *be as in Definition 2.14.*

(1) *For every* $\langle f_0, \ldots, f_{n-1}\rangle \in \prod_{i\in I} |M_i|$ *and every* L*-term* $\tau(x_0, \ldots, x_{n-1})$

$\tau^N([f_0], \ldots, [f_{n-1}]) = [g]$ *where*

$$g(i) := \tau^{M_i}(f_0(i), \ldots, f_{n-1}(i)) \textit{ for all } i \in I.$$

(2) *When* D *is an ultrafilter then*

$$N \models \varphi[[f_0],\ \ldots, [f_{n-1}]] \iff \{i \in I \mid M_i \models \varphi[f_0(i),\ \ldots, f_{n-1}(i)]\} \in D$$
$$\textit{for every } \langle f_0, \ldots, f_{n-1}\rangle \in \prod_{i\in I} |M|.$$

(3) *The natural embedding of* M *into* $\prod^I M/D$ *when* D *is an ultrafilter is an elementary embedding.*

PROOF.

(1) By induction on complexity of terms.
(2) By induction on φ:
   (a) For atomic φ:
      (i) Suppose that $\varphi(\mathbf{x})$ is the formula $\tau_1(\mathbf{x}) = \tau_2(\mathbf{x})$ just use (1) and the definition of satisfaction.
      (ii) Suppose that φ is the formula $R(\tau_0, \dots, \tau_{n-1})$; Again use (1) and definition of satisfaction.
   (b) For $\varphi(\mathbf{x}) = \psi_1(\mathbf{x}) \wedge \psi_2(\mathbf{x})$; By definition of satisfaction we get that

$$S_\ell := \{i \in I \mid M_i \models \psi_\ell[f_0(i), \ \dots, f_{n-1}(i)]\} \in D \text{ for } \ell = 1, 2.$$

   Since D is closed under intersection $S := S_1 \cap S_2 \in D$. Since

$$S = \{i \in I \mid M_i \models \varphi[f_0(i), \ \dots, f_{n-1}(i)]\} \in D$$

   we are done.
   (c) For $\varphi(\mathbf{x}) = \neg\psi(\mathbf{x})$; This is the place where the assumption that D is an *ultra*filter is used. By definition of satisfaction

$$N \models \varphi[[f_0], \dots, [f_{n-1}]] \iff N \not\models \psi[[f_0], \dots, [f_{n-1}]].$$

   By the induction hypothesis

$$N \not\models \psi[[f_0], \dots, [f_{n-1}]] \iff$$
$$S := \{i \in I \mid M_i \models \psi[f_0(i), \dots, f_{n-1}(i)]\} \notin D.$$

   Since D is an ultrafilter $I - S \in D$. Thus

$$\{i \in I \mid M_i \models \neg\psi[f_0(i), \dots, f_{n-1}(i)]\} \in D.$$

   (d) For $\varphi(\mathbf{x}) = \exists y \psi(y; \mathbf{x})$; By definition of satisfaction

$$N \models \varphi[[f_0], \dots, [f_{n-1}]] \iff \text{there exists } f \in \prod_{i \in I} |M_i|$$

such that $N \models \psi[[f]; [f_0], \dots, [f_{n-1}]] \iff$ there exists $f \in \prod_{i \in I} |M_i|$

such that $\{i \in I \mid M_i \models \psi[f(i); f_0(i), \dots, f_{n-1}(i)]\} \in D \iff$
$\{i \in I \mid M_i \models \exists x \psi[x; f_0(i), \dots, f_{n-1}(i)]\} \in D.$

(3) Let $M^* := \{f_a \mid a \in |M|\}$. Show $M^* \preceq \prod^I M/D$. It is clear that M^* is a submodel of $\prod^I M/D$. Using part (2) show that: For every formula $\varphi(x_0, \dots, x_{n-1})$ and all $\langle a_1, \dots, a_n\rangle \in |M|$ we have that

$$\prod^I M/D \models \varphi([f_{a_1}], \dots [f_{a_{n-1}}]) \iff M \models [a_1, \dots, a_n].$$

□

Ultrapowers can be used as a substitute of the compactness theorem to obtain new proofs to the existence of non-standard models of arithmetic or a concrete non-archimedean elementary extension of the reals. However we will show later that they have also some other uses.

EXAMPLES 2.18. The following examples are important:

(1) (While this is not the original construction used by Skolem to obtain a non-standard model of arithmetic, it is ismilar) Let $N := \langle \omega, +, \cdot \rangle$ and D be a non-principle ultrafilter over ω. The structure $\prod^{\omega} N/D$ is a non-standard model of arithmetic. Moreover for every infinite set of prime numbers S there exists $a_S \in \prod^{\omega} N/D$ such that $\prod^{\omega} N/D \models [p \mid a_S] \iff p \in S$ for every prime p (compare with Theorem 5.8):
Fix $\{p_k \mid k < \omega\} = S$. Now let $f_S(n) := \prod_{k<n} p_k$. If $p \in S$ then the set $\{k < \omega \ : \ p \nmid a_k\}$ is finite and can not member of D. Thus $\{n < \omega \ : \ p \mid f(n)\} \in D$ If $p \notin S$ then $\{n < \omega \ : \ p \mid f(n)\} = \emptyset$. This establishes that $a_S := [f_S]$ is as required.
(2) Let $M := \langle \mathbb{R}, +, \cdot, < \rangle$ and D be a non-principle ultrafilter over ω. Check that $\prod^{\omega} M/D$ is a non-Archimedean model of the real numbers.
(3) Let P be the set of prime numbers and let D a non-principle ultrafilter over P. For every prime p let F_p be a field of characteristics p. $\prod^{P} F_p/D$ is a field of characteristics 0.
(4) Let D be a non-principle ultrafilter over ω. The ultrapower $\prod^{\omega} \mathbf{Z}/D$ is an integral domain but not a UFD.
(5) (Kochen 1961) Let F be a field and I a non-empty set. Consider the mapping:

$$D \mapsto I_D \quad \text{when } I_D \text{ is the ideal generated by}$$
$$\{\eta \in \prod^{I} F \mid\} \text{in the ring } \prod^{I} F.$$

   (a) The above is a bijection from the set of ultrafilters over I onto the maximal ideals of the ring $\prod^{I} F$.
   (b) $\prod^{I} F/I_D \cong \prod^{I} F/D$ for every ultrafilter D over I.

Using ultraproducts one can give a purely algebraic proof to the compactness theorem that does not depend on the concept of provability and the completeness theorem.

THEOREM 2.19 (Algebraic proof of the compactness theorem). *Let T be a first-order theory. If for every finite $T' \subseteq T$ there exist an $\mathrm{L}(T)$-structure $M' \models T'$ then T has a model.*

PROOF. Consider

$$I := \{T' \subseteq T \ : \ |T'| < \aleph_0\}.$$

By the assumption, for every $i \in I$ there exists an L-structure $M_i \models i$. We will define an ultrafilter D on I such that $\prod^{I} M_i/D \models T$. For every sentence $\varphi \in T$ let

$$[\varphi] := \{T' \in I \ : \ \varphi \in T'\}.$$

Since $F := \{[\varphi] \ : \ \varphi \in T\}$ has the finite intersection property there is an ultrafilter $D \supseteq F$ over I.

By Los's theorem it suffices to verify that

$$\{i \in I \ : \ M_i \models \varphi\} \in D \quad \text{for all } \varphi \in T :$$

Let $\varphi \in T$ be given. By the definition of F we get that $[\varphi] \in D$. Now if $i \in [\varphi]$ then $M_i \models \varphi$, thus we have shown that

$$[\varphi] \subseteq \{i \in I \ : \ M_i \models \varphi\}.$$

This together with the fact that $[\varphi] \in D$ implies that $\{i \in I \ : \ M_i \models \varphi\} \in D$. □

The idea of the previous proof can be used to characterize elementary classes. Recall

DEFINITION 2.20. An abstract class $\mathcal{K}$ is an *elementary class* iff there exists a first-order theory T such that

$$\mathcal{K} = \mathrm{Mod}(T) \text{ and } \mathrm{L}(\mathcal{K}) = \mathrm{L}(T).$$

The following is a characterization of elementary classes:

THEOREM 2.21. *Let L be a similarity type, and let $\mathcal{K}$ be a class of L-structures. The following are equivalent:*

(1) *$\mathcal{K}$ is an elementary class.*
(2) *$\mathcal{K}$ is closed under elementary equivalence and ultraproducts.*

PROOF. (1) $\Longrightarrow$ (2): Closure under $\equiv$ is trivial, closure under ultraproducts follows from Los's theorem.

(2) $\Longrightarrow$ (1): Let

$$T_{\mathcal{K}} := \bigcap_{M \in \mathcal{K}} \mathrm{Th}(M).$$

We will show that $\mathcal{K} = \mathrm{Mod}(T_K)$. Let $N \models T_K$. Take $T := \mathrm{Th}(N)$, I and D as in the previous proof.

For every $i \in I$ there exists $M_i \in \mathcal{K}$ such that$M_i \models i$: Otherwise, there is a finite $T' \subseteq T$ such that $M \not\models T'$ for every $M \in \mathcal{K}$. Thus $\neg \bigwedge T' \in T_K$ which by the choice of N gives that $\neg \bigwedge T' \in T$ this is impossible since T is consistent.

As in the previous argument show that $\prod^I M_i/D \models \mathrm{Th}(N)$. Since $\mathcal{K}$ is closed under ultraproducts $\prod^I M_i/D \in \mathcal{K}$, since $\mathrm{Th}(\prod^I M_i/D) = \mathrm{Th}(N)$ by the closure of $\mathcal{K}$ under elementary equivalence $N \in \mathcal{K}$. □

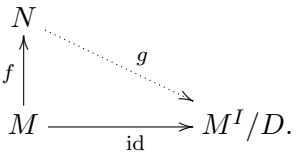
When an ultraproduct is saturated?

2.3. Ultrapowers of models of set theory. Connections with elementary embeddings of transitive classes. reconstructing the UF from the elementary embedding of models of set theory

$M = \langle |M|, \dots, \rangle$ well founded

PROPOSITION 2.22. *If D is a complete u.f. on I then $\Pi^I M/D$ is also well founded.*

THEOREM 2.23 (Scott's characterization of elementary embeddings). *Let M and N be both L-structures. There exists an elementary embedding $f : M \to N$ iff there exists an infinite I and an ultrafilter D on I such that there exists an elementary embedding $g : N \to M^I/D$ so that the diagram*



commutes.

PROOF. □

PROPOSITION 2.24. *If D is a complete u.f. on I M omitting a type p then $\Pi^I M/D$ is also omits p.*

Scott's theorem: There is no measurable in $\mathbf{L}$

DUALITY OF TAKING A SUBMODEL $M \prec H(\chi), \in, \dots$ AND $j : M \to M^U$

2.4. Construction of ultrafilters. We start with a basic result of combinatorial set theory, its proof uses a fundamental lemma due to Engelking and Kalowitz which is of independent importance.

THEOREM 2.25 (Existence of many subsets). *Let λ be an infinite cardinal number. There exists $\{S_i \mid i < 2^\lambda\} \subseteq \mathcal{P}(\lambda)$ such that*

$$|\bigcap_{i\in X} S_i - \bigcup_{j\in Y} S_j| = \lambda \text{ for all finite pairwise disjoint } X, Y \subseteq \lambda.$$

Even the case where X and Y are singletons, the Theorem gives a useful:

COROLLARY 2.26. *Let λ be an infinite cardinal number.*
There exists $F \subseteq \mathcal{P}(\lambda)$ of cardinality 2^λ such that $|S| = \lambda$ for all $S \in F$ and

$$S_1 \neq S_2 \in F \implies S_1 \not\subseteq S_2 \text{ and } S_2 \not\subseteq S_1.$$

Another important application of Theorem 2.25 is the following classical result on the number of ultrafilters.

THEOREM 2.27 (Engelking Karlowitz). *Let I be an infinite set. The number of distinct ultrafilters on I is $2^{2^{|I|}}$.*

PROOF. Denote by λ the cardinality of I. Let $\{S_i \mid i < 2^\lambda\} \subseteq \mathcal{P}(I)$ be as in Theorem 2.25.

For $X \subseteq 2^\lambda$ consider

$$F_X := \{S_i \mid i \in X\} \cup \{I - S_j \mid j \in 2^\lambda - X\}.$$

By the Theorem F_X is a family of subsets of I with the finite intersection property. By Lemma 2.6 and Lemma 2.10 there is an ultrafilter D_X over I extending F_X. Since $X_1 \neq X_2 \implies D_{X_1} \neq D_{X_2}$ the family $\{D_X \mid X \subseteq 2^\lambda\}$ is as required. □

We will use a lemma due to Engelking and Karlowitz which will also be used elsewhere.

LEMMA 2.28 (Engelking Karlowitz lemma). *For every infinite cardinal λ there exists a family $\mathcal{G}$ of 2^λ-many functions from λ to λ such that for all $\{f_k \mid k < n\} \subseteq_{finite} \mathcal{G}$ and every $\langle \gamma_k < \lambda \mid k < n\rangle$ there exists $\xi < \lambda$ satisfying $f_k(\xi) = \gamma_k$ for all $k < n$.*

PROOF. Let $\mathcal{F}$ be the following set of triples

$$\Big\{\big\langle A, \langle C^k \mid k < n\rangle, \langle \gamma^k \mid k < n\rangle\big\rangle \;\Big|\; n < \omega,\ A \subseteq_{finite} \lambda,\ C^k \subseteq A,\ \gamma^k < \lambda,\ k_1 \neq k_2 \implies C^{k_1} \neq C^{k_2}\Big\}.$$

Since λ is infinite fix a well ordering of $\mathcal{F}$:

$$\Big\{\big\langle A_\xi, \langle C^k_\xi \mid k < n_\xi\rangle, \langle \gamma^k_\xi \mid k < n_\xi\rangle\big\rangle \;\Big|\; \xi < \lambda\Big\}.$$

For every $B \subseteq \lambda$ let

$$f_B(\xi) := \begin{cases} \gamma^k_\xi & \text{if } B \cap A_\xi = C^k_\xi \\ 0 & \text{Otherwise.} \end{cases}$$

Now show that $\mathcal{G} := \{f_B \mid B \subseteq \lambda\}$ is as required: Let $n < \omega$, $\{B_k \subseteq \lambda \mid k < n\}$ and $\langle \gamma_k < \lambda \mid k < n\rangle$ be given. Choose a finite $A \subseteq \lambda$ such that

$$k_1 \neq k_2 < n \implies A \cap B_{k_1} \neq A \cap B_{k_2}.$$

By the choice of the enumeration of $\mathcal{F}$ there exists $\xi < \lambda$ be such that

$$n = n_\xi,\ A = A_\xi,\ A \cap B_k = C^k_\xi \text{ and } \gamma_k = \gamma^k_\xi \text{ for all } k < n_\xi.$$

By the definition of f_{B_k} we get that

$$f_{B_k}(\xi) = \gamma_k \quad \text{for all } k < n.$$

□

PROOF OF THEOREM 2.25. Let $\mathcal{G}$ be from the previous lemma.

Fix $\{f_i \mid i < 2^\lambda\} := \mathcal{G}$.
For every $i < 2^\lambda$ let $S_i := \{\xi < \lambda \mid f_i(\xi) = 0\}$. Given X and Y pairwise disjoint finite subsets of λ let $n_X := |X|$ and $n_Y := |Y|$. Let $n := n_X + n_Y$. For every $\alpha < \lambda$ take $\langle \gamma_k^\alpha \mid k < n\rangle := \bar{0}^{n_X} \,\hat{}\, \bar{\alpha}^{n_Y}$. Where $\bar{\alpha}^n$ stands for $\langle \underbrace{\alpha, \ldots, \alpha}_{n\text{-times}} \rangle$. By the lemma for every $\alpha > 0$ there exists $\xi_\alpha \in \bigcap_{i\in X} S_i - \bigcup_{j\in Y} S_j$. Notice that $\alpha_1 < \alpha_2 < \lambda \implies \xi_{\alpha_1} \neq \xi_{\alpha_2}$. □

The following corollary of EK will be used extensively in chapters 12 and 13 to construct many models:

COROLLARY 2.29. *Let λ be an uncountable regular cardinal. For every stationary $S \subseteq \lambda$ there exists a family of stationary sets $\{S_i \subseteq S \mid i < 2^\lambda\}$ such that*

$$i \neq j \implies S_i \not\equiv S_j \mod D_\lambda.$$

good ultrafilters

2.5. Some more basic facts.

DEFINITION 2.30. Let I be an infinite set and D a non-principle ultrafilter on I. Suppose that M is a structure. For ordinals α let $\mathrm{UL}(M, D, \alpha)$ called *limit ultra power of M of length α with respect to D* defined by induction on α:

For $\alpha = 0$ let $\mathrm{UL}(M, D, \alpha) := M$.

For $\alpha = \beta + 1$ let $\mathrm{UL}(M, D, \alpha) := \prod^I \mathrm{UL}(M, D, \beta)/D$.

For α limit let $\mathrm{UL}(M, D, \alpha) := \bigcup_{\beta<\alpha} \mathrm{UL}(M, D, \beta)$.

FCP?

Horn Sentences/prolog? reduced products

pcf theory.

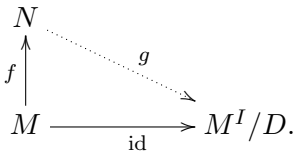
Connections with elementary embeddings of transitive classes. reconstructing the UF from the elementary embedding of models of set theory

$M = \langle |M|, \ldots, \rangle$ well founded

PROPOSITION 2.31. *If D is a complete u.f. on I then $\Pi^I M/D$ is also well founded.*

THEOREM 2.32 (Scott's characterization of elementary embeddings). *Let M and N be both L-structures. There exists an elementary embedding $f : M \to$*

N iff there exists an infinite I and an ultrafilter D on I such that there exists an elementary embedding $g : N \to M^I/D$ so that the diagram



commutes.

PROOF. □

THEOREM 2.33 (Frayne's Lemmma). *Let M and N be both L-structures. $M \equiv N$ iff there is an infinite I and an ultrafilter D on I such that there is an elementary embedding $f : N \to M^I/D$.*

PROOF. □

GIVE ANOTHER PROOF OF ROBINSON'S CONSISTENCY THEOREM USING SCOTT+FRAYNE.

PROOF. EXERCISE?

□

Exercises 3.2.

(1) Let I be an infinite set and $a \in I$. Verify that $\{S \subseteq I \mid a \in S\}$ is an ultrafilter over I. Is the assumption that I is infinite necessary?
(2) Let I be an infinite set

$$D := \{S \subseteq I \ : \ |I_S| < \aleph_0\}$$

is called the *Fréchet filter over* I. Verify that D is a filter but not an ultrafilter.
(3) Let I be an infinite set, show that the filter generated by

$$F := \{I - \{a\} \mid a \in I\}$$

is the Fréchet filter on I.
(4) Suppose D is a proper filter over an infinite set I.
Show that $S \in D \implies |S| \geq \aleph_0$.
(5) Let $\{F_i \mid i < \alpha\}$ be an increasing chain of filters on I show that $\bigcup_{i<\alpha} F_i$ is also a filter over I.
(6) Use transfinite induction, the previous exercise and a well ordering of $\mathcal{P}(I)$ to construct a non-principle ultrafilter over I.
(7) Let L be a similarity type, I a nonempty set, $\{M_i \mid i \in I\}$ a family of L-structures and D ultrafilter over I. Show that for every $S \in D$
   (a) The set $D[S] := \{S \cap S' \mid S' \in D\}$ is an ultrafilter over S.
   (b) $\prod^I M_i/D \cong \prod^S M_i/D[S]$.
(8) Let I be a non-empty set and let $j_0 \in I$ be given.
Show that $D := \{S \subseteq I \ : \ j_0 \in S\}$ is an ultrafilter. It is called *the principle ultrafilter generated by* j_0. .
(9) Let F be the Fréchet filter on I. Show that that if $D \supseteq F$ is an ultra filter on I then it is not principle.
(10) Let I be an infinite set and F a filter on I. Prove

$$\bigcap F = \emptyset \implies D \text{ is non-principle filter, for all } D \supseteq F.$$

(11) Let L be a similarity type, I a nonempty set, $\{M_i \mid i \in I\}$ a family of L-structures and D the principle ultrafilter generated by $j_0 \in I$ over I. Show that $\prod^I M_i/D \cong M_{j_0}$.
(12) Let I be a non-empty set and F a filter on I. Prove

$$\bigcap F = \emptyset \implies D \text{ is non-principle filter, for all } D \supseteq F.$$

(13) The following are dedicated to generalizations of Lemma 2.6 and Theorem 2.25:
   (a) Let λ be an infinite cardinal and let $\kappa \leq \lambda$ be such that $\lambda^{<\kappa} = \lambda$. Prove:

   LEMMA 2.34. *There exists a family $\mathcal{G}$ of cardinality 2^λ of functions from λ to λ such that for all $\alpha < \kappa$, every $\{f_\beta \mid \beta < \alpha\} \subseteq \mathcal{G}$ and every*

$\langle \gamma_\beta < \lambda \mid \beta < \alpha \rangle$ *there exists* $\xi < \lambda$ *satisfying* $f_\beta(\xi) = \gamma_\beta$ *for all* $\beta < \alpha$.

(b) Let λ be an infinite cardinal and let $\kappa \leq \lambda$ be such that $\lambda^{<\kappa} = \lambda$. Prove:

THEOREM 2.35. *There exists* $\{S_i \mid i < 2^\lambda\} \subseteq \mathcal{P}(\lambda)$ *such that*

$$|\bigcap_{i\in X} S_i - \bigcup_{j\in Y} S_j| = \lambda \text{ for all pairwise disjoint } X, Y \subseteq \lambda \text{ of cardinality less than } \kappa.$$

(c)

(14) Let $N := \langle \omega, +, \cdot \rangle$, I an infinite set and D a non-principle ultra filter on I. Prove that N^I/D is a non-standard model of arithmetic.

(15) Suppose $M \equiv N$, show that there exists a set I and ultra filter D on I such that there exists an elementary embedding $f : M \to N^I/D$.

(16) Let I be an infinite set and suppose $\{F_i \mid i \in I\}$ are fields. Let $R := \prod_{i\in I} F_i$, notice that R is a commutative ring but not a field (since it has zero divisors). Let D be an ultra filter on I and consider

$$J_D := \{r \in R \mid \{i \in I \mid r(i) = 0_{F_i}\} \in D\}.$$

Verify that J_D is a maximal ideal of R and that the field R/J_D is isomorphic to $\prod^I F_i/D$. For every maximal ideal $J \subset R$ there exists an ultra filter D on I such that $J = J_D$.

(17)

(18)

(19)

(20)

(21) Let $\langle P, < \rangle$ be well ordered and D a complete u.f. on I. Show that $\Pi^I \langle P, < \rangle / D$ is well-founded.

3. Ehrenfeucht-Fraissé games

Inspired by the back and forth property as a tool to prove that structures are isomorphic (see Exercise 2.44) one can use an infinite game to prove that structures are elementarily equivalent.

Later in section 2.3 we will present a similar characterization of existence of elementary embedding and isomorphism between pairs of models.

DEFINITION 3.1. Let M and N be L-structures. For $n < \omega$ by induction on n define the relation $M \equiv_n N$:

(1) For $n = 0$: $M \equiv_n N$ iff there are no individual constants in L or $M' \cong N'$ where M' (N') is the submodel of M (N) generated by the interpretations of the individual constants.
(2) For $n+1$: $M \equiv_{n+1} N$ iff for every $a \in |M|$ there exists $b \in |N|$ such that $\langle M, a\rangle \equiv_n \langle N, b\rangle$ and for every $b \in |N|$ there exists $a \in |M|$ such that $\langle M, a\rangle \equiv_n \langle N, b\rangle$.

Recall the following simple observation from predicate calculus (proved by induction on formulas using definition of satisfaction):

FACT 3.2 (existence of prenex normal form). *Let M be an L-structure. For every $\varphi \in \mathrm{Sent}(L)$ there exists a sentence ψ such that*

$$\psi = \mathbf{Q}_1 \cdots \mathbf{Q}_n \bigwedge \Gamma \quad \textit{such that} \quad M \models \varphi \iff M \models \psi$$

where Γ is a finite set of atomic or negations of atomic formulas and $\mathbf{Q}_\ell \in \{\forall, \exists\}$. A formula like ψ is called a prenex normal form formula of quantifier depth n

PROPOSITION 3.3. *Let M and N be L-structures and $n < \omega$. If $M \equiv_n N$ then $M \models \psi \iff N \models \psi$ for all prenex normal formulas of quantifier depth n.*

PROOF. By induction on $n < \omega$. □

In exercise 1.2.44 we introduced a generalization of Cantor's characterization of the order type of the rationals called the back and forth property. Let's have another look at the back and forth property:

DEFINITION 3.4. Let M and N be L-structures and α an ordinal. The relation $M \cong^\alpha_p N$ holds iff there exists $I \subseteq {}^{\alpha>}|M| \times {}^{\alpha>}|M|$ such that

(1) $\langle\langle\rangle, \langle\rangle\rangle \in I$, (recall that $\langle\rangle$ is the empty sequence)
(2) $\langle \mathbf{a}, \mathbf{b}\rangle \in I \implies \ell(\mathbf{a}) = \ell(\mathbf{b})$,
(3) if $\langle \mathbf{a}, \mathbf{b}\rangle \in I$ then

$$\mathrm{tp}_{\mathrm{qf}}(\mathbf{a}/\emptyset, M) = \mathrm{tp}_{\mathrm{qf}}(\mathbf{b}/\emptyset, N)$$

for every $a \in |M|$ there exists $b \in |N|$ such that $\langle \mathbf{a}^\frown a, \mathbf{b}^\frown b\rangle \in I$ and for every $b \in |N|$ there exists $a \in |M|$ such that $\langle \mathbf{a}^\frown a, \mathbf{b}^\frown b\rangle \in I$.

LEMMA 3.5. *Let M and N be countable L-structures. If $M \cong_p^\omega N$ then $M \cong N$.*

LEMMA 3.6. *Let M and N be L-structures and $n < \omega$. $M \cong_p^n N$ iff $M \equiv_n N$.*

Exercises 3.3.

(1) Show that $\langle \omega, < \rangle \equiv \langle \omega + \omega + \omega^* \rangle$.
(2) Recall the definition of ordinal exponentiation:
$\omega^{n+1} := \omega^n \omega$ for all $n < \omega$ and $\omega^\omega := \bigcup_{n<\omega} \omega^n$.

Show that $\mathrm{Th}(\langle \omega^n, < \rangle)$ does not have finite axiomatization. Hint: Show that for every sentence φ in the language $\{<\}$ there exists $n < \omega$ such that $\langle \omega^\omega, < \rangle \models \varphi \iff \langle \omega^n, < \rangle \models \varphi$.
(3) Let $\mathbb{Z}_n$ be the additive group of the integers modulo n. Denote by G the direct sum of ω-many copies of $\mathbb{Z}_n$ and by H the direct product of ω-many copies of $\mathbb{Z}_n$. Show that $G \equiv H$.
(4)
(5)
(6)
(7)
(8)
(9)

4. Two applications to algebra

In this section we concentrate in two applications. The first concerns algebraically closed fields and the secod real closed fields.

The first is central to model theory and plays a major role in applications.

The first is derivation of one of the more basic non-trivial facts of algebraic geometry (Hilbert's zeros-theorem) and the second application is presenting a solution to Hilbert's 17-th problem using the model theory of real-closed fields.

Artin's solutions

We will need few concepts from elementary algebraic geometry:

DEFINITION 4.1. Let F be a field and $\mathbf{x}$ a fixed n-tuple of variables (transcendental elements over F).

(1) $V \subseteq F^n$ is *algebraic set* iff there exists a set of polynomials $\Phi \subseteq F[\mathbf{x}]$ such that
$$V = \big\{\mathbf{a} \in F^n \mid (\forall p(\mathbf{x}) \in \Phi)\ [F \models p(\mathbf{a}) = 0]\big\}.$$
For any $S \subseteq F^n$ (not necessarily algebraic) denote by $\mathrm{I}(S)$ the set of *all* polynomials with coefficients from F that vanish on S, i.e.
$$\mathrm{I}(S) := \big\{p \in f[\mathbf{x}]\ :\ (\forall \mathbf{a} \in S)\ [F \models p(\mathbf{a}) = 0]\big\}.$$

(2) For $J \subseteq F[\mathbf{x}]$ let
$$\mathrm{V}(J) := \big\{\mathbf{a} \in F^n \mid (\forall p(\mathbf{x}) \in J)\ [F \models p(\mathbf{a}) = 0]\big\}.$$

PROPOSITION 4.2. *Let F be a field and $\mathbf{x}$ a fixed n-tuple of variables (transcendental elements over F).*

(1) $S_1 \subseteq S_2 \subseteq F^n \implies \mathrm{I}(S_1) \subseteq \mathrm{I}(S_2)$.
(2) *If* $\Phi_1 \subseteq \Phi_2 \subseteq F[\mathbf{x}]$ *then* $\mathrm{V}(\Phi_2) \subseteq \mathrm{V}(\Phi_1)$.
(3) *Let* $I \subseteq F[\mathbf{x}]$ *be an ideal. If* $\Phi \subseteq I$ *generates the ideal* I *then* $\mathrm{V}(I) = \mathrm{V}(\Phi)$.
(4) *For every* $S \subseteq F^n$ *the set* $\mathrm{I}(S)$ *is an ideal of the ring* $F[\mathbf{x}]$.
(5) *For any ideal* I *of* $F[\mathbf{x}]$ *we have that* $\mathrm{I}(\mathrm{V}(I)) \supseteq I$ *holds.*

PROOF. $(1), (2)$ and (5) are trivial. (3) and (4) follow directly from the definitions. □

In order to state the following fundamental result of commutative algebra we need a definition:

DEFINITION 4.3. Let R be a ring. R is a *Noetherian ring* iff every ideal I of R is finitely generated.

It is easy to check:

PROPOSITION 4.4. *Let R be a commutative ring. The ring R is Noetherian iff for every increasing chain of ideals $\{I_n \mid n < \omega\}$ there exists $n_0 < \omega$ such that $I_n = I_{n_0}$ for all n.*

THEOREM 4.5 (Hilbert's basis theorem). *Let R be a commutative ring. If R is Noetherian then also the ring $R[x]$ is Noetherian.*

PROOF. Suppose that R is Noetherian and show that every ideal is finitely generated. Let J be a given ideal of $R[x]$. For every $d < \omega$ let

$$I_d := \{a_d \in R \mid \sum_{i=0}^{d} a_i x^i \in J\}.$$

Clearly I_d is an ideal of R (for all d) and since $p \in J \implies xp \in J$ we have that

$$I_d \subseteq I_{d+1} \text{ for all } d.$$

Since R is Noetherian, by Proposition 4.4 there exists d_0 such that

$$I_{d_0} = I_d \text{ for all } d \geq d_0.$$

Using the assumption that R is Noetherian, each of the ideals I_d is finitely generated. For every $k \leq d_0$ let $r_{k,0}, \ldots, r_{k,n_k} \subseteq R$ such that

$$I_k = \langle r_{k,0}, \ldots, r_{k,n_k} \rangle.$$

Let $f_{k,i} \in R[x]$ such that its leading coefficient is $r_{k,i}$.

Let $S := \{f_{k,i} \mid i \leq n_k, k \leq d_0\}$, it suffices to show that

$$J = \langle S \rangle.$$

For the sake of contradiction suppose that there exists $f \in J$ such that $f \notin \langle S \rangle$. We may assume that f is of minimal degree d with the above property. Denote by a the leading coefficient of f. There are two cases:

Case 1: If $d \leq d_0$ then since $a \in I_d$ we have

$$a = \sum_{i \leq n_d} a_i r_{d,i} \text{ for some } a_i \in R.$$

Thus

$$\deg(f - \sum_{i \leq n_d} a_i f_{d,i}) < d.$$

By the minimality of d we get

$$f - \sum_{i \leq n_d} a_i f_{d,i} = q \text{ for some } q \in \langle S \rangle.$$

Since $\langle S\rangle$ is an ideal we have

$$f = \sum_{i\leq n_d} a_i f_{d,i} + q \in \langle S\rangle.$$

Thus we contradicted the assumption that $f \notin \langle S\rangle$.

Case 2: Suppose $d > d_0$. Since $I_d = I_{d_0}$ as before we have

$$a = \sum_{i\leq n_{d_0}} a_i r_{d_0,i} \text{ for some } a_i \in R.$$

Since we can get that

$$\deg(f - \sum_{i\leq n_{d_0}} a_i f_{d_0,i}) < d.$$

As in the previous case we have contradicted the minimality of d. □

Originally Hilbert proved his theorem for the following corollary that follows easily by induction from the basis theorem:

COROLLARY 4.6. *If F is a field then the ring $F[\mathbf{x}]$ is Noetherian.*

Notice that if $\Phi \subseteq F[\mathbf{x}]$ generates an ideal I then $\mathrm{V}(I) = \mathrm{V}(\Phi)$.

The following is a quick model theoretic proof to a fundamental algebraic result. It was one of the first remarkable results of Robinson in applying mathematical logic to classical algebra.

THEOREM 4.7 (Hilbert's Nullstellensatz). *Let F be an algebraically closed field. If I is an ideal of $F[\mathbf{x}]$ and $g \in F[\mathbf{x}]$ such that $g \in \mathrm{I}(\mathrm{V}(I))$ then there exists a positive integer n such that $g^n \in I$.*

We will derive Hilbert's Nullstellensatz theorem from the following:

THEOREM 4.8 (Hilbert's weak Nullstellensatz). *Let F be an algebraically closed field. If I is a proper ideal of $F[\mathbf{x}]$ then $\mathrm{V}(I) \neq \emptyset$.*

First we present an argument known as *Rabinowotz'z trick* that shows that Theorem 4.8 implies Theorem 4.7:

PROOF. Let y be a new variable (not in $\mathbf{x}$) and suppose that $\{f_1, \ldots, f_k\} \subseteq F[\mathbf{x}]$ are generators of the ideal I. Consider now the polynomials

$$f_1, \ldots, f_k \text{ and } yg - 1.$$

Clearly they all are elements of $F[\mathbf{x}, y]$ and by the assumption on the relationship between $\{f_1, \ldots, f_k\}$ and g they don't have a common zero in F. Let J be the

ideal generated by $\{f_1, \ldots, f_k, yg - 1\}$. By Theorem 4.8 applied to J and F we get that $1 \in J$. Thus there are $p_1, \ldots, p_k, q \in F[\mathbf{x}, y]$ such that

$$p_1 f_1 + \cdots p_k f_k + q(yg - 1) = 1.$$

Take $y := \frac{1}{g}$. We get

$$p_1(\mathbf{x}, \frac{1}{g}) f_1(\mathbf{x}) + \cdots p_k(\mathbf{x}, \frac{1}{g}) f_k(\mathbf{x}) = 1.$$

Let n be large enough such that

$$g^n(\mathbf{x})(p_1(\mathbf{x}, \frac{1}{g}) f_1(\mathbf{x}) + \cdots p_k(\mathbf{x}, \frac{1}{g})) \in F[\mathbf{x}].$$

Thus we have shown that g^n is in the ideal generated by $\{f_1, \ldots, f_k\}$. □

PROOF OF THEOREM 4.8. By Proposition 4.2(2) it is enough to show that if $J \supseteq I$ then $\mathrm{V}(J) \neq \emptyset$. By Zorn's lemma we may assume that I is a maximal ideal.

Let $R := F[\mathbf{x}]/I$. Since maximal ideals are prime the ideal I is a prime ideal and we have that the ring R is an integral domain. Let K be an algebraically closed field containing the field of fractions of R. By the natural embedding

$$F \to F[\mathbf{x}]/I \to K$$

we may assume that F is a subfield of K. Since both are algebraically closed fields of the same characteristics p. By the model-completeness of T_{ACF_p} (corollary 13.12) we get that $F \preceq K$.

By Hilbert's basis theorem there exists a finite set $\Phi \subseteq I$ such that Φ generates the ideal I. Since $\Phi \subseteq I$ by the definition of R there exists $\mathbf{a} \in R$ such that

$$R \models \bigwedge_{p \in \Phi} [p(\mathbf{a}) = 0].$$

Thus

$$K \models \exists x_0 \cdots \exists x_{n-1} \bigwedge_{p \in \Phi} [p(\mathbf{x}) = 0].$$

Since $F \preceq K$ and $\Phi \subseteq F[\mathbf{x}]$ we have

$$F \models \exists x_0 \cdots \exists x_{n-1} \bigwedge_{p \in \Phi} [p(\mathbf{x}) = 0].$$

Let $\mathbf{b} \in F$ be such that $\bigwedge_{p \in \Phi} [p(\mathbf{b}) = 0]$. Clearly the is claim established since by Proposition 4.2(3):

$$\mathbf{b} \in \mathrm{V}(\Phi) = \mathrm{V}(I).$$

□

DEFINITION 4.9. Let R be a ring and I an ideal, the set $\{a \in R \mid \exists n \geq 1 [a^n \in I]\}$ is denoted by $\sqrt{I}$.

An ideal I is said to be a *radical ideal* iff $\sqrt{I} = I$.

PROPOSITION 4.10. *Let R be a ring and I an ideal. Always $\sqrt{I}$ is an ideal of R. If I is a prime ideal then I is radical.*

PROOF. Exercise. □

Often Hilbert's Nullstellensatz theorem is stated as: Let F be an algebraically closed field. If I is a proper ideal of $F[\mathbf{x}]$ then $\mathrm{I}(\mathrm{V}(I))$ is a radical ideal.

Another equivalent formulation of Hilbert's Nullstellensatz theorem is:

THEOREM 4.11. *If F is an algebraically closed filed then*

$$I \mapsto \mathrm{V}(I)$$

is a bijection from the radical ideals of $F[\mathbf{x}]$ on to the irrducible algebraic sets.

PROOF. □

and Artin's solution to Hilbert's 17-th problem (positive functions on real closed fields are sums of squares of rational functions, (note that sometimes you can get simpler forms see Hilbert/Tartar)

Strong Null? (Atiya-MacDonalds?)

PAC-fields

DEFINITION 4.12. Let F be a field n a positive integer and $\mathbf{x} = \langle x_0, \ldots, x_{n-1} \rangle$. The *affine n dimensional space over F* is the set F^n and is denoted by $\mathbb{A}^n(F)$. A subset S of $\mathbb{A}^n(F)$ is called *Zariski closed* (or *algebraic*) iff there exists $\Phi \subseteq F[\mathbf{x}]$ such that $S = \mathrm{V}(\Phi)$.

PROPOSITION 4.13. *The set of Zariski closed sets define a topology on $\mathbb{A}^n(F)$.*

PROOF. Since $\bigcap_{i \in S} V_i = \mathrm{V}(\bigcup_{i \in I} I_i)$ and
$V_1 \cup V_2 = \mathrm{I}(\{pq \mid p \in \mathrm{I}(V_1), q \in \mathrm{I}(V_2)\})$. □

The space $\mathbb{A}^n(F)$ with the above topology is called *Zariski's topology*.

DEFINITION 4.14. Let L_{orf} be the language of ordered fields consisting of $\{+, \cdot, <, 0, 1\}$. The theory of *real closed fields* if denoted by T_{RCF} consists of the axioms of fields T_f together with the following finite axioms (or ordered fields):

(1) $\forall x[\neg(x < x)]$
(2) $\forall x \forall y \forall z[x < y \wedge y < z \to x < z]$
(3) $\forall x \forall y[x = y \vee x < y \vee y < x]$
(4) $\forall x \forall y \forall z[x < y \to x + z < y + z]$
(5) $\forall x \forall y[x > 0 \wedge y > 0 \to x \cdot y > 0]$

(6) $\forall x[x > 0 \to \exists y[y \cdot y = x]]$

and the following infinite set of axioms: $\{\psi_{2n+1} \mid n < \omega\}$ where

$$\psi_m := \forall a_0 \ldots \forall a_m \big[a_m \neq 0 \to \exists x[\sum_{i=0}^{m} a_i \cdot x^i = 0]\big].$$

Using elementary analysis one can show that the field of real numbers is a model of T_{RCF}. The theory T_{RCF} was introduced by Emil Artin as a generalization of the field of real numbers.

Exercises 3.4.

(1)
(2) Derive Theorem 4.11 from Hilbert's Nullstellensatz theorem.
(3) Derive Hilbert's Nullstellensatz theorem from Theorem 4.11.
(4)
(5)
(6)
(7)
(8)
(9)

5. Non-standard analysis*

Chain law, Riemann integral, Bernstein-Robinson (poly. compact operators), Brownian motion, ergodic theorem (? Kame? Benji Weiss)

Jordan curve theorem?

V. Kanovei and M. Reeken, A nonstandard proof of the Jordan curve theorem, Real Anal. Exchange **24** (1998/99), no. 1, 161–169; CNO CMP 1 691 743

N. Bertoglio and R. Chuaqui, An elementary geometric nonstandard proof of the Jordan curve theorem, Geom. Dedicata **51** (1994), no. 1, 15–27; MR 95e:54056

Exercises 3.5.

(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)
(9)

6. When does a class have a structure theory? Shelah's thesis

What are cardinal invariants? Examples: vector spaces, algebraically closed fields, torsion free groups?

DEFINITION 6.1. Let T be a given theory and λ a cardinal number at least $|T| + \aleph_0$, by induction on β define:

$$inv_\beta(\lambda) := \begin{cases} \left\{ \{\lambda_i \ : \ i < |T|\} \ : \ \sum_{i<|T|} \lambda_i = \lambda \right\} & \text{if } \beta = 0 \\ \bigcup_{\gamma<\beta} inv_\gamma(\lambda) & \text{if } \beta \text{ is limit} \\ \left\{ \{inv_\gamma(\lambda_i) \ : \ i < |T|\} \ : \ \sum_{i<|T|} \lambda_i = \lambda \right\} & \text{if } \beta = \gamma + 1. \end{cases}$$

DEFINITION 6.2. T has a *structure theorem of depth* β iff for every λ big enough there exists $F : \mathrm{Mod}(T) \to inv_\beta(\lambda)$ such that

$$F(M) = F(N) \implies M \cong N \quad \text{for all } M, N \in \mathrm{Mod}(T)_\lambda.$$

It is not difficult to prove

PROPOSITION 6.3. *Suppose that T has a structure theorem of depth β. For every α we have*

$$\mathrm{I}(\aleph_\alpha, T) < \beth_{1+\beta}(|\alpha| + \aleph_0).$$

PROOF. ADD LATER □

One of the results of Shelah's work toward proving the "main gap" is the following rather deep fact:

THEOREM 6.4. *Let T be a countable complete first-order theory. If T has a structure theorem of depth β for some β then T has a structure theorem of depth ω_1.*

depth $< \omega_1$.

structure/non-structure,

Historical comment: In 1983 I have suggested to Saharon a "Church-like" thesis concerning abstract structure theory. He presented part of this in [**Sh200**].

Expand on this.

Question: Is ω_1 there because it is $\delta(\aleph_0)$? What is the correct statement for uncountable T?

Insert Shelah's open problem?

Shelah's/Church's thesis

Part 2

Galois Theory

CHAPTER 4

Complete types and indiscernibles

Introduction

Both notions appearing in the name of this chapter were introduced in the fifties and they are central to almost all significant results in model theory from the sixties. The first brig into our bag of tools groups of automorphisms similarly to Galois theory and the second allows us to deal with generalized dimensions and skeleton of models.

In this chapter we study systematically types. Recall from the proof of the Erdős-Rado that for a given structure M and $A \subseteq |M|$, $\mathbf{a} \in |M|$ the set $\mathrm{tp}(\mathbf{a}/A, M)$ is the set of formulas with parameters from A that are simultaneously satisfied by $\mathbf{a}$ in M.

We will show that there is a kind of *Galois theory* for models and for especially nice models (saturated) the type of an element $a \in |M|$ over a set A correspond to the the orbit of a under the natural action of the group $\mathrm{Aut}_A(M)$ on $|M|$.

Indiscernible are a generalization of both set of linearly independent vectors and algebraically independent elements in a field. In this section we will focus in *indiscretnible sequences* and in a later ([]) subject will deal also with *indiscernible set*.

Indiscernible sequences were introduced first by Ehrenfeucht and Mostowski in 1956 to construct models with the maximal number of automorphisms, shortly Morley used them in two important steps (Theorem [] and Theorem []) in proving his categoricity theorem.

In 1971 Shelah used indiscernible sequence to show that unstable theories have the maximal number of models (Theorem []), starting what he named *non structure theory*.

In 1977 Paris and Harrington used a strong finite Ramsey-like theorem (that is true in the standard model of Peano's arithmetic) to construct a model generated by indiscernible sequence (mimicking the Ehrenfeucht Mostowski argument []) to construct a model of PA, thus concluding (using Gödel's second incompleteness theorem) that this finite combinatorial fact is not proveable in PA.

Starting with the early seventies indiscernible sequences and sets took a central role in the development of stability theory and later in full classification theory. See chapters X, Y & Z.

Overview of the structure of this chapter:

Section 7: $\mathrm{D}(T)$ the *diagram of the theory* T is the set of complete types in finitely many variables (over the empty set). It turns out that for a complete theory T in a countable language there is a simple relationship between $\mathrm{I}(\aleph_0, T)$ and the cardinality of the diagram of T. Namely

THEOREM 0.1. *For a first-order theory T in a countable language. If $|\mathrm{D}(T)|$ is uncountable then $\mathrm{I}(\aleph_0, T) = |\mathrm{D}(T)|$.*

Later we will show that on the set $\mathrm{D}(T)$ there is a topology that shares some of the properties of Euclidean topology and we prove a version of the Cantor-Bendixon theorem (see Jech [] page):

THEOREM 0.2. *For a first-order theory T in a countable language. If $|\mathrm{D}(T)|$ is uncountable then $|\mathrm{D}(T)| = 2^{\aleph_0}$.*

Section 1: Saturated models are related to models in similar way to the relationship between algebraically closed field relates and fields. Namely they have lots of automorphisms and if two elements a and b "look the same over a subset A of the model" (namely satisfy the same first-order properties with respect on formulas with parameters from A) then there exists an automorphism of the model mapping a to b which is the identity on the set A.

Section 2: Universal models and the monster model.

Section ??: When dealing with a complete first-order theory T and its monster model $\mathfrak{C}$ we will see that an important object to aid the analysis of $\mathrm{Mod}(T)$ Galois types are the groups $\{\mathrm{Aut}_A(\mathfrak{C}) \mid A \subseteq \mathfrak{C}\}$ and the action of $\mathrm{Aut}_A(\mathfrak{C})$ on the subsets of $\mathfrak{C}$. In section 2 we have seen that for $\mathbf{a} \in |\mathfrak{C}|$ the orbit of $\mathbf{a}$ under the action of $\mathrm{Aut}_A(\mathfrak{C})$ can be identified with $\mathrm{tp}(\mathbf{a}/A)$. We would like to have a similar device for Abstract Elementary Classes. There are several problems in attempting this. Among them: How to define something like $\mathrm{tp}(\mathbf{a}/A, M)$, what is the analogue of saturation? Is there a monster model?

1. Saturated models

The concept of saturated model was identified as a key concept of model theory in the relatively late date of 1962 (in Michael Moreley and Robert Vaught's paper [**MoVa**]). It is interesting to mention that it has several precursors that appeared many years earlier.

Felix Hausdorff in 1914 (see [**Hau**]) studied η_α-orders (which are saturated models of T_{ord} of cardinality $\aleph_\alpha$). This work was extended to models of $\mathrm{Th}(\langle \mathbb{R}, +, \cdot, <, 0, 1\rangle)$ by Paul Erdős, Leonard Gillman and Melvin Henriksen in 1957 (see [**EGH**]).

In 1960 Bjarni Jónsson studied a relative of the notion of saturated models named universal homogeneous-models (see [**Jo60**]).

DEFINITION 1.1. Let M be an L-structure.

(1) M is λ-*saturated* iff for every $A \subseteq |M|$ of cardinality less than λ and any M-type p in one variable over A the type p is realized in M.
(2) M is *saturated* iff M is $\|M\|$-saturated.

It should be clear that for cardinals $\mu < \lambda$ the following implication holds: If M is λ-saturated then M is μ-saturated. Thus when M is saturated it is μ-saturated for every $\mu < \|M\|$. A natural question to consider at this stage is: Why not require M to be $\|M\|^+$-saturated? This turns out to be impossible (see Exercise 1 below).

A related category-theoretic notion is that of model homogenous models, they can be defined for any abstract elementary class:

DEFINITION 1.2. Let $\mathcal{K}$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$. $M \in \mathcal{K}$ is λ-*model-homogenous* model iff For every $N \preceq_{\mathcal{K}} M$ and every $N' \succeq_{\mathcal{K}} N$ of cardinality less than λ there exists a $\preceq_{\mathcal{K}}$-embedding $g : N' \to M$ such that $g \restriction |N| = \mathrm{id}_{|N|}$.

When $\lambda = \|M\|$ we omit the parameter, i.e. M is *model homogenous* means that M is $\|M\|$-model homogenous.

For an arbitrary L-structure M we say that M *is model-homogenous* iff M is a model-homogenous model and is member of $\mathcal{K}$ when $\mathcal{K} := \mathrm{Mod}(\mathrm{Th}(M))$ and $\preceq_{\mathcal{K}}$ is elementary submodel.

The following result which generalizes Hausdorff's uniqueness theorem of η_α-orders is due to Michael Morley and Robert Vaught, it is one of the most important properties of saturated models:

THEOREM 1.3 (Uniqueness). *Suppose that M and N are saturated models such that $M \equiv N$ and $\|M\| = \|N\|$ then $M \cong N$.*

The following can be viewed as a category-theoretic characterization of saturated models in the language of elementary embeddings:

THEOREM 1.4 (characterization of saturation). *Suppose M is a structure such that $\|M\| \geq \lambda > |\mathrm{L}(M)|$. The following are equivalent*

(1) *M is λ-saturated.*
(2) *M is λ-model-homogenous, where $\preceq_{\mathcal{K}}$ is the notion of elementary submodel.*

Condition (2) in Theorem 1.4 is very important in classification theory for non-elementary classes. In the exercises we mention few generalizations.

For "warm up" let's prove something simple about saturation:

PROPOSITION 1.5. *Suppose M is an L-structure, $\varphi(x;\mathbf{y}) \in \mathrm{Fml}(L)$ and let $\mathbf{b} \in |M|$ such that $\ell(\mathbf{b}) = \ell(\mathbf{y})$. If M is λ-saturated and $\varphi(M;\mathbf{b})$ is infinite then $|\varphi(M;\mathbf{b})| \geq \lambda$.*

PROOF. Notice that for $\lambda = \aleph_0$ the statement is trivially true. Thus we may assume that $\lambda > \aleph_0$. Suppose for the sake of contradiction that $|\varphi(M;\mathbf{b})| < \lambda$, denote by μ the cardinality of the set $\varphi(M;\mathbf{b})$. Fix an enumeration $\{a_i \ : \ i < \mu\}$ of the set $\varphi(M;\mathbf{b})$.

Consider $p := \{\varphi(x;\mathbf{b})\} \cup \{x \neq a_i \ : \ i < \mu\}$. Since $\varphi(M;\mathbf{b})$ is infinite the set p is an M-type over the set $\mathbf{b} \cup \{a_i \ : \ i < \mu\}$. Since the p is a type over a set of cardinality $\leq \aleph_0 + \mu < \lambda$ by λ-saturation of M there exists $a \in M \models p$. by the definition of p we get $M \models \varphi[a;\mathbf{b}]$ thus $a \in \varphi(M;\mathbf{b})$ but by the second part of the definition of p entails that $a \neq a_i$ for all $i < \mu$ which is a contradiction to the choice of $\{a_i \ : \ i < \mu\}$. □

The key concept to the proof of the uniqueness theorem (as well as the previous one) is the notion of (M,N)-mapping. The origins of the notion of (M,N)-mapping can be traced back to Cantor's Theorem 2.13, it is a combination of the notions of elementary equivalence and partial isomorphism.

DEFINITION 1.6. Let M and N be both L-structures.

(1) We say that g is *partial function from M to N* iff there exists $A \subseteq |M|$ such that $g : A \to |N|$.
(2) A partial function g from M to N is called *(M,N)-mapping* iff
for every $\mathbf{a} \in \mathrm{Dom}(g)$ and every $\varphi(\mathbf{x}) \in \mathrm{Fml}(L)$ we have that
$$M \models \varphi[\mathbf{a}] \iff N \models \varphi[g(\mathbf{a})].$$
Where $g(\mathbf{a}) = \langle g(a_1), \ldots, g(a_n) \rangle$.

REMARKS 1.7. (1) Note that by Claim 2.17 if $g : M \cong N$ then g is an (M,N)-mapping.

(2) $M \equiv N$ iff the empty mapping is an (M,N)-mapping.
(3) Suppose M and N are L-structures. Let $\mathbf{a}$ be a finite sequence of elements from $|M|$ and let $p := \mathrm{tp}(\mathbf{a}/\emptyset, M)$. If $\mathbf{b} \in |N|$ realizes the type p (namely, $\mathrm{tp}(\mathbf{a}/\emptyset, M) = \mathrm{tp}(\mathbf{b}/\emptyset, N)$) then $f := \{\langle a_i, b_i\rangle \mid i < n\}$ is a (M, N)-mapping. Where $n = \ell(\mathbf{a})$, $\mathbf{a} = \langle a_0, \dots, a_{n-1}\rangle$ and $\mathbf{b} = \langle b_0, \dots, b_{n-1}\rangle$.

LEMMA 1.8. *Suppose g is an (M,N)-mapping then*

(1) $M \equiv N$
(2) *g is injective*
(3) *g^{-1} is an (N,M)-mapping.*

PROOF. (1) and (3) are immediate.

For (2) let $\varphi(x; y)$ be the formula $[x = y]$. □

The extension lemma is:

LEMMA 1.9 (one step extension). *Suppose f is an (M,N)-mapping, $\lambda > |\mathrm{Dom}(f)|$ and N is a λ-saturated model. Then for every $a \in |M|$ there exists g which is an (M,N)-mapping extending f such that $a \in \mathrm{Dom}(g)$.*

PROOF. let $A := \mathrm{Dom}(f)$. Consider the type $p := \mathrm{tp}(a/A, M)$. Now define $q := f(p)$. Where $f(p)$ is the set $\{\varphi(x; f(\mathbf{b})) \ : \ \varphi(x; \mathbf{b}) \in p\}$.

CLAIM 1.10. *q is an N-type over the set $f[A]$.*

PROOF. Let $q^* \subseteq_{finite} q$ be given. Suppose $\{\varphi_i(x; \mathbf{c}_i) \ : \ i < n\} = q^*$ let $\mathbf{b}_i \in A$ be such that $\mathbf{c}_i = f(\mathbf{b}_i)$. Since f is an (M,N)-mapping we have that $M \models \exists x \bigwedge_{i<n} \varphi_i(x; \mathbf{b}_i)$ iff $N \models \exists x \bigwedge_{i<n} \varphi_i(x; f(\mathbf{b}_i))$
iff $N \models \exists x \bigwedge_{i<n} \varphi_i(x; \mathbf{c}_i)$.

By the definition of p we have that a satisfies $M \models \bigwedge_{i<n} \varphi_i[a; \mathbf{b}_i]$ which by the above equivalence and the definition of satisfaction gives

$$N \models \exists x \bigwedge_{i<n} \varphi_i(x; \mathbf{c}_i).$$

Thus q is an N-type over $f[A]$. □

Since $|f[A]| = |A| < \lambda$, by the λ-saturation of N there exists $b \in |N|$ such that b realizes q.

Define $g := f \cup \{\langle a, b\rangle\}$.

CLAIM 1.11. *g is an (M,N)-mapping.*

PROOF. Let $\varphi(\mathbf{x}) \in \mathrm{Fml}(L(M))$ be given, and suppose that $\mathbf{a} \in \mathrm{Dom}(g)$. We have to show that

$$(*) \quad M \models \varphi[\mathbf{a}] \iff N \models \varphi[g(\mathbf{a})].$$

If $\mathbf{a} \in \mathrm{Dom}(f)$ then this is true by the hypothesis on f. Otherwise the sequence must contain at least one occurrence of the element a. Notice that it is sufficient to prove $(*)$ for formulas such that $\varphi(\mathbf{x}) \to \bigwedge_{i<j} x_i \neq x_j$. Thus without loss of generality we may assume that $\mathbf{a} = \langle a \rangle^\frown \mathbf{b}$ when $\mathbf{b} \in \mathrm{Dom}(f)$.

Let $\varphi(\mathbf{x})$ be given. Suppose that $M \models \varphi[\mathbf{a}]$ then we have that $M \models \varphi[a; \mathbf{b}]$, by the definition of p we get $\varphi(x; \mathbf{b}) \in p$. By the definition of q we have that $\varphi(x; f(\mathbf{b})) \in q$. Now by the choice of the element b this gives $N \models \varphi[b; f(\mathbf{b})]$ and by the definition of g we have $N \models \varphi[g(\mathbf{a})]$.

Note that this establishes the equivalence $(*)$ since the above argument holds for every $\varphi(\mathbf{x})$ so if $M \not\models \varphi[\mathbf{a}]$ then by the truth definition we have $M \models \neg\varphi[\mathbf{a}]$ by the above argument now applied to $\neg\varphi$ we get $N \models \neg\varphi[g(\mathbf{a})]$ namely $N \not\models \varphi[g(\mathbf{a})]$. □

□

We need one more fact which is a generalization of Theorem I.3.31:

LEMMA 1.12 (continuity lemma). *Let M and N be L-structures and let δ be a limit ordinal. If $\{f_i : i < \delta\}$ are (M,N)-maps satisfying $i < j < \delta \implies f_i \subseteq f_j$ then $\bigcup_{i<\delta} f_i$ is an (M,N)-map.*

PROOF. Directly from the definition of (M, N)-map. □

The following is the key extension lemma:

LEMMA 1.13 (extension lemma). *Let M and N be L-structures and λ an infinite cardinal number. Suppose f is an (M,N) embedding and $B \subseteq |M|$ such that $\mathrm{dom}\, f \subseteq B$, $|\mathrm{dom}\, f| < \lambda$ and $B \leq \lambda$. If N is λ-saturated then there exists an (M,N)-mapping $\bar{f}$ extending f such that $\mathrm{dom}\, \bar{f} = B$.*

PROOF. Denote by A the set $\mathrm{dom}\, f$. Let $i_0 \leq \lambda$ be such that there exists an enumeration $\{b_i \mid i < i_0\}$ of the set $B - \mathrm{dom}\, f$.

By induction of $i < i_0$ define a chain $\{f_i \mid i < i_0\}$ of (M,N)-mappings such that

(1) $f_0 = f$,
(2) if $\delta < i_0$ is a limit ordinal then $f_\delta = \bigcup_{i<\delta} f_i$,
(3) $i < j < i_0 \implies f_i \subseteq f_j$ and
(4) $\mathrm{dom}\, f_i = A \cup \{b_j \mid j < i\}$.

This is enough: Define $\bar{f}$ as $\bigcup_{i<i_0} f_i$, using Lemma 1.12 it is easy to see that $\bar{f}$ is as required.

The construction: It suffices to define at the successor stage. Suppose f_i is defined. Apply Lemma 1.9 to $f := f_i$ and $a := b_i$ and denote by f_{i+1} the function g. □

PROOF OF THEOREM 1.4. (2) $\implies$ (1): Let $A \subseteq |M|$ be of cardinality less than $\|M\|$ and let p be an M-type over A. By the DLS theorem pick $N \preceq M$ of cardinality less than λ containing the set A. Since $N \preceq M$ and the assumption that p is an M-type over A it follows that p is an N-type over A. Let $N' \succeq N$ be of cardinality $\|N\|$ realizing the type p. Denote by $a \in |N'|$ a realization of p. Since $\|N'\| = \|N\| < \|M\|$ and the mapping $\mathrm{id}_{|N|}$ in an (M,N)-map by assumption (2) there exists an elementary embedding $g : N' \to M$ extending $\mathrm{id}_{|N|}$. It is easy to check that $g(a)$ is an element of M realizing the type p.

(1) $\implies$ (2): Given $N \preceq M$ and $N' \succeq N$ of cardinality less than that of M. Let $f := \mathrm{id}_{|N|}$. Since $N' \succeq N \preceq M$ the function f is (M,N)-mapping. Apply now the extension lemma to f and and $|N'|$ to get an elementary embedding $g : |N'| \to M$ extending f. □

PROOF OF THEOREM 1.3. Let $\lambda = \|M\| = \|N\|$. Fix $\{a_i : i < \lambda\} = |M|$ and $\{b_i : i < \lambda\} = |N|$.

For every $i < \lambda$ define an (M, N)-mapping f_i such that

(1) $i < j \implies f_i \subseteq f_j$,
(2) $a_i \in \mathrm{Dom}(f_{2i+1})$,
(3) $b_i \in \mathrm{Rang}(f_{2i+2})$ and
(4) for every $i < \lambda$ we have that $|\mathrm{Dom}(f_i)| \leq |i|$.

This is enough: Let $F := \bigcup_{i<\lambda} f_i$. By Lemma 1.12 F is an (M,N)-mapping, by (2) $\mathrm{Dom}(F) = |M|$ and by (3) $\mathrm{Rang}(F) = |N|$ thus F is an isomorphism from M onto N.

This is possible: By induction on $i < \lambda$ define f_i:

- For $i = 0$, consider $p = \mathrm{tp}(a_0, \emptyset, M)$ since p is an M-type over $\emptyset$ it is a $\mathrm{Th}(M)$-type. Since $N \equiv M$ the set p is also a $\mathrm{Th}(N)$-type. By saturation of N there exists $b \in N$ realizing p. Clearly $f_0 = \{\langle a_0, b\rangle\}$ is as required.
- For $i = j + 1$, suppose that f_j is defined. There are 3 possibilities:
  (1) there exists $\alpha < j$ such that $j = 2\alpha + 1$,
  (2) there exists $\alpha < j$ such that $j = 2\alpha$, or
  (3) none of the above.

(1) if $j = 2\alpha + 1$, denote by g the (N, M)-mapping f_j^{-1}, consider $q := \mathrm{tp}(b_\alpha, \mathrm{Dom}(g), N)$ by the property of g the set q is an M-type over a subset of cardinality less than λ, by saturation of M there exists an element $a \in |M|$ realizing q. Check that the mapping $f_i := f_j \cup \{\langle a, b_\alpha\rangle\}$ is as required.
(2) if $j = 2\alpha$, let $p := \mathrm{tp}(a_\alpha, \mathrm{Dom}(f_j), M)$. Since f_j is an (M, N)-mapping the set $q := f_j(p)$ is an N-type, by saturation of N it has a realization by some $b \in |N|$. Define f_i to be $f_j \cup \{\langle a_\alpha, b\rangle\}$.
(3) none of the above, let $f_i = f_j$.

- For i limit, let $f_i := \bigcup_{j<i} f_j$. The fact that f_i is an (M, N)-mapping follows from Exercise 22. Now by the induction hypothesis we have

$$|\mathrm{Dom}(f_i)| \leq \sum_{j<i} |\mathrm{Dom}(f_j)| \leq |i| \cdot |i| = |i|.$$

□

The following generalizations of Lemma 1.13, have similar proofs:

LEMMA 1.14. *Let M be a λ-saturated model. If $N_0 \prec M$ and we have N_1 and $f : N_0 \to N_1$ elementary such that $\|N_0\| < \lambda$ and $\|N_1\| \leq \lambda$, then there exists $g : N_1 \to M$ elementary embedding such that $g(f(a)) = a$ for every $a \in |N_0|$.*

Before the statement of the existence theorem let us review some basics of cardinal arithmetic. For cardinal numbers λ and μ, it is easy to show that the cardinal λ^μ is the cardinality of the set of functions from μ to λ which is equal to the cardinality of of the family of subsets of λ of power μ.

NOTATION 1.15. For cardinal numbers λ and κ

$$\lambda^{<\kappa} := \sum_{\mu<\kappa} \lambda^\mu \quad \text{when } \mu \text{ varies on cardinal numbers.}$$

Thus $\lambda^{<\kappa}$ is the number of elements in $\{B \subseteq A : |B| < \kappa\}$ when A is a set of cardinality λ.

EXAMPLES 1.16.

(1) $\aleph_1^{<\aleph_1} = 2^{\aleph_0}$ this is true since using the Cantor-Bernestein theorem it is enough to show

$$\aleph_1^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0}$$

and

$$\aleph_1^{\aleph_0} \geq 2^{\aleph_0}.$$

(2) Similarly to the above we get that when $\lambda := \mu^+$ for some infinite cardinal number μ then $\lambda = \lambda^{<\lambda}$ is equivalent to $\mu^+ = 2^\mu$.
(3) When $\lambda > \aleph_0$ is strongly inaccessible then $\lambda = \lambda^{<\lambda}$ holds.

(4) When GCH holds then for all infinite regular cardinals we have that $\lambda = \lambda^{<\lambda}$ holds.

Recall

For any cardinal number λ (in fact also for an ordinal) we can define the *cofinality of* λ as

$$\operatorname{cf}\lambda := \min\big\{\beta \mid (\exists f : \beta \to \lambda)(\forall \alpha < \lambda)(\exists \gamma < \beta)[f(\gamma) \geq \alpha]\big\}.$$

It is not difficult to see that an uncountable cardinal λ is regular iff $\lambda = \operatorname{cf}\lambda$.

The following can be found in every book covering basic set theory:

FACT 1.17 (Zermelo König theorem). *For infinite cardinal number λ always $\lambda^{\operatorname{cf}\lambda} > \lambda$.*

Thus in models where GCH holds $\lambda^{<\lambda} = \lambda$ iff λ is regular. Without GCH from Zermelo König we get that $\lambda^{<\lambda} = \lambda \implies \lambda$ is regular. However, when $\lambda = \mu^+$ and $2^\mu > \mu^+$ the converse fails.

The existence theorem is:

THEOREM 1.18 (Existence). *Suppose M is an infinite structure. If λ and κ are cardinals such that $\lambda^{<\kappa} = \lambda \geq \|M\| + 2^{|\operatorname{L}(M)|+\aleph_0}$ then there exists $N \succ M$ of cardinality λ which is κ-saturated.*

The proof is similar to the argument we used to prove the Erdos-Rado theorem (see Exercise 25).

LEMMA 1.19. *For every M and λ as above there exists $N \succ M$ of cardinality λ such that for every $A \subseteq |M|$ of cardinality less than κ and every M-type over A is realized in N.*

PROOF. Let $\{A_i : i < \lambda\} = \{B \subseteq |M| : |B| < \kappa\}$. Denote by Φ_i the set $\{p : p \text{ is an } M\text{-type over } A_i\}$. Since every type over the set A_i is a subset of $\{\varphi(x; \mathbf{a}) : \mathbf{a} \in A_i, \varphi(x, \mathbf{y}) \in \operatorname{Fml}(\operatorname{L}(M))\}$ we have that

$$|\Phi_i| \leq 2^{|\operatorname{Fml}(\operatorname{L}(M))|+|A|} \leq 2^{|\operatorname{L}(M)|} \cdot 2^{|A|} \leq \lambda^{<\kappa} = \lambda.$$

Define $\{N_i : i < \lambda\}$ such that

(1) $N_0 = M$,
(2) for every $i < \lambda$ we have that $\|N_i\| \leq \lambda$,
(3) $i < j \implies N_i \preceq N_j$,
(4) all the types from Φ_i are realized in N_{i+1}.

This is enough: It should be clear that $N := \bigcup_{i<\lambda} N_i$ is an elementary extension (use Theorem 1.3.30).

This is possible: The definition of the above chain is by induction on $i < \lambda$:

- for $i = 0$, let $N_0 := M$,
- for $i = j + 1$, apply Theorem 7.18 $|\Phi_i|$-many times to get N_{i+1}.
- for i limit, let $N_i := \bigcup_{j<i} N_j$. In addition to the induction hypothesis we are using Theorem 1.3.30.

□

PROOF OF THEOREM 1.18. Define $\{M_i : i < \lambda\}$ as follows:

(1) $M_0 = M$,
(2) $i < j \Longrightarrow M_i \preceq M_j$,
(3) for every $i < \lambda$ we have that $\|M_i\| \leq \lambda$,
(4) for every $A \subseteq |M_i|$ of cardinality less than κ and every M_i-type over A is realized in M_{i+1}.

This is enough: Let $N := \bigcup_{i<\lambda} M_i$. By Theorem 1.3.30 N is an elementary extension of M.

Why it is κ-saturated? Let $A \subseteq |N|$ be a given set of cardinality less than κ. From Zermelo-König we know that $\mathrm{cf}(\lambda) \geq \kappa$, thus there exists $i < \lambda$ such that $A \subseteq |M_i|$ notice that since $M_i \preceq N$ any N-type over A is actually an M_i-type. Now by requirement (4) all the M_i-types over A are realized in M_{i+1}. □

COROLLARY 1.20. *Let M be an infinite structure. If λ satisfies $\lambda = \lambda^{<\lambda}$ and $\lambda \geq 2^{\|M\|+\aleph_0}$ then M has a proper elementary extension of cardinality λ.*

COROLLARY 1.21 (κ^+-saturated elementary extensions exists). *Let M be an infinite structure. For every $\kappa \geq \aleph_0$ there exists $N \succ M$ which is κ^+-saturated.*

PROOF. Let $\mu := \|M\|$ and $\lambda := \mu^\kappa$. As $\lambda^{<\kappa^+} = \lambda$ an application of Theorem 1.18 produces N as required. □

QUESTION 1.22. Is the assumption $\lambda = \lambda^{<\lambda}$ necessary for existence of a saturated model of cardinality λ?

ALTERNATIVE PROOF TO THEOREM 1.18: Define $\{M_i : i < \lambda^+\}$ as follows:

(1) $M_0 \supsetneqq M$,
(2) $i < j \Longrightarrow M_i \preceq M_j$,
(3) if $i < \lambda^+$ is limit then $M_i = \bigcup_{j<i} M_j$,
(4) $\|M_i\| \leq \lambda$ for every $i < \lambda^+$,

(5) if there exists a type over a subset of cardinality less than κ not realized in M_i then let

$$j(i) := \min\{j \leq i \ : \ \exists B \subseteq |M_j|, \ |B| < \kappa, \\ \exists q \in \mathrm{S}(B) \ \text{ not realized in } M_i\}.$$

Denote by q_i a type witnessing the definition of $j(i)$. The model M_{i+1} realizes the type q_i.

The construction: For limit i take unions. In case there is $i < \lambda^+$ such that (5) fails then we are done since the model M_i is κ-saturated.

It suffices to prove that the construction can't continue λ^+-many steps.

CLAIM 1.23. *There exists $\delta < \lambda^+$ such that M_δ is κ-saturated.*

PROOF. Consider the set

$$S_i := \{\alpha < \lambda^+ \mid j(\alpha) \leq i\}.$$

Using the assumption that $\lambda^{<\kappa} = \lambda \geq \|M\| + 2^{|\mathrm{L}(M)|+\aleph_0}$ and the estimate in the proof of Lemma 1.19 check that the number of types over subsets of $|M_i|$ of cardinality less than κ is $\leq \lambda$, namely we get that $|S_i| \leq \lambda < \lambda^+$. Since λ^+ is a regular cardinal S_i is a bounded subset of λ^+. Let $\beta_i := \sup S_i$. We have the following property:

$$(*) \quad \alpha \in (\beta_i, \lambda^+) \implies j(\alpha) > i.$$

Consider

$$C := \{\delta < \lambda^+ \ : \ \delta \text{ is limit and } (\forall i < \lambda^+)\ [i < \delta \to \beta_i < \delta]\}.$$

Since $i \mapsto \beta_i$ is a function from λ^+ to λ^+ and λ^+ is regular by Corollary 1.1.10 the set C is a closed unbounded subset of λ^+. Let $\mu := \operatorname{cf} \lambda$. Since $\lambda = \lambda^{<\kappa}$ by Zermelo-König's $\mu \geq \kappa$. Since $\mu \leq \lambda < \lambda^+$ by Proposition 1.1.14 the following set is stationary

$$S^{\lambda^+}_\mu := \{\delta < \lambda^+ \ : \ \operatorname{cf} \delta = \mu\}.$$

Take $\delta \in C \cap S^{\lambda^+}_\mu$.

Now show that M_δ is κ-saturated: If not, then there are $B \subseteq M_\delta$ and a type $q \in \mathrm{S}(B)$ omitted by M_δ. Since $\operatorname{cf} \delta = \mu \geq \kappa$ there exists $i < \delta$ such that $B \subseteq |M_i|$. Since $\delta \in C$ we have that $\beta_i < \delta$. Pick $\alpha \in (\beta_i, \delta)$, by $(*)$ we have that $j(\alpha) > i$. This is a contradiction to the definition of $j(\alpha)$. Since q is an M_α-type not realized over a small subset of M_i. $\square$

1.1. Application 1: Elimination of quantifiers.

THEOREM 1.24. *The theory T_{ACF_p} admit elimination of quantifiers.*

PROOF. Let □

THEOREM 1.25. *The theories T_{ord}, T_{ind} and T_{ACF_p} all admit elimination of quantifiers*

PROOF. □

1.2. Application 2: Decideability of divisible abelian groups. In 1950 Wanda Szmielew proved that the theory of divisible abelian groups is decideable. For her proof she used elimination of quantifiers.

Notice that by Exercise 1.2.77 and Vaught's theorem the theory of divisible torsion-free abelian groups is decideable (since it is complete and has a recursive axiomatization). The difficulty occurs potentially by the torsion part.

Below we use saturated models to prsent the exact invariants:

THEOREM 1.26.

PROOF. □

Exercises 4.1.

(1) Suppose that M is an infinite model, let $A = |M|$ and $p(x) := \{x \neq a \ : \ a \in A\}$. Show that p is an M-type over A that is not realized in M.
(2) Suppose that $M \cong N$. If M is saturated than N is saturated.
(3) Suppose that M is a saturated model. Show that if N is a reduct of M to some $L' \subseteq \mathrm{L}(M)$ then N is also saturated.
(4) Suppose that M is an uncountable model of cardinality at least $|\mathrm{L}(M)|$. Show that if M is μ^+-saturated for all $\mu < \|M\|$ then M is saturated.
(5) Let p be a type over subset of a model M. p is called *algebraic* iff $p[M]$ is finite, where $p[M] := \{\mathbf{a} \in |M| \ : \ M \models \varphi[\mathbf{a}; \mathbf{b}], \ \varphi(\mathbf{x}; \mathbf{b}) \in p\}$.

    Let M be λ-saturated. Show for every type p over a set of cardinality less than λ, if p in non-algebaic then p is realized by at least λ-many elements.
(6) Suppose that M is a saturated model. Let $A \subseteq |M|$ be a of cardinality less than $\|M\|$. Show that $N := \langle M, c_a \rangle_{a \in A}$ is also a saturated model. Where N is an expansion of M to $L' := L \cup \{c_a \mid a \in A\}$ by new constants that are interpreted by the corresponding elements of A.
(7) If M and N are elementarily equivalent and both are $\aleph_0$-saturated then $M \cong_p N$.
(8) Let $M \models PA$. If M is $\aleph_0$-saturated then $\|M\| \geq 2^{\aleph_0}$.
(9) If $M \models \mathrm{Th}(\langle \mathbb{R}, +, \cdot \rangle)$ is $\aleph_1$-saturated then $\|M\| \geq 2^{\aleph_0}$.
(10) T_{ord} has a saturated model of cardinality $\aleph_1$ iff $2^{\aleph_0} = \aleph_1$.
(11) T_{ind} has a saturated model of cardinality $\aleph_1$ iff T_{ord} has a saturated model of cardinality $\aleph_1$.
(12) Show that there is an injective homomorphism f from $\langle \mathbb{R}, +, \cdot \rangle$ into M of the previous exercise.
(13) Suppose M is as in Exercise 9. Is there an elementary embedding from the field of real numbers into M[1]?
(14) Let M be a saturated model of T_{ord}. Show that for every $A \subseteq |M|$ such that $|A| < \|M\|$ there exists $b \in |M|$ such that $M \models a < b$ for every $a \in A$.
(15) If M is a saturated model then it is not a Jonsson model. Moreover, always there exists $N \precneqq M$ such that $M \cong N$.
(16) Without using Theorem 1.18 prove
    Theorem. Every infinite $M \models T$ has a proper elementary extension of cardinality $\|M\| + |\mathrm{D}(T)|$ which is $\aleph_0$-saturated.
(17) Let M and N be both L-structures.
    Prove that if there exists f an (M, N)-mapping then $M \equiv N$.
(18) Let M and N be both L-structures. If f an (M, N)-mapping then f^{-1} is an (N, M)-mapping.

[1]This is a much harder exercise than the previous one.

(19) Let M and N be both L-structures. If f an (M, N)-mapping then f is injective.

(20) Suppose f is an (M, N)-mapping with $A = \operatorname{dom} f$. If $\mathbf{a}$ a finite sequence of elements from A and $\varphi(\mathbf{x}; \mathbf{y})$ is a formula such that $\varphi(M; \mathbf{a})$ is a finite set then

$$|\varphi(M; \mathbf{a})| = |\varphi(N; f(\mathbf{a}))|.$$

(21) Let M and N be both L-structures. Suppose that there are $M' \preceq M$ and $N' \preceq N$ such that there is an isomorphism $f : M' \cong N'$. Show that f is an (M, N)-mapping.

(22) Let δ be a limit ordinal. Suppose that $\{f_i \ : \ i < \delta\}$ are (M, N)-mappings such that $i < j \implies f_i \subseteq f_j$.
Show that $\bigcup_{i<\delta} f_i$ is also an (M, N)-mapping.

(23) This is a generalization of the previous exercise. Let M and N be L-structures and suppose $\langle I, < \rangle$ is a directed set. Suppose $\{f_i \mid i \in I\}$ are (M, N)-mappings. If $i < j \implies f_i \subseteq f_j$ for all $i, j \in I$ then $\bigcup_{i \in I} f_i$ is an (M, N)-mapping.

(24) Suppose M is a saturated model. Show that for every M-type p over less than $\|M\|$ parameters either $|p[M]| = \|M\|$ or $|p[M]| < \aleph_0$ where

$$p[M] := \bigcap_{\varphi(\mathbf{x};\mathbf{b}) \in p} \varphi(M; \mathbf{b}).$$

(25) Suppose $\mathbf{a} \in M \preceq N$. If both models are saturated of the same cardinality then there exists $f : M \cong N$ such that $f(\mathbf{a}) = \mathbf{a}$.

(26) Let L be a similarity type and $\mu > |L| + \aleph_0$. Suppose $M_o \preceq M_1$ and $N_0 \preceq N_1$ are L-structures such that N_1 is μ-saturated. If $f : M_0 \to N_0$ is an elementary embedding and $\|N_0\| < \|M_1\| \leq \mu$ then there exists an elementary embedding $f' : M_1 \to N_1$ extending f.

(27) Prove the following generalization of Theorem 1.4:

THEOREM 1.27 (characterization of λ-saturation). *Let L be a similarity type, $\lambda > \aleph_0 + |L|$ and M L-structure.*

The following are equivalent

(a) *M is λ-saturated.*

(b) *M is λ-model-homogenous, where $\preceq_{\mathcal{K}}$ is the notion of elementary submodel.*

(28) Suppose M is a saturated model. Show that for every *definable equivalence relation* $E(\mathbf{x}, \mathbf{y})$ with parameters from M, i.e. there exists a formula $\varphi(\mathbf{x}; \mathbf{y}; \mathbf{z})$ and $\mathbf{c} \in |M|$ such that $n := \ell(\mathbf{x}) = \ell(\mathbf{y})$ and

$$E := \left\{(\mathbf{a}, \mathbf{b}) \in {}^n|M| \times {}^n|M| \ : \ M \models \varphi[\mathbf{a}; \mathbf{b}; \mathbf{c}]\right\}$$

is an equivalence relation.

Prove that either

$$|M/E| = \|M\| \quad \text{or} \quad M/E \text{ is finite.}$$

(29) Suppose M is a λ-saturated model, $A \subseteq |M|$ of cardinality $< \lambda$ and $1 \leq n < \omega$.

Prove that if $|\mathrm{S}^1(A, M)| < \lambda$ then $|\mathrm{S}^{n+1}(A \cup B, M)| < \lambda$ for all $B \subseteq |M|$ of cardinality n.

(30) Show that T_{ord} does not have a saturated model of cardinality $\beth_\omega$. Hint: Solve Exercise 14.

(31) Every countable complete theory has a universal model (see Definition 2.2) of cardinality $\beth_\omega$.

(32) Suppose $M \models T_{ord}$. Show that if $\|M\|$ is singular then M can not be saturated. Moreover when $\lambda := \|M\|$ then M can not be $\mathrm{cf}\, \lambda^+$-saturated.

(33) Prove that the field of complex numbers is a saturated model of T_{ACF_0}.

(34) Prove (without using the previous Exercise) that the complex numbers is a model-homogeneous model of T_{ACF_0}.

(35) Suppose λ is an uncountable cardinal. If $\lambda^{<\lambda} = \lambda$ then λ must be a regular cardinal. Is it possible to derive the regularity of λ from $2^{<\lambda} = \lambda$?

(36) Show that every uncountable algebraically closed field is a saturated model of T_{ACF_p}. Hint: Use an argument similar to the one used in the proof of Theorem 2.38.

(37) Why in the previous exercise we restricted attention to uncountable fields? Is every countable algebraically closed field saturated? Is there a countable algebraically closed field which is as model is saturated?

(38) Show that $\langle \mathbb{R}, < \rangle$ (the reals with the usual ordering) is not a saturated model of T_{ord} of cardinality $\aleph_1$.

(39) Suppose $2^{\aleph_0} > \aleph_1$. Find two $\aleph_1$-saturated non-isomorphic models of T_{ord} of cardinality $2^{\aleph_0}$.

(40) Let $L \subsetneqq L^*$ be two similarity types. Suppose $T \subseteq T^*$ are theories in L and L^*.

If T^* is a complete theory in L and $M \models T$ is a saturated model then there exists $M^* \models T^*$ which is an expansion of M to L^*.

(41) Prove that when M is saturated then for all $A \subseteq |M|$ such that $|A| < \|M\|$ the function f of Exeercise 1.4.21 from $^n|M|$ to the set of ultrafilters of $BA^n_A(M)$ is a bijection. Is the converse true?

(42) Suppose $\mathcal{K}$ is an AEC and $\lambda = \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ has the λ-AP, λ-JEP, every model of cardinality λ has a proper extension ($\mathcal{K}_\lambda$ has the NMM property) and $2^\lambda = \lambda^+$ then there exists in $\mathcal{K}_{\lambda^+}$ a model-homogeneous model. Hint: Use Lemma 2.2.22.

(43) Construct an example of an AEC without the JMP such the uniqueness property for model-homogeneous models fails.

(44) Let $\mathcal{K}$ be AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. For $M \in \mathcal{K}_\lambda$ we denote by

$$\mathrm{D}(M) := \{N/\cong :\ N \prec_{\mathcal{K}} M, \|N\| = \mathrm{LS}(\mathcal{K})\}.$$

Prove that if $M, N \in \mathcal{K}_\lambda$ are both model homogenous when $\lambda > \mathrm{LS}(\mathcal{K})$ and $\mathrm{D}(M) \cap \mathrm{D}(N) \neq \emptyset$ then $M \cong N$.

(45)

2. The monster model, homogenous and special models

In field theory for the study of finite algebraic extensions of the rationals it is common to use the following fact:

FACT 2.1. *If F is a finite dimensional extension of the field $\langle \mathbb{Q}, +, \cdot \rangle$ then there exists a field monomorphism from F into the complex numbers which is the identity on $\mathbb{Q}$. Thus without loss of generality we may assume that all finite field extensions of $\mathbb{Q}$ are subfields of $\mathbb{C}$.*

We will show that a similar fact holds for any complete first-order theory T. That is for any infinite cardinal number $\lambda \geq |\mathrm{L}(T)|$ there exists $\mathfrak{C} \models T$ (of cardinality $\leq 2^{\lambda}$) such that for any $M_1 \preceq M_2 \models T$ of cardinality $\leq \lambda$ there are elementary embeddings $f_\ell : M_\ell \to \mathfrak{C}$ such that $f_1 \subseteq f_2$ and for every $N \succeq M_1$ such that $|N| - |M_1|$ is of cardinality less than λ there is $g : N \to \mathfrak{C}$ extending f_1.

DEFINITION 2.2. Let $\mathcal{K}$ be an abstract class, and suppose that $M \in \mathcal{K}$.

(1) Let λ be a cardinal. M is called *λ-universal model of $\mathcal{K}$* iff for every $N \in \mathcal{K}$ of cardinality λ there exists a $\mathcal{K}$-embedding $g : N \to M$ (namely there exists $N^* \preceq_{\mathcal{K}} M$ such that $\exists g : N \cong N^*$).
(2) M is called a *universal model of $\mathcal{K}$* iff for every $N \in \mathcal{K}$ of cardinality $\|M\|$ there exists a $\mathcal{K}$-embedding $g : N \to M$.
(3) When the class $\mathcal{K}$ is not mentioned we mean that $\mathcal{K} = \{N \mid N \equiv M\}$. we mean that it is universal with respect to the class $\mathrm{Mod}(T)$.

THEOREM 2.3 (Universality). *If M is saturated then M is a universal model. Namely, for every $N \equiv M$ of cardinality less or equal to $\|M\|$ there exists an elementary embedding from N into M. Moreover, let T be complete and suppose $\lambda \geq |T|$. If $M \models T$ is λ-saturated then every $N \models T$ of cardinality $\leq \lambda$ is isomorphic to an elementary sub model of M.*

PROOF. Repeat the argument in the uniqueness proof, just delete clause (3). □

This theorem implies:

COROLLARY 2.4 (Saturated models are universal). *Suppose T is a complete first-order theory and let $\lambda > |T|$. If $M \models T$ is saturated of cardinality λ then for every $N \models T$ of cardinality λ there exists an elementary embedding $g : N \to M$.*

The following is a useful reformulation:

THEOREM 2.5 (Universality with respect to T). *Let T be a complete first order theory and suppose that M is μ-saturated for some μ. Then for every $N \models T$ of cardinality $\leq \mu$ there exists an elementary embedding $g : N \to M$.*

PROOF. Exercise. □

In Corollary 2.4 we have seen that when T is a complete first-order theory and $M \models T$ is a saturated model of cardinality $\lambda \geq |T|$ then N is universal.

In Theorem 1.4 we have shown that saturated model and model-homogeneous models are the same thing (when $\lambda > |T|$ and $\mathcal{K} = \mathrm{Mod}(T)$). So it is only natural to expect that this generalizes to AECs.

Let $\mathcal{K}$ be an AEC, $\lambda > \mathrm{LS}(\mathcal{K})$, Suppose $M \in \mathcal{K}_\lambda$ is model-homogeneous, is M also universal for $\mathcal{K}_\lambda$? This can't be true as AEC generalizes also incomplete theories and we can't expect universality even when $\mathcal{K}$ is elementary but its axiomatization is not a complete theory. So we need an assumption of $\mathcal{K}$ that would be a substitute to the assumption that T is complete. Recall that by Proposition 6.2 since T is complete it has the JMP. It turns out that JMP is such an assumption on $\mathcal{K}$. So the following is a generalization of Corollary 2.4 in several ways.

THEOREM 2.6 (Model homogeneous models are universal). *Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$, let $\mu < \lambda$ be such that $\mu \geq \mathrm{LS}(\mathcal{K})$ and $\mathcal{K}$ has the μ-JEP. If $M \in \mathcal{K}$ is λ-model homogeneous then for every $N \in \mathcal{K}_\lambda$ there exists a $\mathcal{K}$-embedding $g : N \to M$.*

First observe that the definition of homogeneous model implies a stronger property, which is an analog to the extension Lemma 1.13 for saturated models.

PROPOSITION 2.7. *Let $\mathcal{K}$ be an AEC, $\lambda > \mathrm{LS}(\mathcal{K})$, Suppose $M \in \mathcal{K}$ is λ-model-homogeneous and $N^* \preceq_{\mathcal{K}} M$ is of cardinality $< \lambda$. For every $N \in \mathcal{K}_\lambda$ such that $N \succeq_{\mathcal{K}} N^*$ there exists a $\mathcal{K}$-embedding $g : N \to M$ such that $g \restriction N^* = \mathrm{id}_{N^*}$.*

PROOF. Pick an increasing and continuous resolution $\{N_i \mid i < \lambda\} \subseteq K_{<\lambda}$ such that $N_0 := N^*$.

It is enough to construct $\{g_i : N_i \to M \mid i < \lambda\}$ increasing chain of $\mathcal{K}$-embeddings. We do that by induction on $i < \lambda$. For $i = 0$ and $g_0 := \mathrm{id}_{N^*}$, for i limit take $N_i := \bigcup_{i<\lambda} N_i$. Given g_i let $M_i := g_i[N_i]$. By one of the isomorphism axions for AECs let $N^{**} \in \mathcal{K}$ be such that $N^* \succeq_{\mathcal{K}} N^*$ and let $g^* : N_{i+1} \cong N^{**}$ satisfying $g^* \supseteq g_i$. As M is model homogeneous and $N^{**} \succeq_{\mathcal{K}} M_i$ of cardinality $< \lambda$ there exists $h : N^{**} \to M$ such that $h \restriction M_i = \mathrm{id}_{M_i}$. Let $g_{i+1} = h \circ g^*$. Check that it is as required. □

PROOF OF THEOREM 2.6. Using the last proposition and one of the isomorphism axioms (Axiom XX of AEC) it is enough to find $N_0 \preceq_{\mathcal{K}} N$ of cardinality μ isomorphic to a submodel of M.

Pick $N_0 \preceq_{\mathcal{K}} N$ and $M_0 \preceq_{\mathcal{K}} M$ both of cardinality μ. By μ-JMP there exists $N_1 \in \mathcal{K}_\mu$ such that $N_1 \succeq_{\mathcal{K}} N_0$ and there exists $h : N_0 \to N_1$. By model

homogeneouity of M as $\|N_1\| = \mu < \lambda$ there exists an embedding $g : N_1 \to M$ such that $g \restriction M_0 = \mathrm{id}_{M_0}$. As $N_0 \cong g[h[N_0]] \preceq_{\mathcal{K}} M$ we are done. □

THEOREM 2.8 (Model homogeneous models are unique). *Let $\langle \mathcal{K}, \preceq_{\mathcal{K}} \rangle$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$, let $\mu < \lambda$ be such that $\mu \geq \mathrm{LS}(\mathcal{K})$ and $\mathcal{K}$ has the μ-JMP. If $M, N \in \mathcal{K}_\lambda$ are both model homogeneous then f there exists $g : N \cong M$.*

PROOF. Similar to the proof of Theorem 2.6, left as an exercise to the reader. □

As an application of the existence and universality results of saturated models I present alternative proofs to Abraham Robinson's theorems that the amalgamation property and the joint embedding properties hold for the class of models of a complete first-order theory.

COROLLARY 2.9 (Complete first-order theories satisfy the AP). *Suppose T is a complete first-order theory. Let $\mathcal{K}$ be the induced AEC. i.e. $\mathcal{K} := \langle \mathrm{Mod}(T), \preceq \rangle$ where $M \preceq N$ stands for M is an elementary submodel of N. Then the AEC $\mathcal{K}$ has the AP.*

PROOF. Recall it suffices to show that if $M_0 \prec M_f rm{-}e$ and $f : M_0 \to M_1$ is elementary then there are $N \succ M_2$ and an elementary embedding $g : M_1 \to N$ extending f.

Let $\mu := \|M_1\| + \|M_2\| + |\mathrm{L}(T)|$, $\kappa := \mu^+$ and $\lambda := 2^{\mu^+}$. As $\lambda^{<\kappa} = \lambda$ and $\lambda \geq 2^{|\mathrm{L}(T)| + \aleph_0}$ there exists $N \succ M_2$ a κ-saturated of cardinality λ. As $\|M_1\| < \kappa$ by Lemma 1.14 there exists an elementary embedding $g : M_1 \to N$ extending f. □

Robinson's original proof used the compactness theorem (as saturated models where introduced several years later), see Theorem 1.6.8.

We with little more care we can get the stronger result:

COROLLARY 2.10 (Complete first-order theories satisfy the DAP). *Suppose T is a complete first-order theory. Let $\mathcal{K}$ be the induced AEC. Then the AEC $\mathcal{K}$ has the Disjoint Amalgamation Property. Namely, given $M_0 \prec M_\ell \models T$ there exists $N \succ M_2$ and an elementary embedding $g : M_1 \to N$ such that $g \restriction M_0 = \mathrm{id}_{M_0}$ and $g[|M_1|] \cap |M_2| = |M_0|$.*

PROOF. Exercise. □

Using properties of saturated models we can also establish Robinson's result that the Joint Embedding Property holds in the class of models of a complete first-order theory.

COROLLARY 2.11 (Complete first-order theories satisfy the JEP). *Suppose T is a complete first-order theory. Let $\mathcal{K}$ be the induced AEC. Then the AEC $\mathcal{K}$ has the JEP.*

PROOF. Suppose $M, N \models T$. Enough to find $N^* \succ N$ and an elementary embedding $g^* : M \to N^*$.

Let $\mu := \|M\| + \|N\| + |\operatorname{L}(T)|$. By Corollary 1.21 let $N^* \succ N$ be a μ^+-saturated model.

□

Compare this with the proof of Proposition 1.6.2.

Another generalization ...

THEOREM 2.12 (Strong universality). *Let T be a complete first order theory and suppose that M is μ-saturated and $A \subseteq |M|$ of cardinality less than μ. For every $N \models T$ such that there is an (N, M)-map $g : A \to |M|$ if $\|N\| \leq \mu$ then there exists an elementary embedding $g : N \to M$ such that $g \restriction A = \mathrm{id}_A$.*

PROOF. Exercise. □

A related concept to that of saturation is sequential homogeneity:

DEFINITION 2.13. Let M be an L-structure and let $\lambda \leq \|M\|$.

(1) *M is λ-homogenous*[2] iff for every (M, M)-mapping f such that $|\operatorname{dom} f| < \lambda$ and every $a \in |M|$ there exists $b \in |M|$ such that $f \cup \{\langle a, b\rangle\}$ is also an (M, M)-mapping.
(2) *M is λ-strongly-homogenous*[3] iff for every (M, M)-mapping f such that $|\operatorname{dom} f| < \lambda$ there exists $g \in \operatorname{Aut}(M)$ extending f.
(3) *M is homogenous* or *M is strongly-homogenous* iff for every (M, M)-mapping f such that $|\operatorname{dom} f| < \|M\|$ there exists $g \in \operatorname{Aut}(M)$ extending f.

An equivalent statement to "M is λ-homogenous" is the following: For every ordinal $\gamma < \lambda$ and every $\mathbf{a}, \mathbf{b} \in {}^{\gamma}|M|$ if $\operatorname{tp}(\mathbf{a}/\emptyset, M) = \operatorname{tp}(\mathbf{b}/\emptyset, M)$ then for every $a \in |M|$ there exists $b \in |M|$ such that $\operatorname{tp}(\mathbf{a}^\frown a/\emptyset, M) = \operatorname{tp}(\mathbf{b}^\frown b/\emptyset, M)$.

The notions of homogenuity from the previous definition should not be confused with the corresponding notions of *model-homogenuity* (from Definition **??**), they are different.

[2]Also called *M is λ-sequentially homogenous*.

[3]Also called *M is λ-sequentially strongly-homogenous*.

Notice that immediately from Lemma 1.9 we get:

PROPOSITION 2.14. *Every λ-saturated model is λ-homogenous. In particular, if M is saturated then it is strongly homogenous.*

Is the converse of the previous Proposition true? Namely: Are the properties of homogeneity and saturation equivalent? It turns out that in general saturation is a stronger property than sequencial homogenuity.

First let me introduce a generalization of $\mathrm{D}(T)$:

DEFINITION 2.15. Let M be a model, the *diagram of* M is

$$\mathrm{D}(M) := \{\mathrm{tp}(\mathbf{a}/\emptyset, M) \mid \mathbf{a} \in {}^{\omega>}|M|\}.$$

Observe that for L-structures M and N the assumption $\mathrm{D}(M) = \mathrm{D}(N)$ implies the $M \equiv N$.

In Definition **??** we present a parallel notion to that of a diagram of a model for the context of AECs.

Notice that for a model $M \models T$ always $\mathrm{D}(M) \subseteq \mathrm{D}(T)$, but even when $T = \mathrm{Th}(M)$ it is possible that $\mathrm{D}(M) \subsetneq \mathrm{D}(T)$, check that this happens when M is the algebraic closure of $\mathbb{Q}$ and $T = T_{ACF_0}$.

THEOREM 2.16 (Uniqueness of countable homogenous models). *Let M and N be countable L-structures such that $\mathrm{D}(M) = \mathrm{D}(N)$. If both are homogenous then $M \cong N$.*

PROOF. We prove a little more, namely the family

$$F := \{h \mid h \text{ is an } (M, N)\text{-mapping with a finite domain}\}$$

has the back and forth property. By symmetry it is enough to show that given $h \in F$ and $a \in |M|$ there exists $b \in |N|$ such that $h \cup \{\langle a, b\rangle\} \in F$. Let $\mathbf{a}$ be an enumeration of $\mathrm{dom}\, h$. Since $\mathrm{D}(M) \subseteq \mathrm{D}(N)$ there are $\mathbf{b}', b' \in |N|$ such that

$$(*) \quad \mathrm{tp}(\mathbf{a}^\frown a/\emptyset, M) = \mathrm{tp}(\mathbf{b}'^\frown b'/\emptyset, N).$$

Since $\mathrm{tp}(h(\mathbf{a})/\emptyset, N) = \mathrm{tp}(\mathbf{b}'/\emptyset, N)$ by $\aleph_0$-homogeneity of N there is $b \in |N|$ such that

$$(**)\ \mathrm{tp}(\mathbf{b}'^\frown b'/\emptyset, N) = \mathrm{tp}(h(\mathbf{a})^\frown b/\emptyset, N).$$

Placing $(*)$ and $(**)$ together gives $\mathrm{tp}(\mathbf{a}^\frown a/\emptyset, M) = \mathrm{tp}(h(\mathbf{a})^\frown b/\emptyset, N)$, namely $h \cup \{\langle a, b\rangle\} \in F$. □

Let T be some fixed first-order theory in a countable language and $\Gamma \subseteq D(T)$. Consider

$$\mathcal{K}_{T,\Gamma} := \{M \models T \ :\ \mathrm{D}(M) = \Gamma, M \text{ is } \aleph_0\text{-homogenous}\}$$

The class with the notion of $\mathrm{L}(T)$-elementary submodel is an AEC. By the previous theorem it is $\aleph_0$-categorical. In exercise **??** we show that for Γ the set of isolated types in T_{ord} we have $\mathrm{I}(\lambda, \mathcal{K}) = 2^\lambda$ for all $\lambda > \aleph_0$.

REMARK 2.17 (union of homogenous models). If $\{M_n \mid n < \omega\}$ is an elementary chain of $\aleph_0$-homogenous models then also $\bigcup_{n<\omega} M_n$ is $\aleph_0$ homogenous.

Similarly to existence of saturated models we have and analog result to Theorem 1.18, notice that while the assumptions are similar they are not identical!

THEOREM 2.18 (Existence of sequential homogenous models). *Let M be an infinite L-structure and $\lambda \geq \kappa$ are infinite cardinals. Suppose $\lambda \geq \|M\| + |L|$. If λ satisfies $\lambda = \lambda^{<\kappa}$ then there exists $N \supsetneq M$ of cardinality λ which is κ-homogenous.*

PROOF. We will define an increasing continuous chain $\{M_\alpha \mid \alpha < \lambda\}$ such that

(1) $M_0 \supsetneq M$,
(2) $\|M_\alpha\| = \lambda$,
(3) $M_\alpha \precneqq M_{\alpha+1}$,
(4) for every (M_α, M_α)-mapping f such that $|\operatorname{dom} f| < \kappa$ and every $a \in |M_\alpha|$ there is an $(M_{\alpha+1}, M_{\alpha+1})$-mapping $g \supseteq f$ such that $a \in \operatorname{dom} g$.

This is enough, using the assumption that $\operatorname{cf} \lambda \geq \kappa$ verify that $N := \bigcup_{\alpha<\lambda} M_\alpha$ is as required.

Given M_α, as the cardinality of M_α is λ the number of (M_α, M_α)-mappings with domain of cardinality $< \kappa$ is $\lambda^{<\kappa} (= \lambda)$.

Fix $\{(f_i, a_i) \mid i < \lambda\}$ to be an enumeration of the set

$$\{(f, a) \mid a \in |M_\alpha|,\ f \text{ is an } (M_\alpha, M_\alpha)\text{-mapping with } |\operatorname{dom} f| < \kappa\}.$$

Now define $\{N^i_\alpha \mid i < \lambda\}$ increasing and continuous such that

(1) $N^0_\alpha \supsetneq M_\alpha$
(2) $\|N^i_\alpha\| = \lambda$,
(3) $N^i_\alpha \precneqq N^{i+1}_\alpha$,
(4) there is an $(N^{i+1}_\alpha, N^{i+1}_\alpha)$-mapping $g \supseteq f_i$ such that $a_i \in \operatorname{dom} g$.

Clearly defining $M_{\alpha+1}$ as $\bigcup_{i<\lambda} N^i_\alpha$ has the desired properties.

To carry out the construction of $\{N^i_\alpha \mid i < \lambda\}$ by induction on $i < \lambda$, by continuity of the chain it is enough to describe the construction of N^{i+1}_α from $N^i_\alpha,\ f_i$ and a_i. It suffices to prove

CLAIM 2.19. *Let M be an infinite model, $\lambda \geq \|M\| + |\operatorname{L}(M)|$, $a \in |M|$ and f an (M, M)-mapping. There are $N \succ M$ of cardinality λ and $g \supseteq f$ an (N, N)-mapping such that $a \in \operatorname{dom} g$.*

PROOF. Consider

$$p(x) := \{\varphi(x; f(\mathbf{a})) \mid \varphi(x; \mathbf{y}) \in \operatorname{Fml}(\operatorname{L}(M)),\ M \models \varphi[a; \mathbf{a}],\ \mathbf{a} \in \operatorname{dom} f\}$$

Notice that p in an M-type over $\operatorname{range} f$: Given a finite $q \subseteq p$, as $\bigwedge q \in p$ it is enough to show that $M \models \exists x \varphi(x; f(\mathbf{a}))$, but this is immediate as f is an (M, M)-mapping. By Theorem 7.18 there is $N \succ M$ of cardinality λ and $b \in |N|$ realizing p. Now verify that $g := f \cup \{\langle a, b\rangle\}$ is as required. □

□

COROLLARY 2.20. *Let T be a theory and $\lambda \geq |\operatorname{L}(T)|$. If T has an infinite model then T has an $\aleph_0$-homogenous model of cardinality λ. Moreover, if $M \models T$ is of cardinality λ then there exists $N \succeq M$ an $\aleph_0$-homogenous model of cardinality λ.*

PROOF. Apply the previous theorem when $\kappa = \aleph_0$. □

Suppose T is a theory in a countable language. Clearly any $\aleph_0$-saturated model of T has to be of cardinality $\geq |\operatorname{D}(T)|$, thus when $\operatorname{D}(T)$ is uncountable T does not have countable saturated models. On the other hand from the corollary it follows that we can have $\aleph_0$-homogenous models in every cardinality.

The following generalization of Theorem 2.18 has a similar proof:

COROLLARY 2.21. *Suppose $M \prec N$ and Q a new unary predicate. Consider the following $\operatorname{L}(N) \cup \{Q\}$-structure:*

$$\mathfrak{B} := \langle |N|, |M|, F^N, R^N, c^N \rangle_{F,R,c \in \operatorname{L}(N)},$$ *where Q is interpreted by the set $|M|$. If $\lambda \geq \|N\| + |\operatorname{L}(N)|$ and λ satisfies $\lambda = \lambda^{<\kappa}$ then there are $\mathfrak{B}_1 \supsetneq \mathfrak{B}$ and $\mathfrak{B}_2 \supsetneq \mathfrak{B}$ of cardinality λ such that both $\mathfrak{B}_1 \restriction \operatorname{L}(N)$ and $\langle Q^{\mathfrak{B}_2}, F^{\mathfrak{B}_2}, R^{\mathfrak{B}_2}, c^{\mathfrak{B}_2} \rangle_{F,R,c \in \operatorname{L}(N)}$ are κ-homogenous.*

PROOF. Exercise. □

As an application of existence and uniqueness of countable homogenous models (for a countable theory) we present now Vaught's original proof of his two cardinal theorem. There are two ideas in the proof, they appeared earlier in Exercise 2.9 and in the proof of (3) $\implies$ (1) in Theorem 1.13.11. See also Exercise 1.3.50.

PROOF OF VAUGHT'S TWO CARDINAL THEOREM 1.12.9. It is enough to construct $\{M_\alpha \mid \alpha < \omega_1\}$ such that

(1) $M_0 \models T$ with P^{M_0} infinite,
(2) $\alpha < \beta \implies M_\alpha \precneqq M_\beta$,
(3) $\|M_\alpha\| = \aleph_0$ for all $\alpha < \omega_1$,
(4) $P^{M_\alpha} = P^{M_0}$ for all $\alpha < \omega_1$,
(5) M_α is $\aleph_0$-homogenous and
(6) $\mathrm{D}(M_\alpha) = \mathrm{D}(M_0)$.

Clearly $\bigcup_{\alpha<\omega_1} M_\alpha$ is an $\langle \aleph_1, \aleph_0\rangle$-model.

In order to construct a chain as above, by uniqueness of countable homogenous models and the observation that a union of $\aleph_0$-homogenous models must be $\aleph_0$-homogenous, it suffices to find a pair of countable homogenous $M \precneqq N \models T$ such that $P^M = P^N$ and $\mathrm{D}(M) = \mathrm{D}(N)$.

As T admits a $\langle \lambda, \mu\rangle$-model let N^* be such a model. Pick $M^* \preceq N^*$ of cardinality μ such that $|M^*| \supseteq P^{N^*}$. Notice that automatically $P^{M^*} = P^{N^*}$.

Let $\mathfrak{B}^* := \langle |N^*|, |M^*|, P^{N^*}, \ldots\rangle$ be the expansion of N^* to $\mathrm{L}(T) \cup \{Q\}$ where M is a new unary predicate interpreted by the set $|M^*|$. Let $T^* := \mathrm{Th}(\mathfrak{B}^*)$. Note that if $\mathfrak{B}' \models T^*$ then $M' \precneqq N'$ and $P^{M'} = P^{N'}$, where N' is $\mathfrak{B}' \restriction \mathrm{L}(T)$ and $M' := \langle Q^{\mathfrak{B}'}, F^{\mathfrak{B}'}, R^{\mathfrak{B}'}, c^{\mathfrak{B}'}\rangle_{F,R,c\in \mathrm{L}(T)}$.

Define $\{\mathfrak{B}_n \mid n < \omega\}$ as follows:

(1) $\mathfrak{B}_0 \equiv \mathfrak{B}^*$.
(2) $\mathfrak{B}_{2n} \restriction \mathrm{L}(T)$ is a countable homogenous structure,
(3) $\langle Q^{\mathfrak{B}_{2n+1}}, F^{\mathfrak{B}_{2n+1}}, R^{\mathfrak{B}_{2n+1}}, c^{\mathfrak{B}_{2n+1}}\rangle_{F,R,c\in \mathrm{L}(T)}$ also a countable homogenous structure.
(4) $n < k \implies \mathfrak{B}_n \precneqq \mathfrak{B}_k$
(5) every $p \in \mathrm{D}(N'_n)$ is realized in M'_{n+1}.

The construction is possible by Corollary 2.20.

Let $\mathfrak{B}_\omega := \bigcup_{n<\omega} \mathfrak{B}_n$. Let $N := \mathfrak{B}_\omega \restriction \mathrm{L}(T)$ and

$$M := \langle Q^{\mathfrak{B}_\omega}, F^{\mathfrak{B}_\omega}, R^{\mathfrak{B}_\omega}, c^{\mathfrak{B}_\omega}\rangle_{F,R,c\in \mathrm{L}(T)}.$$

By requirements (2) and (3) the models M and N are homogenous. By requirement (5) we have that $\mathrm{D}(M) = \mathrm{D}(N)$. By the uniqueness theorem for homogenous models (Theorem 2.16) we get that $M \cong N$. Since $\mathfrak{B}_\omega \equiv \mathfrak{B}^*$ we have that $M \precneqq N$ and $P^M = P^N$. □

It turns out that while homogeneity is strictly weaker than saturation, under an additional assumption they are equivalent.

THEOREM 2.22. *Let M be an L-structure and suppose $\lambda \geq |L| + \aleph_0$. The model M is λ-saturated iff M is λ-homogenous and λ universal.*

PROOF. Suppose N is λ-saturated. By Theorem 2.3 it is λ-universal. To see that it is λ-homogenous: Let $\mathbf{a}, \mathbf{b} \in {}^{\gamma}|M|$ for some $\gamma < \lambda$ and $a \in |M|$ such that $\mathrm{tp}(\mathbf{a}/\emptyset, M) = \mathrm{tp}(\mathbf{b}/\emptyset, M)$. Consider

$$p(x) := \mathrm{tp}(a/\mathbf{a}, M).$$

Let f be the mapping $f(\mathbf{a}) = \mathbf{b}$, clearly it is an (M, M)-mapping. By the argument in the proof of Claim 2.19 the set $q(x) := \{\varphi(x, f^{-1}(\mathbf{c})) \mid \varphi(x; \mathbf{c}) \in p\}$ is an M-type over $\cup\mathbf{b}$. By λ-saturation there is $b \in |M|$ realizing q. So we found an element b such that $\mathrm{tp}(\mathbf{a}, a/\emptyset, M) = \mathrm{tp}(\mathbf{b}, b/\emptyset, M)$.

Suppose M is λ-homogenous and λ-universal. Let $A \subseteq |M|$ be of cardinality $< \lambda$ and $p \in \mathrm{S}^1(A)$. We may assume that $\mathfrak{C} \succeq M$ is a big saturated model, there is $a \in \mathfrak{C}$ realizing the type p. Let $N \preceq \mathfrak{C}$ be of cardinality λ such that $|N| \supseteq A \cup \{a\}$. By λ-universality of M there exists an elementary embedding $g : N \to M$. Let $\mathbf{a}$ be an enumeration of the set A. Since $\mathrm{tp}(\mathbf{a}/\emptyset, \mathfrak{C}) = \mathrm{tp}(g(\mathbf{a})/\emptyset, \mathfrak{C})$ implies that $\mathrm{tp}(\mathbf{a}/\emptyset, M) = \mathrm{tp}(g(\mathbf{a})/\emptyset, M)$ and $g(a) \in |M|$, by λ-homogenuity there exits $a \in |M|$ such that $\mathrm{tp}(\mathbf{a}, a/\emptyset, M) = \mathrm{tp}(g(\mathbf{a}), g(a)/\emptyset, M)$. We are done since a realizes the type p. □

Before presenting a rather useful generalization of the last theorem we need a new notion.

DEFINITION 2.23. Let M be an L-structure and μ an infinite cardinal number.

(1) We say that M is *μ-weakly universal* iff for every $N \equiv M$ and every $A \subseteq |N|$ of cardinality μ there exists an (N, M)-mapping f such that $\mathrm{dom}\, f = A$.
(2) We say that M is *$< \mu$-weakly universal* iff for every $N \equiv M$ and every $A \subseteq |N|$ of cardinality $< \mu$ there exists an (N, M)-mapping f such that $\mathrm{dom}\, f = A$.

Clearly μ-weakly universality is a strictly weaker property than μ-universality. The following is an improvement of Theorem 2.22:

THEOREM 2.24. *Let M be an L-structure and suppose $\lambda \geq |L|$. Suppose M is λ-homogenous. If M is $< \aleph_0$-universal then M is λ-saturated.*

PROOF. Prove by induction on $\mu \leq \lambda$ that for every

$(*)_\mu$ $A \subseteq |M|$ of cardinality less than μ and every type over A is realized in M.

Like in the proof of Theorem 2.22, given $p \in \mathrm{S}^1(A)$ realized by some $a \in \mathfrak{C}$, pick $N \preceq \mathfrak{C}$ of cardinality $|A| + |L| + \aleph_0$ containing $A \cup \{a\}$.

If $\mu \leq \aleph_0$ then since M is $< \aleph_0$-weakly universal there exists an (N, M)-mapping f such that $\mathrm{dom}\, f = A \cup \{a\}$. Let $\mathbf{a}$ be an enumeration A. As $\mathrm{tp}(\mathbf{a}/\emptyset, M) =$

$\mathrm{tp}(f(\mathbf{a})/\emptyset, M)$, by homogeniuty there exists $b \in M$ such that $\mathrm{tp}(\mathbf{a}, a/\emptyset, M) = \mathrm{tp}(f(\mathbf{a}), b/\emptyset, M)$, check that b realizes p.

If $\mu > \aleph_0$ pick an enumeration $\{a_i \mid i < \mu\}$ of $A \cup \{a\}$ such that $a_0 = a$. Construct $\{f_i \mid f_i \text{ is an } (N, M)\text{-mapping}, i < \mu\}$ such that

(1) $i < j \implies f_i \subseteq f_j$,
(2) $\mathrm{dom}\, f_i = \{a_j \mid j < i\}$ and
(3) if i is limit then $f_i = \bigcup_{j<i} f_j$.

This is possible by the induction hypothesis $(*)_{|i|}$ for all $i < \mu$.

Now take $f := \bigcup_{i<\mu} f_i$. As it is an (N, M) mapping f from $A \cup \{a\}$ into M. As in the previous stage using λ-homogenuity there is $b \in |M|$ such that $\mathrm{tp}(\mathbf{a}, f(a)/\emptyset, M) = \mathrm{tp}(f(\mathbf{a}), b/\emptyset, M)$ and $b \models p$. □

A consequence of the previous proof is the following surprising compactness-like property of homogenous models:

COROLLARY 2.25 (compactness of homogenous models). *Let M be a λ-homogenous model (for some infinite λ) and suppose $A \subseteq |M|$ is of cardinality $< \lambda$. If $p \in \mathrm{S}^1(A)$ has the property that $p \restriction B$ is realized in M for all finite $B \subseteq A$ then p is realized in M.*[4]

REMARK 2.26. *The monster model* $\mathfrak{C}$: From the existence and the strong universality theorems we may assume that for a give complete T there exists a model $\mathfrak{C} \models T$ such that for every model $M \models T$ we have that the model M is an elementary submodel of $\mathfrak{C}$. Thus whenever working with a complete theory T we may assume that all the models of T are elementary submodels of a large saturated model $\mathfrak{C}$ notice that all we need is that $\mathfrak{C}$ is μ-saturated when μ is greater than the cardinalities of all models involved. From the practical point of view we may assume that $\mathfrak{C}$ is a proper class.

The notation $\mathfrak{C}$ is borrowed from field theory. See van der Wearden XXX.

Already in the proof of the Erdős-Rado theorem we realized that counting the number of types over a fixed set of parameters is important, we introduce a special notation for that:

DEFINITION 2.27. Let M be an L-structure, and $A \subseteq |M|$.

(1) For $\mathbf{a} \in |M|$ denote by $\mathrm{tp}(\mathbf{a}/A, M)$ the set
$\{\varphi(\mathbf{x}; \mathbf{b}) \ : \ \varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{L}(M), \ \mathbf{b} \in A, \ M \models \varphi[\mathbf{a}; \mathbf{b}]\}$.
(2) For a positive integer n let $\mathrm{S}^n(A, M)$ be the set
$\{\mathrm{tp}(\mathbf{a}/A, M) \ : \ \mathbf{a} \in {}^n|M|\}$.

[4] $p \restriction B$ is defined as $\{\varphi(x; \mathbf{b}) \in p \mid \mathbf{b} \in B\}$.

(3) When $M = \mathfrak{C}$ denote $\mathrm{S}^n(A, M)$ by $\mathrm{S}^n(A)$. When $n = 1$ denote $\mathrm{S}^1(A, M)$ by $\mathrm{S}(A, M)$

The letter S in the above notation stands for *Stone space* this notation is borrowed from topology (which will be discussed in section 8 below). On the set $\mathrm{S}^n(A, M)$ (when M is saturated and $\|M\| > |A|$) there is a particularly nice topology (see section 8) the connections with topology are not essential for the rest of this book.

We have now sufficient terminology to introduce one of the most important properties of complete first-order theories.

DEFINITION 2.28. Let T be a complete first-order theory and suppose $\mathfrak{C}$ is its monster model.

(1) For an infinite cardinal λ we say that T *is stable in* λ iff for every $A \subseteq |\mathfrak{C}|$ and every $n < \omega$ we have that
$$|A| \leq \lambda \quad \text{implies} \quad |\mathrm{S}^n(A)| \leq \lambda.$$
(2) T is *stable* iff there exists a cardinal λ such that T is stable in λ.
(3) T is called *superstable* iff there exists a cardinal μ such that for every $\lambda \geq \mu$ the theory is stable in λ.

In Chapter 4 (Definition 4.1.2) the concept of stable theory will be reintroduced and studied extensively. Several examples are presented in Exercise 30.

Let me just mention two simplified tests of stability. Firstly in the last definition we may take $n = 1$, namely:

PROPOSITION 2.29. *Let T be a complete theory and $\lambda \geq |T|$. The theory T is stable in λ iff for every A of cardinality λ we have that $|\mathrm{S}(A)| \leq \lambda$.*

PROOF. Exercise 34. □

Instead of counting types over all sets it is enough to consider types only over models:

PROPOSITION 2.30. *Let T be a complete theory and $\lambda \geq |T|$. The theory T is stable in λ iff we have $|\mathrm{S}(M)| \leq \lambda$ for every $M \models T$ of cardinality λ.*

PROOF. Exercise 35. □

Notice that stability is preserved by expansions by not too many constants:

PROPOSITION 2.31. *Suppose T is stable in λ. For every $A \subseteq |\mathfrak{C}|$ of cardinality $\leq \lambda$ also the theory $T_A := \mathrm{Th}(\langle \mathfrak{C}, c_a \rangle_{a \in A})$ is stable in λ*

PROOF. Trivial. □

The notion of saturated model can be relativized:

DEFINITION 2.32. Let $\varphi(x)$ be a formula with one variable in L and suppose $\lambda \geq |L| + \aleph_0$.

(1) An L-structure M is called (φ, λ)-*saturated* iff $\varphi(M)$ is infinite and for every $A \subseteq |M|$ of cardinality less than λ and set of formulas p with parameters from A such that for every finite $q \subseteq p$ there is $\mathbf{a}_q \in \varphi(M)$ realizing q there exists $\mathbf{a} \in \varphi(M) \models p$.
(2) When $\lambda = \|M\|$ we omit the parameter λ and say that M *is* φ*-saturated*.
(3) When the formula $\varphi(\mathbf{x})$ is a unary predicate P in the language L we say that M *is* P*-saturated* .

Prove the following generalization of Theorem 1.4:

THEOREM 2.33. *Let M be a structure, and suppose that λ is a cardinality such that $\|M\| \geq \lambda > |\mathrm{L}(M)| + \aleph_0$. The following are equivalent:*

(1) *M is λ-saturated.*
(2) *For every $N \preceq M$ of cardinality less than λ and every $N' \succeq N$ of cardinality at most λ there exists an elementary embedding $g : N' \to M$ which is the identity on $|N|$.*

The following corollary of Proposition 2.14 is one of the most important uses of saturation of a model and permits development of "Galois theory".

COROLLARY 2.34. *Let M be a saturated model, let $A \subseteq |M|$ such that $|A| < \|M\|$. For every $\mathbf{a}, \mathbf{b} \in {}^{\omega >}|M|$ if $\mathrm{tp}(\mathbf{a}/A, M) = \mathrm{tp}(\mathbf{b}/A, M)$ then there exists $f \in \mathrm{Aut}(M)$ such that $f(\mathbf{a}) = \mathbf{b}$ and $f \restriction A = \mathrm{id}_A$.*

REMARK 2.35. Notice that Corollary 2.34 asserts that for M saturated, $|A| \subseteq |M|$ of cardinality less than $\|M\|$ and all finite $\mathbf{a}, \mathbf{b}$ from $|M|$ the following are equivalent:

(1) $\mathrm{tp}(\mathbf{a}/A, M) = \mathrm{tp}(\mathbf{b}/A, M)$
(2) $\mathbf{b}$ is an element of the orbit of $\mathbf{a}$ under the group action of $\mathrm{Aut}_A(M)$ on the set $|M|$ (see Exercise I.2.66(d)).

Compare Corollary 2.34 with the following fact from field theory:

FACT 2.36 (Kronecker's lemma). *Let F be a field, $p(x) \in F[x]$ irreducible and $E \geq F$ a splitting field containing all the roots of p. For any $a, b \in E$ roots of p there exists $f \in Gal(E/F)$ such that $f(a) = b$.*

A variant of the uniqueness theorem is an important component of the proof of Chang's two cardinal theorem (see Theorem 12.1)

THEOREM 2.37 (Uniqueness). *Let L be a similarity type and let $\varphi(x)$ be a formula. Suppose that M is a saturated model of cardinality λ (in the language L) if N is a φ-saturated structure of cardinality λ then there exists an elementary embedding $g : N \to M$ such that $g[\varphi(N)] = \varphi(M)$.*

PROOF. Exercise. □

There is an interesting connection between closed weak models of set theory and saturation.

PROPOSITION 2.38. *Let M be a λ-saturated model and suppose that χ is a regular cardinal sufficiently large such that $M \in H(\chi)$ and*

$$\langle H(\chi), \in\rangle \models M \text{ is a saturated model}.$$

For $\kappa \leq \lambda$, if $\mathfrak{B} \preceq \langle H(\chi), \in\rangle$ is such that $S \in |\mathfrak{B}|$ for every $S \subseteq |\mathfrak{B}|$ of cardinality less than κ then $M^{\mathfrak{B}}$ is κ-saturated.

PROOF. Exercise. □

So far the only sufficient condition for existence of saturated models of cardinality λ is the assumption that $\lambda = \lambda^{<\lambda}$. In a later chapter (Theorem **??**) it will be shown that for $\lambda > |T| + \aleph_0$ such that $\lambda < \lambda^{<\lambda}$ a theory T has a saturated model of cardinality λ iff T is stable in λ. One may wonder whether some of corollaries of existence of saturated models can be maintained? E.g. is there a universal model of cardinality λ even when $\lambda < \lambda^{<\lambda}$ without any specific assumptions on T? The notion of *special model* was introduced by Chang and Keisler for this purpose.

DEFINITION 2.39. Let M be an L-structure. For infinite cardinal $\lambda \geq |L|$ we say that M *is special over* λ provided there exists $\{M_n \mid n < \omega\}$ such that

(1) $M_n \prec M_{n+1} \prec M$ for ann $n < \omega$,
(2) M_n is $\beth_n(\lambda)^+$-saturated,
(3) $\|M_n\| = \beth_{n+1}(\lambda)$ and
(4) $M = \bigcup_{n<\omega} M_n$.

An elementary chain $\{M_n \mid n < \omega\}$ satisfying the above requirements is called a *specializing sequence for* M.

Unlike saturated models special models exists without any cardinal-arithmetic assumptions.

THEOREM 2.40. *Let T be a first-order theory and $\lambda \geq |T|$. There exists $M \models T$ special over λ. Moreover, if M is an infinite L-structure and $\lambda \geq \max\{\|M\|, |L|\}$ then there exists $N \gtrdot M$ special over λ.*

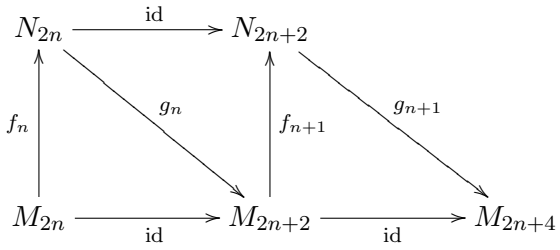
PROOF. Existence Theorem □

THEOREM 2.41. *Suppose $M \equiv N$, $\lambda \geq |\mathrm{L}(M)| + \aleph_0$. If both models are special over λ then $M \cong N$.*

PROOF. Fix $\{N_k \prec N \mid k < \omega\}$ and $\{M_k \prec M \mid k < \omega\}$ both increasing and converging to N and M, also satisfying that M_k is $\beth_k(\lambda)^+$-saturated of cardinality $\beth_{k+1}(\lambda)$ and N_k is $\beth_{k+1}(\lambda)^+$-saturated of cardinality $\beth_{k+2}(\lambda)$.

By induction on $n < \omega$ define $\{g_n, f_n \mid n < \omega\}$ such that

(1) $f_n : M_{2n} \to N_{2n}$ and $g_n : N_{2n} \to M_{2n+2}$ and also
(2) the following diagram commutes:



for all $n < \omega$.

Let $f := \bigcup_{n<\omega} f_n$ and $g := \bigcup_{n<\omega} g_n$. As f is an elementary embedding and $f \circ g^{-1} = \mathrm{id}_M$ and $g \circ f^{-1} = \mathrm{id}_N$ the function f is an isomorphism of M onto N.

For $n = 0$: Use that N_0 is $\|M_0\|^+$-saturated. Using that M_2 is $\|N_0\|^+$-saturated apply Lemma 1.14 to get g_0 as required.

Given f_n and g_n: Define f_{n+1} by applying Lemma 1.14 to g_n and use the same lemma again applied to f_{n+1} to get g_{n+1}. □

DEFINITION 2.42. A structure M is a *special model* iff there exists an elementary chain $\{M_\mu \mid \mu < \|M\|\}$; by convention μ runs over infinite cardinals such that

$$\bigcup_{\mu<\|M\|} M_\mu = M \text{ and } M_\mu \text{ is } \mu^+\text{-saturated for all } \mu.$$

The sequence $\{M_\mu \mid \mu < \|M\|\}$ is called *a specializing sequence for M*.

While T_{ord} can not have a saturated model of cardinality $\beth_\omega$ (see Exercise 1.30) it follows from the next proposition that any theory T has a special model of cardinality $\beth_\omega(|T|)$.

PROPOSITION 2.43. *Let $\lambda > |T| + \aleph_0$. If λ is strong limit then T has a special model of cardinality λ. Thus under the Generalized Continuum Hypothesis T has a special model at every $\lambda > |T| + \aleph_0$.*

This proposition follows easily from

THEOREM 2.44 (existence of special models). *Suppose $\lambda = 2^{<\lambda} > \aleph_0$ and M is an infinite structure such that $\|M\| < \lambda$ then there exists $N \succeq M$ special of cardinality λ.*

PROOF. Case 1: $\exists \mu < \lambda, 2^\mu = \lambda$

Case 2: .... □

Notice that for every μ the cardinal $\lambda := \beth_\omega(\mu)$ is a cardinal satisfying the condition $\lambda = 2^{<\lambda}$ and is greater than μ. Thus by the previous theorem any consistent theory has arbitrarily large special models. This is not true for saturation, e.g. if T is unstable and GCH fails everywhere than T has no saturated models at all.

Similarly to the situation with saturated models, special models are unique and universal:

THEOREM 2.45 (Special models are universal). *Let N be a special model of cardinality $\lambda > |\operatorname{L}(M)|$. If $M \equiv N$ of cardinality $\leq \lambda$ then there exists an elementary embedding $f : M \to N$.*

PROOF. Let $\kappa := \|M\|$. As $\lambda > |\operatorname{L}(M)| + \aleph_0$ we may assume without loss of generality that $\kappa > |\operatorname{L}(M)| + \aleph_0$. Fix an increasing and continuous elementary chain $\{M_\alpha \mid \alpha < \kappa\}$ such that $M = \bigcup_{\alpha<\kappa} M_\alpha$ and $\alpha < \kappa \implies \|M_\alpha\| < \kappa$. Suppose $\{N_\mu \mid \mu < \lambda\}$ is a specializing sequence for N.

For $\alpha < \kappa$ let $h(\alpha) := \|M_\alpha\|$. For every $\alpha < \kappa$ define $f_\alpha : M_\alpha \to N_{h(\alpha)}$ elementary embedding such that for every $\alpha < \beta < \kappa$ the following diagram commutes

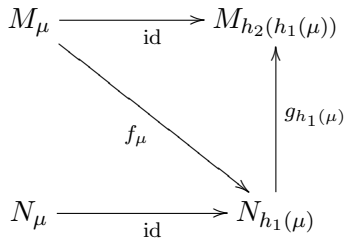
$$\begin{array}{ccc} M_\alpha & \xrightarrow[\text{id}]{} & M_\beta \\ \Big\downarrow{\scriptstyle f_\alpha} & & \Big\downarrow{\scriptstyle f_\beta} \\ N_{h(\alpha)} & \xrightarrow[\text{id}]{} & N_{h(\beta)^+} \end{array}$$

The definition of f_α is possible as $N_{h(\alpha)}$ is universal for models of cardinality $h(\alpha)$ and $_\alpha$ can be extended to f_β since $N_{h(\beta)^+}$ is $\|M_\beta\|^{++}$-saturated..

Now verify that $f := \bigcup_{\alpha<\kappa} f_\alpha$ is as required. □

THEOREM 2.46 (Uniqueness of specials). *Suppose both M and N are special models such that $M \equiv N$. If $\|M\| = \|N\| > |\operatorname{L}(M)|$ then $M \cong N$.*

PROOF. Let $\lambda := \|M\|$. Suppose that $\{M_\mu \mid \mu < \lambda\}$ and $\{N_\mu \mid \mu < \lambda\}$ are specializing sequences of M and N. Let $h_1(\mu) := \|M_\mu\|$ and $h_2(\mu) := \|N_\mu\|$. For every $\mu < \lambda$ we define $f_\mu : M_\mu \to N_{h_1(\mu)}$ and $g_\mu : N_\mu \to M_{h_2(\mu)}$ elementary mappings such that the following diagram commutes



and also we have that $\mu_1 < \mu_2 < \lambda \implies f_{\mu_1} \subseteq f_{\mu_2}$ and $g_{\mu_1} \subseteq g_{\mu_2}$. This is possible by using the saturation of the models in the chains. Now the function $f := \bigcup_{\mu<\lambda} f_\mu$ is an isomorphism from M onto N. □

It is no coincidence that the diagram in the uiniqueness proof looks familiar it is similar to the zig-zag argument used in the proof of Theorem 6.4. As an application of the existence and uniqueness of special models we present a second proof due to C. C. Chang for the Robinson's consistency property (Theorem 1.6.4). This proof completely avoids chasing the arrows in the commutative diagrams as in the first proof.

A PROOF OF THEOREM 1.6.4 USING SPECIAL MODELS. Suppose that T_ℓ theories in L_ℓ (for $\ell = 1, 2$) such that $T_0 := T_1 \cap T_2$ is a complete theory in $L_0 := L_1 \cap L_2$.

As in the first proof of Theorem 1.6.4 it is enough to find $M_\ell \models T_\ell$ for $\ell = 1, 2$ such that $M_1 \restriction L_0 \cong M_2 \restriction L_0$.

If one of the models is finite then since T_0 is complete also the other one is finite of the same cardinality, moreover we get $M_1 \restriction L_0 \cong M_2 \restriction L_0$.

In the case both are infinite, let $\lambda := \max\{|L_1|, |L_2|\}$. Let $M_\ell \models T_\ell$ special models of cardinality $\beth_\omega(\lambda)$. Since $M_1 \restriction L_0$ and $N_1 \restriction L_0$ are both special elementarily equivalent models by uniqueness of special models we are done. □

The following is now immediate:

COROLLARY 2.47. *If $\lambda > |T| + \aleph_0$ is strong limit then T has a universal model of cardinality λ. E.g. If we assume* GCH *holds then T has a universal model in every $\lambda > |T|$.*

(1) Abstract elementary class (continuous, compact)?
(2) If M is a saturated model then it is not a Jónsson model. Moreover M has a proper elementary submodel N isomorphic to M.

(3) amalgamation property
(4) Existence of model-homogeneous structure in abstract elementary classes
(5) Uniqueness of model-homogeneous structure in abstract elementary classes
Need $\mathrm{D}(M) = \mathrm{D}(N)$
(6)

2.1. The monster model. Much of classification theory deals with the question of what are the members of $\mathrm{Mod}(T)$ (up to an isomorphism) for a complete first-order theory T.

Suppose that we are interested in models of cardinality μ for some fixed μ. It would be very convenient to be able to assume that all models of cardinality μ can be viewed as elementary submodels of a certain fixed "homogeneous" model $\mathfrak{C}$.

This is inspired b Andre Weill's use of the complex numbers as a *universal domain*, namely all algebraic extensions of fields of finite transcendence degree of the rationals are isomorphic to subfields of the complex numbers.

Notice that by universality of saturated models if $\lambda > \mu$ and satisfies $\lambda = \lambda^{<\lambda}$ then (the) saturated model $\mathfrak{C} \models T$ of cardinality λ is as required. Unfortunately this depends actually on the existence of arbitrarily large lambda satisfying $\lambda = \lambda^{<\lambda}$ and this is not proveable from ZFC. It turns out that by combining Theorems **??**, **??** and **??** we get:

THEOREM 2.48 (Existence of a monster model). *Let T be a complete first-order theory and suppose $\mu \geq |T|$. There exists $\mathfrak{C} \models T$ of cardinality greater than μ such that*

(1) *Every $N \models T$ of cardinality μ is isomorphic to an elementary submodel of $\mathfrak{C}$. Moreover if $f : N \to \mathfrak{C}$ is an elementary embedding and $N' \succeq N$ of cardinality $\leq \mu$ then there exists an elementary embedding $g : N' \to \mathfrak{C}$ extending f.*
(2) *Suppose $A \subseteq |\mathfrak{C}|$ is of cardinality $\leq \mu$ and $\mathbf{a}$ and $\mathbf{b}$ are sequences of elements from $\mathfrak{C}$ of length $\leq \mu$ such that $\mathrm{tp}(\mathbf{a}/A, \mathfrak{C}) = \mathrm{tp}(\mathbf{b}/A, \mathfrak{C})$ then there exists $f \in \mathrm{Aut}_A(\mathfrak{C})$ such that $f(\mathbf{a}) = \mathbf{b}$.*

PROOF. Take $\mathfrak{C} \models T$ special of cardinality $\beth_\omega(\mu)$. Verify the properties. □

A model satisfying Theorem 2.48 is called the *monster model*.

Using the moster model simplifies the technical details of many theorems and definitions by typically removing at least one quantifier from statements. $\mathfrak{C}$ was introduced in print by Shelah in 1978 without a satisfactory explanation why we may assume its existence.

While I don't like this name, it will be used in the rest of the book often as many people use it. The first person I have heard using this term was Gregory Cherlin, who used it already in 1980. Despite of several obvious reasons not to use this name it is used by model theorists.

Exercises 4.2.

(1) Let T be a countable theory. Suppose that $\mathrm{D}(T)$ is countable, show that for any $M \models T$ and every finite $A \subseteq |M|$ the set $\{\mathrm{tp}(b/A, M) \mid b \in |M|\}$ is countable.
(2) Suppose T is a countable theory. If $\mathrm{I}(\aleph_0, T) \neq 2^{\aleph_0}$ then T has a saturated model of cardinality $\aleph_0$.
(3) Suppose M is a saturated model is necessarily $\mathrm{Aut}(M)$ a transitive group?
(4) Let M be a saturated model and $A \subseteq |M|$ of cardinality less than $\|M\|$. Let p be a complete M-type over A, show that $\mathrm{Aut}_A(M)$ acts transitively on the set $p(M)$ (see Exercise 3.1.41(a)i).
(5) Prove that in the definition of saturation we could replace finite sequence by elements. Namely for a given model and an infinite cardinal λ the following are equivalent:
   (a) For every $A \subseteq |M|$ of cardinality less than λ and every finite sequence of variables $\mathbf{x}$ if $p(\mathbf{x})$ is a type over A which is finitely satisfiable in M then p is realized in M.
   (b) For every $A \subseteq |M|$ of cardinality less than λ and every variable x if $p(x)$ is a type over A which is finitely satisfiable in M then p is realized in M.
(6) Let M be an $\aleph_1$-saturated model in a countable similarity type L, let A be a countable subset of $|M|$, and let n be a positive integer. Suppose that every M - type over A in n variables is realized in M. Prove the following implication

$$|\mathrm{S}^n(A, M)| > \aleph_0 \Longrightarrow |\mathrm{S}^n(A, M)| = 2^{\aleph_0}.$$

(7) Let T be a first-order theory, and let $\lambda \geq |\mathrm{L}(T)|$. Suppose that T is categorical in λ. Show that the model of cardinality λ must be saturated.
(8) Prove Proposition 2.14.
(9) Using the previous exercise conclude that every uncountable algebraically closed field is saturated.
(10) Let M be the algebraic closure of $\mathbb{Q}$ and $T := T_{ACF_0}$, why $\mathrm{D}(M) \subsetneq \mathrm{D}(T)$?
(11) Suppose $M \models T$ is $\aleph_0$-saturated. Show that $\mathrm{D}(M) = \mathrm{D}(T)$.
(12) Let T be a first-order theory and $\Gamma \subseteq \mathrm{D}(T)$. Show that $\mathcal{K}, \leq\rangle$ and $\mathcal{K}^*, \leq\rangle$ are both AECs where $\mathcal{K} := \{M \models T \mid M \text{ is } \aleph_0 \text{ homogenous }\}$, $\mathcal{K}^* := \{M \models T \mid M \text{ is } \aleph_0 \text{ homogenous with } \mathrm{D}(M) = \Gamma\}$ and $\leq$ is elementary submodel.
(13) Let T be a first-order theory in a countable language, $\Gamma \subseteq \mathrm{D}(T)$. Let $\mathcal{K} := \{M \models T \mid M \text{ is } \aleph_0 \text{ homogenous with } \mathrm{D}(M) = \Gamma\}$ Show that $\langle \mathcal{K}, \leq\rangle$ is an AEC when $\leq$ is elementary submodel. Prove that if there are $M, N \in \mathcal{K}_{\aleph_0}$ such that $M \not\cong N$ then $\mathcal{K}_{\aleph_1} \neq \emptyset$.
(14) Prove Corollary 2.21.

(15) Let M be an L-structure and μ an infinite cardinal. Verify that M is μ-homogenous iff for every $\gamma < \mu$, $\mathbf{a}, \mathbf{b} \in {}^{\gamma}|M|$ and every $a \in |M|$:

$$(M, \mathbf{a}) \equiv (N, \mathbf{b}) \implies \text{there exists } b \in |M| \text{ such that } (M, \mathbf{a}, a) \equiv (N, \mathbf{b}, b).$$

Recall that $(M, \mathbf{a})$ stand the expansion of M by the constants $\{c_i \mid i < \gamma\}$ where c_i is interpreted by the element a_i.

(16) A structure M is an *atomic model* iff $\mathrm{tp}(\mathbf{a}/\emptyset, M)$ is isolated for every finite $\mathbf{a} \in |M|$. A set of formulas p such that all the free variables are from a a finite $\mathbf{x}$ is called *isolated type* if there exists a formula $\varphi(\mathbf{x})$ such that

$$M \models \exists \mathbf{x} \varphi(\mathbf{x}) \text{ and } M \models \forall \mathbf{x}[\varphi(\mathbf{x}) \to \psi(\mathbf{x})] \text{ for all } \psi \in p.$$

Prove that if M is an atomic model then M is $\aleph_0$-homogenous.

(17) Derive Proposition 1.9.25 from Theorem 2.16.

(18) Establish the following generalization of Theorem 2.16:

THEOREM 2.49. *Let M and N be L-structures such that $\mathrm{D}(M) = \mathrm{D}(N)$. If both are μ-homogenous for an infinite μ then $M \equiv_{\infty,\omega} N$.*

(19) We say that M is $< \aleph_0$-*universal* iff for every $N \equiv M$ and every finite $A \subseteq |N|$ of cardinality $\leq \|M\|$ there exists an (N, M)-map g such that $\mathrm{dom}\, g = A$. Prove the following variant of Theorem 2.24:

THEOREM 2.50. *Let M be an L-structure and suppose $\lambda \geq |L| + \aleph_0$. Suppose M is λ-homogenous. If M is $< \aleph_0$-universal then M is λ-saturated.*

(20) Let T be a first-order theory. Consider

$$\mathcal{K} := \{M \models T \ : \ M \text{ is } \aleph_0\text{-homogenous}\}$$

Show that $\langle \mathcal{K}, \prec \rangle$is an AEC with $\mathrm{LS}(\mathcal{K}) = \aleph_0 + |\mathrm{L}(T)|$.

(21) Let T be a first-order theory and $\Gamma \subseteq \mathrm{D}(T)$.

$$\mathcal{K}^{\Gamma} := \{M \ : \ \mathrm{D}(M) = \Gamma \text{ and } M \text{ is } \aleph_0\text{-homogenous}\}$$

Show that $\langle \mathcal{K}^{\Gamma}, \prec \rangle$is an AEC with $\mathrm{LS}(\mathcal{K}^{\Gamma}) = \aleph_0 + |\mathrm{L}(T)|$. In case $|\mathrm{L}(T)| \leq \aleph_0$ the class is $\aleph_0$-categorical.

(22) Let

$$\mathcal{K}^{at}_T := \{M \models T \ : \ M \text{ is atomic}\}$$

While $\mathcal{K}^{at}_T$ is $\aleph_0$-categorical. Using Proposition 1.2.62 and Corollary 1.9.30 show that $\mathrm{I}(\lambda, \mathcal{K}^{at}_{T_{ord}}) = 2^{\lambda}$ for all $\lambda > \aleph_0$.

(23) Prove Corollary 2.25.

(24) Let T be a countable first-order theory and $\varphi(x) \in \mathrm{Fml}(\mathrm{L}(T))$. If there are $M_1 \precneqq M_2 \models T$ such that $\varphi(M_1) = \varphi(M_2)$ is infinite then there exists an uncountable $M \models T$ such that $\varphi(M)$ is countable.

(25) Find an alternative proof of Theorem 1.18. Hint:
   (a) Let $N^* \succeq M$ be such that for all $A \subseteq |N^*|$ of cardinality $< \kappa$ and every M-type over A is realized in N^*
   (b) Use Exercise 4.31.

(26) If a complete theory T is stable in λ then T has a saturated model of cardinality λ^+. Hint: Construct a strictly increasing elementary chain of length λ^+ where the models have cardinality λ satisfying a similar requirement to the chain constructed in the proof of Corollary 1.4.26.

(27) Suppose M is an infinite model and λ a regular cardinal such that $\lambda \geq \|M\| + |\operatorname{L}(M)|$. If $\operatorname{Th}(M)$ is λ-stable then there exists $N \gneqq M$ saturated of cardinality λ.

(28) If a complete theory T is stable in λ and $\lambda \geq |T|$ is regular then T has a saturated model of cardinality λ.

(29) The next exercise is a generalization of the previous. Suppose T is stable in $\lambda \geq |T|$. Prove that T has a κ-saturated model of cardinality λ for every regular $\kappa \leq \lambda^+$

(30) This group of exercise is dedicated to several examples of stable/unstable theories:
   (a) Let S be an infinite set and E an equivalence relation on S. Show that $\operatorname{Th}(\langle S, E\rangle)$ is $\aleph_0$-stable.
   (b) Let $E_n := \{\langle \eta, \nu\rangle \mid \eta, \nu \in {}^{\omega}\omega,\ \eta \restriction n = \nu \restriction n\}$. Show that $\operatorname{Th}(\langle {}^{\omega}\omega, E_n\rangle_{n<\omega})$ is stable but not superstable.
   (c) Let $E_n := \{\langle \eta, \nu\rangle \mid \eta, \nu \in {}^{\omega}2,\ \eta \restriction n = \nu \restriction n\}$. Show that $\operatorname{Th}(\langle {}^{\omega}2, E_n\rangle_{n<\omega})$ is superstable but not stable in $\aleph_0$.
   (d) T_{ACF_p} is superstable.
   (e) T_{ind} is unstable.
   (f) T_{ord} is not stable in $\aleph_0$, in fact this theory is unstable.
   (g) Suppose T is a complete theory containing ZF. Show that T is unstable.

(31) In the previous exercise we have encountered *combinatorial* examples to varius properties. In this exercise I mention several *algebraic* examples. The reader should be warned that the verification is at times harder than in the previous exercise, however in a certain sense the previous combinatorial examples are more important for the development of the theory as they are typical examples (this will be made precise in **??**).?
   (a) T_{ACF_0} is $\aleph_0$-stable.
   (b) The theory of separably closed fields (which are not alegbraically closed) is stable but not superstable.
   (c) Presburger's arithmetic i.e. the theory of $\langle \mathbb{Z}, +\rangle$ is superstable but not $\aleph_0$-stable.
   (d) $\operatorname{Th}(\langle \mathbb{R}, +, \cdot\rangle)$ is unstable.

?!

(32) Find an example of a complete T which is not stable but there is $L \subsetneq \operatorname{L}(T)$ such that $T \restriction L$ is stable. Find an example with $L \neq \emptyset$.

(33) Let λ be an infinite cardinal such that for every $\mu < \lambda$ the theory T has a saturated model of cardinality μ. Does this imply that T has a saturated model of cardinality λ?

(34) Prove Proposition 3.1. Hint: Use induction on n and the fact that the cardinal λ^+ is a regular cardinal.

(35) Prove proposition 2.30. Hint: Use Lemma 7.17.

(36) Suppose $\mathbf{a} \in M^1 \preceq M^2$, $\lambda := \|M\| = \|N\| > |\mathrm{L}(M^1)| + \aleph_0$ and there exists an increasing sequence of cardinals $\{\mu_i < \lambda \mid i < \xi\}$ such that $\lambda = \sum_{i<\xi} \mu_i$ with $\mu_0 > |\mathrm{L}(M^1)|$. If there are $\{M_i^\ell \mid i < \xi\}$ elementary chains of saturated models such that $\|M_i^\ell\| = \mu_i$ and $M^\ell = \bigcup_{i<\xi} M_i^\ell$ (for $\ell = 1, 2$) then there exists $f : M \cong N$ such that $f(\mathbf{a}) = \mathbf{a}$.

(37) Prove theorem ???

(38) Prove Theorem 2.37.

(39) Prove Proposition 2.38.

(40) Prove Corollary 2.34.

(41) The following three exercises emphasize important Galois-theoretic aspects of types.

(a) Let M be a saturated model, $A \subseteq |M|$ such that $|A| < \|M\|$, and suppose that p is a complete M-type over A.

(i) Let $G_A := \{f \in \mathrm{Aut}(M) \ : \ (\forall a \in A) f(a) = a\}$, let $b \in |M|$. Show that: b realizes p iff $(\exists f \in G_A) f(b)$ realizes p iff $(\forall f \in G_A)$ $f(b)$ realizes p.

(ii) Let $p(M) := \{b \in |M| \ : \ b \text{ realizes } p\}$. Show that the group G_A acts on the set $p(M)$ when the action is defined as follows: For $f \in G_A$, and $b \in p(M)$ let $f \cdot b := f(b)$.

(iii) For $a \in |M|$ let $\mathrm{orbit}(a)$ be the orbit of a in the action of G_A on the set $|M|$. Show that

$$\mathrm{tp}(a/A, M) = \mathrm{tp}(b/A, M) \iff \mathrm{orbit}(a) = \mathrm{orbit}(b) \quad \text{for all } a, b \in |M|.$$

Where exactly did you use the assumption that M is saturated?

(b) Let M be a saturated model. Construct a bijection from $\mathrm{D}_1(\mathrm{Th}(M))$ onto $\{\mathrm{orbit}(a) \mid a \in |M|\}$ where $\mathrm{orbit}(a)$ is the orbit of a under the action of $\mathrm{Aut}(M)$ on the set $|M|$.

(c) Let M be a saturated model. Consider the action of $\mathrm{Aut}(M)$ on the set $|M|$, for $a \in |M|$ let G_a be the stabilizer of a, namely:

$$G_a := \{f \in \mathrm{Aut}(M) \mid a = f(a)\}.$$

Prove that $[\mathrm{Aut}(M) : G_a] = |p(M)|$ where $p = \mathrm{tp}(a/\emptyset, M)$.

(42) The purpose of this exercise is to show that a saturated model is a "Galois extension" of its small elementary submodels.

For any model M and G subgroup of $\mathrm{Aut}(M)$ let

$$M^G := \{\mathbf{a} \in |M| \ : \ \forall \sigma \in G \ [\sigma(\mathbf{a}) = \mathbf{a}]\}.$$

Suppose M is saturated and $\|M\| > \aleph_0 + |\mathrm{L}(M)|$. If $N \preceq M$ is of cardinality $< \|M\|$ then

$$M^{\mathrm{Aut}_N(M)} = |N|.$$

(43) The purpose of this group of exercises is to introduce *Galois types* that are central for classification theory of AECs (see Definition 1.5).

(a) Let N_1 and N_2 elementary submodels of the monster model $\mathfrak{C}$. Suppose that $A \subseteq |N_1| \cap |N_2|$.
Definition For $\mathbf{a}_\ell \in N_\ell$ let

$$\langle A, N_1, \mathbf{a}_1\rangle E_A \langle A, N_2, \mathbf{a}_2\rangle \iff \text{there are elementary embeddings } f_\ell : N_\ell \to \mathfrak{C} \text{ such that } f_\ell \restriction A = \mathrm{id}_A \text{ and } f_1(\mathbf{a}_1) = f_2(\mathbf{a}_2).$$

Show that
(i) E_A is an equivalence relation.
(ii) $\mathrm{tp}(\mathbf{a}_1/A, N_1) = \mathrm{tp}(\mathbf{a}_2/A, N_2)$ iff $\langle A, N_1, \mathbf{a}_1\rangle E_A \langle A, N_2, \mathbf{a}_2\rangle$
(iii) If $N' \succeq N_1$ then $\langle A, N_1, \mathbf{a}_1\rangle E_A \langle A, N', \mathbf{a}_1\rangle$.

(b) **Definition** Let N_1 and N_2 elementary submodels of the monster model $\mathfrak{C}$. Suppose that $A \subseteq |N_1| \cap |N_2|$. For $\mathbf{a}_\ell \in N_\ell$ let

$$\langle A, N_1, \mathbf{a}_1\rangle E^*_A \langle A, N_2, \mathbf{a}_2\rangle \iff \text{there exists } M \succeq N_\ell \text{ there are elementary embeddings } f_\ell : N_\ell \to M \text{ such that } f_\ell \restriction A = \mathrm{id}_A \text{ and } f_1(\mathbf{a}_1) = f_2(\mathbf{a}_2).$$

Prove that $E_A = E^*_A$.

(c) Let M be a saturated model and let $A \subseteq |M|$ be of cardinality less than $\|M\|$. Recall the notation of Exercise 1.2.66) Consider the action of G_A on ${}^{<\omega}|M|$. Show that

$$\mathbf{a} \sim_A \mathbf{b} \iff \mathrm{tp}(\mathbf{a}/A, M) = \mathrm{tp}(\mathbf{b}/A, M) \quad \text{for all } \mathbf{a}, \mathbf{b} \in |M|.$$

(d) Suppose that $M = \mathfrak{C}$. Then for $\mathbf{a}_1, \mathbf{a}_2 \in \mathfrak{C}$

$$\mathbf{a}_1 \sim_A \mathbf{a}_2 \iff \langle A, N_1, \mathbf{a}_1\rangle E_A \langle A, N_2, \mathbf{a}_2\rangle \quad \text{for any } N_\ell \supseteq \mathbf{a}_\ell.$$

(44) Suppose that M is a λ-saturated structure. Let

$$\mathcal{K} := \{N \preceq M \,:\, \|N\| < \lambda\} \text{ and let } N_1 \preceq_{\mathcal{K}} N_2 \overset{def}{\iff} N_1 \preceq N_2.$$

Show that $\langle \mathcal{K}, \preceq_{\mathcal{K}}\rangle$ has the μ-amalgamation property for all $\mu \leq \lambda$.

(45) Let M be a given structure, $A \subsetneq |M|$ and p is an M-type over A. The type p is not algebraic if an only if p realized by an element of $|N| - |M|$ iff p for any (some) $N \succeq M$.

(46) Let L be a similarity type and M an L-structure. Suppose $\lambda > |L|$ is uncountable. The following are equivalent:
(a) M is λ-saturated
(b) For every $N \preceq M$ of cardinality less than λ, every non-elgebraic type $p \in \mathrm{S}(N)$ is realized in M
(c) For every $N \preceq M$ of cardinality less than λ, every non-elgebraic type $p \in \mathrm{S}(N)$ is realized by λ-many elements of M.

(47) There exists a non-saturated $M \models T_{ord}$ of cardinbality $\beth_\omega$ such that every non-algebraic formula $\varphi(\mathbf{x}; \mathbf{a})$ with parameters from M is reaalized by $\beth_\omega$-many elements of M.

(48) Let M be a special model show that also $M \restriction L'$ is special for every $L' \subseteq \mathrm{L}(M)$.

(49) Suppose λ is a strong limit cardinal greater than $|L|$. Let M be a special model in L of cardinality λ. Show that for every $T \supseteq \mathrm{Th}(M)$ such that $|T| < \lambda$ the structure M can be expanded to a model of T.

(50) Try to address the same problem when the assumption that M is special is replaced by M is saturated. The case when $\|M\|$ is a regular cardinal is not harder than previous exercise, prove it. The case the cardinality of M is singular is true, but I do not know a proof of it without using material presented in later chapters.

(51) Let T be a complete theory. Suppose there exists $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T)$, $M \models T$ and $\{\mathbf{a}_n \mid n < \omega\} \subseteq |M|$ such that

$$M \models \varphi[\mathbf{a}_n;\mathbf{a}_k] \iff n < k \text{ for all } n,k < \omega,$$

then for any linearly ordered set $\langle I, <_I\rangle$ there are $\{\mathbf{b}_i \mid i \in I\} \subseteq |\mathfrak{C}|$ such that

$$\mathfrak{C} \models \varphi[\mathbf{b}_i;\mathbf{b}_j] \iff i <_I j \text{ for all } i,j \in I.$$

(52) This is a generalization of Exercise 30(g). Suppose that T, M, φ and $\{\mathbf{a}_n \mid n < \omega\}$ are as in the previous exercise. Prove that T must be unstable.

(53) We say that $\varphi(\mathbf{x};\mathbf{y})$ has the *independence property* if for every $M \models T$ and every $n < \omega$ there exists $\{\mathbf{a}_i \mid i < n\} \subseteq {}^{\ell(\mathbf{y})}|M|$ such that

$$M \models \exists \mathbf{x} \bigwedge_{i \in n} \varphi(\mathbf{x};\mathbf{a}_i)^{i \in w} \text{ for every } w \subseteq n.$$

Show that if $\varphi(\mathbf{x};\mathbf{y})$ has the independence property then for every $\lambda \geq \aleph_0$ there exists a set of cardinality λ such that $|\mathrm{S}^{\ell(\mathbf{x})}(A)| = 2^\lambda$. Thus if a theory contains a formula with the independence property then T is unstable.

(54) Let $A \subseteq B$ be subsets of $\mathfrak{C}$. Suppose M is a model containing A (always we assume also that $M \preceq \mathfrak{C}$). If M is $|B|$-saturated and $|A| < |B|$ then there exists f a $(\mathfrak{C},\mathfrak{C})$-mapping such that $f : B \to \mathfrak{C}$ and $f \restriction A = \mathrm{id}_A$.

(55) Let $\mathcal{K}$ be an AEC and let $\lambda > \mathrm{LS}(\mathcal{K}) + \aleph_0$. Suppose that $\mathcal{K}_\mu$ has the JMP for some $\mathrm{LS}(\mathcal{K}) + \aleph_0 \leq \mu < \lambda$. If $M, N \in \mathcal{K}_\lambda$ are both model-homogeneous then $M \cong N$.

(56) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}}\rangle$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$, suppose there exists $\mu \leq \lambda$ such that $\mathcal{K}$ is μ-categorical. If $M, N \in \mathcal{K}_\lambda$ are both model homogeneous then there exists $g : N \cong M$.

(57) Let $\langle \mathcal{K}, \preceq_{\mathcal{K}}\rangle$ be an AEC and $\lambda > \mathrm{LS}(\mathcal{K})$, suppose there exists $\mu \leq \lambda$ such that $\mathcal{K}$ is μ-categorical. If $M \in \mathcal{K}_\lambda$ is model homogeneous then for every $N \in \mathcal{K}_\lambda$ there exists a $\mathcal{K}$-embedding $g : N \to M$.

(58) M is λ-homogenous iff for every ordinal $\gamma < \lambda$ and every $\mathbf{a}, \mathbf{b} \in {}^{\gamma}|M|$ if $\mathrm{tp}(\mathbf{a}/\emptyset, M) = \mathrm{tp}(\mathbf{b}/\emptyset, M)$ then for every $a \in |M|$ there exists $b \in |M|$ such that $\mathrm{tp}(\mathbf{a}^\frown a/\emptyset, M) = \mathrm{tp}(\mathbf{b}^\frown b/\emptyset, M)$.

(59)

3. Indiscernibles and Ehrenfeucht-Mostowski models

DEFINITION 3.1. An L-structure M is called *rigid* iff M has no nontrivial automorphism, namely $\mathrm{Aut}(M) = \{\mathrm{id}_M\}$.

EXAMPLES 3.2. Below we consider a few examples of rigid models and than non-rigid.

(1) The standard model of arithmetic $\langle \omega, +, \cdot, 0, 1\rangle$, the ring of integers $\langle \mathbb{Z}, +, ;0, 1\rangle$, the field of real numbers $\langle \mathbb{R}, +, \cdot, 0, 1\rangle$, and any well ordered set $\langle P, <\rangle$ are examples of rigid structures.
(2) An n dimensional vector space has $n!$ many automorphisms induced by permutations of one fixed base and in general the vector space has $\leq |V| + |F| + \aleph_0$-many automorphisms, the field of complex numbers $\langle \mathbb{C}, +, \cdot, 0, 1\rangle$ has $2^{2^{\aleph_0}}$ many automorphisms. If M is a saturated model and $\|M\| > |\mathrm{D}(T)|$ where T is the theory of M then $|\mathrm{Aut}(M)| \geq \|M\|$. The rational numbers $\langle \mathbb{Q}, <\rangle$ with the usual order have $2^{\aleph_0}$ many automorphisms.

Andrzej Ehrenfeucht and Andrzej Mostowski in their paper [**EM**] answered affirmatively the following question:

QUESTION 3.3. Given a first-order theory T and a cardinal $\lambda \geq |T|$, does there exist a non-rigid model for T of cardinality λ?

I will motivate their approach by the following analysis of $\mathrm{Aut}(V)$ when V is an F-vector space. It is a fact of linear algebra that there exists a basis B for V. Let $\{b_i \mid i < \lambda\}$ be an enumeration of B, where $\lambda = \dim_F V$. If σ is a permutation of λ then there exists $\bar{\sigma} \in \mathrm{Aut}(V)$ given by

$$\bar{\sigma}(v) := \sum_{k<n} a_k b_{\sigma(i_k)} \text{ where } \{a_k \mid k < n\} \subseteq F \text{ such that } v = \sum_{k<n} a_k b_{i_k}.$$

Clearly if σ_1 and σ_2 are different permutations of λ then $\bar{\sigma}_1 \neq \bar{\sigma}_2$. Thus if V is an infinite dimensional vector space then $|\mathrm{Aut}(V)| = 2^{\dim_F V}$.

Now given $M \models T$ without knowing much (or anything) about T can this argument be repeated? Certainly not as M might be rigid. However to find a model M as above instead of working in a given model is much easier.

A technical device introduced by them in their solution turned out to be of great value for many unrelated problems.

DEFINITION 3.4. Let L be a similarity type, M an L structure and $\langle I, <_I\rangle$ a linearly-ordered set. Suppose $A \subseteq |M|$ and $\mathbf{J} := \langle \mathbf{a}_i \mid i \in I\rangle \subseteq |M|$, where $\ell(\mathbf{a}_i) = n < \omega$ for all $i \in I$. $\mathbf{J}$ is called a *sequence of indiscernibles over A in M* iff for every $n < \omega$ every $i_0 <_I \cdots <_I i_{n-1} \in I$ and every $j_0 <_I \cdots <_I j_{n-1} \in I$

we have that

$$\mathrm{tp}(\mathbf{a}_{i_0}, \dots, \mathbf{a}_{i_{n-1}}/A, M) = \mathrm{tp}(\mathbf{a}_{j_0}, \dots, \mathbf{a}_{j_{n-1}}/A, M).$$

When $A = \emptyset$ we delete reference to A.

A natural example for a sequence of indiscernibles is a sequence of linearly independent vectors. As the following example will show in a vector space we have additional interesting properties.

PROPOSITION 3.5. *Let F be a field and V a vector space over F. Suppose W is a subspace of V and $\{a_i \mid i < \alpha\}$ a basis for V such that there exists $\beta < \alpha$ satisfying such that $\{a_i \mid i < \beta\}$ is a basis for W. Then $\mathbf{J} := \{a_i \mid \beta \leq i < \alpha\}$ is a sequence of indiscernibles over the set W.*

PROOF. Let $n < \omega$ and $\beta \leq i_0 < \cdots < i_{n-1} < \alpha$ and any $\beta \leq j_0 < \cdots < j_{n-1} < \alpha$ be given. Take σ to be a permutation of $\{a_i \mid i < \alpha\}$ such that $\sigma \restriction \{a_i \mid i < \beta\} = \mathrm{id}_{\{a_i | i < \beta\}}$ and $\sigma(a_{i_k}) = a_{j_k}$ for all $k < n$. Let $\bar{\sigma}$ be the automorphism of V induced by σ. Clearly $\bar{\sigma} \restriction W = \mathrm{id}_W$.
By Claim I.2.17 and the definition of σ we have

$$\begin{aligned} V \models \varphi[a_{i_0}, \dots, a_{i_{n-1}}; \mathbf{b}] \quad &\iff V \models \varphi[\bar{\sigma}(a_{i_0}), \dots, \bar{\sigma}(a_{i_{n-1}}); \mathbf{b}] \\ &\iff V \models \varphi[a_{j_0}, \dots, a_{j_{n-1}}; \mathbf{b}] \end{aligned}$$

for every formula $\varphi(x_0, \dots, x_{n-1})$ and every $\mathbf{b} \in W$.
Thus

$$\mathrm{tp}(a_{i_0}, \dots, a_{i_{n-1}}/W, V) = \mathrm{tp}(a_{j_0}, \dots, a_{j_{n-1}}/W, V).$$

□

Notice that the assumption that the sequences are increasing was not used.

This leads us to a stronger notion of indiscernibles. Namely to that of a *set of indiscernibles*:

DEFINITION 3.6. Let L be a similarity type, M an L structure and $\langle I, <_I \rangle$ a linearly-ordered set. Suppose $A \subseteq |M|$ and $\mathbf{J} := \{\mathbf{a}_i \mid i \in I\} \subseteq |M|$, where $\ell(\mathbf{a}_i) = n < \omega$ for all $i \in I$. $\mathbf{J}$ *is called a set of indiscernibles over A in M* iff for every $n < \omega$ every $\{i_0, \dots, i_{n-1}\} \subseteq I$ and every $\{j_0, \dots, j_{n-1}\} \subseteq I$ (using the set convention we implicitly have that $i_k \neq i_m$ for all $k < m < n$) we have that

$$\mathrm{tp}(\mathbf{a}_{i_0}, \dots, \mathbf{a}_{i_{n-1}}/A, M) = \mathrm{tp}(\mathbf{a}_{j_0}, \dots, \mathbf{a}_{j_{n-1}}/A, M).$$

When $A = \emptyset$ we delete A.

Clearly every set of indiscernibles is a sequence of indiscernibles. Notice that the argument of Proposition 3.5 actually shows that $\mathbf{J}$ is a set of indiscernibles over W.

A natural question to ask is when a sequence of indiscerrnibles is not a set of indiscernibles? Little reflection shows that having a linear order in the similarity type is an obstacle:

PROPOSITION 3.7. *Let L be a similarity type, M an L structure, $\langle I, <_I\rangle$ a linearly-ordered set and suppose that $\mathbf{I} := \langle \mathbf{a}_i \mid i \in I\rangle \subseteq |M|$ is a sequence of indiscernibles. If there exists a first-order formula $\varphi(\mathbf{x};\mathbf{y})$ in L with $\ell(\mathbf{x}) = \ell(\mathbf{y}) = \ell(\mathbf{a}_i)$ for all $i \in I$ such that φ induces a linear order on at least two elements of $\mathbf{I}$ then $\mathbf{I}$ is not a set of indiscernibles.*

PROOF. Suppose that $i_1 \neq i_2 \in I$ such that φ induces a linear order on $\{\mathbf{a}_{i_1}, \mathbf{a}_{i_2}\}$, without loss of generality we may assume that $M \models \varphi[\mathbf{a}_{i_1}; \mathbf{a}_{i_2}]$.

Had $\mathbf{I}$ been a set of indiscernibles we would have

$$\mathrm{tp}(\mathbf{a}_{i_1}, \mathbf{a}_{i_2}/\emptyset, M) = \mathrm{tp}(\mathbf{a}_{i_2}, \mathbf{a}_{i_1}/\emptyset, M).$$

So in particular we have

$$M \models \varphi[\mathbf{a}_{i_1}; \mathbf{a}_{i_2}] \iff M \models \varphi[\mathbf{a}_{i_2}; \mathbf{a}_{i_1}],$$

but this contradicts the assumption that $M \models \varphi[\mathbf{a}_{i_1}; \mathbf{a}_{i_2}]$ and the hypothesis that φ induces a linear order on $\{\mathbf{a}_{i_1}, \mathbf{a}_{i_2}\}$. □

REMARK 3.8. The notion we introduced in Definition 3.4 plays also a central role in set theory (little we present in below). In set theory texts what we call here *indiscernible sequence* often is called *indiscernible set*. As we will see in Chapter 4 one of the characterizations of stability of a theory is that every indiscernible sequence is also an indiscernible set. So we must use here different terms.

Since a set of linearly independent vectors in a vector space is a set of indiscernibles it is reasonable to seek capturing the notion of a basis (or a maximally independent set of vectors). This motivates the following:

DEFINITION 3.9. Let M be an L-structure, $A \subseteq |M|$, $\langle I, <\rangle$ a linearly ordered set. A sequence $\mathbf{I} := \langle \mathbf{b}_i \mid i \in I\rangle \subseteq |M|$ is called a *maximal sequence of indiscernibles over A in M* iff there is no sequence of indiscernibles $\mathbf{J} \supsetneq \mathbf{I}$ in M over A.

By replacing "sequence" with "set" we get the notion of *maximal set of indiscernibles over A in M* .

The following is an important extension result:

THEOREM 3.10 (stretching of an indiscernible sequence). *Suppose $\mathbf{I} := \{\mathbf{a}_i \mid i \in I\}$ is an indiscernible sequence over A, when I is linearly ordered without a last element. Then for any l.o. J such that $\langle I, <_I\rangle \subseteq \langle J, <_J\rangle$ there exists $\mathbf{J} := \{\mathbf{b}_j \mid j \in J\} \subseteq \mathfrak{C}$ indiscernible over A such that $\mathbf{a}_i = \mathbf{b}_i$ for all $i \in I$.*

PROOF. By expanding $\mathrm{L}(T)$ with constants for the elements of A we may assume that $A = \emptyset$. Let $\{\mathbf{d}_j \mid j \in J\}$ be new constants.

Let $p_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) := \mathrm{tp}(\mathbf{a}_1, \ldots, \mathbf{a}_n/\emptyset, \mathfrak{C})$. By the assumption on $\mathbf{I}$ we have that

$$\langle \mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n} \rangle \models p_n \quad \text{for all } i_1 < \cdots < i_n.$$

Consider

$$T^* := T \cup \{d_{j_1} \neq d_{j_2} \mid j_1 \neq j_2 \in J\} \cup \{p_n(\mathbf{d}_{j_1}, \ldots, \mathbf{d}_{j_n}) \mid n < \omega, j_1 < \ldots < j_n \in J\}$$

Given $T_0 \subseteq T^*$ let $\{\mathbf{d}_{j_1}, \ldots, \mathbf{d}_{j_n}\}$ be the new constants in T_0. Check that

$$(M, \mathbf{a}_1, \ldots, \mathbf{a}_n) \models T_0.$$

By the compactness theorem there exists $M^* \models T^*$. For any $j \in J$ let $\mathbf{e}_j := \mathbf{d}_j^{M^*}$.

Since for every $n < \omega$ and any $i_1 < \cdots i_n \in J$ we have that

$$\mathrm{tp}(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}/\emptyset, \mathfrak{C}) = \mathrm{tp}(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_n}/\emptyset, \mathfrak{C}) \quad \text{for all } j_1 < \cdots j_n \in I$$

the function $f := \{\langle \mathbf{e}_i, \mathbf{a}_i \rangle \mid i \in I\}$ is a $(\mathfrak{C}, \mathfrak{C})$-mapping. Since $\mathfrak{C}$ is a monster model there exists $f^* \in \mathrm{Aut}(\mathfrak{C})$ be an extending f. Check that when we define $\mathbf{b}_j$ as $f^*(\mathbf{e}_j)$ for all $j \in J$ the sequence $\{\mathbf{b}_j \mid j \in J\}$ is as required. □

An easy application of Zorn's lemma gives:

PROPOSITION 3.11. *Let M be an L-structure, $A \subseteq |M|$ be given and $\langle I, < \rangle$ is a linearly ordered set. For every $\mathbf{I} := \{\mathbf{b}_i \mid i \in I\} \subseteq |M|$ sequence of indiscernibles over A in M there exists a maximal sequence of indiscernibles $\mathbf{J} \supseteq \mathbf{I}$ in M over A.*

PROOF. Exercise. □

PROPOSITION 3.12. *Let M be an L-structure, $A \subseteq |M|$ be given and $\langle I, < \rangle$ is a linearly ordered set. Suppose $\mathbf{I} := \langle \mathbf{b}_i \mid i \in I \rangle \subseteq |M|$ is an infinite sequence of indiscernibles over A in M. If M is λ-saturated and $\lambda > |A| + |I|$ then $\mathbf{I}$ is a not a maximal sequence of indiscernibles over A in M.*

PROOF. Consider

$$p(\mathbf{x}) := \{\varphi(\mathbf{a}; \mathbf{b}_{i_0}, \ldots, \mathbf{b}_{i_{n-1}}, \mathbf{x}) \leftrightarrow \varphi(\mathbf{a}; \mathbf{b}_{j_0}, \ldots, \mathbf{b}_{j_{n-1}}, \mathbf{x}) \mid n < \omega, \mathbf{a} \in A,$$
$$i_0 <_I \cdots <_I i_{n-1}, j_0 <_I < \cdots <_I j_{n-1} \in I, \varphi(\mathbf{y}) \in L(M)\}.$$

□

PROPOSITION 3.13. *Let M be an L-structure, $A \subseteq |M|$ be given and $\langle I, < \rangle$ be a linearly ordered set. Suppose $\mathbf{I} := \langle \mathbf{b}_i \mid i \in I \rangle \subseteq |M|$ is an infinite set*

of indiscernibles over A in M. $\mathbf{I}$ *is a maximal set of indiscernibles over A in M iff the type*

$$p(\mathbf{x}) := \{\varphi(\mathbf{x}; \mathbf{b}_{i_0}, \dots, \mathbf{b}_{i_{n-1}}; \mathbf{a}) \leftrightarrow \varphi(\mathbf{x}; \mathbf{b}_{j_0}, \dots, \mathbf{b}_{j_{n-1}}; \mathbf{a}) \mid n < \omega, \mathbf{a} \in A, \\ \{i_0, \dots, i_{n-1}\}, \{j_0, \dots, j_{n-1}\} \subseteq I, \varphi(\mathbf{x}; \mathbf{y}; \mathbf{z}) \in L(M)\}$$

is omitted in M.

PROOF. Exercise **??**. □

Actually a stronger fact then the last proposition is true:

THEOREM 3.14. *Let M be an L-structure, $A \subseteq |M|$ be given and $\langle I, < \rangle$ is a linearly ordered set. Suppose $\mathbf{I} := \langle \mathbf{a}_i \mid i \in I \rangle \subseteq |M|$ is an infinite sequence of indiscernibles over A in M. If M is λ-homogeneous and $\lambda > |A| + |I|$ then $\mathbf{I}$ is a not a maximal sequence of indiscernibles over A in M.*

PROOF. Exercise **??**. □

A related notion:

DEFINITION 3.15. Let M be an L-structure, $A \subseteq |M|$ be given. The model M is *minimal over* A iff there is no *proper* elementary submodel N of M such that $|N| \supseteq A$.

In a later section (**??.??**) we will discuss the existence of such a model. However we can state a nice necessary requirement:

PROPOSITION 3.16. *Let M be an L-structure, $A \subseteq |M|$ be given. If the model M is minimal over A then there is no infinite* $\mathbf{I}$ *set of indiscernibles over A in M.*

PROOF. Otherwise, suppose that M is minimal over A and $\mathbf{I}$ is an infinite sequence of indiscernibles over A in M. We may assume that $\mathbf{I}$ is a maximal set of indiscernibles over A in M. Fix $\{\mathbf{a}_i \mid i < \mu\} = \mathbf{I}$. Consider

$$p(\mathbf{x}) := \{\varphi(\mathbf{x}; \mathbf{a}_{i_0}, \dots, \mathbf{a}_{i_{n-1}}) \leftrightarrow \varphi(\mathbf{x}; \mathbf{a}_{j_0}, \dots, \mathbf{a}_{j_{n-1}}) \mid \\ i_0 < \cdots < i_{n-1}, j_0 < \cdots < j_{n-1} < \mu, \varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{L}(M)\}.$$

Since $\mathbf{I}$ is a maximal set of indiscernibles over A in M the type p is omitted in M. Now define:

$$p'(\mathbf{x}) := \{\varphi(\mathbf{x}; \mathbf{a}_{i_0}, \dots, \mathbf{a}_{i_{n-1}}) \leftrightarrow \varphi(\mathbf{x}; \mathbf{a}_{j_0}, \dots, \mathbf{a}_{j_{n-1}}) \mid \\ 0 < i_0 < \cdots < i_{n-1}, 0 < j_0 < \cdots < j_{n-1} < \mu, \varphi(\mathbf{x}; \mathbf{y}) \in L(M)\}.$$

Clearly $\mathbf{a}_0 \models p'$. By Exercise **??** let $f \in \mathrm{Aut}_A(\mathfrak{C})$ be such that f is a bijection from $\mathbf{I}$ onto $\mathbf{I} - \{\mathbf{a}_0\}$. Clearly $f(p) = p'$. ?????????????????????? □

The basic existence result is:

THEOREM 3.17 (Ehrenfeucht-Mostowski). *If T has an infinite model then for every linearly ordered set $\langle I <_I \rangle$ and every positive integer ℓ there exists $M_I \models T$ such that there is an sequence of indiscernibles $\langle \mathbf{a}_i \in {}^{\ell}|M| \, : \, i \in I \rangle$.*

PROOF. Let $\{\mathbf{c}_i \mid i \in I\}$ be a set ℓ-tuples of constants not in $\mathrm{L}(T)$. Clearly it suffices to show that the theory

$$T_I := T \cup \{\mathbf{c}_i \neq \mathbf{c}_j \mid i \neq j \in I\} \cup \Big\{\varphi(\mathbf{c}_{i_0}, \dots, \mathbf{c}_{i_{n-1}}) \leftrightarrow \varphi(\mathbf{c}_{j_0}, \dots, \mathbf{c}_{j_{n-1}}) \mid$$
$$n < \omega, \ i_0 <_I \cdots <_I i_{n-1} \in I, \ \ j_0 <_I \cdots <_I j_{n-1} \in I, \ \varphi(\mathbf{x}) \in \mathrm{L}(T)\Big\}$$

is consistent.

Let $M'_I \models T_I$, take $M_I := M'_I \restriction \mathrm{L}(T)$ and $\{\mathbf{c}_i^{M_I} \mid i \in I\}$ ia sequence of indiscernibles.

Using the the finite character of consistency it is enough to show that for every finite $\Gamma \subseteq \mathrm{Fml}(\mathrm{L}(T))$ the theory $T_I(\Gamma)$ is consistent where $T_I(\Gamma)$ is the following theory:

$$T \cup \{\mathbf{c}_i \neq \mathbf{c}_j \mid i \neq j \in I\} \cup \Big\{\varphi(\mathbf{c}_{i_0}, \dots, \mathbf{c}_{i_{n-1}}) \leftrightarrow \varphi(\mathbf{c}_{j_0}, \dots, \mathbf{c}_{j_{n-1}}) \mid$$
$$n < \omega, \ i_0 <_I \cdots <_I i_{n-1} \in I, \ \ j_0 <_I \cdots <_I j_{n-1} \in I, \ \varphi(\mathbf{x}) \in \Gamma\Big\}.$$

Using the compactness theorem the consistency of $T_\omega(\Gamma)$ implies that of $T_I(\Gamma)$.

We argue by induction on $n = |\Gamma| < \omega$:
For $n = 0$; The theory $T_\omega(\emptyset)$ has a model since T has an infinite model.
For $n > 0$; Suppose that

$$\Gamma = \Gamma_0 \cup \big\{\varphi(\mathbf{x}_0, \dots, \mathbf{x}_{m-1})\big\}.$$

Namely $n = |\Gamma| = |\Gamma_0| + 1$.

Since I is a linearly ordered set another use of the compactness theorem shows that it is enough to prove the following

LEMMA 3.18. *Let $\{\mathbf{c}_k \mid k < \omega\}$ be ℓ-tuples of constants not in $\mathrm{L}(T)$. If $M \models T$ and $\{\mathbf{a}_k \mid k < \omega\} \subseteq |M|$ such that $(M, \mathbf{a}_k)_{k<\omega} \models T_\omega(\Gamma_0)$ then there is an infinite $S \subseteq \omega$ such that $(M, \mathbf{a}_k)_{k \in S} \models T_\omega(\Gamma)$.*

PROOF. For $k_0 < \cdots < k_{m-1} < \omega$ define

$$F(k_0, \dots, k_{m-1}) := \begin{cases} 1 & \text{if } (M, \mathbf{a}_k)_{k<\omega} \models \varphi[\mathbf{c}_{k_0}, \dots, \mathbf{c}_{k_{m-1}}] \\ \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the infinite Ramsey's theorem is the statement

$$(*) \quad \aleph_0 \to (\aleph_0)^m_2.$$

An application of $(*)$ to the coloring F gives an infinite $S \subseteq \omega$ and $t \in \{0,1\}$ such that we have that $F(k_0, \ldots, k_{m-1}) = t$ for every $k_0 < \cdots < k_{m-1} \in S$.
Thus we have

$$(M, \mathbf{a}_k)_{k\in S} \models \varphi[\mathbf{c}_{i_0}, \ldots, \mathbf{c}_{i_{m-1}}] \iff (M, \mathbf{a}_k)_{k\in S} \models \varphi[\mathbf{c}_{j_0}, \ldots, \mathbf{c}_{j_{m-1}}]$$

for all $i_0 < \cdots < i_{m-1} \in S$ and every $j_0 < \cdots < j_{m-1} \in S$. □

□

In the previous proof it is possible to replace the use of Ramsey's theorem with the Erdős-Rado theorem. This can be done by replacing Lemma 3.18 with the following lemma (using the notation from the proof of Theorem 3.17):

LEMMA 3.19. *Suppose* $\Gamma = \Gamma_0 \cup \{\varphi(\mathbf{x}_0, \ldots, \mathbf{x}_{m-1})\}$. *If* $T_\omega(\Gamma_0)$ *has a model, then* $T_\omega(\Gamma)$ *has a model.*

PROOF. Notice that by the compactness theorem: If $T_\omega(\Gamma_0)$ has a model then $T_\lambda(\Gamma_0)$ has a model for every cardinal number λ. Let $\lambda := \beth^+_m$. By the previous observation fix $(M, \mathbf{a}_i)_{i<\lambda} \models T_\lambda(\Gamma_0)$. For $\xi_0 < \cdots < \xi_{m-1} < \lambda$ define

$$F(\xi_0, \ldots, \xi_{m-1}) := \begin{cases} 1 & \text{if } (M, \mathbf{a}_i)_{i<\lambda} \models \varphi[\mathbf{c}_{\xi_0}, \ldots, \mathbf{c}_{\xi_{m-1}}] \\ 0 & \text{otherwise.} \end{cases}$$

An application of $(\beth_n)^+ \to (\aleph_1)^m_2$ to the coloring F gives an uncountable $S \subseteq \lambda$ such that

$$(M, \mathbf{a}_i)_{i\in S} \models \varphi[\mathbf{c}_{\xi_0}, \ldots, \mathbf{c}_{\xi_{m-1}}] \iff (M, \mathbf{a}_i)_{i\in S} \models \varphi[\mathbf{c}_{\zeta_0}, \ldots, \mathbf{c}_{\zeta_{m-1}}]$$

for every $\xi_0 < \cdots < \xi_{m-1} \in S$ and every $\zeta_0 < \cdots < \zeta_{m-1} \in S$.

Fix $\{i_n \mid n < \omega\} \subseteq S$ an increasing subset. Thus $\langle M \restriction \mathrm{L}(T), \mathbf{a}_{i_n}\rangle_{i<\omega}$ is a model of $T_\omega(\Gamma)$. □

The notion of Ehrenfeucht-Mostowski models was used by Morley in his innovative paper where he proved his categoricity theorem quite differently to the original usage in the Ehrenfeucht and Mostowski paper.

The notion of a stable theory can be relativized to a model:

DEFINITION 3.20. Let M be an L-structure and let $\lambda \geq |L|$. We say that M *is stable in* λ iff for every $n < \omega$ we have that

$$(*)_n \quad |A| \leq \lambda \implies |\,\mathrm{S}^n(A, M)| \leq \lambda \quad \text{for all } A \subseteq |M|.$$

Recall that $\mathrm{S}^n(A, M) := \{\mathrm{tp}(\mathbf{a}/A, M) \mid \mathbf{a} \in {}^n|\,M|\}$.

Notice that for a complete T and a monster model $\mathfrak{C} \models T$
the theory T is stable in λ iff $\mathfrak{C}$ is stable in λ.

It turns out that Ehrenfeucht-Mostowski models generated by well ordered indiscernibles are stable:

THEOREM 3.21. *Let α be an ordinal, suppose that T has built-in Skolem functions and $\mathbf{I} := \{a_i \mid i < \alpha\}$ is a sequence of indiscernibles (ordered by the $\in$ relation on α). The model $M := \mathrm{EM}(\mathbf{I})$ is λ-stable for every $\lambda \geq |\mathrm{L}(T)| + \aleph_0$.*

PROOF. Let $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ be given. Clearly it is enough to prove the following Claims:

CLAIM 3.22. *Let M be any model (we don't need here that $\mathrm{L}(M)$ has Skolem functions). If $(*)_1$ holds then M is stable in λ.*

PROOF. Show by induction on $n < \omega$ that $(*)_n$ holds: For the sake of contradiction suppose that $|A| = \lambda$ and $|\mathrm{S}^{n+1}(A, M)| > \lambda$. Pick $S \subseteq \mathrm{S}^{n+1}(A, M)$ of cardinality λ^+. For every $p \in S$ let

$$q_p := \{\exists x_n \varphi(x_0, \ldots, x_{n-1}, x_n; \mathbf{a}) \mid \varphi(x_0, \ldots, x_{n-1}, x_n; \mathbf{a}) \in p\}.$$

Take $\bar{q}_p \in \mathrm{S}^n(A, M) \supseteq q_p$. Consider the mapping $p \mapsto \bar{q}_p$. By $(*)_n$ there are $S^* \subseteq S$ of cardinality λ^+ and $q^* \in \mathrm{S}^n(A, M)$ such that

$$p \in S^* \implies \bar{q}_p = q^*.$$

Let $\langle b_0, \ldots, b_{n-1} \rangle \in |M|$ realizing the type q^*. Since S^* has cardinality λ^+ also the set $\{p(b_0, \ldots, b_{n-1}, x_n) \mid p \in S^*\}$ has cardinality λ^+.
Thus $|\mathrm{S}^1(A \cup \{b_0, \ldots, b_{n-1}\}, M)| \geq \lambda^+$ this is a contradiction to $(*)_1$. □

CLAIM 3.23. *Let $M := \mathrm{EM}(\mathbf{I})$ and λ be as in the statement of Theorem 3.21. The model M and λ satisfy requirement $(*)_1$.*

PROOF. Let $A \subseteq \mathrm{EM}(\mathbf{I})$ be a given set of cardinality λ. For every $b \in \mathrm{EM}(\mathbf{I})$ fix a term $\tau_b(x_1, \ldots, x_{n_b})$ and a sequence $\mathbf{i}(b) := \langle i_0(b), \ldots, i_{n_b - 1}(b) \rangle \in I$ such that $b = \tau_b[a_{i_0(b)}, \ldots, a_{i_{n_b-1}(b)}]$. We denote by $\mathbf{a}_{\mathbf{i}(b)}$ the sequence $\langle a_{i_0(b)}, \ldots, a_{i_{n_b-1}(b)} \rangle$.

Let $C_A := \{i_k(b) \mid k < n_b, b \in A\}$. For $b_1, b_2 \in \mathrm{EM}(\mathbf{I})$ define

$$b_1 \sim_A b_2 \iff \tau_{b_1} = \tau_{b_2} \wedge n_{b_1} = n_{b_2} \wedge \forall \xi \in C_A \forall k < n_{b_1} [i_k(b_1) \leq \xi \leftrightarrow i_k(b_2) \leq \xi].$$

Notice that the above relation is an equivalence relation.

SUB CLAIM 3.24. *If $b_1 \sim_A b_2$ then $\mathrm{tp}(b_1/A, \mathrm{EM}(\mathbf{I})) = \mathrm{tp}(b_2/A, \mathrm{EM}(\mathbf{I}))$.*

PROOF. Suppose $\varphi(x; \mathbf{a})$ is a formula with parameters from A. Let $\bar{\tau}_{\mathbf{a}}$ be sequence of terms and $\bar{\mathbf{i}}_{\mathbf{a}}$ be an increasing sequence of ordinals from $C(A)$ such

that $\mathbf{a} = \bar{\tau}(\bar{\mathbf{i}}_{\mathbf{a}})$. Suppose $\mathbf{i}_{b_1}$ and $\mathbf{i}_{b_2}$ are sequences of ordinals from α and τ_{b_ℓ} are terms such that $b_\ell = \tau_\ell(\mathbf{i}_{b_\ell})$.

As $b_1 \sim_A b_2$, the sequences $\mathbf{i}_{b_1}{}^\frown\bar{\mathbf{i}}_{\mathbf{a}}$ and $\mathbf{i}_{b_2}{}^\frown\bar{\mathbf{i}}_{\mathbf{a}}$ have the same order type, by indiscernibility of $\mathbf{I}$. Notice that the definition of the equivalence relation we get $\tau_{b_1} = \tau_{b_2}$ we denote this term by τ_b, so we get choice of notation

$$\mathrm{EM}(\mathbf{I}) \models \varphi[b_1; \mathbf{a}] \iff \mathrm{EM}(\mathbf{I}) \models \varphi[\tau_b[\mathbf{a}_{\mathbf{i}(b_1)}]; \bar{\tau}_{\mathbf{a}}[\mathbf{a}_{\mathbf{i}(\mathbf{a})}]]$$

by indiscernibility

$$\mathrm{EM}(\mathbf{I}) \models \varphi[\tau_b[\mathbf{a}_{\mathbf{i}(b_1)}]; \bar{\tau}_{\mathbf{a}}[\mathbf{a}_{\mathbf{i}(\mathbf{a})}]] \iff \mathrm{EM}(\mathbf{I}) \models \varphi[\tau_b[\mathbf{a}_{\mathbf{i}(b_2)}]; \bar{\tau}_{\mathbf{a}}[\mathbf{a}_{\mathbf{i}(\mathbf{a})}]]$$

and

$$\mathrm{EM}(\mathbf{I}) \models \varphi[\tau_b[\mathbf{a}_{\mathbf{i}(b_2)}]; \bar{\tau}_{\mathbf{a}}[\mathbf{a}_{\mathbf{i}(\mathbf{a})}]] \iff \mathrm{EM}(\mathbf{I}) \models \varphi[b_2; \mathbf{a}].$$

□

By the Sub Claim $|\,\mathrm{S}^1(A, \mathrm{EM}(\mathbf{I}))| \leq |\,\mathrm{EM}(\mathbf{I})/\sim_A|$ it suffices to show that $\mathrm{EM}(\mathbf{I})/\sim_A$ has cardinality less or equal to λ. Since $\lambda \geq |\,\mathrm{L}(T)| + \aleph_0$ it is enough to show that the equivalence relation $\approx_{A,n}$ has at most λ equivalence classes on ${}^n\alpha$ where

$$\langle i_0, \ldots, i_n\rangle \approx_{A,n} \langle j_0, \ldots, j_n\rangle \overset{def}{\iff} \forall \xi \in C_A \forall k < n[i_k \leq \xi \leftrightarrow j_k \leq \xi].$$

Since α is well ordered for every $i < \alpha$ let

$$i^* := \begin{cases} \min\{\xi \in C_A \mid \xi \geq i\} & \text{if there exists } \xi \in C_A \text{ greater or equal to } i \\ \infty & \text{otherwise.} \end{cases}$$

It is straightforward from the definition of the relation $\approx_{A,n}$ to see that:

$$\langle i_0, \ldots, i_n\rangle \approx_{A,n} \langle j_0, \ldots, j_n\rangle \iff i^*_k = j^*_k \quad \forall k < n.$$

□

Since for every $i < \alpha$ we have that $i^* \in C_A \cup \{\infty\}$ the cardinality of ${}^n\alpha \,/\approx_{A,n}$ is bounded by $|C_A| + 1 \leq (\sum_{a\in A} n_a) + 1 \leq \lambda \cdot \aleph_0 + 1 = \lambda$. As

$$|\,\mathrm{S}^1(A, \mathrm{EM}(\mathbf{I}))| \leq |\,\mathrm{EM}(I)/\sim_A| \leq \lambda \cdot \aleph_0 \cdot \sum_{n<\omega} |\,{}^n\alpha \,/\approx_{A,n}| \leq \lambda.$$

This completes the proof of Theorem 3.21. □

The following important corollary of Theorem 3.21 will be used in the proof of Morley's theorem:

COROLLARY 3.25 (Categoricity implies Stability). *Let T be a first-order theory. If T is categorical in some $\mu > |\,\mathrm{L}(T)| + \aleph_0$ then T is stable in every $\lambda \geq |\,\mathrm{L}(T)| + \aleph_0$ such that $\lambda < \mu$.*

In particular, if T is categorical in an uncountable cardinal and $\mathrm{L}(T)$ is at most countable then T is $\aleph_0$-stable.

PROOF. Since T is categorical (by Theorem 1.3.26) T must be complete. Suppose for the sake of contradiction that T is not stable in λ as above. Namely there exists $A \subseteq \mathfrak{C}$ of cardinality λ such that $|\mathrm{S}(A)| \geq \lambda^+$.

By the upward Löwenheim-Skolem there exists $M \models T$ of cardinality μ such that $M \supseteq A$ and $|S(A, M)| \geq \lambda^+$. Using Theorem 1.14.13 pick T^{sk} to be an expansion of T with Skolem functions of cardinality $|\mathrm{L}(T)| + \aleph_0$. Using Theorem 1.14.13 there is a model $M^{sk} \models T^{sk}$ which is generated by a sequence of indiscernibles $\mathbf{I} := \{a_i \mid i < \mu\}$, namely $M^{sk} = \mathrm{EM}(\mathbf{I})$. Now by categoricity in μ the model $M^{sk} \restriction \mathrm{L}(T)$ is isomorphic to M, but this is a contradiction to Theorem 3.21. □

The main importance of the Ehrenfeucht-Mostowski models is that they behave really well, especially as a functor from the category of linearly ordered sets into the category $\mathrm{Mod}(T^{sk})$. Recall (Definition 1.14.21) that when T has Skolem Functions then for any $\mathbf{I} \subseteq |M|$, when $M \models T$, $\mathrm{EM}(\mathbf{I})$ stands for the submodel of M generated by the set $\mathbf{I}$.

Theorem 3.27 below summarizes some of the most important properties.

But first introduce an important notion:

DEFINITION 3.26. Suppose T has built-in Skolem functions, $M \models T$. Let $\langle I, <_I \rangle$ and $\langle J, <_J \rangle$ be linearly ordered sets. Suppose that $\mathbf{I} := \langle a_i \mid i \in I \rangle \subseteq |M|$ and $\mathbf{J} := \langle b_j \mid j \in J \rangle \subseteq |M|$ be sequence of indiscernibles. *The sequences* $\mathbf{I}$ *and* $\mathbf{J}$ *have the same blue-print* iff for every increasing $i_1 <_I \cdots <_I i_n \in I$ and every increasing $j_1 <_I \cdots <_I j_n \in J$ we have that

$$\mathrm{EM}(\mathbf{I}) \models \varphi[a_{i_1}, \ldots, a_{i_n}] \iff \mathrm{EM}(\mathbf{J}) \models \varphi[b_{j_1}, \ldots, b_{j_n}] \text{for all atomic } \varphi(\mathbf{x}) \in \mathrm{L}(T).$$

THEOREM 3.27 (Ehrenfeucht-Mostowski). *Suppose T has built-in Skolem functions. Let $\langle I, <_I \rangle$ and $\langle J, <_J \rangle$ be linearly ordered sets.*
Suppose that $\mathbf{I} := \langle a_i \mid i \in I \rangle$ and $\mathbf{J} := \langle b_j \mid j \in J \rangle$ be sequence of indiscernibles with the same blue-print.

(1) *If $\mathbf{I} \subseteq \mathbf{J}$ then $\mathrm{EM}(\mathbf{I}) \preceq \mathrm{EM}(\mathbf{J})$.*
(2) *If $g : I \to J$ is order preserving then there exists an elementary embedding $\bar{g} : \mathrm{EM}(\mathbf{I}) \to \mathrm{EM}(\mathbf{J})$ such that $\bar{g}(a_i) = b_{g(i)}$ for all $i \in I$.*
(3) *If $g \in \mathrm{Aut}(\langle I, <_I \rangle)$ then there exists $\bar{g} \in \mathrm{Aut}(\mathrm{EM}(\mathbf{I}))$ such that $\bar{g}(a_i) = a_{g(i)}$ for all $i \in I$.*
(4) *If I is infinite and there is an order preserving function $g : I \to J$ then we have*

$$p \in \mathrm{S}^n(\emptyset, \mathrm{EM}(\mathbf{J})) \implies p \in \mathrm{S}^n(\emptyset, \mathrm{EM}(\mathbf{I})) \quad \textit{for all types and every } n < \omega.$$

PROOF. (1) Since T is model-complete (see Remark 1.14.17).

(2) Since $\mathbf{I}$ generates the structure $\mathrm{EM}(\mathbf{I})$ for every $b \in |\mathrm{EM}(\mathbf{I})|$ there exist a term $\tau_b(x_1, \ldots, x_{n_b})$ and a sequence $i_1, \ldots, i_{n_b} \in I$ such that $b = \tau_b[a_{i_1}, \ldots, a_{i_{n_b}}]$.

Define $\bar{g}(b) := \tau_b[b_{g(i_1)}, \ldots, b_{g(i_{n_b})}]$. It is enough to show that $\bar{g}$ is a well defined injection.

Suppose $\tau(x_1, \ldots, x_n)$ is term and $j_1 < \cdots < j_n \in I$ such that

$$\tau_b[a_{i_1}, \ldots, a_{i_{n_b}}] = \tau[a_{j_1}, \ldots, a_{j_n}].$$

We want to show that

$$(*) \quad \tau_b[b_{g(i_1)}, \ldots, b_{g(i_{n_b})}] = \tau[a_{g(j_1)}, \ldots, a_{g(j_n)}].$$

Pick $\{\xi_1, \ldots, \xi_m\}$ to be an increasing enumeration of the set $\{i_1, \ldots, i_{n_b}\} \cup \{j_1, \ldots, j_n\}$. By adding dummy variables to the terms τ_b and τ we may assume that both are in the variables $x_1, \ldots, x_m$. Now denote by $\varphi(x_1, \ldots, x_m)$ the formula $\tau_b(x_1, \ldots, x_m) = \tau(x_1, \ldots, x_m)$. Since $\mathrm{EM}(\mathbf{I}) \models \varphi[a_{\xi_1}, \ldots, a_{\xi_m}]$, g is order preserving and $\mathbf{I}, \mathbf{J}$ have the same blue print it follows that also $\mathrm{EM}(\mathbf{J}) \models \varphi[b_{g(\xi_1)}, \ldots, b_{g(\xi_m)}]$.

By the definition of φ this entails that $(*)$ holds.

To show that $\bar{g}$ is injective, is similar. Suppose $c_1 \neq c_2$, let $c_1 = \tau_1[a_{i_1}, \ldots, a_{i_{n_1}}]$ and $c_2 = \tau_2[a_{j_1}, \ldots, a_{j_{n_2}}]$. Define $\varphi(\mathbf{x})$ to be $\tau_1(\mathbf{x}) \neq \tau_2(\mathbf{x})$, since $\mathbf{I}$ and $\mathbf{J}$ have the same blue-print we get from $\mathrm{EM}(\mathbf{I}) \models \varphi[\mathbf{a}]$ that $\bar{g}(c_1) \neq \bar{g}(c_2)$.

(3) Apply (2) to the case $I = J$.

(4) Suppose $\mathbf{b} \in \mathrm{EM}(\mathbf{J}) \models p$. We will construct $\mathbf{b}^* \in \mathrm{EM}(\mathbf{I})$ realizing p.

Let $\bar{\tau}(\mathbf{x}) := \langle \tau_0(\mathbf{x}), \ldots, \tau_{n-1}(\mathbf{x}) \rangle$ and $j_0 < \cdots < j_{m-1} \in J$ be such that $\mathbf{b} = \bar{\tau}[b_{j_0}, \ldots, b_{j_{m-1}}]$. Since I is infinite there are $j'_0 < \cdots < j'_{m-1} \in g[I]$.

Define $\mathbf{b}' := \bar{\tau}[b_{j'_0}, \ldots, b_{j'_{m-1}}]$. Using indiscernibility of $\mathbf{J}$ show that also $\mathbf{b}'$ realizes p. Now using (2) show that $\mathbf{b}^* := \bar{g}^{-1}(\mathbf{b}')$ realizes p. □

Using Theorem 3.27 we can now solve Question 3.3:

COROLLARY 3.28 (models with many automorphisms). *Let T be a first-order theory that has an infinite model. For every $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ T has a non-rigid model of cardinality λ.*

PROOF. Take $T^{sk} \supseteq T$ to be an expansion of T with built-in Skolem functions. By Theorem 3.17 there is a model $\mathrm{EM}(\mathbf{I}) \models T^{sk}$ where $\mathbf{I}$ is a sequence of indiscernibles of order type $I = \eta + \lambda$ (where η is the order type of the rationals).

Let $M := \mathrm{EM}(\mathbf{I}) \restriction \mathrm{L}(T)$. Clearly $|\operatorname{Aut}(M)| \geq |\operatorname{Aut}(\mathrm{EM}(\mathbf{I}))|$. By Theorem 3.27(3) we have $|\operatorname{Aut}(\mathrm{EM}(\mathbf{I}))| \geq |\operatorname{Aut}(\langle I, <_I \rangle)| \geq \aleph_0$. □

By adding a technical twist to the choice of I is the above argument we could make the model to have 2^λ many automorphisms. See Exercise 4. For this we need the following:

PROPOSITION 3.29. *For every infinite cardinal number λ there exists a linearly ordered set $\langle P, < \rangle$ of cardinality λ such that $|\operatorname{Aut}(\langle P, < \rangle)| = 2^\lambda$.*

PROOF. Let the order type of $\langle P, < \rangle$ be $(1 + \eta) \cdot \lambda$ namely let

$$P := \lambda \times \{q \in \mathbb{Q} \mid 0 \leq q < 1\}$$

and the relation $<$ is the lexicographic ordering on P.

Consider

$$F := \{g : \lambda \to 2 \mid \forall \delta < \lambda [\delta \text{ limit } \to f(\delta) = 0\}.$$

Clearly $|F| = 2^\lambda$. For every $f \in F$ let $g_f \in \operatorname{Aut}(\langle P, < \rangle)$ be defined as follows: Given $\langle \alpha, q \rangle \in P$ let $\delta \leq \alpha$ be the unique limit ordinal and n be the unique natural number such that $\alpha = \delta + n$.

$$g_f(\alpha, q) := \begin{cases} \langle \delta + 2m, 2q \rangle & \text{if .....} \\ \\ \langle \delta + 2m + 2, 2(q - \frac{1}{2}) \rangle & \text{otherwise.} \end{cases}$$

□

Another very important corollary (due to Morley) will be used in Section 7 to omit types:

THEOREM 3.30 (omitting types). *Suppose M has built-in Skolem functions and let $T := \mathrm{Th}(M)$. Let $\{p_n \in \mathrm{D}_n(T) \mid n < \omega\}$ be a fixed set of types all realized in M such that $p_{n+1} \subseteq p_n$ for all $n < \omega$.*

Let $\{c_n \mid n < \omega\}$ be new constants let T' be a theory in $\mathrm{L}(M) \cup \{c_n \mid n < \omega\}$ containing

- *T and*
- *$\{p_n(c_0, \ldots, c_{n-1}) \mid n < \omega\}$*

If $N' \models T'$ and $N := \mathrm{EM}(\{c_n^{N'} \mid n < \omega\})$ then

$$[M \text{ omits } p \implies N \text{ omits } p] \quad \text{for every } \mathrm{L}(T)\text{-type } p.$$

PROOF. For the sake of contradiction suppose that $p(\mathbf{y})$ is a type omitted by M and realized by $\mathbf{a} \in |N|$. Since N is the Skolem hull of $\{c_n^{N'} \mid n < \omega\}$ there exists a sequence of terms $\langle \tau_0(x_0, \ldots, x_{n-1}), \ldots, \tau_{k-1}(x_0, \ldots, x_{n-1}) \rangle$ in $\mathrm{L}(T)$ such that $k = \ell(\mathbf{y}) = \ell(\mathbf{a})$ and $\mathbf{a} = \tau(c_0, \ldots, c_{n-1})$.

Since $\mathbf{a} \models p(\mathbf{y})$ we have that

$$N \models \varphi(\tau(c_0, \ldots, c_{n-1}) \quad \text{for all } \varphi(\mathbf{y}) \in p.$$

Since $N \models p_n(c_0, \ldots, c_{n-1})$ and there exists $\mathbf{b} = \langle b_0, \ldots, b_{n-1} \rangle \in M$ realizing p_n. We get that

$$N \models \varphi[\tau(b_0, \ldots, b_{n-1}] \quad \text{for all } \varphi(\mathbf{y}) \in p.$$

Thus $\mathbf{b} \models p$ a contradiction to assumption. □

Often the previous theorem is used to construct large models that omit

COROLLARY 3.31 (Sufficient condition for omitting types). *Let T be a theory with built-in Skolem functions and suppose that Γ is a set of T-types. If there exists $M \models T$ such that $M = \mathrm{EM}(\mathbf{I})$ when $\mathbf{I}$ is an infinite sequence of indiscernibles and M omits p for all $p \in \Gamma$ then there are models of T or arbitrary large cardinality omitting all the types from Γ.*

PROOF. Stretch $\mathbf{I}$ to $\mathbf{J}$ and apply Theorem 3.27(4). □

The following is a generalization of the notion of indiscernible sequence obtained from Definition 3.4 by removing two of the universal quantifiers. It is interesting to notice that this form is essentially the form of indiscernibles used by Ramsey in his original article (where he introduced the theorem that started the field of Ramsey theory) [**?**].

DEFINITION 3.32. Let L be a similarity type, M an L structure and $\langle I, <_I \rangle$ a linearly-ordered set. Suppose $A \subseteq |M|$ and $\mathbf{J} := \langle \mathbf{a}_i \mid i \in I \rangle \subseteq |M|$, where $\ell(\mathbf{a}_i) = n < \omega$ for all $i \in I$. Let $n_1 < \omega$ and $\Delta^* \subseteq \mathrm{Fml}(\mathrm{L}(M))$ be given. $\mathbf{J}$ is an (n_1, Δ^*) *sequence of indiscernibles over A in M* iff every $i_0, <_I \ldots <_I i_{n_1-1} \in I$ and every $j_0 <_I \ldots <_I j_{n_1-1} \in I$ we have that

$$\mathrm{tp}_{\Delta^*}(\mathbf{a}_{i_0}, \ldots, \mathbf{a}_{i_{n_1-1}}/A, M) = \mathrm{tp}_{\Delta^*}(\mathbf{a}_{j_0}, \ldots, \mathbf{a}_{j_{n_1-1}}/A, M).$$

When $A = \emptyset$ we delete A.

The notion of (n_1, Δ^*) *set of indiscernibles over A in M* is defined similarly to Definition 3.6.

CLAIM 3.33.

PROOF. □

THEOREM 3.34 (Skolem hulls of well ordered sets are stable).

PROOF. □

THEOREM 3.35. *Let T be a complete theory. Suppose A and $\mathbf{I}$ are subsets of the monster model such that $\mathbf{I}$ is an infinite sequence of indiscernibles over the set A and M is a model containing A. If M is $|A|^+$-saturated then there exists an infinite sequence of indiscernibles $\mathbf{J} \subseteq M$ over the set A such that for every finite $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1} \in \mathbf{J}$ there are $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1} \in \mathbf{I}$ such that*

$$\mathrm{tp}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}/A) = \mathrm{tp}(\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}/A).$$

PROOF. □

THEOREM 3.36. *Let $\mathbf{I} := \langle \mathbf{a}_\alpha \mid \alpha < \omega + \omega \rangle$ be a sequence of indiscernibles over a set A. Then $\langle \mathbf{a}_{\omega+n} \mid n < \omega \rangle$ is an indiscernible sequence over $A \cup \{\mathbf{a}_n \mid n < \omega\}$.*

PROOF. See exercise 12. □

Another useful fact about sequence of indiscernibles is

THEOREM 3.37 (extending sequences of indiscernibles). *Let T be a complete theory. If $\mathbf{I} = \{\mathbf{a}_i : i < \alpha\}$ is a sequence of indiscernibles from the monster model then for every limit $\beta > \alpha$ there exists a sequence $\{\mathbf{a}_j : \alpha \leq j < \beta\}$ such that $\{\mathbf{a}_i : i < \beta\}$ is a sequence of indiscernibles.*

PROOF. Let $\{\mathbf{c}_i : i < \beta\}$ be new constants and consider the theory

$$T^* := T \cup \{\varphi(\mathbf{c}_{i_0}, \ldots, \mathbf{c}_{n-1}) \mid \varphi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T)), \varphi(\mathbf{x}) \in \mathrm{tp}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}), n < \omega\}.$$

As for any finite subset of T^* we can use finitely many of the $\mathbf{a}_n$'s to make an expansion of $\mathfrak{C}$ to be a model of that finite theory, by the compactness theorem there exists $N^* \models T^*$. Let $\mathbf{b}_i^* := \mathbf{c}_i^{N^*}$. Define a function f from $\{\mathbf{b}_i^* \mid i < \alpha\}$ to $\{\mathbf{a}_i \mid i < \alpha\}$ given by $f(\mathbf{b}_i^*) = \mathbf{a}_i$.

By indiscenibility and the property of the $\mathbf{b}_i^*$s f is an $(N^* \restriction \mathrm{L}(T), \mathfrak{C})$-mapping. By universality of $\mathfrak{C}$ let $g \in \mathrm{Aut}(\mathfrak{C})$ be an extension of f. Now for all $i < \beta$ let $\mathbf{b}_i := f(\mathbf{b}_i^*)$, check that it has the required properties.

□

COROLLARY 3.38 (extending sequences of indiscernibles over sets). *Let T be a complete theory. If $\mathbf{I} = \{\mathbf{a}_i : i < \alpha\}$ is a sequence of indiscernibles over a set A then for every limit $\beta > \alpha$ there exists a sequence $\{\mathbf{a}_j : \alpha \leq j < \beta\}$ such that $\{\mathbf{a}_i : i < \beta\}$ is a sequence of indiscernibles over the set A.*

PROOF. Exercise. □

The following is an interesting and a useful observation:

PROPOSITION 3.39. *Let M be a given infinite structure, $\lambda > |\mathrm{L}(M)|$. Suppose that M is λ-homogeneous. For every $A \subseteq |M|$ of cardinality $< \lambda$, if $\mathbf{I} := \{\mathbf{a}_n \mid n < \omega\} \subseteq |M|$ is a sequence of indiscernibles over the set A then there exist a sequence of indiscernibles $\mathbf{J} \subseteq |M|$ extending $\mathbf{I}$ of cardinality λ.*

PROOF. Exercise 13. □

The following is a useful technical fact connecting indiscernibles and the monster model:

PROPOSITION 3.40. *Let $\mathbf{I} = \{\mathbf{a}_n \mid n < \omega\}$ be an indiscernible sequence over A. If $\mathbf{b} \models \mathrm{tp}(\mathbf{a}_0.A)$ then there exists an indiscernible $\mathbf{J} = \{\mathbf{b}_n \mid n < \omega\}$ over A such that $\mathbf{b}_0 = \mathbf{b}$ and*

$$\mathrm{tp}(\mathbf{a}_0 \dots \mathbf{a}_n/A) = \mathrm{tp}(\mathbf{b}_0 \dots \mathbf{b}_n/A) \text{ for all } n < \omega.$$

PROOF. Since $\mathrm{tp}(\mathbf{b}/A) = \mathrm{tp}(\mathbf{a}_0/A)$ there exists $f \in \mathrm{Aut}_A(\mathfrak{C})$ such that $f(\mathbf{a}_0) = \mathbf{b}$. Verify that $\mathbf{b}_n := f(\mathbf{a}_n)$ is as required. □

Ehrenfeucht and Mostowski's notion of indiscernible sequence have found interesting applications in set theory.

First we introduce some terminology and a new partition relation. Recall from 1.**??** that for a linearly-ordered set X and $n < \omega$ the set of subsets of X of cardinality n can be identified with the increasing sequences of length n from X and this set is denoted by $[X]^n$. When X is a set of ordinals and α an ordinal let $[X]^\alpha$ stand for the set of increasing sequences of elements of X of order-type α. By $[X]^{<\alpha}$ denote the set $\bigcup_{\beta<\alpha}[X]^\beta$. So in particular $[X]^{<\omega}$ stands for the finite increasing sequences of elements from X.

We write

$$\lambda \to (\mu)^{<\alpha}_\kappa$$

for statement for every (coloring) $F : [\lambda]^{<\alpha} \to \kappa$ there exists $X \subseteq \lambda$ of cardinality μ such that F is constant on $[X]^\beta$ for all $\beta < \alpha$.

Similarly,

$$\lambda \to (\mu)^{\alpha}_\kappa$$

stands for statement for every (coloring) $F : [\lambda]^\alpha \to \kappa$ there exists $X \subseteq \lambda$ of cardinality μ such that F is constant on $[X]^\alpha$.

Here is now a sharp weakening of the previous relation:

$$\lambda \to [\mu]^{\alpha}_\kappa$$

stands for statement: for every $F : [\lambda]^\alpha \to \kappa$ there exists $i < \kappa$ and there exists $X \subseteq \lambda$ of cardinality μ such that $i \notin F"[X]^\alpha$.

REMARK 3.41. Notice that for $\kappa = 2$ we have that

$$\lambda \to [\mu]^\alpha_\kappa \iff \lambda \to (\mu)^\alpha_\kappa.$$

For $\kappa \geq 3$ the partition relation $\lambda \to [\mu]^\alpha_\kappa$ is much weaker than $\lambda \to (\mu)^\alpha_\kappa$.

The negation of the square bracket relation is of particular interest, in part due to its connection with the existence of Jónsson's algebras:

PROPOSITION 3.42. *For $\lambda > \aleph_0$ the relation $\lambda \not\to [\lambda]^n_\lambda$ for a positive $n < \omega$ implies existence of a Jónsson Algebra of cardinality λ.*

PROOF. Let $F : [\lambda]^n \to \lambda$ a witness to the above partition relation. It is immediate form our definitions that $M := \langle \lambda, F \rangle$ is a Jonsson Algebra. □

It is not hard to show that under GCH, Jonsson Algebras exists in every successor cardinal:

PROPOSITION 3.43. *If $2^\mu = \mu^+$ then*

$$(1) \quad \mu^+ \not\to [\mu^+]^2_{\mu^+}.$$

PROOF. □

REMARK 3.44. It is immediate from the definitions that $\lambda \not\to [\mu]^\alpha_\kappa$ for any $\kappa \geq 2$ implies that $\lambda \not\to (\mu)^\alpha_2$.

Notice that the relation $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$ is a strengthening of Sierpinsk's "anti-Ramsey" result (Theorem 1.4.13).

Todorcevic [] established that (1) holds for μ regular ? in ZFC, his method is an important set-theoretic argument outside the scope of this book. ?!

The issue of existence of Jónsson algebra was introduced in this book in Exercise 1.3.39, it is relatively easy to find such an algebra of cardinality $\aleph_n$ for all $n < \omega$. For a long period the best known result for $\lambda < \aleph_\omega$ was Proposition 3.43 which uses GCH, in 1984 Tryba and Woodin (independently) found a ZFC proof for existence of a Jónsson Algebra in λ if $\lambda = \mu^+$ and μ is regular and little more. See Theorem 2.13.13. Short time Later Todorcevic obtained the ultimate: A proof that (1) holds in ZFC whenever μ is regular.

THEOREM 3.45 (Erdős and Rado). *If there exists a cardinal λ such that $\lambda \to (\aleph_0)^\omega_2$ then AC fails.*

PROOF. We think of $[\omega]^\omega$ as the set of increasing sequences of natural numbers. For $\eta \neq \nu \in [\omega]^\omega$ define

$$\eta \equiv \nu \iff \exists n < \omega \forall k [k \geq n \to \eta[k] = \nu[k]].$$

Define coloring F using representatives ... □

We prove a much stronger statement:

THEOREM 3.46 (Erdős, Hajnal, Rado and Maté). *It is a theorem of ZFC that for every pair of cardinals $\lambda \geq \kappa \geq \aleph_0$*

$$(*)_{\lambda,\kappa} \quad \lambda \not\to [\kappa]^{\kappa}_{2^\kappa} \quad \textit{holds.}$$

PROOF. We think of $[\lambda]^\kappa$ as the collection of all subsets of λ of cardinality κ.

First we establish $(*)_{\lambda,\kappa}$ for $\lambda = \kappa$ and than derive from it that $(*)_{\lambda,\kappa}$ holds for all $\lambda > \kappa$.

Fix $\langle X_i \mid i < 2^\kappa \rangle$ an enumeration of $[\kappa]^\kappa$ such that every element of $[\kappa]^\kappa$ appears 2^κ-many times.

For every $i < 2^\kappa$ pick $Y_i \in [X_i]^\kappa$ such that $Y_j \cap Y_i = \emptyset$ for all $j < i$ (this is possible as $[X_i]^\kappa$ contains 2^κ-many elements).

Now for every $Y \in [\kappa]^\kappa$ define

$$F(Y) := \begin{cases} 0 & \text{if } \forall i < 2^\kappa [Y \neq Y_i] \\ \\ ordertype(\{j < i \mid X_j = Y_i\}, \in) & \text{otherwise.} \end{cases}$$

For any $X \in [\kappa]^\kappa$ and every $i < 2^\kappa$ let α be such that $X = X_\alpha$ and $ordertype((\{j < \alpha \mid X_j = Y_i\}, \in) = i$. Namely, $F(Y_\alpha) = i$. So we established that for every $X \in [\kappa]^\kappa, i < 2^\kappa$ there exists $Y \subseteq X$ such that $(Y) = i$. I.e. the coloring F of $[\kappa]^\kappa$ witnesses that $\kappa \not\to [\kappa]^\kappa_{2^\kappa}$.

For $\lambda \geq \kappa$: Let $\mathcal{F} \subseteq [\lambda]^\kappa$ a family of sets satisfying

$$X \neq Y \in \mathcal{F} \implies |X \cap Y| < \kappa \text{ and } (\forall X \in [\lambda]^\kappa - \mathcal{F})\ (\exists Y \in \mathcal{F})[|X \cap Y| = \kappa].$$

Such $\mathcal{F}$ exists by Zorn's Lemma.

For every $X \in \mathcal{F}$ fix F_X be a coloring witnessing $(*)_{\kappa,\kappa}$.

We are ready to define $F : [\lambda]^\kappa \to 2^\kappa$: Given $Y \in [\lambda]^\kappa$

$$F(Y) := \begin{cases} 0 & \text{if } \neg(\exists X \in \mathcal{F})[Y \subseteq X] \\ \\ f_X(Y) & \exists X \in \mathcal{F}[Y \subseteq X]. \end{cases}$$

As $|Y| = \kappa$ by the property of $\mathcal{F}$ if there exists $X \in \mathcal{F}$ such that $Y \subseteq X$ then it must be unique, thus F is a well defined coloring.

Given any $H \in [\lambda]^\kappa$ by maximality of $\mathcal{F}$ there exists $X \in \mathcal{F}$ such that $|S \cap X| = \kappa$. Thus

$$F(Y) = F_X(Y) \text{ for all } Y \in [H \cap X]^\kappa.$$

By the choice of F_X we have $F_X"[H \cap X]^\kappa = 2^\kappa$, so together with the previous observation we get

$$F"[H]^\kappa \supseteq F"[H \cap X]^\kappa = F_X"[H \cap X]^\kappa = 2^\kappa.$$

□

As we work in ZFC the above partition relation is interesting only for $\alpha \leq \omega$.

DEFINITION 3.47. For a cardinal μ, the *μ-Erdős cardinal* is defined as

$$\kappa(\mu) := \min\{\lambda \mid \lambda \to (\mu)^{<\omega}_2\}.$$

DEFINITION 3.48. A cardinal $\lambda \geq \aleph_0$ is called *Ramsey* iff

$$\lambda \to (\lambda)^{<\omega}_2.$$

PROPOSITION 3.49.

$$\lambda \to (\mu)^{<\omega}_2 \iff \lambda \to (\mu)^{<\omega}_\kappa$$

for κ ....

It is not difficult to show that $\aleph_0$ is not a Ramsey cardinal (Exercise **??**). Clearly a Ramsey cardinal must be weakly-compact (recall Definition **??**). Thus it is impossible to find a ZFC proof of existence of Ramsey cardinals. What about the existence of Erdős cardinals?

THEOREM 3.50 (Silver). *Let $\mu \geq \aleph_0$ be a given cardinal and $\lambda \geq \mu$. The following are equivalent.*

(1) $\lambda \to (\mu)^{<\omega}_2$.
(2) *For any L-structure M when L is at most countable, if $\|M\| \geq \lambda$ then there exists $I \subseteq |M|$ of cardinality μ which is an indiscernible sequence over $\emptyset$.*
(3)
(4)

LEMMA 3.51. *Silver's Lemma characterizing Erdős card in terms of existence of indiscernible seq*

PROOF. □

DEFINITION 3.52. $0^\#$

THEOREM 3.53 (Kunen). *If there exists λ satisfying*

$$\lambda \to (\omega_1)^{<\omega}_2$$

then $0^\#$ exits

THEOREM 3.54 (Kunen). *$0^\#$ exits iff there exists a non-trivial automorphism of the constructible universe.*

Exercises 4.3.

(1) Let $M \preceq N$. If $\mathbf{I} \subseteq |M|$ is a sequence of indiscernibles in M then also $\mathbf{I}$ is a sequence of indiscernibles in N.
(2) Let $M \preceq N$ and $A \subseteq |M|$. Suppose $\{\mathbf{a}_n \mid n < \omega\} \subseteq |N|$ is a sequence of indiscernibles over A. If M is $(|A| + |\operatorname{L}(T)| + \aleph_0)^+$-saturated then there exists $\mathbf{J} = \{\mathbf{b}_n \mid n < \omega\} \subseteq |M|$ an indiscernible sequence over A such that for every $i_1 < \ldots i_n < \omega$ and every $j_1 < \ldots < j_n < \omega$ we have that

$$\operatorname{tp}(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}/A, N) = \operatorname{tp}(\mathbf{b}_{j_1}, \ldots, \mathbf{b}_{j_n}/A, M).$$

(3) Suppose T has an infinite model. Show that for every $\lambda \geq |\operatorname{L}(T)| + \aleph_0$ there exists $M \models T$ of cardinality λ such that there exists $N \precneqq M$ isomorphic to M. Hint: $\langle \omega, < \rangle \cong \langle \omega - \{0\}, < \rangle$.
(4) Show that if T admits an infinite model then there exists $M \models T$ of cardinality λ such that $|\operatorname{Aut}(M)| = 2^\lambda$ for every $\lambda \geq |T| + \aleph_0$.
(5) For every infinite structure M such that $\|M\| \geq |\operatorname{L}(M)|$ there exists a proper elementary extension N and a chain $\{N_k \mid k < \omega\}$ such that $N = N_0$, $N_k \succ N_{k+1}$ for all $k < \omega$ and $N \cong N_k$.
(6) Now using the solution of Exercise 3, we can obtain a much stronger version of the assertion of Exercise 3.

    For every infinite structure M such that $\|M\| \geq |\operatorname{L}(M)|$ there exists a proper elementary extension N and a chain $\{N_k \mid k < \omega\}$ such that $N = N_0$, $N_k \succ N_{k+1}$ for all $k < \omega$ and $M = \bigcap_{k<\omega} N_k$.
(7) Let $\lambda > \aleph_0$. Suppose that M is a λ-saturated structure. Show that for every non-algebraic $\varphi(\mathbf{x}; \mathbf{a})$ with parameters from M there exists a sequence of indiscernibles $\mathbf{I} \subseteq \varphi(M; \mathbf{a})$ of cardinality λ.
(8) Is it possible to remove the saturation assumption from the previous exercise?
(9) Suppose that T is a first-order theory such that $\operatorname{L}(T)$ contains a binary relation $<$ and there exists $M \models T$ such that M has an infinite subset linearly ordered by $<^M$. Prove that for every infinite cardinal $\lambda > |\operatorname{L}(T)|$ the theory T is not λ-categorical.
(10) Let $\lambda > \aleph_0$. Suppose that M is a λ-saturated structure. Show that for every non-algebraic type p with a set of parameters from M of cardinality $< \lambda$ there exists a sequence of indiscernibles $\mathbf{I} \subseteq p[M]$ of cardinality λ.
(11) Prove the following:

    THEOREM 3.55. *$a \notin \operatorname{acl}_{\mathfrak{C}}(A) \iff$ there exists an infinite indiscernible sequence over A containing a.*

(12) Let $\langle I, <_I \rangle$ be an infinite linearly ordered set and suppose $s \in I$ such that $\{t \in I \mid s <_I t\}$ is infinite. If $\langle \mathbf{a}_t \mid t \in I \rangle$ is a sequence of indiscernibles over A then $\langle \mathbf{a}_t \mid t \in I \wedge t \geq s \rangle$ is an indiscernible sequence over the set $A \cup \{\mathbf{a}_t \mid t \in I \wedge t <_I s\}$.

(13) Prove Proposition 3.39. Hint: Use the method of the proof of Theorem 2.24.

(14) If M is a saturated model then $|\operatorname{Aut}(M)| = 2^{\|M\|}$.

(15) Let T be a complete theory $\varphi(\mathbf{x};\mathbf{y})$ is an $\mathrm{L}(T)$-formula, and suppose $M \models T$. A set of formulas $p(\mathbf{x})$ is *k-contradictory* iff for every $q \subseteq p$ we have

$$|q| = k \implies T \vdash \neg\exists\mathbf{x} \bigwedge q(\mathbf{x}).$$

Let $\mathbf{I} \subseteq M$ be a sequence of indiscernibles and $p := \{\varphi(\mathbf{x};\mathbf{a}) \mid \mathbf{a} \in \mathbf{I}\}$. Show if p is not a T-type then there exists $k < \omega$ such that p is k-contradictory.

(16) Let T be a theory, $\varphi(\mathbf{x};\mathbf{y})$ an $\mathrm{L}(T)$-formula and $M \models T$. If there exists an infinite $\mathbf{I} \subseteq {}^{\ell(\mathbf{y})}|M|$ such that $\{\varphi(\mathbf{x};\mathbf{a}) \mid \mathbf{a} \in \mathbf{I}\}$ is k-contradictory for some $k < \omega$ then for every $\lambda \geq \aleph_0$ there exists $N \models T$ with an infinite indiscernible sequence $\{\mathbf{a}_i \mid i < \lambda\} \subseteq {}^{\ell(\mathbf{y})}|N|$ such that $\{\varphi(\mathbf{x};\mathbf{a}_i) \mid i < \lambda\}$ is k-contradictory.

(17) If $\langle I, <_I\rangle$ is a saturated structure and $\mathbf{I} := \langle a_i \mid i \in I\rangle$ is an indiscernible sequence then $\mathrm{EM}(\mathbf{I})$ is saturated.

(18) Let L be a similarity type, M be an infinite L-structure with buit-in Skolem functions. Suppose $\lambda \geq |L| + \aleph_0$, $I := \langle \lambda^+, \in\rangle$. Suppose $\mathbf{I} := \langle a_i \mid i \in I\rangle$ is an indiscernible sequence in M. For every $\alpha \leq \lambda^+$ let $M_\alpha := \mathrm{EM}(\{a_i \mid i < \alpha\})$.

If there are limit $\delta_1 < \delta_2 < \lambda^+$ such that .... then $|M_{\lambda^+}| - |M_{\delta_2}|$ ....

(19) If $\mathbf{I}$ is an infinite indiscernible sequence over A then $\mathbf{I}$ is an indiscernible sequence over $\operatorname{acl}(A)$

(20) This is a straightening of Exercise 2.51. Let T be complete and $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T)$. Suppose that there exists $\{\mathbf{a}_n \mid n < \omega\} \subseteq |\mathfrak{C}|$ such that

$$\mathfrak{C} \models \varphi[\mathbf{a}_n;\mathbf{a}_k] \iff n < k \text{ for every } n, k < \omega.$$

Prove that for every linearly ordered set I there is an indiscernible sequence $\mathbf{I} = \{\mathbf{b}_i \mid i \in I\} \subseteq |\mathfrak{C}|$ such that

$$\mathfrak{C} \models \varphi[\mathbf{b}_i;\mathbf{b}_j] \iff i <_I j \text{ for every } i, j \in I.$$

(21) Suppose that $\varphi(\mathbf{x};\mathbf{y})$ has the independence property. Prove that for every infinite I there are M and an indiscernible sequence $\mathbf{I} := \{\mathbf{a}_i \mid i \in I\} \subseteq {}^{\ell(\mathbf{y})}|M|$ such that for every $w \subseteq I$ there exists $\mathbf{b}_w \in {}^{\ell(\mathbf{x})}|M|$ satisfying

$$M \models \varphi[\mathbf{b}_w;\mathbf{a}_i] \iff i \in w.$$

(22) Let $\langle I_\ell, < \ell\rangle$ be infinite linearly ordered sets. Suppose $\mathbf{I}_\ell := \langle a^\ell_i \mid i \in I\rangle$ are indiscernible sequences. Prove that

$$\langle I_1, <_1\rangle \equiv_{\infty,\lambda} \langle I_2, <_2\rangle \implies \mathrm{EM}(\mathbf{I}_1) \equiv_{\infty,\lambda} \mathrm{EM}(\mathbf{I}_2).$$

(23) $\aleph_0$ is not a Ramsey cardinal.

(24) Suppose that $M \precneqq N$ and $A \subseteq |M|$. Show that for every finite $\mathbf{b} \in |N| - |M|$ such that $\operatorname{tp}(\mathbf{b}/M)$ is not algebraic there exists an A-indiscernible sequence $\{\mathbf{a}_n \mid n < \omega\}$ such that $\mathbf{a}_0 = \mathbf{b}$.

(25) Let I be a linearly ordered set without a last element.

Suppose $\{\mathbf{a}_i : i \in I\}$ is an indiscernible sequence over A, for any strictly increasing $f : I \to I$, there exists $\bar{f} \in \mathrm{Aut}_A(\mathfrak{C})$ of such that $\bar{f}(\mathbf{a}_i) = \mathbf{a}_{f(i)}$ for all $i \in I$.

CHAPTER 5

Galois types and monster models in Abstract Elementary Classes

Introduction

While it was Lindenbaum who in the thirties discovered the notion of complete types, only in the mid fifties the notion of a consistent set of first-order formulas was discovered to be important. E.g Ryll-Nard characterization of $\aleph_0$-categoricity (Theorem 9.38), Los's perfect set argument (Theorem 7.23). The full importance of the notion of complete types was used heavely in Morley in his proof of his categoricity theorem. One very important aspect of complete (first-order types) is summerized in the following

FACT 0.1. *Suppose M is an uncountable saturated model such that $\|M\| > |\mathrm{L}(M)|$. Let $A \subseteq |M|$ and $a, b \in |M|$ be such that $|A| < \|M\|$. Then the following holds*

$$\mathrm{tp}(a/A, M) = \mathrm{tp}(b/A, M) \iff \exists f \in \mathrm{Aut}_A(M)[f(a) = b].$$

It turns out that set of formulas can not play a similar role to what they did for us in elementary classes.

Another related difficulty is with the notion of saturated model. Already in small fragments of $L_{\omega_1,\omega}$ we face the need to omit types. Notice that $M \models \neg\exists \mathbf{x}[\bigwedge_{n<\omega} \varphi_n(\mathbf{x})]$ iff M omits the type $\{\varphi_n(\mathbf{x}) \mid n < \omega\}$. Thus it does not make sense to expect to deal with even $\aleph_0$-saturated models.

The main problem is with the notion of saturated model, since the primary example of an abstract elementary class is that of $\mathcal{K} = \mathrm{EC}(T, \{p\})$ saturated models do not exist in $\mathcal{K}$. However the notion of model-homogeneous structure may exists in $\mathcal{K}$ and is well behaved. The definition of type is tailored to fit perfectly with the concept of model-homogenous structure.

It appears that a substitute involving amalgamation is more productive.

In this section we will introduce generalizations of the concepts of first-order types and first-order saturation in such a way that an analog for $(*)$ holds.

In the following section the definitions are introduced. Shelah's original concepts require little less than full amalgamation of the class (at the cost of increased complexity).

In this chapter we assume that

HYPOTHESIS 0.2. The class $\mathcal{K}$ has the amalgamation property. For many results it is enough to assume that $\mathcal{K}_\mu$ has the amalgamation property for some $\mu \geq \mathrm{LS}(\mathcal{K})$.

In the first section we introduce some of the most basic definitions.

In the second section we show that saturated models are homogeneous, that is generalizing a classical theorem of elementary classes.

In sections 3 and four we develop the basic theory of minimal types (generalizing properties of strongly minimal sets in $\aleph_0$-stable theories).

1. Types in Abstract Elementary Classes

DEFINITION 1.1 (Pre-types). Let $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$.

$$\mathcal{K}^3_\lambda := \{(a, M, N) \mid M, N \in \mathcal{K}_\lambda,\ M \leq_{\mathcal{K}} N,\ a \in |N| - |M|\}$$

an element of $\mathcal{K}^3_\lambda$ is called *pre-type*.

Shortly we will define type as an equivalence class of a pre-type.

We first define an equivalence relation on elements of $\mathcal{K}^3_\lambda$.

DEFINITION 1.2. Let (a, M, N_1) and $(b, M, N_2) \in \mathcal{K}^3_\lambda$, $(a, M, N_1) \sim (b, M, N_2)$, iff there exists $N \in \mathcal{K}_\lambda$ and $\mathcal{K}$-embeddings $h_\ell : N_\ell \underset{M}{\to} N$ such that $h_1(a) = h_2(b)$. Namely the following diagram commutes:

$$\begin{array}{ccc} N_1 & \xrightarrow{h_1} & N \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle h_2} \\ M & \xrightarrow[\mathrm{id}]{} & N_2 \end{array}$$

REMARK 1.3. By the isomorphism axioms like in (Lemma **??**) we may assume that $h_2 = \mathrm{id}_{N_2}$. I.e.

$$(a, M, N_1) \sim (b, M, N_2) \iff \exists N \in \mathcal{K}_\lambda [N \succeq_{\mathcal{K}} N_2 \wedge (\exists h : N_1 \underset{M}{\to} N) h(a) = b].$$

LEMMA 1.4. *Suppose that $\mathcal{K}_\lambda$ has the amalgamation property. For $M \in \mathcal{K}_\lambda$ the relation $\sim$ is an equivalence relation on $\mathcal{K}^3_\lambda$.*

PROOF. Reflexivity and Symmetry are clear. Let us see that Transitivity holds. Suppose that

$$(a, M, N_1) \sim (b, M, N_2) \quad \text{and} \quad (b, M, N_2) \sim (c, M, N_3).$$

By definition of $\sim$ and the first assumption there are $N \in \mathcal{K}_\lambda$ and $h : N_1 \underset{M}{\to} N$ such that $N \geq_{\mathcal{K}} N_2$ and $h(a) = b$. From the second assumption there are $N' \in \mathcal{K}_\lambda$ and $h' : N_3 \underset{M}{\to} N'$ such that $N' \geq_{\mathcal{K}} N_2$ and $h'(c) = b$.

I.e. the two diagrams

$$\begin{array}{ccc} N_1 & \xrightarrow[h]{} & N \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M & \xrightarrow[\mathrm{id}]{} & N_2 \end{array} \qquad \begin{array}{ccc} N_2 & \xrightarrow[\mathrm{id}]{} & N' \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle h'} \\ M & \xrightarrow[\mathrm{id}]{} & N_3 \end{array}$$

commute. By the λ-AP, there exists $N^* \in \mathcal{K}_\lambda$ and an embedding $f : N \underset{N_2}{\to} N^*$ such that $N^* \geq N'$ and $f(b) = c$. I.e. the following diagram commutes.

$$\begin{array}{ccc} N & \xrightarrow[f]{} & N^* \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ N_2 & \xrightarrow[h']{} & N' \end{array}$$

Pasting diagrams together, we obtain that the diagram

$$\begin{array}{ccccc} N_1 & \xrightarrow[h]{} & N & & \\ {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & \searrow^{f} & \\ M & \xrightarrow[\mathrm{id}]{} & N_2 & & N^* \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & \nearrow_{\mathrm{id}} & \\ N_3 & \xrightarrow[h']{} & N' & & \end{array}$$

is commutative.

Take $h_1 := f \circ h$ and $h_3 := h'$. Since $h_\ell : N_\ell \underset{M}{\to} N^*$ such that $h_1(a) = b = h_3(c)$ we proved that $(a, M, N_1) \sim (c, M, N_3)$. □

So what if the λ-AP does not hold? Given any symmetric and reflexive binary relation the transitive closure of that relation is an equivalence relation (see Definition 15.1.2). Let $\sim^*$ be the transitive closure of $\sim$ on $\mathcal{K}^3_\lambda$. Thus a restatement of the last lemma is that the λ-AP implies that $\sim^* = \sim$ on $\mathcal{K}^3_\lambda$.

It is now natural to set the next few definitions.

DEFINITION 1.5 (Galois types). Suppose that $\mathcal{K}$ has the λ-AP.

(1) For $(a, M, N) \in \mathcal{K}^3_\lambda$, we let
$$\text{ga-tp}(a/M, N) = (M, N, a)/\sim^* .$$
This is the *Galois-type of a over M in N*;
(2) For $M \in \mathcal{K}_\lambda$, we let
$$\mathrm{gS}(M) = \{ \text{ga-tp}(a'/M, N') \ : \ (a, M, N') \in \mathcal{K}^3_\lambda \}.$$
(3) $\mathcal{K}$ is *λ-Galois-stable* iff
$$M \in \mathcal{K}_\lambda \implies |\mathrm{gS}(M)| \leq \lambda.$$
(4) Given $p \in \mathrm{gS}(M)$ and $N \in \mathcal{K}$, we say that p is *realized* by $a \in N$, if there exists $N' \geq_{\mathcal{K}} N$ of cardinality λ containing $|M| \cup \{a\}$ such that $(a, M, N') \in \mathcal{K}^3_\lambda$ and $(a, M, N') \in p$.
Like in the first-order case we will write $a \models p$ for a is a realization of p.

(5) Let $M^* \in \mathcal{K}$ and suppose $(a, M, N) \in \mathcal{K}^3_\lambda$ such that $M \leq M^*$. We say that (a, M, N) *is strongly realized in* M^* provided there exists $N' \leq M^*$ and $b \in N' - M$ such that $(a, M, N) \sim (b, M, N')$.

Unlike in the first-order case we don't have a natural topology on $\mathrm{gS}(M)$ but still it has many interesting properties.

Our next definition may look at first as a delirium:

DEFINITION 1.6. Let $\lambda > \mathrm{LS}(\mathcal{K})$ and $M^* \in \mathcal{K}_{\geq \lambda}$. We say that M^* is *λ-Galois-saturated* iff for every $M \leq_{\mathcal{K}} M^*$ of cardinality $< \lambda$ every $p \in \mathrm{gS}(M)$ is realized in M^*.

M^* is *λ-strongly-Galois-saturated* if for every $M \prec M^*$ such that $\|M\| < \lambda$ and every $(a, M, N) \in \mathcal{K}^3_\mu$ (where $\mu := \|N\|$) there are $N' \prec M^*$ of cardinality μ and $b \in N' - M$ such that $(a, M, N) \sim (b, M, N')$.

Recall

DEFINITION 1.7. Let $M \in \mathcal{K}$ be of cardinality $> \mathrm{LS}(\mathcal{K})$. We say that M is *model homogenous* iff for every $N \leq \mathcal{K} M$ and any $N' \geq \mathcal{K} N$ of cardinality $< \|M\|$ there exists $f : N' \underset{N}{\to} M$.

LEMMA 1.8. *Let $\mathcal{K}$ be an AEC. If $M \in \mathcal{K}_{>\mathrm{LS}(\mathcal{K})}$ is model homogenous then for any $N <_{\mathcal{K}} M$ and any $\mathcal{K}$-embedding $h : N \to N'$ such that $N' \in \mathcal{K}_{<\|M\|}$ there exists a $\mathcal{K}$-embedding $f : N' \to M$ such that $f \supseteq h^{-1}$.*

PROOF. Use the isomorphism axioms. □

It is a corollary of fact of field theory that any uncountable algebraically closed is model homogenous when $\mathcal{K}$ is a class of fields and $\leq_{\mathcal{K}}$ is the subfield relation. This follows from the following extension theorem.

THEOREM 1.9. *Suppose E is an uncountable algebraically closed field. For any $F_0 \leq E$ and any $F_1 \geq F_0$ such that $|F_1| < |E|$ there exists an injective field homomorphism $f : F_1 \to E$ such that $f \restriction F_0 = \mathrm{id}_{F_0}$.*

PROOF. Similar to the proof of Theorem I.2.38. □

Using the compactness theorem it is not difficult to establish

FACT 1.10. *Let M be an uncountable structure such that $\|M\| > |\mathrm{L}(M)|$, $T := \mathrm{Th}(M)$ and $\mathcal{K} = \mathrm{Mod}(T)$ where $\preceq$ is the usual notion of elementary substructure. Then the following are equivalent*

(1) *M is saturated.*
(2) *M is model homogenous.*

THEOREM 1.11 (characterization of model homog). *Let $\lambda > \mathrm{LS}(\mathcal{K})$ and $M \in \mathcal{K}$. M is λ-model homoegeous iff for every $\mu < \lambda$, every $N \preceq_{\mathcal{K}} M$ of cardinality μ and every $N' \succeq_{\mathcal{K}} N$ of cardinality μ there exists $f : N' \xrightarrow[N]{} M$.*

PROOF. The direction $\Longrightarrow$ is trivial.

For the other direction. Argue by induction of $\|N'\|$: For $\|N'\| = \mu$ this is the assumption.

For $\|N'\| > \mu$, using Theorem 1.2.9 pick a resolution of N': $\{N'_\alpha \mid \alpha < \|N'\|\} \subseteq \mathcal{K}_{<\|N'\|}$ such that $N' = \bigcup_{\alpha<\|N'\|} N'_\alpha$ and apply the induction hypothesis to define an increasing chain $f_\alpha : N'_\alpha \to M$ such that $f_0 = \mathrm{id}_N$. Finally the function $f := \bigcup_{\alpha<\|N'\|} f_\alpha$ is as required (using Exercise 2.2.7). □

2. Galois saturation and model-homogenuity are the same

The following Theorem came as a great surprise to me convincing me that the notion of Galois-types is the correct one. The original statement with a flawed proof appeared in [**Sh 300**]. The first correct proofs appeared years later in [**Sh 576**] and in [**Gr2**].

THEOREM 2.1. *Suppose $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is an AEC, $\lambda > \mathrm{LS}(\mathcal{K})$ and $\mathcal{K}_{<\lambda}$ has the amalgamation property. Let $M^* \in \mathcal{K}$ be given. The model M^* is λ-Galois-saturated iff M^* is λ-model homogenous.*

REMARK 2.2. While this is a generalization of Theorem 1.4, the proof of the non-trivial direction presented in section 1 can't be used here as we can't handle arbitrary sets of elements (only models). Thus an entirely different argument is needed.

PROOF. For the direction $\Longleftarrow$ the argument is essentially like the first-order: Suppose M^* is λ-model-homogenous. Let $N \preceq_{\mathcal{K}} M^*$ of cardinality $< \lambda$ let $p \in \mathrm{gS}(N)$ be given. Pick $(a, N, N_1) \in p$. Since M^* is model-homogenous there exist a $\mathcal{K}$-embedding $f : N_1 \underset{N}{\to} M^*$. Check that $f(a) \models p$.

For the direction $\Longrightarrow$ the standard first-order argument fails, as it requires considering types over arbitrary sets (not only over models). We will do something drastically different.

Suppose $N \leq_{\mathcal{K}} M^*$ and $N' \geq_{\mathcal{K}} N$ are given such that $\mu := \|N'\| < \lambda$. Using Theorem 1.11 we may assume that $\|N\| = \mu$. Fix $\{a_i \mid i < \mu\} = |N'|$. By induction on $i < \mu$ define $\langle N^i_\ell, f_i \mid i \leq \mu, \ell \in \{1, 2\} \rangle$ increasing and continuous such that

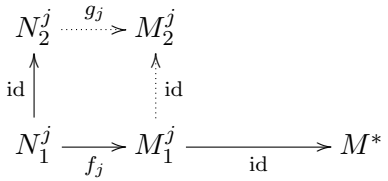
(1) $N^0_1 = N$, $N^0_2 = N'$, $f_0 = \mathrm{id}_N$;
(2) $N^i_1 \leq_{\mathcal{K}} N^i_2$ and $\|N^i_2\| = \mu$;
(3) $f_i : N^i_1 \to M^*$;
(4) $a_i \in N^{i+1}_1$.

By the limit axioms we have that $N^\mu_1 \leq_{\mathcal{K}} N^\mu_2$, $N' \leq_{\mathcal{K}} N^\mu_2$ and trivially $N' \subseteq N^\mu_1$, thus by coherence we get $N' \leq_{\mathcal{K}} N^\mu_1$.

Using Exercise 2.2.7 f_μ is a $\mathcal{K}$-embedding of N^μ_1 into M^*. Using the second isomorphism axiom (see Remark 1.6) and $f_\mu \supseteq f_0 = \mathrm{id}_N$ we have that $f_\mu \restriction N'$ is as required, i.e. it is a $\mathcal{K}$-embedding and $f_\mu \restriction N' : N' \underset{N}{\to} M^*$. It suffices to carry out an inductive construction as above.

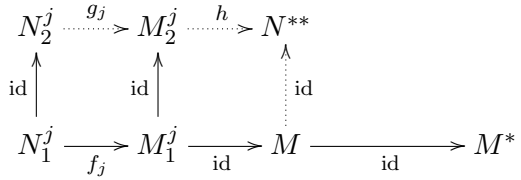
By the continuity requirement it suffices to define for $i = j + 1$. Let $M^j_1 := f_j[N^j_1]$.

By the fundamental extension Lemma (2.1.7) choose $M_2^j \geq_{\mathcal{K}} M_1^j$ and $g_j : N_2^j \cong M_2^j$ such that $g_j \supseteq f_j$ Namely the diagram

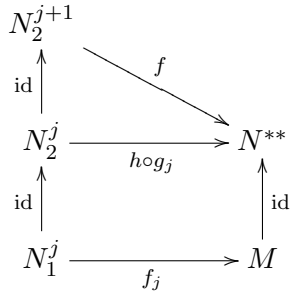


commutes.

If $a_j \in N_1^j$ then do nothing (i.e. take $f_{j+1} := f_j$, $N_1^{j+1} := N_1^j$ and $N_2^{j+1} := N_2^j$). Otherwise, consider the pre-type $(g_j(a_j), M_1^j, M_2^j)$. By saturation there are $M \leq_{\mathcal{K}} M^*$ of cardinality μ and $b \in M$ such that $(b, M_1^j, M) \sim (g_j(a_j), M_1^j, M_2^j)$. Let $N^{**} \in \mathcal{K}_\mu$ and $h : M_2^j \underset{M_1^j}{\to} N^{**}$ such that $N^{**} \geq_{\mathcal{K}} M$ and $h(g_j(a_j)) = b$. Namely the diagram

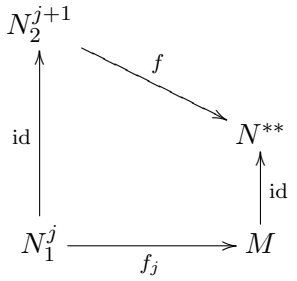


commutes. Since $h \circ g_j$ is a $\mathcal{K}$-embedding from N_2^j into N^{**} using the fundamental extension lemma, there are $N_2^{j+1} \in \mathcal{K}_\mu$ and $f : N_2^{j+1} \cong N^{**}$ such that $N_2^{j+1} \geq_{\mathcal{K}} N_2^j$ and $f \supseteq h \circ g_j$. This translates to the following diagram:



Commutes. Let $N_1^{j+1} := f^{-1}[M]$. Since $M \leq_{\mathcal{K}} N^{**}$ by Remark 2.1.6 we get that $N_1^{j+1} \leq_{\mathcal{K}} N_2^{j+1}$.

Since $f(a_j) = b$ and $b \in M$ we get that $a_j \in N_1^{j+1}$ and from $f \supseteq f_j$ and the fact that



commutes we get that $N_1^{j+1} \geq_{\mathcal{K}} N_1^j$. Now take $f_{j+1} := f \restriction N_1^{j+1}$, by the previous diagram we have that $f_{j+1} \supseteq f_j$.

We still need to show that $f_{j+1} : N_1^{j+1} \to M^*$. By the choice of N_1^{j+1} we have that $f_{j+1}[N_1^{j+1}] = f[N_1^{j+1}] = M \leq_{\mathcal{K}} M^*$. □

Notice that we did not use in the above argument the amalgamation property and we have stated it to make the notion of Galois-types possible. I have chosen the above formulation based on aesthetic grounds and in order to clarify the analogy between the new notions and the classical first-order concepts.

The more general theorem is:

THEOREM 2.3. *Suppose $\langle \mathcal{K}, \leq_{\mathcal{K}} \rangle$ is an AEC and $M \in \mathcal{K}_{>\mathrm{LS}(\mathcal{K})}$. The following conditions are equivalent*

(1) *M is model homogenous;*
(2) *for every $N \leq_{\mathcal{K}} M$ of cardinality $< \|M\|$ and every $(a, N, N') \in \mathcal{K}^3_{\|N\|}$ there are $b \in M$ and $M^* \leq_{\mathcal{K}} M$ of cardinality $\|N\|$ such that $M^* \geq \{b\} \cup |N|$ and $(b, N, M^*) \sim (a, N, N')$.*

PROOF. Exercise. □

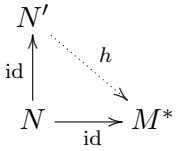
REMARK 2.4. Shelah in [**Sh 576**] calls a model $M \in \mathcal{K}_{>\mathrm{LS}(\mathcal{K})}$ satisfying property (2) of the previous theorem *strongly saturated*, in order to be consistent with our terminology we use the term *strong-Galois-Saturation*. Theorem 2.3 is stronger as we do not require the relation $\sim$ of pre-types to be a transitive relation.

3. α-limit models a substitute to saturation

The argument for the non-trivial direction from Theorem 2.1 can be used to get an important variant of first-order saturation that will be exploited in future chapters. I have decided to introduce it here not only in order to clarify the previous argument and provide you with a good exercise.

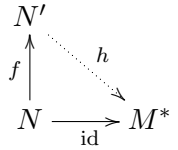
DEFINITION 3.1. . Let $\mathcal{K}$ be an AEC.

(1) Let κ be a cardinal $\geq \text{LS}(\mathcal{K})$. We say M^* is *κ-universal over N* iff for every $N' \in \mathcal{K}_\kappa$ with $N \leq_{\mathcal{K}} N'$ there exists a $\mathcal{K}$-embedding $h : N' \underset{N}{\to} M^*$ such that:



(2) We say M^* is *universal over N* or M^* is *a universal extension of N* iff M^* is $\|N\|$-universal over N.

REMARK 3.2. If M^* is universal over N then for any $N' \in \mathcal{K}_{\|N\|}$ and any $\mathcal{K}$-embedding $f : N \to N'$ there exists a $\mathcal{K}$-embedding $h : N' \underset{N}{\to} M^*$ such that:



The following is also easy to prove:

PROPOSITION 3.3. *Let $\mathcal{K}$ be an AEC and $\mu \geq \text{LS}(\mathcal{K})$. Suppose $\mathcal{K}_\mu$ has the AP. If for every $N \in \mathcal{K}_\mu$ there exists $M \in \mathcal{K}_\mu$ such that $M \geq_{\mathcal{K}} N$ and M is universal over N then $\mathcal{K}$ is μ-Galois stable.*

PROOF. Exercise. □

Surprisingly, the converse is true.

THEOREM 3.4 (Existence). *Let $\mathcal{K}$ be an AEC, $\mu \geq \text{LS}(\mathcal{K})$, $\mathcal{K}_\mu$ is without maximal models and suppose it is Galois-stable in μ. If $\mathcal{K}$ has the amalgamation property then for every $N \in \mathcal{K}_\mu$ there exists $M^* \gneq_{\mathcal{K}} N$, universal over N of cardinality μ.*

Before starting the proof we need

LEMMA 3.5. *Suppose $\mathcal{K}_\mu$ has the AP. Let $M \in \mathcal{K}_\mu$, $\Gamma \subseteq \mathrm{gS}(M)$. If $|\Gamma| \leq \mu^+$ then there exists $M' \succeq_{\mathcal{K}} M$ realizing all the types from Γ.*

PROOF. Let $\{(a_i, M, N_i) \in \mathcal{K}^3_\mu \mid i < |\Gamma|\}$ be such that for evry $p \in \mathrm{gS}(M)$ there exists $i < |\Gamma|$ satisfying $(a_i, M, N_i) \in p$. By induction of $i < |\Gamma|$ define $\{M_i \mid i < |\Gamma|\} \subseteq \mathcal{K}_\mu$ and $\{f_i : N_i \underset{M}{\to} M_{i+1} \mid i < |\Gamma|\}$to be increasing and continuous: Take $M_0 = M$ for limit i take unions and at sucessor stage apply μ-AP to get $M_{i+1} \succeq_{\mathcal{K}} M_i$ and $f_i : N_i \underset{M}{\to} M_{i+1}$. The model $M' := \bigcup_{i<\mu} M_i$ is as required.

□

PROOF OF THEOREM 3.4. Using μ-Galois stability, the NMM property and the μ-AP define $\{M_i \mid i < \mu\} \subseteq \mathcal{K}_\mu$ to be a strictly increasing and continuous $\preceq_{\mathcal{K}}$-chain such that

$$M_0 = N \text{ and } p \in \mathrm{gS}(M_i) \implies p \text{ is realized in } M_{i+1} \text{for all } i < \mu.$$

Now let $M^* := \bigcup_{i<\mu} M_i$.

CLAIM 3.6. *M^* is universal over M.*

PROOF. Let $N' \succeq_{\mathcal{K}} N$ be given of cardinality μ. Pick $\{a_i \mid i < \mu\}$ an enumeration of $|N'|$. By induction on $i < \mu$ define $\langle N^i_\ell, f_i \mid i \leq \mu, \ell \in \{1, 2\}\rangle$ increasing and continuous such that

(1) $N^0_1 = N$, $N^0_2 = N'$, $f_0 = \mathrm{id}_N$;
(2) $N^i_1 \preceq_{\mathcal{K}} N^i_2$ and $\|N^i_2\| = \mu$;
(3) $f_i : N^i_1 \to M_i$;[1]
(4) $a_i \in N^{i+1}_1$.

Clearly this is sufficient since $(\bigcup_{i<\mu} f_i) \restriction N$ is as required. By continuity it is enough to describe the construction at successor stages.

If $a_j \in N^j_1$ let $N^{j+1}_\ell := N^j_\ell$.

Otherwise, denote by $M^j_0 \prec_{\mathcal{K}} M_j$ the image of N^j_0 under f_j, let $g \supseteq f_j$ and $\overline{M}^j_1 \succ_{\mathcal{K}} M^j_0$ be such that $g : N^j_2 \cong \overline{M}^j_1$.

[1]Notice that this is a little stronger requirement than what we had in (3) of Theorem 2.1.

By the amalgamation property there are M_1^j and $\overline{g} : \overline{M}_1^j \to M_1^j$ whose restriction is the identity over M_1^j such that the diagram

$$\begin{array}{ccccccc} N_1^j & \xrightarrow{g} & \overline{M}_1^j & \xrightarrow{\overline{g}} & M_1^j & & \\ \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ N_1^j & \xrightarrow[f_j]{} & M_0^j & \xrightarrow[\mathrm{id}]{} & M_j & \xrightarrow[\mathrm{id}]{} & M^* \end{array}$$

commutes.

Since $a_j \in N_2^j - N_1^j$, $g(a_j) \in \overline{M}_1^j - M_0^j$. We can choose $\bar{g}$ such that $\overline{g}(g(a_j)) \notin M_j$. Let $p := \text{ga-tp}(\overline{g}(g(a_j))/M_j, M_1^j)$. By the construction of $\langle M_i \rangle_{i<\mu}$ there exists $b \in M_{j+1}$ realizing the type p. By definition of Galois-types we have $(M_j, M_1^j, \overline{g}(g(a_j))) \sim (M_j, M_{j+1}, b)$. Thus there are $N^{**} \in \mathcal{K}_\mu$ and mappings h_1, h_2 such that the diagram

$$\begin{array}{ccc} M_1^j & \xrightarrow{h_2} & N^{**} \\ \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle h_1} \\ M_j & \xrightarrow[\mathrm{id}]{} & M_{j+1} \end{array}$$

commutes and

$$(*) \quad h_2(\overline{g}(g_j(a_j))) = h_1(b).$$

We may assume that $h_2 = \mathrm{id}_{M_1^j}$. Thus by gluing the last two diagrams together we get that the diagram

$$\begin{array}{ccccccc} N_1^j & \xrightarrow{g \circ \overline{g}} & M_1^j & \xrightarrow{\mathrm{id}} & N^{**} & & \\ \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle h_1} & & \\ N_1^j & \xrightarrow[f_j]{} & M_j & \xrightarrow[\mathrm{id}]{} & M_{j+1} & \xrightarrow[\mathrm{id}]{} & M^* \end{array}$$

commutes. Now pick $N_2^{j+1} \succ_{\mathcal{K}} N_1^j$ and $h \supseteq g \circ \overline{g}$ such that $h : N_2^i \cong N^{**}$.

So we have that

$$\begin{array}{ccccc} N_2^{i+1} & & \xrightarrow{\quad h \quad} & & \\ \uparrow{\scriptstyle \mathrm{id}} & & & & \\ N_2^j & & \xrightarrow[g \circ \overline{g}]{} & & N^{**} \\ \uparrow{\scriptstyle \mathrm{id}} & & & & \uparrow{\scriptstyle h_1} \\ N_0^j & & \xrightarrow[f_j]{} & & M_{j+1} \end{array}$$

commutes. Let $N_1^{j+1} := h^{-1}[h_1[M_{j+1}]]$.
Notice that by the above diagram, we get that $N_1^{j+1} \supseteq N_1^j$. Now we are ready to define the mapping f_i, let $f_{j+1} := h_1^{-1} \circ (h \restriction N_1^j)$. It is a $\mathcal{K}$-embedding of N_2^{j+1} to M_{j+1} that extends f_j.

Verify that $\bigcup_{i<\mu} f_i$ is an embedding of N' over N into M^*. □

□

The following notion is central as it plays in AECs a role similar to that of saturated models.

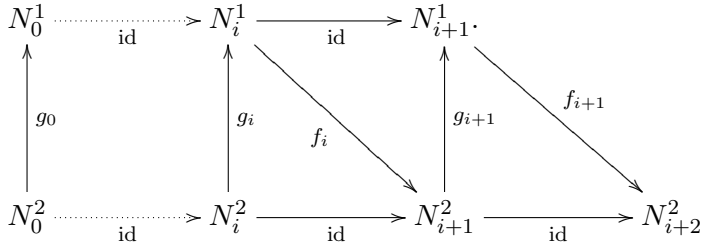
DEFINITION 3.7. Let $\mathcal{K}$ be an AEC, $\mu \geq \mathrm{LS}(\mathcal{K})$ and $\alpha < \mu^+$ a limit ordinal. For $N, M \in \mathcal{K}_\mu$ we say that M *is* α*-limit over* N provided $N \leq_{\mathcal{K}} M$ and there exists $\{N_i \mid i \leq \alpha\} \subseteq \mathcal{K}_\mu$ increasing and continuous in $\leq_{\mathcal{K}}$ such that N_{i+1} is universal over N_i for every $i < \alpha$, $N_0 = N$ and $N_\alpha = M$.

THEOREM 3.8 (Easy Uniqueness). *Let $\mathcal{K}$ be an AEC, $\mu \geq \mathrm{LS}(\mathcal{K})$ and $N \in \mathcal{K}_\mu$.*

(1) *Let $\alpha < \mu^+$ be a limit ordinal. If $M_1, M_2 \in \mathcal{K}_\mu$ are both (μ, α)-limits over N then there there exists $g : M_2 \cong_N M_{!}$.*
(2) *. Suppose $\alpha_\ell < \mu^+$ are both limit ordinals such that $\mathrm{cf}\,\alpha_1 = \mathrm{cf}\,\alpha_2$. If $M_\ell \in \mathcal{K}_\mu$ is α_ℓ-limit over N (for $\ell = 1, 2$) then there exists $g : M_2 \cong_N M_1$.*

PROOF. (1) Fix $\{N_i^\ell \mid i \leq \alpha\}$ witnessing that M_ℓ is α-limit over N.

By induction on $i \leq \alpha$ define $f_i : N_i^1 \to N_{i+1}^2$ and $g_i : N_i^2 \to N_i^1$ such that the following diagram commutes:



For $i = 0$ let $g_0 := \mathrm{id}_N$, recall that $N_0^1 = N_0^2 = N$.

For i limit let $g_i := \bigcup_{j<i} g_j$.

Given $g_i : N_i^2 \to N_i^1$ using the universality of N_{i+1}^2 over N_i^2 there exists f_i such that

$$\begin{array}{ccc} N_i^1 & & \\ {\scriptstyle g_i}\uparrow & \searrow^{f_i} & \\ N_i^2 & \xrightarrow[\mathrm{id}]{} & N_{i+1}^2 \end{array}$$

commutes.

Given $f_i : N_i^1 \to N_{i+1}^2$ using the universality of N_{i+1}^1 over N_i^1 there exists $g_{i+1} : N_{i+1}^2 \to N_{i+1}^1$ such that

$$\begin{array}{ccc} N_{i+1}^2 & & \\ {\scriptstyle f_i}\uparrow & \searrow^{g_{i+1}} & \\ N_i^1 & \xrightarrow[\mathrm{id}]{} & N_{i+1}^1 \end{array}$$

commutes.

Clearly $g := g_\alpha$ is a $\mathcal{K}$-embedding from M^2 to M^1. As $f := \bigcup_{i<\alpha} f_i$ is the inverse of f and its domain is M^1 we have that $g : M^2 \cong_N M^1$.

(2) Fix $\{N_i^\ell \leq \alpha_\ell\}$ witnessing that M_ℓ is α_ℓ-limit over N (for $\ell = 1, 2$). Let $\alpha := \operatorname{cf} \alpha_1 = \operatorname{cf} \alpha_2$. Fix a cofinal $f_\ell : \alpha \to \alpha_\ell$ such that in the case that $\alpha > \aleph_0$ then for limit $i < \alpha$ we have that both $f_\ell(i)$ (for $\ell = 1, 2$) are also limit.

Note that for $\ell = 1, 2$ we have that $\{N_{f_\ell(i)}^\ell \mid i \leq \alpha\}$ witness that M^ℓ is α-limit over N.

Apply part (1) to the above sequences.

□

It is a major and interesting problem if the above uniqueness property holds also when $\operatorname{cf} \alpha_1 \neq \operatorname{cf} \alpha_2$. We will show later[2] that for elementary classes of the form $\langle \operatorname{Mod}(T), \prec \rangle$ when T is complete, the uniqueness property for limit models is equivalent to super stability of T.

The fact that the notion of limit model is a generalization of saturated models follows from the following:

PROPOSITION 3.9. *Let M be an L-structure of cardinality $\mu \geq \aleph_0 + |L|$. Suppose that μ is regular. Consider $\langle \mathcal{K}_M, \leq_{\mathcal{K}} \rangle$, when $\mathcal{K}_M := \{N : N \equiv M\}$ and $\leq_{\mathcal{K}}$ the elementary sub model relation.*

[2]See Theorem 8.1.62.

If $N \precneqq M$ such that M is μ-limit over N then M is saturated.

PROOF. Suppose $\{N_i \mid i < \mu\}$ witnessing that M is μ-limit over N. Let $A \subseteq |M|$ of cardinality $< \mu$ and p an A-type. Let $i_0 < \mu$ be such that $A \subseteq |N_{i_0}|$. Take $N' \succneqq N_{i_0}$ be of cardinality μ realizing p. By universality of N_{i_0+1} over N_{i_0} p is realized in N_{i_0+1}, hence in M. □

We make the following hypothesis.

HYPOTHESIS 3.10. $\mathcal{K}$ has the λ-AP ($\lambda \geq \mathrm{LS}(\mathcal{K})$).

For the reader who consider's the definition of $\mathrm{tp}(a, M, N)$ suspicious we suggest to jump ahead and read the statement of Theorem **??**.

We now define a substitute for extension of types.

DEFINITION 3.11. Let $M_0, M_1 \in \mathcal{K}$ be such that $M_0 \preceq_{\mathcal{K}} M_1$. Suppose $p_0 \in \mathrm{gS}(M_0)$ and $p_1 \in \mathrm{gS}(M_1)$. We say that p_1 *extends* p_0 , written $p_0 \leq p_1$, if there exists $N \in \mathcal{K}_\lambda$, $M_1 \preceq_{\mathcal{K}} N$ and $a \in N - M_1$ such that

$$p_0 = \text{ga-tp}(a/M_0, N) \quad \text{and} \quad p_1 = \text{ga-tp}(a/M_1, N).$$

Sometimes we write $p_0 = p_1 \restriction M_0$ for $p_0 \leq p_1$.

PROPOSITION 3.12. *Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ has the λ-AP then for every $p_0 \in \mathrm{gS}(M_0)$, $M_0 \in \mathcal{K}_\lambda$ there exists $p_1 \in \mathrm{gS}(M_1)$ extending p_0 for all $M_1 \succ_{\mathcal{K}} M_0$ of cardinality λ.*

PROOF. Exercise. □

The following definition is here just to get a taste of the definitions to come.

DEFINITION 3.13. We say that a type $p \in \mathrm{gS}(M)$ has *finite character* (or *local character*), if given any directed system

$$\langle M_t \ : \ t \in I \rangle$$

of $\mathcal{K}$ models such that $\bigcup \{M_t \ : \ t \in I\} = M$, and any type $p' \in \mathrm{gS}(M)$ satisfying

$$p \restriction M_t = p' \restriction M_t, \quad \text{for every } t \in I,$$

we have $p = p'$.

A more local version of the above property turned to be a major area of research in recent years. It is called *tameness*:

DEFINITION 3.14. Let $\mathcal{K}$ be an AEC with $\chi > \mu \geq \mathrm{LS}(\mathcal{K})$ and α an ordinal. We say that $\mathcal{K}$ is α-(μ, χ)-*tame* provided for every $M \in \mathcal{K}_\chi$ and every $p \neq q \in \mathrm{gS}^\alpha(M)$ there exists $N \leq_{\mathcal{K}} M$ of cardinality μ such that $p \restriction N \neq p \restriction N$.

When $\alpha = 1$ we write (μ, χ)-tame for short.

When $\mathcal{K}$ is α-(μ, χ)-tame for all χ we say $\mathcal{K}$ is μ-*tame*.

When $\mathcal{K}$ is χ-tame for some $\chi \geq \mathrm{LS}(\mathcal{K})$ we say$\mathcal{K}$ *is tame*.

It turns out that the assumption that $\mathcal{K}$ is tame is strong enough to allow development of an asymptotic classification theory.

CHAPTER 6

More on Types

Introduction

In this chapter we study

Overview of the structure of this chapter:

Section 1: Definability and the Lascar group.
Section 3: Game theoretic characterization of elementary embedding and isomorphism.
Section 4: Saturation of ultraproducts.
Section 5: The Keisler-Shelah's theorem.
Section 6: More on model complete theories.
Section 3: Indiscernibles and Ehrenfeucht-Mostowski models.
Section 9: We study countable models, introduce the main existence theorem Henkin's omitting types theorem and derive several consequences.

The following is a typical application:

COROLLARY 0.1 (Keisler and Morley). *For every countable* $M \models$ ZF *there exists a* $N \supsetneq M$ *of cardinality* $\aleph_1$ *such that* $\omega^N = \omega^M$.

Section 7: More on omitting types.
Section 8: Complete types as a topological space. We show for example that the extended omitting-types theorem is a consequence of the Baire-category theorem.
Section 10: More on existence, omitting types, and the completeness theorem.
Section 11: The Paris Harrington's theorem
Section 12: In the sixties and early seventies some of the most important theorems of model theory were of what is called *two-cardinal theorems*. This part of model theory is closer to set theory than most other parts of this book.

Let T be a theory such that its language includes a unary predicate P *admits a* $\langle \lambda, \mu \rangle$*-model* iff there is $M \models T$ of cardinality λ such that $|P^M| = \mu$.

It turns out that the assumption that T admits a $\langle \lambda, \mu \rangle$-model has several implications and restrictions on pairs of cardinals $\kappa \leq \chi$ such

that T admits also a $\langle \chi, \kappa \rangle$-model. In this section we prove several easy results and consider some examples.

Section 12: We prove an upward version of vaught's two-cardinal theorem called Chang's two-cardinal theorem and present several other non-trivial results that can be viewed as applications of the methods of earlier parts of this chapter.

Section 13: Chang's conjecture and Jónsson algebras

1. Definability and the Lascar group

DEFINITION 1.1. Let M be an L-structure and $A \subseteq |M|$.

(1) Suppose $A \subseteq |M|, n < \omega$ and $S \subseteq {}^n|M|$. The set S is *definable over* A iff there are $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{Fml}(L)$ and $\mathbf{a} \in A$ such that $S = \varphi(M;\mathbf{a})$.
(2) An element $\mathbf{b} \in |M|$ is *definable over* A iff there are $\varphi(\mathbf{x};\mathbf{y})$ and $\mathbf{a} \in A$ such that $\{\mathbf{b}\} = \varphi(M;\mathbf{a})$.
(3) $\mathrm{dcl}_M(A) := \{\mathbf{b} \in |M| \ : \ \mathbf{b} \text{ is definable over } A\}$.
(4) $\mathrm{acl}(A)$ stands for $\mathrm{acl}_{\mathfrak{C}}(A)$ and $\mathrm{dcl}(A)$ for $\mathrm{dcl}_{\mathfrak{C}}(A)$.

A related concept and a more useful one is introduced in the following:

DEFINITION 1.2. Let M be an L-structure and $A \subseteq |M|$.

(1) Let $\mathbf{b} \in |M|$ be given. We say that $\mathbf{b}$ *is algebraic over* A iff there are $\mathbf{a} \in A, \varphi(\mathbf{x};\mathbf{y}) \in \mathrm{Fml}(L)$ and there exists a positive integer n such that $M \models \varphi[\mathbf{b};\mathbf{a}] \wedge \exists^{\leq n}\mathbf{y}\varphi(\mathbf{y};\mathbf{a})$.
(2) *The algebraic closure of* A *in* M is denoted by $\mathrm{acl}_M(A)$ and it is the set $\{\mathbf{b} \in |M| \ : \ \mathbf{b} \text{ is algebraic over } A\}$.
(3) $\mathrm{acl}(A)$ stands for $\mathrm{acl}_{\mathfrak{C}}(A)$.

In the field of complex numbers the number i is an algebraic element over $\emptyset$ (since $\mathbb{C} \models i^2 + 1 = 0 \wedge \exists^2 y[y^2 + 1 = 0]$) but i is not definable over $\emptyset$ since the mapping[1] $z \mapsto \bar{z}$ is an automorphism of the field of complex numbers.

The terminology of being algebraic over a set is motivated by field theory. However the concepts are not identical. It turns out that for an algebraically closed field F an element a is algebraic (in the sense of field-theory) iff a is algebraic over $\emptyset$ (in the sense of Definition 1.2).

The following Lemma should be obvious

LEMMA 1.3. *Let M be a given structure and suppose that $A \subseteq M$ and $\mathbf{b} \in |M|$.*

(1) $\mathrm{dcl}_M(A) \subseteq \mathrm{acl}_M(A)$.
(2) *If $\mathbf{b}$ is definable over A then for every $f \in Aut_A(M)$ the function f is the identity on $\mathbf{b}$.*
(3) *Let B be another subset of M. If $A \subseteq B$ then $\mathrm{dcl}_M(A) \subseteq \mathrm{dcl}_M(B)$ and $\mathrm{acl}_M(A) \subseteq \mathrm{acl}_M(B)$.*

PROOF. Exercise. □

The last example can be generalized to

[1] This is what is called conjugacy, namely $a + ib$ is mapped to $a - ib$

PROPOSITION 1.4. *Let* $\mathbf{b}$ *and* A *be given. The following are equivalent:*

(1) $\mathbf{b} \in \mathrm{acl}(A) - \mathrm{dcl}(A)$.
(2) *There exists* $\mathbf{b}' \neq \mathbf{b}$ *such that for every* $\varphi(\mathbf{x}; \mathbf{a})$ *with* $\mathbf{a} \in A$ *if* $\models \exists^{<\omega}\mathbf{x}\varphi(\mathbf{x}; \mathbf{a})$ *then we have* $\models \varphi[\mathbf{b}; \mathbf{a}]$ *iff* $\models \varphi[\mathbf{b}'; \mathbf{a}]$.

PROOF. □

Like in field theory, there is an important connection between (M_1, M_2)-mapping and the extension to the (model-theoretic) algebraic closure of a set (see Exercise 5).

The great importance of these concepts (mainly that of acl induces in some cases a dependence relation) will be employed in Section 3.9 and in its extensions. At this stage the reader may view what we are doing as exercises involving the simple concepts.

DEFINITION 1.5. Let W be a non-empty set, and suppose that $\mathrm{cl} : \mathcal{P}(W) \to \mathcal{P}(W)$.

(1) We say that cl *is a closure operation on the set* W iff
   (a) For every $A \subseteq W$ we have that $A \subseteq \mathrm{cl}(A)$
   (b) *monotonicity:* For every $A, B \subseteq W$ we have that $A \subseteq B$ implies that $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
   (c) *transitivity:* For every $A, B \subseteq W$ and every $a \in W$ we have that from $A \subset \mathrm{cl}(B)$ and $a \in \mathrm{cl}(A)$ follows that $a \in \mathrm{cl}(B)$.
(2) cl *is a finitary closure operation on the set* W iff in addition of being a closure operation it has *the finite character* namely: For every $a \in W$ and for every $A \subseteq W$ we have that $a \in \mathrm{cl}(A)$ implies that there exists a finite $A_0 \subseteq A$ such that $a \in \mathrm{cl}(A_0)$.

EXAMPLE 1.6. Let F be a filed, let W be an F-vector space and for $A \subseteq W$ define $\mathrm{cl}(A)$ to be the subspace of W generated by A. It is an elementary exercise in linear algebra to verify that this cl is a finitary closure operation. More on this example and its generalization see Appendix 16.

EXAMPLE 1.7. Let W be a topological space and take cl to be the topological closure. Clearly it is a closure operation. Usually it is not finitary. E.g. When W is $\mathbb{R}$ with the usual Euclidean topology them cl is not finitary since $\sqrt{2} \in \mathrm{cl}(\mathbb{Q})$ but there is no finite $A \subseteq \mathbb{Q}$ such that $\sqrt{2} \in A$. However when W is topological space with the discrete topology (every subset is closed) then obviously cl has the finite character.

EXAMPLE 1.8. Let W be an algebraically closed field, let F be W's prime field. For every $A \subset W$ let $\mathrm{cl}(A) := \{b \in W \ : \ b$ there exists a nonzero polynomial q with coefficients from $F(A)$ such that $q(b) = 0\}$. It is an easy exercise of field-theory to prove that cl is a a closure operation with the finite character.

EXAMPLE 1.9. Let T be a complete first-order theory, suppose $W := T$. For every $\Phi \subseteq W$ define $\mathrm{cl}(\Phi)$ to be $\{\psi \in \mathrm{Sent}(\mathrm{L}(T)) \ : \ \Phi \vdash \psi\}$. It is an elementary exercise of predicate calculus to show that cl is a closure operation that has the finite character.

The following is an interesting property of closure that is often referred to as *idempotence* property:

PROPOSITION 1.10. *Let* cl *be a closure on the set* W*. For every* $A \subseteq W$ *we have that* $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

PROOF. Exercise. □

The following lemma contains two additional examples of closure operations with the finite character. Due to the particular importance of this I have decided to name this fact as a Lemma rather than including it in the previous example.

LEMMA 1.11. *Let* M *be a given model. Both* dcl *and* acl *are finitary closure operations on* $|M|$.

PROOF. Check that both functions have the finite character and satisfy requirements (1) & (2). We need only to verify (3):

dcl*:* Suppose that $a \in \mathrm{dcl}(A)$ and $A \subseteq \mathrm{dcl}(B)$. Let $\varphi(a; \mathbf{y})$ and $\mathbf{b} \in A$ be such that $M \models \varphi[a; \mathbf{b}] \wedge \forall x[\varphi(x; \mathbf{b}) \to x = a]$, and suppose $\mathbf{b} = \langle b_0, \ldots, b_{m-1} \rangle$. Since $A \subseteq \mathrm{dcl}(B)$ there $\psi_i(y; \mathbf{c}_i)$ for $(i = 0, k-1)$ where $\mathbf{c}_i \in C$ such that $M \models \psi_i[b_i; \mathbf{c}_i] \wedge \bigwedge_{i<k} \forall y[\psi_i(y; \mathbf{c}_i) \to b_i = y]$.

acl*:*

□

Recall (from Exercise 5) that the relation $\approx$ defined as $\varphi(\mathbf{x}; \mathbf{a}) \approx \psi(\mathbf{x}; \mathbf{b})$ iff $\models \forall \mathbf{x}[\varphi(\mathbf{x}; \mathbf{a}) \leftrightarrow \psi(\mathbf{x}; \mathbf{b})]$ is an equivalence relation on formulas with parameters from $\mathfrak{C}$.

In what follows by $\varphi(\mathbf{x}; \mathbf{a})/\approx$ we denote the equivalence class of $\varphi(\mathbf{x}; \mathbf{a})$.

DEFINITION 1.12. Let A and $\varphi(\mathbf{x}; \mathbf{b})$ be given.

$$\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A} := \big\{\varphi(\mathbf{x}; F(\mathbf{b}))/\approx : \ F \in \mathrm{Aut}_A(\mathfrak{C})\big\}.$$

THEOREM 1.13. *(Svenonius) Let* A *be an arbitrary subset of* $\mathfrak{C}$*,* $\varphi(\mathbf{x}; \mathbf{y})$ *a formula and suppose that* $\mathbf{b} \in {}^{\ell(\mathbf{y})}\mathfrak{C}$*. The following are equivalent*

(1) $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| = 1$

(2) $\varphi(\mathbf{x};\mathbf{b})$ *is equivalent to* a formula with parameters from A *(i.e there exist* $\mathbf{a} \in A$ *and a formula* $\psi(\mathbf{x};\mathbf{z})$ *such that* $\varphi(\mathbf{x};\mathbf{b}) \approx \psi(\mathbf{x};\mathbf{a})$*).*

PROOF. The direction (1)⇐(2) is Proposition 1.2.31.

(2)⇐(1): Suppose that $\varphi(\mathbf{x};\mathbf{b})$ and A are such that $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| = 1$. Take

$$p(\mathbf{y}) := \mathrm{tp}(\mathbf{b}/A).$$

The assumption $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| = 1$ means that

$$p(\mathbf{y}) \vdash \forall \mathbf{x}[\varphi(\mathbf{x};\mathbf{y}) \leftrightarrow \varphi(\mathbf{x};\mathbf{b})].$$

Thus there is a finite set $\psi_1(\mathbf{y};\mathbf{c}_1), \ldots, \psi_n(\mathbf{y};\mathbf{c}_n) \in p$ such that

$$\{\psi_1(\mathbf{y};\mathbf{c}_1), \ldots, \psi_n(\mathbf{y};\mathbf{c}_n)\} \vdash \forall \mathbf{x}[\varphi(\mathbf{x};\mathbf{y}) \leftrightarrow \varphi(\mathbf{x};\mathbf{b})].$$

Namely

$$\exists \mathbf{y}[\bigwedge_{k=1}^{n} \psi_k(\mathbf{y};\mathbf{c}_k) \wedge \varphi(\mathbf{x};\mathbf{y})] \approx \varphi(\mathbf{x};\mathbf{b}).$$

Namely $\exists \mathbf{y}[\bigwedge_{k=1}^{n} \psi_k(\mathbf{y};\mathbf{z}_k) \wedge \varphi(\mathbf{x};\mathbf{y})]$ with the parameters from A is equivalent to $\varphi(\mathbf{x};\mathbf{b})$. □

Motivated by the previous theorem the following is a natural generalization of the notion of *being definable over* A:

DEFINITION 1.14. We say that $\varphi(\mathbf{x};\mathbf{b})$ is *over* A iff $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| = 1$.

We say that $\varphi(\mathbf{x};\mathbf{b})$ is *almost over* A iff $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| < \aleph_0$.

EXAMPLES 1.15. (1) Consider the case when the theory is T_{ACF_0} and $A = \mathbb{Q}$. For the following three cases of $\varphi(\mathbf{x};\mathbf{b})$ we have:

$$x^2 + 17x - 3 = 0 \quad \text{is over } A$$
$$x^2 + \sqrt{7}x + 5 = 0 \quad \text{is almost over } A$$
$$x^2 + \pi x - 3 = 0 \quad \text{is not almost over } A$$

(2) Suppose $E(x;y)$ is a definable equivalence with parameters from A. If $\mathfrak{C}/E$ is finite then $E(x;b)$ is almost over A for any b,

It turns out that the last example is generic, while not every formula almost a set is an equivalence relation, every formula that is almost over a set A depends on a definable equivalence relation with finitely many classes. Namely, it is possible to characterize the formulas that are almost over a set A using definable equivalence relations with finitely many equivalence classes with parameters from A.

REMARK 1.16. For any formula $\varphi(\mathbf{x};\mathbf{b})$ we can associate naturally a definable equivalence relation $E_\varphi(\mathbf{x};\mathbf{y})$ with the parameters $\mathbf{b}$. Namely the relation

$$E_\varphi(\mathbf{x};\mathbf{y}) := [\varphi(\mathbf{x};\mathbf{b}) \leftrightarrow \varphi(\mathbf{y};\mathbf{b})]$$

is an equivalence relation its two equivalence classes are $\varphi(\mathfrak{C};\mathbf{b})$ and $\neg\varphi(\mathfrak{C};\mathbf{b})$.

DEFINITION 1.17.

$$\begin{aligned}\mathrm{FE}^n(A) :=& \{E(\mathbf{x};\mathbf{y};\mathbf{a}) \mid \ell(\mathbf{x}) = \ell(\mathbf{y}) = n,\ \mathbf{a} \in A,\\ & E(\mathbf{x};\mathbf{y};\mathbf{z}) \in \mathrm{Fml}(\mathrm{L}(T)),\ E(\mathbf{x};\mathbf{y};\mathbf{a}) \text{ is an equivalence relation and}\\ & {}^n|\,\mathfrak{C}\,|/E(\mathbf{x};\mathbf{y};\mathbf{a}) \text{ is finite}\}.\end{aligned}$$

When $E \in \mathrm{FE}^n(A)$ denote by $n(E)$ the number of equivalence classes. S

DEFINITION 1.18. Let $E(\mathbf{x};\mathbf{y}) \in \mathrm{FE}^n(A)$. We say that $\varphi(\mathbf{x};\mathbf{b})$ *depends on* $E(\mathbf{x};\mathbf{y})$ iff

$$\mathfrak{C} \models \forall\mathbf{x}\forall\mathbf{y}[E(\mathbf{x};\mathbf{y}) \to [\varphi(\mathbf{x};\mathbf{b}) \leftrightarrow \varphi(\mathbf{y};\mathbf{b})]].$$

REMARK 1.19. Let $\varphi(\mathbf{x};\mathbf{b})$ and A be given. If $\varphi(\mathbf{x};\mathbf{b})$ depends on $E(\mathbf{x};\mathbf{y})$ for some $E(\mathbf{x};\mathbf{y}) \in \mathrm{FE}^{\ell(\mathbf{x})}(A)$ then since $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| \leq n(E)$ we have that $\varphi(\mathbf{x};\mathbf{b})$ is almost over A.

Also the converse is true, it is the contents of the next theorem. Notice the similarity to the Chang-Kueker-Makkai definability theorem. The argument is a variation on the argument of Theorem 1.13

DEFINITION 1.20. Given $\varphi(\mathbf{x};\mathbf{b})$ and $E(\mathbf{x};\mathbf{y};\mathbf{a}) \in \mathrm{FE}^{\ell(\mathbf{x})}(A)$. We say that $\varphi(\mathbf{x};\mathbf{b})$ is *depends on* E iff

$$E(\mathbf{x};\mathbf{y};\mathbf{a}) \vdash [\varphi(\mathbf{x};\mathbf{b}) \leftrightarrow \varphi(\mathbf{y};\mathbf{b})].$$

The next is an example of a Galois-theoretic argument:

THEOREM 1.21 (Shelah). *Let $\varphi(\mathbf{x};\mathbf{b})$ and A be given. If $\varphi(\mathbf{x};\mathbf{b})$ is almost over A then there exists $E(\mathbf{x};\mathbf{y}) \in \mathrm{FE}^{\ell(\mathbf{x})}(A)$ such that $\varphi(\mathbf{x};\mathbf{b})$ depends on $E(\mathbf{x};\mathbf{y})$.*

PROOF. Let $n := |\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| < \omega$. Suppose $\{\mathbf{b}_i \mid i < n\}$ be such that

$$\{\varphi(\mathbf{x};\mathbf{b}_i)/\approx :\ i < n\} = \Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}.$$

Let

$$E^*(\mathbf{x};\mathbf{y}) := \bigwedge_{i<n} [\varphi(\mathbf{x};\mathbf{b}_i) \leftrightarrow \varphi(\mathbf{y};\mathbf{b}_i)].$$

Clearly E^* is an equivalence relation and $n(E^*) \leq 2^n$. As $f \in \mathrm{Aut}_A(\mathfrak{C}) \implies f(E^*) = E^*$, by Sevenonius's theorem there exists $E \in \mathrm{FE}(A)$ equivalent to E^*. □

PROOF. Denote by $p(\mathbf{y})$ the type $\mathrm{tp}(\mathbf{b}/A)$. Let $n := |\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| < \omega$. Thus the following set is inconsistent:

$$(1) \quad \{ \bigwedge_{k<k'\leq n} \neg\forall\mathbf{x}[\varphi(\mathbf{x};\mathbf{y}_k) \leftrightarrow \varphi(\mathbf{x};\mathbf{y}_{k'})]\} \bigcup_{k\leq n} p(\mathbf{y}_k).$$

However the next set is consistent

$$(2) \quad \{ \bigwedge_{k<k'<n} \neg\forall\mathbf{x}[\varphi(\mathbf{x};\mathbf{y}_k) \leftrightarrow \varphi(\mathbf{x};\mathbf{y}_{k'})]\} \bigcup_{k<n} p(\mathbf{y}_k).$$

By (1) there exists $\psi(\mathbf{y};\mathbf{a}) \in p$ such that

$$(3) \quad \{ \bigwedge_{k\neq k'<n} \neg\forall\mathbf{x}[\varphi(\mathbf{x};\mathbf{y}_k) \leftrightarrow \varphi(\mathbf{x};\mathbf{y}_{k'})]\} \cup \{\psi(\mathbf{y}_k;\mathbf{a}) \ : \ k<n\}$$

is inconsistent. By (2) and (3) we have that

$$(*) \quad \models \exists\mathbf{y}_0 \ldots \exists\mathbf{y}_{n-1}\Big[\bigwedge_{k<n} \psi(\mathbf{y}_k;\mathbf{a}) \wedge \bigwedge_{k<k'<n-1} \neg\forall\mathbf{x}[\varphi(\mathbf{x};\mathbf{y}_k) \leftrightarrow \varphi(\mathbf{x};\mathbf{y}_{k'})] \\ \wedge \forall\mathbf{y}[\psi(\mathbf{y};\mathbf{a}) \rightarrow \bigvee_{k<n} \forall\mathbf{x}[\varphi(\mathbf{x};\mathbf{y}_k) \leftrightarrow \varphi(\mathbf{x};\mathbf{y})]\Big].$$

Let $E(\mathbf{x};\mathbf{z};\mathbf{a}) := \forall\mathbf{y}[\psi(\mathbf{y};\mathbf{a}) \rightarrow [\varphi(\mathbf{x};\mathbf{y}) \leftrightarrow \varphi(\mathbf{z};\mathbf{y})]]$. Clearly E is an equivalence relation with parameters from A and also $\varphi(\mathbf{x};\mathbf{b})$ depends on E. The fact that $E \in \mathrm{FE}^{\ell(\mathbf{x})}(A)$ follows from

CLAIM 1.22. $n(E) \leq 2^{n-1}$.

PROOF. Fix some $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}$ witnessing $(*)$. Given $\mathbf{c} \in {}^{\ell(\mathbf{x})}|\mathfrak{C}|$ the class $\mathbf{c}/E$ is determined by $S \subseteq n$ such that

$$\models \bigwedge_{i\in S} \varphi(\mathbf{c};\mathbf{b}_i) \wedge \bigwedge_{j\in n-S} \neg\varphi(\mathbf{c};\mathbf{b}_j).$$

Hence $n(E) \leq 2^n$. □

□

PROPOSITION 1.23. $\mathrm{FE}^m(A)$ *is closed under conjunction.*

PROOF. By induction it is enough to show that

$$E_1 \wedge E_2 \in \mathrm{FE}^m(A) \quad \text{for every } E_1, E_2 \in \mathrm{FE}^m(A).$$

All is needed to show is that ${}^m|\mathfrak{C}|/E_1 \wedge E_2$ has finitely many equivalence classes. However it is clear that $n(E_1 \wedge E_2) \leq n(E_1) \cdot n(E_2)$. □

LEMMA 1.24. *Let*

$$\Phi := \{\varphi(\mathbf{x};\mathbf{b}) \mid \varphi(\mathbf{x};\mathbf{b}) \textit{ is almost over } A\}.$$

The set Φ is a fragment. Namely, it is closed under all propositional connectives and first-order quantification.

PROOF. We will verify that Φ is closed under$\wedge$, $\neg$ and $\exists$.

Closed under $\wedge$: Let $\varphi_\ell(\mathbf{x};\mathbf{b}_\ell) \in \Phi$ be given and suppose $E_\ell \in \mathrm{FE}^m(A)$ such that

$$\varphi_\ell(\mathbf{x};\mathbf{b}_\ell) \quad \text{depends on } E_\ell \text{ for } \ell = 1,2.$$

It is easy to see that

$$E_1 \wedge E_2 \vdash [\varphi_1(\mathbf{x};\mathbf{b}_1) \wedge \varphi(\mathbf{x};\mathbf{b}_2) \leftrightarrow \varphi_1(\mathbf{y};\mathbf{b}_1) \wedge \varphi(\mathbf{y};\mathbf{b}_2)].$$

By Proposition 1.23 we have that $E_1 \wedge E_2 \in \mathrm{FE}^m(A)$.

Closed under $\neg$: Let $\varphi(\mathbf{x};\mathbf{b}) \in \Phi$ depending on E. Clealry we have that

$$E \vdash [\neg\varphi(\mathbf{x};\mathbf{b}) \leftrightarrow \neg\varphi(\mathbf{y};\mathbf{b})].$$

Closed under $\exists$: Let $\varphi(x,\mathbf{x};\mathbf{b}) \in \Phi$ and $E(x,\mathbf{x};y,\mathbf{y}) \in \mathrm{FE}^m(A)$ be such that

$$E \vdash [\varphi(x,\mathbf{x};\mathbf{b}) \leftrightarrow \varphi(y,\mathbf{y};\mathbf{b})].$$

Let

$$E^*(\mathbf{x};\mathbf{y}) := [\forall x \exists y E(x,\mathbf{x};y,\mathbf{y}) \wedge \forall y \exists x E(x,\mathbf{x};y,\mathbf{y})].$$

Since it is immediate to see that

$$E^* \vdash [\exists z \varphi(z,\mathbf{x};\mathbf{b}) \leftrightarrow \exists z \varphi(z,\mathbf{y};\mathbf{b})]$$

is E^* an equivalence relation, the fact that $E^* \in \mathrm{FE}^m(A)$ follows from

$$n(E^*) \leq 2^{n(E)}.$$

□

THEOREM 1.25. *Let $\varphi(\mathbf{x};\mathbf{b})$ and A be given. The following are equivalent:*

(1) $\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}$ *is not a proper class, i.e.* $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| < \| \mathfrak{C} \|$,
(2) $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| < \aleph_0$, *(i.e. $\varphi(\mathbf{x};\mathbf{b})$ is almost over A).*
(3) *There exists $E \in \mathrm{FE}^{\ell(\mathbf{x})}(A)$ such that $\varphi(\mathbf{x};\mathbf{b})$ depends on E.*

PROOF. (1)⇒(2): Suppose for the sake of contradiction that $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| \geq \aleph_0$. Let $\{f_n \in \mathrm{Aut}_A(\mathfrak{C}) \mid n < \omega\}$ be such that

$$(*) \quad \models \neg\forall\mathbf{x}[\varphi(\mathbf{x};f_n(\mathbf{b})) \leftrightarrow \varphi(\mathbf{x};f_k(\mathbf{b}))] \quad \text{for all } n \neq k < \omega.$$

For any cardinal κ let $\{c_i \mid i < \kappa\}$ be a set of new constants. Let $p(\mathbf{x}) := \mathrm{tp}(\mathbf{b}/A)$, now let

$$\begin{aligned} T^*_\kappa :=& T \cup \{\neg\forall\mathbf{x}[\varphi(\mathbf{x};\mathbf{c}_i) \leftrightarrow \varphi(\mathbf{x};\mathbf{c}_j)] \mid i \neq j < \kappa\} \\ & \cup \{p(\mathbf{c}_i) \mid i < \kappa\} \cup \{\mathbf{c}_i \neq \mathbf{c}_j \mid i \neq j < \kappa\}. \end{aligned}$$

By $(*)$ the theory T^*_κ is consistent. By the definition of T^*_κ we have that $|\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}| \geq \kappa$, for all κ.

(2)⇒(1) is trivial.

(2) $\Longrightarrow$ (3) is Theorem 1.21.

(3) $\Longrightarrow$ (2) is Remark 1.19. □

DEFINITION 1.26. A type p is *almost over* A iff $\varphi(\mathbf{x};\mathbf{a})$ is almost over A for all $\varphi(\mathbf{x};\mathbf{a}) \in p$.

DEFINITION 1.27. Suppose $\mathbf{b} \in {}^{m}|\,\mathfrak{C}\,|,\ A \subseteq |\,\mathfrak{C}\,|$

$$\mathrm{stp}(\mathbf{b}/A) := \{E(\mathbf{x};\mathbf{b}) \mid E \in \mathrm{FE}^m(A),\ \ell(\mathbf{x}) = \ell(\mathbf{b}) = m\}.$$

$\mathrm{stp}(\mathbf{b}/A)$ is called the *strong type of* $\mathbf{b}$ *over* A.

DEFINITION 1.28.

LEMMA 1.29. *If* $\mathbf{a} \models p$ *and* p *is almost over* A *then*

$$\mathrm{stp}(\mathbf{a}/A) \vdash p.$$

PROOF. Let $\varphi(\mathbf{x};\mathbf{b}) \in p$ be given, $m = \ell(\mathbf{x})$. Since $\varphi(\mathbf{x};\mathbf{b})$ is almost over A there is $E \in \mathrm{FE}^m(A)$ such that

$$E(\mathbf{x};\mathbf{y}) \vdash \varphi(\mathbf{x};\mathbf{b}) \leftrightarrow \varphi(\mathbf{y};\mathbf{b}).$$

Since $\mathbf{a} \models p$ we have that

$$(*) \qquad \models \varphi[\mathbf{a};\mathbf{b}]$$

Suppose $\mathbf{d} \models \mathrm{stp}(\mathbf{a}/A)$. By the definition of strong type we have

$$\models E[\mathbf{d};\mathbf{a}]$$

so

$$\models \varphi[\mathbf{d};\mathbf{b}] \leftrightarrow \varphi[\mathbf{b};\mathbf{b}].$$

Using $(*)$:

$$\models \varphi[\mathbf{d};\mathbf{b}]$$

i.e. $\mathbf{d}$ realizes p. □

The next is another example of use of a Galois-theoretic argument:

COROLLARY 1.30. *If* $E(\mathbf{x};\mathbf{y};\mathbf{c})$ *is an equivalence relation with finitely many equivalence classes which is almost over the set* A *then there exists* $E^*(\mathbf{x};\mathbf{y}) \in \mathrm{FE}^{\ell(\mathbf{x})}(A)$ *such that*

$$\models \forall\mathbf{x}\forall\mathbf{y}[E^*(\mathbf{x};\mathbf{y}) \to E(\mathbf{x};\mathbf{y};\mathbf{c})].$$

PROOF. Let $\mathbf{c}_0, \ldots, \mathbf{c}_{n-1}$ be such that $\{E(\mathbf{x};\mathbf{y};\mathbf{c}_k) \mid k < n\}$ is a complete set of representatives of $\Gamma_{E(\mathbf{x};\mathbf{y};\mathbf{c}),A}$.

As for every $k < n$ there exists $f_k \in \mathrm{Aut}_A(\mathfrak{C})$ such that $f(\mathbf{c}) = \mathbf{c}_k$ we have that $m := n(E(\mathbf{x};\mathbf{y};\mathbf{c})) = n(E(\mathbf{x};\mathbf{y};\mathbf{c}_k))$ Define

$$E'(\mathbf{x};\mathbf{y};\mathbf{c}_0,\dots,\mathbf{c}_{n-1}) := \bigwedge_{k<n} E(\mathbf{x};\mathbf{y};\mathbf{c}_k).$$

Clearly $n(E') \leq m^k < \aleph_0$ and

$$\models \forall\mathbf{x}\forall\mathbf{y}[E'(\mathbf{x};\mathbf{y};\mathbf{c}_0,\dots,\mathbf{c}_{n-1}) \to E(\mathbf{x};\mathbf{y};\mathbf{c})].$$

By definition of E' if $f \in \mathrm{Aut}_A(\mathfrak{C})$ then $f(E') = E'$, thus $|\Gamma_{E',A}| = 1$. Theorem 1.13 gives an $E^*(\mathbf{x};\mathbf{y}) \in \mathrm{FE}^{\ell(\mathbf{x})}(A)$ equivalent to $E'(\mathbf{x};\mathbf{y};\mathbf{c}_0,\dots,\mathbf{c}_{n-1})$. □

EXAMPLE 1.31. Almost over a field F in field theory means algebraic over F

DEFINITION 1.32. Let A be a subset of the monster model $\mathfrak{C}$. An automorphism $g \in \mathrm{Aut}_A(\mathfrak{C})$ is a *strong automorphism over* A iff there exists a model $M \preceq \mathfrak{C}$ such that $|M| \supseteq A$ and $g \in \mathrm{Aut}(M)$. The subgroup of $\mathrm{Aut}_A(\mathfrak{C})$ generated by

$$\{g \in \mathrm{Aut}_A(\mathfrak{C}) \mid g \text{ is a strong automorphism}\}$$

is called the *group of strong automorphisms over* A and denoted by $\mathrm{SAut}_A(\mathfrak{C})$. When A is the empty set we write $\mathrm{SAut}(\mathbf{C})$

Notice that $g \cdot h \cdot g^{-1} \in \mathrm{SAut}_A(\mathfrak{C})$ for every $g \in \mathrm{Aut}_A(\mathfrak{C})$ and every $h \in \mathrm{SAut}_A(\mathfrak{C})$[2] thus

$$\mathrm{SAut}_A(\mathfrak{C}) = \{g_0 \cdots g_n \mid n < \omega,\ g_k \text{ is a strong automorphism over } A\}$$

is a normal subgroup of $\mathrm{Aut}_A(\mathfrak{C})$. The following definition makes sense.

DEFINITION 1.33. *Lascar group* over a set A is

$$\mathrm{Gal}_A(\mathfrak{C}) := \mathrm{Aut}_A(\mathfrak{C})/\mathrm{SAut}_A(\mathfrak{C}).$$

DEFINITION 1.34. We say that $\mathbf{a}$ and $\mathbf{b}$ have the same *Lascar strong type over* A iff there exists $g \in \mathrm{SAut}_A(\mathfrak{C})$ such that $f(\mathbf{a}) = \mathbf{b}$. In symbols: $\mathrm{lstp}(\mathbf{a}/A) = \mathrm{lstp}(\mathbf{b}/A)$. So $\mathrm{lstp}(\mathbf{a}/A)$ stands for the orbit of $\mathbf{a}$ under the group $\mathrm{SAut}_A(\mathfrak{C})$.

DEFINITION 1.35.

(1) Let $n < \omega$ and $A \subseteq \mathfrak{C}$. A relation $E \subseteq \mathfrak{C}^n \times \mathfrak{C}^n$ is *A-invariant* iff

$$E(\mathbf{a};\mathbf{b}) \iff E(f(\mathbf{a});f(\mathbf{b})) \text{ for all } f \in \mathrm{Aut}_A(\mathfrak{C}).$$

(2) An equivalence relation $E \subseteq \mathfrak{C}^n \times \mathfrak{C}^n$ is *finite* iff $|\,\mathfrak{C}^n/E| < \aleph_0$. The relation E is *bounded* iff there exists a cardinal number such that $|\,\mathfrak{C}^n/E| < \lambda$.

[2]This is Exercise 1.2.69.

(3) Let $A \subseteq \mathfrak{C}$ and $\mathbf{a} \in \mathfrak{C}$. The *Lascar type of* $\mathbf{a}$ *over* A is denoted by $\mathrm{lstp}(\mathbf{a}/A)$ it is the equivalence class of $\mathbf{a}$ when the equivalence relation is defined by

$$\mathrm{lstp}(\mathbf{a}/A) = \mathrm{lstp}(\mathbf{b}/A) \stackrel{def}{\iff} E(\mathbf{a};\mathbf{b}) \text{ for all bounded and definable } A\text{-invariant equivalence relations } E.$$

(4) Let $A \subseteq \mathfrak{C}$ and $\mathbf{a} \in \mathfrak{C}$. The *strong type of* $\mathbf{a}$ *over* A is denoted by $\mathrm{stp}(\mathbf{a}/A)$ it is the equivalence class of $\mathbf{a}$ when the equivalence relation is defined by

$$\mathrm{stp}(\mathbf{a}/A) = \mathrm{stp}(\mathbf{b}/A) \stackrel{def}{\iff} \mathfrak{C} \models E[\mathbf{a};\mathbf{b}] \text{ for all bounded equivalence relations definable over } A.$$

REMARK 1.36. Let $E(\mathbf{x};\mathbf{y};\mathbf{z})$ be a formula such that $\ell(\mathbf{x}) = \ell(\mathbf{y})$ and for some $\mathbf{c}$ we have that $E(\mathbf{x};\mathbf{y};\mathbf{c})$ is an equivalence relation. An application of the compactness theorem gives

$$E(\mathbf{x};\mathbf{y};\mathbf{c}) \text{ is bounded} \iff \mathfrak{C}^{\ell(\mathbf{x})}/E \text{ is finite.}$$

We notice the following simple observation:

LEMMA 1.37. *Let $E(\mathbf{x};\mathbf{y})$ be a bounded A-invariant equivalence relation. If $\mathbf{I}$ is a infinite sequence of indiscernibles over A then $\mathfrak{C} \models E[\mathbf{a};\mathbf{b}]$ for all $\mathbf{a},\mathbf{b} \in \mathbf{I}$.*

PROOF. Otherwise, there are $\mathbf{I}$ and $\mathbf{a},\mathbf{b} \in \mathbf{I}$ such that $\mathfrak{C} \models \neg E[\mathbf{a},\mathbf{b}]$.
Let $\lambda := |\mathfrak{C}^n/E|$. By compactness and Ramsey's theorem there exists a A-sequence of indiscernibles $\mathbf{J} \supseteq \mathbf{I}$ of cardinality λ^+ since $\mathfrak{C} \models \neg E[\mathbf{a};\mathbf{b}]$ by indiscernibility we get that $\mathfrak{C} \models \neg E[\mathbf{c};\mathbf{d}]$ for all $\mathbf{c},\mathbf{d} \in \mathbf{J}$, contradiction to the definition of λ. □

NOTATION 1.38. Let A be a set and α an ordinal. For $\mathbf{a},\mathbf{b} \in {}^{\alpha}|\mathfrak{C}|$ denote by $E_A(\mathbf{a},\mathbf{b})$ the following statement:
There are $n < \omega$ and $\{\mathbf{a}_k \mid k \leq n\} \subseteq {}^{\alpha}|\mathfrak{C}|$ such that

(1) $\mathbf{a} = \mathbf{a}_0$ and $\mathbf{b} = \mathbf{a}_n$.
(2) For every $k < n$ there exists a sequence of indiscernibles $\mathbf{I}_k$ over A such that $\mathbf{a}_k, \mathbf{a}_{k+1} \in \mathbf{I}_k$ for all $k < n$.

REMARK 1.39. Notice that E_A is an A-invariant equivalence relation over A.

THEOREM 1.40. *Let A and $\mathbf{a},\mathbf{b} \in {}^{\alpha}|\mathfrak{C}|$ be given. Suppose that both $\mathrm{tp}(\mathbf{a}/A)$ and $\mathrm{tp}(\mathbf{b}/A)$ are not algebraic. Then*

$$\mathrm{lstp}(\mathbf{a}/A) = \mathrm{lstp}(\mathbf{b}/A) \iff E_A(\mathbf{a},\mathbf{b}).$$

PROOF. ($\Rightarrow$): It suffices to show that E_A is a bounded relation. For the sake of contradiction suppose that E_A is unbounded. By Lemma 1.37 the relation E_A is unbounded. Let $\{\mathbf{a}_i \mid i < \beth_{(2^{|T|+|A|})^+}\}$ be such that $i \neq j \implies \neg E_A(\mathbf{a}_i, \mathbf{a}_j)$. By Corollary 1.39 there exists a sequence of indiscernibles $\langle \mathbf{b}_n \mid n < \omega \rangle$ over A such that there are $i_1 < i_2 < \beth_{(2^{|T|+|A|})^+}$ such that

$$\mathrm{tp}(\mathbf{b}_0\mathbf{b}_1/A) = \mathrm{tp}(\mathbf{a}_{i_1}\mathbf{a}_{i_2}/A).$$

Since E_A is A-invariant, an application of $f \in \mathrm{Aut}_A(\mathfrak{C})$ such that $f(\mathbf{a}_{i_1}\mathbf{a}_{i_2}) = \mathbf{b}_0\mathbf{b}_1$ gives that

$$(*) \quad \neg E_A(\mathbf{b}_0, \mathbf{b}_1).$$

Notice that since $\mathbf{b}_0$ and $\mathbf{b}_1$ are members of the same sequence of indiscernibles over A by the definition of E_A we get that $E_A(\mathbf{b}_0, \mathbf{b}_1)$ must hold, in contradiction to $(*)$.

($\Leftarrow$): Suppose that $E_A(\mathbf{a}; \mathbf{b})$ holds. Let $E(\mathbf{x}; \mathbf{y})$ be a given bounded A-invariant equivalence relation. Let $\mathbf{a} = \mathbf{a}_0, \ldots, \mathbf{a}_n = \mathbf{b}$ from the definition of E_A. Since $\mathbf{a}_k$ and $\mathbf{a}_{k+1}$ are elements of the same infinite sequence of indiscernibles over A, by Lemma 1.37 we have

$$E(\mathbf{a}; \mathbf{a}_1), E(\mathbf{a}_1; \mathbf{a}_2), \cdots, E(\mathbf{a}_{n-1}; \mathbf{b}).$$

Thus by transitivity of E we get that $E(\mathbf{a}; \mathbf{b})$ holds. □

LEMMA 1.41. *For any n-tuples $\mathbf{a}$ and $\mathbf{b}$ The following are equivalent:*

(1) $\mathrm{stp}(\mathbf{a}/\operatorname{acl}(A)) = \mathrm{stp}(\mathbf{b}/\operatorname{acl}(A))$,
(2) $\mathfrak{C} \models E[\mathbf{a}; \mathbf{b}]$ *for all* $E(\mathbf{x}; \mathbf{y}) \in \mathrm{FE}^n(A)$.

PROOF. □

DEFINITION 1.42. *Strong type* $\mathrm{stp}(\mathbf{a}/A)$

acl(A)

def(A)

characterize orbits induced by $Aut(\mathfrak{C})$.

Exercises 6.1.

(1) $\mathrm{acl}(A) = \{\mathbf{c} \in \mathfrak{C} \mid$ the set $\{f(\mathbf{c}) \mid f \in \mathrm{Aut}_A(\mathfrak{C})\}$ is finite$\}$.
(2) $\mathrm{dcl}(A) = \{\mathbf{c} \in \mathfrak{C} \mid \mathbf{c}$ is fixed by all the elements of $\mathrm{Aut}_A(\mathfrak{C})\}$.
(3) Suppose M is a structure and $a \in |M|$, $A \subseteq |M|$. Show that $a \in \mathrm{dcl}_M(A)$ iff there exists $\mathbf{b} \in A$ and there exists a formula $\varphi(x; \mathbf{y})$ such that
$M \models \varphi[a; \mathbf{b}] \wedge \forall x[\varphi(x; \mathbf{b}) \to x = a]$.
(4) Suppose M is a structure, $A \subseteq |M|$. Show that for every $N \succ M$ we have that both $\mathrm{dcl}_M(A) = \mathrm{dcl}_N(A)$ and $\mathrm{acl}_M(A) = \mathrm{acl}_N(A)$ hold. Moreover for $a \in |M|$ we have that $a \in \mathrm{dcl}_M(A)$ iff $a \in \mathrm{dcl}_N(A)$ and $a \in \mathrm{acl}_M(A)$ iff $a \in \mathrm{acl}_N(A)$.
(5) Prove the following

THEOREM 1.43. *Suppose M and N L-structures. Let $A \subseteq |M|, B \subseteq |N|$ be given. If $f : A \to B$ is an (M, N)-mapping then there exists an (M, N)-mapping $\overline{f} : \mathrm{acl}_M(A) \to \mathrm{acl}_N(B)$ extending f.*

(6) Let $E_1, E_2 \in \mathrm{FE}^n(A)$and let $E(\mathbf{x}; \mathbf{y}) := E_1 \wedge E_2$.
Show that $n(E) \leq n(E_1)n(E_2)$.
(7) If $\varphi(\mathbf{x}; \mathbf{b})$ depends on $E \in \mathrm{FE}^n(A)$ then $2|\Gamma_{\varphi,A}| \leq n(E)$.
(8) Suppose that $\varphi(\mathbf{x}; \mathbf{b})$ is almost over A and $\{\varphi(\mathbf{x}; \mathbf{b}_i) \mid i < m\}$ is a complete set of representatives of $\Gamma_{\varphi(\mathbf{x};\mathbf{b}),A}$.
Let $\psi := \bigvee_{i<m} \varphi(\mathbf{x}; \mathbf{b}_i)$ and $\psi' := \bigwedge_{i<m} \varphi(\mathbf{x}; \mathbf{b}_i)$. Show that both ψ and ψ' are over A.
(9) Suppose that $E(\mathbf{x}; \mathbf{y})$ is a first-order formula with parameters from $B \subseteq |\mathfrak{C}|$ that defines an equivalence relation on ${}^{\ell(\mathbf{x})}|\mathfrak{C}|$. If $E \notin \mathrm{FE}^{\mathbf{x}}(B)$ then for every λ there exists $\{\mathbf{a}_i \mid i < \lambda\} \subseteq {}^{\ell(\mathbf{x})}|\mathfrak{C}|$ satisfying $i \neq j \implies \mathfrak{C} \models \neg E[\mathbf{a}_i; \mathbf{a}_j]$.
(10) let F be an algebraically closed field and $a \in F$. Prove that a is algebraic (in the sense of field theory) iff a is algebraic over $\emptyset$ (in the sense of model theory).
(11) A definable equivalence relation $E(\mathbf{x}; \mathbf{y})$ with parameters from A is called *Bounded* iff the cardinality of $|\mathfrak{C}|/E$ is less than that of the cardinality of $|\mathfrak{C}|$.
Prove that E is bounded iff $E \in \mathrm{FE}^{\ell(\mathbf{x})}(A)$.
(12)
(13)

2. Using models of set theory to establish consistency of a first-order theory

In this section I will present a powerful technique that has many applications. The idea is as follows. Given a first-order theory T with a property $Pr1$, usually some statement of existence of model(s) of certain type and our goal is to establish that T has a model satisfying another requirement, say $Pr2$. We will construct a first-order theory $T^* \supseteq T$ in an expansion of $\mathrm{L}(T)$ we will have two steps: First show that if T^* is consistent than there exists $M \models T$ satisfying $Pr2$ and second, show that T^* is consistent.

Suppose $\{M_i \mid i \in I\}$ is a set of models that witness that T has $Pr1$. We will take a "sufficiently large" χ such that $T, \{M_i \mid i \in I\}, I$ are elements of $H(\chi)$ and the structure $\langle H(\chi), \in\rangle$ reflect "several relevant properties" of the models M_i and T. Consider the structure $\mathfrak{B} := \langle H(\chi), \in, \omega, T, I, i \mapsto M_i, \ldots\rangle$ by the undefinability of well-ordering there exists $\mathfrak{B}^* \equiv \mathfrak{B}$ such that $\langle \omega^{\mathfrak{B}^*}, \in^{\mathfrak{B}^*}\rangle$ is not-well founded. Namely there exists $\{a_n \mid n < \omega\} \subseteq \omega^{\mathfrak{B}^*}$ such that $\mathfrak{B}^* \models [a_{n+1} < a_n]$ for every $n < \omega$.

We will use the a_n's to find sets $X_n \in \mathfrak{B}^*$ so that given a finite $T_n \subseteq T^*$ for some $i \in I^{\mathfrak{B}^*}$ and some $\mathbf{a} \in X_n$ we have that $\langle M_i^{\mathfrak{B}^*}, \mathbf{a}\rangle \models T_n$.

As an example we prove

THEOREM 2.1 (Vaught's gap $\beth_\omega$). *Let T be a first-order theory with a unary predicate P in $\mathrm{L}(T)$. Suppose that for every $n < \omega$ there exists $M_n \models T$ such that $\|M_n\| \geq \beth_n(|P^{M_n}|)$ and P^{M_n} is infinite. Then for any cardinals $\lambda \geq \mu \geq |T|$ there exists $N \models T$ such that $\|N\| = \lambda$ and P^N has cardinality μ.*

PROOF. Let T^{Sk} be an expansion of T with built in Skolem functions. Let $\{M_n \mid n < \omega\}$ be model of T satisfying the assumption on T. Without loss of generality we may assume that $M_n \models T^{Sk}$ for all $n < \omega$. Let $F(n) := M_n$, let χ be large enough so that $T^{Sk}, \{M_n \mid n < \omega\}, F, \mathrm{L}(T), \mathrm{Fml}(\mathrm{L}(T^{Sk}))$ are elements of $H(\chi)$ and $\langle H(\chi), \in\rangle$ reflects the following facts:

(1) $\langle H(\chi), \in\rangle \models [F(n) \models_{\mathrm{L}(T^{Sk})} T^{Sk}$ for every $n < \omega$,
(2) $\langle H(\chi), \in\rangle \models [\beth_n(\mu_n)^+ \to (\mu_n^+)^{n+1}_{2^{\mu_n}}]$ for every $n < \omega$, where $\mu_n \in H(\chi)$ and $\langle H(\chi), \in\rangle \models \mu_n = |P^{F(n)}|$.
(3) $\langle H(\chi), \in\rangle \models \forall n[n \in \omega \to \forall X[X \in F(n) \wedge |X| \geq \beth_n(|P^{F(n)}|)^+ \to \exists Y \subseteq X[|Y| \geq (|P^{F(n)}|)^+ \wedge \mathbf{a}, \mathbf{b} \in [Y]^{n+1} \to \mathrm{tp}_{\mathrm{L}(T^{Sk})}(\mathbf{b}/P^{F(n)}) = \mathrm{tp}_{\mathrm{L}(T^{Sk})}(\mathbf{a}/P^{F(n)})]]$.
(4)

Let $\{c_i \mid i < \alpha\}$ be new constant symbols and let T^*_α be the theory consisting of

- T^{Sk},

- $\{\mathbf{c}_i \neq c_j \mid i \neq j < \alpha\}$,
- set of sentences asserting that P is satisfied by infinitely many elements and
- $\{\forall y_0 \dots \forall y_{m-1}[\bigwedge_{k<m} P(y_k) \to [\varphi(c_{i_0}, \dots, c_{i_{n-1}}; \mathbf{y}) \leftrightarrow \varphi(c_{j_0}, \dots, c_{j_{n-1}}; \mathbf{y})] \mid n < \omega, i_0 < \dots i_{n-1}, j_0 < \dots j_{n-1} < \alpha, \varphi(\mathbf{x}; \mathbf{y} \in \mathrm{Fml}(\mathrm{L}(T^{Sk}))\}$.

By the compactness theorem T^*_ω is consistent iff T^*_α consistent for all infinite α.

We will prove:

CLAIM 2.2 (First Claim). *If T^*_λ is consistent then there exists $N \models T$ of cardinality λ such that $|P^N| = \mu$.*

and

CLAIM 2.3 (Second Claim). *T^*_ω is consistent.*

PROOF OF THE FIRST CLAIM. Let $N^* \models T^*_\lambda$ of cardinality λ such that also P^{N^*} has cardinality λ. Fix $M^* \preceq N^* \restriction \mathrm{L}(T^{Sk})$ of cardinality μ such that P^{M^*} has cardinality μ and M^* contains the set $\{c_i^{N^*} \mid i < \mu\}$. Denote by A_α the set $\{c_i^{N^*} \mid i < \alpha\}$.

Let N be $\mathrm{EM}(A_\lambda \cup P^{M^*}) \restriction \mathrm{L}(T)$.

We will show that $P^N = P^{M^*}$. For the sake of contradiction suppose that there exists $b \in P^N - P^{M^*}$. Let $q := \mathrm{tp}(b/A_\mu \cup P^{M^*})$. Let $\tau(\mathbf{x}; \mathbf{y})$ be a term in $\mathrm{L}(T^{Sk})$ and $\mathbf{a} \in A_\lambda \cup P^{M^*}$ and $\mathbf{c} \in P^{M^*}$ such that $b = \tau(\mathbf{a}; \mathbf{c})$.

By the choice of b and the definition of q we have

$$(*) \quad P(x) \in q \quad \text{and} \quad \{b \neq d \mid d \in P^{M^*}\} \subseteq q.$$

Since A_λ is an indiscernible sequence over P^{M^*} and A_μ has no last element there exists $\mathbf{a}' \in A_\mu$ such that $\mathrm{tp}(\mathbf{a}/P^{M^*}) = \mathrm{tp}(\mathbf{a}'/P^{M^*})$, thus there is $f \in \mathrm{Aut}_{P^{M^*}}(\mathfrak{C})$ such that $f(\mathbf{a}) = \mathbf{a}'$. Now an application of f on $(*)$ gives us:

$$(**) \quad \tau(\mathbf{a}'; \mathbf{c}) \models \{P(x)\} \cup \{b \neq d \mid d \in P^{M^*}\}.$$

As $\mathbf{a}', \mathbf{c} \in |M|$ we have that M^* must contain $\tau(\mathbf{a}'; \mathbf{c})$, thus we found an element of M realizing the type $\{P(x)\} \cup \{b \neq d \mid d \in P^{M^*}\}$

□

PROOF OF THE SECOND CLAIM. □

□

Exercises 5.3.

(1) Let T be a first-order theory. Show that for every $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ there are a family $\{M_n \models T \mid n < \omega\}$ and $M \models T$ all of cardinality λ satisfying

$$(*) \quad M = \bigcap_{n<\omega} M_n \quad \text{and} \quad M_{n+1} \precneqq M_n \text{ for all } n < \omega.$$

Thus when T is $\aleph_0$-categorical in a countable language then for every countable $M \models T$ there exists a family $\{M_n \mid n < \omega\}$ of countable models of T satisfying $(*)$.

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3. Game theoretic characterization of elementary embedding and isomorphism

Below we describe two games that are variants on the Ehernfeucht-Fraisse games games and characterizes existence of elementary embeddings and isomorphism between models.

DEFINITION 3.1. Let M and N be L-structures, α an ordinal and χ a cardinal number. The game $GE^{\alpha}_{\chi}(M,N)$ is between two players I and II its length is α. Player I chooses an ordinal $i_\beta < \chi$ and a sequence $\langle a^\beta_i \in |M| \ : \ i < i_\beta\rangle$; Player II responds by choosing a sequence $\langle b^\beta_i \in |N| \ : \ i < i_\beta\rangle$. The game ends after playing α moves.

Player II is a winner iff

$$\mathrm{tp}(\langle b^\beta_i \ : \ i < i_\beta, \beta < \alpha\rangle/\emptyset, N) = \mathrm{tp}(\langle a^\beta_i \ : \ i < i_\beta, \beta < \alpha\rangle/\emptyset, M).$$

Player I wins iff player II have lost.

LEMMA 3.2. *Let M nad N be L-structures, denote by $\alpha := \|M\|$. Player II has a winning strategy in the game $GE^{\alpha}_{\omega}(M,N)$ iff there exists an elementary embedding from M to N.*

First we need a definition that is used in the description of the winning condition for the game characterizing isomorphism:

DEFINITION 3.3. Let M be a given L-structure, χ an infinite cardinal and $\mathbf{a} := \langle \ldots, a_i, \ldots\rangle_{i\in I}$ be an infinite sequence of elements from M. The set $\mathrm{TPS}^{\chi}(\mathbf{a}, M)$ is the following family of sets of formulas in the free variables $\{y_n \mid n < \omega\} \cup \{x_i \mid i \in I\}$ where:

$$\Gamma \in \mathrm{TPS}^{\chi}(\mathbf{a}, M) \iff |\Gamma| \leq \chi \text{ and there exists } \mathbf{b} \in |M| \text{ such that}$$
$$\varphi(\mathbf{y}; x_{i_0}, \ldots, x_{i_\varphi}) \in \Gamma \implies M \models \varphi[\mathbf{b}; a_{i_0}, \ldots, a_{i_\varphi}].$$

DEFINITION 3.4. The game $GI^{\alpha}_{\chi}(M,N)$ is also a two player game. The players carry out the same moves as in the previous game. The condition for victory for player II is:

$$\mathrm{TPS}^{\chi}(\langle b^\beta_i \ : \ i < i_\beta, \beta < \alpha\rangle, N) = \mathrm{TPS}^{\chi}(\langle a^\beta_i \ : \ i < i_\beta, \beta < \alpha\rangle, M).$$

LEMMA 3.5. *Let M nad N be L-structures. Player II has a winning strategy in the game $GI^{\omega}_{\aleph_0}(M,N)$ iff $M \cong N$.*

Exercises 6.3.

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4. Saturation of ultraproducts

The the following was a central problem during the sixties:

QUESTION 4.1. Let M be a given model and D a non-principle unlrafilter on an infinite set I. Is the ultrapower M^I/D saturated?

DEFINITION 4.2. Let μ be an uncountable cardinal. A filter D is *μ-incomplete* iff there exists $\{S_i \mid i < \alpha < \mu\} \subseteq D$ such that $\bigcap_{i<\alpha} S_i \notin D$. When $\mu = \aleph_1$ we say that D is *countably incomplete*.

THEOREM 4.3. *Let L and $\{M_i \mid i \in I\}$ be L-structures, and λ be such that $\lambda \geq |L|$. If D in λ^+-incomplete then the ultraproduct $\prod_{i\in I} M_i/D$ is λ^+-saturated.*

PROOF. □

COROLLARY 4.4. *Assuming* GCH.

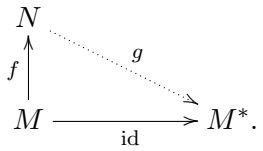
5. Keisler-Shelah's theorem*

6. More on model complete theories*

This section deals with an application of saturated models to model completion of a theory. This depends on section 1.13 from Chapter 1. We will present Blum's test for model completion with few applications.

An immediate application to the existence theorem together with Robinson's test for model completion (Theorem 1.13.19) is a useful test due to Simon Kochen for model-completeness of a theory:

LEMMA 6.1. *A theory T is model complete iff for every $M, N \models T$, every $M^* \supseteq M$ a $\|N\|^+$-saturated model of T and every monomorphism $f : M \to N$ there exists a monomorphism $g : N \to M^*$ such that the diagram*



commutes.

PROOF. For the non-trivial direction: We verify $(*)$ from Theorem 1.13.19. Let M and N be models of T and $f : M \to N$ a monomorphism. By Theorem 1.18 there exists $M^* \succ M$ of cardinality $2^{\|N\|}$ which is $\|N\|^+$-saturated. By the hypothesis of this proposition there exists a monomorphism $g : N \to M^*$ such that the above diagram commutes. □

The following is Lenore Blum's useful characterization of model completion:

THEOREM 6.2 (Blum's test for model-completion).

PROOF. □

Exercises 5.6.

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7. Shelah's Generalization of Ehrenfeucht-Mostowski models

The technology developed in section 3 is sufficient for dealing with the original question that Ehrenfeucht and Mostowski dealt with namely constructing models for first-order theories with large groups of automorphisms. In the next chapter when dealing with Morley's categoricity theorem we will see that these models are instrumental in proving that categoricity is a cardinal $> |\mathrm{L}(T)| + \aleph_0$ implies stability in all smaller cardinals.

For certain applications (among them classification theory for AECs and the study of non-superstable first-order theories) we use a generalization of the notions and theorems of section 3 and this is the goal of the present section.

DEFINITION 7.1. Suppose T is first-order complete theory, let $L^* \supseteq \mathrm{L}(T)$. A theory T^* in L^* is called *conservative extension of* T iff $T^* \supseteq T$ and for every $\mathrm{L}(T)$-structure M if $M \models T$ and $\|M\| \geq |\mathrm{L}(T)| + \aleph_0$ then there exists an expansion M^* of M to L^* such that $M^* \models T^*$.

We will make a serious use of the following simple observation:

REMARK 7.2. Suppose L is a given language, $L_2 \supseteq L_1 \supseteq L$, T_1 is a theory in L_1 and Γ is set of finitary types in L_1 (without parameters).

If T_2 is a conservative extension of T_1 then

$$\mathrm{PC}(T_1, \Gamma, L) = \mathrm{PC}(T_2, \Gamma, L).$$

The following is an important example of a conservative extension:

THEOREM 7.3. *Let T be a given consistent theory. Define $T_0 := T$ and by induction on $n < \omega$:*

$$T_{n+1} := T_n \cup \{\forall \mathbf{x}[\varphi(\mathbf{x}; y) \to \varphi(\mathbf{x}; F_{\varphi(\mathbf{x};y)}(\mathbf{x})) \mid \varphi(\mathbf{x}; y) \in \mathrm{L}(T_n)\}.$$

Now let $T^{sk} := \bigcup_{n<\omega} T_n$.

The theory T^{sk} is a conservative extension of T.

PROOF. Exercise. □

A combination of Remark 7.2 with the previous theorem gives:

THEOREM 7.4. *Given a class of L-structures $\mathcal{K}$ such that $\mathcal{K} = \mathrm{PC}(T_1, \Gamma, L)$ we may assume that T_1 has built-in Skolem functions.*

The contents of the following theorem is that when dealing with the class $\mathrm{PC}(T_1, \Gamma, L)$ with uncountable $\mathrm{L}(T_1)$ we may assume without loss of generality that there exists a countable $L_1^c \subseteq \mathrm{L}(T_1)$ such that $\mathrm{L}(T_1) - L_1^c$ consists of constant symbols only.

THEOREM 7.5. *Suppose T is a complete first-order theory. There exists a conservative extension T_1 of T and a countable $L^c_1 \subseteq \mathrm{L}(T_1)$ such that*

- *For every finite $\mathbf{x}$ there exists a relation symbol $R(\mathbf{x}; y) \in L^c_1$ and a function symbol $F(\mathbf{x}; z) \in L^c_1$.*
- *For every formula $\varphi(\mathbf{x}) \in L_1$ and every term $\tau(\mathbf{x}) \in L_1$ there are constant symbols $c_\varphi, d_\tau \in L_1$.*

Satisfying

$$\forall \mathbf{x}[\varphi(\mathbf{x}) \leftrightarrow R(\mathbf{x}; c_\varphi)],\ \forall \mathbf{x}[\tau(\mathbf{x}) = F(\mathbf{x}; d_\tau)] \in T_1.$$

PROOF. Exercise. □

7.1. Index models.

DEFINITION 7.6. Let I be an $\mathrm{L}(I)$-structure, $A \subseteq I$ and $\mathbf{t}$ a finite sequence of elements of I.

$$\mathrm{atp}(\mathbf{t}/A, I) := \{\varphi(\mathbf{x}; \mathbf{s}) \mid \ell(\mathbf{x}) = \ell(\mathbf{t}), \mathbf{s} \in A, I \models \varphi[\mathbf{t}; \mathbf{s}],\ \varphi(\mathbf{x}; \mathbf{y}) \text{ is an atomic (or a negation of an atomic) formula in } \mathrm{L}(I)\}.$$

REMARK 7.7. Observe that $\mathrm{atp}(\mathbf{t}/A, I)$ is what we also denote by $\mathrm{tp}_{\mathrm{qf}}(\mathbf{t}/A, I)$.

DEFINITION 7.8. Let I be an index model, and suppose $\mathbf{s}, \mathbf{t} \in I$, $A \subseteq I$.

$$\mathrm{S} \sim \mathbf{t} \mod A \iff \mathrm{atp}(\mathbf{s}/A, I) = \mathrm{atp}(\mathbf{t}/A, I).$$

When $A = \emptyset$ we omit it and when the identity of I is clear we omit it. Thus, $\mathbf{s} \sim \mathbf{t}$ stands for $\mathrm{atp}(\mathbf{s}/\emptyset, I) = \mathrm{atp}(\mathbf{t}/\emptyset, I)$.

Compare the last equivalence relation with the relations introduced in the proof of Claim 3.23.

The next is a generalization of the notion of indiscernible sequence:

DEFINITION 7.9. Let $\Delta \subseteq \mathrm{Fml}(L)$, $\mathfrak{C}$ monster model for some L-theory, $n < \omega$ and $A \subseteq |\mathfrak{C}|$.

$\mathbf{I} := \langle \mathbf{a}_s \in \mathfrak{C} \mid s \in I \rangle$ is a (Δ, n)-*indiscernible sequence pover A* iff

(1) $\ell(\mathbf{a}_s) = \ell(\mathbf{a}_t)$ for all $s, t \in I$ and
(2) $\mathbf{s} \sim \mathbf{t} \implies \mathrm{tp}_\Delta(\mathbf{a}_{s_o}, \ldots, \mathbf{a}_{s_{n-1}}/A, \mathfrak{C}) = \mathrm{tp}_\Delta(\mathbf{a}_{t_o}, \ldots, \mathbf{a}_{t_{n-1}}/A, \mathfrak{C})]$ for all $\mathbf{s}, \mathbf{t} \in {}^n\mathrm{I}$.

We say that $\mathbf{I}$ is *non-trivial* iff $s \neq t \implies \mathbf{a}_s \neq \mathbf{a}_t$.

DEFINITION 7.10. Supose I is an index model.

$$K(I) := \{J \text{ is an } \mathrm{L}(I)\text{-structure } \mid \forall \mathbf{s} \in J \exists \mathbf{t} \in I[\mathrm{atp}(\mathbf{s}/\emptyset, J) = \mathrm{atp}(\mathbf{t}/\emptyset, I)]\}.$$

EXAMPLES 7.11. There are two important examples for index models

(1) I is a linear order, i.e. $\mathrm{L}(I)$ consits of the binary relation $<$. Denote by K_{ord} the class $K(\omega)$
(2) For an infinite cardinal λ let $I_\lambda := \langle {}^{\omega >}\lambda, \lhd, <_{lex}, h(\cdot,\cdot), P_n \rangle_{n<\omega}$ where
${}^{\omega >}\lambda$ is the set of finite and ω sequences of ordinals less than λ,
$\eta \lhd \nu$ stand for η is an initial segment of ν,
$\eta <_{lex} \nu$ stand for $\eta \neq \nu \wedge \exists n < \omega [\eta[n] < \nu[n] \wedge \eta \restriction n = \nu \restriction n]$,
$h(\eta,\nu)$ is the largest common initial segment to η and ν,
$P_n(\eta) \iff \ell(\eta) = n$.

DEFINITION 7.12.

DEFINITION 7.13.

8. D(T) as a topological space*

Viewing D(T) as a topological space has several advantages: We will get more uniform and conceptual proofs to Theorems 7.24 and 9.15. We will see that the Cantor-Bendixon rank of closed subsets is useful in finding important invariants of the theory.

For a standard reference in topology see [**Du**]. Recall the following basic concepts from general topology:

DEFINITION 8.1.

(1) Let X be a set and let $\mathcal{F}$ be a family of subsets of X. The ordered pair $(X, \mathcal{F})$ is called a *topological space* iff
   (a) $\emptyset$ and X are elements of $\mathcal{F}$
   (b) $\mathcal{F}$ is closed under finite intersections and
   (c) $\mathcal{F}$ is closed under arbitrary union
   The elements of $\mathcal{F}$ are *open sets*, a set $S \subseteq X$ is called *closed* iff $X - S \in \mathcal{F}$. A set $S \subseteq X$ is called *clopen* iff it is open and closed. For any $S \subseteq X$ the $\bar{S}$ *closure of* S is the set $\bigcup\{C \mid C \supseteq S,\ C \text{ is closed}\}$.
   For a set $S \subseteq X$ *the induced topology on* S is $\mathcal{F}_S := \{S \cap U \mid U \in \mathcal{F}\}$.
(2) Let $(X, \mathcal{F})$ be a topological space. A family $\mathcal{F}' \subseteq \mathcal{F}$ is called an *open cover of* X iff $X = \bigcup \mathcal{F}'$. The space $(X, \mathcal{F})$ is *compact* iff for every cover $\mathcal{F}'$ there exists a finite $\mathcal{F}'' \subseteq \mathcal{F}'$ open cover of X.
(3) Let $(X, \mathcal{F})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{F}$ is called a *basis of the space* $(X, \mathcal{F})$ iff for every $S \in \mathcal{F}$ there is $\mathcal{B}_S \subseteq \mathcal{B}$ such that $\bigcup \mathcal{B}_S = S$.
(4) Let $(X, \mathcal{F})$ be a topological space, and $x \in X$. A family $\mathcal{B} \subseteq \mathcal{F}$ is called a *a local basis of the space* $(X, \mathcal{F})$ *at* x iff for every U_x open containing the element x there is $\mathcal{S}_x \in \mathcal{B}$ such that $x \in S_x \subseteq U_x$.
(5) Let $(X, \mathcal{F})$ be a topological space. The space is called *second countable* iff there exists a countable basis of X.
(6) Let $(X, \mathcal{F})$ be a topological space. The space is called *first countable* iff every $x \in X$ there is a countable local basis $\mathcal{B}_x$.
(7) Let $(X, \mathcal{F})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{F}$ is called a *subbasis of the space* $(X, \mathcal{F})$ iff $\{\bigcap \mathcal{B}' \subseteq \mathcal{B} \ : \ |\mathcal{B}'| < \aleph_0\}$ is a basis for $(X, \mathcal{F})$.
(8) A topological space $(X, \mathcal{F})$ is called *Hausdorff* iff for every $x_1 \neq x_2 \in X$ there are pairwise disjoint $U_1, U_2 \in \mathcal{F}$ such that $x_\ell \in U_\ell$ for $\ell = 1, 2$. The space is called *normal* iff for every S_1 and S_2 pairwise disjoint closed sets there are open sets $U_\ell \supseteq S_\ell$ for $\ell = 1, 2$ such that $U_1 \cap U_2 = \emptyset$. The notion of regular space is an interpolant of Hausdorff and norma. A $(X, \mathcal{F})$ be a topological space is called *regular* iff for every $x \in X$ and every closed $S \subseteq X$ such that $x \notin S$ there pairwise disjoint open sets U_1 and U_2 such that $x \in U_1$ and $S \subseteq U_2$.

(9) Let $(X, \mathcal{F})$ be a Hausdorff space. The space is *locally compact* iff for every $x \in X$ there exists an open set $U \ni x$ such that $\bar{U}$ is compact in the induced topology.
(10) The ordered pair (X, d) is called a *metric space* iff $d : X \times X \to \{a \in \mathbb{R} \mid a > 0\}$ satisfying:

$$d(x, y) = 0 \text{ iff } x = y,$$

$$d(x, y) = d(y, x) \text{ for all } x, y \in X \text{ and}$$

$$d(x, z) \leq d(x, y) + d(y, z) \text{ for every } x, y, z \in X.$$

(11) Let (X, d) be a metric space. The space is said to be *ultrametric* iff the metric satisfies the following strong form of the triangle inequality:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \text{ for every } x, y, z \in X.$$

(12) Let (X, d) be a metric space. For any $a \in X$ and $r \in \mathbb{R}$ *the open ball with radius r centered at a* is the set $B(a, r) := \{y \in X \mid d(a, y) < r\}$.
(13) A topological space $(X, \mathcal{F})$ is called *metrizeable* iff there exists $d : X \times X \to \{a \in \mathbb{R} \mid a > 0\}$ such that (X, d) is a metric space and $\{B(a, r) \mid a \in X,\ r \in \mathbb{R}, r > 0\}$ is a basis of the space $(X, \mathcal{F})$.

FACT 8.2 (Urysohn metrization, Corollary IX.9.2 in [**Du**]). *Let $(X, \mathcal{F})$ be a regular topological space. If the space is second countable then the space is metrizeable.*

DEFINITION 8.3. Let (X, d) be a metric space. The space is *complete* iff every Cauchy sequence in X converges. A topological space $(X, \mathcal{F})$ is *topologically complete space* iff it is metrizable by a complete metric. A topological space $(X, \mathcal{F})$ is *Baire* iff for every any countable $\mathcal{F}' \subseteq \mathcal{F}$ such that $D \in \mathcal{F}' \implies D$ is dense. We have $\bigcap \mathcal{F}'$ is dense.

FACT 8.4 (Baire category theorem, Theorems XI.10.1 and XIV.4.1 in [**Du**]). *Let $(X, \mathcal{F})$ be a topological space.*

(1) *If $(X, \mathcal{F})$ is Hausdorff and locally compact then it is a Baire space.*
(2) *If $(X, \mathcal{F})$ is topologically complete space then it is a Baire space.*

The fact that a space is a Bair space is often used in another form. But first we need a definition

DEFINITION 8.5. Let $(X, \mathcal{F})$ be a topological space.

(1) A subset S of the space X is *nowhere dense* iff $X - \bar{S}$ is dense.
(2) A countable union of nowhere dense is called *meager* (also *of the first category*)

PROPOSITION 8.6. *If a topological space $(X, \mathcal{F})$ is Bair then X is not meager.*

PROOF. Let $\{S_n \mid n < \omega\}$ be nowhere dense subsets. The sets $D_n := X - \bar{S}_n$ are dense open sets. Since the space is Bair the set $\bigcap_{n<\omega} D_n$ is dense and in particular not empty. Thus we have

$$X - \bigcup_{n<\omega} \bar{S}_n = \bigcap_{n<\omega} X - \bar{S}_n = \bigcap_{n<\omega} D_n \neq \emptyset.$$

□

DEFINITION 8.7. Let $(X, \mathcal{F})$ be a topological space. $S \subseteq X$ is *nowhere dense* if $X - \bar{S}$ is dense.

So what is the connection with model theory?

DEFINITION 8.8. Let M be a saturated model and $A \subseteq |M|$ such that $|A| < \|M\|$. For any $n < \omega$ and $\varphi(\mathbf{x}; \mathbf{a})$ where $\ell(\mathbf{x}) = n$ and $\mathbf{a} \in A$, let $U_{\varphi(\mathbf{x};\mathbf{a})} := \{p \in \mathrm{S}^n(A, M) \mid \varphi(\mathbf{x}; \mathbf{a}) \in p\}$.

Notice that for M, A and $\varphi(\mathbf{x}; \mathbf{a})$ as above
$M \models \exists \mathbf{x} \varphi(\mathbf{x}; \mathbf{b})$ iff $U_{\varphi(\mathbf{x};\mathbf{a})} \neq \emptyset$.

PROPOSITION 8.9. *Let M, A and $\varphi(\mathbf{x}; \mathbf{a})$ be as in Definition 8.8.*
Let $\mathcal{B} := \{U_{\varphi(\mathbf{x};\mathbf{a})} \mid \mathbf{a} \in A, \varphi(\mathbf{x}; \mathbf{y}) \in L(M), M \models \exists \mathbf{x} \varphi(\mathbf{x}; \mathbf{a})\}$. The family $\mathcal{B}$ of subsets of $\mathrm{S}^n(A, M)$ is closed under finite intersections and finite unions. $\mathcal{B}$ is containing $\emptyset$ and $\mathrm{S}^n(A, M)$ as elements.

PROOF. Exercise. □

The topological space induced by the above basis $\mathcal{B}$ on $\mathrm{S}^n(A, M)$ is called the *Stone space*. It is the contents of the next theorem that the compactness theorem is essentially the statement that the stone space of types is a compact topological space.

THEOREM 8.10. *The Stone space $\mathrm{S}^n(A, M)$ has the following properties:*

(1) *$\mathrm{S}^n(A, M)$ is Hausdorff, moreover it is regular.*
(2) *$\mathrm{S}^n(A, M)$ is compact.*
(3) *If A and $\mathrm{L}(M)$ are at most countable then the space is second countable.*
(4) *The space is Baire.*
(5) *If $|A| + |\mathrm{L}(M)| \leq \aleph_0$ then it is metrizeable and separable.*

PROOF. (1) If $p \neq q \in \mathrm{S}^n(A, M)$ then let $\varphi(\mathbf{x}; \mathbf{a}) \in p$ be such that $\neg\varphi(\mathbf{x}; \mathbf{a}) \in q$. Thus clearly $p \in U_{\varphi(\mathbf{x};\mathbf{a})}$ and $q \in U_{\neg\varphi(\mathbf{x};\mathbf{a})}$ and $U_{\varphi(\mathbf{x};\mathbf{a})} \cap U_{\neg\varphi(\mathbf{x};\mathbf{a})} = \emptyset$. Let $S \subsetneq \mathrm{S}^n(A, M)$ closed and $p \notin S$.
Since $\mathrm{S}^n(A, M) - S$ is open and $p \in \mathrm{S}^n(A, M) - S$ there is $\varphi(\mathbf{x}; \mathbf{a})$ such that $p \in U_{\varphi(\mathbf{x};\mathbf{a})}$ and $U_{\varphi(\mathbf{x};\mathbf{a})} \subseteq \mathrm{S}^n(A, M) - S$. Clearly $U_1 := U_{\varphi(\mathbf{x};\mathbf{a})}$ and $U_2 := U_{\neg\varphi(\mathbf{x};\mathbf{a})}$ are as required.

(2) Suppose for the sake of contradiction that $\{U_{\varphi_i(\mathbf{x};\mathbf{a}_i)} \mid i < \alpha\}$ is a cover such that $\mathbf{a}_i \in A$ and

$$(*) \quad \bigcup_{i\in I} U_{\varphi_i(\mathbf{x};\mathbf{a}_i)} \subsetneq \mathrm{S}^n(A,M) \quad \text{for all finite } I \subseteq \alpha.$$

By $(*)$ for every finite $I \subseteq \alpha$ there exists $p_I \in \mathrm{S}^n(A,M)$ such that $p_I \notin \bigcup_{i\in I} U_{\varphi_i(\mathbf{x};\mathbf{a}_i)}$ which means (since p_I is a complete type) that $\neg\varphi_i(\mathbf{x};\mathbf{a}_i) \in p_I$ for all $i \in I$.
By the compactness theorem the set $q := \{\neg\varphi_i(\mathbf{x};\mathbf{a}_i) \mid i < \alpha\}$ is consistent. By the $(\aleph_0+|A|)^+$-saturation of M there is $p_q \in \mathrm{S}^n(A,M)$ extending q. Since $p_q \supseteq q$ we have that $p_q \notin \bigcup_{i<\alpha} U_{\varphi_i(\mathbf{x};\mathbf{a}_i)}$. Contradiction to being a cover.

(3) In this case the above basis is countable.

(4) By Fact 8.4 and parts $(1), (2)$.

(5) By Fact 8.2 and parts $(1), (3)$. □

Recall the following standard concept:

DEFINITION 8.11. Let X be a topological space and $S \subseteq X$.

(1) $p \in X$ is an *accumulation point of* S iff for every open U such that $p \in U$ there exists $x \in U \cap S - \{p\}$. Denote by S' the *derived set of* S it is $\{p \in X \mid p$ is an accumulation point of $S\}$.
(2) An element $p \in X$ is an *isolated point of* S iff $p \in S - S'$.

REMARK 8.12. A type $p \in \mathrm{S}^n(A,M)$ is an isolated point iff there exists a formula $\varphi(\mathbf{x};\mathbf{a}) \in p$ isolating the type p (in the sense of Definition 9.8). Notice that if $\Gamma \subseteq \mathrm{S}^n(A,M)$ is meager then p is not isolated for all $p \in \Gamma$.

For the purpose of the following application we will consider types with infinitely many variables:

DEFINITION 8.13. Let α be an ordinal, $\mathbf{x}_\alpha := \{x_\beta \mid \beta < \alpha\}$ a sequence of variables and Suppose that $\alpha < \|M\|$.

(1) For $\mathbf{b} \in {}^{\alpha}\mathrm{M}$ let $\mathrm{tp}(\mathbf{b}/A,M)$ be the set:

$$\{\varphi(x_{i_0},\dots x_{i_{n-1}};\mathbf{a}) \ : \ M \models \varphi[b_{i_0},\dots b_{i_{n-1}};\mathbf{a}], n<\omega, i_0 < \cdots i_{n-1} < \alpha\}.$$

(2) $\mathrm{S}^\alpha(A,M) := \{\mathrm{tp}(\mathbf{b}/A,M) \mid \mathbf{b} \in {}^{\alpha}\mathrm{M}\}$.

Similarly to the topology on $\mathrm{S}^n(A,M)$ the set $\mathcal{B}_\alpha$ is a basis for the space $\mathrm{S}^\alpha(A,M)$, where $\mathcal{B}_\alpha$ is defined as:

$$\{U_{\varphi(\mathbf{x};\mathbf{a})} \mid \mathbf{a} \in A, \mathbf{x} \text{ is a finite sequence of variables from } \mathbf{x}_\alpha, \ M \models \exists\mathbf{x}\varphi(\mathbf{x};\mathbf{a})\}.$$

Notice that the statements of Proposition 8.10 still hold (after minor adjustments) for the space $\mathrm{S}^\alpha(A,M)$ e.g. for parts (3) and (5) of Proposition 8.10 we require that

$|\alpha| \leq \aleph_0$. We will use the space $\mathrm{S}^\omega(A, M)$ to get an alternative proof to Henkin's omitting types theorem (Theorem 9.15).

THEOREM 8.14. *Let T and A be countable sets and n a positive integer. If $\Gamma \subseteq \mathrm{S}^n(A, M)$ is meager then there exists a countable $N \models T$ containing A omitting all the types from Γ.*

PROOF. First observe a simple property of the extension of the natural maps between the Boolean algebra (see Exercise 1.10g).

LEMMA 8.15. *For $\beta < \alpha$ let $\pi_{\alpha,\beta} : \mathrm{S}^\alpha(A, M) \to \mathrm{S}^\beta(A, M)$ be the projection map. The maps $\pi_{\alpha,\beta}$ are continuous and open. For all $\alpha > \beta > \gamma$ we have that $\pi_{\alpha,\gamma} = \pi_{\beta,\gamma} \circ \pi_{\alpha,\beta}$.*

PROOF. □

Thus $\Gamma^* := \pi^{-1}_{\omega,n}[\Gamma]$ is meager subset of $S^\omega(A, M)$. By Bair category theorem there exists $q \in \mathrm{S}^\omega(A, M) - \Gamma^*$. Let $\mathbf{a} \in {}^\omega|M|$ be a realization of q. We need to show that the set $\{a_k \mid k < \omega\}$ is a universe of an elementary submodel of M omitting all the types in Γ.

All the types from Γ are omitted: Suppose $\mathbf{b}$ is a sequence of length n from $\{a_k \mid k < \omega\}$. Let $m < \omega$ be such that $\mathbf{b} = \langle a_{k_1}, \ldots, a_{k_n} \rangle$ is a sub sequence of $\{a_k \mid k < m\}$. Since $\pi_{\omega,m}(q) \notin \pi^{-1}_{m,n}[\Gamma]$ and $\mathrm{tp}(\mathbf{b}/A, M) = \pi_{\omega,m}(q)$ we get that $\mathrm{tp}(\mathbf{b}/A, M) \notin \Gamma$. □

The following example will be used in later chapters (see **??**) to connect model theory and algebraic geometry over algebraically closed fields.

First let me introduce an entirely different point of view of Zariski's topology:

DEFINITION 8.16. Let R be a commutative ring with 1. The *spectrum of* R[3] is the set of all prime ideal of R, in symbols:

$$Spec(R) := \{I \mid \ I \text{ is a prime ideal of } R\}.$$

For any $S \subseteq R$, let $C(S) := \{I \in Spec(R) \mid I \supseteq S\}$.

PROPOSITION 8.17. *Let R be a commutative ring with 1, $\langle Spec(R), \{Spec(R) - C(S) \mid S \subseteq R\}\rangle$ is a topological space.*

(1) *The space is T_1 but not Hausdorff.*
(2) *A basis for the topology are $\{Spec(R) - C(f) \mid f \in R\}$.*
(3) *$Spec(R)$ is compact.*

[3]Also known as the *prime spectrum of R*.

PROOF. The first part is easy, note that we use the 1 the second part follows from Hilbert's basis theorem. The last part uses that $1 \in R$. □

The above topology is sometimes called *Zariski's spectrum*.

There is a very interesting connection between the Stone space $\mathrm{S}(F)$ of types over a field and the set of prime ideals of $F[x]$.

DEFINITION 8.18. Let F be an infinite field and $\mathbf{x}$ and n-tuple of variables. Suppose that $\mathfrak{C}$ is a monster model for F, i.e. $F \preceq \mathfrak{C}$. For every $p \in \mathrm{S}^n(F)$ let $I_p := \{q \in F[\mathbf{x}] \ : \ [q(\mathbf{x}) = 0] \in p\}$.

PROPOSITION 8.19. *Suppose F is an algebraically closed field. The mapping $p \mapsto I_p$ is a continuous bijection from the space $\mathrm{S}^n(F)$ onto the space $Spec(F[\mathbf{x}])$.*

PROOF. Notice that for a prime ideal I of $F[\mathbf{x}]$ by quantifier elimination for T_{ACF} the set

$$p_I := \Big\{\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{Fml}(T_{ACF}) \ \Big| \ \mathbf{a} \in F, \\ \{q(\mathbf{x}) = 0 \mid q(\mathbf{x}) \in I\} \cup \{\neg[q(\mathbf{x}) = 0] \mid q(\mathbf{x}) \notin I\} \vdash \varphi(\mathbf{x};\mathbf{a})\Big\}$$

is an element of $\mathrm{S}^n(F)$ and we have that $p = p_{I_p}$ for all $p \in \mathrm{S}^n(F)$ as well as $I_{p_I} = I$ for all $I \in Spec(F[\mathbf{x}])$. □

REMARK 8.20. Notice that all we need for existence of a bijection between the complete types over a field and the set of prime ideals is that $\mathrm{Th}(F)$ admits elimination of quantifiers. This will be exploited when we will deal in section **??**.**??** with *differentially closed fields*.

The ideal I_p can be used to give a quick characterization for saturated models of T_{ACF}.

PROPOSITION 8.21. *Suppose $M \models T_{ACF}$. The model M is λ-saturated if and only if the field M has transcendence degree λ.*

PROOF. Suppose M is λ-saturated. Define by induction on $\alpha < \lambda$ an element $a_\alpha \in M$ such that a_α realizes the type

$$\{f(x) \neq 0 \mid f \in F_\alpha[x]\}, \text{ where } F_\alpha \text{ is the subfield of } M \text{ generated by } \{a_\beta \mid \beta < \alpha\}.$$

Suppose that the model M has transcendence degree λ. Let $A \subseteq |M|$ of cardinality $< \lambda$, let F be the subfield of M generated by A. It is enough to show that any $p \in \mathrm{S}^x(F)$ is realized in M. Given such p, consider I_p. If $I_p = \emptyset$ then x is transcendental over F and we are done since for any $a \in M$ if a is transcendental over F then $F(x) \cong_F F(a)$.

If I_p is not empty then as $F[x]$ is a PID there exists $f \in I_p$ generating it. As M is algebraically closed there is $a \in M$ such that $M \models f(a) = 0$. Using elimination of quantifiers a realizes p. □

REMARK 8.22. Note that the last proposition is totally unnecessary as by Theorems 3.1.9 and 4.1.4 one instantly can see that any uncountable model of T_{ACF} is saturated and as the isomorphism types of models of a fixed characteristic are determined by the transcendence degree a countable model is saturate iff it has transcendence degree $\aleph_0$.

Exercises 6.8.

(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)

9. The topology of Lascar's groups

10. More on existence, omitting types, and the completeness theorem

The following definitions are motivated by P. Cohen's forcing technique (from set-theory).

DEFINITION 10.1. Let $(P, \leq)$ be a partially ordered set.

(1) A subset $D \subseteq P$ is called *directed* iff for every $p, q \in D$ there exists $r \in D$ such that $p \leq r \wedge q \leq r$.
(2) A subset $D \subseteq P$ is called λ-*directed* iff for every $Q \subseteq D$ of cardinality $< \lambda$ there exists $r \in D$ such that $p \leq r$ for all $p \in Q$.
(3) $D \subseteq P$ is called *dense* iff for every $p \in P$ there exists $q \in D$ such that $q \geq p$.
(4) $D \subseteq P$ is *open* (or *upward closed*) iff for every $p \in P$ there exists $q \in D$ such that $p \leq q$.
(5) $D \subseteq P$ is *downward closed* iff for every $p, q \in P$ the following implication holds: $p \in D \implies \forall q \leq p \to q \in D$.
(6) Let F be a family of dense open subsets of P. A subset G of P is called F- *generic* iff G is directed, downward closed, and for every $D \in F$ we have that $D \cap G \neq \emptyset$.

REMARK 10.2.

(1) The choice of the words "dense" and "open" is based on a topological motivation. In the topology induced by the partial order the sets introduced in $(2)\&(3)$ are dense and open (respectively).
(2) A poset is directed iff it is $\aleph_0$-directed.
(3) What is called dense in Definition 2.1(2) is called unbounded in Definition I.8.1.
(4) When $(P, \leq)$ is a linearly ordered set, or has a maximal elements the above notions are not very interesting. From now on we will assume that $(P, \leq)$ satisfy a non-triviality requirement:

HYPOTHESIS 10.3. For every $p \in P$ there exist $q_1, q_2 \in P$ such that $q_l \geq p$ $(l = 1, 2)$ and q_1, q_2 do not have a common extension (in such a case we say that q_1 and q_2 are *incompatible elements*).

The fundamental question to ask in this framework is: Given a poset $\langle P, \leq \rangle$ and a family of dense open sets $\mathcal{F}$. Does there exist an $\mathcal{F}$ - generic subset of P? Actually we would like to be able to answer a harder question: For which posets $\langle P, \leq \rangle$ and which cardinal number λ we have that: For every family $\mathcal{F}$ of cardinality at most λ of dense open sets there exists an $\mathcal{F}$ generic subset of P. It should be clear that this question is trivial when $(P, \leq)$ is a linearly ordered set. Certainly the above must be less or equal to $2^{|P|}$. The following example shows that in general we can not expect $= 2^{|P|}$.

PROPOSITION 10.4. *There exists a countable poset that has a family of dense open sets of the cardinality of the continuum for which a generic set does not exist.*

PROOF. Let $P = \{f : n \to 2 \mid n < \omega\}$ with the natural ordering (function extension). For every $f : \omega \to 2$ consider $D_f = \{g \in P \ : \ g \leq f\}$, and for $n < \omega$ let $D_n = \{f \in P \mid n \in \operatorname{Dom} f\}$. Define $\mathcal{F} = \{D_f, D_n \mid f : \omega \to 2, n < \omega\}$. It is easy to see that F is a family of dense open sets. Suppose that there exists a $G \subseteq P$ which is $\mathcal{F}$ - generic. Since G is directed $h := \bigcup G$ is a function from a subset X of ω onto 2. Since $n < \omega \Longrightarrow G \cap D_n \neq \emptyset$. For every $n < \omega$ we have that $n \in \operatorname{Dom} h$. Namely $X = \omega$ or $h : \omega \to 2$. Since G is $\mathcal{F}$ - generic in particular we have that $G \cap D_h \neq \emptyset$. Let $f \in G \cap D_h$. By the definition of the ordering on P and the definition of h from $f \in G$ we have that $f \subseteq h$. But D_h is the set of all functions which are not restrictions of h. I.e. $f \in D_h$ contradiction to the assumption that there exists a function $f \in G \cap D_h$. □

The following can be viewed as a generalization of Proposition 2.3:

PROPOSITION 10.5. *Let $\langle P, \leq \rangle$ be as in Hypothesis 2.2, and let $\mathcal{F}$ be a non-empty family of dense open subsets of P. If $G \subseteq P$ is $\mathcal{F}$-generic then $P - G$ is dense open.*

PROOF. Let $p \in P$ be given. By Hypothesis 2.2 there exist an incompatible pair q_l ($l = 1, 2$). Let $D \in \mathcal{F}$; By the density of D (and genericity of G) there exists $r \in G \cap D$ such that $r \geq q_l$. Clearly r and q_l are incompatible. Since G is directed $q_l \in G$. Thus $q_l \in P - G$. Using the fact that G is downward closed given $q \in P - G$ for all $r \in P$ such that $r \geq q$ we have $r \in G$. Which establishes that $P - G$ is open. □

The second part of the next Corollary generalizes Proposition 2.3:

COROLLARY 10.6. *Let $(P, \leq)$ be as in Hypothesis 2.2.*

(1) *Suppose $\mathcal{F}$ is a non-empty family of dense open subsets of P. If $\exists G \subseteq P$ $\mathcal{F}$-generic then $\exists D \subseteq P$ dense open such that $G \cap D \neq \emptyset$.*
(2) *If $\mathcal{F}$ is the family of all dense open subsets of P then $\exists G \subseteq P$ which is $\mathcal{F}$-generic.*

PROOF. Follows from Proposition 2.4. □

In the early sixties Paul Cohen introduced the above ideas to set theory and have proved the following fundamental result (which we will not use in this book). Suppose V is a transitive class (namely the $\in$ relation is transitive) containing the ordinals and $(P, \leq)$ is a poset in V, and suppose $G \subseteq P$ be $\{D \in V | D \subseteq P$ is dense and open$\}$ - generic; By $V[G]$ denote the minimal transitive class containing

$V \cup \{G\}$. Then if $\langle V, \in\rangle \models \mathrm{ZF}$ then $\langle V[G], \in\rangle \models \mathrm{ZF}$. He found two posets P_l and P_l such that $V[G^{P_1}] \models \mathrm{ZF} + \mathrm{AC}$ and $V[G^{P_1}] \models \mathrm{ZF} + \mathrm{CH}$. Were G^{P_1} stands for a generic subset of P_1. .....THE INDUSTRY OF FORCING .....

The next is a fundamental and easy positive result.

LEMMA 10.7. *Let $\langle P, \leq\rangle$ be an infinite poset. If $\mathcal{F}$ is a countable family of dense open subsets of P then there exists an $\mathcal{F}$-generic subset of P.*

PROOF. Suppose $\{D_n \ : \ n < \omega\}$ is an enumeration of F. For every $n < \omega$ define $p_n \in P$ as follows.

(1) $p_n \in D_n$, and
(2) $p_{n+1} \geq p_n$.

This is enough: Let $G := \{q \in P : (\exists n < \omega) q \leq p_n\}$. Certainly G is directed. It is obvious that $\bigcup_{n<\omega} G \cap Dn \neq \emptyset$. This is possible: By induction on $n < \omega$: Pick $p_0 \in D_0$ be an arbitrary element. Assume that $p_n \in D_n$ is defined, by the density of D_{n+1} there exists $p_{n+1} \in D_{n+1}$ such that $p_{n+1} \geq p_n$. □

The obvious question to ask at his stage: Is Lemma 2.3 the best possible? Namely, would it be possible to remove the assumption of countability of the family $\mathcal{F}$ in Lemma 2.6? Little reflection shows that without some additional assumptions on the poset P this is false. This brings us to

DEFINITION 10.8. Let $\langle P, \leq\rangle$ be a poset, and let λ be an infinite cardinal number. We say that P is λ-*complete* iff for every $\mu < \lambda$ and every $\leq$-increasing sequence $\{p_i \ : \ i < \mu\}$ (i.e. $i < j < \mu \implies p_i \leq p_j$) the $\sup_{i<\mu} p_i$ exists and belong to P.

REMARKS 10.9. (1) Any poset is $\aleph_0$-complete.

(2) What we call $\aleph_1$ - complete is often called σ - complete.

EXAMPLE 10.10. Let T be a consistent theory, let λ be an infinite cardinal number, and let $C = \{c_\alpha \ : \ \alpha < \lambda\}$ be new constants (not in $\mathrm{L}(T)$).
Let

$$P = \{q \subseteq \mathrm{Sent}(\mathrm{L}(T) \cup C) \mid q \cup T \text{ is consistent, and } |q| < \lambda\}$$

the ordering is the subset relation. It is clear that if ¡= then P is - complete.

Using the notion of - completeness we can formulate a generalization of Lemma 2.6.

THEOREM 10.11. *Let be a cardinal number, and let $\langle P, \leq\rangle$ be a poset. If $\langle P, \leq\rangle$ is - complete then for every family $\mathcal{F}$ of cardinality at most of dense open subsets of P there exists an $\mathcal{F}$-generic subset of P.*

PROOF. Suppose that $\{D_i \ : \ i < \omega\} = \mathcal{F}$. By induction on $i < \omega$ define $p_i \in P$ such that

(1) $p_i \in D_i$, and
(2) $p_i \geq p_j$ for every $j < i$.

For $i = 0$ pick p_0 to be an arbitrary element of D_0. Suppose that $\{p_j \ : \ j < i\}$ are well defined. By the - completeness denote $p := Sup_{j<i} p_j \in P$.

By the density of D_i there exists $p_i \in D_i$ such that $p_i \geq p$. It is easy to check that $G := \{q \in P \ : \ (\exists i <) q \leq p_i\}$ is F-generic. □

In order to convince the reader that Theorem 2.6 is powerful we use it below to derive the "hard" direction of Gödel's completeness theorem.

PROOF. of Theorem I.3.1 $(a) \Longrightarrow (b)$ Suppose T is a consistent theory. By Zorn's lemma we may assume without loss of generality that T is complete. Let be an arbitrary cardinal number which is at least $|\,\mathrm{L}(T)|$, let $C = \{c_i \ : \ i < \omega\}$ be a set of new constant symbols (not in $\mathrm{L}(T)$), and let $L' = \mathrm{L}(T) \cup C$. Define

$$P = \{q \subseteq \mathrm{Sent}(L') \ : \ |q| < \aleph_0 \text{ and } T \cup q \text{ is consistent}\}.$$

For every formula $\varphi(x) \in \mathrm{Fml}(\mathrm{L}(T))$ such that $T \models \exists x \varphi(x)$ let

$$D_\varphi := \{q \in P \mid \text{there exists } i < \omega \text{ such that } c_i \text{ does not appear in } q, \text{ and } [\exists x \varphi(x) \to \varphi(c_i)] \in q\}.$$

$D\varphi$ is dense: Given $p \in P$ suppose that there is no $q \in D_\varphi$ such that $q \supseteq$p. Namely for every $i <$ we have that $\{[\exists x \varphi(x) \to \varphi(c_i)]\} \cup T \cup p$ is not consistent. Since $|p| <$ there exists $i_0 < \omega$ such that c_{i_0} does not appear in p. Hence in particular $\{[\exists x \varphi(x) \to \varphi(c_{i_0})]\} \cup T \cup p$ is not consistent. By the consistency of $T \cup p$ (remember that $p \in P$) this is possible only if $T \cup p \models [\exists x \varphi(x) \to \varphi(c_{i_0})]$ which by propositional logic entails that $T \cup p \models \exists x \varphi(x) \wedge \varphi(c_{i_0})$. Since c_{i_0} does not appear in $T \cup p$ we may replace the constant by a universally quantified variable. Namely we have $T \cup p \models \forall y [\exists x \varphi(x) \wedge \varphi(y)]$ when y does not appear in φ . Since $\exists x \varphi(x)$ is a sentence we have $T \cup p \models [\exists x \varphi(x) \wedge \forall y \varphi(y)]$ which by the duality between universal and existential quantifiers give $T \cup p \models [\exists x \varphi(x) \wedge \exists y \varphi(y)]$ which is a contradiction to the consistency of $T \cup p$. For every function symbol $F(x)$ of $\mathrm{L}(T)$ and every $c \in$ let d be a constant symbol of L' which does not appear in c, now define $D_{F,c,d} := \{q \in P \ : \ [F(c) = d] \in q\}$. It follows from the axioms of predicate calculus that $D_{F,c}$ is dense. Now apply Theorem 2.6 to the following family of dense sets $\mathcal{F} := \{D_\varphi, D_{F,c,d} \ : \ \varphi \in \mathrm{Fml}(\mathrm{L}(T)), F(x)$ a function symbol, $c \in (x)\mathbf{C}(L')$, and $d \in \mathbf{C}(L')\}$. Let G be an $\mathcal{F}$-generic set. Define $T' = \cup G$. From the definition of P and the finite character of consistency it is clear that $T \cup T'$ is consistent. Actually $T^* \supseteq T$ since T' contains all $\mathrm{L}(T)$-sentences of the form $\exists x \varphi(x)$ when $T \models \exists x \varphi(x)$ (G intersects $D\varphi$). Let T^* be a completion of T' in L'. For any c, d constant symbols ofL' define $c \approx d$ iff

$T^* \models c = d$. It is easy to show that $\approx$ is an equivalence relation on $\mathbf{C}(L')$ which is compatible with the function and relation symbols of $\mathrm{L}(T)$. For $c \in \mathbf{C}(L')$ let denote its $\approx$ equivalence class by $[c]$. Now define $A = \{[c]_{\approx} \ : \ c \in \mathbf{C}(L')\}$, it forms a universe of a model of T^* when the relations and functions are defined using representatives of the equivalence classes (notice that this is well defined by the logical axioms of predicate calculus governing equality). At the end take a reduct to $\mathrm{L}(T)$. The details are left to the reader. □

If the only thing we could do with Lemma 2.3 or Theorem 2.6 is just to get the above proof of the completeness theorem (which by the way is very similar to that of Henkin) than we would not present it here. I claim that we have a very powerful uniform method to build models from constants (notice that in the model constructed in the last proof every element corresponds (to at least one) constant in L'). We can "milk" the generic set by intersecting it with more dense open sets. Another example is a proof to Theorem 1.6 (generalizing Henkin's omitting types theorem to omit countable many types in several variables simultaneously).

PROOF. of Theorem 1.6 Let T be a countable theory, and let $\{p_n \ : \ n < \omega\}$ be a set of T- types (p_n is a $k(n)$ - type)) such that none of the types is isolated. Let C be a countable set of new constant symbols. Define $P = \{q \subseteq \mathrm{Sent}(\mathrm{L}(T) \cup C) \ : \ |q| < \aleph_0,$ and $q \cup T$ is consistent$\}$ when the order relation is the subset relation. For every $n < \omega$ and every $d \in C$ of length $k(n)$ let $D_{n,d} := \{q \in P|$ there exists $\psi(x) \in p_n$ such that $\psi(d) \in q\}$. $D_{n,d}$ is dense: Let $p \in P$ be given, suppose that $D_{n,d}$ is not dense for some $n < \omega$ and $d \in k(n)C$. Namely for every $\psi(x) \in p_n$ we have that $\psi(d) \cup p \cup T$ is inconsistent. By the consistency of $p \cup T$ this is possible only if for every $\psi(x) \in p_n$ $p \cup T \models \psi(d)$. Since q is finite we have that $T \models \bigwedge p \to \psi(d)$. Since the constants d do not appear in T we may replace them in $\bigwedge p \to \psi(d)$ by new variables x. Let $\bigwedge p(x) \to \psi(x)$ be that new formula. Now we have that $T \models \forall x[\bigwedge p(x) \to \psi(x)]$ for every $\psi(x) \in p_n$. Namely $\bigwedge p(x)$ isolates the type p_n contradiction to the assumption. Let $\mathcal{F}$ be the family of dense sets from the proof of Theorem I.3.1. Define $F' := F \cup \{D_{n,d} \ : \ n < \omega, d \in k(n)C\}$. Let G be $\mathcal{F}'$ - generic (exists by Lemma 2.3), and let T^* be a completion of $\cup G$. As before $A := \{[c] \ : \ c \in C\}$ is a universe of a model M of T^*. We need to show that for $n < \omega$ the type p_n is omitted in M. Suppose p_n is realized, then there exists d such that $\psi \in p_n \Longrightarrow M \models \psi[d]$. Namely $T^* \models \psi[d]$, but since $D_{n,d}$ is dense there exists $\psi(x) \in p_n$ such that $\psi(d) \in \bigcup G$. □

Exercises 6.10.

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11. The Paris Harrington's theorem*

For many years Gödel's incompleteness theorem of arithmetic was criticized as to be too philosophical and there was a quest for a more "mathematical" looking statement that may be independent of Peano's arithmetic.

In 1977 Jeff Paris and Leo Harrington have managed to prove that a version of the finite Ramsey's theorem is independent of Peano's arithmetic while it is clearly provable in ZFC.

Their method at first was considered to be very promising and there was hope that it will be adopted to prove independence of other hard problems in finite mathematics (maybe $\mathrm{P} = \mathrm{NP}$?).

12. More on two cardinal theorems*

In this section we discuss several two cardinal theorems that can be viewed as generalizations of the the upward Löwenheim Skolem Theorem (for theories). The central theorems of this section are theorems of Chang, Jensen, Vaught and Morely.

The flavor of Jensen's two cardinal theorem is more set-theoretic than model-theoretic (it depends on a new axiom), in recent books it have shifted out from model theory texts into set theory texts. However due to the importance of the two-dimensional construction used in the proof (for simpler - see Theorems 1.6.4 and 2.13.19 and for more elaborate extensions see Chapter **??**).

We start with Chang's two cardinal theorem which is a converse to Vuaght's two cardinal theorem (Theorem 12.9):

THEOREM 12.1 (Chang's two cardinal theorem). *Let T be a first-order theory. If $\lambda = \lambda^{<\lambda} > |\mathrm{L}(T)|$ then $\langle \aleph_1, \aleph_0 \rangle \to \langle \lambda^+, \lambda \rangle$.*

In section 1 we have introduced a localized version of saturation that will be used here. Recall Definition 2.32.

We will use special cases of Theorems 1.18 and 2.37:

FACT 12.2. *Let L be a given similarity type that contains a unary predicate P.*

(1) *Suppose that M is an L structure such that P^M is infinite.*
If $\lambda \geq \|M\| + 2^{|L|}$ and $\lambda = \lambda^{<\lambda}$ then there exists a saturated model $N \succ M$ of cardinality λ.
(2) *Suppose M is a saturated L-structure. If $N \equiv M$ is a P-saturated of cardinality $\|M\|$ then there exists an elementary embedding $g : N \to M$ such that $g[P^N] = P^M$.*

Before starting the argument of the proof of Theorem 12.1 let me explain the idea and the main difficulty.

Certainly it is enough to define an elementary chain $\{M_\alpha \mid \alpha < \lambda^+\}$ which is increasing and continuous.

Such that

(1) $M_0 \models T$,
(2) $\|M_\alpha\| = |P^{M_\alpha}| = \lambda$, for every $\alpha < \lambda^+$
(3) $P^{M_\alpha} = P^{M_{\alpha+1}}$, for every $\alpha < \lambda^+$ and
(4) $M_\alpha \subsetneq M_{\alpha+1}$, for every $\alpha < \lambda^+$.

Given such a chain of models clearly $M := \bigcup_{\alpha<\lambda^+} M_\alpha$ will be a $\langle \lambda^+, \lambda\rangle$ model.

The difficulty is how to choose define the above chain at the successor stage. Namely, suppose we are given with a model M_α such that $\|M_\alpha\| = |P^{M_\alpha}| = \lambda$ we need to be able to find $M_{\alpha+1}$ which is a proper elementary extension of M_α of cardinality λ without extending the predicate.

PROOF. □

Suppose that M is a structure of cardinality μ^+ such that a unary predicate P is interpreted in M by a set of cardinality μ. Since the number of all finite subsets of P^M is also μ there is a bijection

$$f : \mathrm{S}_{<\aleph_0}(P^M) \to P^M.$$

let $E \subseteq P^M \times P^M$ defined by

$$\langle x, y\rangle \in E \iff \exists s \in \mathrm{S}_{<\aleph_0}(P^M)[x \in s \wedge f(s) = y].$$

Namely for $b \in P^M$ the set $\{a \in P^M \mid \langle a, b\rangle \in E\}$ codes the finite set of elements mapped by f to b.

By the downward Löwenheim-Skolem theorem, there is $N \preceq \langle M, E\rangle$ of cardinality μ such that $P^N = P^M$. Consider now $\mathfrak{B} := \langle M, |N|, E\rangle$ such that $\mathrm{L}(\mathfrak{B}) = \mathrm{L}(M) \cup \{Q, E\}$ when Q is a new unary predicate interpreted by $|N|$.

Notice that by the choice of E we have that

$$\mathfrak{B} \models \forall z_0 \cdots \forall z_{n-1}[\bigwedge_{i<n} P(z_i) \wedge \bigwedge_{i<n} \varphi(z_i; \mathbf{x}) \to \exists y[\bigwedge_{i<n} (z_i E y) \wedge \forall v[vEy] \to \varphi(v; \mathbf{x})]$$

holds for every $\varphi(x; \mathbf{y}) \in \mathrm{L}(\mathfrak{B})$ and every $n < \omega$.

Since $\lambda = \lambda^{<\lambda}$ there exists $\mathfrak{A} \models \mathrm{Th}(\mathfrak{B})$ saturated of cardinality λ.

Now define an increasing continuous chain $\{M_\alpha \mid \alpha < \lambda^+\}$ such that

(1) M_α is (P, λ)-saturated (i.e. for every 1-type containinig $P(x)$ with $< \lambda$-many parameters is realized in M_α).
(2) $P^{M_\alpha} = P^{M_{\alpha+1}}$.

LEMMA 12.3. *Assume Chang's trick. If $\{M_i : i < \alpha\}$ is an increasing and continuous chain of (λ, P)-saturated models then also $\bigcup_{i<\alpha} M_i$ is (λ, P)-saturated.*

PROOF. □

Later (Theorem 5.**??**) we will show that Chang's trick is not necessary for superstable T.

PROOF. □

The Chang's theorem leaves open several questions:
Questions:

(1) Is it necessary to assume that λ is singular? At least, are there models of ZFC where $\langle \aleph_1, \aleph_0 \rangle \to \langle \aleph_{\omega+1}, \aleph_\omega \rangle$ holds?
(2) Suppose λ is regular. Is the cardinal-arithmetic assumption on λ necessary?
(3) In the proof we where expanding the language L, is this necessary?

The first question was answered by Jensen using the set-theoretic principle know as square:

DEFINITION 12.4. Let λ an infinite cardinal. The principle *square* λ is denoted by $\square_\lambda$. It is the following principle:

There exists a sequence $\langle C_\delta \mid \delta < \lambda^+,\ \delta \text{ limit} \rangle$ such that for every limit $\delta < \lambda^+$:

(1) C_δ is closed and unbounded subset of δ,
(2) if $\mathrm{cf}(\delta) < \lambda$ then $|C_\delta| < \lambda$, and
(3) if ξ is a limit point of C_δ then $C_\xi = \xi \cap C_\delta$.

THEOREM 12.5 (Jenssen 1972). *Let T be a first-order theory. Assume the generalized continuum hypothesis. Suppose $\lambda > |\mathrm{L}(T)|$ is singular and $\square_\lambda$ holds. Then $\langle \aleph_1, \aleph_0 \rangle \to \langle \lambda^+, \lambda \rangle$.*

Mention Litman's observation and maybe Ben-David and Magidor (Rami?). [**BG**].

It is known that Chang's two-cardinal theorem is not provable from ZFC alone:

FACT 12.6 (Mitchell?). *If $Con(\mathrm{ZFC} + \exists \kappa [\kappa$ is weakly compact $])$ then $Con(\mathrm{ZFC} + \neg \exists \tau [\tau$ is an $\aleph_2$ Aronszajn tree $])$*

THEOREM 12.7 (Vaught 1965). *Let L be a language containing a unary predicate P. Suppose that M is an L-structure such that $\|M\| \geq \beth_\omega(|P^M| + |L|)$ then for every $\lambda \geq |L| + \aleph_0$ there exists $N \equiv M$ of cardinality λ and $|P^N| \leq |L|$.*

PROOF. □

Vaught gap $\beth_\omega$ and Morley's $\beth_{\omega_1}$.

without expanding the language see [**Sh72**] and [**Gr2**]

Exercises 6.12.

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13. Chang's conjecture and Jónsson algebras*

In this section we discuss existence of Jónsson algebras algebras and a generalization of Vaught's two cardinal theorem. In an unexpected way the two are related via a Ramsey-like partition relation.

Chang's conjecture is introduced in the paragraph after Theorem 1.12.10.

The following strengthening of Vaught's two cardinal theorem is natural:

DEFINITION 13.1. The statement that for every model M of type $\langle \aleph_2, \aleph_1 \rangle$ (in a countable language) there exists $N \preceq M$ of type $\langle \aleph_1, \aleph_0 \rangle$ is called *Chang's conjecture*. This can be generalized: Suppose that $\lambda_1, \lambda_2, \chi_1$ and χ_2 are cardinals such that $\lambda_1 > \lambda_2$ and $\lambda_2 \geq \chi_1 > \chi_2 \geq \aleph_0$ holds.
We denote by $\langle \lambda_2, \lambda_2 \rangle \twoheadrightarrow \langle \chi_1, \chi_2 \rangle$ the statement:
For every model M of cardinality λ_1 (in a countable language) with a predicate P^M of cardinality λ_2 there exists $N \preceq M$ of cardinality χ_1 such that $|P^N| = \chi_2$.

According to a legend, several times in the mid sixties C.C. Chang proposed to his graduate students in UCLA the above conjecture as a thesis problem. They all failed to prove it.

Later it turned out that Chang's conjecture is independent of ZFC. Currently Chang's conjecture is of more interest to set-theorists than to model-theorists. Following this trend I will only mention some of the basic facts about its status.

It is interesting that when $\mathrm{Th}(M)$ is $\aleph_0$-stable then the strongest form of Chang's Conjecture is a ZFC theorem. We will establish this as a bi-product of our proof of Morley's categoricity theorem, see Theorem 7.6.

FACT 13.2.

(1) *(Silver)* $Con(\mathrm{ZFC}+\exists\kappa[\kappa$ *is a Ramsey cardinal* $])$ *implies that* $Con(\mathrm{ZFC}+$ *Chang's conjecture holds* $])$.
(2)

EXAMPLE 13.3.

FACT 13.4. *Shelah/Rips on Kurosh*

PROPOSITION 13.5. *Connect Chang's conjecture with Jónsson algebras*

PROPOSITION 13.6. *Suppose that λ is an infinite cardinal number. there exists a similarity type L of cardinality at most λ and an L-structure of cardinality λ^+ which is a Jónsson algebra*

It is interesting to notice that Chang's conjecture is equivalent to a partition theorem which is a "weak Ramsey" theorem.

First we need:

DEFINITION 13.7 (square bracket symbol). Let λ, μ and κ be cardinals. Suppose n is a positive integer.

$$\lambda \to [\mu]^n_\kappa \overset{def}{\iff} \text{For every } F : [\lambda]^n \to \kappa \text{ there are } i_0 \in \kappa, \text{ and } S \subseteq \lambda \text{ of cardinality } \mu \text{ such that } i_0 \notin F\big[[S]^n\big].$$

In such a case we say that *the set S omits the color i_0* .

This partition relation is a weak Ramsey relation due to the following:

PROPOSITION 13.8. *Let λ, μ and n be as in the previous definition.*

$$\lambda \to [\mu]^n_2 \iff \lambda \to (\mu)^n_2$$

PROOF. Easy exercise. □

The following partition relation is a generalization of the above partition relation:

DEFINITION 13.9. Let λ, μ, κ_0 and κ_1 be cardinals. Suppose n is a positive integer.

$$\lambda \to [\mu]^n_{\kappa_1,\kappa_0} \overset{def}{\iff} \text{For every } F : [\lambda]^n \to \kappa_1 \text{ there exists } S \subseteq \lambda \text{ of cardinality } \mu \text{ such that } |\{i_0 < \kappa_1 \ : \ i_0 \notin F\text{“}[S]^n\}| = \kappa_0.$$

As usual $\lambda \to [\mu]^{<\omega}_{\kappa_1,\kappa_0}$ holds iff the domain of the coloring F is the set $[\lambda]^{<\omega}$.

Fredrick Rowbottom in his 1964 Ph.D. thesis discovered (among other things) connections between large cardinals and fundamental model theoretic questions. One of his more surprising results is the following:

THEOREM 13.10 (Rowbottom [**Ro**]). *Suppose that $\lambda_1, \lambda_2, \chi_1$ and χ_2 are cardinals such that $\lambda_1 > \lambda_2$ and $\lambda_2 \geq \chi_1 > \chi_2 \geq \aleph_0$ holds. The statement $\langle \lambda_2, \lambda_2 \rangle \twoheadrightarrow \langle \chi_1, \chi_2 \rangle$ is equivalent to $\lambda_1 \to [\chi_1]^{<\omega}_{\lambda_2,\chi_2}$*

Using a simple inductive argument Erdős and Hajnal in 1974 have shown that:

PROPOSITION 13.11. *$\aleph_2 \to [\aleph_1]^2_{\aleph_1,\aleph_0}$ is equivalent to $\aleph_2 \to [\aleph_1]^{<\omega}_{\aleph_1,\aleph_0}$*

Thus we get:

COROLLARY 13.12 (Erdős and Hajnal [**EH**]). *Chang's conjecture is equivalent to $\aleph_2 \to [\aleph_1]^2_{\aleph_1,\aleph_0}$*

To get L finite is much more difficult.

The following was proved independently by Tryba [**?**] and Woodin in 1984:

THEOREM 13.13 (exsitence of Jónsson algebra for successors of regulars). *Suppose that λ is an uncountable regular cardinal number. there exists a finite similarity type L and an L-structure of cardinality λ^+ which is a Jónsson algebra.*

Before proving the above theorem let's recall from Definition 1.3.24 the notions of Jónsson algebra and Jónsson model. Clearly every Jónsson algebra is a Jónsson model and the converse is false.

From the point of view of Theorem 13.13 it is natural to consider:

DEFINITION 13.14. A cardinal number μ is a *Jónsson cardinal* iff there exists a Jónsson algebra of cardinality μ in a countable similarity type.

By Exercise 1.3.39 the cardinals $\aleph_n$ (for every $n < \omega$) are Jónsson. By Theorem 13.13 all successors of regulars are Jónsson.

A natural question is why did we define the concept only for algebras, why not models? Is there a parallel concept for Jónsson models? The reason is the following observation:

PROPOSITION 13.15. *Let $\mu > \aleph_0$. If there exists a Jónsson model of cardinality μ in a countable language then μ is a Jónsson cardinal.*

PROOF. Let M be the model of cardinality μ. By Theorem 1.14.13 there exists an expansion M^{sk} of M with Skolem functions such that $|\mathrm{L}(M^{sk})| = \aleph_0$. Let $\{F_n \mid n < \omega\} = \mathrm{L}(M^{sk}) - \mathrm{L}(M)$. Check that $\mathcal{A} := \langle |M|, F_n^{M^{sk}} \rangle_{n<\omega}$ is a Jónsson algebra of cardinality μ. □

Now we are ready to prove Theorem 13.13:

PROOF. By Proposition 13.15 it is enough to find a Jónsson model of cardinality λ^+ in a countable language. Let $S := \{\delta < \lambda^+ \mid \operatorname{cf} \delta = \lambda\}$. Since S is stationary by Theorem 1.19 there exists a partition $\bar{S} := \{S_\alpha \mid \alpha < \lambda^+\}$ of S into stationary sets. Since $\bar{S}$ is a partition we have a function $f : S \to \lambda^+$ such that $\beta \in S \implies \beta \in S_{f(\beta)}$. Let χ be a regular large enough so that $\{\bar{S}, S, \lambda^+, f\} \cup \lambda^+ \subseteq H(\chi)$ and $\langle H(\chi), \in \rangle$ reflects all relevant information on the above mentioned sets.

Pick $M \prec \langle H(\chi), \in \rangle$ of cardinality λ^+ such that $M \supseteq \lambda^+ \cup \{f, S\}$. Now fix $g : \lambda^+ \to |M|$ a bijection. Let

$$N := \langle |M|, \lambda^+, \in, f, g, s \rangle$$

where λ^+ is a unary predicate interpreted by the set of ordinals less than λ^+, f and g are unary functions interpreted by the corresponding functions and the constant s is interpreted by the set S.

CLAIM 13.16. *N is a Jónsson model.*

PROOF. Let $N' \preceq N$ of cardinality λ^+. Since $\|N'\| = \lambda^+$ (using the function g) also the interpretation of λ^+ in N' has cardinality λ^+. In order to get $N' = N$ (using g) it suffices to show that $\lambda^+ \subseteq |N'|$.

Let $C := \{\delta < \lambda^+ \mid \delta = \sup |N'| \cap \delta\}$. Since $\lambda^{+N'}$ has cardinality λ^+ and is a regular cardinal the mapping $\delta \mapsto \sup |N'| \cap \delta$ is a function from λ^+ into λ^+, by Theorem 1.1.9 the set C is a club.

It is enough to show:

LEMMA 13.17. $C \cap S \subseteq |N'|$

Why? Let $\alpha < \lambda^+$ be given. Since S_α is stationary $C \cap S_\alpha \neq \emptyset$ and by the lemma there exists $\delta \in C \cap S_\alpha \cap |N'|$. By the definition of f we get that $\alpha = f(\delta)$. Since $\delta \in |N'|$ and N' is closed under f we get hat $\alpha \in |N'|$.

PROOF. For the sake of contradiction suppose that there exists $\delta \in C \cap S_\alpha - N'$. Let

$$\alpha := \min\{\alpha \in N' \mid \alpha > \delta\}.$$

Since $\delta \notin N'$ and N' is a weak model of set theory we get that also $\delta + 1 \notin N'$ thus α must be a limit ordinal.

CLAIM 13.18. *There exists a club $D \subseteq \alpha$ such that $\delta \in D \implies \operatorname{cf} \delta < \lambda$.*

PROOF. Define a continuous increasing elementary chain $\{M_\zeta \preceq \langle \alpha, \in\rangle \mid \zeta < \lambda\}$ such that $\bigcup_{\zeta<\lambda} |M_\zeta| = \alpha$ and $\zeta < \lambda \implies \|M_\zeta\| < \lambda$. Let $\alpha_\zeta := \sup |M_\zeta|$ for $\zeta < \lambda$. Check that $D := \{\alpha_\zeta : |M_\zeta| = \alpha_\zeta, \zeta < \lambda\}$ is as required. □

Since $\delta \in C$ we have $\delta = \sup |N'| \cap \delta$, by $\delta \notin N'$ and the choice of α we have $|N'| \cap [\delta, \alpha) = \emptyset$. Thus we get that

$$(*) \quad \delta = \sup |N'| \cap \alpha.$$

Since $\langle |N'|, \in\rangle \preceq \langle H(\xi), \in\rangle$ and $\{S, \alpha\} \subseteq |N'|$ we get

$$N' \models D \text{ is unbounded in } \alpha.$$

By $(*)$ we get that $\delta = \sup |N'| \cap \alpha = \sup |N'| \cap D$. Thus the subset $|N'| \cap D$ of D is bounded, since D is closed its supremum $\delta \in D$. This is a contradiction to the choice of δ as an element of S. □

□

Tryba's original proof proved a slightly stronger theorem than Theorem 13.13. Before stating it we need a definition: For an uncountable regular μ and a stationary subset S of μ we say that S *does not reflect* iff for every limit $\alpha < \mu$

there is a club $C \subseteq \alpha$ such that $\alpha \cap S = \emptyset$. Notice that the contents of Claim 13.18 is that $S^{\lambda^+}_{\lambda}$ does not reflect. It is not difficult to see that the above proof yields:

If μ is an uncountable regular such that there exists a non-reflecting stationary $S \subseteq \mu$ then there exists a Jónsson algebra of cardinality μ.

Notice that there is a natural connection between the square bracket relation when $\kappa = \mu = \lambda$ and existence of Jónsson algebras:

PROPOSITION 13.19. *Let $\mu > \aleph_0$. If $\mu \not\to [\mu]^{<\omega}_{\mu}$ then there exists a Jónsson algebra of cardinality μ*

PROOF. Exercise. □

Stevo Todorcević in 1984 proved the following amazing strenghtening of Tryba and Woodin's result (previously was shown by Erdős and Hajnal [**?**] to hold under $2^{\lambda} = \lambda^+$):

THEOREM 13.20 (Todorcević [**To**]). *If $\lambda \geq \aleph_0$ is regular then $\lambda^+ \not\to [\lambda^+]^{<\omega}_{\lambda^+}$.*

Another deep result (that depends on pcf-theory) is:

THEOREM 13.21 (Shelah [**Sh g**]). *There exists a Jónsson algebra of cardinality $\aleph_{\omega+1}$.*

Much more is known about the existence of Jónsson algebras at certain cardinalities and Chang's conjecture, for more the reader is referred to Aki Kanamori's book [**Ka94**].

Exercises 6.13.

(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)

CHAPTER 7

Morley's Theorem

Introduction

The main theorem of this chapter is Morley's theorem:

THEOREM 0.1 (Morley [**Mo1**]). *Let T be a first-order theory in a countable language. If there exists a cardinal $\lambda > \aleph_0$ such that T is categorical in λ then T is μ-categorical for every $\mu > \aleph_0$.*

In what follows I give an outline and explanation of the structure of the proof of Morley's Theorem:

Let T be a fixed first-order theory in a countable language.

Suppose that $\lambda > \aleph_0$ is such that T is categorical in λ. By Vaught's theorem (Theorem 1.3.26) the theory T is complete, thus (by Theorem 4.2.48) we may assume that all the models are elementary submodels of the monster model $\mathfrak{C}$ and all sets and elements are from the universe of $\mathfrak{C}$.

In the first section I discuss several dimension functions in fields outside logic.

Motivated by dimension functions in the second section I introduce a rank function $\mathrm{R}[\cdot]$ from sets of type-definable subsets[1] of $\mathfrak{C}$ (with parameters from $\mathfrak{C}$) into $\{-1\} \cup \mathrm{On} \cup \{\infty\}$ some basic properties are proved among other things it is shown that for a theory T in a countable language, T is $\aleph_0$-stable iff $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \omega_1$.

Using Theorem 2.3.34 From the assumption that T is categorical in λ we conclude that the model of cardinality λ is isomorphic to a reduct of the Skolem-Hull of a sequences of indiscernibles ordered by λ. Using Theorem 4.3.34 it follows that T is $\aleph_0$-stable. Using the rank function R we derive that T must be stable in every cardinality. Using that it is shown that the model of cardinality λ is $\aleph_1$-saturated.

[1] See Definition 7.30.

Let $\mu > \aleph_0$ be given. We have to show that any $M, N \models T$ of cardinality μ are isomorphic. Using Theorem 4.1.3 (since T is complete) it suffices to show that every model of cardinality μ is saturated.

We need two technical tools that appear to be of great importance also in other situations. Assuming only that T is stable in $\aleph_0$ we show

- Let M be an arbitrary model (elementary submodel of the monster model). For any set $A \subseteq |M|$ and any uncountable set $\mathbf{I} \subseteq |M|$ such that $|\mathbf{I}| > |A|$ and any infinite cardinality λ such that $|\mathbf{I}| > \lambda \geq |A|$ there exists $\mathbf{J} \subseteq \mathbf{I}$ of cardinality λ^+ such that $\mathbf{J}$ is a sequence of indiscernibles over A (ordered by λ^+).
- For every A there exists a constructible model model over A.

Now suppose that $\mu > \aleph_0$ and $M \models T$ of cardinality μ are given. If M is not saturated, let $p \in \mathrm{S}(A)$ (for some $A \subseteq |M|$ of cardinality $< \mu$) a type not realized in M. Let $\lambda := |A| + \aleph_0$ we pick $\mathbf{I} := \{a_i \ : \ i < \lambda^+\} \subseteq M$ which is a sequence of indiscernibles over A. Using a downward Lowenheim-Skolem type of argument we find a countable $\mathbf{I}_0 \subseteq \mathbf{I}$ (without a last element) and construct a type p_0 over a countable set $A_0 \subseteq A$ such that p_0 is not isolated over $A_0 \cup \mathbf{I}_0$. Using the compactness theorem find $\mathbf{J} := \{b_j \ : \ j < \lambda\}$ an indiscernible over A_0 such that there is a countable ordinal α such that $\{b_j \ : \ j < \alpha\} = \mathbf{I}_0$. Let N be a constructible model over $A_0 \cup \mathbf{J}$. All we are left to do is to show that p_0 is not realized in N, this is done by induction on the length of the isolating sequence of N. Now this contradicts an earlier observation that the model of cardinality λ must be $\aleph_1$-saturated.

In further sections I will present two additional proofs of Morley's theorem. Both proofs avoid using the rank function. The second proof is due to Baldwin & Lachlan and the third proof is Keisler's. Baldwin & Lachlan's method of attack is instead of transferring non-saturation of models among uncountable cardinals to prove outright that any pair of uncountable models (of the same cardinality) are isomorphic, the argument is similar to the argument of Stienitz's theorem (See Theorem 1.2.38) than the above. The importance of this approach is in the introduction of the notion of dimension theory that generalizes various algebraic dimension theories.

Keisler's proof uses his two cardinal theorem (Theorem 1.12.11) to transfer results between uncountable cardinals. Its advantage over the first proof is not using the rank function, its advantage over the second proof that it does not depend on "Galois Theory" at al just on definability (in the set-theoretic sense) of saturation.. Also it is more elegant than the other two arguments. The first two proofs are more important, since they introduced very useful divices to the area and alowing interesting generalizations, while Keisler's is not known to have any extensions.

The Baldwin Lachlan proof introduces a connection between uncountable categoricity and inexistence of two-cardinal models:

THEOREM 0.2. *Suppose T is a complete $\aleph_0$-stable theory. The following are equivalent*

(1) *T is categorical in some $\lambda > \aleph_0$,*
(2) *T is categorical in every $\lambda > \aleph_0$,*
(3) *for every $M \models T$ and every non-algebraic formula $\varphi(\mathbf{x}; \mathbf{a})$ with parameters from M. $\varphi(M; \mathbf{a}) \subsetneqq \varphi(N; \mathbf{a}) \iff M \precneqq N$ for every N.*

Morley's paper containing his categoricity theorem was considered by the experts at the time of publication to be the most difficult result in model theory. Indeed Morley used essentially everything was known in pure model theory at his time (except ultraproducts). One of the leading model theorists of the sixties told me that he had to spend six months to understand Morely's original paper.

One of the more difficult aspects of Morley's paper was the introduction of the rank function. For many years even till the late seventies people placed high premium on proofs that avoids usage of any rank function.

Around 1970 Keisler found an argument that avoids completely mention of a rank function. Many people considered this argument of Keisler as a first step in systematic elimination of the rank function from Model Theory. In the last twenty years this opinion was reversed and now the majority consider rank functions indispensable.

Remark to the expert: For the first proof of Morley's theorem I am using the rank $\mathrm{R}[p, L, 2]$ instead of the conventional (and common approach in the litrature) use of Morley's rank. The advantage of this approach is that the argument and technique can be generalized in several ways to non-elementary classes, E.g. Theorems **??**.**??**, while arguments involving Morley's rank can not be generalized due to the failure of the compactness theorem. and **??**.**??**.

1. Dimension in model theory

Probably the most important concept introduced by Morley for the proof of his categoricity theorem is the notion of an ordinal-valued rank function assigned to type-definable sets in a model. It is a generalization of the dimension of a variety in algebraic geometry.

Within a relatively short period of time after Morley's categoricity theorem several new rank functions were discovered. Most of the rank functions can be viewed as generalizations of algebraic-geometric ranks and generalizations of a set theoretic rank (the Cantor-Bendixon rank). The rest of this section is dedicated to an exposition of some familiar rank (-dimension) functions.

One of the more powerful ideas in all of mathematics is the introduction (By Kronecker ?) of the concept of Dimension. I assume that your are familiar with the notions of *dimension of a vector space* and with the notion of *transcendence degree* . The usual definition are by the cardinality of a minimal set of generators. However there is a dual point of view (which is equivalent in the finite dimensional case):

DEFINITION 1.1 (topological definitions of dimension).

(1) Let V be a vector space over F. Define $\dim_F V \geq n$ by induction on $n < \omega$:
   (a) Always $\dim_F V \geq 0$.
   (b) For $n = k+1$; $\dim_F V \geq n$ holds iff there exists a subspace $W \subsetneq V$ such that $\dim_F W \geq k$.
   (c) For finitely generated vector space V, let $\dim_F V := Sup\{n \mid \dim_F V \geq n\}$.
(2) Let E be an algebraically closed field with F the algebraic closure of the prime field of E.
   (a) Always $tr.degE \geq 0$.
   (b) For $n = k + 1$; $tr.degE \geq n$ holds iff there exists an algebraically closed subfield $E' \subsetneq E$ such that $tr.degE' \geq k$.
   (c) For finitely generated algebraically closed field E, let
   $$tr.degE := Sup\{n \mid tr.degE \geq n\}.$$

There are many more examples of similar definitions of dimension functions in algebraic geometry and commutative algebra. Recall that for a field F, $n < \omega$ a set $V \subseteq F^n$ is called *algebraic set* iff there is a set $\Phi \subseteq F[x_1, \ldots, x_n]$ such that $V = \{\mathbf{a} \in F^n \mid \forall p \in \Phi[p(\mathbf{a}) = 0]\}$. An algebraic set V is called *a variety* iff there are no algebraic sets $W_\ell \subsetneq V$ (for $\ell = 1, 2$) such that $V = W_1 \cup W_2$. Given a variety $V \subseteq F^n$ it follows from the definitions that

$$\mathrm{I}(V) := \{p(\mathbf{x}) \in F[x_1, \ldots, x_n] \mid \forall \mathbf{a} \in V[p(\mathbf{a}) = 0]\}$$

is a prime ideal of the ring $F[x_1, \ldots, x_n]$.

DEFINITION 1.2.

(1) Let F be a field and $V \subseteq F^n$ a variety. By induction on $m < \omega$ define when $\dim V \geq m$:
   (a) Always $\dim V \geq 0$.
   (b) For $m = k + 1$; $\dim V \geq m$ holds iff there exists a variety $W \subsetneq V$ such that $\dim W \geq k$.
   (c) For a variety $V \subseteq F^n$ let $\dim V := \sup\{m \mid \dim V \geq m\}$. It follows from Hilbert's basis theorem that always $\dim V$ is a natural number.
(2) Let R be a commutative ring (with identity), and let $P \subseteq R$ be a prime ideal. By induction on $n < \omega$ define when *the height of* P $ht(P) \geq n$:
   (a) Always $ht(P) \geq 0$.
   (b) For $n = k + 1$; $ht(P) \geq n$ iff there exists a prime ideal P' of R satisfying that $P' \subsetneq P$ and $ht(P') \geq k$.
   (c) $ht(P) := \sup\{n \mid ht(P) \geq n\}$.
   (d) *The Krull dimension of* R is denoted by $\dim(R)$ and is defined as $\sup\{ht(P) \mid P \text{ is a prime ideal of } R\}$.

One of the fundamental results of commutative algebra asserts a surprising connection between the geometric dimension of a variety and the Krull dimension of its coordinate ring:

FACT 1.3 (Theorem 11.25 in [**AM**]). *Let F be an algebraically closed field. and let $V \subseteq F^n$ be a variety. Let $\Gamma(V)$ be* the coordinate ring of V - *it is $F[x_1, \ldots, x_n]/\mathrm{I}(V)$ and $F(V)$ be its field of fractions. We have the following identity*

$$\dim F(V) = \dim V.$$

See Chapter 11 of [**AM**] and Chapter 5 of [**Mat**].

Georg Cantor in XXX was interested in computing the cardinalities of uncountable subsets of the reals, his deepest result in this direction is the following:

FACT 1.4 (Cantor-Bendixon Theorem). *Let S be a closed subset of the real numbers. If $|S| > \aleph_0$ then $|S| = 2^{\aleph_0}$.*

For a proof see Theorem 14 in [**Je78**]. The key concept in the proof is an ordinal-valued rank function known as the *Cantor-Bendixon rank*.

DEFINITION 1.5. Let A be a closed subset of $\mathbb{R}$.

(1) A real number $r \in A$ is an *accumulation point of* A iff there exists $\{a_n \mid n < \omega\} \subseteq A$ such that $r = \lim a_n$. By A' denote the set of accumulation points of A.
(2) Define by induction on ordinals when $\mathrm{CB}(A) \geq \alpha$:
   (a) Always $\mathrm{CB}(A) \geq 0$,

(b) $\mathrm{CB}(A) \geq \alpha + 1$ iff there exists a closed $B \subsetneq A'$ such that $\mathrm{CB}(B) \geq \alpha$,
(c) for α limit $\mathrm{CB}(A) \geq \alpha$ holds iff $\mathrm{CB}(A) \geq \beta$ for every $\beta < \alpha$.
(d) The *Cantor-Bendixon rank* of the closed set A is defined by $\mathrm{CB}(A) := \sup\{\alpha \mid \mathrm{CB}(A) \geq \alpha\}$.

Since the set of nationals is a dense countable subset of the reals the reals don't have an uncountable decreasing sequence of closed sets we have:

PROPOSITION 1.6. *If A is a closed subset of $\mathbb{R}$ then $\mathrm{CB}(A) < \omega_1$.*

Morley in his proof introduced a rank function $\mathrm{RM}(\cdot)$ defined on *type definable sets* (A set $D \subseteq {}^n\mathfrak{C}$ is type definable iff there exists a type p with free variables from $x_0, \ldots, x_{n-1}$ such that $D = \{\mathbf{a} \in \mathfrak{C} : \mathbf{a} \models p\}$). It is a simultaneous generalization of the algebraic-geometric dimension of a variety and that of the Cantor-Bendixon rank.

For the proof of Morley's theorem to be presented below we have chosen to work with a different rank than the original Morley's rank. It is the rank $\mathrm{R}[p, L, 2]$ from Chapter **??**, it does not have the ultrametric property of Morley's rank (see Definition 11.5) but has the uniqueness property over every set (not only models). This rank has important applications for non-elementary classes. (see [**Gr1**])

2. A rank function

We assume that T is a fixed complete first-order theory, and $\mathfrak{C}$ is its monster model.

One of the more powerful ideas in Morley's proof of his categoricity theorem was an introduction of a rank function that assigns to type-definable subsets of $\mathfrak{C}$ an ordinal. The idea is to assign an ordinal to sets that will measure their complexity. Unlike proof theoretic complexity or syntactical complexity that depends on the number of quantifiers required to define a set, we have a notion that is more geometric in nature.

There are several dozens of interesting such rank functions that all share some properties in common. While it is trivial to invent such a function (e., g. let the ranks of S be 1 if it is not-empty and 0 if it is empty), not every rank-function is of value. What we are looking at is to find a rank-function that has some "interesting properties"? Well I could introduce a list of such properties, however this might be look too abstract to the reader and also most of the rank functions were discovered in a different process. What I will do I will introduce first a very simple example of such function, list its important properties and use it in the proof of the categoricity theorem. Our first function is not what Morley used, I will explain elsewhere the differences.

DEFINITION 2.1. Let p be a set of formulas in $\mathbf{x}$ (maybe with parameters from $\mathfrak{C}$). By induction we define for every ordinal α when the relation $\mathrm{R}^{\mathbf{x}}[p] \geq \alpha$ holds.

$\mathrm{R}^{\mathbf{x}}[p] \geq 0$ iff p is consistent.

$\mathrm{R}^{\mathbf{x}}[p] \geq \alpha + 1$ iff for every finite $q \subseteq p$ there are a formula $\psi(\mathbf{x}; \mathbf{y})$ and $\mathbf{b} \in \mathfrak{C}$ such that both $\mathrm{R}^{\mathbf{x}}[q \cup \{\psi(\mathbf{x}; \mathbf{b})\}] \geq \alpha$ and $\mathrm{R}^{\mathbf{x}}[q \cup \{\neg\psi(\mathbf{x}; \mathbf{b})\}] \geq \alpha$ hold.

For α limit, the inequality $\mathrm{R}^{\mathbf{x}}[p] \geq \alpha$ holds iff $\mathrm{R}^{\mathbf{x}}[p] \geq \beta$ holds for all $\beta < \alpha$.

DEFINITION 2.2. Let p be a set of formulas in $\mathbf{x}$ (maybe with parameters from $\mathfrak{C}$).

If p is inconsistent we set $\mathrm{R}^{\mathbf{x}}[p] := -1$

If there exists an ordinal β such that $\mathrm{R}^{\mathbf{x}}[p] \leq \beta$ we set $\mathrm{R}^{\mathbf{x}}[p] := \alpha$ when α satisfies $\mathrm{R}^{\mathbf{x}}[p] \geq \alpha$ holds and $\mathrm{R}^{\mathbf{x}}[p] \geq \alpha + 1$ fails.

If for every ordinal β we have $\mathrm{R}^{\mathbf{x}}[p] \geq \beta$ then $\mathrm{R}^{\mathbf{x}}[p] := \infty$

Since the free variables of a given set of formulas p can be computed from p, in order to avoid complicated typing by abuse of notation we omit the superscript from R, namely write $\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a})]$ instead of $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})]$. Note that the last definition is one of the few places where the superscript $\mathbf{x}$ is necessary.

REMARK 2.3. Form the definition it is clear that given a set of formulas p there is always a finite subset q of p such that $\mathrm{R}[p] = \mathrm{R}[q]$. This is a fundamental property of the rank and for future reference we call this property *The finite character of the rank*.

Since in model-theory it is more important to study definable sets than the syntactic objects that define them. We introduce the following relation that already appeared implicitly in the exercises involving the Lindenbaum algebra of definable sets.

DEFINITION 2.4. Let $\mathbf{x}$ be a finite sequence of variables. For sets of $\mathbf{x}$-formulas p and q the symbol $p \vdash q$ stands for "for every $\mathbf{a} \in {}^{\ell(\mathbf{x})}|\mathfrak{C}|$ if $\mathbf{a}$ realizes p then $\mathbf{a}$ realizes q".

One may argue that $p \models q$ is a better notation for the last relation, since $p \vdash q$ is widely used I prefer to follow the majority over introducing non-standard notation.

The following properties are fundamental

LEMMA 2.5. *Let p and q be sets of $\mathbf{x}$-formulas.*

(1) $p \vdash \mathbf{x} = \mathbf{x}$,
where $\mathbf{x} = \mathbf{x}$ *stands for the formula* $x_1 = x_1 \wedge x_2 = x_2 \wedge \cdots x_n = x_n$.
(2) *If* $p \subseteq q$ *then* $q \vdash p$.
(3) *If* p *is finite then* $p \vdash \bigwedge p$ *and* $\bigwedge p \vdash p$.
(4) *If* $p \vdash \varphi(\mathbf{x}; \mathbf{a})$ *then there exists a finite* $q \subseteq p$ *such that* $q \vdash \varphi(\mathbf{x}; \mathbf{a})$.

PROOF. Parts 1,2 and 3 are left as an exercises.

(4): Let $\{c_1, \ldots, c_n\}$ be new constant symbols. From $p \vdash \varphi(\mathbf{x}; \mathbf{a})$ it follows that for every expansion $\mathfrak{C}'$ of $\mathfrak{C}$ to $L(\mathfrak{C}) \cup \{c_1; \ldots, c_n\}$ we have that if $\mathfrak{C}' \models p(\mathbf{c})$ then $\mathfrak{C}' \models \varphi(\mathbf{c}; \mathbf{a})$. Since $T = \mathrm{Th}(\mathfrak{C})$ we have that $T \cup p(\mathbf{c}) \vdash \varphi(\mathbf{c}; \mathbf{a})$. Now by the finite character of provability (see Remark 2.20) there exists a finite $q \subseteq p$ such that $T \cup q(\mathbf{c}) \vdash \varphi(\mathbf{c}; \mathbf{a})$. By (3) and repeated use of the deduction theorem ($|q|$-many times) we get that $T \vdash \bigwedge q(\mathbf{c}) \to \varphi(\mathbf{c}; \mathbf{a})$. Since $\{c_1, \ldots, c_n\}$ are new constant symbols we have (using the axioms of predicate calculus) that $T \vdash \forall \mathbf{x}[\bigwedge q(\mathbf{x}) \to \varphi(\mathbf{x}; \mathbf{a})]$. Namely there is a finite $q \subseteq p$ such that $q \vdash p$. □

LEMMA 2.6 (Monotonicity lemma). *Suppose that* p *and* q *are sets of formulas with free variables from the sequence* $\mathbf{x}$. *If* $p \vdash q$ *then* $\mathrm{R}^{\mathbf{x}}[p] \leq \mathrm{R}^{\mathbf{x}}[q]$.

PROOF. If $\mathrm{R}[q] = -1$ then q is inconsistent and by Lemma 2.5(2) there is a finite inconsistent $p_0 \subseteq p$. By The definition of the rank we have that $\mathrm{R}[p] = \mathrm{R}[p_0] = -1$ too.

Now show by induction on $\alpha \in \mathrm{On}$ that

$$\mathrm{R}[p] \geq \alpha \implies \mathrm{R}[q] \geq \alpha \text{ for every } p \vdash q :$$

Suppose $\alpha = 0$ then the above argument shows that p is consistent and $p \vdash q$ implies that q is consistent.

Suppose α is limit then by the definition of the rank $\mathrm{R}[p] \geq \beta$ for all $\beta < \alpha$. Since $\beta < \alpha$ by the induction hypothesis

$$\mathrm{R}[p] \geq \beta \text{ and } p \vdash q \implies \mathrm{R}[q] \geq \beta.$$

Using the definition of the rank again we get that $\mathrm{R}[q] \geq \alpha$.

Suppose $\alpha = \beta + 1$, $\mathrm{R}[p] \geq \beta + 1$ and $p \vdash q$. Given a finite $q_0 \subseteq q$, by lemma 2.5(4) let p_0 be a finite subset of p such that $p_0 \vdash q_0$. Using the assumption that $\mathrm{R}[p] \geq \beta + 1$ to $p_0 \subseteq p$: There exists $\psi_{p_0}(\mathbf{x}; \mathbf{b}_{p_0})$ such that

$$\mathrm{R}[p_0 \cup \{\psi_{p_0}(\mathbf{x}; \mathbf{b}_{p_0})\}] \geq \beta \text{ and } \mathrm{R}[p_0 \cup \{\neg\psi_{p_0}(\mathbf{x}; \mathbf{b}_{p_0})\}] \geq \beta.$$

Clearly

$$p_0 \cup \{\psi_{p_0}(\mathbf{x}; \mathbf{b}_{p_0})\} \vdash q_0 \cup \{\psi_{p_0}(\mathbf{x}; \mathbf{b}_{p_0})\} \text{ and } p_0 \cup \{\neg\psi_{p_0}(\mathbf{x}; \mathbf{b}_{p_0})\} \vdash q_0 \cup \{\neg\psi_{p_0}(\mathbf{x}; \mathbf{b}_{p_0})\}.$$

An application of the induction hypothesis gives

$$\mathrm{R}[q_0 \cup \{\psi_{p_0}(\mathbf{x}; \mathbf{b}_{p_0})\}] \geq \beta \text{ and } \mathrm{R}[q_0 \cup \{\neg\psi_{p_0}(\mathbf{x}; \mathbf{b}_{p_0})\}] \geq \beta.$$

Thus by the definition of $\mathrm{R}[\cdot]$ we have that $\mathrm{R}[q_0] \geq \beta + 1$. □

From the monotonicity lemma one concludes a weaker form of monotonicity that is used often:

PROPOSITION 2.7. *Let p and q be sets of formulas (maybe with parameters) with the same free variables $\mathbf{x}$. If $p \subseteq q$ then $\mathrm{R}[q] \leq \mathrm{R}[p]$. In particular, $\mathrm{R}[p] \leq \mathrm{R}[\mathbf{x} = \mathbf{x}]$ for every type p in $\mathbf{x}$.*

PROOF. Notice that $p \subseteq q \implies q \vdash p$ and apply the monotonicity lemma. □

COROLLARY 2.8 (Finite character). *For any set of formulas (with parameters from $\mathfrak{C}$) p there exists a finite $p_o \subseteq p$ such that $\mathrm{R}[p] = \mathrm{R}[p_0]$.*

PROOF. If p is inconsistent by Lemma 2.5(4) there exists $p_0 \subseteq p$ which is also inconsistent. Thus by the definition of R we have $\mathrm{R}[p] = \mathrm{R}[P_0] = -1$.

Suppose that $\mathrm{R}[p] = \infty$. Let p_0 be an arbitrary finite subset of p. By Proposition 2.7 $\mathrm{R}[p_0] \geq \mathrm{R}[p]$.

The third possibility is: $0 \leq \mathrm{R}[p] < \infty$. Let $\alpha := \mathrm{R}[p]$. By the definition of the rank $\mathrm{R}[p] \not\geq \alpha + 1$. Thus there exists a finite $p_0 \subseteq p$ such that

$$(*) \quad \mathrm{R}[\bigwedge p_0] \not\geq \alpha + 1.$$

Since $p \vdash p_0$ by Proposition 2.7

$$(**) \quad \mathrm{R}[p] \leq \mathrm{R}[p_0].$$

Combining $(*)$ with $(**)$ gives

$$\alpha + 1 \not\geq \mathrm{R}[p_0] \geq \mathrm{R}[p] = \alpha, \text{ thus } \mathrm{R}[p_0] = \alpha.$$

□

DEFINITION 2.9. Suppose that p is a set of formulas with free variables from the sequence $\mathbf{x}$. *The domain of the type p* is denoted by $\mathrm{Dom}(p)$ it is the set of parameters appearing in p. Namely $\mathrm{Dom}(p) := \bigcup\{\cup\mathbf{a} \ : \ \varphi(\mathbf{x}; \mathbf{a}) \in p\}$.

Where $\cup\mathbf{a}$ stands for the set of elements from the sequence $\mathbf{a}$, namely if $\mathbf{a} = \langle a_0, \ldots, a_{n-1}\rangle$ then $\cup\mathbf{a} := \{a_k \mid k < n\}$.

The following lemma show that the rank is *invariant under automorphisms of the universe*:

LEMMA 2.10 (Invariance). *Suppose that p is a set of formulas with free variables from the sequence $\mathbf{x}$. If $f \in \mathrm{Aut}(\mathfrak{C})$ then $\mathrm{R}[p] = \mathrm{R}[f(p)]$.*
Where $f(p) := \{\varphi(\mathbf{x}; f(\mathbf{a})) \ : \ \varphi(\mathbf{x}; \mathbf{a}) \in p\}$.

PROOF. Exercise. □

The previous Lemma is often used in the following form:

COROLLARY 2.11. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be sequences of elements of the monster model of the same arity. If* $\mathrm{tp}(\mathbf{a}/\emptyset) = \mathrm{tp}(\mathbf{b}/\emptyset)$ *then for every formula* $\varphi(\mathbf{x};\mathbf{y})$ *such that* $\ell(\mathbf{y}) = \ell(\mathbf{a})$ *we have that* $\mathrm{R}[\varphi(\mathbf{x};\mathbf{a})] = \mathrm{R}[\varphi(\mathbf{x};\mathbf{b})]$.

PROOF. The hypothesis that $\mathrm{tp}(\mathbf{a},\emptyset) = \mathrm{tp}(\mathbf{b},\emptyset)$ implies that the function g defined as $g(\mathbf{a}) = \mathbf{b}$ is an $(\mathfrak{C},\mathfrak{C})$-mapping, pick $f \in \mathrm{Aut}(\mathfrak{C})$ extending g and apply Lemma 2.10. □

In the next section we will prove the following fundamental theorem that justify interest in the function $\mathrm{R}[\cdot]$.

THEOREM 2.12. *Let* T *be a complete theory in a countable language. The following conditions are equivalent:*

(1) T *is stable in* $\aleph_0$.
(2) *For every finite* $\mathbf{x}$ *we have that* $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \infty$.

For future reference the following property will be called *uniqueness of extension*.

LEMMA 2.13 (Uniqueness). *Let* p *be a consistent set of formulas in* $\mathbf{x}$ *with the property that* $\mathrm{R}[p] < \infty$. *Suppose that* $p \subseteq q_l$ *for* $l = 1, 2$ *are types such that* $\mathrm{Dom}\, p \subseteq A$, *and both* $q_l \in \mathrm{S}^{\mathbf{x}}(A)$ *are of the same ordinal rank as* p. *Then* $q_1 = q_2$.

PROOF. Suppose for the sake of contradiction that $q_1 \neq q_2$. Since both are complete types there exists $\varphi(\mathbf{x};\mathbf{a}) \in q_1$ such that $\neg\varphi(\mathbf{x};\mathbf{a}) \in q_2$.

Let α be the ordinal such that $\alpha = \mathrm{R}[p] = \mathrm{R}[q_l]$. By the monotonicity lemma applied twice we have that $\mathrm{R}[p] \geq \mathrm{R}[p \cup \{\varphi(\mathbf{x};\mathbf{a})\}] \geq \mathrm{R}[q_1]$.

Similarly we get that $\mathrm{R}[p] \geq \mathrm{R}[p \cup \{\neg\varphi(\mathbf{x};\mathbf{a})\}] \geq \mathrm{R}[q_2]$.

Since all quantities are equal to α combing the last two lines gives: $\mathrm{R}[p \cup \{\varphi(\mathbf{x};\mathbf{a})\}] \geq \alpha$ and $\mathrm{R}[p \cup \{\neg\varphi(\mathbf{x};\mathbf{a})\}] \geq \alpha$. Which by the definition of R gives that $\mathrm{R}[p] \geq \alpha + 1$ contradiction to the hypothesis that $\mathrm{R}[p] = \alpha$. □

Exercises 7.2.

(1) Suppose $p \vdash q$. Show that if q is finite then there exists a finite $p_0 \subseteq p$ such that $p_0 \vdash q$.
(2) Suppose $p \vdash q$. Show that $f(p) \vdash f(q)$ for every $f \in \mathrm{Aut}(\mathfrak{C})$.
(3) Let $\varphi(\mathbf{x}; \mathbf{a})$ be such that $1 \leq n_\varphi := |\varphi(\mathfrak{C}; \mathbf{a})| < \aleph_0$. Then

$$\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a})] \leq \log_2 n_\varphi.$$

(4) Let p be a type. $\mathrm{R}[p] < \omega$ iff p is algebraic.
(5) Let p be a finite set of formulas. Verify that $\mathrm{R}[p] = \mathrm{R}[\bigwedge p]$.
(6) Let p be a complete type in $\mathbf{x}$. Show that there exists $\varphi(\mathbf{x}; \mathbf{a}) \in p$ such that $\mathrm{R}[p] = \mathrm{R}[\varphi(\mathbf{x}; \mathbf{a})]$.
(7) Let f be a *definable bijection* of $\mathfrak{C}$ (i.e. there exists a formula $\psi(x, y; \mathbf{c})$ such that $f = \{\langle a, b\rangle \mid \mathfrak{C} \models \psi[a, b; \mathbf{c}]\}$) show that $\mathrm{R}^{\mathbf{x}}[p] = \mathrm{R}[p(f(\mathbf{x}))]$ holds for every p.
(8) Let G be an infinite group Suppose that $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \infty$ (in the monster model for $\mathrm{Th}(G)$). If H is a *proper* subgroup of G definable by the formula $\psi(\mathbf{x}; \mathbf{b})$ then $\mathrm{R}[\psi(\mathbf{x}; \mathbf{b})] < \mathrm{R}[\mathbf{x} = \mathbf{x}]$. Hint: For $a \in G$ the coset Ha is also a definable set (with parameters a and $\mathbf{b}$) and the mapping $x \mapsto xa$ is a definable bijection (with the parameter a).
(9) Let G be an infinite group Suppose that $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \infty$ (in the monster model for $\mathrm{Th}(G)$). If H is a proper subgroup of G with infinite index definable by the formula $\psi(\mathbf{x}; \mathbf{b})$ then $\mathrm{R}[\psi(\mathbf{x}; \mathbf{b})] + \omega < \mathrm{R}[\mathbf{x} = \mathbf{x}]$.
(10) Let G be a group. If $\mathrm{Th}(G)$ is $\aleph_0$-stable then there is no $\{H_n \leq G \mid n < \omega\}$ such that H_n is a definable subgroup of G and $H_{n+1} \subsetneq H_n$ for all $n < \omega$.
(11) In this exercise we present a slightly different definition of a rank function. Prove that the function defined below is equal to the function from Definition 2.2.

DEFINITION 2.14. Let $\varphi(\mathbf{x}; \mathbf{a})$ be a formula with parameters from $\mathfrak{C}$. By induction on $\alpha \in \mathrm{On}$ define when $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})] \geq \alpha$:
(a) $R^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})] \geq 0$ iff $\mathfrak{C} \models \exists \mathbf{x} \varphi(\mathbf{x}; \mathbf{a})$.
(b) For α limit we denote by $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})] \geq \alpha$ the statement: For every β less than α we have that $\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a})] \geq \beta$ holds.
(c) $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})] \geq \alpha + 1$ iff there exists a formula $\psi(\mathbf{x}; \mathbf{b})$ (with $\mathbf{b}$ a sequence of elements from $\mathfrak{C}$) such that $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}))] \geq \alpha$ and $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a}) \wedge \neg\psi(\mathbf{x}; \mathbf{b}))] \geq \alpha$.

The function $\mathrm{R}^{\mathbf{x}}[\cdot]$ appears in Shelah's **[Sh c]** as $\mathrm{R}^{l(\mathbf{x})}[\cdot, L, 2]$.

DEFINITION 2.15. Let $\varphi(\mathbf{x}; \mathbf{a})$. Define $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})] \in \{-1\} \cup \mathrm{On} \cup \{\infty\}$:
(a) $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})] = -1$ iff $\mathfrak{C} \models \neg\exists \mathbf{x} \varphi(\mathbf{x}; \mathbf{a}))$.
(b) $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})] = \alpha$ (for some $\alpha \in \mathrm{On}$) iff $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})] \geq \alpha$ and it is not true that $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x}; \mathbf{a})] \geq \alpha + 1$.

(c) $\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] = \infty$ iff for every ordinal α we have that $\mathrm{R}^{\mathbf{x}}(\varphi(\mathbf{x};\mathbf{a})) \geq \alpha$ holds.

It turns out that it is also interesting to assign ranks to sets of formulas.

DEFINITION 2.16. Let $\mathbf{x}$ be a finite sequence of variables, and suppose that p is a set of $\mathbf{x}$-formulas maybe with parameters (from $\mathfrak{C}$).

$$\mathrm{R}^{\mathbf{x}}[p] := \min\{\mathrm{R}^{\mathbf{x}}[\bigwedge q] \ : \ q \subseteq_{finite} p\}.$$

Where we consider the class $\{-1\} \cup \mathrm{On} \cup \{\infty\}$ well-ordered by the obvious order (it is the order-compactification of On).

(12) The purpose of the following exercises is to show that instead of defining $\mathrm{R}[p]$ for types we could define
$\mathrm{R} : \mathrm{Fml}(\mathrm{L}(T)) \times {}^{\omega >}\mathfrak{C} \to \{-1\} \cup \mathrm{On} \cup \{\infty\}$ first and than extends the definition of $\mathrm{R}[\varphi(\mathbf{x};\mathbf{a})]$ and than to extend it to types. This approach makes the finite character to be trivial but make proving some other properties little more difficult.

DEFINITION 2.17. Let $\varphi(\mathbf{x};\mathbf{a})$ be a formula with parameters from $\mathfrak{C}$. By induction on $\alpha \in \mathrm{On}$ define when $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha$:

- $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq 0$ iff $\mathfrak{C} \models \exists \mathbf{x} \varphi(\mathbf{x};\mathbf{a})$.
- For α limit we denote by $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha$ the statement: For every β less than α we have that $\mathrm{S}[\varphi(\mathbf{x};\mathbf{a})] \geq \beta$ holds.
- $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha + 1$ iff there exists a formula $\psi(\mathbf{x};\mathbf{b})$ (with $\mathbf{b}$ a sequence of elements from $\mathfrak{C}$) such that $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{b}))] \geq \alpha$ and $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{b}))] \geq \alpha$.
- $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] = \infty$ iff for all $\alpha \in \mathrm{On}$ $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha$.
- $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] = \alpha$ for some $\alpha \in \mathrm{On}$. iff $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha$ and it is not the case that $\mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha + 1$.
- For a set of formulas p with free variables among $\mathbf{x}$ let

$$\mathrm{S}^{\mathbf{x}}[p] := \min\{\mathrm{S}^{\mathbf{x}}[\bigwedge q] \mid q \subseteq_{\text{finite}} p\}.$$

(a) Show that if $\mathfrak{C} \models \forall \mathbf{x}[\psi(\mathbf{x};\mathbf{a}) \to \varphi(\mathbf{x};\mathbf{b})]$
then $\mathrm{S}^{\mathbf{x}}[\psi(\mathbf{x};\mathbf{a})] \leq \mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{b})]$.
(b) Show by induction on α that

$$\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha \implies \mathrm{S}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha$$

.
(c) Prove that

$\mathrm{R}^{\mathbf{x}}[p] = \mathrm{S}^{\mathbf{x}}[p]$ for any p with free variables among $\mathbf{x}$.

(13)
(14) The purpose of the following exercises is to present another function $\mathrm{R}_* : \mathrm{Fml}(\mathrm{L}(T)) \times {}^{\omega >}\mathfrak{C} \to \{-1\} \cup \mathrm{On} \cup \{\infty\}$ that is equal to $\mathrm{R}[\cdot]$.

DEFINITION 2.18. Let p be a set of formulas (maybe with parameters from $\mathfrak{C}$) with free variables among $\mathbf{x}$. By induction on $\alpha \in \mathrm{On}$ define when $\mathrm{R}^{\mathbf{x}}_*[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha$:

- $\mathrm{R}^{\mathbf{x}}_*[p] \geq 0$ iff $\exists \mathbf{a} \in \mathfrak{C} \models p$.
- For α limit, $\mathrm{R}^{\mathbf{x}}_*[p] \geq \alpha$ iff for every $\beta < \alpha$ we have that $\mathrm{R}^{\mathbf{x}}_*[p] \geq \beta$ holds.
- $\mathrm{R}^{\mathbf{x}}_*[p] \geq \alpha + 1$ iff there exists a formula $\psi(\mathbf{x};\mathbf{b})$ (with $\mathbf{b}$ a sequence of elements from $\mathfrak{C}$) such that $\mathrm{R}^{\mathbf{x}}_*[p \cup \{\psi(\mathbf{x};\mathbf{b})\}] \geq \alpha$ and $\mathrm{R}^{\mathbf{x}}_*[p \cup \{\neg\psi(\mathbf{x};\mathbf{b})\}] \geq \alpha$.
- $\mathrm{R}^{\mathbf{x}}_*[p] = \infty$ iff for all $\alpha \in \mathrm{On}$ $\mathrm{R}^{\mathbf{x}}_*[p] \geq \alpha$.
- $\mathrm{R}^{\mathbf{x}}_*[p] = \alpha$ for some $\alpha \in \mathrm{On}$. iff $\mathrm{R}^{\mathbf{x}}_*[p] \geq \alpha$ and it is not the case that $\mathrm{R}^{\mathbf{x}}_*[p] \geq \alpha + 1$.

(a) Recall that when p and q are set of formulas in the variables $\mathbf{x}$ then $p \vdash q$ stands for the statement: for every $\mathbf{a} \in \mathfrak{C}$ if a realizes p then $\mathbf{a}$ realizes q. Show that

$$p \vdash q \implies \mathrm{R}^{\mathbf{x}}_*[p] \leq R^{\mathbf{x}}_*[q].$$

(b) $\mathrm{R}^{\mathbf{x}}_*[\cdot]$ has the finite character, namely for every set of formulas p in the variables $\mathbf{x}$ there exists a finite $q \subseteq p$ such that $\mathrm{R}^{\mathbf{x}}_*[q] = \mathrm{R}^{\mathbf{x}}_*[p]$.

(c) Show by induction on α that

$$\mathrm{R}^{\mathbf{x}}[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha \implies \mathrm{R}^{\mathbf{x}}_*[\varphi(\mathbf{x};\mathbf{a})] \geq \alpha$$

.

(d) Prove that

$\mathrm{R}^{\mathbf{x}}[p] = \mathrm{R}^{\mathbf{x}}_*[p]$ for any p with free variables among $\mathbf{x}$.

(15)

(16) Suppose $\mathrm{R}[\varphi(\mathbf{x};\mathbf{a})] = \alpha$ and $\mathrm{tp}(\mathbf{a}/B)$ is not algebraic. Show that there exists a B-indiscernible sequence $\{\mathbf{a}_n \mid n < \omega\}$ with $\mathbf{a}_0 = \mathbf{a}$ such that

$$\mathrm{R}[\varphi(\mathbf{x};\mathbf{a}_n)] = \alpha \text{ for every } n < \omega.$$

(17)

(18)

(19)

(20)

(21)

3. $\aleph_0$-stability

Recall that by Corollary 4.3.25 categoricity in some $\lambda > \aleph_0$ implies that T is $\aleph_0$-stable.

The two fundamental results of this section are a proof of Theorem 2.12 and the fact that $\aleph_0$-stability implies stability in every infinite cardinality.

Before continuing any further we need to establish a technical fact about stability that was asserted as an exercise in Chapter 2.

PROPOSITION 3.1. *Let T be a complete theory and $\lambda \geq |T|$. The theory T is stable in λ iff for every A of cardinality λ we have that $|\mathrm{S}^1(A)| \leq \lambda$.*

PROOF. It is enough to show that if for every A of cardinality λ we have that $|\mathrm{S}^1(A)| \leq \lambda$ then for every $n < \omega$ we have that $|\mathrm{S}^n(A)| \leq \lambda$.

We establish this by induction of n. Suppose for the sake of contradiction that $|\mathrm{S}^n(A)| \leq \lambda$ but $|\mathrm{S}^{n+1}(A)| > \lambda$. Pick $\{p_i \mid i < \lambda^+\} \subseteq \mathrm{S}^{n+1}(A)$. Now for every $p \in \mathrm{S}^{n+1}(A)$ consider

$$q_p := \{\exists x_0 \varphi(x_0, \ldots, x_n; \mathbf{a}) \mid \varphi(\mathbf{x}; \mathbf{a}) \in p\}.$$

Clearly q_p is a consistent set of formulas in n variables, pick $q_p^* \in \mathrm{S}^n(A) \supseteq p_q$. Now consider

$$i \mapsto q^*_{p_i}.$$

The domain of this mapping is λ^+ while the range is a set of cardinality $\leq \lambda$. By the pigeonhole principle there are $q \in \mathrm{S}^n(A)$ and $S \subseteq \lambda^+$ of cardinality λ^+ such that $i \in S \implies q^*_{p_i} = q$. Let $\mathbf{b}$ be a sequence realizing the type q. Substitute $\mathbf{b}$ for the variables $x_1, \ldots, x_n$ in p_i, namely consider $\{p_i(x_0, \mathbf{b}) \mid i \in S\}$. Notice that since $i \neq j \in S \implies p_i \neq p_j$, trivially $p_i(x_0, \mathbf{b}) \neq p_j(x_0, \mathbf{b})$.

I claim that for $i \in S$ we have that $p_i(x_0, \mathbf{b}) \in \mathrm{S}^1(A \cup \mathbf{b})$: Given $\{\psi_k(x_0, \mathbf{b}, \mathbf{a}_k) \mid k < m\} \subseteq p_i(x_0, \mathbf{b})$, enough to show that $\mathfrak{C} \models \exists x_0 \varphi(x_0, \mathbf{b}, \mathbf{a})$, where $\varphi(x_0, \ldots, x_n; \mathbf{a}) := \bigwedge_{k<m} \psi_k(x_0, \ldots, x_n; \mathbf{a}_k)$. Since $\{\psi_k(x_0, \ldots, x_n; \mathbf{a}_k) \mid k < m\} \subseteq p_i(x_0, \ldots, x_n)$ and p_i is a complete type we have that $\varphi(x_0, \ldots, x_n; \mathbf{a}) \in p_i(x_0, \ldots, x_n)$. By the choice of q_i and the assumption that $i \in S$ this give that

$$\exists x_0 \varphi(x_0, \ldots, x_n; \mathbf{a}) \in q.$$

Since $\mathbf{b} \models q$ we have that $\mathfrak{C} \models \exists x_0 \varphi(x_0, \mathbf{b}, \mathbf{a})$.

Thus we found a subset of $\mathrm{S}^1(A \cup \mathbf{b})$ of cardinality λ^+, this is impossible since we have assumed that $|\mathrm{S}^1(A)| \leq \lambda$ for *every* A of cardinality λ and λ is infinite. □

Instead of proving Theorem 2.12 we prove little more:

THEOREM 3.2. *Let T be a complete theory in a countable language. The following conditions are equivalent:*

(1) *T is stable in $\aleph_0$.*
(2) *For every finite $\mathbf{x}$ we have that $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \omega_1$.*
(3) *For every finite $\mathbf{x}$ we have that $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \infty$.*

The proof of Theorem 2.12 is divided into three steps: (1) implies (2), (2) implies (3) which is trivial and (3) implies (1), instead the last step we establish a stronger useful fact in Theorem 3.3.

PROOF. (1) $\implies$ (2): Suppose that T is stable in $\aleph_0$ and $\mathrm{R}[\mathbf{x} = \mathbf{x}] \geq \omega_1$. The idea of this proof is to obtain a contradiction to the hypothesis of $\aleph_0$-stability by constructing a countable subset A of the monster model such that $|\mathrm{S}^{\ell(\mathbf{x})}(A)| = 2^{\aleph_0}$. We follow the structure of the proof of Theorem 2.7.24:

For every $\eta \in {}^{\omega>}2$ define a finite $p_\eta \subseteq \mathrm{Fml}(\mathrm{L}(\mathrm{CD}(\mathfrak{C}))$ and $\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)$ with the following properties:

(1) $p_{\langle\rangle} = \{\mathbf{x} = \mathbf{x}\}$
(2) p_η is a $\mathfrak{C}$-type and if $\varphi \in p_\eta$ then $\mathrm{FV}(\varphi) = \mathbf{x}$
(3) For every $\eta \in {}^{\omega>}2$ there exists $\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)$ such that $p_{\eta^\frown 0} = p_\eta \cup \{\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)\}$ and $p_{\eta^\frown 1} = p_\eta \cup \{\neg\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)\}$
(4) for every $\eta \in {}^{\omega>}2$, if ν is an initial segment of η then $p_\nu \subseteq p_\eta$
(5) $\mathrm{R}[p_\eta] \geq \omega_1$.

This is enough: Clearly if a family $\{p_\eta \ : \ \eta \in {}^{\omega>}2\}$ satisfying (1)-(4) exists then define $A := \{\mathbf{a}_\eta \ : \ \eta \in {}^{\omega>}2\}$ and for $\eta \in {}^{\omega}2$ let $p_\eta = \bigcup_{n<\omega} p_{\eta\restriction n}$, now as in the proof of Theorem 2.7.24 we can show that $|\mathrm{S}^{\ell(\mathbf{x})}(A)| = 2^{\aleph_0}$.

*This is possible:*By induction on $\ell(\eta)$.

For $n = \ell(\eta) = 0$, let $p_{\langle\rangle} := \{\mathbf{x} = \mathbf{x}\}$.

For $n \geq 1$, suppose that $\{p_\eta, \psi_\eta(\mathbf{x}; \mathbf{a}_\eta) \ : \ \eta \in {}^{n>}2\}$ are defined. Given $\eta \in {}^{n-1}2$ since $\mathrm{R}[p_\eta] \geq \omega_1$ for every $i < \omega_1$ let $\varphi_{\eta,i}(\mathbf{x}; \mathbf{a}_{\eta,i})$ be such that $\mathrm{R}[p_\eta \cup \varphi_{\eta,i}(\mathbf{x}; \mathbf{a}_{\eta,i})] \geq i$ and $\mathrm{R}[p_\eta \cup \neg\varphi_{\eta,i}(\mathbf{x}; \mathbf{a}_{\eta,i})] \geq i$. Since the number of possible formulas is countable there are an uncountable $S_\eta \subseteq \omega_1$ and a formula $\psi_\eta(\mathbf{x}; \mathbf{y}_\eta)$ such that for every $i \in S_\eta$ we have that $\varphi_{\eta,i}(\mathbf{x}; \mathbf{y}_{\eta,i}) = \psi_\eta(\mathbf{x}; \mathbf{y}_\eta)$. For every $i \in S_\eta$ define now $q_{\eta,i}$ to be the type $\mathrm{tp}(\mathbf{a}_{\eta,i}, \mathrm{Dom}\, p_\eta)$. Now by $\aleph_0$ stability there exists and uncountable $S^*_\eta \subseteq S_\eta$ and $q_\eta \in \mathrm{S}(\mathrm{Dom}\, p_\eta)$ such that for every $i \in S^*_\eta$ we have that $q_{\eta,i} = q_\eta$. Let $\mathbf{a}_\eta$ be a realization of q_η. Now define $p_{\eta^\frown 0} := p_\eta \cup \{\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)\}$ and $p_{\eta^\frown 1} := p_\eta \cup \{\neg\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)\}$.

All we need to check is that $\mathrm{R}[p_{\eta^\frown \ell}] \geq \omega_1$ for $\ell \in \{0, 1\}$. Let's verify this for $\ell = 0$ ($\ell = 1$ is left to the reader): By the definition of R and uncountability of

S^*_η it is enough to show that for every $i \in S^*_\eta$ we have that $\mathrm{R}[p_\eta \cup \{\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)] \geq i$. Let $i \in S^*_\eta$.

$$\begin{aligned} \mathrm{R}[p_\eta \cup \{\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)\}] = & \text{ (apply Lemma 2.10 to } q_\eta = q_{\eta,i}) \\ \mathrm{R}[p_\eta \cup \{\psi_\eta(\mathbf{x}; \mathbf{a}_{\eta,i})\}] = & \text{ (since } \varphi_{\eta,i}(\mathbf{x}; \mathbf{y}_{\eta,i}\}) = \psi_\eta(\mathbf{x}; \mathbf{y}_\eta)) \\ & \mathrm{R}[p_\eta \cup \{\varphi_{\eta,i}(\mathbf{x}; \mathbf{a}_{\eta,i})\}] \geq i. \end{aligned}$$

□

(3)$\Longrightarrow$(1) of Theorem 2.12 follows from the following stronger and important result:

THEOREM 3.3. *Let T be a complete theory. If $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \infty$ then T is stable in every $\lambda \geq |T|$.*

PROOF. Suppose that T is unstable in some infinite λ. Let A be a subset of the monster model of cardinality λ such that $|\,\mathrm{S}(A)| > \lambda$.
Pick $\{p_i \ : \ i < \lambda^+\} \subseteq \mathrm{S}(A)$. By $\aleph_0$-stability for every $i < \lambda^+$ there exists an ordinal α_i such that $\mathrm{R}[p_i] = \alpha_i$. By the finite character of the rank (Remark 2.3) for every $i < \lambda^+$ there exists $q_i \subseteq_{finite} p_i$ such that $\mathrm{R}[q_i] = \alpha_i$.

Since

$$\{q_i \ : \ i < \lambda^+\} \subseteq \{\Phi \ : \ \Phi \text{ is a finite set of formulas with parameters from } A\}$$

we have that

$$|\{q_i \ : \ i < \lambda^+\}| \leq \aleph_0 + |A| = \aleph_0 + \lambda = \lambda.$$

The mapping $i \mapsto q_i$ is a function whose domain is λ^+ and its range is a set of cardinality no more than λ, by the pigeonhole principle there exists $S \subseteq \lambda^+$ of cardinality λ^+ and a finite q such that $i \in S \Longrightarrow q_i = q$.

Pick $i_1 \neq i_2 \in S$, by the definition of q_i and the hypothesis on q we have $\alpha_{i_1} = \mathrm{R}[p_{i_1}] = \mathrm{R}[q_{i_1}] = \mathrm{R}[q] = \mathrm{R}[q_{i_2}] = \mathrm{R}[p_{i_2}] = \alpha_{i_1}$. Thus p_{i_1} and p_{i_2} are both same-rank completions of the type q to the set A. By the uniqueness property (Lemma 2.13) we get that $p_{i_1} = p_{i_2}$ contradiction. □

From Theorem 2.12 and Theorem 3.3. We derive the following important:

COROLLARY 3.4. *Let T be a complete theory in a countable language. If T is stable in $\aleph_0$ then T is stable in every $\lambda \geq \aleph_0$.*

The first proof of Corollary is an application of the characterization of $\aleph_0$-stability. There is an alternative more direct proof that does not use the rank, it uses only the perfect set argument[2]:

[2]Like in the prof of Theorem 1.7.21.

ALTERNATIVE PROOF OF COROLLARY 3.4. For the sake of contradiction suppose that there exists an infinite set B such that $|\mathrm{S}(B)| > |B|$. For a finite set $\Phi \subseteq \{\varphi(x;\mathbf{b}) \mid \mathbf{b} \in B, \varphi(x;\mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))\}$, let

$$\mathrm{S}_\Phi(B) := \{p \in \mathrm{S}(B) \mid p \supseteq \Phi\}.$$

We will define $\{p_\eta \in \eta \in {}^{\omega>}2\}$ and $\{\psi_\eta(x;\mathbf{a}_\eta) \mid \eta \in {}^{\omega>}2\}$ where $\mathbf{a}_\eta \in B$ such that

(1) $p_{\langle\rangle} = \{x = x\}$,
(2) $|\mathrm{S}_{p_\eta}| > |B|$,
(3) $p_{\eta\hat{}0} = p_\eta \cup \{\psi_\eta(x;\mathbf{a}_\eta)\}$ and $p_{\eta\hat{}1} = p_\eta \cup \{\neg\psi_\eta(x;\mathbf{a}_\eta)\}$.

This is enough as we can take $A := \{\mathbf{a}_\eta \mid \eta \in {}^{\omega>}2\}$ and for $\eta \in {}^{\omega}2$ let $p_\eta := \bigcup_{n<\omega} p_{\eta\restriction n}$ and q_η a completion of p_η to the set A. As usual we get $\eta \neq \nu \in {}^{\omega}2 \implies q_\eta \neq q_\nu$. Thus T is not $\aleph_0$-stable.

The construction of the formulas is by induction on $n < \omega$:

For $n = 0$ as $\mathrm{S}_{\{x=x\}}(B) = \mathrm{S}(B)$, by the assumption on B we have that $p_{\langle\rangle}$ satisfies requirement (2).

Suppose $\eta \in {}^{n}2$ and p_η is defined, we want to find $\psi_\eta(x;\mathbf{a}_\eta)$ to satisfy requirement (3). If such a formula does not exists then for every formula $\varphi(x;\mathbf{y})$ and every $\mathbf{a} \in B$ there exists $\ell(\varphi(\mathbf{x};\mathbf{a})) \in \{0,1\}$ such that $|\mathrm{S}_{p_\eta \cup \{\varphi^{\ell(x;\mathbf{a})}\}}| \leq |B|$ and $|\mathrm{S}_{p_\eta \cup \{\varphi^{1-\ell(\varphi(\mathbf{x};\mathbf{a}))}(x;\mathbf{a})\}|} > |B|$.

Let $\Gamma_\eta^{\varphi(x;\mathbf{a})} := \mathrm{S}_{p_\eta \cup \{\varphi^{\ell(\varphi(\mathbf{x};\mathbf{a}))}(x;\mathbf{a})\}}$ and let

$$\Gamma_\eta := \bigcup_{\mathbf{a}\in B, \varphi(x;\mathbf{y})\in \mathrm{Fml}(\mathrm{L}(T))} \Gamma_\eta^{\varphi(x;\mathbf{a})}.$$

As the cardinality of the index set is $\leq \aleph_0 \cdot |B|$ the cardinality of Γ_η is at most $|B|$, thus by the inductive assumption the cardinality of $\mathrm{S}_{p_\eta}(B) - \Gamma_\eta$ is greater than $|B|$, thus there are $p_1 \neq p_2 \in \mathrm{S}_{p_\eta}(B) - \Gamma_\eta$.

Now pick any $\psi_\eta(x;\mathbf{a}_\eta) \in p_1$ such that $\neg\psi_\eta(x;\mathbf{a}_\eta) \in p_2$, this is a formula satisfying (3). □

The hypothesis of Theorem 3.3 appears to be interesting for it own sake, so we name it:

DEFINITION 3.5. A complete first-order theory T is called *totally transcendental* iff $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \infty$ for all finite $\mathbf{x}$.

We can summarize some of the previous results in the new terminology as:

COROLLARY 3.6. *Let T be a complete theory in a countable language. The following are equivalent:*

(1) *T is stable in every $\lambda \geq \aleph_0$.*
(2) *T is stable in $\aleph_0$.*
(3) *T is totally transcendental.*

Clearly when a theory T is $\aleph_0$-stable it has a countable language. It turns out that the notion of being totally transcendental is useful for some theories in uncountable languages (such that $|T| < 2^{|T|}$).

Morley in his original paper used a different rank function than $\mathrm{R}[\cdot]$. I have decided to make use of $\mathrm{R}[\cdot]$ in my presentation of Morley's theorem over the original *Morley's rank* due to several reasons. Among them a technical simplicity and the rank's applicability to several generalizations. Morley's original function is very important as it generalizes the notion of dimension of an algebraic set over an algebraically closed field.

DEFINITION 3.7 (Morley's Rank). Let p be a set of formulas in a finite sequence of variables $\mathbf{x}$.

(1) By induction on ordinals define when $\mathrm{RM}^{\mathbf{x}}[p] \geq \alpha$:
    $\alpha = 0$; $\mathrm{RM}^{\mathbf{x}}[p] \geq 0$ iff p is consistent.
    α is limit; $\mathrm{RM}^{\mathbf{x}}[p] \geq \alpha$ iff $\mathrm{RM}^{\mathbf{x}}[p] \geq \beta$ for all $\beta < \alpha$.
    $\alpha = \beta + 1$; for every finite $p_0 \subseteq p$ there are complete types $\{q_k \mid k < \omega\}$ such that
    (a) $\mathrm{RM}^{\mathbf{x}}[p_0 \cup q_k] \geq \beta$ for all $k < \omega$ and
    (b) for every $k_1 \neq k_2 < m$ there exists $\psi(\mathbf{x}; \mathbf{b})$ such that $\psi(\mathbf{x}; \mathbf{b}) \in q_{k_1}$ and $\neg\psi(\mathbf{x}; \mathbf{b}) \in q_{k_2}$.
(2) If p is inconsistent we set $\mathrm{RM}^{\mathbf{x}}[p] := -1$, if for every α we have that $\mathrm{RM}^{\mathbf{x}}[p] \geq \alpha$ we denote this by $\mathrm{RM}^{\mathbf{x}}[p] = \infty$.
    If none of the above holds then there is a unique α such that $\mathrm{RM}^{\mathbf{x}}[p] \geq \alpha$ but $\mathrm{RM}^{\mathbf{x}}[p] \not\geq \alpha + 1$ and we denote this by $\mathrm{RM}^{\mathbf{x}}[p] = \alpha$. The function $\mathrm{RM}^{\mathbf{x}}[\cdot]$ is called *Morley's rank*. Following the convention with $\mathrm{R}^{\mathbf{x}}$ we usually will omit the subscript $\mathbf{x}$ and will write $\mathrm{RM}[p]$ instead of $\mathrm{RM}^{\mathbf{x}}[p]$.

PROPOSITION 3.8. *Let p be a set of formulas with free variables from $\mathbf{x}$. Always $\mathrm{RM}\,\mathbf{x}[p] \leq \mathrm{R}^{\mathbf{x}}[p]$.*

PROOF. Trivial. □

Like the function $\mathrm{R}[\cdot]$ Morley's rank also satisfies the monotonicity and the finite character properties. However it has a couple of important properties that is not shared with the former:

THEOREM 3.9 (ultrametric property). *For every p*

$$\mathrm{RM}^{\mathbf{x}}[p \cup \{\varphi(\mathbf{x}; \mathbf{a}) \vee \psi(\mathbf{x}; \mathbf{b})\}] = \mathrm{Max}\left\{ \mathrm{RM}^{\mathbf{x}}[p \cup \{\varphi(\mathbf{x}; \mathbf{a})\}], \mathrm{RM}^{\mathbf{x}}[p \cup \{\psi(\mathbf{x}; \mathbf{b})\}] \right\}$$

PROOF. Exercise? □

THEOREM 3.10 (extension property). *Let p be a set of formulas with free variables from $\mathbf{x}$. If $\mathrm{RM}^{\mathbf{x}}[p] < \infty$ then for any set A there exists $q \in \mathrm{S}^{\mathbf{x}}(A)$ such that*

$$\mathrm{RM}^{\mathbf{x}}[p] = \mathrm{RM}^{\mathbf{x}}[p \cup q].$$

PROOF. Let $\alpha = \mathrm{RM}^{\mathbf{x}}[p] < \infty$.

Consider

$$\Gamma := \{\psi(\mathbf{x}; \mathbf{a}) \mid \mathrm{RM}^{\mathbf{x}}[p \cup \neg\psi(\mathbf{x}; \mathbf{a})] < \alpha\}.$$

If $p \cup \Gamma$ is inconsistent then there exists a finite $\Gamma_0 \subseteq \Gamma$ such that

$$p \cup \bigwedge \Gamma_0 \text{ is inconsistent.}$$

Thus

$$p \vdash \bigvee_{\psi \in \Gamma_0} \neg\psi(\mathbf{x}; \mathbf{a})$$

Trivially

$$p \vdash p \cup \{ \bigvee_{\psi \in \Gamma_0} \neg\psi(\mathbf{x}; \mathbf{a})\}$$

By Monotonicity and the ultrametric property we get the following contardiction

$$\alpha \leq \mathrm{RM}^{\mathbf{x}}[p \cup \{ \bigvee_{\psi \in \Gamma_0} \neg\psi(\mathbf{x}; \mathbf{a})\}] \leq \mathrm{Max}_{\psi \in \Gamma_0} \mathrm{RM}^{\mathbf{x}}[p \cup \{\neg\psi(\mathbf{x}; \mathbf{a})\}] < \alpha.$$

□

DEFINITION 3.11 (Multiplicity). Let p be a set of formulas with free variables from $\mathbf{x}$. If $\mathrm{RM}^{\mathbf{x}}[p] = \beta$ is an ordinal then $\mathrm{Mult}^{\mathbf{x}}[p] := m$ where m is the maximal integer such that for every finite $p_0 \subseteq p$ there are complete types $\{q_k \mid k < m\}$ such that

(1) $\mathrm{RM}^{\mathbf{x}}[p_0 \cup q_k] \geq \beta$ for all $k < m$ and
(2) for every $k_1 \neq k_2 < m$ there exists $\psi(\mathbf{x}; \mathbf{b})$ such that $\psi(\mathbf{x}; \mathbf{b}) \in q_{k_1}$ and $\neg\psi(\mathbf{x}; \mathbf{b}) \in q_{k_2}$.

COROLLARY 3.12 (Uniqueness of extension). *Suppose $\mathrm{RM}^{\mathbf{x}}[p] = \alpha$ and $\mathrm{Mult}^{\mathbf{x}}[p] = 1$ then for every $A \supseteq \mathrm{dom}\, p$ there exists a unique $q \in \mathrm{S}^{\mathbf{x}}(A)$ extending p such that $\mathrm{RM}[q] = \alpha$.*

Similarly to Theorem 3.2 we can prove

THEOREM 3.13. *Let T be a complete theory in a countable language. The following conditions are equivalent:*

(1) *T is stable in $\aleph_0$.*
(2) *For every finite $\mathbf{x}$ we have that $\mathrm{RM}^{\mathbf{x}}[\mathbf{x} = \mathbf{x}] < \omega_1$.*

(3) *For every finite* $\mathbf{x}$ *we have that* $\mathrm{RM}^{\mathbf{x}}[\mathbf{x} = \mathbf{x}] < \infty$.

Exercises 7.3.

(1) Use Proposition 2.2.30 and Exercise 2.**??** to give an alternative proof to Corollary 3.4

(2) The purpose of this exercise is to show that a localized version of Theorem 2.12 holds: ?

?!

THEOREM 3.14. *Let T be a complete theory in a countable language. Suppose p is a finite set of formulas in the free variables $\mathbf{x}$ The following conditions are equivalent:*

(a) *For every countable $A \supseteq \operatorname{Dom} p$ the set $\{q \in \mathrm{S}(A) \ : \ q \supseteq p\}$ is countable,*
(b) *we have that $\mathrm{R}[p] < \omega_1$ and*
(c) *we have that $\mathrm{R}[p] < \infty$.*

(3) If $p \vdash q$ then $\mathrm{RM}[p] \leq \mathrm{RM}[q]$.

(4) Verify the finite character of Morley's rank. Namely given a set of formulas p in the variables $\mathbf{x}$ show that there exists a finite $q \subseteq p$ such that $\mathrm{RM}[p] = \mathrm{RM}[q]$.

(5) Let p be a consistent set of formulas with free variables from $\mathbf{x}$. We have

$$p \text{ is algebraic } \iff \mathrm{RM}^{\mathbf{x}}[p] = 0.$$

(6) The purpose of this exercise is to present an alternative proof for $(3) \implies (2)$ of Theorem3.2: Let T be a countable first order theory.

(a) Suppose $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \infty$ show that the set $\{\mathrm{R}[p] \mid p \text{ is a consistent type}\}$ is an initial segment of the ordinals.
(b) For every $\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{L}(T)$ the following is true

$$|\{\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a})] : \mathbf{a} \in \mathfrak{C} \text{ and } \mathfrak{C} \models \exists \mathbf{x} \varphi(\mathbf{x}; \mathbf{a})\}| \leq |\mathrm{D}(T)|.$$

(c) Use the above to conclude that

$$\mathrm{R}^{\mathbf{x}}[\mathbf{x} = \mathbf{x}] < \infty \implies \mathrm{R}[\mathbf{x} = \mathbf{x}] < \omega_1.$$

(7) Suppose T is an $\aleph_0$-stable theory in a countable language. For fixed $n < \omega$ and A let

$$S := \{\mathrm{R}[p] \mid p \in \mathrm{S}^n(A),\ p \text{ is not algebraic and } A \text{ a set in the moster model } \}.$$

Let $\alpha_0 := \min S$ and $p \in \mathrm{S}^n(A_0)$ such that $\alpha_0 = \mathrm{R}[p]$. Prove that for any $B \supseteq A_0$ and non-algebraic $q_1, q_2 \in \mathrm{S}^n(B)$ we have that both q_1 and q_2 extend p_0 then $q_1 = q_2$.

(8) In this exercise we outline yet another proof of $(1) \implies (2)$ from Theorem 2.7.24. The argument is quite different than the above exercise or the original argument in the text. Use Lopez-Escobar's undefinability of ω_1 in $L_{\omega_1,\omega}$ (see Theorem 2.**??**) together with models of weak set theory (see section 10 of chapter 1). Hint: Do the next exercise.

(9) Let T be a complete first-order theory in a countable language. Consider

$$\mathcal{K} := \{M \models T \ : \ M \text{ is atomic}\}$$

Show that T if $\aleph_0$-stable then $\langle \mathcal{K}, \prec \rangle$ is an AEC with $\mathrm{LS}(\mathcal{K}) = \aleph_0$.

(10) Given a theory T in a countable language let α be sufficiently large so that $\mathbf{V}_\alpha$ contains T, an $\aleph_1$-saturated model of T and the definition of $\mathrm{R}^{\mathbf{x}}[\cdot]$. Suppose $\mathrm{R}^{\mathbf{x}}[\mathbf{x} = \mathbf{x}] \geq \omega_1$, show that if

$$\mathfrak{B} \equiv_{\omega_1,\omega} \langle \mathbf{V}_\alpha, \in, \omega_1 \rangle \text{ with } \omega_1^{\mathfrak{B}} \text{ is not well founded}$$

then there is a countable set in a model of T (in $\mathfrak{B}$) with continuum many types over it.

(11) Show directly from field-theoretic facts that T_{ACF_p} is a λ-stable theory for every $\lambda \geq \aleph_0$.
Hint: Use the proof of Theorem 1.2.38.

(12) Use Theorem 1.12.11 to find an alternative proof to $\aleph_0$-stability implies λ-stability for every λ. Hint: Show that λ instability implies $\aleph_0$-instability, use an expansion by an appropriate model of set theory.

(13)

(14) Show that as an abelian group $\langle \mathbb{Z}, +, 0 \rangle$ is not $\aleph_0$-stable.

(15) Is there a theory T in a countable language such that $\mathrm{D}(T)$ is uncountable but T is $\aleph_0$-stable?

(16) Is T_{ord} stable?

(17) Is ZF stable?

(18) For every $\alpha < \omega_1$ there is a complete $\aleph_0$-stable theory such that $\mathrm{R}[\mathbf{x} = \mathbf{x}] \geq \alpha$.

(19) Suppose that T is $\aleph_0$-stable. Show that for every regular $\mu > \aleph_0$ there exists a saturated $M \models T$ of cardinality μ.

(20) Suppose that T is $\aleph_0$-stable. Show that for every $\mu > \aleph_0$ there exists a universal $M \models T$ of cardinality μ.

(21) Suppose that T is $\aleph_0$-stable. Does T necessarily have a countable saturated model?

(22) Suppose T is $\aleph_0$-stable, show that necessarily a union of an elementary chain of saturated models must be saturated.

(23) Prove Theorem 4.1 using the argument used to prove Theorem **??**.

(24)

(25)

4. Existence of indiscernibles, non-splitting and cohiers

The main Theorem of this section is a theorem about existence of an indiscernible sequence in a given model. Unlike Ehrenfeucht-Mostowski theorem we manage to show (under suitable assumptions on T) that for any $M \models T$ one can find an indiscernible sequence of in M. This amount to a strong generalization of the result of linear algebra that among a "large" set of vectors one can find a "large" subset of linearly independent vectors. The main new concept of this section is a Galois-Theoretic notion concerning types and sets. Suppose that p is a complete type over the set B and A is a subset of B. Clearly p is one of many different completions to the set B of $p \restriction A$. Is there a natural sense to say that p is determined by $p \restriction A$? It turns out that there are several such notions, in this section we introduce the simplest, namely *p does not split over the set A*.

We start with the existence theorem.

THEOREM 4.1. *Suppose M is an uncountable structure with a countable language such that $\mathrm{Th}(M)$ is $\aleph_0$-stable.*

For any $A, \mathbf{I} \subseteq |M|$ such that there is an uncountable regular cardinal λ satisfying $|A| < \lambda \leq |\mathbf{I}|$ there exists $\mathbf{J} \subseteq \mathbf{I}$ of cardinality λ which is a sequence of indiscernibles over the set A.

PROOF. By an application of the pigeon hole principle we may assume that there exists $n < \omega$ such that $\mathbf{a} \in \mathbf{I} \implies \ell(\mathbf{a}) = n$ and $|\mathbf{I}| = \lambda$..

We call a type p in n-variables a *large type* iff $\{\mathbf{a} \in \mathbf{I} \mid \mathbf{a} \text{ realizes } p\}$ has cardinality λ.

Observe that for every B of cardinality $< \lambda$ there exists a large $p \in\in \mathrm{S}^n(B)$: Consider $\mathbf{a} \mapsto \mathrm{tp}(\mathbf{a}/B)$, a function from $\mathbf{I}$ to $\mathrm{S}^n(B)$. From $\aleph_0$-stability and Corollary 3.4 we have that $|\mathrm{S}^n(B)| \leq |B| + \aleph_0 < \lambda$, as the domain of the mapping has cardinality λ an application of the pigeonhole principle gives $\mathbf{J} \subseteq \mathbf{I}$ of cardinality λ and $p \in \mathrm{S}^n(B)$ such that $\mathbf{a} \in \mathbf{J} \implies \mathbf{a} \models p$.

This observation implies that the following set is not empty:

$$S := \big\{ R[p] \ : \ B \subseteq |M| \text{ of cardinality } < \lambda, B \supseteq A \text{ and } p \in \mathrm{S}^n(B) \text{ is large } \big\}.$$

CLAIM 4.2 (Extension property for large types). *For any $C \supseteq B \supseteq A$ such that $|C| < \lambda$. If $p \in \mathrm{S}^n(B)$ is large then there exists $q \in \mathrm{S}^n(C) \supseteq p$ also large.*

PROOF. Given B, C and p as above. Let $\mathbf{J} \subseteq \mathbf{I}$ of cardinality λ such that $\mathbf{a} \in \mathbf{J} \implies \mathbf{a} \models p$.

For every $\mathbf{a} \in \mathbf{J}$ let $q_{\mathbf{a}} := \mathrm{tp}(\mathbf{a}/C) \in \mathrm{S}^n(C)$. By $\aleph_0$-stability, using Corollary 3.4, we have that $|\mathrm{S}^n(C)| \leq |C| + \aleph_0 < \lambda$. Since λ is regular, by the pigeon hole principle (applied to $\mathbf{a} \mapsto q_{\mathbf{a}}$) there exists $q \in \mathrm{S}^n(C)$ and $\mathbf{J}' \subseteq \mathbf{J}$ of cardinality λ such that $q_{\mathbf{a}} = q$ for every $\mathbf{a} \in \mathbf{J}'$. $\square$

By $\aleph_0$-stability for every type p we have that $\mathrm{R}[p]$ is an ordinal and by the observation S is a non-empty set. Let $\alpha_0 := \min S$ and let $p_0 \in \mathrm{S}^n(B_0)$ such that $\mathrm{R}[p_0] = \alpha_0$. Let $\mathbf{J} \subseteq \mathbf{I}$ of cardinality λ realizing p_0. For any $C \supseteq B_0$ of cardinality $< \lambda$ by the extension property of large types there exists a type $q \in \mathrm{S}^n(C) \supseteq p_0$ large. As q is large, by minimality of α_0 we have that $\mathrm{R}[q] \geq \alpha_0$, by Monotonicity as $q \supseteq p_0$ we get $\mathrm{R}[q] = \alpha_0$.

Define by induction on $i < \lambda$ an element $\mathbf{a}_i \in \mathbf{I} - \{\mathbf{a}_j \mid j < i\}$ such that $p_i := \mathrm{tp}(\mathbf{a}_i/B_0 \cup \{\mathbf{a}_j \mid j < i\}) \supseteq p_0$ and p_i is large. Notice that as α_0 is the minimal element of S, using monotonicity we have that $\mathrm{R}[p_i] = \alpha_0$ for all $i < \lambda$.

Now we show that $\mathbf{J} := \{\mathbf{a}_i \mid i < \lambda\}$ is an indiscernible sequence over B_0:

Observe first $i < j \implies p_i \subseteq p_j$: Denote $B_i := B_0 \cup \{\mathbf{a}_j \mid j < i\}$. As both p_i and $p_j \restriction B_i$ are same rank complete extensions (to B_i) of p_0 we have by the uniqueness property (Lemma 2.13) that $p_i = p_j \restriction B_i$ which means that $p_i \subseteq p_j$.

Now by induction on $n < \omega$ prove

$$(*)_n \quad \mathrm{tp}(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}/B_0) = \mathrm{tp}(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}/B_0)$$

For all $i_1 < \cdots < i_n < \lambda$ and every $j_1 < \cdots < j_n < \lambda$.

Note that the case $n = 1$ follows from $i < j \implies p_i \subseteq p_j$.

Suppose that $(*)_n$ but $(*)_{n+1}$ fails. W.l.o.g. we may assume that $i_{n+1} \leq j_{n+1}$. Since $(*)_{n+1}$ fails there are two $n+1$-tuples such that there exists a formula $\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}; \mathbf{y})$ and $\mathbf{b} \in B_0$ such that

$$M \models \varphi[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{n+1}}; \mathbf{b}] \text{ and } M \models \neg\varphi[\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_{n+1}}; \mathbf{b}].$$

Translating this to types we have

$$\varphi(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}, \mathbf{x}_{n+1}; \mathbf{b}) \in p_{i_{n+1}} \text{ and}$$

$$(**) \quad \neg\varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{x}_{n+1}; \mathbf{b}) \in p_{j_{n+1}}.$$

By $(*)_n$ there exists $f \in \mathrm{Aut}_{B_0}(\mathfrak{C})$ such that $f(\mathbf{a}_{i_k}) = \mathbf{a}_{j_k}$ for $1 \leq k \leq n$. Using invariance we have that $\mathrm{R}[f(p_{i_{n+1}})] = \mathrm{R}[p_{i_{n+1}}] = \alpha_0$.

As $f \restriction B_0 = \mathrm{id}_{B_0}$ we have $f(p_0) = p_0$. Using $f(\mathbf{a}_{i_k}) = \mathbf{a}_{j_k}$ we have $p_0 \cup \{\varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{x}_{n+1}; \mathbf{b})\} \subseteq f(p_{i_{n+1}})$. By monotonicity we have $\mathrm{R}[p_0 \cup \{\varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{x}_{n+1}; \mathbf{b})\}] \geq \alpha_0$.

From $(**)$ we get $\mathrm{R}[p_0 \cup \{\neg\varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{x}_{n+1}; \mathbf{b})\}] \geq \alpha_0$. The last statement give together $\mathrm{R}[p_0] \geq \alpha_0 + 1$. □

An analysis of the last proof when we have $(*)_n \wedge \neg(*)_{n+1}$ is the motivation of the next definition that turns out to be central in many situations even when $\mathrm{R}[\mathbf{x} = \mathbf{x}] = \infty$ or the language is uncountable or even when T is not first-order.

DEFINITION 4.3. Let p be a type in $\mathbf{x}$ and suppose A is a subset of the monster model. We say that p *splits over* A iff there are $\mathbf{a}, \mathbf{b} \in \mathrm{Dom}\, p$ and a formula $\varphi(\mathbf{x}; \mathbf{y})$ such that

(1) $\mathrm{tp}(\mathbf{a}/A) = \mathrm{tp}(\mathbf{b}/A)$
(2) $\varphi(\mathbf{x}; \mathbf{a}) \in p$ and $\neg\varphi(\mathbf{x}; \mathbf{b}) \in p$.

While this definition is unlike anything we have seen so far, it belongs to a very important family of notions[3] that identify a nice behavior of a p type relatively to a set A.

So what exactly is splitting all about? The simplest way to understand it is by using our "Galois Theory". Recall that by Theorem 2.2.48 (2) for two finite n-tuples $\mathbf{a}$ and $\mathbf{b}$ we know that $\mathrm{tp}(\mathbf{a}/A) = \mathrm{tp}(\mathbf{b}/A)$ iff $\mathrm{orbit}_A(\mathbf{a}) = \mathrm{orbit}_A(\mathbf{b})$ where $\mathrm{orbit}_A(\mathbf{a})$ is the orbit of the element $\mathbf{a}$ under the action of the group $\mathrm{Aut}_A(\mathfrak{C})$. Now, suppose that $p \in \mathrm{S}(B)$ and let $\mathbf{c} \models q$ by the above q can be identified with $\mathrm{orbit}_B(\mathbf{c})$, in the case $A \subseteq B$ since the group $\mathrm{Aut}_A(\mathfrak{C})$ is bigger than $\mathrm{Aut}_B(\mathfrak{C})$ in general $\mathrm{orbit}_B(\mathbf{c}) \subsetneq \mathrm{orbit}_A(\mathbf{c})$. By the next Proposition it follows that q does not split over A iff the orbits of $\mathbf{c}$ under the groups $\mathrm{Aut}_A(\mathfrak{C})$ and $\mathrm{Aut}_B(\mathfrak{C})$ are equal.

PROPOSITION 4.4 (Galois-Theoretic characterization of splitting). *Let A and B be sets, $p \in \mathrm{S}^n(B)$ be given and let $\mathbf{c} \models p$.*

(1) *If $\mathrm{orbit}_A(\mathbf{c}) = \mathrm{orbit}_B(\mathbf{c})$ then p does not split over A.*
(2) *Suppose M is a saturated model and $A \subseteq |M|$ such that $\|M\| > |A|$, for $p \in \mathrm{S}(|M|)$. The following are equivalent*
   (a) *p does not split over A,*
   (b) *$f(p) = p$ for all $f \in \mathrm{Aut}_A(M)$.*

PROOF. Exercise. □

Similarly to the function $\mathrm{R}^{\mathbf{x}}[\cdot]$ it appears that the notion of "p does not splits over A" has some interesting properties which are shared with $\mathrm{R}^{\mathbf{x}}[p] = \alpha$, this is done in the next the lemmas and some additional properties are presented in the exercises.

[3]Here is a partial list of such notions: *p forks over A*, *p divides over A*, *p finitely satisfiable in A*, *p strongly-splits over A* and *p weakly divides over A*.

REMARK 4.5 (Monotonicity of non-splitting). Let p be a type. If p does not split over some A then p does not split over B for every $B \supseteq A$.

The following is an example of a non-splitting type, it also gives an important connection between the function $\mathrm{R}[\cdot]$ and the notion of non-splitting.

THEOREM 4.6. *Let p be an $\mathbf{x}$-type and $A \subseteq \operatorname{Dom} p$.*
If $\mathrm{R}^{\mathbf{x}}[p] = \mathrm{R}^{\mathbf{x}}[p \restriction A] \in \operatorname{On}$ then p does not split over the set A.

PROOF. Let $\alpha := \mathrm{R}[p]$. Let $q := p \restriction A$, suppose that p splits over A. Let $\mathbf{a} \in \operatorname{Dom} p$, $\mathbf{b} \in \operatorname{Dom} p$ be such that $\operatorname{tp}(\mathbf{a}/A) = \operatorname{tp}(\mathbf{b}/A)$ and $\varphi(\mathbf{x};\mathbf{y})$ is a formula such that we have: $\varphi(\mathbf{x};\mathbf{a}) \in p$ and $\neg\varphi(\mathbf{x};\mathbf{b}) \in p$. By the assumption and the monotonicity of the rank we have

$$\alpha = \mathrm{R}[q] \leq \mathrm{R}[q \cup \{\varphi(\mathbf{x};\mathbf{a})\}] \leq \mathrm{R}[p] = \alpha$$

and

$$(\#) \quad \alpha = \mathrm{R}[q] \leq \mathrm{R}[q \cup \{\neg\varphi(\mathbf{x};\mathbf{b})\}] \leq \mathrm{R}[p] = \alpha.$$

Thus we found $\mathbf{a}, \mathbf{b} \in \operatorname{Dom} p$ such that

$$(*) \quad \operatorname{tp}(\mathbf{a}/A) = \operatorname{tp}(\mathbf{b}/A) \text{ and}$$

$$(**) \quad \mathrm{R}[q \cup \{\neg\varphi(\mathbf{x};\mathbf{b})\}] = \mathrm{R}[q \cup \{\varphi(\mathbf{x};\mathbf{a})\}] = \alpha.$$

By $(*)$ there is an automorphism $f \in \operatorname{Aut}_A(\mathfrak{C})$ such that $f(\mathbf{a}) = \mathbf{b}$. Now apply f to $(**)$ using the invariance of the rank we get:

$$\alpha = \mathrm{R}[f(q)] = \mathrm{R}[f(q \cup \{\varphi(\mathbf{x};\mathbf{a})\})].$$

Since $\operatorname{Dom} q = A$ and $f \restriction A = \mathrm{id}_A$ we get

$$\alpha = \mathrm{R}[q] = \mathrm{R}[q \cup \{\varphi(\mathbf{x}; f(\mathbf{a}))\})].$$

Since $f(\mathbf{a}) = \mathbf{b}$ we have

$$(\#\#) \quad \alpha = \mathrm{R}[q] = \mathrm{R}[q \cup \{\varphi(\mathbf{x};\mathbf{b})\})].$$

Now an application of the definition of R to $(\#)$ and $(\#\#)$ gives $\mathrm{R}[q] \geq \alpha + 1$ which is a contradiction to the hypothesis that $\mathrm{R}[q] = \alpha$. □

COROLLARY 4.7 (Finite character of non-splitting). *If T is $\aleph_0$-stable then for every complete type p there exists a finite $A \subset \operatorname{Dom} p$ such that p does not split over A.*

PROOF. Since T is $\aleph_0$-stable Theorem 2.12 implies that $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \infty$, thus there exists an ordinal α such that $\alpha = \mathrm{R}[p]$. By the finite character there is a finite $q \subseteq p$ such that $\mathrm{R}[q] = \alpha$. By Theorem 4.6 p does not split over the set $A := \operatorname{Dom} q$. □

LEMMA 4.8 (Invariance). *Let p be a type and A a set. The type p splits over A if and only if $f(p)$ splits over $f(A)$ for every $f \in \operatorname{Aut}(\mathfrak{C})$.*

PROOF. Exercise. □

The following is a sufficient condition for existence of sequence of indiscernibles. It is a general method that will be used extensively elsewhere.

LEMMA 4.9. *Let $A \subseteq B$ and $\mathbf{I} := \{\mathbf{a}_i \ : \ i < \alpha\}$, for $i < \alpha$ let $p_i := \mathrm{tp}(\mathbf{a}_i/A_i)$ where $A_i := B \cup \{\mathbf{a}_j \ : \ j < i\}$. If for every $i < j < \alpha$ we have that $p_i \subseteq p_j$ and for every $i < \alpha$ the type p_i does not split over A then $\mathbf{I}$ is a sequence of indiscernibles over the set B.*

PROOF. By induction on $n < \omega$ show that for every $\varphi(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}; \mathbf{y})$, every $\mathbf{b} \in B$, every $i_1 < \cdots < i_n < \alpha$ and every $j_1 < \cdots < j_n < \alpha$ we have that

$$\mathfrak{C} \models \varphi[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}; \mathbf{b}] \iff \mathfrak{C} \models \varphi[\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}; \mathbf{b}].$$

For $n = 1$; The statement follows from the hypothesis that $p_i \subseteq p_j$ for every $i < j < \alpha$.

For $n + 1$; Let

$$i_1 < \cdots < i_n < i < \alpha \quad \text{and} \quad j_1 < \cdots < j_n < j < \alpha \quad \text{be given.}$$

We may assume without loss of generality that $j \leq i$. By the induction hypothesis we have that

$$\mathrm{tp}(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}/B) = \mathrm{tp}(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}/B).$$

Since the type p_i is complete and does not split over A we have

$$(*) \quad \varphi(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}; \mathbf{x}_{n+1}) \in p_i \quad \text{iff} \quad \varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}; \mathbf{x}_{n+1}) \in p_i.$$

Let $\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}; \mathbf{b})$ be a given formula with parameters from B.
By p_i's definition we have that

$$\mathfrak{C} \models \varphi[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}, \mathbf{a}_i; \mathbf{b}] \quad \text{iff} \quad \varphi(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}, \mathbf{x}_{n+1}; \mathbf{b}) \in p_i.$$

Using $(*)$ we get that

$$\mathfrak{C} \models \varphi[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}, \mathbf{a}_i; \mathbf{b}] \quad \text{iff} \quad \varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{x}_{n+1}; \mathbf{b}) \in p_i.$$

Since $j_1 < \cdots < j_n < j \leq i$ and $p_j \subseteq p_i$

$$\varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{x}_{n+1}; \mathbf{b}) \in p_i \quad \text{iff} \quad \varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{x}_{n+1}; \mathbf{b}) \in p_j.$$

Thus,

$$\mathfrak{C} \models \varphi[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}, \mathbf{a}_i; \mathbf{b}] \quad \text{iff} \quad \varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{x}_{n+1}; \mathbf{b}) \in p_j.$$

Since

$$\varphi(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{x}_{n+1}; \mathbf{b}) \in p_j \quad \text{iff} \quad \mathfrak{C} \models \varphi[\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}, \mathbf{a}_j; \mathbf{b}]$$

we are done. □

The following theorem can be viewed as an abstract generalization of the fact that in any vector space V of cardinality greater than its field of scalars and for every $A \subseteq V$ there is an uncountable set $\mathbf{I}$ of vectors in V that are outside the subspace spanned by A.

In the proof we will use elementary properties of stationary sets, like Fodor's lemma (Theorem 1.1.18). Later in Theorem 4.6.5 we will present a generalization of Theorem **??** with an entirely different proof that does not mention stationary sets. Needless to say that I think that the proof technique to be presented below is important.

SECOND PROOF FOR THEOREM 4.1. Pick $\{\mathbf{a}_i \mid i < \lambda^+\} \subseteq \mathbf{I}$. We may assume that there exists $n < \omega$ such that $i < \lambda^+ \implies \ell(\mathbf{a}_i) = n$. Using the Downward Löwenheim-Skolem-Tarski theorem and the Tarski-Vaught chain theorem we can define $\{M_i \preceq M \mid i < \lambda^+\}$ such that

(1) $|M_0| \supseteq A$,
(2) $i < j < \lambda \implies M_i \preceq M_j$,
(3) $\|M_i\| \leq |i| + |A| + \aleph_0$ for all $i < \lambda$,
(4) $\mathbf{a}_i \in |M_{i+1}|$ for all $i < \lambda$,
(5) if $\delta < \lambda$ is a limit ordinal then $M_\delta = \bigcup_{i<\delta} M_i$ and
(6) for every $\mathbf{b} \in |M|$ and every $i < \lambda$ there exists $\mathbf{b}' \in |M_{i+1}|$ realizing $\mathrm{tp}(\mathbf{b}/|M_i|, M)$.

Notice that we are using stability in $|M_i|$ of T to carry out requirement (6), this follows from the the assumption that T is stable in $\aleph_0$ (by Corollary 3.4).

For every $i < \lambda$ we set $p_i := \mathrm{tp}(\mathbf{a}_i/|M_i|, M)$.

Since T is $\aleph_0$-stable by Corollary 4.7 for every $i < \lambda$ there exists a finite $B_i \subseteq |M_i|$ such that p_i does not split over the set B_i.

Let $S := \{\delta < \lambda \mid \delta \text{ is a limit ordinal}\}$. It is well known that S is a club, hence a stationary subset of λ.

Now consider $f : S \to \lambda$ defined as follows:

$$f(\delta) := \min\{j \ : \ B_\delta \subseteq |M_j|\}.$$

Since B_δ is finite and the chain is continuous the function f is pressing down, namely $f(\delta) < \delta$ for all $\delta \in S$. By Fodor's theorem there are $i_0 < \lambda$ and a stationary $S^* \subseteq S$ such that $f(\delta) = i_0$ for every $\delta \in S^*$.

For every $\delta \in S^*$ consider the mapping

$$\delta \mapsto B_\delta.$$

By the choice of S^* and i_0 the above mapping is a function into the finite subsets of $|M_{i_0}|$. Since the number of finite subsets of $|M_{i_0}|$ is $\|M_{i_0}\|$ and $|S^*| = \lambda$ there

are $S' \subseteq S^*$ of cardinality λ and a finite set $B \subseteq |M_{i_0}|$ such that
$\delta \in S' \implies B_\delta = B$.

Now for every $\delta \in S'$ consider the mapping

$$\delta \mapsto p_\delta \restriction |M_{i_0+1}|.$$

Since the domain has cardinality λ and the range ($= S^n(|M_{i_0+1}|)$) by stability has cardinality $\leq \|M_{i_0+1}\|$ an application of the pigeon-hole principle gives $S^{**} \subseteq S'$ of cardinality λ and a type $q \in S^n(|M_{i_0+1}|)$ such that
$\delta \in S^{**} \implies p_\delta \restriction |M_{i_0+1}| = q$.

Let $S^\# := \{\delta \in S^{**} \mid \delta > i_0\}$. Pick $\{i_\xi \mid \xi < \lambda\}$ and increasing enumeration of the set S'.

Define $\mathbf{J} := \{\mathbf{a}_{i_\xi} \mid \xi < \lambda\}$.

By Lemma 4.9 it is enough to prove the following:

CLAIM 4.10. *For every $\xi < \lambda$ the type p_{i_ξ} does not split over M_{i_0} and for if $\xi_1 < \xi_2 < \lambda$ then $p_{i_{\xi_1}} \subseteq p_{i_{\xi_2}}$.*

PROOF. Since $\delta \in S^\#(\subseteq S') \implies p_\delta$ does not split over $B(\subseteq |M_{i_0}|)$ by monotonicity of non-splitting the type p_δ does not split over $|M_{i_0}|$.

Let $\delta_1 < \delta_2 \in S^\#$. If $p_{\delta_1} \not\subseteq p_{\delta_2}$ then there is

$$(*) \quad \varphi(x;\mathbf{b}) \in p_{\delta_1} \text{ such that } \neg\varphi(x;\mathbf{b}) \in p_{\delta_2}.$$

Since $\delta_2 > i_0$ by requirement (6) there exists $\mathbf{b}' \in |M_{i_0+1}|$ such that

$$(**) \quad \mathrm{tp}(\mathbf{b}/|M_{i_0}|) = \mathrm{tp}(\mathbf{b}'/|M_{i_0}|).$$

Since p_{δ_1} and p_{δ_2} do not split over $|M_{i_0}|$ from $(*)$ and $(**)$ we get that also

$$(\#) \quad \varphi(x;\mathbf{b}') \in p_{\delta_1} \text{ and } \neg\varphi(x;\mathbf{b}') \in p_{\delta_2}.$$

But $(\#)$ contradicts that for $\delta_1, \delta_2 \in S^{**}$ we have that $p_{\delta_1} \restriction |M_{i_0}| = p_{\delta_2} \restriction |M_{i_0}|$. □

□

I have selected to start with this particular argument since its ideas are sufficiently rich that several extensions of that argument will give more powerful results. Later we will see that it is possible to weaken the assumption of $\aleph_0$-stability by λ-stability. When *splitting* is replaced by *forking* (see Theorem **??**). Also the latter part of the argument will be used is deriving the *order property* from instability (see Theorem **??**).

Using the full power of $\aleph_0$-stability gives an easier proof using properties of the rank:

THIRD PROOF FOR THEOREM 4.1. Pick $\{\mathbf{a}_i \mid i < \lambda\} \subseteq \mathbf{I}$. We may assume that there exists $n < \omega$ such that $i < \lambda^+ \implies \ell(\mathbf{a}_i) = n$. Using the Downward Löwenheim-Skolem-Tarski theorem and the Tarski-Vaught chain theorem we can define $\{M_i \preceq M \mid i < \lambda\}$ such that

(1) $\|M_0\| \supseteq A$,
(2) $i < j < \lambda \implies M_i \preceq M_j$,
(3) $\|M_i\| \leq |A| + |i| + \aleph_0$ for all $i < \lambda$,
(4) $\mathbf{a}_i \in |M_{i+1}|$ for all $i < \lambda$, and
(5) if $\delta < \lambda$ is a limit ordinal then $M_\delta = \bigcup_{i<\delta} M_i$.

Now consider $f : S \to \lambda$ defined as follows:

$$f(\delta) := \min\{j \ : \ \mathrm{R}^n[p_\delta] = \mathrm{R}^n[p_\delta \restriction |M_j|]\}.$$

By the finite character of R we get that $\delta \in S \implies f(\delta) < \delta$. By Fodor's lemma there exists a stationary $S^* \subseteq S$ and $j_0 < \lambda$ such that

$$\delta \in S^* \implies \mathrm{R}^n[p_\delta] = \mathrm{R}^n[p_\delta \restriction |M_{j_0}|].$$

Now consider

$$\delta \mapsto p_\delta \restriction |M_{j_0}|.$$

By stability the range of this function (which is a subset of $\mathrm{S}^n(|M_j|)$) has cardinality $< \lambda$, however the domain has cardinality λ, thus by the pigeon hole principle there are $S^{**} \subseteq S^*$ of cardinality λ and $p \in \mathrm{S}^n(|M_{j_0}|)$ such that

$$\delta \in S^{**} \implies p_\delta \restriction |M_{j_0}| = p.$$

Using the uniqueness property of R (Lemma 2.13) we get that $i < j \in S^{**} \implies p_i \subseteq p_j$ for all $i \geq j_0$. By Exercise 3 we get that p_i does not split over $|M_{j_0}|$ for all $i \in S^{**}$ such that $i \geq j_0$.

By Lemma 4.9 $\{\mathbf{a}_i \mid i \in S^{**} \wedge i \geq j_0\}$ is an inidscernible sequence over $|M_{j_0}|$. □

In section 9 I will present yet another proof of Theorem 4.1.

I have selected the proof of $(1) \implies (2)$ in Theorem 2.12 due to its clarity and since it provides an application of the various properties of the rank functions. There are several other proofs. One that makes a novel use of weak models of set theory and uses undefinability of the well ordering by $L_{\omega_1,\omega}$ is of particular importance.

ALTERNATIVE PROOF OF $(1) \implies (2)$ IN THEOREM 2.12. Suppose that T is in a countable language and is $\aleph_0$-stable. Let $\mathfrak{C}$ be a monster model for T. For sake of contradiction suppose that $\mathrm{R}[\mathbf{x} = \mathbf{x}] \geq \omega_1$.

Consider the sentence

$$\psi := \forall\alpha[\alpha \in \Omega \to \mathrm{R}[\mathbf{x} = \mathbf{x}] \geq \alpha]\wedge\forall\alpha[\alpha \in \Omega \to \bigwedge_{q\subseteq_{\text{finite}}\mathrm{L}(T)} R(\wedge q) \geq \alpha \to \mathfrak{C} \models \exists\mathbf{x}[\wedge q]$$

$$\bigwedge_{q\subseteq_{\text{finite}}\mathrm{L}(T)} \forall\alpha[\alpha \in \Omega\wedge R(\wedge q) \geq \alpha{+}1 \to \bigvee_{\psi(\mathbf{x};y)\in\mathrm{L}(T)} \exists\mathbf{b}\ [R(\wedge q\wedge\psi(\mathbf{x};\mathbf{b})) \geq \alpha\wedge R(\wedge q\wedge\neg\psi(\mathbf{x};\mathbf{b})) \geq \alpha]\big]$$

Let α be an ordinal large enough so that all the above sets are elements of $\mathbf{V}_\alpha$ and $\langle\mathbf{V}_\alpha, \in\rangle \models \psi$.

Let $B := \langle\mathbf{V}_\alpha, \in, \omega_1, \mathfrak{C}, \mathrm{Fml}(\mathrm{L}(T)), \mathrm{L}(T), \mathrm{R}[\cdot], \varphi\rangle_{\varphi\in\mathrm{L}(T)}$ where $\mathrm{L}(\mathfrak{B})$ is an extension of $\mathrm{L}(T)$ with $\omega_1, \mathrm{Fml}(\mathrm{L}(T))$ and $\mathrm{L}(T)$ stand for unary predicates, $\Omega^{\mathfrak{B}}$ is interpreted by the set ω_1 and $\varphi(\mathbf{x})$ are individual constants for every formula of $\mathrm{L}(T)$. By the undefinability of well-ordering there exists $\mathfrak{B}^* \models \psi$ such that there exists $\{a_n \mid n < \omega\} \subseteq \Omega^{B^*}$ strictly decreasing and the sets of $\mathrm{L}(T)$-formulas of $\mathfrak{B}^*$ is the same as that of $\mathfrak{B}$.

By the third part of ψ there are $\{\mathbf{b}_n \mid n < \omega\} \subseteq \mathfrak{B}^*$ and $\varphi_n(\mathbf{x};\mathbf{y}_n) \in \mathrm{Fml}(\mathrm{L}(T))$ such that for every $\eta \in {}^n2$

$$\mathfrak{B}^* \models R[p_\eta \cup \{\psi_n(\mathbf{x};\mathbf{b}_n)\}] \geq a_n \wedge R[p_\eta \cup \{\neg\psi_n(\mathbf{x};\mathbf{b}_n)\}] \geq a_n$$

where $p_\eta := \{\psi_k(\mathbf{x};\mathbf{b}_k)^{\eta[k]} \mid k < n\}$.

As usual verify that for $\eta \neq \nu \in {}^\omega2 \implies p_\eta \neq p_\nu$, where $p_\eta := \bigcup_{n<\omega} p_{\eta\restriction n}$. Thus $|\mathrm{S}^{\mathbf{x}}(\{\cup\mathbf{b}_n \mid n < \omega\})| = 2^{\aleph_0}$. □

The rest of this section is not necessary for a proof of Morley's theorem, however since it shades some light on the notion of non-splitting it is included here.

A property of types that implies that they do not split is the notion that Shelah calls *semi-definability* and the Parisian school named *coheir*. Since the French terminology is used more widely, I will use it too.

DEFINITION 4.11. Let A be a set and p a type.

(1) p is *finitely satisfiable in* A iff for every finite $p_0 \subseteq p$ there exists $\mathbf{a} \in A$ such that $\mathfrak{C} \models \bigwedge p_0[\mathbf{a}]$.
(2) Suppose p is finitely satisfiable in A. For $B \supseteq \mathrm{dom}\, p$ a type $q \in \mathrm{S}(B)$ is a *coheir of* p iff $q \supseteq p$ and q is finitely satisfiable in A.

PROPOSITION 4.12. *If p is finitely satisfiable over A then p does not split over A.*

PROOF. Otherwise let $\mathbf{b}, \mathbf{c} \in \mathrm{dom}\, p$ and a formula $\varphi(\mathbf{x};\mathbf{y})$ such that

$$\mathrm{tp}(\mathbf{b}/A) = \mathrm{tp}(\mathbf{c}/A) \text{ and } \{\varphi(\mathbf{x};\mathbf{b}), \neg\varphi(\mathbf{x};\mathbf{c})\} \subseteq p.$$

By the finite satisfiability of p over A there exists $\mathbf{a} \in A$ such that

$$(*) \quad \mathfrak{C} \models \varphi[\mathbf{a}; \mathbf{b}] \wedge \neg\varphi[\mathbf{a}; \mathbf{c}].$$

Notice that from $\mathrm{tp}(\mathbf{b}/A) = \mathrm{tp}(\mathbf{c}/A)$ it follows that $\mathfrak{C} \models \varphi[\mathbf{a}; \mathbf{b}] \leftrightarrow \varphi[\mathbf{a}; \mathbf{c}]$ but this contradicts $(*)$. $\square$

Exercises 7.4.

(1) A type p splits over the set A if and only if $f(p)$ splits over the set $f[A]$ for every $f \in \mathrm{Aut}(\mathfrak{C})$ that takes B to B.
(2) Suppose $A \subseteq B$ and let $q := \mathrm{tp}(\mathbf{c}/B)$. Show that q does not split over A iff $f(\mathbf{c}) \models q$ for every $f \in \mathrm{Aut}_A(\mathfrak{C})$.
(3) Let $A \subseteq B \subseteq C$, let $p \in \mathrm{S}^n(C)$ and let $q \in \mathrm{S}^n(B)$.
Suppose $q \restriction A \subseteq p$ and $\mathrm{R}^n[q] = \mathrm{R}^n[p] = \mathrm{R}^n[p \restriction A] < \infty$. Show that p does not split over A and $q \subseteq p$.
(4) Suppose p splits over the set A. Find $f \in \mathrm{Aut}_A(\mathfrak{C})$ such that $f(p) \neq p$.
(5) Let $p = \mathrm{tp}(\mathbf{c}/A)$ and $B \subseteq A$. The type p does not split over B iff

$$\mathrm{tp}(\mathbf{a}_1/B) = \mathrm{tp}(\mathbf{a}_2/B) \implies \mathrm{tp}(\mathbf{c}^\frown\mathbf{a}_1/\emptyset) = \mathrm{tp}(\mathbf{c}^\frown\mathbf{a}_2/\emptyset) \text{ for all } \mathbf{a}_1, \mathbf{a}_2 \in A.$$

(6) Let $p = \mathrm{tp}(\mathbf{c}/A)$ and $B \subseteq A$. The type p does not split over B iff $f(p) = p$ for every $f \in \mathrm{Aut}_B(\mathfrak{C})$
(7) The following group of exercises establishes several fundamental properties of non-splitting:
   (a) (Monotonicity) Let $A \subseteq B \subseteq C$. If $p \in \mathrm{S}(C)$ splits over B then p splits over A.
   (b) (Uniqueness property) Let $A \subseteq B \subseteq C$. Suppose that for every finite sequence of elements $\mathbf{c} \in C$ there is a sequence $\mathbf{b} \in B \models \mathrm{tp}(\mathbf{c}/A)$. If $p \in \mathrm{S}(B)$ does not split over A and for $\ell = 1, 2$ $q_\ell \in \mathrm{S}(C)$ are extensions of p and (both) do not split over A then $q_1 = q_2$.
   (c) (Extension property) Let $A \subseteq B \subseteq C$. Suppose that for every finite sequence of elements $\mathbf{c} \in C$ there is a sequence $\mathbf{b} \in B \models \mathrm{tp}(\mathbf{c}/A)$. If $p \in \mathrm{S}(B)$ does not split over A then there exists $q \in \mathrm{S}(C)$ an extension of p that does not split over the set A.
   (d) (Stationarity over models) Let $A \subseteq |M|$. If M is $|A|^+$-saturated then for every $p \in \mathrm{S}(|M|)$ that does not split over A and every $C \supseteq |M|$ there exists a unique type $q \in \mathrm{S}(C)$ extending p and not splitting over A.
   (e) (Bounded number of extensions) Let $A \subseteq C$, denote

$$\Gamma^C_A := \{p \in \mathrm{S}(C) \mid p \text{ does not split over } A\}.$$

   Suppose that there exists an $|A|^+$-saturated model M such $|M| \subseteq C$ and $|M| \supseteq A$. Then $|\Gamma^C_A| \leq 2^{|M|}$ for every C.
(8) The next group of exercises establishes some fundamental properties of cohiers:
   (a) (Monotonicity) Let $A \subseteq B \subseteq C$. Suppose $p \in \mathrm{S}(C)$ is finitely satifiable in A then
      (i) p is finiteley satisfiable in B
      (ii) $p \restriction B$ is finitely satisfiable in A.
   (b) (Extension property) Let $A \subseteq B \subseteq C$. Suppose that $p \in \mathrm{S}(B)$ is finitely satisfiable in A then there exists $q \in \mathrm{S}(C)$ a cohier of p.

(c) (Stationarity over models) Let $A \subseteq |M|$. If M is $|A|^+$-saturated then for every $p \in \mathrm{S}(|M|)$ if p is a cohier of $p \restriction A$ then for every $C \supseteq |M|$ there exists a unique type $q \in \mathrm{S}(C)$ extending p and q coheir of $p \restriction A$.

(d) (Bounded number of extensions) Let $A \subseteq C$, denote

$$\Gamma_A^C := \{p \in \mathrm{S}(C) \mid p \text{ is a coheir of } p \restriction A\}.$$

Suppose that there exists an $|A|^+$-saturated model M such $|M| \subseteq C$ and $|M| \supseteq A$. Then $|\Gamma_A^C| \leq 2^{|M|}$ for every C.

(9) Suppose that there exist a strictly increasing chain $\{B_i \mid i < \mu\}$ of sets and a type $p \in \mathrm{S}(\bigcup_{i<\mu} B_i)$ such that $p \restriction B_{i+1}$ splits over B_i for all $i < \mu$. Show that there exist a strictly increasing and *continuous* chain $\{B'_i \mid i < \mu\}$ of sets and a type $p' \in \mathrm{S}(\bigcup_{i<\mu} B'_i)$ such that $p' \restriction B'_{i+1}$ splits over B'_i for all $i < \mu$.

(10) Suppose that there exist a strictly increasing chain $\{B_i \mid i < \mu\}$ of sets and a type $p \in \mathrm{S}(\bigcup_{i<\mu} B_i)$ such that $p \restriction B_{i+1}$ splits over B_i for all $i < \mu$. If $\operatorname{cf} \mu > |\mathrm{L}(T)| + \aleph_0$ then there are $\varphi(\mathbf{x}; \mathbf{y})$, an increasing and continuous chain $\{B'_i \subseteq \operatorname{dom} p \mid i < \mu\}$ and there are $\{\mathbf{a}'_i, \mathbf{b}'_i \in B'_{i+1} \mid i < \mu\}$ such that for every $i < \mu$
   (a) $\operatorname{tp}(\mathbf{a}'_i / B'_i) = \operatorname{tp}(\mathbf{b}'_i / B'_i)$ and
   (b) $\varphi(\mathbf{x}; \mathbf{a}'_i) \wedge \neg\varphi(\mathbf{x}; \mathbf{b}'_i) \in p$.

(11) Let $A \subseteq B$. Suppose $p \in \mathrm{S}(B)$. Show that p split over A iff there exists $f \in \operatorname{Aut}_A(\mathfrak{C})$ such that $p \cup f(p)$ is not consistent.

(12)

(13) Suppose $M \precneqq N$ and $p \in \mathrm{S}(N)$ splits over M. Show that there are $\{\mathbf{a}_n \mid n < \omega\}$, $\{\mathbf{b}_n \mid n < \omega\}$ both $|M|$-indiscernible sequences and there is $\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{L}(T)$ such that $\varphi(\mathbf{x}; \mathbf{a}_0) \wedge \neg\varphi(\mathbf{x}; \mathbf{b}_0) \in p$ and satisfying that $\operatorname{tp}(\mathbf{a}_0 / |M|) = \operatorname{tp}(\mathbf{b}_0 / |M|)$.

(14) Suppose T is an $\aleph_0$-stable. If $\mathbf{I}$ is an infinite indiscernible sequence over a set A then for any finite B there exists a finite $\mathbf{J} \subseteq \mathbf{I}$ such that $\mathbf{I} - \mathbf{J}$ is an indiscernible sequence over $A \cup B$.

(15) Suppose T is an $\aleph_0$-stable. Show that if $\{M_n \mid n < \omega\}$ is an elementary chain of $\aleph_1$-saturated models then also $\bigcup_{n<\omega} M_n$ is an $\aleph_1$-saturated model.

(16) Suppose T is an $\aleph_0$-stable. Show that if $\{M_i \mid i < \alpha\}$ is an elementary chain of saturated models all of the same cardinality then also $\bigcup_{i<\alpha} M_i$ is a saturated model.

5. Prime, primary and atomic models

A very natural notion introduced by Morley for his categoricity theorem is that of a "prime model over a set" which can be viewed as a generalization of algebraic closure in field theory.

Recall that given a complete first order theory T we work inside its monster model $\mathfrak{C}$ and when we write M or N it is assumed that they are elementary sub models of $\mathfrak{C}$ and sets A and B are assumed to be subsets of $|\mathfrak{C}|$, sequence $\mathbf{a}$ and $\mathbf{b}$ are assumed to be from $|\mathfrak{C}|$.

DEFINITION 5.1. Let A be subset a of the monster model. A model $M \supseteq A$ *is a prime model over* A iff for every N and every (M, N)-mapping $g : A \to N$ there exists an elementary embedding $\bar{g} : M \to N$ extending g.

REMARK 5.2. Notice that the above definition is using the monster model. It is not hard that the condition is equivalent to the following statement:

Suppose $M \models T$ and $A \subseteq |M|$. We say that M is prime over A iff for every $N \models T$ and any $f : A \to |N|$ which is an (M, N)-mapping there exists an elementary embedding $f^* : M \to N$ extending f.

Thus M is a prime model over a set A iff it is the minimal object in the category of models containing A where morphisms are elementary embeddings.

Note that the concept M is prime over $\emptyset$ was introduced in Definition 2.9.26.

It is a basic exercise (elementary field theory) to show that in the theory T_{ACF_p} for every subset A of the monster model there exists a prime model M_A over A. Check that M_A is the algebraic closure of $\mathbb{F}_p(A)$ where $\mathbb{F}_p$ is the prime field.

Notice that for a complete theory T a model $M \models T$ is algebraically-prime (see Definition 13.2) iff M is prime over $\emptyset$ (in the sense of the last definition).

PROPOSITION 5.3. *If M is prime over A then $\|M\| \leq |A| + |\mathrm{L}(T)|$.*

PROOF. Exercise. □

Morley managed to show that prime models exists for a large class of theories, namely:

THEOREM 5.4 (Existence of prime models). *Let T be a countable complete first-order theory. If T is stable in $\aleph_0$ then for every A there exists a prime model M_A over A.*

Before starting the proof we need the following technical fact:

LEMMA 5.5 (isolated types are dense). *Suppose that T is $\aleph_0$-stable. For any $\mathbf{a}$ and A such that $\mathbf{a} \in A$ and every $\varphi(\mathbf{x};\mathbf{y})$ such that $\models \exists \mathbf{x}\varphi(\mathbf{x};\mathbf{a})$ there exists $q \in \mathrm{S}^{\ell(\mathbf{x})}(A)$ containing the formula $\varphi(\mathbf{x};\mathbf{a})$ and q is isolated by a formula with parameters from A.*

Notice that the statement is nothing more than that the theory T_A (-the expansion of T with constants for the elements of A) is an atomic theory (recall Definition 2.9.32). This lemma is an extension of Lemma 2.9.33 for uncountable A. In the topological language of section 2.8 the statement of the lemma can be restated as "The set of isolated types is dense in the space $\mathrm{S}^n(A)$".

PROOF. Consider the following set S:

$$\{\mathrm{R}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c})] \ : \ \psi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T),\ \mathbf{c} \in A,\ \mathfrak{C} \models \exists \mathbf{x}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c})]\}.$$

By $\aleph_0$-stability Theorem 2.12 implies that the set S is a set of ordinals.
Let $\alpha := \min S$ and pick $\psi(\mathbf{x};\mathbf{c})$ such that $\mathrm{R}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c})] = \alpha$. Now let $\mathbf{b}$ be such that $\models \varphi[\mathbf{b};\mathbf{a}] \wedge \psi[\mathbf{b};\mathbf{c}]$. Define q as $\mathrm{tp}(\mathbf{b}, A)$. The fact that $\varphi(\mathbf{x};\mathbf{a}) \in q$ follows from the choice of $\mathbf{b}$. All we have to show is that q is isolated. In fact we show that $\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}) \vdash q$: Otherwise there exists $\rho(\mathbf{x};\mathbf{d}) \in q$ such that

$$\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}) \nvdash \rho(\mathbf{x};\mathbf{d}).$$

Thus both $\models \exists \mathbf{x}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}) \wedge \neg\rho(\mathbf{x};\mathbf{d})]$ and $\models \exists \mathbf{x}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}) \wedge \rho(\mathbf{x};\mathbf{d})]$ hold. By the definition of the set S and the minimality of α we get that $\mathrm{R}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}) \wedge \neg\rho(\mathbf{x};\mathbf{d})] \geq \alpha$ and $\mathrm{R}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}) \wedge \rho(\mathbf{x};\mathbf{d})] \geq \alpha$. The last two by the definition of R gives that $\mathrm{R}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c})] \geq \alpha + 1$. Which is a contradiction to the choice of α. □

Before constructing prime models we introduce the concept of *primary model*[4] we will show that it is easy to construct and to show that it is prime.

Recall the following:

DEFINITION 5.6. Let p be a consistent set of formulas (i.e. a type) in the variables $\mathbf{x}$, and let A be any set. We say that p *is isolated over* A iff there exists a formula $\varphi(\mathbf{x};\mathbf{y})$ and a sequence $\mathbf{a}$ of elements of A such that

$$\varphi(\mathbf{x};\mathbf{a}) \vdash p.$$

When $A \subseteq \mathrm{dom}\, p$ instead of writing p is isolated over A we write p *is isolated*.

Notice that if p is a complete type then p is isolated iff there exists $\varphi(\mathbf{x};\mathbf{a}) \in p$ such that $\varphi(\mathbf{x};\mathbf{a}) \vdash p$.

[4]Also known as *constructible model*.

DEFINITION 5.7. Let M be a model and A a subset of its universe. The model M *is primary over* A iff there exists an ordinal α and an enumeration $\{a_i : i < \alpha\}$ of the set $|M| - A$ such that the type

$$p_i := \mathrm{tp}(a_i, A \cup \{a_j : j < i\}) \text{ is isolated for every } i < \alpha.$$

The sequence $\{a_i : i < \alpha\}$ is called *an isolating sequence for* M *over* A.

More generally, given sets $A \subseteq B$. We say that B is a *constructible set over* A iff there exists $\{a_i : i < \alpha\}$ an enumeration of the set $B - A$ such that $\{a_i : i < \alpha\}$ is an isolating sequence over A.

LEMMA 5.8. *Let M be a model and A a subset of its universe. If M is primary over A then M is prime over A.*

PROOF. Suppose $\{a_i : i < \alpha\}$ is an isolating sequence of M over A. Let $g : A \to N$ be an (M, N)-mapping. We will construct an (M, N)-mapping $\bar{g} : M \to N$ extending g by induction on $i < \alpha$:
For every $i < \alpha$ let $g_i : A \cup \{a_j : j < i\} \to N$ be such that

(1) $g_0 = g$,
(2) g_i is an (M, N)-mapping,
(3) if $i < j < \alpha$ then $g_i \subseteq g_j$ and
(4) if i is limit then $g_i = \bigcup_{j<i} g_j$.

This is enough: Let $\bar{g} := \bigcup_{i<\alpha} g_i$. Check that it is as required.

This is possible: Define g_0 as g, due to condition (4) all we have to do is to take care of the successor case. Namely let $i < \alpha$ suppose g_i is as above we have to find an (M, N)-mapping g_{i+1} extending g_i such that $\mathrm{Dom}\, g_{i+1} = \mathrm{Dom}\, g_i \cup \{a_i\}$. Since $p_i = \mathrm{tp}(a_i, A \cup \{a_j : j < i\})$ is isolated over the set $A \cup \{a_j : j < i\}$ there exists $\varphi_i(x; \mathbf{a}) \in p_i$ isolating it. Since g_i is (M, N)-mapping from

$$\varphi_i(x; \mathbf{a}) \vdash p_i \quad \text{we get}$$
$$\varphi_i(x; g_i(\mathbf{a})) \vdash g_i(p_i). \quad (*)$$

Since $\models \exists x \varphi_i(x; \mathbf{a})$ also $N \models \exists x \varphi_i(x; g_i(\mathbf{a}))$ pick $b_i \in N$ such that $N \models \varphi_i[b_i; g_i(\mathbf{a})]$ an application of $(*)$ gives

$$b_i \text{ realizes the type } g_i(p_i).$$

Thus $g_{i+1} := g_i \cup \{\langle a_i, b_i \rangle\}$ is an (M, N)-mapping as required[5]. □

Using the previous lemma we derive Theorem 5.4 from the following

THEOREM 5.9 (existence of a primary model). *Let T be a countable complete first-order theory. If T is $\aleph_0$-stable then for every A there exists a primary model M over A.*

[5]This is similar to the proof of Claim 4.1.11.

Before starting the proof we remind the reader of some basic facts concerning ordinal multiplication. Let α_1 and α_2 be ordinals $\alpha_1 + \alpha_2$ is the order-type of the set $S_1 \cup S_2$ where S_l are pairwise disjoint well-ordered sets of order-type α_l and the order $<$ of $S_1 \cup S_2$ is defined as $x < y$ iff $x \in S_1 \wedge y \in S_2$ or $x, y \in S_l \wedge x <_l y$.

We will be using Fact **??**:4.20.

PROOF. Let $\lambda := |A| + \aleph_0$ let $\alpha := \lambda \cdot \omega$. It is enough to define $\mathbf{I} := \{a_i \ : \ i < \alpha\}$ such that $\mathbf{I}$ is an isolating sequence of a model over the set A. Namely we have to accomplish two things: $\mathbf{I} \cup A$ is a model and show that for every $i < \alpha$ the type of a_i is isolated over all the previous elements.

By induction on $n < \omega$ define $\{a_i \ : \ \lambda \cdot n \leq i < \lambda \cdot (n+1)\}$ such that

(1) $\mathrm{tp}(a_i/A_i)$ is isolated for every $i < \lambda \cdot n$, where $A_i := A \cup \{a_j \ : \ j < i\}$
(2) for every $n < \omega$, every formula $\varphi(x; \mathbf{y})$ and every $\mathbf{a} \in A_{\lambda \cdot n}$ if $\models \exists x \varphi(x; \mathbf{a})$ then there is an ordinal j such that $\lambda \cdot n \leq j < \lambda \cdot (n+1)$ and $\models \varphi[a_j; \mathbf{a}]$.

Clearly the above is enough.

The construction: Suppose that $\{a_i \ : \ i < \lambda \cdot n\}$ are defined. Since $\lambda \geq |T|$ we can fix $\{\varphi_\xi(x; \mathbf{b}_\xi) \ : \ \xi < \lambda\}$ to be an enumeration of all formulas with parameters from $A_{\lambda \cdot n}$ such that for every $\xi < \lambda$ we have that $\models \exists x \varphi_\xi(x; \mathbf{b}_\xi)$. By Lemma 5.5 there exists an isolated complete type $q \in \mathrm{S}(A_{\lambda \cdot n + \xi})$ containing $\varphi_\xi(x; \mathbf{b}_\xi)$. Take a_ξ to be a realization of q. □

The following property of primary models is used in the prof of the last step of Morley's theorem:

THEOREM 5.10 (Primary models are atomic). *Let M be primary over A. Every type of a finite sequence from M is isolated over A. Moreover, if B is constructible over A then $\mathrm{tp}(\mathbf{c}/A)$ is isolated for every $\mathbf{c} \in B$.*

We will argue by induction on the length of the isolating sequence of B over A. Before turning to the proof we establish a Lemma that will take care of the induction step:

LEMMA 5.11 (Concatenation Lemma). *Let $\mathbf{a}$ and $\mathbf{b}$ be finite sequences of elements, A is the set of elements in $\mathbf{a}$. If $\mathrm{tp}(\mathbf{a}/\emptyset)$ and $\mathrm{tp}(\mathbf{b}/A)$ are isolated iff $\mathrm{tp}(\mathbf{a}^\frown\mathbf{b}/\emptyset)$ is an isolated type.*

PROOF. ($\Longrightarrow$) Suppose $\varphi(\mathbf{x})$ and $\psi(\mathbf{y}; \mathbf{x})$ are such that

$$\varphi(\mathbf{x}) \vdash \mathrm{tp}(\mathbf{a}/\emptyset) \text{ and } \psi(\mathbf{y}; \mathbf{a}) \vdash \mathrm{tp}(\mathbf{b}/A).$$

Now we show that

$$\varphi(\mathbf{x}) \wedge \psi(\mathbf{y}; \mathbf{x}) \vdash \mathrm{tp}(\mathbf{a}^\frown\mathbf{b}/\emptyset) :$$

Given $\chi(\mathbf{x};\mathbf{y}) \in \mathrm{tp}(\mathbf{a}\hat{}\mathbf{b}/\emptyset)$ as $\chi(\mathbf{a};\mathbf{y}) \in \mathrm{tp}(\mathbf{b}/A)$ we have

$$\models \psi(\mathbf{y};\mathbf{a}) \to \chi(\mathbf{a};\mathbf{y}).$$

Thus

$$\forall \mathbf{y}[\psi(\mathbf{y};\mathbf{x}) \to \chi(\mathbf{x};\mathbf{y})] \in \mathrm{tp}(\mathbf{a}/\emptyset).$$

Since $\varphi(\mathbf{x}) \vdash \mathrm{tp}(\mathbf{a}/\emptyset)$ we get

$$\models \forall \mathbf{x}\big[\varphi(\mathbf{x}) \to \forall \mathbf{y}[\psi(\mathbf{y};\mathbf{x}) \to \chi(\mathbf{x};\mathbf{y})]\big]$$

which gives us

$$\models \forall \mathbf{x}\forall \mathbf{y}[\varphi(\mathbf{x}) \wedge \psi(\mathbf{y};\mathbf{x}) \to \chi(\mathbf{x};\mathbf{y})]$$

which is what we want.

($\Longleftarrow$) Suppose

$$\varphi(\mathbf{x};\mathbf{y}) \vdash \mathrm{tp}(\mathbf{a}\hat{}\mathbf{b}/\emptyset) \quad (*).$$

I claim that

$$\varphi(\mathbf{a};\mathbf{y}) \vdash \mathrm{tp}(\mathbf{b}/A) :$$

Let $\psi(\mathbf{a};\mathbf{y}) \in \mathrm{tp}(\mathbf{b}/A)$ be given. If there exists $\mathbf{c}$ such that $\models [\varphi(\mathbf{a};\mathbf{c}) \wedge \neg\psi(\mathbf{a};\mathbf{c})]$, by $(*)$ we get that $\neg\psi(\mathbf{x};\mathbf{y}) \in \mathrm{tp}(\mathbf{a}\hat{}\mathbf{b}/\emptyset)$ and in particular $\neg\psi(\mathbf{a};\mathbf{y}) \in \mathrm{tp}(\mathbf{b}/A)$ which is a contradiction.

It is enough to show that $\exists \mathbf{x}[\varphi(\mathbf{x};\mathbf{b})] \vdash \mathrm{tp}(\mathbf{b}/\emptyset)$: Given $\psi(\mathbf{y}) \in \mathrm{tp}(\mathbf{b}/\emptyset)$ we have

$$\models \forall \mathbf{x}\forall \mathbf{y}[\varphi(\mathbf{x};\mathbf{y}) \to \psi(\mathbf{y})].$$

Using predicate calculus this gives

$$\models \forall \mathbf{y}[\exists \mathbf{x}\varphi(\mathbf{x};\mathbf{y}) \to \psi(\mathbf{y})].$$

□

PROOF OF THEOREM 5.10. Suppose $\mathbf{I} = \{a_i \mid i < \alpha\}$ is an isolating sequence of B over A. We may assume (by expanding the language by constants for the elements of A) that $A = \emptyset$.

We will show by induction on $\beta \leq \alpha$ that for any finite $\mathbf{a} \in \{a_i \mid i < \beta\}$ we have that $\mathrm{tp}(\mathbf{a}/\emptyset)$ is isolated: When $\beta = 1$ then this follows from the assumption that $\mathbf{I}$ is an isolating sequence, when β is limit then the induction hypothesis gives what we want.

Suppose $\beta = j + 1$ and $\mathbf{a} = \langle a_{i_1}, \ldots, a_{i_n}, a_j\rangle$ where $i_1 < \cdots i_n < j$. As $\mathrm{tp}(a_j/\{a_i \mid i < j\})$ is isolated there are a finite $\mathbf{b}^* \in \{a_i \mid i < j\}$ and $\varphi(x;\mathbf{y}) \in \mathrm{L}(T)$ such that

$$\varphi(x;\mathbf{b}^*) \vdash \mathrm{tp}(a_j/\{a_i \mid i < j\}).$$

By the induction hypothesis $\mathrm{tp}(\mathbf{b}^*\hat{}\langle a_{i_1}, \ldots, a_{i_n}\rangle/\emptyset)$ is isolated.

Now an application of the Lemma , by making the substitution: $\mathbf{a} := \mathbf{b}^{*\frown}\langle a_{i_1}, \ldots, a_{i_n}\rangle$ and $\mathbf{b} := a_j$ gives that $\mathrm{tp}(a_j{}^\frown\langle a_{i_1}, \ldots, a_{i_n}\rangle)$ is isolated. □

This property of primary models is important so we give it a formal name:

DEFINITION 5.12. Let M be an L-structure and $A \subseteq B$ subsets of its universe. We say that B is *atomic over the set* A iff $\mathrm{tp}(\mathbf{b}/A, M)$ is isolated over A for every $\mathbf{b}$ finite sequence of elements of B.

The set B is called *atomic* iff B is atomic over $\emptyset$.

A structure M is an *atomic model* iff $|M|$ is atomic over $\emptyset$.

PROPOSITION 5.13 (atomic over is a transitive relation). *Suppose $A \subseteq B \subseteq C$. If C is atomic over B and B is atomic over A then C is atomic over A.*

PROOF. Exercise. □

The following property of atomic models turns to be important for the model theory of $L_{\omega_1,\omega}$-theories.

THEOREM 5.14. *Let T be a complete theory in a countable language. If T has an infinite atomic model then for every countable A (subset of $\mathfrak{C}$) there exists an atomic $N \models T$ containing A.*

PROOF. Suppose $M \models T$ is an infinite atomic. Fix $\{a_n \mid n < \omega\}$ an enumeration of A.

By induction on $n < \omega$ define

$$f_n : \{a_k \mid k < n\} \to M \quad (\mathfrak{C}, M)\text{-mapping.}$$

Since $f := \bigcup_{n<\omega} f_n$ is an elementary embedding taking A into M we are done. □

The results and notions of this section were developed by Morley for his proof, they are based on the following early results:

THEOREM 5.15 (Uniqueness of countable atomic models, Vaught). *Let T be a countable first-order complete theory. If M is a countable atomic model of T then it is prime over $\emptyset$ and if also $N \models T$ is countable atomic then $M \cong N$.*

PROOF. Use the argument from the proof of previous theorem. □

Using Henkin's omitting-types theorem we get:

THEOREM 5.16 (Vaught's uniqueness theorem). *Let T be a complete theory in a countable language. If both M and N are countable prime (over $\emptyset$) then $M \cong N$.*

PROOF. This follows from Proposition 9.25 and Theorem 9.28. □

This naturally brings us to the following question:

QUESTION 5.17. Let T be a complete T and A any subset of the monster model, is the following implication true: If M and N are prime over A then $M \cong N$?

Clearly Vaught's uniqueness theorem, provides an affirmative answer to the question when both the language of T and A are at most countable.

Later we'll establish that when T is $\aleph_0$-stable this is true for all A and also for some stable T when $|T|$ is countable. However it is possible to establish uniqueness of primary models without additional assumptions on the theory.

The rest of this section will be not used in the proof of Morley's theorem.

THEOREM 5.18 (Uniqueness of primary models, Ressayre). *Let A be a set and suppose that f is an $(\mathfrak{C}, \mathfrak{C})$-mapping such that $\operatorname{dom} f = A$. If M is primary over A and N is primary over $f[A]$ then there exists $f^* : M \cong N$ extending f.*

Before starting the proof we have to establish some simple facts concerning subsets of constructible models.

DEFINITION 5.19 (Closed subset of an isolating sequence). Suppose $\{a_i \mid i < \alpha\}$ an isolating sequence over the set A. For every $i < \alpha$ let $\varphi_i(x; \mathbf{y}_i)$ be a formula and $\mathbf{b}_i$ a finite sequence of elements of $A_i := A \cup \{a_j \mid j < i\}$ such that

$$\varphi_i(x; \mathbf{b}_i) \vdash \operatorname{tp}(a_i/A_i).$$

A subset U of α is called *closed* iff $\mathbf{b}_i \in A \cup \{a_j \mid j \in U\}$ for every $i \in U$.

Notice that if $\{a_i \mid i < \alpha\}$ is an isolating sequence over the set A, then trivially for every $\beta < \alpha$ the set β is closed. Moreover, if $\beta < \alpha$ and $U \subseteq \alpha$ is closed then also $\beta \cup U$ is closed.

LEMMA 5.20. *Let A be a subset of the monster model. If $\{a_i \mid i < \alpha\}$ is an isolating sequence over A then for every $i < \alpha$ there exists a finite closed $U \subseteq \alpha$ such that $i \in U$.*

PROOF. Suppose $\{a_i \mid i < \alpha\}$ is an isolating sequence over A. Argue by induction on α:

If $\alpha = 1$ then as $\operatorname{tp}(a_0/A)$ is isolated, clearly $U := \{0\}$ is closed.

If α is limit then this holds by the induction hypothesis.

If $\alpha = \beta + 1$; by the induction hypothesis it is enough to show that there exists a finite closed U such that $\beta \in U$.

Suppose $i_1 < \cdots < i_n < \beta$ and $\varphi(x; \mathbf{y})$ are such that such that

$$\varphi(x; a_{i_1}, \ldots, a_{i_n}) \vdash \mathrm{tp}(a_\beta / A_\beta).$$

By induction hypothesis there are finite closed $U_k \subseteq \beta$ (for $1 \leq k \leq n$) such that $i_k \in U_k$ for all k. Verify that $U := U_1 \cup \cdots U_n \cup \{\beta\}$ is a closed set as required. □

LEMMA 5.21. *Let A be a subset of the monster model. If $\{a_i \mid i < \alpha\}$ is an isolating sequence over A then $\{a_i \mid i < \alpha\} \cup A$ is constructible over $A \cup \{a_i \mid i \in U\}$ for any closed $U \subseteq \alpha$.*

PROOF. By expanding the language with constants for the elements of A we may assume that $A = \emptyset$. Suppose U is closed and let $C := \{a_i \mid i \in U\}$. It is enough to show that

$$(*)_i \quad \varphi_i(x; \mathbf{b}_i) \vdash \mathrm{tp}(a_i / C \cup A_i) \quad \text{for every } i < \alpha \text{ such that } i \notin U.$$

If $(*)_i$ fails for some $i < \alpha$ then there exists an increasing $V := \{j_1, \ldots, j_n\} \subseteq U$ such that $j_1 > i$, a formula $\rho(x; \mathbf{x}; \mathbf{z})$ and $\mathbf{d} \in A_i$ such that

$$\rho(x; a_{j_1}, \ldots, a_{j_n}; \mathbf{d}) \in \mathrm{tp}(a_i / C \cup A_i)$$

and

$$(**) \quad \models \exists x[\varphi_i(x; \mathbf{b}_i) \wedge \neg \rho(x; a_{j_1}, \ldots, a_{j_n}; \mathbf{d})].$$

By the previous lemma we may assume that V is closed. Since $\{a_j \mid j \leq j_n\}$ is an isolating sequence over A_i, by the moreover part of Theorem 5.10 there exists a formula $\psi(\mathbf{x}; \mathbf{y})$ and $\mathbf{a} \in A_i$ such that

$$\psi(\mathbf{x}; \mathbf{a}) \vdash \mathrm{tp}(a_{j_1}, \ldots, a_{j_n} / A_i).$$

As

$$\varphi_i(x; \mathbf{b}_i) \wedge \psi(\mathbf{x}; \mathbf{a}) \vdash \mathrm{tp}(a_j \hat{} \langle a_{j_1}, \ldots, a_{j_n} \rangle / A_i).$$

This is a contradiction to $(**)$. □

PROOF. □

It is important to recognize that the notion of atomic model over a set was introduced by Vaught few years before Morley introduced the notion of primary model (for his categoricity theorem)¿ More on atomic models (especially countable) see Exercise 10 and 9.40.

Exercises 7.5.

(1) Let M and N be prime models over the set A. Show that $\|M\| = \|N\|$.
(2) Suppose that $|\,\mathrm{L}(T)|$ is at most countable. Prove that if $M \models T$ is uncountable and prime over some $A \subseteq |M|$ then $\|M\| = |A|$.
(3) Suppose $F \leq E$ are fields such that E is algebraic over F. Show that E is atomic over F.
(4) Let $T := T_{ACF_p}$. Show that for every A ($\subseteq |\,\mathfrak{C}\,|$) the algebraic closure of A is a prime model over A.
(5) Is the algebraic closure from the previous question necessarily primary over A?
(6) Let T be an $\aleph_0$-stable countable theory and let M be an uncountable model of T. Show that for every $A \subseteq |M|$ of cardinality $< \|M\|$ there is $N \preceq M$ a primary model over A.
(7) Let T be countable theory. Suppose that there exists a countable $M \models T$ which is prime over $\emptyset$ and is also saturated. Show that T must be categorical in $\aleph_0$.
(8) Suppose T is complete and we work inside a monster model $\mathfrak{C}$. Prove the following:

PROPOSITION 5.22. *Let M be a model and A a subset of its universe. M is prime over A iff for any N containing A there is an elementary mapping $g : M \to N$ such that $g \restriction A = \mathrm{id}_A$.*

(9) Suppose T is a complete first-order in a countable language. Show that any countable atomic $M \models T$ is prime over $\emptyset$.
(10) Suppose T is complete and we work inside a monster model $\mathfrak{C}$. Prove the following theorem of Vaught:

THEOREM 5.23 (Uniqueness of countable atomic models). *Suppose M_1 and M_2 are both prime atomic models over a set A. If both models are countable and $M_1 \equiv M_2$ then there exists $f : M_1 \cong_A M_2$.*

(11) Suppose T is a complete first-order theory in a countable language. If both $M, N \models T$ are countable and atomic over $\emptyset$ then $M \cong N$.
(12) Suppose $A \subseteq B \subseteq C$. If C is atomic over B and B is atomic over A then C is atomic over A.
(13) Prove that the notions appearing in Remark 5.2 and Definition 5.1 are the same.
(14) Let T be $\aleph_0$-stable in a countable language and $\varphi(x)$ a formula in the language of T. If there exists $M \models T$ such that $\|M\| > |\varphi(M)| \geq \aleph_0$. Show that there exists $N \preceq M$ of cardinality $\aleph_1$ such that $|\varphi(N)| = \aleph_0$.
(15) Let T be a first-order theory and let $\mathcal{K}$ be the class

$$\{M \models T \ : \ M \text{ is an atomic model}\}.$$

Show that $\langle \mathcal{K}, \prec \rangle$ is an AEC. What is $\mathrm{LS}(\mathcal{K})$?
(16)
(17)

6. Every model is saturated

In this section we show that if a countable theory T is categorical in an uncountable cardinality λ then all uncountable models are saturated thus T is categorical in every uncountable cardinality.

We start with a simple observation:

THEOREM 6.1. *Let T be a countable first-order theory. If T is categorical in some $\lambda > |\mathrm{L}(T)|$ then the model of cardinality λ is $\aleph_1$ saturated.*

PROOF. Since T is categorical in $\lambda > |T|$ by Corollary 4.3.25 it follows that T is $\aleph_0$-stable.

Define an elementary chain $\{M_i : i < \aleph_1\}$ such that

(1) if $i < \aleph_1$ then $\|M_i\| = \lambda$,
(2) if $i < j < \aleph_1$ then $M_i \preceq M_j$,
(3) if $i < \aleph_1$ then every $p \in \mathrm{S}(M_i)$ is realized in M_{i+1}.

Notice that since $\aleph_0$-stability implies λ-stability it is possible to fulfill the second requirement. Note also that from the third requirement it follows that the chain is strictly increasing.

Let $N := \bigcup_{i<\aleph_1} M_i$. To show that the model N is as required we need to show that for a given countable $A \subseteq |N|$ and a type p over A. The type p is realized in N: Since $|N|$ is a union of a strictly increasing chain of sets of length $\aleph_1$ there is $i < \aleph_1$ such that $A \subseteq M_i$ and by the second requirement p is realized in M_{i+1}. □

A similar argument to the one used in the last proof can be used to show

THEOREM 6.2. *Let T be a countable first-order theory. If T is categorical in some $\lambda > |\mathrm{L}(T)|$ then the model of cardinality λ is saturated.*

PROOF. See exercise 1. □

The purpose of the next theorem is to transfer non-saturation from one cardinality to λ (the categoricity cardinal).

THEOREM 6.3 (Transfer of non-saturation). *Suppose T is an $\aleph_0$-stable theory. Let λ and μ be uncountable cardinals. If T has an a model of cardinality μ which is not saturated then T has a model of cardinality λ that is not even $\aleph_1$-saturated.*

PROOF. Let M be a non-saturated model of cardinality μ. That is there is $A \subseteq |M|$ of cardinality less than μ and a consistent set of formulas p with whose

parameters are from the set A which is not realized in M. Denote by κ the cardinal $|A| + \aleph_0$. By Theorem 3.3 T is stable in κ thus by Theorem **??** there exists $\mathbf{I} := \{a_i \mid i < \kappa^+\} \subseteq |M|$ a sequence of indiscernibles over A.

Notice that since *p is a type over A and not realized in M.* we actually have even a stronger property: *The type p is not isolated over $A \cup \mathbf{I}$.* Otherwise, there is a consistent $\varphi(x; \mathbf{b})$ such that $\mathbf{b} \in A \cup \mathbf{I}$ and $\varphi(x; \mathbf{b}) \vdash p$. Since $\models \exists x \varphi(x; \mathbf{b})$ there is $a \in M$ such that $\models \varphi[a; \mathbf{b}]$ thus $a \models p$.

Let $\mathbf{I}_0 := \{a_i \mid i < \omega\}$.

The following is a Downward Löwenheim-Skolem argument:

LEMMA 6.4. *There is countable $A_0 \subset A$ such that $p \restriction A_0$ is not isolated over $A_0 \cup \mathbf{I}_0$.*

PROOF. By induction on $n < \omega$ define $B_k \subseteq A$ such that

(1) $B_k \subseteq B_{k+1}$ for all k,
(2) for every $k < \omega$ B_k is countable,
(3) for every $k < \omega$ the type $p \restriction B_{k+1}$ is not isolated over $\mathbf{I}_0 \cup B_k$.

This is enough: Define $A_0 := \bigcup_{k<\omega} B_k$. Suppose for the sake of contradiction that $p \restriction A_0$ is isolated over $\mathbf{I}_0 \cup A_0$. Thus there exists $k < \omega$ and there are $\mathbf{a} \in B_k$, $\mathbf{c} \in \mathbf{I}_0$ and a formula $\varphi(x; \mathbf{y}; \mathbf{z})$ such that $\varphi(x; \mathbf{c}; \mathbf{a}) \vdash p \restriction A_0$. This is impossible by requirement (3).

The construction: Observe that since p is not realized in M and $\mathbf{I} \cup A$ is a subset of $|M|$ the type p is not isolated over $\mathbf{I} \cup A$. Otherwise there are $\mathbf{a} \in A$, $\mathbf{c} \in \mathbf{I}$ and a formula $\varphi(x; \mathbf{y}; \mathbf{z})$ such that

$$(\dagger) \quad \varphi(x; \mathbf{c}; \mathbf{a}) \vdash p.$$

Since M is a model there exist $e \in |M|$ such that $M \models \varphi[e; \mathbf{c}; \mathbf{a}]$. Thus by $(\dagger)$ the element e realizes p in M, in contradiction to the hypothesis that p is omitted in M.

The definition of $\{B_k \ : \ k < \omega\}$ is by induction on k. For $k = 0$ take $B_0 := \emptyset$. Since p is not isolated over $\mathbf{I} \cup A$ it is not isolated over $\mathbf{I}_0 \cup B_k$. That is for every $\varphi(x; \mathbf{y})$ and every finite $\mathbf{c} \in \mathbf{I}_0 \cup B_k$ there is $\psi(x; \mathbf{a}_{\varphi(x;\mathbf{c})}) \in p$ such that

$$\models \exists x[\varphi(x; \mathbf{c}) \wedge \psi(x; \mathbf{a}_{\varphi(x;\mathbf{c})})] \text{ and } \models \exists x[\varphi(x; \mathbf{c}) \wedge \neg\psi(x; \mathbf{a}_{\varphi(x;\mathbf{c})})].$$

Let

$$B_{k+1} := B_k \cup \{\mathbf{a}_{\varphi(x;\mathbf{c})} \mid \mathbf{c} \in \mathbf{I}_0 \cup B_k, \varphi(x; \mathbf{y}) \in \mathrm{L}(T), \models \exists x \varphi(x; \mathbf{c}) \text{ is consistent}\}.$$

Check that B_{k+1} is as required. □

Using the compactness theorem show that the following theory is consistent:

$$T_\lambda := T \cup \{c_i \neq c_j \ : \ i < j < \lambda\} \cup$$

$$\{\varphi(c_{i_0}, \dots, c_{i_{n-1}}; \mathbf{a}) \leftrightarrow \varphi(c_0, \dots, c_{n-1}; \mathbf{a}) \ : \ i_0 < \dots < i_{n-1} < \lambda,$$

$$\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T)), \mathbf{a} \in A_0\} \cup \{a_i = c_i \mid i < \omega\}.$$

An application of the compactness theorem to T_λ gives that there is a sequence of indiscernibles (in $\mathfrak{C}$) $\mathbf{J} := \{a_i \mid i < \lambda\}$ over A_0. Notice that $\mathbf{I}_0$ are the first ω-elements of $\mathbf{J}$. For more details see Corollary 3.38

Since T is $\aleph_0$-stable by Theorem 5.4 there exists a primary model N over $\mathbf{J} \cup A_0$. It suffices to prove

CLAIM 6.5. *The model N is not $\aleph_1$-saturated, since the type $p \restriction A_0$ is omitted in N.*

I will present two proofs the first a short argument using Theorem 5.10 and the second little different.

FIRST PROOF OF CLAIM 6.5. Suppose for the sake of contradiction that $b \in N$ is an element realizing $p_0 \restriction A_0$. By Theorem 5.10 there exists $\varphi(x; \mathbf{y}; \mathbf{z})$ a formula from $\mathrm{L}(T)$ and there are $\mathbf{a} \in A_0$ and $\mathbf{j}$ a finite sequence of ordinals $\mathbf{j}$ from λ such that

$$(*) \quad \varphi(x; \mathbf{a}; \mathbf{a}_\mathbf{j}) \vdash p_0 \restriction A_0.$$

Suppose $\mathbf{j} = \langle i_1, \dots, i_n, j_{n+1}, \cdots, j_m \rangle$ an increasing sequence such that $i_n < \omega$ and $j_{n+1} \geq \omega$. As ω has no last element we pick $i_{n+1} < \ldots < i_m < \omega$ such that $i_n < i_{n+1}$. As $\mathbf{J}$ is a sequence of indiscernible we get that

$$\mathrm{tp}(\mathbf{a}_\mathbf{j}/A_0) = \mathrm{tp}(a_{i_0}, \dots, a_{i_n}, a_{i_{n+1}}, \dots, a_{i_m}/A_0)$$

there is $f \in \mathrm{Aut}_{A_0 \cup \{a_{i_0}, \dots, a_{i_n}\}}(\mathfrak{C})$ such that $f(\langle a_{j_{n+1}}, \dots, a_{j_m} \rangle) = \langle a_{i_{n+1}}, \dots, a_{i_m} \rangle$. An application of f to $(*)$ gives

$$(**) \quad \varphi(x; \mathbf{a}; a_{i_0}, \dots, a_{i_n}, a_{i_{n+1}}, \dots, a_{i_m}) \vdash p_0 \restriction A_0.$$

Namely $p_0 \restriction A_0$ is isolated over $A_0 \cup \mathbf{I}_0$, this is a contradiction to Lemma 6.4 . $\square$

SECOND PROOF OF CLAIM 6.5. Let $\alpha < \lambda^+$ and suppose that $\{b_i \ : \ i < \alpha\}$ is an isolating sequence of N over $\mathbf{J} \cup A_0$. We need to show that no element of N realizes the type $p \restriction A_0$.

Observe that no element of $\mathbf{J} \cup A_0$ can realize $p \restriction A_0$.

Show for every $i < \alpha$ that b_i does not realize $p \restriction A_0$:

SUB CLAIM 6.6. *$p \restriction A_0$ is not isolated over $\mathbf{J} \cup A_0$.*

PROOF. Suppose that $b \models p \restriction A_0$ and there exists $\varphi(x; \mathbf{c}; \mathbf{a}) \in \mathrm{tp}(b_0/\mathbf{J} \cup A_0)$ such that

$$(*) \quad \varphi(x; \mathbf{c}; \mathbf{a}) \vdash \mathrm{tp}(b_0/\mathbf{J} \cup A_0) \vdash p \restriction A_0.$$

where $\mathbf{a} \in A_0$ and $\mathbf{c} = \langle c_{j_0}, \dots, c_{j_{n-1}} \rangle$ a sequence of elements of $\mathbf{J}$ such that $j_0 < \cdots < j_{n-1} < \lambda$. Since $\mathbf{J}$ extends an the infinite sequence of indiscernibles $\mathbf{I}_0$ (that had no last element) we can find $j'_0 < \cdots < j'_{n-1} < \beta_0$ such that $\mathbf{d} := \langle a_{j'_0}, \dots, a_{j'_{n-1}} \rangle$ (in $\mathbf{I}_0$) satisfying that $\mathrm{tp}(\mathbf{d}/A_0) = \mathrm{tp}(\mathbf{c}/A_0)$. Thus there exists $f \in \mathrm{Aut}_{A_0}(\mathfrak{C})$ such that $f(\mathbf{c}) = \mathbf{d}$. An application of f to $(*)$ gives

$$\varphi(x; f(\mathbf{c}); \mathbf{a}) \vdash p \restriction f(A_0).$$

By $f \restriction A_0 = \mathrm{id}_{A_0}$ and the other assumption on f we get

$$\varphi(x; \mathbf{d}; \mathbf{a}) \vdash p \restriction A_0.$$

Namely $p \restriction A_0$ is isolated over $A_0 \cup \mathbf{I}_0$, this is a contradiction to Lemma 6.4. □

For $i = 0$; Immediate from the sub claim. Suppose that $i > 0$ and $b_i \models p \restriction A_0$.
Let $\varphi_i(x; \mathbf{c}_i; \mathbf{b}_i; \mathbf{a}_i) \in \mathrm{tp}(b_i/\mathbf{J} \cup \{b_j \mid j < i\} \cup A_0)$ such that

$$(**)_i \quad \varphi_i(x; \mathbf{c}_i; \mathbf{b}_i; \mathbf{a}_i) \vdash \mathrm{tp}(b_i/\mathbf{J} \cup \{b_j \mid j < i\} \cup A_0).$$

Where $\mathbf{a} \in A_0$, $\mathbf{c}_i \in \mathbf{J}$ and $\mathbf{b}_i \in \{b_j \mid j < i\}$.
Let $j^i_0 < \cdots < j^i_{n_i-1} < \lambda$ and $f(i)_0 < \cdots < f(i)_{m_i-1} < i$ such that
$\mathbf{c}_i = \langle c_{j^i_0}, \dots, c_{j^i_{n_i-1}} \rangle$ and $\mathbf{b}_i = \langle b_{f(i)_0}, \dots, b_{f(i)_{m_i-1}} \rangle$.

Since $i > 0$ for every $n < n_i$ there is
$\varphi_{f(i)_n}(x; \mathbf{c}_n; \mathbf{b}_n; \mathbf{a}_n) \in \mathrm{tp}(b_{f(i)}/\mathbf{J} \cup \{b_j \mid j < f(i)\} \cup A_0)$ isolating the type. Since for every $n < n_i$ we have that $f(f(i))_n < f(i)_n$ and there is no infinite descending sequence of ordinals there exists $k < \omega$ such that

$$\varphi_{f^k(i)_n}(x; \mathbf{c}_n; \mathbf{b}_n; \mathbf{a}_n) \in \mathrm{tp}(b_{f^k(i)}/\mathbf{J} \cup A_0)$$

Where $f^k(i)$ is $f(\cdots(i)\cdots)$ k-times. Now define

$$\psi(x; \mathbf{c}; \mathbf{a}) := \exists y_0 \cdots \exists y_k \bigwedge_{j<k} \varphi_{f^j(i)_n}(y_k; \mathbf{c}_n; \mathbf{b}_n; \mathbf{a}_n).$$

□

An alternative way to show this is by observing that Theorem 5.10 contradicts SubClaim 6.6. □

Similarly to Corollary 2.9.39 we can deal with expansions by constants. Recall that when T is a complete theory and $A \subseteq |\mathfrak{C}|$ then by T_A denote the following expansion of T: $\mathrm{Th}(\langle \mathfrak{C}, c_a \rangle_{a \in A})$.

COROLLARY 6.7 (Uncountable categoricity is preserved by expansions). *Let T be a countable theory and $A \subseteq \mathfrak{C}$. If T is categorical in some uncountable cardinal then T_A is categorical in every $\lambda > |A| + \aleph_0$.*

PROOF. Exercise. □

The following is a byproduct of the proof of Morley's theorem

COROLLARY 6.8. *Let T be a countable complete theory. If there exists $\lambda > \aleph_0$ such that every model of T of cardinality λ is $\aleph_1$-saturated then T is categorical in every uncountable cardinality.*

THEOREM 6.9. *If T is categorical in some $\lambda > |T|$ then the model of cardinality λ is $|T|^+$-saturated.*

The proof of Theorem 6.3 can be adapted to get an omitting-types theorem.

THEOREM 6.10 (Omitting-types for $\aleph_0$-stable theories). *Let T be a complete $\aleph_0$-stable theory and $p(\mathbf{x})$ be a set of formulas in finitely many variables $\mathbf{x}$. If T has an uncountable model omitting p then for every uncountable μ T has a model of cardinality μ omitting p.*

PROOF. Exercise. □

Using the technique from Claim 6.5 we get the following additional structural result:

THEOREM 6.11. *Suppose that T is in a countable language and is categorical in an uncountable cardinal, $M \models T$ and $\varphi(\mathbf{x};\mathbf{a})$ a non-algebraic formula with parameters from M. There is no infinite set $\mathbf{I} \subseteq |M|$ such that $\mathbf{I}$ are indiscernibels over the set $\varphi(M;\mathbf{a})$.*

PROOF. For the sake of contradiction suppose that such a set $\mathbf{I}$ exists,. Let $\mathbf{J} \supseteq \mathbf{I}$ be a set of indiscernibles of cardinality $\|M\|^+$. Pick M^* an $F^t_{\aleph_0}$ primary over $\varphi(M,\mathbf{a}) \cup \mathbf{J}$ (exists since T is $\aleph_0$ stable). Denote by A the set $\varphi(M;\mathbf{a})$, and by p the following set of formulas $\{\varphi(\mathbf{x};\mathbf{a}) \wedge \mathbf{x} \neq \mathbf{b} \mid \mathbf{b} \in A\}$, it is a consistent type since $\varphi(M,\mathbf{a})$ is infinite. We will obtain a contradiction to $\aleph_1$-categoricity by showing that M^* omits p (which show that M^* is not saturated): Suppose there exists an element b in M^* realizing p. Since M^* is primary over $A \cup \mathbf{J}$ there are a formula $\psi(\mathbf{x};\mathbf{y};\mathbf{z})$, $\mathbf{c} \in A$ and $\mathbf{j} \in J$ such that

$$(*) \quad \psi(\mathbf{x};\mathbf{c};\mathbf{j}) \vdash \mathrm{tp}(\mathbf{b}/A).$$

Since $\mathbf{I}$ is an infinite subset of $\mathbf{J}$ there exists a finite sequence $\mathbf{i}$ of elements of $\mathbf{I}$ (of the same length as $\mathbf{j}$) such that $\mathrm{tp}(\mathbf{i}/A) = \mathrm{tp}(\mathbf{j}/A)$. Let F be an (M^*, M^*)-mapping such that $F \restriction A = \mathrm{id}_A$ and $F(j) = i$. An application of F to $(*)$ gives

$$(**) \quad \psi(\mathbf{x};\mathbf{c};\mathbf{i}) \vdash \mathrm{tp}(\mathbf{b}/A).$$

Since $M^* \models \exists \mathbf{x} \psi(\mathbf{x};\mathbf{c};\mathbf{i})$ (exemplified by b) also $M \models \exists \mathbf{x} \psi(\mathbf{x};\mathbf{c};\mathbf{i})$ let $d \in M$ such that $M \models \psi[\mathbf{d};\mathbf{c};\mathbf{i}]$ by $(**)$ $d \models \mathrm{tp}(\mathbf{b}/A)$, and in particular p is realized by $\mathbf{d}$ (since $p \subseteq \mathrm{tp}(\mathbf{b}/A)$). This is impossible since it is clear from the definition of p that p is omitted by M. □

Exercises 7.6.

(1) Prove Theorem 6.2. Hint: Distinguish between two cases: λ regular and λ singular. For λ singular show that for every $\mu < \lambda$ there is a μ^+-saturated model of cardinality λ.

(2) An outline for an alternative proof to Lemma 6.4:
Let $\mathfrak{B} := \langle H(\chi), \in, p, A, \mathbf{I}, \varphi \rangle_{\varphi \in \mathrm{Fml}(L)}$ when χ is a regular cardinal large enough such that $H(\chi)$ contains the sets p, A and $\mathbf{I}$ and $\langle H(\chi), \in \rangle$ reflects the fact that "the type p is not isolated over $A \cup \mathbf{I}$". $p, A, \mathbf{I}$, are unary predicates and φ are individual constants.

Show that if $M \preceq \mathfrak{B}$ is countable then the sets p^M, A^M and $\mathbf{I}^M$ are as required.

(3) Prove:

THEOREM 6.12. *Let T be a complete first-order theory in a countable language. The following are equivalent:*
(a) *T is categorical in some uncountable cardinal.*
(b) *T is categorical in every uncountable cardinal.*
(c) *All uncountable models of T are $\aleph_1$-saturated.*

(4)

(5) Suppose T is a complete $\aleph_0$-stable theory in a countable language. If there are $M \models T$ and $\varphi(\mathbf{x}) \in \mathrm{Fml}(\mathrm{L}(T))$ such that $\|M\| > |\varphi(M)| \geq \aleph_0$ then for every $\lambda \geq \aleph_0$ there exists $N \models T$ such that $\|N\| = \lambda$ and $|\varphi(N)| = \aleph_0$.

Hint: Consider the type $p := \{\varphi(\mathbf{x}) \wedge \mathbf{x} \neq \mathbf{a} \mid \mathbf{a} \in \varphi(M)\}$, it is omitted by M.

(6)

(7) Prove Theorem 6.10.

(8) Does the the conclusion of Theorem 6.10 holds also for $\mu = \aleph_0$?

(9) Suppose that T is $\aleph_0$-stable theory, let A be a subset of the monster model and $\Gamma \subseteq \mathrm{S}^n(A)$. Prove the following generalization of Theorem 6.10: If there exists $M \models T$ such that $\|M\| > |A| + \aleph_0$ omitting all types from Γ then for every $\mu \geq |A| + \aleph_0$ there exists a model $N \models T$ of cardinality μ omitting all types in Γ.

(10) Let T be $\aleph_0$-stable in a countable language and $\varphi(x)$ a formula in the language of T. If there exists $M \models T$ such that $\|M\| > |\varphi(M)| + \aleph_0$. Show that for any $\lambda \geq \|M\|$ there exists $N \supsetneq M$ of cardinality λ such that $\varphi(N) = \varphi(M)$.

(11) Let T be a complete first-order theory in a countable language which is $\aleph_0$-stable. If there exists λ such that all models of T of cardinality λ are $\aleph_1$-saturated then T is categorical in all uncountable cardinals.

(12)

7. Chang's Conjecture is true for $\aleph_0$-stable theories

Shortly after Vaught established his two cardinal theorem 12.9, C. C. Chang asked if instead finding an $(\aleph_1, \aleph_0)$-model elementarily equivalent to a two cardinal model can this model be an elementary submodel?

Namely

CONJECTURE 7.1 (Chang). *Let L be at most countable containing a unary predicate P. If M is an L-structure such that $\|M\| = \aleph_2$ and $|P^M| = \aleph_1$ then there exists $N \prec M$ such that $\|N\| = \aleph_1$ and $|P^N| = \aleph_0$.*

According to a legend, C. C. Chang assigned his conjecture as a thesis topic to several students in the early sixties. Until

THEOREM 7.2 (Rowbottom 1964). *Chang's conjecture fails in* **L**.

THEOREM 7.3 (Silver []). *If there exists an Erdős cardinal then Chang's conjecture is true.*

and

THEOREM 7.4 (Donder []). *If the statement of Chang's conjecture holds then there exists an ω_1-Erdős cardinal in* **K**.

DEFINITION 7.5. Chang Conjecture and STRONG CHange C

STATE CONSISTENCY RESULTS (SILVER, KOEPKE, ω_1-RAMSEY AND HUGE)

As an application of the ideas used to prove Morley's theorem we will prove

THEOREM 7.6 (Chang's conjecture holds for $\aleph_0$-stable theories). *Suppose T is an $\aleph_0$-stable theory in a countable language containing a unary predicate P. If $M \models T$ is a two-cardinal model, i.e. $\|M\| > |P^M| \geq \aleph_0$ then for any μ, λ such that $\aleph_0 \leq \mu \leq |P^M|$ and $\mu \leq \lambda < \|M\|$ there exists $N \preceq M$ of cardinality λ such that $|P^N| = \mu$.*

PROOF. By $\aleph_0$ stability (using Theorem 4.1) there exists $\{a_i \mid i < \lambda^+\} \subseteq |M| - P^M$ an indiscernible sequence over P^M. Let $A_0 \subseteq P^M$ be of cardinality μ, take $M_0 \preceq M$ of cardinality μ such that M_0 contains $\{a_i \mid i < \omega\} \cup A_0$. Let N^* be a primary model over $P^{M_0} \cup \{a_i \mid i < \lambda\}$. As primary models are also prime (Lemma 5.8) there exists an elementary embedding $f : N^* \to M$. To show that $N := f[N^*]$ is as required, it suffices to prove

CLAIM 7.7. $P^N = P^{M_0}$.

PROOF. Suppose for the sake of contradiction that there exists $b \in P^N - P^{M_0}$. As N is primary model over $A := P^{M_0} \cup \{a_i \mid i < \lambda\}$ (by Theorem 5.10) there are $\varphi(x; \mathbf{y}; \mathbf{z}), \mathbf{c} \in P^{M_0}$ and $\mathbf{a} \in \{a_i \mid i < \lambda\}$ such that

$$(*) \quad \varphi(x; \mathbf{c}; \mathbf{a}) \vdash \mathrm{tp}(b/A).$$

By the choice of b we have that the formula $P(x)$ is an element of $\mathrm{tp}(b/A)$ and that $\{x \neq d \mid d \in P^{M_0}\} \subseteq \mathrm{tp}(b/A)$.

Since $\{a_i \mid i < \lambda^+\}$ is an indiscernible sequence over P^{M_0} and $\{a_i \mid i < \omega\} \subseteq |M_0|$ there exists $\mathbf{a}' \in \{a_i \mid i < \omega\}$ such that $\mathrm{tp}(\mathbf{a}/P^{M_0}) = \mathrm{tp}(\mathbf{a}'/P^{M_0})$. Let $f \in \mathrm{Aut}_{P^{M_0}}(\mathfrak{C})$ such that $f(\mathbf{a}) = \mathbf{a}'$. Apply f on $(*)$ and we get

$$(**) \quad \varphi(x; \mathbf{c}; \mathbf{a}') \vdash P(x) \cup \{x \neq d \mid d \in P^{M_0}\}.$$

As $\mathfrak{C} \models \exists x \varphi(x; \mathbf{c}; \mathbf{a}')$ and $\mathbf{c}, \mathbf{a}' \in M_0$ we get that there exists $b' \in M_0$ such that $M_0 \models \varphi[b'; \mathbf{c}; \mathbf{a}']$, now by $(**)$ we get $M_0 \models P(b') \cup \{b' \neq d \mid d \in P^{M_0}\}$ which is a contradiction. □

□

The method of proof of the last theorem can be used to establish a super strong two-cardinal theorem and also a strong version of Chang's conjecture.

THEOREM 7.8 (The strongest possible two cardinal theorem). *Suppose T is $\aleph_0$-stable. If T admits a $\langle \kappa^+, \kappa \rangle$-model for some κ then for every $\mu \leq \lambda$ infinite T admits a $\langle \lambda, \mu \rangle$-model.*

THEOREM 7.9 (Strong Chang's conjecture). *Suppose T is $\aleph_0$-stable. If $M \models T$ is a $\langle \kappa^+, \kappa \rangle$-model for some κ then for every $\mu \leq \lambda < \kappa$ infinite there exists $N_{\lambda,\mu} \preceq M$ of cardinality λ such that $P^{N_{\lambda,\mu}}$ has cardinality μ.*

I leave both proofs as exercise to the reader.

EXERCISE

Let T be first-order in a countable language. Suppose T is λ-categorical in some $\lambda > \aleph_0$

There are no $M \precneqq N \models T$, $\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))$ and $\mathbf{a} \in |M|$ such that $\varphi(M; \mathbf{a}) \subsetneqq \varphi(N; \mathbf{a})$.

8. Quasi-minimal formulas and an omitting types Theorem

I present here a key step towards the Baldwin-Lachlan proof of Morley's theorem which has value of its own.

We (in Theorem I.6.13) constructed a theory T_0 in a countable language and a type p_0 such that $\mathrm{EC}(T_0, p_0)$ has models in all infinite cardinals $\leq 2^{\aleph_0}$ but has no model of cardinality $2^{\aleph_0}$. It is easy to see that T_0 is unstable. So it is natural to wonder if such a stable example exists.

Let T be a consistent first-order theory and Γ a set of T-types (without parameters. We know that if $M \in \mathrm{EC}(T, \Gamma)$ then it is possible that there is no $N \in \mathrm{EC}(T, \Gamma)$ such that $\|N\| > \|M\|$.

First I will show using the methods used to prove Morley's theorem that when T is an $\aleph_0$-stable theory (in a countable language) then such an example does not exists. Later we present a stronger result due to Baldwin and Lachlan.

THEOREM 8.1. *Suppose T is an $\aleph_0$-stable theory in a countable language and Γ is a set of T-types. If $\mathrm{EC}(T, \Gamma)$ has an uncountable model then for every $\mu > \aleph_0$ there exists $N \in \mathrm{EC}(T, \Gamma)$ of cardinality μ.*

PROOF. As $M \in \mathrm{EC}(T, \Gamma)$ and $N \prec M$ implies that $N \in \mathrm{EC}(T, \Gamma)$. It is enough to show that given $M \in \mathrm{EC}(T, \Gamma)$ of cardinality $\lambda > \aleph_0$ then for any $\mu > \lambda$ there exists $N \in \mathrm{EC}(T, \Gamma)$ of cardinality μ.

Using $\aleph_0$-stability fix $\{a_i \mid i < \omega\} \subseteq |M|$ an indiscernible sequence. Using Theorem 3.37 there exists $\{a_i \mid \omega_{\leq} i < \mu\} \subseteq |\mathfrak{C}|$ such that $\mathbf{I} := \{a_i \mid i < \mu\}$ is an indiscernible sequence. Using $\aleph_0$-stability let N be a primary model over $\mathbf{I}$.

It suffices to prove that

CLAIM 8.2. $N \in \mathrm{EC}(T, \Gamma)$.

For the sake of contradiction suppose that there are $p(\mathbf{x}) \in \Gamma$ and $b \in |N|$ such that $\mathbf{b} \models p$. As N is primary it is atomic thus there exists a formula $\varphi(\mathbf{x}; \mathbf{y})$ and $i_1 < \cdots i_n < \mu$ such that

$$\varphi(\mathbf{x}; a_{i_1}, \ldots, a_{i_n}) \vdash \mathrm{tp}(b/\mathbf{I}) \text{ and } \mathfrak{C} \models \exists \mathbf{x} \varphi(\mathbf{x}; a_{i_1}, \ldots, a_{i_n}).$$

As $p \subseteq \mathrm{tp}(b/\mathbf{I})$ we have

$$(*) \quad \varphi(\mathbf{x}; a_{i_1}, \ldots, a_{i_n}) \vdash p.$$

Pick $j_1 < \ldots < j_n < \omega$, by indicsernibility there exists $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(a_{i_k}) = a_{j_k}$ for every $1 \leq k \leq n$. An application of f to $(*)$ gives that

$$(**) \quad \varphi(\mathbf{x}; a_{j_1}, \ldots, a_{j_n}) \vdash p.$$

Now as $\mathfrak{C} \models \exists \mathbf{x} \varphi(\mathbf{x}; a_{i_1}, \ldots, a_{i_n})$ an application of f gives $\mathfrak{C} \models \exists \mathbf{x} \varphi(\mathbf{x}; a_{j_1}, \ldots, a_{j_n})$ there exist $\mathbf{b}' \in |M|$ such that $M \models \varphi[\mathbf{b}'; a_{j_1}, \ldots, a_{j_n}]$ using $(**)$ we get that $\mathbf{b}' \models p$ a contradiction to M omitting p. □

THEOREM 8.3 (Baldwin-Lachlan's omitting types theorem). *Suppose T is $\aleph_0$-stable in a countable language and let Γ be a set of T-types.*

For every uncountable $M \in \mathrm{EC}(T, \Gamma)$ there exists $N \in \mathrm{EC}(T, \Gamma)$ such that $N \gneqq M$.

First we consider an important generalization of the notion of strongly minimal formula, this we will extend later when dealing with AECs (see section **??**).

DEFINITION 8.4. Let M be an uncountable model, for a formula $\varphi(\mathbf{x}; \mathbf{y})$ and $\mathbf{a} \in |M|$ we say $\varphi(\mathbf{x}; \mathbf{a})$ *is M-large* provided $|\varphi(M; \mathbf{a})| > \aleph_0$.

Suppose $\varphi(\mathbf{x}; \mathbf{a})$ is M-large, $\varphi(\mathbf{x}; \mathbf{a})$ is called *quasi-minimal over M* provided for every $\chi(\mathbf{x}; \mathbf{b})$ (with $\mathbf{b} \in |M|$) either

$$|\varphi(M; \mathbf{a}) \cap \psi(M; \mathbf{b})| \leq \aleph_0 \text{ or } |\varphi(M; \mathbf{a}) \cap \neg\psi(M; \mathbf{b})| \leq \aleph_0$$

Similarly to Theorem 9.9 we can establish:

THEOREM 8.5 (Density of q.m. formulas). *Suppose T is $\aleph_0$-stable and $M \models T$ uncountable. If $\varphi(\mathbf{x}; \mathbf{a})$ is M-large then there exists a quasi-minimal formula $\varphi^*(\mathbf{x}; \mathbf{a}^*)$ with $\mathbf{a}^* \in |M|$ such that $\varphi^*(\mathbf{x}; \mathbf{a}^*) \vdash \varphi(\mathbf{x}; \mathbf{a})$.*

PROOF. Suppose that M and $\varphi(\mathbf{x}; \mathbf{a})$ are such that $\mathbf{a} \in |M|$ with $|\varphi(M; \mathbf{a})| > \aleph_0$ and there is no q.m. formula $\varphi^*(\mathbf{x}; \mathbf{a}^*)$ as above.

For every $\eta \in {}^{\omega >}2$ define $\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)$ with $\mathbf{a}_\eta \in |M|$ such that both

$$\bigwedge p_\eta \wedge \psi_\eta(\mathbf{x}; \mathbf{a}_\eta) \text{ and } \bigwedge p_\eta \wedge \neg\psi_\eta(\mathbf{x}; \mathbf{a}_\eta)$$

are realized by uncountably many elements of M, where

$$p_\eta := \{\varphi(\mathbf{x}; \mathbf{a})\} \cup \{\psi_{\eta \restriction k}^{\eta[k]}(\mathbf{x}; \mathbf{a}_{\eta \restriction k}) \mid k < \ell(\eta)\}.$$

Now as usual we contradict $\aleph_0$-stability, since $|\mathrm{S}^{\mathbf{x}}(A)| = 2^{\aleph_0}$ when $A := \{\mathbf{a}_\eta \mid \eta \in {}^{\omega >}2\}$. □

PROOF OF 8.3. As M is uncountable and $\mathrm{Fml}(\mathrm{L}(T))$ is countable there exists a formula $\varphi(\mathbf{x})$ such that $\varphi(M)$ is uncountable.

By the density of q.m. formulas let $\varphi^*(\mathbf{x}; \mathbf{a}^*)$ such that $\varphi^*(\mathbf{x}; \mathbf{a}) \vdash \varphi(\mathbf{x})$ for some $\mathbf{a}^* \in |M|$. Consider

$$p(\mathbf{x}) := \{\psi(\mathbf{x}; \mathbf{d}) \ : \ \psi(\mathbf{x}; \mathbf{z}) \in \mathrm{L}(T), \mathbf{d} \in |M| \text{ and } \varphi^*(\mathbf{x}; \mathbf{a}^*) \wedge \psi(\mathbf{x}; \mathbf{d}) \text{ is } M\text{-large}\}.$$

CLAIM 8.6. $p \in \mathrm{S}^{\mathbf{x}}(|M|)$

PROOF. Since $\varphi^*(\mathbf{x};\mathbf{a}^*)$ is quasi minimal clearly $\psi(\mathbf{x};\mathbf{d}) \in p$ or $\neg\psi(\mathbf{x};\mathbf{d}) \in p$ for all ψ and $\mathbf{b} \in |M|$. Enough to show that p is consistent. Given $\{\psi_k(\mathbf{x};\mathbf{d}_k) \mid k < n\} \subseteq p$ we have that $\bigwedge_{k<n} \psi_k(\mathbf{x};\mathbf{d}_k) \in p$. Otherwise $\bigvee_{k<n} \neg\psi_k(\mathbf{x};\mathbf{d}_k) \in p$, but

$$\psi_k(\mathbf{x};\mathbf{d}_k) \in p \implies \varphi^*(\mathbf{x};\mathbf{a}^*) \wedge \neg\psi_k(\mathbf{x};\mathbf{d}_k) \text{ is countable}$$

and a finite union of countable sets is also countable. □

As p can't be realized by an element of M, there exists $\mathbf{b} \in \mathfrak{C} - |M|$ such that $\mathbf{b} \models p$.

Let N be primary over $|M| \cup \mathbf{b}$. As $N \gtrneq M$ it is enough to establish

CLAIM 8.7. $N \in \mathrm{EC}(T,\Gamma)$.

PROOF. For the sake of contradiction, suppose that there is $\mathbf{c} \in |N|$ such that $\mathbf{c} \models q$ for some $q \in \Gamma$. By Theorem 5.10 N is atomic over $|M| \cup \{\mathbf{b}\}$, i.e. there are $\mathbf{a} \in M$ and a formula $\psi(\mathbf{z};\mathbf{y};\mathbf{x})$ such that

$$\psi(\mathbf{z};\mathbf{a};\mathbf{b}) \vdash \mathrm{tp}(\mathbf{c}/M \cup \mathbf{b}) \text{ and } \mathfrak{C} \models \exists\mathbf{z}\psi(\mathbf{z};\mathbf{a};\mathbf{b})$$

In particular we have

$$\psi(\mathbf{z};\mathbf{a};\mathbf{b}) \vdash q.$$

I.e. for every $\chi(\mathbf{z}) \in q$

$$\mathfrak{C} \models \forall\mathbf{z}[\psi(\mathbf{z};\mathbf{a};\mathbf{b}) \to \chi(\mathbf{z})].$$

Namely,

$$(1) \quad \chi^*(\mathbf{x}) := \forall\mathbf{z}[\psi(\mathbf{z};\mathbf{a};\mathbf{x}) \to \chi(\mathbf{z})] \in p.$$

Since

$$\mathfrak{C} \models \exists\mathbf{z}\psi(\mathbf{z};\mathbf{a};\mathbf{b})$$

we get

$$(2) \quad \exists\mathbf{z}\psi(\mathbf{z};\mathbf{a};\mathbf{x}) \in p.$$

By (1) and quasi minimality we get that $D_\chi := \neg\chi^*(M) \cap \varphi^*(M;\mathbf{a}^*)$ is at most countable for all $\chi(\mathbf{z}) \in q$ and using (2) we get $D := \neg\exists\mathbf{z}\psi(\mathbf{z};\mathbf{a};M) \cap \varphi^*(M;\mathbf{a}^*)$ is also at most countable.

Since $\varphi^*(M;\mathbf{a}^*)$ is uncountable there exists

$$\mathbf{b}' \in \varphi^*(M;\mathbf{a}^*) - (\bigcup_{\chi\in q} D_\chi \cup D).$$

Since $\mathfrak{C} \models \exists\mathbf{z}\psi(\mathbf{z};\mathbf{a};\mathbf{b}')$ there exists $\mathbf{c}' \in {}^{\mathbf{z}}|M|$ witnessing this, so by (1) and the choice of $\mathbf{c}'$ (Definition of D_χ) we get that $M \models [\psi(\mathbf{c}';\mathbf{a};\mathbf{b}') \to \chi(\mathbf{c}')]$ for all $\chi \in q$. Namely we found an element $\mathbf{c}' \in |M|$ realizing q this is a contradiction to $M \in \mathrm{EC}(T,\{q\})$.

□

This concludes the proof of the omitting types theorem. □

Notice that the only non-trivial property of $\aleph_0$-stability used in the previous argument is the fact that for every $A \subseteq |M|$ there exists a primary model $N \prec M$ over A.

Below we introduce the property known by the name *Vaughtian Pair* and will be used in the next section (see Theorem 9.1) to characterize when a first-order theory is categorical in an uncountable cardinal.

DEFINITION 8.8 (Vaughtian Pair). Let T be a theory and $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))$ be given. We say that (M,N) *is Vaughtian pair for* φ iff there exists $\mathbf{a} \in {}^{\ell(\mathbf{y})}|M|$ such that

$$M \precneqq N \models T \quad \text{and} \quad \varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a}) \text{ is infinite.}$$

We say that T *admits a Vaughtian pair* iff there are models M, N, $\mathbf{a} \in |M|$ and a formula φ as above.

Notice that if $\mathrm{L}(T)$ has a unary predicate P and there exists $M \models T$ such that $\|M\| > |P^M| \geq |\mathrm{L}(T)| + \aleph_0$ then T has a Vaughtian pair for P.

The following two corollaries will be used in the next section:

COROLLARY 8.9. *Suppose T is $\aleph_0$-stable. If there are $M \models T$, $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))$ and $\mathbf{a} \in |M|$ such that $\|M\| > |\varphi(M;\mathbf{a})| = \aleph_0$ then there exist $N \succneqq M$ such that $\varphi(N;\mathbf{a}) = \varphi(M;\mathbf{a})$.*

PROOF. Consider

$$p(\mathbf{x}) := \{\varphi(\mathbf{x};\mathbf{a}) \wedge \mathbf{x} \neq \mathbf{b} \mid \mathbf{b} \in {}^{\ell(\mathbf{x})}\varphi(M;\mathbf{a})\}.$$

Clearly M omits p. As $T_A := \mathrm{Th}(M,c)_{c\in\varphi(M;\mathbf{a})}$ is also $\aleph_0$-stable an application of Lachlan's theorem produces $N \succneqq M$ omitting p. We are done, since whenever $N \succ M$ such that N omits the type p then $\varphi(N;\mathbf{a}) = \varphi(M;\mathbf{a})$. □

Note that the last Corollary provides us with a very strong upward two-cardinal theorem generalizing Chang's two cardinal theorem:

COROLLARY 8.10 (Upward two-cardinal theorem for $\aleph_0$-stable theories). *Suppose M is an $(\aleph_1,\aleph_0)$-model in a countable language. If $\mathrm{Th}(M)$ is $\aleph_0$-stable then for every $\lambda > \aleph_1$ there exists $N \succneqq M$ which is a $(\lambda,\aleph_0)$-model.*

PROOF. Use the previous corollary to build an elementary chain of length λ. □

THEOREM 8.11 (Categoricity implies no Vaughtian pairs). *Let T be first-order in a countable language. Suppose T is λ-categorical in some $\lambda > \aleph_0$ then there are no $M \precneqq N \models T$, $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))$ and $\mathbf{a} \in |M|$ such that $\varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a})$ and infinite.*

PROOF. Otherwise, by the Löwenheim-Skolem theorems there are $M \precneqq N$ both of cardinality λ, $\mathbf{a} \in |M|$ and $\varphi(\mathbf{x};\mathbf{y})$ such that $\varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a})$ is infinite. By induction on $\alpha < \lambda^+$ define $\{M_\alpha \mid \alpha < \lambda^+\}$ such that

(1) $M_0 = M$,
(2) $M_\alpha \precneqq M_\beta$ for all $\alpha < \beta$,
(3) $\|M_\alpha\| = \lambda$,
(4) $\varphi(M_\alpha;\mathbf{a}) = \varphi(M;\mathbf{a})$

Given M_α by λ categoricity $M_\alpha \cong M$ and by Lemma 1.2.12 there exists $M_{\alpha+1} \gneqq M_\alpha$ isomorphic to N as required.

Let $M^* := \bigcup_{\alpha<\lambda^+} M_\alpha$. By the properties of the elementary chain we have

$$\|M^*\| = \lambda^+ \text{ and } \varphi(M^*;\mathbf{a}) = \varphi(M;\mathbf{a}) \text{ is of cardinality } \leq \lambda.$$

An application of Vaught's two cardinal theorem[6] gives us $N^* \equiv M^*$ of cardinality $\aleph_1$ such that $|\varphi(N^*;\mathbf{a})| = \aleph_0$.

As T is categorical in $\lambda > \aleph_0$ By Corollary 3.25, the theory T is $\aleph_0$-stable. An application of the previous corollary gives: There exists $N^{**} \gneqq N^*$ of cardinality λ such that $\varphi(N^{**};\mathbf{a}) = \varphi(N^*;\mathbf{a})$ is countable. This is a contradiction to the assumption that T is λ-categorical. (see Exercise 1.5.30)[7] □

COROLLARY 8.12 (The Hanf number of $\aleph_0$-stable theory is $\aleph_1$). *Suppose T is $\aleph_0$-stable and let Γ be a set of T-types. If there exists an uncountable $M \in \mathrm{EC}(T,\Gamma)$ then $\mathrm{EC}(T,\Gamma)$ has models in all infinite cardinals.*

Clearly the omitting types theorem implies the Corollary. There is a direct and simple argument to prove the corollary without using the notion of quasi-minimal formulas.

PROOF OF 8.12. Using Theorem 4.1 pick $\{a_i \mid i < \omega_1\} \subseteq |M|$ an indiscernible sequence over $\emptyset$. For any cardinal $\lambda \geq \omega_1$ let$\{b_j \mid j < \lambda\} \subseteq \mathfrak{C}$ be an indiscernible sequence such that $b_i = a_i$ for all $i < \omega_1$.

Let N be primary over $\{b_j \mid j < \lambda\}$. For the sake of contradiction suppose that there exists $\mathbf{c} \in |N|$ such that $\mathbf{c} \models q$ for some $q \in \Gamma$. As N is atomic over $\{b_j \mid j < \lambda\}$ there are $\psi(\mathbf{x};\mathbf{y})$ and $\mathbf{b} \in \{b_j \mid j < \lambda\}$ such that

$$\psi(\mathbf{x};\mathbf{b}) \vdash \mathrm{tp}(\mathbf{c}/N).$$

In particular we have

$$(*) \quad \psi(\mathbf{x};\mathbf{b}) \vdash q.$$

[6]Instead of Vaught's theorem we could also use here the stronger result: Theorem 7.6.

[7]It is possible to avoid using Theorem 8.3 here by replacing it with an argument like in the proof of Claim 6.5.

We may assume that $\mathbf{b} = \langle b_{j_1}, \dots, b_{j_n} \rangle$, by indiscernibly there are $i_1 < \cdots < i_n < \omega_1$ such that $\mathrm{tp}(\mathbf{b}/\emptyset) = \mathrm{tp}(\mathbf{b}^*/\emptyset)$ where $\mathbf{b}^* := \langle b_{i_1}, \dots, b_{i_n} \rangle$. Let $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(b) = \mathbf{b}^*$. Apply f to $(*)$ to get

$$\psi(\mathbf{x}; \mathbf{b}^*) \vdash q.$$

As $\mathbf{b}^* \in M$ and $M \models \exists \mathbf{x} \psi(\mathbf{x}; \mathbf{b}^*)$ there exists $c^* \in |M|$ realizing q contradiction to the assumption that M omits all types from Γ. $\square$

Exercises 7.8.

(1) Suppose that T admits a Vaughtian pair for some φ then for every $\lambda \geq |\operatorname{L}(T)|$ there are $M \precneqq N$ Vaughtian pair for φ with both models of cardinality λ.
(2) Let M be an uncountable structure with $|\operatorname{L}(M)| \leq \aleph_0$. For every $\varphi(\mathbf{x};\mathbf{y}) \in \operatorname{Fml}(\operatorname{L}(M))$ and every $\mathbf{a} \in {}^{\ell(\mathbf{y})}|M|$ such that $\|M\| > |\varphi(M;\mathbf{a})| \geq \aleph_0$ there exists $N \equiv M$ such that $N \supseteq \cup\mathbf{a}$, $\|N\| = \aleph_1$ and $|\varphi(N;\mathbf{a})| = \aleph_0$.
(3) Suppose N is an L-structure, $\varphi(\mathbf{x};\mathbf{y}) \in \operatorname{Fml}(L)$, and $\mathbf{a} \in {}^{\ell(\mathbf{y})}|N|$ such that $\lambda := |\varphi(N;\mathbf{a})| < \|N\|$. If $\lambda \geq |L| + \aleph_0$ then there exists $M \precneqq N$ of cardinality λ such that M and N is a Vaughtian pair.
(4) Suppose T is $\aleph_0$-stable. If T admits a Vaughtian pair then for every $\lambda \geq \mu \geq |T|$ there exists a Vaughtian pair $M \precneqq N \models T$ such that $\|M\| = \mu$ and $\|N\| = \lambda$.
(5)
(6)

9. Strongly minimal sets and the Baldwin-Lachlan proof

In this section I present another proof of Morely's theorem. In this proof saturated models are not used at all. Instead it is structured similarly to Steinitz's proof of the categoricity on T_{ACF_p} in uncountable cardinals (see Theorem I.2.38). We will be introducing the notion of strongly minimal sets which allows us to define a dimension simultaneously generalizing dimension function from linear algebra and field theory.

As a bi-product we will obtain a new and interesting characterization of $\aleph_1$-categorical theories.

THEOREM 9.1 (Baldwin-Lachlan 1971). *Let T be a complete first-order theory in a countable language. If T is $\aleph_0$-stable then the following are equivalent.*

(1) *There exists an uncountable λ such that T in λ-categorical.*
(2) *For every uncountable μ the theory T is μ-categorical.*
(3) *For every $M \preceq N \models T$ and every $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))$ there is no $\mathbf{a} \in {}^{\ell(\mathbf{y})}|M|$ such that $\varphi(M;\mathbf{a})$ is infinite and $\varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a})$.*

Clearly (2) implies (1).

The direction (1) implies (3) is Corollary 8.11.

In this section we focus on the hard direction $(3) \implies (2)$.

REMARK 9.2. Recall that by Proposition 3.1.5, if M is saturated then $|\varphi(M;\mathbf{a})| \geq \aleph_0 \implies \|M\| = |\varphi(M;\mathbf{a})|$ for every $\mathbf{a} \in M$ and every formula $\varphi(\mathbf{x};\mathbf{y})$.

The assumption that a theory T is not having a Vaughtian pair implies that every $M \models T$ such that $\|M\| > |T|$ we have $|\varphi(M;\mathbf{a})| \geq \aleph_0 \implies \|M\| = |\varphi(M;\mathbf{a})|$ for every $\mathbf{a} \in M$ and every formula $\varphi(\mathbf{x};\mathbf{y})$.

PROPOSITION 9.3. *Suppose T has no Vaughtian Pair. Let $M \models T$ and $\varphi(\mathbf{x};\mathbf{a})$ be a non-algebraic formula with parameters from M. If M is prime over $\varphi(M;\mathbf{a}) \cup \mathbf{a}$ then M is also minimal over $\varphi(M;\mathbf{a}) \cup \mathbf{a}$, i.e. if $N \preceq M$ contains the set $\varphi(M;\mathbf{a}) \cup \mathbf{a}$ then $N = M$.*

Moreover, if in addition T is $\aleph_0$-stable then for every $\mathbf{a} \in |M|$ and $\varphi(\mathbf{x};\mathbf{a}) \in \mathrm{L}(T)$ such that $\varphi(\mathbf{x};\mathbf{a})$ is not algebraic, then M is prime and minimal over $\varphi(M;\mathbf{a}) \cup \mathbf{a}$.

PROOF. Clearly not having a Vaughtian pair implies the first part.

Enough to show that the model M is prime over $D := \varphi(M;\mathbf{a}) \cup \mathbf{a}$. Using $\aleph_0$-stability by Theorem 5.4 there exists N^* prime over D. Let $f : N^* \to M$ be

an elementary embedding such that $f \restriction D = \mathrm{id}_D$. Let $N := f[N^*]$. If $N \subsetneqq M$ then $N \precneqq M$ is a Vaughtian pair. $\square$

In this section we use extensively the following notation: For $n < \omega$ and a formula $\varphi(\mathbf{x})$ we write $M \models \exists^n \mathbf{x} \varphi(\mathbf{x})$ to denote the statement $|\varphi(M)| \geq n$, recall that there is a first-order sentence equivalent to $\exists^n \mathbf{x} \varphi(\mathbf{x})$. See Exercise 1.1.4.

Not having Vaughtian Pairs in a theory simplifies the question when $\varphi(\mathbf{x}; \mathbf{a})$ is algebraic, by providing a uniform bound n_φ that depends only on $\varphi(\mathbf{x}; \mathbf{y})$ and not the parameters $\mathbf{a}$:

THEOREM 9.4 (Uniform test for being algebraic). *Suppose T has no Vaughtian Pair then for every $\varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{L}(T)$ there exists $n_\varphi < \omega$ such that*

$$\mathfrak{C} \models \exists^{n_\varphi} \mathbf{x} \varphi(\mathbf{x}; \mathbf{a}) \implies \varphi(\mathbf{x}; \mathbf{a}) \textit{ is not algebraic for every } \mathbf{a}.$$

PROOF. Otherwise, for every $n < \omega$ there exists $\mathbf{a}_n \in |\mathfrak{C}|$ such that

$$n \leq |\varphi(\mathfrak{C}; \mathbf{a}_n)| < \aleph_0.$$

Pick N such that $N \supseteq \bigcup_{n<\omega} \cup \mathbf{a}_n$, notice that we have

$$n \leq |\varphi(N; \mathbf{a}_n)| < \aleph_0 \text{ for every } n < \omega.$$

The idea is to use the above information to find $N \precneqq M \models T$ and $\mathbf{a} \in {}^{\ell(\mathbf{y})}|N|$ such that $\varphi(M; \mathbf{a}) = \varphi(N; \mathbf{a})$ is infinite. We establish this by expanding $\mathrm{L}(T)$ with a unary predicate P and constants $\mathbf{c}$ such that the new theory T^* will extend T and if $M^* \models T^*$ then P^{M^*} will be a universe of N as required, when $M := M^* \restriction \mathrm{L}(T)$.

Let L^* be an expansion of $\mathrm{L}(T)$ by a new unary predicate P and a finite sequence of constants $\mathbf{c} = \langle c_i \mid i < \ell(\mathbf{y}) \rangle$.

Let T^* consists of the following sets of sentences

(1) T

(2) $\{\exists^{\geq n} \mathbf{x} \varphi(\mathbf{x}; \mathbf{c}) \mid n < \omega\}$

(3) $\exists x[\neg P(x)] \wedge \bigwedge\limits_{i<\ell(\mathbf{y})} P(c_i)$

(4) $\{\forall \mathbf{z}[\bigwedge\limits_{i<\ell(\mathbf{z})} P(z_i) \wedge \exists x \psi(x; \mathbf{z}) \to \exists x[P(x) \wedge \psi(x; \mathbf{z})]] \ : \ \psi(x; \mathbf{z}) \in \mathrm{L}(T)\}$

(5) $\forall \mathbf{x}[\varphi(\mathbf{x}; \mathbf{c}) \to \bigwedge\limits_{i<\ell(\mathbf{x})} P(x_i)]$.

By the definition of T^* we have the following: If $M^* \models T^*$ than $M := M^* \restriction \mathrm{L}(T) \models T$, by (4) and (3) the set P^{M^*} induces a proper $\mathrm{L}(T)$-elementary

submodel N of M with $\mathbf{a} := \mathbf{c}^{M^*} \in N$. Using (2) the set $\varphi(M;\mathbf{a})$ is infinite. By (5) we have that $\varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a})$, thus we obtained a Vaughtian pair for T (by (1)).

Let T_0 be a finite subset of T^*. Let n_0 be the largest integer such that

$$\exists^{\geq n_0}\mathbf{x}\varphi(\mathbf{x};\mathbf{a}_{n_0}) \in T_0.$$

Let $M \supsetneqq N$, now define M^* an expansion of M to L^*: Let $\mathbf{c}^{M^*} := \mathbf{a}_{n_0}$ and $P^{M^*} := |N|$. As M^* satisfies all statements from $(3), (4)$ and (5) clearly $M^* \models T_0$. □

A SECOND PROOF TO THEOREM 9.4. Otherwise, for every $n < \omega$ there exists $\mathbf{a}_n \in |\,\mathfrak{C}\,|$ such that

$$n \leq |\varphi(\mathfrak{C};\mathbf{a}_n)| < \aleph_0.$$

Let $\lambda := (2^{|T|})^+$. Notice that since $|T|$ is infinite $\lambda^{\aleph_0} = \lambda$ and $\lambda > 2^{\aleph_0}$. Let $M \models T$ be of cardinality λ containing $\{\mathbf{a}_n \mid n < \omega\}$.

Clearly $n \leq |\varphi(M;\mathbf{a}_n)| < \aleph_0$, let D be a non-principle ultrafilter over ω. Let $N := M^\omega/D$. Let $\mathbf{a}^* \in N$ be the D-equivalence class of $\langle \mathbf{a}_n \mid n < \omega\rangle$, now we have

$$|\varphi(N;\mathbf{a}^*)| \leq \prod_{n<\omega} |\varphi(M;\mathbf{a}_n)| \leq \aleph_0^{\aleph_0} = 2^{\aleph_0}$$

Since $\|N\| = \lambda^{\aleph_0} = \lambda > |\varphi(N;\mathbf{a}^*)|$ any $N_1 \prec N$ of cardinality $2^{|T|}$ that contains $\varphi(N;\mathbf{a}^*)$ is a Vaughtian pair. □

Here is an outline to establish part (3) of Theorem 9.1 from the assumptions of $\aleph_0$-stability and no Vaughtian pairs: Let $\mu > \aleph_0$ and $M, N \models T$ be given of cardinality μ. We will find a very special non-algebraic formula $\varphi(\mathbf{x};\mathbf{a})$ with parameters from M and $\mathbf{a}' \in |N|$ such that $\mathrm{tp}(\mathbf{a}/\emptyset, M) = \mathrm{tp}(\mathbf{a}'/\emptyset, N)$, namely an (M,N)-mapping $f(\mathbf{a}) = \mathbf{a}'$. Let $D := \varphi(M;\mathbf{a})$ and $D' := \varphi(N;\mathbf{a}')$, using the special choice of $\varphi(\mathbf{x};\mathbf{a})$ we will show that there exists a surjective (M,N)-mapping $f^* : D \to D'$ extending f. By the previous proposition M is prime over D, thus there exists an elementary embedding $f^{**} : M \to N$ extending f^*, since and N is minimal over D' we get that $f^{**} : M \cong N$.

The rest of this section is dedicated to the selection of $\varphi(\mathbf{x};\mathbf{a})$ which has the above properties. A sufficient requirement from D to have the required property is being *strongly minimal* to be defined below.

One of the fundamental ideas of model theory was extracted by William Marsh [**Mar 1966**] from Morley's proof of his categoricity theorem is the notion of a strongly minimal formula or a strongly minimal set. It is a corresponding object to that of an irreducible variety in algebraic geometry.

DEFINITION 9.5 (strong minimality). Let T be a complete first-order theory.

(1) Let p be a consistent set of formulas in finitely many free variables $\mathbf{x}$ perhaps with parameters from the monster model $\mathfrak{C}$. The type p is called an *algebraic type* iff $|p(\mathfrak{C})| < \aleph_0$ where
$$p(\mathfrak{C}) \text{ is the set } \{\mathbf{c} \in \mathfrak{C} \ : \ \mathbf{c} \models p\}.$$
(2) Let $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T)$ and $\mathbf{a} \in {}^{\ell(\mathbf{y})}|\mathfrak{C}|$. The formula $\varphi(\mathbf{x};\mathbf{a})$ is called an *algebraic formula* iff $\{\varphi(\mathbf{x};\mathbf{a})\}$ is an algebraic type.
(3) Let $M \models T$ and $n < \omega$. Suppose D is an infinite subset of ${}^{n}|M|$. We say that D is a *M-minimal set* iff for every $\psi(\mathbf{x};\mathbf{y})$ (with $\ell(\mathbf{x}) = n$ and $\mathbf{b} \in {}^{\ell(\mathbf{y})}|M|$) either
$$|D \cap \psi(M;\mathbf{b})| < \aleph_0 \text{ or } |D \cap \neg\psi(M;\mathbf{b})| < \aleph_0$$
Let $\mathbf{a} \in |M|$ we say that the formula $\varphi(\mathbf{x};\mathbf{a})$ *is M-minimal* provided $\varphi(M;\mathbf{a})$ is M-minimal.
(4) Suppose $\varphi(\mathbf{x};\mathbf{a})$ is non-algebraic. The formula $\varphi(\mathbf{x};\mathbf{a})$ is *strongly minimal formula* provided $\varphi(\mathfrak{C};\mathbf{a})$ is $\mathfrak{C}$-minimal.
(5) Let M be a model and p a set of formulas in $\mathbf{x}$ with parameters from M. We say that p *is M-minimal* iff $p[M]$ is M-minimal.
(6) A non-algebraic type p is called a *strongly minimal type* provided $p(\mathfrak{C})$ is $\mathfrak{C}$-minimal set.
(7) Let T be a complete theory. We say that T is a *strongly minimal theory* provided the formula $x = x$ is strongly minimal.

Notice that while in general the following is not true: Given $\{S_i \mid i \in I\}$ a family of sets such that $\bigcap_{i\in I} S_i$ is finite then there exists a finite $I_0 \subseteq I$ such that
$$\bigcap_{i\in I_0} S_i = \bigcap_{i\in I} S_i.$$

By the compactness theorem it holds when S_i's are first-order definable subsets of $\mathfrak{C}$! It is the contents of the following proposition that when dealing with a family of sets which are all first-order definable sets induced by a complete type, we can find I_0 as above such that $|I_0| = 1$.

PROPOSITION 9.6. *If $p \in \mathrm{S}^{\mathbf{x}}(A)$ is algebraic then there exists $\varphi(\mathbf{x};\mathbf{a}) \in p$ such that*
$$\varphi(\mathbf{x};\mathbf{a}) \vdash p.$$

PROOF. Exercise. □

The following is very useful:

LEMMA 9.7 (Invariance). *Let M be a model, $f \in \mathrm{Aut}(M)$ and p a set of formulas in $\mathbf{x}$ with parameters from M.*

$$p \text{ is } M\text{-minimal} \iff f(p) \text{ is } M\text{-minimal}.$$

In particular, $\varphi(\mathbf{x};\mathbf{a})$ is strongly minimal iff $\varphi(\mathbf{x};f(\mathbf{a}))$ is strongly minimal for every $f \in \mathrm{Aut}(\mathfrak{C})$. Thus, given a non-algebraic type p, if f is $(\mathfrak{C},\mathfrak{C})$-mapping such that $\mathrm{dom}\, p \subseteq \mathrm{dom}\, f$ then

$$p \text{ is strongly minimal} \iff f(p) \text{ is strongly minimal}.$$

PROOF. Exercise. □

Notice that the notion of strong minimality is a global notion (i.e. it depends of all models of T), in the sense that it depends on the full class of models of T (as its definition is carried out in $\mathfrak{C}$). While the notion of M-minimality is local.

PROPOSITION 9.8. *Let $n < \omega$ and suppose $p \in \mathrm{S}^n(A)$ for some A. The type p is algebraic iff there exists an algebraic $\varphi(\mathbf{x};\mathbf{a}) \in p$ such that $\varphi(\mathbf{x};\mathbf{a}) \vdash p$.*

PROOF. Exercise. □

It appears that it is not always possible to find strongly minimal sets (or formulas), however under "reasonable assumptions" on the theory T, strongly minimal sets exists. The following theorem is a special case of a more general result (see Section **??** of Chapter **??**).

THEOREM 9.9 (Density of M-minimal sets). *Let T be $\aleph_0$-stable and $M \models T$. For every $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T)$, every $\mathbf{a} \in |M|$ such that $\varphi(\mathbf{x};\mathbf{a})$ is not algebraic there are $\mathbf{a}^* \in M$ and a formula $\varphi^*(\mathbf{x};\mathbf{z})$ such that $\varphi^*(\mathbf{x};\mathbf{a}^*)$ is M-minimal and $\varphi^*(\mathbf{x};\mathbf{a}^*) \vdash \varphi(\mathbf{x};\mathbf{a})$.*

REMARK 9.10. Minimal formulas are an analog to prime numbers in arithmetic. The statement of of the theorem can be viewed as a generalization of the fact "every integer greater than one has a prime divisor", where subset is an analogue to divisor.

PROOF. Suppose not. Let $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T)$ be such that for every $\mathbf{a} \in {}^{\ell(\mathbf{y})}|M|$ if $\varphi(\mathbf{x};\mathbf{a})$ is not algebraic then it has no strongly minimal extension. Namely there exists $\psi(\mathbf{x};\mathbf{b})$ (with parameters from the monster model) such that both $\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{b})$ and $\varphi(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{b})$ are not algebraic and have no strongly minimal extensions. We will use this assumption to construct: For every $\eta \in {}^{\omega>}2$ there exists a finite type p_η in $\mathbf{x}$ with parameters from M with the following properties:

(1) $p_{\langle\rangle} := \{\varphi(\mathbf{x};\mathbf{a})\}$.
(2) $\nu \lhd \eta \Longrightarrow p_\nu \subseteq p_\eta$,
(3) $\eta \in {}^{\omega>}2 \Longrightarrow p_\eta$ is not algebraic and
there is no M-minimal $\varphi^*(\mathbf{x};\mathbf{a}^*) \vdash p_\eta$.

(4) For every $n < \omega$ and every $\eta \in {}^{n}2$ there are $\psi_\eta(\mathbf{x};\mathbf{y}_\eta) \in \mathrm{Fml}(\mathrm{L}(T))$ and $\mathbf{b}_\eta \in {}^{\ell(\mathbf{y}_\eta)}|M|$ such that

$$p_{\eta^\frown \ell} := p_\eta \cup \{\psi_\eta(\mathbf{x};\mathbf{y}_\eta)^\ell\} \text{ for } \ell \in \{0,1\}.$$

This is enough: Let

$$A := \{\mathbf{b}_\eta \mid \eta \in {}^{\omega>}2\},$$

for $\eta \in {}^{\omega}2$, let

$$p_\eta := \bigcup_{n<\omega} p_{\eta\restriction n}.$$

Clearly p_η are types over A, and from requirement (3) we have that $\eta \neq \nu \implies p_\eta \neq p_\nu$. Namely $|\mathrm{S}^{\ell(\mathbf{x})}(A)| = 2^{\aleph_0}$ in contradiction to the $\aleph_0$-stability of T.
This is possible: Define p_η by induction on $\ell(\eta)$:

For $\ell(\eta) = 0$ let $p_\eta := \{\varphi(\mathbf{x};\mathbf{a})\}$.

Suppose p_η is a finite type in $\mathbf{x}$ with parameters from M. By part (2) of the inductive assumption $\chi(\mathbf{x};\mathbf{c}) := \bigwedge p_\eta$ is not M-minimal. Let $\mathbf{b}_\eta$ (from M) and $\psi_\eta(\mathbf{x};\mathbf{z}_\eta)$ be witness for that. Namely we have that

$$(*) \quad \chi(\mathbf{x};\mathbf{c}) \wedge \psi_\eta(\mathbf{x};\mathbf{b}_\eta) \text{ and } \chi(\mathbf{x};\mathbf{c}) \wedge \neg\psi_\eta(\mathbf{x};\mathbf{b}_\eta) \text{ are not algebraic.}$$

Check that $\psi_\eta(\mathbf{x};\mathbf{b}_\eta)$ is as required. □

REMARK 9.11 (Density of strongly minimal formulas). Notice that if we assume that M is $\aleph_0$-saturated in the previous theorem we get *density of strongly minimal formulas*. In particular, if we replace M by $\mathfrak{C}$ we get that any infinite definable subset of $\mathfrak{C}$ contains a strongly minimal subset.

The following is useful

THEOREM 9.12 (M-minimal formulas are strongly minimal). *Suppose T has no Vaughtian Pairs. If $\varphi(\mathbf{x};\mathbf{a})$ is M-minimal then $\varphi(\mathbf{x};\mathbf{a})$ is strongly minimal.*

PROOF. If $\varphi(\mathbf{x};\mathbf{a})$ is M-minimal but not strongly minimal then there exists a formula $\psi(\mathbf{x};\mathbf{z})$ and $\mathbf{b} \in \mathfrak{C}$ such that

$$\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{b}) \text{ and } \varphi(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{b}) \text{ are not algebraic.}$$

Using the notation in the statement of Theorem 9.4 let

$$n := \max\{n_{\varphi(\mathbf{x};\mathbf{y})\wedge\psi(\mathbf{x};\mathbf{z})}, n_{\varphi(\mathbf{x};\mathbf{y})\wedge\neg\psi(\mathbf{x};\mathbf{z})}\}$$

We have

$$\mathfrak{C} \models \exists^n\mathbf{x}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{b})] \wedge \exists^n\mathbf{x}[\varphi(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{b})].$$

Thus

$$M \models \exists\mathbf{y}[\exists^n\mathbf{x}[\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{y})] \wedge \exists^n\mathbf{x}[\varphi(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{y})]].$$

Namely there exists $\mathbf{b}' \in |M|$ such that both $\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{b}')$ and $\varphi(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{b}')$ are non-algebraic, a contradiction to the assumption that $\varphi(\mathbf{x};\mathbf{a})$ is M-minimal. □

COROLLARY 9.13. *Suppose T is $\aleph_0$-stable and has no Vaughtian Pairs. For every $M \models T$ and a non-algebraic $\varphi(\mathbf{x};\mathbf{a})$ with $\mathbf{a} \in |M|$ there exists $\varphi^*(\mathbf{x};\mathbf{a}^*)$ a strongly minimal formula with parameters from M such that $\varphi^*(\mathbf{x};\mathbf{a}^*) \vdash \varphi(\mathbf{x};\mathbf{a})$.*

PROOF. Use Theorem 9.9 and Theorem 9.12. □

The following is an example for one of the nice properties strongly minimal formulas have.

PROPOSITION 9.14 (Extension property for strongly minimal types). *Suppose p is a strongly minimal type in $\mathbf{x}$. For every set A there exists a unique type $q \in \mathrm{S}^{\mathbf{x}}(A)$ such that $p \cup q$ is non-algebraic.*

PROOF. *Uniqueness*: Suppose that there exist $q_1 \neq q_2 \in \mathrm{S}^{\mathbf{x}}(A)$ such that $p \cup q_\ell$ ($\ell = 1, 2$) non-algebraic. Since the types are distinct there exists a formula $\psi(\mathbf{x};\mathbf{y})$ and a sequence $\mathbf{a} \in A$ such that $\psi(\mathbf{x};\mathbf{a}) \in q_1$ and $\neg\psi(\mathbf{x};\mathbf{a}) \in q_2$. Since $p \cup q_l$ are not algebraic we have in particular that $p \cup \{\psi(\mathbf{x};\mathbf{a})\}$ and $p \cup \{\neg\psi(\mathbf{x};\mathbf{a})\}$ are not algebraic which contradicts the assumption that p is strongly minimal.

Existence: Let

$$P := \{\Gamma \ : p \cup \Gamma \text{ is not algebraic,} \\ \Gamma \text{ is a set of formulas with parameters from } A \text{ in } \mathbf{x}\}.$$

Clearly $P \neq \emptyset$, and is partially ordered by $\subseteq$. By Zorn's Lemma it is enough to prove

CLAIM 9.15. *Every chain in $(P, \subseteq)$ has an upper bound in P.*

PROOF. Let $C \subseteq P$ be a chain. It suffices to show that $\Gamma^* := \bigcup C \in P$.

If $\Gamma^* \notin P$ then $p \cup \Gamma^*$ is algebraic, thus there exists a finite $\Gamma_0 \subseteq \Gamma^*$ such that $p \cup \Gamma_0$ is algebraic, as there is $\Gamma \in C$ such that $\Gamma_0 \subseteq \Gamma$ we obtained a contraditction. □

□

REMARK 9.16. Here is an entirely different argument of the extension property: given a strongly minimal type p and a set A pick $I \subseteq p[\mathfrak{C}]$ of cardinality μ^+ when $\mu := 2^{|T|+|A|+|\operatorname{dom} p|}$. By an application of the pigeon hole principle to

$$\mathbf{c} \in I \mapsto \operatorname{tp}(\mathbf{c}/A \cup \operatorname{dom} p).$$

There exists $I_0 \subseteq I$ of cardinality μ^+ such that

$$\mathbf{c}, \mathbf{c}' \in I_0 \implies \operatorname{tp}(\mathbf{c}/A \cup \operatorname{dom} p) = \operatorname{tp}(\mathbf{c}'/A \cup \operatorname{dom} p).$$

Check that $q := \operatorname{tp}(\mathbf{c}/A)$ for $\mathbf{c} \in I_0$ is as required.

COROLLARY 9.17 (Extension property for strongly minimal formulas). *Let $\varphi(\mathbf{x}; \mathbf{b})$ be strongly minimal. For every $A \supseteq \cup \mathbf{b}$ there exists a unique non-algebraic $p \in \mathrm{S}(A)$ containing $\varphi(\mathbf{x}; \mathbf{b})$.*

PROOF. Immediate. □

COROLLARY 9.18 (Uniqueness property). *Suppose that p is a strongly minimal type in $\mathbf{x}$. For every $A \supseteq \operatorname{dom} p$ and every $p_1, p_2 \in \mathrm{S}^{\mathbf{x}}(A)$ if $p_\ell \supseteq p$ and p_ℓ is non-algebraic then $p_1 = p_2$.*

COROLLARY 9.19 (extension of $(\mathfrak{C}, \mathfrak{C})$ mappings). *Suppose that $\varphi(\mathbf{x})$ is a strongly minimal formula and f is an $(\mathfrak{C}, \mathfrak{C})$-mapping. If $\models \varphi[\mathbf{b}]$ and $\mathbf{b}$ is not algebraic over $\operatorname{dom} f$ then for any $\mathbf{b}'$ non algebraic such that $\models \varphi[\mathbf{b}']$ then $f \cup \{(\mathbf{b}, \mathbf{b}')\}$ is a $(\mathfrak{C}, \mathfrak{C})$-mapping.*

PROOF. Let $q \in \mathrm{S}^{\mathbf{x}}(\operatorname{dom} f)$ be the non-algebraic extension of $\varphi(\mathbf{x})$. Since $\mathbf{b} \notin \operatorname{acl}(\operatorname{dom} f)$ and $\models \varphi[\mathbf{b}]$ there infinitely many elements of $\varphi[\mathfrak{C}]$ realizing $\operatorname{tp}(\mathbf{b}/\operatorname{dom} f)$, so clearly $\mathbf{b} \models q$. By the uniqueness of extension we have that $\mathbf{b}' \models q$, thus $f \cup \{(\mathbf{b}, \mathbf{b}')\}$ is as required. □

The next uniqueness property is crucial to establish the exchange principle for acl among realizations of a strongly minimal set.

COROLLARY 9.20 (Uniqueness property of sequences). *Suppose that $\varphi(\mathbf{x})$ is a strongly minimal. For every $n < \omega$ if $\{\mathbf{a}_i \mid i < n\}, \{\mathbf{b}_i \mid i < n\} \subseteq \varphi[\mathfrak{C}] - \operatorname{acl}(\emptyset)$ are such that*

$$\mathbf{a}_i \notin \operatorname{acl}(\{\mathbf{a}_j \mid j < i\}) \text{ and } \mathbf{b}_i \notin \operatorname{acl}(\{\mathbf{b}_j \mid j < i\}) \text{ for all } i < n$$

then

$$\operatorname{tp}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}/\emptyset) = \operatorname{tp}(\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}/\emptyset).$$

PROOF. By induction on $n < \omega$. Enough to verify for $n = 2$: Since the elements $\mathbf{a}_0$ and $\mathbf{b}_0$ are not algebraic, by Corollary 9.18

$$\operatorname{tp}(\mathbf{a}_0/\emptyset) = \operatorname{tp}(\mathbf{b}_0/\emptyset).$$

Let

$$f \in \operatorname{Aut}(\mathfrak{C}) \text{ such that } f(\mathbf{a}_0) = \mathbf{b}_0.$$

As $\mathbf{a}_1 \in \varphi[\mathfrak{C}] - \operatorname{acl}(\{\mathbf{a}_0\})$ and $\mathbf{b}_1 \in \varphi[\mathfrak{C}] - \operatorname{acl}(\{\mathbf{b}_0\})$ using uniqueness of extension we get that there exists $f^* \supseteq f \restriction (\mathbf{a}_0)$ such that

$$f^* \in \operatorname{Aut}(\mathfrak{C}) \text{ and } f(\mathbf{a}_1) = \mathbf{b}_1.$$

Thus $f^* \in \operatorname{Aut}(\mathfrak{C})$ such that $f(\mathbf{a}_0, \mathbf{a}_1) = \langle \mathbf{b}_0, \mathbf{b}_1 \rangle$. □

Another useful corollary of extension and invariance is the a generalization of the following fact from linear algebra: if V_ℓ (for $\ell = 1, 2$) are both F-vector spaces and $I_\ell \subseteq V_\ell$ are linearly independent sets then any bijection $f : I_1 \to I_2$ is F-linear.

COROLLARY 9.21 (Extension of $(\mathfrak{C}, \mathfrak{C})$-mappings). *Suppose p is a strongly minimal type, f a $(\mathfrak{C}, \mathfrak{C})$-mapping such that $\operatorname{dom} f \supseteq \operatorname{dom} p$. If $\{\mathbf{a}_i \mid i < \alpha\} \subseteq p[\mathfrak{C}]$ and $\{\mathbf{b}_i \mid i < \alpha\} \subseteq f(p)[\mathfrak{C}]$ such that*

$$\mathbf{a}_i \notin \operatorname{acl}(\{\mathbf{a}_j \mid j < i\}) \text{ and } \mathbf{b}_i \notin \operatorname{acl}(\{\mathbf{b}_j \mid j < i\}) \text{ for all } i < \alpha.$$

Then there exists $f^* \supseteq f$ an $(\mathfrak{C}, \mathfrak{C})$-mapping such that

$$f^*(\mathbf{a}_i) = \mathbf{b}_i \text{ for all } i < \alpha.$$

PROOF. Exercise. □

An alternative way to prove the uniqueness of extensions of strongly minimal formulas is by using the following proposition together with Corollary 3.12.

PROPOSITION 9.22. *Let $\varphi(\mathbf{x}; \mathbf{a})$ be non-algebraic*

$$\varphi(\mathbf{x}; \mathbf{a}) \text{ is strongly minimal } \iff \operatorname{RM}[\varphi(\mathbf{x}; \mathbf{a})] = 1 \text{ and } \operatorname{Mult}[\varphi(\mathbf{x}; \mathbf{a})] = 1$$

PROOF. □

PROPOSITION 9.23 (non splitting). *Suppose $\varphi(\mathbf{x}; \mathbf{a})$ is a strongly minimal formula. For every $A \supseteq \mathbf{a}$ and every type $p \in \mathrm{S}^{\mathbf{x}}(A)$ if p is not algebraic then p doe not split over $\cup\mathbf{a}$.*

PROOF. Otherwise, there are $\psi(\mathbf{x}; \mathbf{b}) \in p$ and $\neg\psi(\mathbf{x}; \mathbf{c}) \in p$ such that $\operatorname{tp}(\mathbf{b}/\cup \mathbf{a}) = \operatorname{tp}(\mathbf{c}/\cup \mathbf{a})$. Let $f \in \operatorname{Aut}_{\cup\mathbf{a}}(\mathfrak{C})$ such that $f(\mathbf{b}) = \mathbf{c}$. Let $q := f(p)$, as p is not algebraic also q is not algebraic. Since $\psi(\mathbf{x}; \mathbf{b}) \in p$ we have $\psi(\mathbf{x}; \mathbf{c}) \in q$. As both

$$\varphi(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{c}) \text{ and } \varphi(\mathbf{x}; \mathbf{a}) \wedge \neg\psi(\mathbf{x}; \mathbf{c})$$

are not algebraic we obtained a contradiction to the assumption that $\varphi(\mathbf{x}; \mathbf{a})$ is strongly minimal.

□

COROLLARY 9.24. *Let $\psi(\mathbf{x}; \mathbf{b})$ be a strongly minimal formula. Let p be a type. If $p \cup \varphi(\mathbf{x}; \mathbf{b})$ is non-algebraic then p does not fork over $\bigcup \mathbf{b}$. Moreover p is stationary over $\bigcup \mathbf{b}$.*

PROOF. Use the extension property of non-forking. □

DEFINITION 9.25. Let A be a given set. *The algebraic closure of A* is

$$\operatorname{acl}(A) := \{\mathbf{c} \ : \ \operatorname{tp}(\mathbf{c}/A) \text{ is algebraic } \}.$$

Notice that by Proposition 6.2 we can rewrite the algebraic closure of A in the following way: $\mathrm{acl}(A) = \{\mathbf{c} : \text{there exists } \mathbf{a} \in A \text{ and there exists an algebraic formula } \varphi(\mathbf{x}; \mathbf{a}) \text{ such that } \models \varphi[\mathbf{c}; \mathbf{a}]\}$.

You may wonder why we use acl (which stands for algebraic closure) to denote the set of elements which are algebraic over the set A. In field theory there is a parallel notion of being algebraic (which is the origin of the usage of the word algebraic above). Recall that for fields $F \leq E$ and an element $a \in E$ we have: a is algebraic over F iff $[F(a) : F] < \aleph_0$.

Suppose $F \subseteq E \subseteq K$ be a given triple of fields. The following fact is not difficult to prove: Let $X \subseteq E$ be a set of algebraic elements over F (i.e. $a \in X \implies a$ is a root of a polynomial with coefficients from F), if $\gamma \in K$ is algebraic over $F(X)$ then γ is algebraic over F (w.l.o.g we may assume that X is finite and use the multiplication formula for dimensions i.e. $[F(X)(\gamma) : F] = [F(X)(\gamma) : F(X)] \cdot [F(X) : F]$). The above indicates that algebraicity in field theory is a transitive relation namely if an element is algebraic over a set X and every element of the set X is algebraic over F then γ is algebraic over F. The purpose of the following Proposition is to show that this kind of transitivity holds for the model theoretic notion of being algebraic (which is the justification of using $\mathrm{acl}(A)$ instead of $al(A)$). Namely we will verify below the transitivity property of pregeometries (which implies by exercise (3) that for every set A we have that $\mathrm{acl}(A) = \mathrm{acl}(\mathrm{acl}(A))$ holds).

PROPOSITION 9.26 (transitivity of acl). *Let X and Y be subsets of the model $\mathfrak{C}$, and suppose a is an element of $\mathfrak{C}$. If $a \in \mathrm{acl}(X)$ and $X \subseteq \mathrm{acl}(Y)$ then $a \in \mathrm{acl}(Y)$.*

PROOF. Let $b_0, \cdots, b_{n-1} \in X$ and a formula $\varphi(x; y_0, \cdots, y_{n-1})$ such that $a \in \varphi(\mathfrak{C}; b_0, \cdots, b_{n-1})$ and the set is finite. Since $X \subseteq \mathrm{acl}(Y)$ for $k < n$ let $\mathbf{c}_k \in Y$ and $\psi_k(y_k; \mathbf{z}_k)$ such that $\psi_k(\mathfrak{C}; \mathbf{c}_k)$ is finite and $b_k \in \psi_k(\mathfrak{C}; \mathbf{c}_k)$.

Let $m := |\varphi(\mathfrak{C}; b_0, \cdots, b_{n-1})|$, now consider the formula $\varphi^*(x; \mathbf{c}_0, \cdots, \mathbf{c}_{n-1})$:

$$\exists y_0 \exists y_1 \ldots \exists y_{n-1} [\bigwedge_{k<n} \psi_k(y_k; \mathbf{c}_k) \wedge \varphi(x; y_0, \cdots, y_{n-1}) \wedge \exists^m x \varphi(x; y_0, \ldots, y_{n-1})].$$

Clearly $a \in \varphi^*(\mathfrak{C}; \mathbf{c}_0, \cdots, \mathbf{c}_{n-1})$, since the cardinality of the set $\varphi(\mathfrak{C}; \mathbf{c}_0, \cdots, \mathbf{c}_{n-1})$ is bounded by $m \cdot \Pi_{k<n} |\psi_k(\mathfrak{C}; \mathbf{c}_k)|$ we have established that $a \in \mathrm{acl}(\{\mathbf{c}_0, \cdots, \mathbf{c}_{k-1}\}) \subseteq \mathrm{acl}(Y)$. □

Similarly to the situation in field theory we can extend elementary embeddings to acl, more precisely we have

LEMMA 9.27 (Extension of (M,N) mapping to the closure). *Let M and N be L-structures and $A \subseteq |M|,\ B \subseteq |N|$. If $f : A \to B$ is a surjective (M,N)-mapping then there exists $\bar{f} : \operatorname{acl}(A) \to \operatorname{acl}(B)$ surjective (M,N)-mapping extending f.*

PROOF. Consider

$$P := \{g \mid g \supseteq f,\ g : A' \to B' \ (M,N)\text{-mapping} \\ A \subseteq A' \subseteq \operatorname{acl}(A),\ B \subseteq B' \subseteq \operatorname{acl}(B)\}.$$

By Zorn's lemma there exists $g^* \in P$ maximal with respect to $\subseteq$.

It is enough to prove

CLAIM 9.28. $\operatorname{dom} g^* = \operatorname{acl}(A)$.

PROOF. Otherwise, there exists $\mathbf{a} \in \operatorname{acl}(A) - \operatorname{dom} g^*$, as $\mathbf{a} \in \operatorname{acl}(\operatorname{dom} g^*)$ there exists $\varphi(\mathbf{x};\mathbf{b})$ with parameters from $\operatorname{dom} g^*$ such that

$$(1) \quad \models \varphi[\mathbf{a};\mathbf{b}] \text{ and}$$
$$(2) \quad \varphi(\mathbf{x};\mathbf{b}) \vdash \operatorname{tp}(\mathbf{a}/\operatorname{dom} g^*).$$

By (1), as $M \models \exists \mathbf{x} \varphi(\mathbf{x};\mathbf{b})$ by elementricity of g^* we get $N \models \exists \mathbf{x} \varphi(\mathbf{x}; g^*(\mathbf{b}))$. Let $\mathbf{a}' \in |N|$ such that $N \models \varphi[\mathbf{a}'; g^*(\mathbf{b})]$. Using (2) we get

$$\varphi(\mathbf{x}; g^*(\mathbf{b})) \vdash \operatorname{tp}(\mathbf{a}'/\operatorname{range} g^*).$$

Since $g^*(\operatorname{tp}(\mathbf{a}/\operatorname{dom} g^*)) = \operatorname{tp}(\mathbf{a}'/\operatorname{range} g^*)$ we get that $g^{**} := g^* \cup \{\langle \mathbf{a}, \mathbf{a}' \rangle\} \in P$ contradicts the maximality of g^*. □

□

Here is an alternative:

LEMMA 9.29. *Let M_1 and M_2 be models and $A_\ell \subseteq |M_\ell|$. If g is an surjective (M_1,M_2)-map $g : A_1 \to A_2$ then there exists g^* an (M_1,M_2)-map extending g, such that $g^* : \operatorname{acl}_{M_1}(A_1) \to \operatorname{acl}_{M_2}(A_2)$ and is onto $\operatorname{acl}_{M_2}(A_2)$.*

PROOF. Consider

$$\mathcal{F} := \{f : A_1' \to A_2' \mid f \text{ is an } (M_1,M_2) \text{ mapping extending } g, A_\ell \subseteq A_\ell' \subseteq \operatorname{acl}_{M_\ell}(A_\ell)\}.$$

By Zorn's lemma pick $g^* \in \mathcal{F}$ a maximal element. We show that this maximal element has the properties of the function promised.

For the sake of contradiction suppose that $A_1' := \operatorname{dom} g^* \subsetneq \operatorname{acl}_{M_1}(A_1)$. Let $a \in \operatorname{acl}_{M_1}(A_1) - A_1'$. Since $A_1' \supseteq A_1$ and a is algebraic over A_1 the element a is algebraic over A_1'. Let

$$n_0 := \min\{|\psi(M_1,\mathbf{b})| \ :\ M_1 \models \psi[a;\mathbf{b}], \mathbf{b} \in A_1', \psi(x;\mathbf{y}) \in \operatorname{Fml}(\mathrm{L}(M_1))\}.$$

Let $\psi(x; \mathbf{b})$ be a witness for n_0. Since g^* is an (M_1, M_2)-mapping we get

$$n_0 = |\psi(M_2; g^*(\mathbf{b}))|.$$

Let $a' \in \psi(M_2; g^*(\mathbf{b}))$.

We obtain a contradiction to the maximality of g^* by showing that $g^* \cup \{\langle a, a' \rangle\}$ is an (M_1, M_2)-mapping.

It suffices to show that

$$M_1 \models \chi[a; \mathbf{c}] \implies M_2 \models \chi[a'; g^*(\mathbf{c})]$$
$$\text{for all } \mathbf{c} \in \operatorname{dom} A_1' \text{ and every } \chi(x; \mathbf{y}).$$

Suppose $M_1 \models \chi[a, \mathbf{c}]$ holds. By the choice of a we have that

$$M_1 \models \varphi[a] \wedge \psi[a; \mathbf{b}] \wedge \chi[a; \mathbf{c}].$$

By the choice of n_0 we get $M_1 \models \forall x[\varphi(x) \wedge \psi(x; \mathbf{b}) \to \chi(x; \mathbf{c})]$. Thus by the choice of a' and elementricity of g^* we get that $M_2 \models \chi[a'; g^*(\mathbf{c})]$.

Why $\operatorname{range}(g^*) = \operatorname{acl}_{M_2}(A_2)$?

Let $a \in \operatorname{acl}_{M_2}(A_2)$ be given. By algebraicity of a there are $n < \omega, \varphi(x; \mathbf{z})$ and $\mathbf{b} \in A_2$ such that

$$n = |\varphi(M_2; \mathbf{b})| \text{ and } a \in \varphi(M_2; \mathbf{b}).$$

Since g is onto A_2 by the assumption that g is an (M_1, M_2)-mapping we have

$$n = |\varphi(M_1; g^{-1}(\mathbf{b}))|.$$

Since $\varphi(M_1; g^{-1}(\mathbf{b})) \subseteq \operatorname{acl}_{M_1}(A_1)$ we have that $\varphi(M_1; g^{-1}(\mathbf{b})) \subseteq \operatorname{dom} g^*$, as $g^*[\varphi(M_1; g^{-1}(\mathbf{b}))] = \varphi(M_2; \mathbf{b})$ we have $a \in \operatorname{range}(g^*)$. □

□

COROLLARY 9.30. *Suppose M and N are models, $\mathbf{a} \in |M|$ and $\varphi(\mathbf{x}; \mathbf{a})$ is strongly minimal. If f is an (M, N)-mapping such that $\mathbf{a} \in \operatorname{dom} f$, $\mathbf{I} \in \varphi(M; \mathbf{a})$ a basis, $\mathbf{J} \in \varphi(N; f(\mathbf{a}))$ a basis such that $|\mathbf{I}| \leq |\mathbf{J}|$ then there exists an (M, N)-mapping g taking $\varphi(M; \mathbf{a})$ to $\varphi(\varphi(N; f(\mathbf{a}))$ satisfying $g \supseteq f$.*

PROOF. □

The main reason to the great importance of strongly minimal sets is the fact that the above notion allows us to defines a closure operator which has a pre-geometry structure among the elements of the strongly minimal set. Pre-geometries are a simultaneous generalization of linear dependence and algebraic dependence (in vector spaces and algebraically closed fields respectively). Some basic facts about pre-geometries are presented in 16.

THEOREM 9.31 (There is a pre-geometry on strongly minimal sets). *Let $\varphi(\mathbf{x})$ be a strongly minimal formula, with $\mathbf{a} \in |M|$ and let $D := \varphi(M)$. For every $X \subseteq D$ let $\mathrm{cl}(X) := \mathrm{acl}(X) \cap D$. Under the above assumptions (D, cl) is a pre-geometry I.e.*

(1) $X \subseteq \mathrm{cl}(X)$
(2) *finite character: for every $\mathbf{b} \in \mathrm{cl}(X)$ there exists a finite subset X_0 of X such that $\mathbf{b} \in \mathrm{cl}(X_0)$.*
(3) $\mathrm{cl}(\cdot)$ *is transitive.*
(4) *Exchange principle: For all $\mathbf{c}, \mathbf{b} \in D$ and every $X \subseteq D$ if $\mathbf{c} \notin \mathrm{cl}(X)$ but $\mathbf{c} \in \mathrm{cl}(X \cup \{\mathbf{b}\})$ then $\mathbf{b} \in \mathrm{cl}(X \cup \{\mathbf{c}\})$.*

PROOF. It is enough to prove part (4). For the sake of contradiction suppose there are $\mathbf{b}, \mathbf{c} \in D$ and $X \subseteq D$ such that $\mathbf{c} \in \mathrm{cl}(X \cup \{\mathbf{b}\})$ but $\mathbf{b} \notin \mathrm{cl}(X \cup \{\mathbf{c}\})$. Notice that by monotonicity of cl using the second assumption we get that $\mathbf{c} \notin \mathrm{cl}(X)$.

Without loss of generality we may assume that we have constant symbols for the elements of X and $\mathbf{a}$, thus we have $\mathbf{c} \in \mathrm{cl}(\mathbf{b}) - \mathrm{cl}(\emptyset)$ and $\mathbf{b} \notin \mathrm{cl}(\mathbf{c})$.

As both elements are not algebraic and $\mathbf{b} \notin \mathrm{cl}(\mathbf{c})$, by Corollary 9.20 we have that

$$(*) \quad \mathbf{b}'\mathbf{c}' \models p \text{ for all non algebraic } \mathbf{b}', \mathbf{c}' \text{ such that } \mathbf{b}' \notin \mathrm{cl}(\mathbf{c}'),$$

where $p(\mathbf{x}, \mathbf{y}) := \mathrm{tp}(\mathbf{bc}/\emptyset)$.

Pick $S \subseteq \varphi(\mathfrak{C}) - D$ infinite consisting of non-algebraic elements only and fix $\mathbf{b}' \in \varphi(\mathfrak{C}) - \mathrm{acl}(S)$.

By $(*)$

$$\mathbf{b}'\mathbf{c}' \models p \text{ for every } \mathbf{c}' \in S.$$

This is a contradiction to the assumption that $\mathbf{c}$ is algebraic over $\mathbf{b}$ since the set S is infinite and p includes a formula of the form $\exists^n \mathbf{y} \psi(\mathbf{y}; \mathbf{x}) \wedge \psi(\mathbf{y}; \mathbf{x})$ for some $n < \omega$. □

ANOTHER PROOF OF PART (4). By expanding the language by constants interpreting the elements of $X \cup \{a\}$ it is enough to show: if $c, b \in D$ such that $c \in \mathrm{acl}(b) - \mathrm{acl}(\emptyset)$ then $b \in \mathrm{acl}(c)$. For the sake of contradiction suppose that there are $c, b \in D$ such that $c \in \mathrm{acl}(b) - \mathrm{acl}(\emptyset)$ but $b \notin \mathrm{acl}(c)$. Since $c \in \mathrm{acl}(b)$ there is a formula $\psi(x; y)$ and a positive integer n such that $\models \psi[c, b] \wedge \exists^n y \psi[y; b]$. Since $b \notin \mathrm{acl}(c)$ the formula $\psi(c; z) \wedge \exists^n y \psi(y; z)$ (which is realized by b) is not algebraic.

CLAIM 9.32. *There exists a finite $S_0 \subseteq D$ such that*

$$b' \in D - S_0 \Longrightarrow \mathfrak{C} \models \psi[c; b'] \wedge \exists^{\leq n} y \psi(y; b').$$

PROOF. Suppose there is no finite S_0 as above. Namely (by strong minimality of D) we have that the set $\{b' \in D \ : \models \psi[c;b'] \wedge \exists^{\leq n} y\psi[y;b']\}$ is finite, let l be its cardinality. Clearly

$$\models \exists x_0 \dots \exists x_{l-1} \forall x \Big[[\bigwedge_{i<j} x_i \neq x_j \wedge \bigwedge_{i<l} \varphi(x_i;\mathbf{a})] \to \psi(c,x_i) \wedge \exists^{\leq n} y\psi(y,x_i)] \\ \wedge [\varphi(\mathbf{x};\mathbf{a}) \to \psi(c;\mathbf{x}) \wedge \exists y \psi(y;x)] \to \bigvee_{i<l} x = x_i \Big].$$

Since $b \in \varphi(\mathfrak{C};\mathbf{a})$ the formula $\varphi(x;\mathbf{a}) \to \psi(c;\mathbf{x}) \wedge \exists^n y\psi(y;\mathbf{x})]$ is an algebraic formula satisfied by b contradiction to the hypothesis that $b \notin \mathrm{acl}(c)$.

□

Let $k := |S_0|$. Now using the definition of S_0 we have

$$\models \exists z_0 \cdots \exists z_{k-1} \forall z \Big[\bigwedge_{i<k} z \neq z_i \wedge \varphi(z;\mathbf{a}) \to [\psi(c;z) \wedge \exists^n y\psi(y;z)] \Big].$$

Since $c \notin acl(\emptyset)$ the following formula

$$\varphi^*(w) := \exists z_0 \dots \exists z_{k-1} \forall z \Big[[\bigwedge_{i<k} z \neq z_i] \wedge \varphi(z;\mathbf{a}) \to [\psi(w,z) \wedge \exists^n y\psi(y;z)] \Big]$$

can not be algebraic (since it is realized by c). Namely there exists $C \subseteq \varphi^*(\mathfrak{C})$ infinite. Let $c_0, \dots, c_n$ be distinct elements of C.
Since for every $i \leq n \models \varphi[c_i]$ there exists a finite $S_0^* \subseteq D$ such that

$$(*) \quad b' \in D - S_0^* \Longrightarrow \models \bigwedge_{i \leq n} \psi[c_i;b'] \wedge \exists^n y\psi(y;b')$$

S_0^* as above exists by taking it to be the set of witnesses for elements satisfying $\exists z_0 \dots \exists z_{k-1} in \varphi(w)$. I.e. $|S_0^*| \leq k \geq (n+1)$. Now pick $b^* \in D - S_0^*$, by (*) we have that $\models \bigwedge_{i \leq n} \psi[c_i,b^*] \wedge \exists^n y\psi(y;b^*)$ holds. Notice that both sub formulas in the last conjunction are contradictory. □

In light of Theorem 9.31 when dealing with a strongly minimal formula $\varphi(\mathbf{x};\mathbf{a})$ when we say $\mathbf{I} \subseteq \varphi(M;\mathbf{a})$ $\mathbf{I}$ *is independent* we mean in the above pregeometry induced by being algebraic on the set $\varphi(M;\mathbf{a})$ namely we have that $\mathrm{tp}(\mathbf{b}/\mathbf{I} - \{\mathbf{b}\})$ is not algebraic for every $\mathbf{b} \in \mathbf{I}$.

REMARK 9.33. Notice that the hypothesis that $\varphi(\mathbf{x};\mathbf{a})$ is strongly minimal was used only to prove the exchange principle.

Using the uniqueness and extension property of strongly minimal formulas we can prove:

LEMMA 9.34. *Suppose M an N are elementarily equivalent, $\mathbf{a} \in |M|$ and $\mathbf{a}' \in |N|$ are such that $\mathrm{tp}(\mathbf{a}/\emptyset,M) = \mathrm{tp}(\mathbf{a}'/\emptyset,N)$ and $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(M)$ a formula such that $\varphi(\mathbf{x};\mathbf{a})$ is strongly minimal.*

If $\mathbf{I} \subseteq \varphi(M;\mathbf{a})$ *and* $\mathbf{J} \subseteq \varphi(N,\mathbf{a}')$ *are both independent then for any bijection* $f : \mathbf{I} \to \mathbf{J}$*,* $f \cup \{\langle \mathbf{a}, \mathbf{a}' \rangle\}$ *is an* (M,N)*-mapping.*

PROOF. Notice that also $\varphi(\mathbf{x};\mathbf{a}')$ is strongly minimal as there exists $g \in \mathrm{Aut}(\mathfrak{C})$ such that $g(\mathbf{a}) = \mathbf{a}'$ and strong minimality is preserved by automorphisms.

Let $\{\mathbf{a}_i \mid i < \alpha\} = \mathbf{I}$ and $\{\mathbf{b}_i \mid i < \alpha\} = \mathbf{J}$, where $f(\mathbf{a}_i) = \mathbf{b}_i$ for all $i < \alpha$.

It is enough to show that

$$\mathrm{tp}(\mathbf{a}_i/\{\mathbf{a}_j \mid j < i\} \cup \mathbf{a}) = \mathrm{tp}(\mathbf{b}_i/\{\mathbf{b}_j \mid j < i\} \cup \mathbf{a}') \text{ for all } i < \alpha.$$

□

PROOF OF (3). Let $\mu > \aleph_0$ and $M, N \models T$ be given of cardinality μ. By $\aleph_0$-stability there exists M_0 prime over $\emptyset$. We may assume that $M_0 \prec M$ and there exists an elementary embedding $f : M_0 \to N$.

By Corollary 9.13 there exists $\varphi(\mathbf{x};\mathbf{a})$ with parameters from M_0 which is strongly minimal and so is also $\varphi(x; f(\mathbf{a}))$. Let $G_1 := G_{M,\varphi(\mathbf{x};\mathbf{a})}$ and $G_2 := G_{N,\varphi(\mathbf{x};f(\mathbf{a}))}$ the pre-geometries from Theorem 9.31.

By Proposition 9.3 as M is prime over $\varphi(M;\mathbf{a})$, by uncountability of λ we get that $|\varphi(M;\mathbf{a})| = \lambda$ (see Exercise 5.2), the same argument show that $|\varphi(N; f(\mathbf{a}))| = \lambda$.

Let $\mathbf{I}_\ell \subseteq G_\ell$ bases of the corresponding pre-geometries. Since $G_\ell = \mathrm{cl}(\mathbf{I}_\ell)$ (by the finite character) and uncountability of λ we get that necessarily $|\mathbf{I}_\ell| = \lambda$.

Using previous theorems we can now extend f to any bijection of $\mathbf{I}_1$ onto $\mathbf{I}_2$, the last mapping can be extended to an (M,N)-mapping from $\mathrm{cl}(\mathbf{I}_1)$ onto $\mathrm{cl}(\mathbf{I}_2)$. As M is prime over $\mathrm{cl}(\mathbf{I}_1)$ there exists an (M,N)-mapping $\bar{f} M \to N$ such that $\bar{f}[\varphi(M;\mathbf{a})] = \varphi(N; f(\mathbf{a}))$, note that if $\bar{f}$ is not onto N then $\bar{f}[M]$ is a proper elementary submodel of N and $\varphi(\mathbf{x}; f(\mathbf{a}))$ contradicts the assumption that T has no Vaughtian Pairs. □

Using the technique from Claim 6.5 we get the following additional structural result:

THEOREM 9.35. *Let* M *and* $\varphi(\mathbf{x};\mathbf{a})$ *a non-algebraic formula with parameters from* M*. There is no infinite set* $\mathbf{I} \subseteq |M|$ *such that* $\mathbf{I}$ *are indiscernibles over the set* $\varphi(M;\mathbf{a})$*.*

PROOF. For the sake of contradiction suppose that such a set $\mathbf{I}$ exists,. Let $\mathbf{J} \supseteq \mathbf{I}$ be a set of indiscernibles of cardinality $\|M\|^+$. Pick M^* an $F^t_{\aleph_0}$ primary over $\varphi(M,\mathbf{a}) \cup J$ (exists since T is $\aleph_0$ stable). Denote by A the set $\varphi(M;\mathbf{a})$, and

by p the following set of formulas $\{\varphi(\mathbf{x};\mathbf{a}) \wedge \mathbf{x} \neq \mathbf{b} \mid \mathbf{b} \in A\}$, it is a consistent type since $\varphi(M,\mathbf{a})$ is infinite. We will obtain a contradiction to $\aleph_1$-categoricity by showing that M^* omits p (which show that M^* is not saturated): Suppose there exists an element b in M^* realizing p. Since M^* is $F^t_{\aleph_0}$ primary over $A \cup J$ there are a formula $\psi(\mathbf{x};\mathbf{y};\mathbf{z})$, $\mathbf{c} \in A$ and $\mathbf{j} \in J$ such that

$$(*) \quad \psi(\mathbf{x};\mathbf{c};\mathbf{j}) \vdash \mathrm{tp}(\mathbf{b}/A).$$

Since $\mathbf{I}$ is an infinite subset of $\mathbf{J}$ there exists a finite sequence $\mathbf{i}$ of elements of $\mathbf{I}$ (of the same length as $\mathbf{j}$) such that $\mathrm{tp}(\mathbf{i}/A) = \mathrm{tp}(\mathbf{j}/A)$. Let F be an (M^*, M^*)-mapping such that $F \restriction A = \mathrm{id}_A$ and $F(j) = i$. An application of F to $(*)$ gives

$$(**) \quad \psi(\mathbf{x};\mathbf{c};\mathbf{i}) \vdash \mathrm{tp}(\mathbf{b}/A).$$

Since $M^* \models \exists \mathbf{x}\psi(\mathbf{x};\mathbf{c};\mathbf{i})$ (exemplified by b) also $M \models \exists \mathbf{x}\psi(\mathbf{x};\mathbf{c};\mathbf{i})$ let $d \in M$ such that $M \models \psi[\mathbf{d};\mathbf{c};\mathbf{i}]$ by $(**)$ $d \models \mathrm{tp}(\mathbf{b}/A)$, and in particular p is realized by $\mathbf{d}$ (since $p \subseteq \mathrm{tp}(\mathbf{b}/A)$). This is impossible since it is clear from the definition of p that p is omitted by M. □

THEOREM 9.36. *Let T be a countable $\aleph_1$ categorical theory. For every model M there exists a unique N which is a proper elementary extension of M and is prime and minimal (namely $M \preceq N$ and $N \neq M$, for every $N' \succ M$ there exists an elementary embedding G of N into N' which is the identity over M, and if N^* is a proper elementary submodel of N which contains M then $N^* = M$.*

PROPOSITION 9.37 (Density).

PROOF. □

DEFINITION 9.38. If the formula $x = x$ is strongly minimal we say that the *theory is strongly minimal*.

The following is an important property of theories introduced by Keisler for the study of the question when an ultrapower of a model with respect to a regular ultrafilter is saturated (see **??**).

DEFINITION 9.39. A formula $\varphi(\mathbf{x};\mathbf{y})$ does not have the *finite cover property* iff there exists a positive integer n_φ such that for every $p \subseteq \{\varphi(\mathbf{x};\mathbf{a}) \mid \mathbf{a} \in \mathfrak{C}\}$ the following holds:

If q is consistent for all $q \subseteq p$ of cardinality $\leq n_\varphi$ then p is consistent.

Often the acronym *f.c.p.* is used.

We say that a *theory fails to have the f.c.p.* iff every formula $\varphi(\mathbf{x};\mathbf{a})$ does not have the f.c.p.

Thus the failure of the f.c.p. is a strengthening of the compactness theorem.

PROPOSITION 9.40. *Suppose that $\psi(\mathbf{x};\mathbf{a})$ is strongly minimal.*
Every $\varphi(\mathbf{x};\mathbf{b})$ such that $\vdash \varphi(\mathbf{x};\mathbf{b}) \to \psi(\mathbf{x};\mathbf{a})$ does not have the f.c.p.

PROOF. □

COROLLARY 9.41. *If T is strongly minimal then T does not have the f.c.p.*

Exercises 7.9.

(1) Use the compactness theorem to prove: If $p \in \mathrm{S}^{\mathbf{x}}(A)$ is algebraic then there exists $\varphi(\mathbf{x};\mathbf{a}) \in p$ such that
$$\varphi(\mathbf{x};\mathbf{a}) \vdash p.$$
(2) Let $p \in \mathrm{S}^{\mathbf{x}}(A)$ be algebraic and
$$n_0 := \min\{n < \omega \mid \mathfrak{C} \models \exists^n \mathbf{x}\varphi(\mathbf{x};\mathbf{a}), \varphi(\mathbf{x};\mathbf{a}) \in p\}.$$
Suppose that $\varphi(\mathbf{x};\mathbf{a}) \in p$ is such that $\mathfrak{C} \models \exists^{n_0}\mathbf{x}\varphi(\mathbf{x};\mathbf{a})$. Prove that
$$\varphi(\mathbf{x};\mathbf{a}) \vdash p.$$
(3) Suppose $\mathbf{a} \in |M|$. Show that $\varphi(\mathbf{x};\mathbf{a})$ is strongly minimal iff $\varphi(\mathbf{x};\mathbf{a})$ in N-minimal for all $N \prec M$.
(4) Let $\varphi(\mathbf{x};\mathbf{a})$ with $\mathbf{a} \in M \models T$ and M is $\aleph_0$-saturated. Show that $\varphi(\mathbf{x};\mathbf{a})$ is M-minimal iff $\varphi(\mathbf{x};\mathbf{a})$ is strongly minimal.
(5) Write up a complete proof confirming Remark 9.11.
(6) For every set of formulas p and $f \in \mathrm{Aut}(\mathfrak{C})$ the type p is algebraic iff $f(p)$ is an algebraic type.

Moreover, in the above we don't need to require that $f \in \mathrm{Aut}(\mathfrak{C})$ it suffixes to assume that f is a $(\mathfrak{C},\mathfrak{C})$-mapping with $\mathrm{dom}\, p \subseteq \mathrm{dom}\, f$.
(7) A formula $\varphi(\mathbf{x};\mathbf{b})$ is strongly minimal iff $\varphi(\mathbf{x};f(\mathbf{b}))$ is strongly minimal for all (some) $f \in \mathrm{Aut}(\mathfrak{C})$.
(8) Suppose $M \models T$ is $\aleph_0$-saturated and T is $\aleph_0$-stable. If $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T)$ and $\mathbf{a} \in |M|$ are such that $\varphi(M;\mathbf{a})$ is infinite then there are $\varphi^*(\mathbf{x};\mathbf{z})$ and $\mathbf{a}^* \in |M|$ such that $\varphi^*(\mathbf{x};\mathbf{a}^*)$ is strongly minimal and
$$M \models \forall\mathbf{x}[\varphi^*(\mathbf{x};\mathbf{a}^*) \to \varphi(\mathbf{x};\mathbf{a})].$$
(9) Let $\varphi(\mathbf{x};\mathbf{y})$ and $\mathbf{a},\mathbf{b} \in {}^{\ell(\mathbf{y})}\mathfrak{C}$ be given such that $\mathrm{tp}(\mathbf{a}/\emptyset) = \mathrm{tp}(\mathbf{b}/\emptyset)$ then
$$\varphi(\mathbf{x};\mathbf{a}) \text{ is algebraic } \iff \varphi(\mathbf{x};\mathbf{b}) \text{ is algebraic.}$$
(10) Let $\varphi(\mathbf{x};\mathbf{y})$ and $\mathbf{a},\mathbf{b} \in {}^{\ell(\mathbf{y})}\mathfrak{C}$ be given such that $\mathrm{tp}(\mathbf{a}/\emptyset) = \mathrm{tp}(\mathbf{b}/\emptyset)$ then
$$\varphi(\mathbf{x};\mathbf{a}) \text{ is strongly minimal } \iff \varphi(\mathbf{x};\mathbf{b}) \text{ is strongly minimal.}$$
(11) A formula $\varphi(\mathbf{x};\mathbf{b})$ is algebraic iff $\mathrm{R}[\varphi(\mathbf{x};\mathbf{b})] < \omega$.
(12) Suppose that p is an $\mathbf{x}$-type and A a set containing the parameters of p. Show that if p is not algebraic then there exists a non-algebraic $q \in \mathrm{S}^{\mathbf{x}}(A)$ extending p. Hint: There exists $S \subseteq p[\mathfrak{C}]$ of cardinality $(2^{|A|+|T|})^+$.
(13) Suppose $\varphi(\mathbf{x};\mathbf{b})$ is strongly minimal and $A \supseteq \cup\mathbf{b}$. Show that if $p \in \mathrm{S}^{\ell(\mathbf{x})}(A)$ is the non-algebraic type extending $\varphi(\mathbf{x};\mathbf{b})$ then p does not split over the set $\cup\mathbf{b}$.
(14) Suppose $A \subseteq |M|$, $p \in \mathrm{S}(A)$ and M is $|A|^+$-saturated. The type p is algebraic iff $\varphi[M]$ is finite.
(15) Suppose $\varphi(\mathbf{x};\mathbf{b})$ is a strongly minimal and $A \supseteq \cup\mathbf{b}$ then
$$p := \{\psi(\mathbf{x};\mathbf{a}) \in \mathrm{Fml}(\mathrm{L}(T)) \mid \varphi(\mathbf{x};\mathbf{b}) \wedge \neg\psi(\mathbf{x};\mathbf{a}) \text{ is algebraic}\}$$
is the non-algebraic complete type over A containing $\varphi(\mathbf{x};\mathbf{b})$.

(16) Let M be $\aleph_0$-saturated, $\mathbf{b} \in |M|$. If $\varphi(\mathbf{x};\mathbf{b})$ is strongly minimal then there exists an infinite sequence of indiscernibles $\mathbf{I} \subseteq M$ based on $\bigcup \mathbf{b}$ such that $\models \varphi[\mathbf{a};\mathbf{b}]$ holds for every $\mathbf{a} \in \mathbf{I}$.

(17) Prove theorem 6.5. The following hints hopefully outline two different proofs.
   (a) Observe that if $|\operatorname{D}(T)| = \aleph_0$ then for every finite sequence of elements $\mathbf{a}$ the expansion $T_{\mathbf{a}}$ by new constants has a countable diagram. Use this fact and repeat the argument in the proof of Theorem 6.4.
   (b) Since $|\operatorname{D}(T)| = \aleph_0$ the theory T has an atomic model M, show that the parameters of the strongly minimal formula can be found in M.

(18) Show that $\operatorname{acl}(A) = \operatorname{acl}(\operatorname{acl}(A))$ for every set A.

(19) Show that for every set A $\operatorname{acl}(A) \supseteq A$.

(20) Show that for any A and B we have $A \subseteq B \implies \operatorname{acl}(A) \subseteq \operatorname{acl}(B)$.

(21) Suppose $A \subseteq M$, show that $\operatorname{acl}(A) \subseteq M$.

(22) Show that in the argument in the proof of Theorem **??** we could find the countable set A inside M (Hint: Use Exercise 2.**??**).

(23) Suppose that $\varphi(\mathbf{x};\mathbf{a})$ is strongly minimal. Show that for infinite $\mathbf{I} \subseteq \varphi(\mathfrak{C};\mathbf{a})$, the set $\mathbf{I}$ is independent iff $\mathbf{I}$ is a sequence of indiscernibles over $\cup\mathbf{a}$.

(24) Suppose T is strongly minimal. Let M be a model and denote by G_M the pregeometry induced by the above formula on $|M|$. Prove that

$$\dim G_M = \|M\| \implies M \text{ is a saturated model.}$$

(25) Suppose every non-algebraic type contains a strongly minimal formula. If for every uncountable M and every strongly minimal $\varphi(\mathbf{x};\mathbf{a})$ with $\mathbf{a} \in |M|$ we have that $\|M\| = |\varphi(M;\mathbf{a})|$ then T is categorical in every uncountable cardinal.

(26) Verify that every $\aleph_0$-stable theory satisfies the assumption of the previous exercise.

(27) Let $T := \operatorname{Th}(\langle \mathbb{R}, +, \cdot \rangle)$. Show that for every $\lambda > \aleph_0$ the theory T is not categorical in λ.

(28) Answer the previous question without using results from this chapter.

(29) Suppose $p \in \operatorname{S}^{\mathbf{x}}(A)$ is a non-algebraic type and $\varphi(\mathbf{x};\mathbf{a}) \in p$ strongly minimal. If $\mathbf{I} = \{\mathbf{a}_i \mid i < \alpha\}$ is an algebraically independent set of realizations of p then $\mathbf{I}$ is an indiscernible sequence.

(30) Suppose $\varphi(x)$ is strongly minimal. Show that if $\mathbf{I} := \{a_i \mid i < \alpha\}$ such that $\mathfrak{C} \models \varphi[a_i]$ and $a_i \notin \operatorname{acl}(\{a_j \mid j < i\})$ for every $i < \alpha$ then $\mathbf{I}$ is a sequence of indiscernibles.

(31)

(32)

10. Some properties of $|T|^+$-categorical theories

The aim of this section is not only to offer an exercise in using the ideas from the proof of Morley's theorem. There are two additional reasons for the presence of this section. First to offer some fundamental facts about $\aleph_1$-categorical (for countable) theories. This is the beginning of a highly developed theory (see Zilber [Zi] etc. XXXX). The second is to prepare the ground for generalizing Morley's theorem for uncountable theories (namely proving Loś's conjecture for uncountable languages). In this section we deal with countable and uncountable theories separately. At the first half of this section we assume that the theory T is countable and $\aleph_1$-categorical, at the second half we will remove the assumption of countability of T but still assume categoricity in $|T|^+$.

THEOREM 10.1. *Suppose T is in a countable language. If T is categorical in an uncountable cardinal λ then T has no Vaughtian Pair.*

PROOF. Suppose that T has a Vaughtian Pair, using the Löwenheim-Skolem Theorems we may assume that T has a Vaughtian Pair $M \precneqq N$, $\mathbf{a} \in |M|$ and a non-algebraic $\varphi(\mathbf{x};\mathbf{a})$ such that $\varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a})$ such that $\|M\| = \|N\| = \lambda$. Using λ-categoricity we can find a Vaughtian Pair M', N' such that $\|M\| = \lambda$ and $\|N\| = \lambda^+$. Now by Vaught's two-cardinal Theorem there exists a Vaughtian Pair (M_0, N_0) such that $\|M_0\| = \aleph_0$ and $\|N_0\| = \aleph_1$.

Recall that categoricity in $\lambda > \aleph_0$ implies that T is categorical in $\aleph_1$ and this imply that the model of cardinality $\aleph_1$ is saturated, however this is a contradiction as N_0 omits the type

$$\{\varphi(x;\mathbf{a}) \wedge x \neq b \mid b \in |M_0|\}.$$

□

THEOREM 10.2. *Suppose T is $\aleph_0$-stable. If T has no Vaughtian Pair then for every N, $A \subseteq |N|$, $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T)$ and $\mathbf{a} \in A$ such that $\varphi(\mathbf{x};\mathbf{a})$ is not algebraic then N is the unique prime model over $\varphi(N;\mathbf{a}) \cup A$.*

PROOF. By $\aleph_0$-stability there exists M^* prime over $\varphi(N;\mathbf{a}) \cup A$. Thus there exists an elementary embedding $f : M \to N$ such that $f \restriction (\varphi(N;\mathbf{a}) \cup A) = \mathrm{id}$. Let $M := f[M^*] \prec N$. If $M \precneqq N$ then (M, N) is a Vaughtian Pair. □

HYPOTHESIS 10.3. T is a countable $\aleph_1$-categorical.

The ideas used to prove the following theorem were borrowed from the proofs of Theorem 1.13.11 and Theorem 2.12.9.

THEOREM 10.4 (No Vaughtian pairs). *Let M be a proper elementary submodel of N, and $\mathbf{a} \in |M|$. If $\varphi(x;\mathbf{a})$ is not algebraic then $\varphi(M;\mathbf{a}) \neq \varphi(N;\mathbf{a})$.*

PROOF. Suppose not. Let M be a proper elementary submodel of N, $\mathbf{a} \in M$ and $\varphi(x;\mathbf{a})$ non-algebraic such that $\varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a})$. By the upward and downward Löwenheim-Skolem theorems we may assume in addition that

$$\|M\| = \|N\| = \aleph_0.$$

Why? Let $M^* := \langle |N|, |M|, F, R, c, a\rangle_{F,R,c\in \mathrm{L}(T)}$. I.e. the language of M^* is an expansion of $\mathrm{L}(T)$ by a unary predicate whose interpretation is the set $|M|$, and a finite sequence of new constants interpreted by a.
Let $T^* := \mathrm{Th}(M^*)$, observe that T^* has the following properties:

(1) $T^* \supseteq T$ (since $\mathrm{Th}(N) \models T$)
(2) $\forall x[\varphi(x;\mathbf{a}) \to x \in |M|] \in T^*$ (since $\varphi(M;\mathbf{a}) \supseteq \varphi(N;\mathbf{a})$)
(3) For every formula $\exists y\varphi(y;x) \in \mathrm{L}(T)$ we have
$\forall x[x \in |M| \wedge \exists y\varphi(y;x) \to \exists z[z \in |M| \wedge \varphi(z;x)]] \in T^*$ (since $M \preceq N$)
(4) $\exists x[x \in |M|] \in T^*$ (since $|M|$ is a proper subset of $|N|$) By Lowenheim-Skolem there exists $N' \models T^*$ of cardinality $\aleph_0$; its reduct and the interpretation of the unary predicate in N' supply a pair of models as required.

Recall Theorem 2.9.1 if $|\mathrm{D}(T)| \leq \aleph_0$ then every model has a countable saturated elementary extension. Since uncountable categoricity implies stability we have in particular that the diagram of T is countable thus by repeating the argument in the proof of Theorem 2.9.1 one can assume that

(1) M and N are countable saturated,
(2) $\mathbf{a} \in M \preceq N$, $M \subsetneq N$ and
(3) $\varphi(M;\mathbf{a}) = \varphi(N;\mathbf{a})$.

By induction on $\alpha < \omega_1$ we define $N_\alpha \models T$ with the following properties

(1) $\gamma < \alpha \Longrightarrow N_\gamma$ is a proper elementary submodel of N_α
(2) if α limit then $N_\alpha = \bigcup_{\gamma<\alpha} N_\gamma$
(3) N_α is a countable saturated model of T
(4) $\varphi(N_0;\mathbf{a}) = \varphi(N_\alpha;\mathbf{a})$ for all α.

This is enough: Let $N^* := \bigcup_{\alpha<\omega_1} N_\alpha$. Note that by requirements 4 & 3 we have $|\varphi(N^*,\mathbf{a})| = |\varphi(N,\mathbf{a})| = \aleph_0$.
Consider the set of formulas $p := \{\varphi(x;\mathbf{a}) \wedge x \neq b \ : \ b \in \varphi(N^*;\mathbf{a})\}$, since $\varphi(x;\mathbf{a})$ is not algebraic clearly p is a type, from the definition of p it follows that N^* omits p. Since the set of parameters of p is countable and $\|N^*\| = \aleph_1$ the model N^* is not saturated. This is a contradiction to $\aleph_1$ categoricity. This is possible: For $\alpha = 0$; let $N_0 := M$. For α limit; let $N_\alpha := \bigcup_{\gamma<\alpha}$ the model N_α satisfies requirement 3 by Theorem 2.9.4 and requirement 4 by the induction hypothesis. For $\alpha = \gamma + 1$; since N_γ is saturated by Theorem 2.9.3 there exists $G : N\gamma \to M$ and let $N_{\gamma+1}$ be the preimage of N by an extension of G.

□

REMARK 10.5. Due to the similarity between the second half of above argument to the one in Vaught's two cardinal theorem ($\langle \lambda^+, \lambda \rangle \to \langle \aleph_1, \aleph_0 \rangle$) often a pair of models as in the statement of Theorem 10.4 is called *Vaughtian pair*.

COROLLARY 10.6. *Let M be a model. If $\varphi(x; \mathbf{a})$ is not algebraic over M then M is prime and minimal over the set $\varphi(M; \mathbf{a})$. (Namely if $N \supseteq \varphi(M; \mathbf{a})$ then there is an embedding of M into N which the identity over $\varphi(M, \mathbf{a})$ and there is no M^* proper submodel of M containing $\varphi(M; \mathbf{a})$)*

PROOF. Let $A := \varphi(M; \mathbf{a})$. By Theorem 5.4 there exists $N \supseteq A$ prime over A, since also $M \supseteq A$ there exists an elementary embedding $F : N \to M$ such that $F \restriction A = \mathrm{id}_A$ we may assume that $N \subseteq M$. Clearly $N \preceq M$ (they are elementary submodels of $\mathfrak{C}$), why $N = M$? Suppose not, namely N is a proper elementary submodel of M. Since $\varphi(M; \mathbf{a}) = A = \varphi(N; \mathbf{a})$ we have contradicted Theorem 5.2. The minimality of M follows similarly, suppose there exists a proper submodel M^* of M containing $\varphi(M, \mathbf{a})$). Clearly $\varphi(M; \mathbf{a}) = A = \varphi(M^*; \mathbf{a})$ is a contradiction to Theorem 5.2. □

The proof of Theorem 10.4 depends on the countability of $\mathrm{L}(T)$ and categoricity in $\aleph_1$ with some effort the theorem can be generalized also for uncountable languages:

THEOREM 10.7 (No Vaughtian pairs in uncountable languages). *Suppose T is a theory categorical in an uncountable cardinality greater that $|T|$. If N is a proper elementary submodel of M and $\varphi(x; \mathbf{a})$ is not algebraic then $\varphi(M; \mathbf{a}) \neq \varphi(N; \mathbf{a})$.*

Remark it is not difficult to prove the above for countable $|T|$. However the technology introduced so far is insufficient to prove the last theorem. Its proof is delayed to Chapter XX.

THEOREM 10.8. *(Baldwin) If T is $\aleph_1$-categorical then* $\mathrm{RM}[\mathbf{x} = \mathbf{x}] < \omega$.

In this part of the section we generalize the previous results to uncountable theories. Inspection of the arguments of Theorems 5.2 and 5.4 show dependence in a crucial manner on Morley's theorem (categoricity implies all models of cardinality greater than $|T|$ are saturated). Since we did not prove yet that Loś conjecture holds we need a substitute to the use of saturation in the above proofs. An inspection shows that in both theorems the types we want to realize are of a particular form namely $p = \{\varphi(\mathbf{x}; \mathbf{a}) \wedge \mathbf{x} \neq \mathbf{b} \mid \mathbf{b} \in A\}$. Below we will introduce a name for types of the above form, and then we will prove that if T is $|T|^+$-categorical then all models realize these types.

DEFINITION 10.9. Let L be a similarity type, M an L-structure, $\varphi(\mathbf{x}; y)$ an L-formula. Suppose $A \subseteq |M|$. Denote by $n := \ell(\mathbf{x})$, for $\mathbf{b}$ an n-tuple of elements from M.

(1) $\mathrm{tp}_\varphi(\mathbf{b}/A, M)$ is the set

$$\{\varphi^t(\mathbf{x};\mathbf{a}) \ : \ \mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A}, t \in \{0,1\}, M \models \varphi^t[\mathbf{b};\mathbf{a}]\}.$$

We use here the standard convention that $\varphi^t(\mathbf{x};\mathbf{a})$ is $\varphi(\mathbf{x};\mathbf{a})$ when $t = 0$ and $\varphi^t(\mathbf{x};\mathbf{a})$ is $\neg\varphi(\mathbf{x};\mathbf{a})$ when $t = 1$. When $M = \mathfrak{C}$ we will not mention it as a parameter, namely we will write $\mathrm{tp}_\varphi(\mathbf{b}/A)$ for $\mathrm{tp}_\varphi(\mathbf{b}/A, \mathfrak{C})$.

(2) We call a set of formulas p a $\varphi(\mathbf{x};\mathbf{y})$-*type* iff the elements of p consists only of the form $\varphi(\mathbf{x};\mathbf{a})$ or $\neg\varphi(\mathbf{x};\mathbf{a})$ when $\mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A}$.

(3) Let be an infinite cardinality. We say that M is iff for every $A \subseteq |M|$ of cardinality less than and every p which is a φ- type over A consistent with $\mathrm{Th}(M)$ there exists a sequence of elements of M realizing p. We say that M is iff M is $(|M|, \varphi)$ saturated. M is iff for every formula $\varphi(\mathbf{x};\mathbf{y})$, M is φ-saturated.

(4) For $\varphi(\mathbf{x};\mathbf{y})$ and a set A as above we will use the following notation to denote the set of all φ types over A which are realized in M it is the set $\{tp_\varphi(\mathbf{b}/A, M) : b \in {}^{\ell(\mathbf{x})}|M|\}$.

In order to prove the analog of Theorem 5.2 for uncountable languages first we will verify couple of facts which will show that as in the case for countable theories categoricity implies that every model is locally saturated.

THEOREM 10.10. *Let T be a first-order theory and suppose that there exists a cardinal $\mu > |\mathrm{L}(T)|$ such that T is categorical in μ.*

(1) *For every $\lambda \geq |\mathrm{L}(T)|$ such that $\lambda < \mu$, T is stable in λ.*
(2) *The model of cardinality μ is locally saturated.*

PROOF. (1) Let $M \models T$ be a model of cardinality μ. Let $T^* \supseteq T$ be an expansion of T of cardinality $|\mathrm{L}(T)|$ such that T^* has Skolem Functions. By Corollary II.6.XXXX the Skolem hull (in T^*) $\mathrm{EM}(\mu)$ of a linearly order set of order type μ is stable in every. Since the number of types does not increase when taking reducts, also $\mathrm{EM}(\mu) \restriction \mathrm{L}(T)$ is stable. Suppose there is a set A of cardinality (for some $|T| < \mu$) such that $|\mathrm{S}(A)| >$, pick $N \supseteq A$ of cardinality μ such that N contains $+$ elements realizing distinct elements of $\mathrm{S}(A)$. But by categoricity in μ we have $\mathrm{EM}(\mu) \restriction \mathrm{L}(T) \cong M \cong N$ which is a contradiction.

(2) Since T is stable, by Theorem X.YY for every A and every formula $\varphi(\mathbf{x};\mathbf{y})$ we have that $|\mathrm{S}^{l(\mathbf{x})}_\varphi(A)| \leq |A| + \aleph_0$. Let M be a model of cardinality μ. For every such that $|\mathrm{L}(T)| < \mu$ we will define an elementary chain $\{N_i : i < \mu^+\}$ with the following properties:

(1) $N_0 = M$,
(2) $i < j \Longrightarrow N_i$ is a proper elementary submodel of N_j
(3) $i < \mu^+ \Longrightarrow \|N_i\| = \mu$
(4) if i is limit then $N_i = \cup_{j<i} N_j$, and the main requirement

(5) $i < \mu^+ \implies$ for every $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T)$, every element of $\mathrm{S}_\varphi^{\ell(\mathbf{x})}(|N_i|)$ is realized in N_{i+1}.

This is possible: It is easy to see that since T is stable, it is possible to carry out a construction by transfinite induction satisfying the above properties. This is enough: Let $A \subseteq M$ be such that $|A| < \|M\|$, and $\varphi(\mathbf{x};\mathbf{y})$ be given. Let $\mu := |A|$, let $\{N_i : i < \mu^+\}$ be a chain satisfying (a)-(e). By (c) we have that also $N := \bigcup_{i<\mu^+} N_i$ has cardinality μ^+ . By categoricity there exists $F : M \cong N$, let $B \subseteq |N|$ be the image of A. Since the chain $\{N_i : i < \mu^+\}$ is increasing and the cofinality of μ^+ is greater than $|B|$ there exists $i < \mu^+$ such that $B \subseteq |N_i|$.
Clearly for every $p \in \mathrm{S}_{\varphi(\mathbf{x})}(A, M)$ we have that p is realized in M iff $F(p)$ is realized in N. By requirement (e) we have that $F(p)$ is realized in $N_{i+1} \preceq N$ for every $p \in \mathrm{S}_\varphi^{l(\mathbf{x})}(A, M)$. □

11. The Baldwin Lachlan proof

In this section we present an alternative proof to Morley's theorem due to Baldwin and Lachlan. It is an important proof since it clarify the analogy between Morley's theorem and Steinitz's characterization of algebraically closed fields.

We begin with Morley's characterization of categorical theories in terms of $\aleph_0$-stability and Vaughtian pairs.

THEOREM 11.1 (Morley 1967). *Let T be a countable theory. If T is stable in $\aleph_0$ then the following are equivalent:*

(1) *T is categorical in some $\lambda > \aleph_0$.*
(2) *T is categorical in every $\lambda > \aleph_0$.*
(3) *T does not have a Vaughtian pair .*

PROOF. $(2) \implies (1)$ is trivial. $(1) \implies (3)$ is Theorem 10.4. We left to prove $(3) \implies (2)$, this will be done after a couple of Lemmas.

LEMMA 11.2. *Suppose T stable in $\aleph_0$ and have no Vaughtian pairs. For any model M and $A := \varphi(M)$ non-algebraic definable set with parameters from $|M|$ we have that M is prime and minimal over A.*

PROOF. *M is prime over A*: By theorem 5.4 take M_A prime model over the set A. Thus there is an elementary embedding $f : M_A \to M$ which is the identity over A. Let $M' := f[M_A]$. Since $A \subseteq M' \subseteq M$ we have $A = \varphi(M') = \varphi(M)$. Thus by the hypothesis that T does not have a Vaughtian pair it follows that $M' = M$.

Minimality: Suppose $N \preceq M$ such that $|N| \supseteq A$.
We have $A = \varphi(N) = \varphi(M)$. Since T has no Vaughtian pair we get that $N = M$. □

LEMMA 11.3. *Let M and N be models. Suppose $\varphi(\mathbf{x})$ is strongly minimal, denote by $G_1 := \varphi(M)$ and $G_2 := \varphi(N)$. If $\mathbf{I}_\ell \subseteq G_\ell$ are algebraically independent sets and $g : \mathbf{I}_1 \to \mathbf{I}_2$ injective then g is an (M, N)-map.*

PROOF. Fix $\{\mathbf{a}^i_\ell \mid i < \mu\} = \mathbf{I}_\ell$ for $\ell = 1, 2$. Suppose $g(\mathbf{a}^i_1) = \mathbf{a}^i_2$. By induction on $n < \omega$ show that

$$(*) \quad M \models \psi[\mathbf{a}^{j_0}_1, \dots, \mathbf{a}^{j_{n-1}}_1] \iff N \models \psi[\mathbf{a}^{j_0}_2, \dots, \mathbf{a}^{j_{n-1}}_2]$$
$$\text{for all } j_0 < \cdots < j_{n-1} < i < \mu \text{ and every } \psi(\mathbf{x}).$$

For $n = 0$ we get that $\operatorname{tp}(\mathbf{a}^{j_0}_1/\emptyset, M) = \operatorname{tp}(\mathbf{a}^{j_0}_2/\emptyset, N)$ since φ is strongly minimal (see Corollary 9.18). Since $\mathbf{a}^i_\ell$ is algebraically independent over $\{\mathbf{a}^j_\ell \mid j < i\}$ an application of of Corollary 9.18 gives the induction step. □

Now to the last step in the proof of $(3) \implies (2)$: Let $M, N \models T$ be given of cardinality $\lambda > \aleph_0$. Using the Baldwin Lachlan characterization we notice that elementary classes are quite different from Abstract Elementary Classes even when they are $\aleph_1$-categorical.

THEOREM 11.4 (Shelah). *Let T be a first-order theory in a countable language. The statement T* is $\aleph_1$-categorical *is absolute.*

PROOF. □

Exercises 7.11.

(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)

12. Keisler's rank-free proof of Morley's theorem

Morley's original proof is one of most difficult results of mathematical logic in the sixties. I was told by one of the leaders of the field at the time that it took him six months to comprehend Morley's proof. It appears that the usage of an ordinal valued rank function on type-definable sets was one of the more complex issues.

In the late sixties Keisler observed that his two cardinal theorem could be used to prove an *upward Morley's Theorem*, see Theorem 12.1 below. Quickly Silver observed that the same argument show that $\aleph_0$-stability implies superstabilty, see Theorem 12.2.

In this section I present Keisler's and Silver's arguments that circumvent completely any rank function. This argument convinced several people that model theory and stability theory in particular could be developed without reference to rank functions. This view was held by some till the early eighties when people finally realized that ranks are unavoidable.

This section is dedicated to an outline of Keisler's elegant argument. I have elected to include it due to three reasons: It is really nice, provides an opportunity to see further connections between the fundamental concepts and since ideas of this argument can be used to prove new theorems.

The overall strategy is the same: Assume that T is categorical in some $\lambda > \aleph_0$, prove that all uncountable models are saturated.

It suffices to prove:

THEOREM 12.1 (Transfering saturation up). *Let T be n a countable language. If every model of T of cardinality $\aleph_1$ is saturated than all uncountable models of T are saturated.*

THEOREM 12.2. *If T is $\aleph_0$-stable then T is stable in every $\mu > \aleph_0$.*

COROLLARY 12.3 (Morley's upward categoricity theorem). *If T is a countable $\aleph_1$-categorical theory then it is categorical in every uncountable cardinal.*

The key result Keisler used was Keisler's two-cardinal theorem (Theorem 2.12.11). Instead, I will use the Keisler-Morley (Thoerem 2.9.19) which was verified immediately after Henkin's omitting-types theorem.

PROOF OF 12.1. Suppose M is an uncountable model and not saturated. Thus there are λ, $A \subseteq |M|$ and $q \in \mathrm{S}(A)$ such that

$$\aleph_1 \leq |A| = \lambda < \|M\| \text{ and } q \text{ not realized in } M.$$

Fix a bijection from A to q denoted by $b \mapsto \varphi_b(x)$.

Now define the following relations:

$$\begin{aligned} P &:= A, \\ R &:= \{(b,c) \in A \times A \mid c \text{ occurs in } \varphi_b(x)\}, \\ S &:= \{(b,c) \in A \times A \mid (M,a)_{a\in A} \models \varphi_b[c]\}. \end{aligned}$$

Let $\mathfrak{B} := \langle M, P, R, S\rangle$, i.e. it is an expansion of M to $\mathrm{L}(T) \cup \{P, R, S\}$.

For every $\varphi(x) \in q$ let

$$\psi_\varphi(y) := \forall z[S(y,z) \leftrightarrow \varphi(z)].$$

we have

$$\mathfrak{B} \models \psi_{\varphi_b}[b] \text{ for all } b \in A.$$

As q is finitely satisfiable in M we get that for every $n < \omega$

$$(*) \quad \mathfrak{B} \models \forall y_0 \cdots \forall y_{n-1}[\bigwedge_{i<n} P(y_i) \to \exists x[\bigwedge_{i<n} S(y_i, x)]].$$

Since q is not realized in M we have

$$(**) \quad \mathfrak{B} \models \neg\exists x \forall y[P(y) \to S(y,x)].$$

By Keisler's two cardinal theorem there are $\mathfrak{B}_0 \prec \mathfrak{B}$ countable and $\mathfrak{B}_1 \succ \mathfrak{B}_0$ of cardinality $\aleph_1$ such that $P^{\mathfrak{B}_0} = P^{\mathfrak{B}_1}$.

Let $A_0 := A \cap P^{\mathfrak{B}_0}$ and $N_\ell := \mathfrak{B}_\ell \restriction \mathrm{L}(M)$.

By the definition of the formulas ψ_φ we get that the relation S codes satisfaction of φ_b. Namely, as $\mathfrak{B}_\ell \models \psi_{\varphi_b}[b]$ holds for all $b \in A_0$ we get

$$(***) \quad \mathfrak{B}_\ell \models \forall z[S(b,z) \leftrightarrow \varphi_b(z)] \text{ for all } b \in A_0.$$

As $\mathfrak{B}_0 \prec \mathfrak{B}$ using $(***)$ we get that $\langle N_0, a\rangle_{a\in A_0} \prec \langle M, a\rangle_{a\in A_0}$.

Since every $\varphi \in q$ contains finitely many parameters from the set A we get that the set $\{a \in M \mid (b,a) \in R\}$ is finite, since $\langle N_0, a\rangle_{a\in A_0} \prec \langle M, a\rangle_{a\in A_0}$ for $b \in P^{\mathfrak{B}_0}$ the set is equal to $\{a \in N_0 \mid (b,a) \in R\}$, thus in particular a subset of A_0. This establishes that

$$b \in P^{N_0} \implies \varphi_b(x) \text{ is a formula in } \mathrm{L}(M) \cup A_0.$$

Let $q_0 := \{\varphi_b(x) \in q \mid b \in A_0\}$. By $(*)$ q_0 is an N_0-type over A_0.

By $(**)$ the type q is not realized in N_1.

Thus N_1 is a model of cardinality $\aleph_1$ not realizing q_0 which is over the countable set A_0. This is a contradiction to the assumption that every model of cardinality $\aleph_1$ is saturated. □

PROOF OF 12.2. Suppose T is unstable in μ. Let A be such that

$$|A| = \mu \text{ and } |\,\mathrm{S}(A)| > \mu.$$

Pick $\Gamma \subseteq \mathrm{S}(A)$ of cardinality μ^+.

Let M be a model of cardinality μ^+ containing A and realizing all the types from Γ.

Fix a bijection from A onto $\{\varphi(x) \mid \varphi \in \mathrm{Fml}(\mathrm{L}(\langle M, a\rangle_{a\in A}))\}$ denoted by $b \mapsto \varphi_b(x)$.

Let P, R and S be defined as in the previous proof. Fix $Q \subseteq |M|$ of cardinality μ^+, such that $a_1 \neq a_2 \in Q \implies \mathrm{tp}(a_0/A, M) \neq \mathrm{tp}(a_2/A, M)$ and an injection $F : |M| \to Q$.

Consider

$$\mathfrak{B} := \langle M, P, Q, R, S, F\rangle.$$

Notice that by the choice of Q we have

$$(\#) \quad \begin{aligned} \mathfrak{B} \models &\forall x_1 \forall x_2 [Q(x_1) \wedge Q(x_2) \wedge x_1 \neq x_2 \to \\ &\exists y [P(y) \wedge \neg [S(y, x_1) \leftrightarrow S(y, x_2)]]]. \end{aligned}$$

By Keisler's theorem there are a countable $\mathfrak{B}_0 \prec \mathfrak{B}$ and $\mathfrak{B}_1 \succ \mathfrak{B}_0$ of cardinality $\aleph_1$ such that $P^{\mathfrak{B}_0} = P^{\mathfrak{B}_1}$. Since F is injective we get that $Q^1 := Q^{\mathfrak{B}_1}$ has cardinality $\aleph_1$. Let $A_0 := P^{\mathfrak{B}_0}$. Using he relation R we get that for every $b \in A_0$ the formula $\varphi_b(x)$ has all its parameters from A_0

By the definition of the formulas ψ_φ we get that the relation S codes satisfaction of φ_b. Namely, as $\mathfrak{B}_1 \models \psi_{\varphi_b}[b]$ holds for all $b \in A_0$ we get

$$\mathfrak{B}_1 \models \forall z [S(b, z) \leftrightarrow \varphi_b(z)] \text{ for all } b \in A_0.$$

Now by $(\#)$ for $a_1 \neq a_2 \in Q^1 \implies \mathrm{tp}(a_1/A_0, N_1) \neq \mathrm{tp}(a_2/A_0, N_1)$. Since A_0 is countable and Q^1 uncountable we contradicted $\aleph_0$-stability. □

The following result was established in Section 5 using the rank function, now I will give a rank-free and more elementary argument to

THEOREM 12.4 (existence of primary models). *Let T be a complete $\aleph_0$-stable. If $M \models T$ then for every subset A of $|M|$ the theory of $\langle M, a\rangle_{\mathbf{a}\in A}$ is atomic thus there exists a primary model $N \prec M$ over A.*

PROOF. First we use directly $\aleph_0$-stability to establish the density of isolated types, i.e. Lemma 5.5:

Let $\mathbf{a} \in A$ and $\varphi(\mathbf{x};\mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))$ be given such that $M \models \exists \mathbf{x} \varphi(\mathbf{x};\mathbf{a})$.

For the sake of contradiction suppose there are no $\psi(\mathbf{x};\mathbf{b})$ and $p \in \mathrm{S}(A) \supseteq \{\varphi(\mathbf{x};\mathbf{a})\}$ such that $\psi(\mathbf{x};\mathbf{b})$ isolates p.

For every $\eta \in {}^{\omega>}2$ there exists $\psi_\eta(\mathbf{x};\mathbf{b}_\eta)$ with parameters from A such that

$$M \models \exists \mathbf{x}[\bigwedge_{i<\ell(\eta)} \psi^{\eta[i]}_{\eta\restriction i}(\mathbf{x};\mathbf{b}_{\eta\restriction i}) \wedge \varphi(\mathbf{x};\mathbf{a})].$$

Now let $B := \{\cup \mathbf{b}_\eta \mid \eta \in {}^{\omega>}2\}$ and for $\eta \in {}^{\omega>}2$ let

$$p_\eta(\mathbf{x}) := \{\psi^{\eta[n]}_{\eta\restriction n}(\mathbf{x};\mathbf{b}_{\eta\restriction n}) \mid n < \omega\}.$$

For $\eta \in {}^{\omega}2$ let $q_\eta \in \mathrm{S}(A) \supseteq p_\eta$. Since $\eta \neq \nu \in {}^{\omega}2 \implies q_\eta \neq q_\nu$ we contradicted stability in $\aleph_0$.

This takes care of a proof Lemma 5.5.

To find an isolating sequence $\{a_i \mid i < \alpha\} \subseteq |M|$ such that $N := A \cup \{a_i \mid i < \alpha\}$ is a primary model over A, we follow the argument from the proof of Theorem 5.9. Set $\lambda := \aleph_0 + |A|$ and $\alpha := \lambda \cdot \omega$.

When selecting a_i we are more careful. If $p_i \in \mathrm{S}(A_i)$ is isolated by $\psi_i(x;\mathbf{b}_i)$ since $M \models \exists x \psi_i(x;\mathbf{b}_i)$ pick $a_i \in |M|$ witnessing this. □

The other important fact from section 4 that used the rank was the proof of Theorem **??**.

PROOF OF THEOREM **??**. Let A, I be given such that $|A| + \aleph_0 \leq \lambda < |\mathbf{I}|$.

CLAIM 12.5. *Let A be a set and $\mathbf{a}$ a finite sequence of elements from A. Suppose $\varphi(\mathbf{x};\mathbf{a})$ is such that $\{\mathbf{b} \in \mathbf{I} \ : \models \varphi[\mathbf{b};\mathbf{a}]\}$ has cardinality $> \lambda$ then there exists a formula $\varphi^*(\mathbf{x};\mathbf{b})$ with parameters from A such that*

$|\varphi^*(\mathbf{I};\mathbf{a}^*)| > \lambda$,

$\varphi^*(\mathbf{x};\mathbf{a}^*) \vdash \varphi(\mathbf{x};\mathbf{a})$ *and*

for every $\psi(\mathbf{x};\mathbf{c})$ either

$|\varphi^*(\mathbf{I};\mathbf{a}^*) \cap \psi(\mathbf{I};\mathbf{b})| \leq \aleph_0$ *or* $|\varphi^*(\mathbf{I};\mathbf{a}^*) \cap \neg\psi(\mathbf{I};\mathbf{b})| \leq \aleph_0$

we call φ^* as above *minimal for* $\mathbf{I}$

PROOF. Otherwise, T is not $\aleph_0$-stable. □

CLAIM 12.6. *Let $B \supseteq \mathbf{a}$ if $p_1, p_2 \in \mathrm{S}(B)$ both realized by infinitely many elements of $\mathbf{I}$ and both extends $\varphi^*(\mathbf{x}; \mathbf{a})$ then $p_1 = p_2$.*

PROOF. If the types are different, since they are complete over B there exists $\psi(\mathbf{x}; \mathbf{b}) \in p_1$ such that $\neg\psi(\mathbf{x}; \mathbf{b}) \in p_2$. Since both types are not algebraic we got that the sets $\varphi^*(\mathbf{I}; \mathbf{a}^*) \cap \psi(\mathbf{I}; \mathbf{b})$ and $\varphi^*(\mathbf{I}; \mathbf{a}^*) \cap \neg\psi(\mathbf{I}; \mathbf{b})$ are both infinite in contradiction to minimality of φ^*. □

CLAIM 12.7. *Suppose $p \in \mathrm{S}(B) \supseteq \{\varphi^*(\mathbf{x}; \mathbf{a}^*)\}$ and p is realize by λ^+-many elements of $\mathbf{I}$ then p does not split over $\mathbf{a}$. ?* ?!

PROOF. Otherwise there are $\psi(\mathbf{x}; \mathbf{y})$ and $\mathbf{b}, \mathbf{c} \in B$ such that $\psi(\mathbf{x}; \mathbf{b}) \in p$, $\neg\psi(\mathbf{x}; \mathbf{c}) \in p$ and $\mathrm{tp}(\mathbf{b}/ \cup \mathbf{a}) = \mathrm{tp}(\mathbf{c}/ \cup \mathbf{a})$.

Let $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(\mathbf{b}) = \mathbf{c}$ and $f(\mathbf{a}) = \mathbf{a}$.

We get that $\psi(x; f(\mathbf{b})) \in f(p)$, namely $\psi(\mathbf{x}; \mathbf{c}) \in f(p)$ □

For every $\mathbf{b} \in \mathbf{I}$ let $\varphi_{\mathbf{b}}(\mathbf{x}; \mathbf{a}_{\mathbf{b}})$ be a formula with parameters from A be such that $\models \varphi[\mathbf{b}; \mathbf{a}_{\mathbf{b}}]$.

By the pigeonhole principle we may assume that $|\mathbf{I}| = \lambda^+$ and for every $\mathbf{b} \in \mathbf{I}$ we have that $\varphi_{\mathbf{b}}(\mathbf{x}; \mathbf{a}_{\mathbf{b}}) = \varphi(\mathbf{x}; \mathbf{a})$

By the Claim let φ^* be minimal for $\mathbf{I}$ such that $\varphi^* \vdash \varphi$.

By induction on $i < \lambda^+$ define $\mathbf{b}_i \in \mathbf{I}$ such that $\mathrm{tp}(\mathbf{b}_i/A_i)$ is realized by λ^+-many elements of $\mathbf{I}$, where $A_i := A \cup \{\mathbf{b}_j \mid j < i\}$.

This is possible by λ^+-stability. □

****

PROOF OF 12.2. Suppose T is unstable in μ. Let A be such that

$$|A| = \mu \text{ and } |\mathrm{S}(A)| > \mu.$$

Pick $\Gamma \subseteq \mathrm{S}(A)$ of cardinality μ^+. Let χ be a regular sufficiently large so that

$$A, \mathrm{S}(A), \Gamma, T, \mathrm{L}(T) \in H(\chi)$$

and

$$\langle H(\chi), \in \rangle \models \text{“}$$

Now take

$$\mathfrak{B}^* := \langle H(\chi), \in, \Gamma, A, T, \mathrm{L}(T) \rangle.$$

Now take $\mathfrak{B} \preceq \mathfrak{B}^*$ of cardinality μ^+ containing the above sets. □

We need to verify:

PROOF OF 12.3. □

PROOF OF 12.4. □

PROOF. □

13. Morely's rank and the local rank

LEMMA 13.1 (Ultrametric Property). *For every p, Δ, and formulas $\{ \psi_\ell(\mathbf{x}; \mathbf{b}_\ell) \mid \ell < n \}$, we have*

$$\mathrm{R}[\, p \cup \bigvee_{\ell<n} \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta] = \max_{\ell<n} \mathrm{R}[\, p \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta].$$

PROOF. By Monotonicity, for every $\ell < n$ we have

$$\mathrm{R}[\, p \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta] \leq \mathrm{R}[\, p \cup \bigvee_{\ell<n} \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta].$$

Hence,

$$\max_{\ell<n} \mathrm{R}[\, p \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta] \leq \mathrm{R}[\, p \cup \bigvee_{\ell<n} \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta].$$

To prove the reverse inequality, we show by induction that for every ordinal α and every type p,

$$\mathrm{R}[\, p \cup \bigvee_{\ell<n} \psi_l(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta] \geq \alpha \quad \text{implies} \quad \max_{\ell<n} \mathrm{R}[\, p \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta] \geq \alpha.$$

When $\alpha = 0$ or α is a limit ordinal, the implication is easy. For the successor stage suppose, for contradiction, that

$$(*)\ \mathrm{R}[\, p \cup \bigvee_{\ell<n} \psi_l(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta] \geq \alpha+1, \quad \text{but} \quad \max_{\ell<n} \mathrm{R}[\, p \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta] \not\geq \alpha+1.$$

Then, for every $l < n$ we have $\mathrm{R}[\, p \cup \psi_l(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta] \leq \alpha$. Choose a finite $r_\ell \subseteq p$ such that

$$\mathrm{R}[\, r_\ell \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\, , \Delta] \leq \alpha$$

and let $r := \bigcup_{\ell<n} r_\ell$. Then $r \subseteq p$ is finite, so, by (*) and the definition of the rank, there exist $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ and $\{\mathbf{a}_i \mid i < \omega\}$ with $\ell(\mathbf{a}_i) = \ell(\mathbf{y})$ such that $\{\varphi(\mathbf{x}; \mathbf{a}_i) \mid i < \omega\}$ is k-contradictory and for every $i < \omega$

$$\mathrm{R}[\, r \cup \bigvee_{\ell<n} \psi(\mathbf{x}; \mathbf{b}_\ell) \cup \varphi(\mathbf{x}; \mathbf{a}_i)\, , \Delta] \geq \alpha.$$

By induction hypothesis, for every $i < \omega$

$$\max_{\ell<n} \mathrm{R}[\, r \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell) \cup \varphi(\mathbf{x}; \mathbf{a}_i)\, \Delta] \geq \alpha,$$

so there exists $\ell_i < n$ such that

$$\mathrm{R}[\, r \cup \psi_{\ell_i}(\mathbf{x}; \mathbf{b}_{\ell_i}) \cup \varphi(\mathbf{x}; \mathbf{a}_i)\, , \Delta] \geq \alpha.$$

By the pigeonhole principle, we may assume that $\ell_i = \ell^* < n$ is fixed for all $i < \omega$ and

$$\mathrm{R}[\, r \cup \psi_{\ell^*}(\mathbf{x}; \mathbf{b}_{\ell^*}) \cup \varphi(\mathbf{x}; \mathbf{a}_i)\, , \Delta] \geq \alpha,$$

for every $i < \omega$. By definition of the rank,

$$\mathrm{R}[\, r \cup \psi_{\ell^*}(\mathbf{x}; \mathbf{b}_{\ell^*})\, , \Delta] \geq \alpha + 1,$$

and by Monotonicity,

$$\mathrm{R}[\, r_{\ell^*} \cup \psi_{\ell^*}(\mathbf{x}; \mathbf{b}_{\ell^*})\,, \Delta] \geq \alpha + 1.$$

But this contradicts the choice of r_{ℓ^*}. Therefore,

$$\max_{\ell<n} \mathrm{R}[\, p \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell)\,, \Delta] \geq \alpha + 1.$$

□

The proof of the following lemma is similar to that of Theorem 1.11.

LEMMA 13.2 (Extension Property). *Let p be a type, Δ and Φ be sets of formulas and $k < \omega$. Suppose that $\mathrm{R}[\, p, \Delta] < \infty$. Then, for every set B there exists a type $q \in S_\Phi(B)$ such that*

$$\mathrm{R}[\, p, \Delta] = \mathrm{R}[\, p \cup q, \Delta].$$

PROOF. We may assume that Φ is closed under conjunction. Suppose that $\mathrm{R}[p, \Delta] = \alpha$. Consider

$$\Gamma := \{\neg\psi(\mathbf{x}; \mathbf{b}) \mid \mathbf{b} \in B, \psi(\mathbf{x}; \mathbf{y}) \in \Phi,\ \mathrm{R}[p \cup \psi(\mathbf{x}; \mathbf{b}), \Delta] < \alpha\}.$$

Let us show that $p \cup \Gamma$ is consistent. If $p \cup \Gamma$ were inconsistent, there would be $\{\neg\psi_i(\mathbf{x}; \mathbf{b}_i) \mid i < n\} \subseteq \Gamma$ such that $p \cup \{\neg\psi_i(\mathbf{x}; \mathbf{b}_i) \mid i < n\}$ is inconsistent. But then, $p \vdash \bigvee_{i<n} \psi_i(\mathbf{x}; \mathbf{b}_i)$. By Monotonicity and Lemma 2.39, we have

$$\alpha = \mathrm{R}[p, \Delta] \leq \mathrm{R}[p \cup \bigvee_{i<n} \psi_i(\mathbf{x}; \mathbf{b}_i), \Delta] = \max_{i<n} \mathrm{R}[\, p \cup \psi_i(\mathbf{x}; \mathbf{b}_i)\,, \Delta] < \alpha,$$

which is, of course, a contradiction.

Choose $q \in S_\Phi(B)$ extending Γ. If $\mathrm{R}[p \cup q, \Delta] < \alpha$, then by Finite Character, there exists $\psi(\mathbf{x}; \mathbf{b}) \in q$ such that $\mathrm{R}[p \cup \psi(\mathbf{x}; \mathbf{b}), \Delta] < \alpha$. Hence, by definition of Γ, we must have $\neg\psi(\mathbf{x}; \mathbf{b}) \in \Gamma \subseteq q$, which is a contradiction. □

14. Some properties of $\aleph_0$-stable theories

LEMMA 14.1. *Suppose T is an $\aleph_0$-stable. Let $\mathbf{I}$ be an infinite indiscernible sequence over a set B. For any finite $\mathbf{a}$ there exists a finite $\mathbf{J_a} \subseteq \mathbf{I}$ such that $\mathbf{I} - \mathbf{J_a}$ is an indiscernible sequence over $B \cup \mathbf{a}$*

THEOREM 14.2. *Let T be an $\aleph_0$-stable theory. If $\{M_n \mid n < \omega\}$ an increasing chain of saturated models then $\bigcup_{n<\omega} M_n$ is saturated.*

PROOF. □

CHAPTER 8

Basics of Stability

Introduction

In this chapter we introduce several important properties of first-order theories and characterizations of them.

One of the most important theorems in the theory is the equivalence of instability with the order-property (see Theorem 5.11 below).

We discuss several rank functions and theorems that are analogous to the characterization of $\aleph_0$-stability in terms of $\mathrm{R}[\mathbf{x} = \mathbf{x}] < \infty$ (Theorem 2.12) enables us to characterize stability, super-stability and simple theories.

In the early days much of the focus was

DEFINITION 0.1 (The stability function). For $n < \omega$ and $\mu \geq |T|$ let

$$f^n_T(\mu) := \sup\{|\,\mathrm{S}^n(M)| \ : \ M \models T, \|M\| = \mu\}.$$

$$f_T(\mu) := \sup_{n<\omega} f^n_T(\mu).$$

EXERCISE 0.2. $f_T(\mu) = f^n_T(\mu) = f^1_T(\mu)$ for all $n < \omega$.

It is also shown that if there is a regular $\lambda > |T|$ such that $\mathrm{I}(\lambda, T) < 2^\lambda$ then T must be superstable.

1. Local Types

In this chapter like in other parts of the book we work with a complete T and by Remark 2.2.26 we may assume that there is a monster model $\mathfrak{C}$ for T. Whenever we will mention models they will be assumed to be elementary submodels of $\mathfrak{C}$ and all sets and elements are assumed to come from $\mathfrak{C}$, unless stated differently.

Few theorems are about a given fixed model M and do not depend on the first-order theory of M.

It turns out that it is important to study not only whether a theory is stable or not but also to look at local version of the notion of stability. We need few definitions.

DEFINITION 1.1. Let M be a structure, Let Δ be a set of formulas from $\mathrm{L}(M)$, $A \subseteq |M|$ and $\mathbf{b} \in |M|$

$$\begin{aligned}\mathrm{tp}_\Delta(\mathbf{b}/A, M) :=& \{\varphi(\mathbf{x};\mathbf{a}) \ : \ \varphi(\mathbf{x};\mathbf{y}) \in \Delta, \ \mathbf{a} \in A, \ M \models \varphi[\mathbf{b};\mathbf{a}]\} \cup \\ & \{\neg\varphi(\mathbf{x};\mathbf{a}) \ : \ \varphi(\mathbf{x};\mathbf{y}) \in \Delta, \ \mathbf{a} \in A, \ M \models \neg\varphi[\mathbf{b};\mathbf{a}]\}.\end{aligned}$$

When $M = \mathfrak{C}$ we omit the parameter M and in case $\Delta = \mathrm{L}(M)$ we omit the Δ.

The last definition is a generalization of the a definition of a type we had in Definition 3.2.28, this permits us to consider local behavior at a formula. Using it we can define *local stability*.

DEFINITION 1.2. Let M be a structure and $\Delta \subseteq \mathrm{Fml}(\mathrm{L}(M))$. Let T be a fixed complete first-order theory and $\mathfrak{C}$ its monster model.

(1) Let A be a subset of $\mathfrak{C}$ and m a positive integer.

$$\mathrm{S}^m_\Delta(A, M) := \{\mathrm{tp}_\Delta(\mathbf{b}/A, M) \ : \ \mathbf{b} \in {}^m|M|\}.$$

Write $\mathrm{S}_\Delta(A, M)$ for $S^1_\Delta(A, M)$; Write $\mathrm{S}^m(A, M)$ for $\mathrm{S}^m_{\mathrm{L}(T)}(A, M)$; Write $\mathrm{S}^m_\Delta(A)$ for $\mathrm{S}^m_\Delta(A, \mathfrak{C})$. So $\mathrm{S}(A)$ stands for set of complete the 1-types in the language of T with parameters from A.

(2) The special case where $\Delta = \{\varphi(\mathbf{x};\mathbf{y})\}$ is important. Instead of writing $\mathrm{tp}_\Delta(\mathbf{b}/A, M)$ we will write $\mathrm{tp}_\varphi(\mathbf{b}/A, M)$ and if $p \in \mathrm{S}^m_\varphi(A)$ we will often write that p is a *φ-type over A* .

(3) Let λ be a cardinality. We say that M *is Δ-stable in λ* or is *(Δ, λ)-stable* iff for every positive $m < \omega$ and every $A \subset |M|$

$$|A| \leq \lambda \quad \text{implies} \quad |S^m_\Delta(A, M)| \leq \lambda.$$

When $M = \mathfrak{C}$ this is a property of the theory T, when Δ is the entire set of formulas of $\mathrm{L}(T)$ we omit the parameter Δ.

(4) T is *superstable* iff there exists a cardinality μ such that for every $\lambda \geq \mu$ the theory T is stable in λ.

In the next proposition it is shown that in Definition 1.2(2) we may assume that the integer $m = 1$.

PROPOSITION 1.3. *Let λ be an infinite cardinality and let Δ be a set of formulas. Suppose that for every $A \subseteq \mathfrak{C}$ we have that*

$$|A| \leq \lambda \quad \textit{implies} \quad |\mathrm{S}^1_\Delta(A, M)| \leq \lambda.$$

Then T is (Δ, λ)-stable.

PROOF. We have to show that for every $m < \omega$ and every $A \subseteq \mathfrak{C}$ the following holds

$$(*)_m \quad |A| \leq \lambda \quad \text{implies} \quad |\mathrm{S}^m_\Delta(A, M)| \leq \lambda.$$

For the sake of contradiction suppose that it is false and let m_0 be the minimal integer such that $(*)_{m_0}$ fails. Let A be a set of cardinality $\leq \lambda$ that witness this. Namely we have $|\mathrm{S}^{m_0}_\Delta(A, M)| > \lambda$ and $|\mathrm{S}^{m_0-1}_\Delta(A, M)| \leq \lambda$. By the hypothesis $m_0 \geq 2$.

For every $p \in \mathrm{S}^{m_0}_\Delta(A, M)$ let

$$p' := \{\exists x_{m_0-1} \varphi(x_0, \ldots, x_{m_0-1}) \ : \ \varphi(x_0, \ldots, x_{m_0-1}) \in p\}.$$

For every p denote by p^* an extension of p' to the set $S^{m_0-1}_\Delta(A, M)$. Consider now the mapping

$$p \mapsto p^*.$$

By the assumption the domain of this mapping is a set of cardinality at least λ^+ while the range has cardinality at most λ. Thus by the pigeon hole principle there exists $S \subseteq \mathrm{S}^{m_0}_\Delta(A, M)$ of cardinality λ^+ and there is $q^* \in \mathrm{S}^{m_0-1}_\Delta(A, M)$ such that $p \in S \Longrightarrow p^* = q^*$.

Let $\mathbf{b} \in {}^{m_0-1}\mathfrak{C}$ be a realization of q^*. Let $S^* := \{p(\mathbf{b}, x_{m_0-1}) \ : \ p \in S\}$. Namely S^* is the set of types of S where for the first $m_0 - 1$ variables we substitute the sequence $\mathbf{b}$. Since S has cardinality λ^+ also S^* has cardinality λ^+ but clearly $S^* \subseteq \mathrm{S}^1_\Delta(A \cup \mathbf{b})$ which contradicts the assumption of the proposition. □

While considering specific examples in classification theory can be misleading I will present three explicit examples:

EXAMPLES 1.4.

(1) Let $M := \langle {}^\omega 2, E_n \rangle_{n<\omega}$. When the binary relations E_n are defined as follows:

$$\eta E_n \nu \overset{def}{\iff} \eta \restriction n = \nu \restriction n \quad \text{for all } \eta, \nu \in {}^\omega 2\,.$$

The theory $\mathrm{Th}(M)$ is superstable but not $\aleph_0$-stable.

(2) Let $N := \langle {}^{\omega}\omega, E_n\rangle_{n<\omega}$. When the binary relations E_n are defined as follows:

$$\eta E_n \nu \overset{def}{\iff} \eta \restriction n = \nu \restriction n \quad \text{for all } \eta, \nu \in {}^{\omega}\omega\,.$$

The theory $\mathrm{Th}(N)$ is stable but not superstable. The simplest way to show that this theory is not superstable is to note that for every $\mu \geq \aleph_0$ the cardinal $\lambda := \beth_\omega(\mu)$ satisfies $\lambda^{\aleph_0} > \lambda$ and there is a set A (in the monster model) of cardinality λ such that $|\,\mathrm{S}^1(A)| = \lambda^{\aleph_0}$.

(3) Let R be a commutative ring and M a left R module. let L_M be the language of r-modules consisting of the binary operation $+$ and for every $r \in R$ a unary function symbol (standing for scalar multiplication by r).

FACT 1.5 (See M. Prest's book [**M**]). *The theory of M is stable.*

(4) There are two radically different examples of unstable theories. They are T_{ind} and T_{ord}.

In what follows we will need also a localized version of the concept of splitting of types.

DEFINITION 1.6. Let Δ_1 and Δ_2 be sets of formulas and let A be a subset of the monster model. A type p is said to (Δ_1, Δ_2)-*splits over* A iff there are $\mathbf{a}, \mathbf{b} \in \mathrm{Dom}\, p$ and there is a formula $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta_2$ such

$$\mathrm{tp}_{\Delta_1}(\mathbf{a}/A) = \mathrm{tp}_{\Delta_1}(\mathbf{b}/A) \quad \text{such that} \quad \varphi(\mathbf{x}; \mathbf{a}) \in p \quad \text{and } \neg\varphi(\mathbf{x}; \mathbf{b}) \in p.$$

Notice that when $\Delta_1 = \Delta_2 = \mathrm{Fml}(L)$ then the above coincides with the concept introduced in Definition 4.3.

Similarly to λ-saturation what can introduce the following local version:

DEFINITION 1.7. Let M be an a given structure and $\Delta \subseteq \mathrm{Fml}(\mathrm{L}(M))$ and λ an infinite cardinal. We say that M is (Δ, λ)-*saturated* iff for every $A \subseteq |M|$ of cardinality $< \lambda$ for every

$$p \subseteq \{\varphi(\mathbf{x}; \mathbf{a}) \mid \varphi(\mathbf{x}; \mathbf{y}) \in \Delta, \mathbf{a} \in A\}$$

if for every finite $q \subseteq p$ there is $\mathbf{b} \in |M|$ such that $M \models \bigwedge q[\mathbf{b}]$ then p is realized in M.

M is called λ-*locally saturated* iff M is (λ, Δ)-saturated for every finite $\Delta \subseteq \mathrm{Fml}(\mathrm{L}(M))$.

N is called *locally saturated* iff M is $\|M\|$-locally saturated.

Clearly every saturated model is locally saturated. It turns out that it is easier to construct locally saturated models than saturated. In the next section we will prove the following characterization:

THEOREM 1.8. *Let T be a complete first-order theory. T is stable iff for every $\lambda \geq |\mathrm{L}(T)| + \aleph_0$ there exists $M \models T$ locally saturated of cardinality λ.*

In the next result we show that when Δ is finite then there exists a formula ψ_Δ with more variables such that Δ-types can be coded by $\{\psi\}$-types. Namely every Δ-type is semantically equivalent to a ψ_Δ-type.

THEOREM 1.9. *Suppose $\Delta = \{\varphi_k(\mathbf{x}; \mathbf{y}_k) \mid k < n\}$ and A is a set with at least two elements. Let $\psi_\Delta(\mathbf{x}; \mathbf{y}_0, \dots, \mathbf{y}_{n-1}; z, z_0, \dots, z_{2n-1})$ be the following formula*

$$\bigwedge_{i<j<2n} z_i \neq z_j \wedge \bigvee_{i<2n} z = z_i \wedge \bigwedge_{i<n} [z = z_i \to \varphi(\mathbf{x}; \mathbf{y}_i)] \wedge \bigwedge_{n \leq i < 2n} [z = z_i \to \neg\varphi(\mathbf{x}; \mathbf{y}_{i-n})].$$

If A has at least $2n$-many elements then for every Δ-type p there exists a $\{\psi_\Delta\}$-type q such that $p[\mathfrak{C}] = q[\mathfrak{C}]$.

PROOF. Immediate. □

Exercises 8.1.

(1)
(2) The purpose of this set of exercises is to establish that similarly to XXX local types and local splitting also relate to a Galois group.
   (a) Let M be an L-structure, $A \subseteq |M|$ and Δ a set of L-formulas.

   $$\mathrm{Aut}_A^\Delta(M) := \{f : |M| \to |M| \ : \ f \text{ is a bijection and}$$
   $$M \models \varphi[\mathbf{a}] \iff M \models \varphi[f(\mathbf{a})] \text{ for every } \varphi(\mathbf{x}) \in \Delta\}.$$

   Verify that $\mathrm{Aut}_A^\Delta(M)$ is a subgroup of the permutation group of the set $|M|$.
   (b)
   (c) Let $A \subseteq |M|$. If M is $(|A| + \aleph_0)^+$-saturated then f
   (d)
(3)
(4)

2. Infinitely many Rank Functions

In this section we introduce several more rank functions. The first is a common generalization for the functions $\mathrm{R}[p]$ (Definition 2.2.16) and $\mathrm{RM}[p]$ from Definition 3.7. In an analogous way to Theorem 2.12 we will use the new rank function to characterize the classes of stable and superstable theories.

In Definition 2.32 below we introduce another rank that will be useful to analyze superstable theories and its relatives will be used to define the class of simple theories.

Recall that a type p is a Δ-type if for every $\varphi(\mathbf{x};\mathbf{a}) \in p$ we have that either $\varphi(\mathbf{x};\mathbf{y}) \in \Delta$ or $\neg\varphi(\mathbf{x};\mathbf{y}) \in \Delta$. Two Δ-type p and q are said to be *explicitly contradictory* if there exist $\varphi(\mathbf{x};\mathbf{y}) \in \Delta$ and $\mathbf{a} \in {}^{\ell(\mathbf{y})}\mathfrak{C}$ such that $\varphi(\mathbf{x};\mathbf{a}) \in p$ and $\neg\varphi(\mathbf{x};\mathbf{a}) \in q$ or vice versa.

The following is useful

LEMMA 2.1. *For every finite set of formulas Δ there exists a formula ψ_Δ such that for any Δ-type q there exists a $\{\psi_\Delta\}$-type q' such that $q[\mathfrak{C}] = q'[\mathfrak{C}]$.*

PROOF. Suppose $\Delta = \{\varphi_i(\mathbf{x};\mathbf{y}_i) \mid i < n\}$.

The formula $\psi_\Delta(\mathbf{x};\mathbf{y}_0,\dots,\mathbf{y}_{n-1};z;z_0,\dots,z_{2n-1})$ to be the conjunction of

$$\bigwedge_{i<j<2n} [z_i \neq z_j] \wedge \bigvee_{i<2n} [z = z_i]$$

with

$$\bigwedge_{i<n} [z = z_i \to \varphi_i(\mathbf{x};\mathbf{y}_i)] \wedge \bigwedge_{i<n} [z = z_{n+i} \to \neg\varphi_i(\mathbf{x};\mathbf{y}_i)].$$

Note that by the construction of ψ_Δ for any $\varphi(\mathbf{x};\mathbf{a}) \in q$ there are $\mathbf{a}^*$, b and $\mathbf{b}$ such that

$$\mathfrak{C} \models \forall\mathbf{x}[\varphi(\mathbf{x};\mathbf{a}) \leftrightarrow \psi_\Delta(\mathbf{x};\mathbf{a}^*;b;\mathbf{b})].$$

□

DEFINITION 2.2. Let p be a set of formulas in $\mathbf{x}$, $\Delta \subseteq \{\varphi(\mathbf{x};\mathbf{y}) : \varphi \in \mathrm{Fml}(\mathrm{L}(T))\}$ and λ a cardinal (can be finite) or ∞.

(1) $\mathrm{R}[p,\Delta,\lambda] \geq \alpha$ is defined by induction on α:
   (a) $\mathrm{R}[p,\Delta,\lambda] \geq 0$ iff p is consistent,
   (b) for α limit, $\mathrm{R}[p,\Delta,\lambda] \geq \alpha$ iff $\mathrm{R}[p,\Delta,\lambda] \geq \beta$ for every $\beta < \alpha$ and
   (c) $\mathrm{R}[p,\Delta,\lambda] \geq \alpha+1$ iff for every finite $q \subseteq p$ and every $\mu < \lambda$ there are $\{q_i : i \leq \mu\}$ *explicitly contradictory Δ-types* such that $\mathrm{R}[p \cup q_i,\Delta,\lambda] \geq \alpha$ for every $i \leq \mu$. (When $\lambda = \infty$ we interpret it as no restriction on μ).

(2) For an ordinal α we denote by $\mathrm{R}[p, \Delta, \lambda] = \alpha$ the statement

$$\mathrm{R}[p, \Delta, \lambda] \geq \alpha \quad \text{and} \quad \mathrm{R}[p, \Delta, \lambda] \not\geq \alpha + 1.$$

(3) If $\mathrm{R}[p, \Delta, \lambda] \geq \alpha$ for every ordinal α this is denoted by $\mathrm{R}[p, \Delta, \lambda] = \infty$.
(4) $\mathrm{R}[p, \Delta, \infty] \geq \alpha$ stands for $\mathrm{R}[p, \Delta, \mu] \geq \alpha$ holds for every cardinal μ.
(5) in the case $\mathrm{R}[p, \Delta, \mu] < \infty$ we write $\mathrm{R}[p, \Delta, \mu] = \alpha$ for the statement "$\mathrm{R}[p, \Delta, \mu] \geq \alpha$ and it is false that $\mathrm{R}[p, \Delta, \mu] \geq \alpha + 1$".
(6) $\mathrm{R}[p, \Delta, \infty] = \alpha$ stands for there exists a cardinal μ_0 such that $\mathrm{R}[p, \Delta, \mu] = \alpha$ holds for every cardinal $\mu \geq \mu_0$.

There are several important values of λ:

EXAMPLE 2.3.

(1) *Morley's rank* $\mathrm{RM}[p]$ is the function $\mathrm{R}[p, L, \aleph_1]$
(2) $\mathrm{R}[p, L, 2]$ is the function $\mathrm{R}[p]$ from Definition 2.2.16.
(3) We will show that $\mathrm{R}[\mathbf{x} = \mathbf{x}, L, \infty] < \infty$ is equivalent to T is superstable.

LEMMA 2.4 (Invariance). *For any set of formulas p and $f \in \mathrm{Aut}(\mathfrak{C})$, $\mathrm{R}[p, \Delta, \lambda] = \mathrm{R}[f(p), \Delta, \lambda]$.*

PROOF. Immediate. □

LEMMA 2.5 (Monotonicity).

(1) *$p \vdash q$ implies $\mathrm{R}[p, \Delta, \lambda] \leq \mathrm{R}[q, \Delta, \lambda]$,*
(2) *$\mu < \lambda$ implies $\mathrm{R}[p, \Delta, \lambda] \leq \mathrm{R}[p, \Delta, \mu]$ and*
(3) *$\Delta_1 \subseteq \Delta_2$ implies that $\mathrm{R}[p, \Delta_1, \lambda] \leq \mathrm{R}[p, \Delta_2, \lambda]$*

PROOF. (1) By induction on α show that $\mathrm{R}[p, \Delta, \lambda] \geq \alpha \Rightarrow \mathrm{R}[q, \Delta, \lambda] \geq \alpha$.

(2), (3) are both immediate from inspecting the definition of the rank. □

COROLLARY 2.6. (1) *If both p and q are sets of formulas with free variables from $\mathbf{x}$ then $p[\mathfrak{C}] = q[\mathfrak{C}]$ implies that $\mathrm{R}[p, \Delta, \lambda] = \mathrm{R}[q, \Delta, \lambda]$.*
(2) *If Δ is a finite set of formulas then there exists a formula ψ_Δ such that $\mathrm{R}[p, \Delta, \lambda] = \mathrm{R}[p, \psi_\Delta, \lambda]$ for every p and every $\lambda \leq \infty$.*

PROOF. (1) Apply monotonicity to $p \vdash q$ and $q \vdash p$.

(2) Use Theorem 1.9. □

Since there are no properly descending sequences of ordinals we get

PROPOSITION 2.7. *If $\alpha := \mathrm{R}[p, \Delta, \infty] < \infty$ then there exists a cardinal λ such that $\mathrm{R}[p, \Delta, \lambda] = \alpha$.*

PROOF. By the second monotonicity property we have that $\lambda \mapsto \mathrm{R}[p, \Delta, \lambda]$ is not increasing. All values of this function (using monotonicity again) are $\geq \alpha$. By the assumption that $\alpha = \mathrm{R}[p, \Delta, \infty] < \infty$ and the definition of the rank we get that there exists a cardinal λ as required. □

PROPOSITION 2.8. *For a countable and complete T the following are equivalent*

(1) *T is $\aleph_0$-stable*
(2) $\mathrm{R}[\mathbf{x} = \mathbf{x}, L, 2] < \omega_1$
(3) $\mathrm{R}[\mathbf{x} = \mathbf{x}, L, 2] < \infty$
(4) $\mathrm{R}[\mathbf{x} = \mathbf{x}, L, \aleph_1] < \omega_1$
(5) $\mathrm{R}[\mathbf{x} = \mathbf{x}, L, \aleph_1] < \infty$.

PROOF. □

Note that conditions $(2) - (5)$ of Proposition 2.8 make sense also for uncountable theories.

The following is a generalization of $\aleph_0$-stability:

DEFINITION 2.9. We say that T is *totally transcendental* iff

$$\mathrm{R}[\mathbf{x} = \mathbf{x}, L, 2] < \infty.$$

LEMMA 2.10 (Finite character). *Given $\Delta \subseteq \{\varphi(\mathbf{x}; \mathbf{y}) \mid \varphi \in \mathrm{Fml}(\mathrm{L}(T))\}$, a cardinal λ, and a set of formulas p, there is a finite subset $q \subseteq p$, such that* $\mathrm{R}[p, \Delta, \lambda] = \mathrm{R}[q, \Delta, \lambda]$.

PROOF. By definition and Lemma 2.5(1). □

As noted above the function $\mathrm{R}^{\mathbf{x}}[p.\Delta, 2]$ is a generalization of the function $\mathrm{R}[p]$ used in the proof of Morley's theorem in Chapter 6. one property it shares with R is the uniqueness of extension.

LEMMA 2.11 (Uniqueness). *Let p be an $\mathbf{x}$-type, Δ a set of formulas and A a set. Suppose $\alpha := \mathrm{R}[p, \Delta, 2] < \infty$. If $q_1, q_2 \in \mathrm{S}^{\mathbf{x}}_{\Delta}(A)$ are such that*

$$\mathrm{R}[p \cup q_\ell, \Delta, 2] = \alpha \text{ for } \ell = 1, 2$$

then $q_1 = q_2$.

PROOF. Otherwise there are $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ and $\mathbf{a} \in A$ such that $\varphi(\mathbf{x}; \mathbf{a}) \in q_1$ and $\neg\varphi(\mathbf{x}; \mathbf{a}) \in q_2$. By the finite character we may assume that p is finite. By monotonicity we get that

$$\mathrm{R}[p \cup \varphi(\mathbf{x}; \mathbf{a}), \Delta, 2] = \mathrm{R}[p \cup \neg\varphi(\mathbf{x}; \mathbf{a}), \Delta, 2] = \alpha.$$

Thus $\mathrm{R}[p, \Delta, 2] \geq \alpha + 1$, this is a contradiction to the assumption on α. □

The following special case of the Lemma is useful to state:

COROLLARY 2.12 (Uniqueness). *Let Δ be a set of formulas and p is a Δ-type such that $\alpha := \mathrm{R}[p, \Delta, 2] < \infty$ and A be a set. If $q_1, q_2 \in \mathrm{S}^{\mathbf{x}}_{\Delta}(A) \supseteq p$ such that $\mathrm{R}[q_1, \Delta, 2] = \mathrm{R}[q_2, \Delta, 2] = \alpha$ then $q_1 = q_2$.*

The last corollary allows us similarly to the proof of Theorem 6.3.3 to get that boundless of this rank implies strong form of stability in all cardinals.

COROLLARY 2.13. *Let Δ be a set of formulas and A. If $\mathrm{R}[\mathbf{x} = \mathbf{x}, \Delta, 2] < \infty$ then $|\mathrm{S}^{\mathbf{x}}_{\Delta}(A)| \leq |A|$ for every infinite A*

PROOF. Suppose A is an infinite set of cardinality λ such that $|\mathrm{S}^{\mathbf{x}}_{\Delta}(A)| > \lambda$. Fix $\{p_i \mid i < \lambda^+\}$ a set of distinct elements of $\mathrm{S}^{\mathbf{x}}_{\Delta}(A)$. By the finite character of the rank for every $i < \lambda^+$ there exists a finite $q_i \subseteq p_i$ such that

$$\alpha_i := \mathrm{R}[q_{p_i}, \Delta, 2] = \mathrm{R}[p_i, \Delta, 2].$$

Consider $f(i) := q_{p_i}$. As the domain of this function is λ^+ and its range finite sets of formulas with parameters from A the cardinality of that range is $\lambda \cdot \aleph_0 = \lambda$. By an application of the pigeon hole principle there exists $S \subseteq \lambda^+$ of cardinality λ^+ and a set of formulas q such that

$$i \in S \implies q_{p_i} = q.$$

Take $i \neq j \in S$. There is $\varphi(\mathbf{x}; \mathbf{a}) \in p_i$ such that $\neg\varphi(\mathbf{x}; \mathbf{a}) \in p_j$. Thus we have

$$q \subseteq q \cup \{\varphi(\mathbf{x}; \mathbf{a})\} \subseteq p_i.$$

Similarly,

$$q \subseteq q \cup \{\neg\varphi(\mathbf{x}; \mathbf{a})\} \subseteq p_j.$$

Using monotonicity of the rank (twice) we get that

$$\mathrm{R}[q \cup \{\varphi(\mathbf{x}; \mathbf{a})\}, \Delta, 2] = \alpha = \mathrm{R}[q \cup \{\neg\varphi(\mathbf{x}; \mathbf{a})\}, \Delta, 2].$$

An application of the definition of R gives that $\mathrm{R}[q, \Delta, 2] \geq \alpha + 1$. □

An important property of the function $\mathrm{R}[p, \Delta, \lambda]$ for infinite λ is:

THEOREM 2.14. *For any finite collection of formulas (perhaps with parameters) and $\lambda \geq \aleph_0$ we have*

$$\mathrm{R}[\{\bigvee_{1 \leq \ell \leq n} \psi_\ell\}, \Delta, \lambda] = \mathrm{Max}_{1 \leq \ell \leq n} \mathrm{R}[\{\psi_\ell\}, \Delta, \lambda].$$

An inductive argument on n shows that it sufficess to verify the above for $n = 2$. For the proof we use actually show a stronger statement:

THEOREM 2.15 (Ultrametric property). *For $\lambda \geq \aleph_0$ and all p we have*

$$\mathrm{R}[p \cup \{\psi_1 \vee \psi_2\}, \Delta, \lambda] = \mathrm{Max}_{1 \leq \ell \leq 2} \mathrm{R}[p \cup \{\psi_\ell\}, \Delta, \lambda].$$

PROOF. Since

$$p \cup \{\psi_\ell\} \vdash p \cup \{\psi_1 \vee \psi_2\}$$

by Monotonicity we have that

$$\mathrm{Max}_{1\leq\ell\leq 2}\,\mathrm{R}[p \cup \{\psi_\ell\}, \Delta, \lambda] \leq \mathrm{R}[p \cup \{\psi_1 \vee \psi_2\}, \Delta, \lambda].$$

We left to show that for every α

$(*)_\alpha$ $\quad \mathrm{R}[p \cup \{\psi_1 \vee \psi_2\}, \Delta, \lambda] \geq \alpha \implies \mathrm{Max}_{1\leq\ell\leq 2}\,\mathrm{R}[p \cup \{\psi_\ell\}, \Delta, \lambda] \geq \alpha$ holds for every type p.

Since $p \cup \{\psi_1 \vee \psi_2\}$ consistent implies that there is $\ell = 1, 2$ such that $p \cup \{\psi_\ell\}$ is consistent we know that $(*)_0$ holds.

Let $\alpha_0 := \min\{\alpha \mid \neg(*)_\alpha\}$.

By the definition of the function R and minimality of α_0 there exists β such that $\alpha_0 = \beta + 1$.

Since $\neg(*)_{\beta+1}$ there exists a type p such that for $\ell = 1, 2$ there are $\mu_\ell < \lambda$ and finite $q_\ell \subseteq p$ such that there are no explicitly contradictory Δ-types $\{r^\ell_j\}_{j\leq\mu_\ell}$ such that $\mathrm{R}[q_\ell \cup \{\psi_\ell\} \cup r^\ell_j, \Delta, \lambda] \not\geq \beta$ for all $j \leq \mu_\ell$.

Let $\mu^* := 2\,\mathrm{Max}\{\mu_1, \mu_2\} + 1$. Since λ is infinite we have that $\mu^* < \lambda$. Let $q := q_1 \cup q_2$. By the monotonicity we have that $\mathrm{R}[q \cup \{\psi_1 \vee \psi_2\}, \Delta, \lambda] \geq \beta + 1$. Thus there are $\{s_j\}_{j\leq\mu^*}$ explicitly contradictory Δ-types such that

$$(*) \quad \mathrm{R}[q \cup s_j \cup \{\psi_1 \vee \psi_2\}, \Delta, \lambda] \geq \beta, \text{ for all } j \leq \mu^*.$$

Since $\beta < \alpha_0$, by $(*)_\beta$ we get that[1]

$$\mathrm{Max}_{1\leq\ell\leq 2}\,\mathrm{R}[q \cup s_j \cup \{\psi_\ell\}, \Delta, \lambda] \geq \beta, \text{ for all } j \leq \mu^*.$$

Thus for every $j \leq \mu^*$ there is $\ell(j) \in \{1, 2\}$ such that

$$\mathrm{R}[q \cup s_j \cup \{\psi_{\ell(j)}\}, \Delta, \lambda] \geq \beta.$$

By the definition of μ^* an application of the pigeonhole principle to

$$j \mapsto \ell(j)$$

gives that there are $\ell_0 \in \{1, 2\}$ and $S \subseteq \mu^*$ of cardinality $\geq \mu_{\ell_0}$ such that

$$j \in S \implies \mathrm{R}[q \cup s_j \cup \{\psi_{\ell_0}\}, \Delta, \lambda] \geq \beta.$$

This implies that

$$\mathrm{R}[q \cup \{\psi_{\ell_0}\}, \Delta, \lambda] \geq \beta + 1$$

a contradiction to the assumption that

$$\mathrm{Max}_{1\leq\ell\leq 2}\,\mathrm{R}[q_\ell \cup \{\psi_\ell\}, \Delta, \lambda] \not\geq \beta + 1.$$

□

[1] We are using $(*)_\beta$ applied to $p := q \cup s_j$.

The ideal of "small" subtests in $p[\mathfrak{C}]$: Fix an $\mathbf{x}$-type, Δ set of formulas, an infinite cardinal λ and an ordinal α. Consider

$$I^{\alpha}_{\mathrm{R},p} := \{\varphi(\mathbf{x};\mathbf{a}) \mid \mathrm{R}^{\mathbf{x}}[p \cup \{\varphi(\mathbf{x};\mathbf{a})\}, \Delta, \lambda] < \alpha, \mathbf{a} \in \mathfrak{C}\}$$

By the monotonicity property of the rank we get that $I^{\alpha}_{\mathrm{R},p}$ is downward closed, namely given $\varphi(\mathbf{x};\mathbf{a}) \in I^{\alpha}_{\mathrm{R},p}$ then also $\psi(\mathbf{x};\mathbf{b}) \in I^{\alpha}_{\mathrm{R},p}$ for every $\psi(\mathbf{x};\mathbf{b})$ such that $\mathfrak{C} \models \forall \mathbf{x}[\psi(\mathbf{x};\mathbf{b}) \to \varphi(\mathbf{x};\mathbf{a})]$.

By the ultra metric property we get that $I^{\alpha}_{\mathrm{R},p}$ is closed under finite union. Namely $I^{\alpha}_{\mathrm{R},p}$ is an ideal of the Lindenbaum boolean algebra $\mathrm{BA}^{\mathbf{x}}(\mathfrak{C})$.

A fundamental question that will occupy us is for what values of α, it is a proper ideal? Does the boolean algebra $\mathrm{BA}^{\mathbf{x}}(\mathfrak{C})/I^{\alpha}_{\mathrm{R},p}$ contain interesting information on p?

A further key question: Is there a cardinal χ such that $I^{\alpha}_{\mathrm{R},p}$ satisfies the χ-chain condition?

A related idea to the above is the notion of α-equivalent formulas:

DEFINITION 2.16. Let p, Δ and $\lambda \geq \aleph_0$ be given. Suppose $\alpha := \mathrm{R}[p, \Delta, \lambda]$ is an ordinal. For any $\varphi(\mathbf{x};\mathbf{a})$ and $\psi(\mathbf{x};\mathbf{b})$ we say that $\varphi(\mathbf{x};\mathbf{a})$ *is α-equivalent to* $\psi(\mathbf{y};\mathbf{b})$ iff

$$\mathrm{R}[p \cup \{\varphi(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{b}) \vee \neg\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{b})\}, \Delta, \lambda] < \alpha.$$

Denoted by $\varphi \sim_{\alpha} \psi$ when the identity of Δ and λ is clear.

LEMMA 2.17. *Let p, Δ, λ and α be as in above the definition. The relation $\sim_{\alpha}$ is an equivalence relation.*

PROOF. Exercise, using the ultra metric property. □

DEFINITION 2.18. Let p, Δ, λ and α be as in the last definition. We say that $\varphi(\mathbf{x};\mathbf{a})$ is *α-strongly minimal* provided

$$\mathrm{R}[p \cup \{\varphi(\mathbf{x};\mathbf{a})\}, \Delta, \lambda] = \alpha$$

and either

$$\mathrm{R}[p \cup \{\varphi(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{b})\}, \Delta, \lambda] < \alpha \text{ or } \mathrm{R}[p \cup \{\varphi(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{b})\}, \Delta, \lambda] < \alpha$$

for every formula $\psi(\mathbf{x};\mathbf{b})$.

REMARK 2.19. A formula $\varphi(\mathbf{x};\mathbf{a})$ is 0-strongly minimal iff $|\varphi[\mathfrak{C}]| = 1$. A formula $\varphi(\mathbf{x};\mathbf{a})$ is 1-strongly minimal iff $\varphi(\mathbf{x};\mathbf{a})$ is strongly minimal in the sense of Definition 9.5.

An important application of the ultrametric property is the following:

THEOREM 2.20 (Extension property of the R rank). *For $\lambda \geq \aleph_0$, Δ set of formulas without parameters, set of formulas p in $\mathbf{x}$ such that $\mathrm{R}^{\mathbf{x}}[p, \Delta, \lambda] = \alpha < \infty$.*

For every set $A \supseteq \mathrm{dom}(p)$, there exists a type $q \in \mathrm{S}^{\mathbf{x}}(A) \supseteq p$ such that $\mathrm{R}^{\mathbf{x}}[q, \Delta, \lambda] = \alpha$.

Moreover, given p, Δ, λ and α as above for any set B there exists $q \in \mathrm{S}^{\mathbf{x}}_{\Delta}(B)$ such that $\mathrm{R}^{\mathbf{x}}[p \cup q, \Delta, \lambda] = \alpha$.

For the proof of this theorem we will use the following:

LEMMA 2.21. *Let*

$$\Gamma := \big\{\neg\psi(\mathbf{x}; \mathbf{a}) \ : \ \mathrm{R}[\{\psi(\mathbf{x}; \mathbf{a})\}, \Delta, \lambda] < \alpha, \psi(\mathbf{x}; \mathbf{y}) \in \Delta, \mathbf{a} \in A\big\}.$$

For any $\mathbf{x}$-type p such that $\mathrm{R}[p, \Delta, \lambda] = \alpha$ the set $p \cup \Gamma$ is consistent.

PROOF. Otherwise, there exists a finite $\{\neg\psi_k(\mathbf{x}; \mathbf{a}_k) \mid k < n\} \subseteq \Gamma$ such that $p \cup \{\neg\psi_k(\mathbf{x}; \mathbf{a}_k) \mid k < n\}$ is inconsistent. Namely,

$$p \vdash \neg(\bigwedge_{k<n} \neg\psi_k(\mathbf{x}; \mathbf{a}_k))$$

By DeMorgan,

$$p \vdash \bigvee_{k<n} \psi(\mathbf{x}; \mathbf{a}_k).$$

An application of monotonicity gives

$$(1) \quad \alpha = \mathrm{R}[p, \Delta, \lambda] \leq \mathrm{R}[\bigvee_{k<n} \psi(\mathbf{x}; \mathbf{a}_k), \Delta, \lambda].$$

But by the ultra metric property and the assumption that $\neg\psi_i(\mathbf{x}; \mathbf{a}_i) \in \Gamma$ we get

$$(2) \quad \mathrm{R}[\bigvee_{k<n} \psi(\mathbf{x}; \mathbf{a}_k), \Delta, \lambda] = \mathrm{Max}_{k<n}\, \mathrm{R}[\psi(\mathbf{x}; \mathbf{a}_k), \Delta, \lambda] < \alpha.$$

$(1) + (2)$ gives $\alpha < \alpha$.

For the moreover part: Let $A := B \cup \mathrm{dom}\, p$. By the first part find $q^* \in \mathrm{S}^{\mathbf{x}}(A)$ extending p with rank α.

Using monotonicity of the rank verify that

$$q := \{\psi(\mathbf{x}; \mathbf{b}) \mid \mathbf{b} \in B, \psi(\mathbf{x}; \mathbf{y}) \in \Delta\, \psi(\mathbf{x}; \mathbf{b}) \in q^*\}$$

is as required. □

PROOF. By the previous lemma $p \cup \Gamma$ is consistent. Let $q \in \mathrm{S}(A)$ be any extension of $p \cup \Gamma$. I claim that $\mathrm{R}[q, \Delta, \lambda] = \alpha$.

Otherwise, by monotonicity and the finite character there exists a finite $q_0 \subseteq q$ and a finite such that $\beta := \mathrm{R}[q_0, \Delta, \lambda] < \alpha$.

Denote $\psi(\mathbf{x};\mathbf{a}) := \bigwedge q_0$, since q is a complete type $\psi(\mathbf{x};\mathbf{a}) \in q$. As the sets $\psi(\mathfrak{C};\mathbf{a})$ and $q_0[\mathfrak{C}]$ are equal we have $\mathrm{R}[\psi(\mathbf{x};\mathbf{a}),\Delta,\lambda] = \beta < \alpha$, thus $\neg\psi(\mathbf{x};\mathbf{a}) \in \Gamma \subseteq q$. This contradicts the consistency of q. □

Notice that in the proof of the Extension Theorem we did not use the definition of the rank function but only some of its properties. This allows us to formulate the following generalization:

THEOREM 2.22 (General Extension Theorem). *Suppose f is an ordinal-valued function from types to ordinals satisfying the following:*

(1) *(finite character) for all p if $\alpha = f(p)$ then there exists a finite $p_0 \subseteq p$ such that $f(p_0) = \alpha$,*
(2) *(monotonicity) for p and q both $\mathbf{x}$-types, if $p \vdash q$ then $f(p) \leq f(q)$*
(3) *(ultra metric property) for any $\psi(\mathbf{x};\mathbf{a})$ and $\varphi(\mathbf{x};\mathbf{b})$ we have*

$$f(\{\psi(\mathbf{x};\mathbf{a}) \vee \varphi(\mathbf{x};\mathbf{b})\}) = \mathrm{Max}\{\{\psi(\mathbf{x};\mathbf{a})\}), f(\{\varphi(\mathbf{x};b)\})\}.$$

then for any $\mathbf{x}$-type p such that $f(p) = \alpha < \infty$ any set of formulas Δ and every $A \supseteq \mathrm{dom}\, p$ there exists $q \in \mathrm{S}^{\mathbf{x}}_{\Delta}(A)$ such that $f(p \cup q) = \alpha$.

PROOF. Consider

$$\Gamma := \big\{\neg\psi(\mathbf{x};\mathbf{a}) \ : \ f(\{\psi(\mathbf{x};\mathbf{a})\}) < \alpha, \psi(\mathbf{x};\mathbf{y}) \in \Delta, \mathbf{a} \in A\big\}.$$

CLAIM 2.23. *$p \cup \Gamma$ is consistent.*

PROOF. Otherwise there exists $\{\neg\psi_\ell \mid \ell < n\} \subseteq \Gamma$ such that $p \cup \{\neg\psi_\ell \mid \ell < n\}$ is inconsistent. Namely if $\mathbf{a} \models p$ than $\mathbf{a} \not\models \{\neg\psi_\ell \mid \ell < n\}$. Thus $p \vdash \bigvee_{\ell<n} \psi_\ell(\mathbf{x};\mathbf{a}_\ell)$.

By Monotonicity we have $f(p) \leq f(\{\bigvee_{\ell<n} \psi_\ell(\mathbf{x};\mathbf{a}_\ell)\})$, using the ultra metric property and the assumption that $f(\psi_\ell) < \alpha$ we get that

$$f(p) \leq \mathrm{Max}_{\ell<n}\, f(\{\psi_\ell(\mathbf{x};\mathbf{a}_\ell)\}) < \alpha.$$

A contradiction to $f(p) = \alpha$. □

It suffices to prove

CLAIM 2.24. *If $q \in \mathrm{S}^{\mathbf{x}}_{\Delta}(A)$ such that $q \cup p \cup \Gamma$ is consistent then $f(p \cup q) = \alpha$.*

PROOF. Otherwise (using monotonicity) $f(p \cup q) < \alpha$. By the finite character there exists a finite $q_0 \subseteq q$ such that $f(p \cup q_0) = f(p \cup q)$. Let $\psi(\mathbf{x};\mathbf{a}) := \bigwedge q_0$. Since q is complete, we have that $\psi(\mathbf{x};\mathbf{a}) \in q$. Note that monotonicity implies $f(q_0) = f(\psi(\mathbf{x};\mathbf{a}))$. By the definition of Γ we have that $\neg\psi(\mathbf{x};\mathbf{a}) \in \Gamma$, is a contradiction to the consistency of $q \cup \Gamma$. □

□

THEOREM 2.25 (Connected). *Let p be a finite set of formulas in $\mathbf{x}$. If $\mathrm{R}[p, \Delta, \lambda] = \alpha < \infty$ then for every $\beta < \alpha$ there exists a Δ-type q such that $\mathrm{R}[p \cup q, \Delta, \lambda] = \beta$.*

PROOF. Suppose for the sake of contradiction that there are a finite p and ordinals $\beta < \alpha$ such that $\mathrm{R}[p, \Delta, \lambda] = \alpha < \infty$ and

$$(*) \quad \mathrm{R}[p \cup q, \Delta, \lambda] \geq \beta \implies \mathrm{R}[p \cup q, \Delta, \lambda] \geq \beta + 1 \text{ for every } \Delta\text{-type } q.$$

Using $(*)$ we will contradict the hypothesis that $\mathrm{R}[p, \Delta, \lambda] < \infty$ by showing that for every Δ-type q and every $\gamma \geq \beta$

$$\mathrm{R}[p \cup q, \Delta, \lambda] \geq \beta \text{ implies } \mathrm{R}[p \cup q, \Delta, \lambda] \geq \gamma.$$

We proceed by induction on $\gamma \geq \beta$.
For $\gamma = \beta$ there is nothing to prove. For γ limit use the inductive hypothesis.
For $\gamma = \zeta + 1 > \beta$, let q be a Δ-type such that $\mathrm{R}[p \cup q, \Delta, \lambda] \geq \beta$. By $(*)$ we have that $\mathrm{R}[p \cup q, \Delta, \lambda] \geq \beta + 1$. Using the definition of R, for every finite subset r_0 of q, for every $\mu < \lambda$, there exists $\{q_i \mid i < \mu\}$ a set of explicitly contradictory Δ-types such that

$$(**) \quad \mathrm{R}[p \cup r_0 \cup q_i, \Delta, \lambda] \geq \beta \quad \text{for every} \quad i < \mu.$$

An application of the induction hypothesis to $(**)$ gives us that

$$\mathrm{R}[p \cup r_0 \cup q_i, \Delta, \lambda] \geq \zeta \quad \text{for every} \quad i \leq \mu.$$

This, using the definition of R once more, gives us that

$$\mathrm{R}[p \cup q, \Delta, \lambda] \geq \zeta + 1 (= \gamma).$$

□

The main idea in the proof of Theorem 3.2.12 was that if $\mathrm{R}[\mathbf{x} = \mathbf{x}, L, 2] \geq \omega_1$ then there exists a certain binary tree consisting of formulas and parameters with consistent branches (indexed by ${}^{\omega_{\geq}}2$). This idea can be generalized for all the different rank functions we have seen so far (in fact the idea will be carried out also all the rank functions to be introduced later). At first we do that for the simplest rank I.e. $\mathrm{R}[p, \varphi, 2]$.

Recall that for a formula φ and $t \in \{0, 1\}$ we use

$$\varphi^t := \begin{cases} \varphi & \text{if } t = 0 \\ \neg\varphi & \text{if } t = 1. \end{cases}$$

DEFINITION 2.26. Suppose p is a set of formulas in $\mathbf{x}$ and let $\varphi(\mathbf{x};\mathbf{y})$ be given. For an ordinal α let

$$\begin{aligned}\Gamma_p(\varphi,\alpha) :=&\{\psi(\mathbf{x}_\eta;\mathbf{a}) \ : \ \psi(\mathbf{x};\mathbf{a})\in p,\ \eta\in {}^{\alpha}2\}\\ &\cup\{\varphi(\mathbf{x}_\eta;\mathbf{y}_{\eta\restriction\beta})^{\eta[\beta]} \ : \ \beta<\alpha,\eta\in {}^{\alpha}2\}.\end{aligned}$$

$\Gamma(\varphi,\alpha)$ stands for $\Gamma_{\mathbf{x}=\mathbf{x}}(\varphi,\alpha)$.

REMARK 2.27. Note that in the proof of the hard direction of Theorem 3.2.12, from the assumption $\mathrm{R}[\mathbf{x}=\mathbf{x},L,2]\geq\omega_1$ we concluded that there are $\{\varphi_{\eta\restriction n}(\mathbf{x}_\eta;\mathbf{y}_{\eta\restriction n}) \ : \ n<\omega,\eta\in {}^{\omega}2\}$ such that

$$\{\varphi_{\eta\restriction n}(\mathbf{x}_\eta;\mathbf{y}_{\eta\restriction n})^{\eta[n]} \ : \ n<\omega,\eta\in {}^{\omega}2\}$$

is consistent.

The following is a localized version of this.

LEMMA 2.28 (Tree characterization). *Let p be a type in $\mathbf{x}$ and let $\varphi(\mathbf{x};\mathbf{y})$ be a formula. Then, for every $n<\omega$, the following are equivalent:*

(1) $\Gamma_p(\varphi,n)$ *is consistent;*
(2) $\mathrm{R}[p,\varphi,2]\geq n$.

PROOF. By the compactness theorem and the finite character of the rank we may assume that p is finite.

If $n=0$, the assertion is trivial. Suppose that for all finite types p, $\mathrm{R}[p,\varphi,2]\geq n$ is equivalent to the consistency of $\Gamma_p(\varphi,n)$. We now prove the equivalence for $n+1$.

If $\mathrm{R}[p,\varphi,2]\geq n+1$, there are two explicitly contradictory $\{\varphi\}$-types q_1 and q_2 such that $\mathrm{R}[p\cup q_1,\varphi,2]\geq n$ and $\mathrm{R}[p\cup q_2,\varphi,2]\geq n$. Since the types are explicitly contradictory, there is a sequence $\mathbf{a}$ such that $\varphi(\mathbf{x};\mathbf{a})\in q_1$ and $\neg\varphi(\mathbf{x};\mathbf{a})\in q_2$. By monotonicity, $\mathrm{R}[p\cup\{\varphi(\mathbf{x};\mathbf{a})\},\varphi,2]\geq n$ and $\mathrm{R}[p\cup\{\neg\varphi(\mathbf{x};\mathbf{a})\},\varphi,2]\geq n$.

Enough to find $\{\mathbf{b}_\eta,\mathbf{a}_{\eta\restriction l} \mid l<n+1 \mid \eta\in {}^{n+1}2, l<n+1\}$ realizing $\Gamma_p(\varphi,n+1)$.

An application of the inductive assumption to $\mathrm{R}[p\cup\{\varphi(\mathbf{x};\mathbf{a})\},\varphi,2]\geq n$ and $\mathrm{R}[p\cup\{\neg\varphi(\mathbf{x};\mathbf{a})\},\varphi,2]\geq n$ gives us

$$\{\mathbf{b}^t_\eta,\mathbf{a}^t_{\eta\restriction l} \mid l<n \mid \eta\in {}^{n}2, l<n\}\models\Gamma_{p\cup\{\varphi^t(\mathbf{x};\mathbf{a})\}}(\varphi,n).$$

For $\eta\in {}^{n+1}2$, let $t\in\{0,1\}$ and $\nu\in {}^{n}2$ be such that $\eta=t\hat{}\nu$. Let $\mathbf{b}_\eta:=\mathbf{b}^t_\nu$, $\mathbf{a}_{\langle\rangle}:=\mathbf{a}$ and $\mathbf{a}_{\eta\restriction(l+1)}:=\mathbf{a}^t_{\nu\restriction l}$ for all $l<n$.

Verify that the above realize $\Gamma_p(\varphi,n+1)$.

Suppose that $\Gamma_p(\varphi, n+1)$ is consistent. Let $\mathbf{a} := \mathbf{a}_{\langle\rangle}$. By the definition of Γ we get that both

$$\Gamma_{p\cup\{\varphi(\mathbf{x};\mathbf{a})\}}(\varphi, n) \text{ and } \Gamma_{p\cup\{\neg(\varphi(\mathbf{x};\mathbf{a})\}}(\varphi, n) \text{ are consistent.}$$

An application of the inductive hypothesis to both trees gives that $\mathrm{R}[p \cup \{\varphi(\mathbf{x};\mathbf{a})\}, \varphi, 2] \geq n$ and $\mathrm{R}[p \cup \{\neg\varphi(\mathbf{x};\mathbf{a})\}, \varphi, 2] \geq n$. By the definition of the rank we get $\mathrm{R}[p, \varphi, 2] \geq n+1$. □

In the previous lemma we could replace n by any ordinal α, but for α infinite we get an inessential notion:

PROPOSITION 2.29. *Let p, φ be as above we have*

$$\mathrm{R}[p, \varphi, 2] < \infty \implies \mathrm{R}[p, \varphi, 2] < \omega.$$

Moreover,

$$\mathrm{R}[p, \varphi, 2] \geq \omega \quad \textit{iff} \quad \Gamma_p(\varphi, \alpha) \textit{ is a consistent set}$$

for all infinite α.

PROOF. By the compactness theorem. □

An interesting corollary is

THEOREM 2.30. $\mathrm{R}[p, \varphi, 2] < \omega$ *iff* $\mathrm{R}[p, \varphi, \infty] < \infty$.

PROOF. Clearly $\mathrm{R}[p, \varphi, 2] < \omega \implies \mathrm{R}[p, \varphi, \infty] < \infty$.

By Proposition 2.7, there is a cardinal λ such that $\mathrm{R}[p, \varphi, \infty] = \mathrm{R}[p, \varphi, \lambda]$. If $\mathrm{R}[p, \varphi, 2] \geq \omega$, Lemma 2.28, along with the Compactness Theorem, implies that $\Gamma_p(\varphi, \lambda)$ is consistent.

Let $\{\mathbf{a}_\eta \mid \eta \in {}^{\lambda>}2\}$ be realizations of $\mathbf{y}_\eta$'s from the tree. Thus, for every $\eta \in {}^{\lambda>}2$, $p_\eta := p \cup \{\varphi(\mathbf{x}; \mathbf{a}_{\eta\restriction\beta})^{\eta[\beta]} \mid \beta < \ell(\eta)\}$ is consistent. Since $\mathrm{R}[p, \varphi, \lambda] < \infty$, $\alpha_0 := \min\{R[p_\eta, \varphi, \lambda] \mid \eta \in {}^{\lambda>}2\}$ is a well-defined ordinal. Let $\eta_0 \in {}^{\lambda>}2$ be such that $\mathrm{R}[p_{\eta_0}, \varphi, \lambda] = \alpha_0$.

For $\gamma < \lambda$, consider $q_\gamma := p_{\eta_0{}^\frown\bar{0}^\gamma{}^\frown 1}$, where $\bar{0}^\gamma$ is a sequence of 0's of length γ. Obviously q_{γ_1} contradicts q_{γ_2} for $\gamma_1 \neq \gamma_2 < \lambda$ and by minimality of α_0, for every $\gamma < \lambda$, $\mathrm{R}[q_\gamma, \varphi, \lambda] \geq \alpha_0$. This implies that $\mathrm{R}[p_{\eta_0}, \varphi, \lambda] \geq \alpha_0 + 1$, which is a contradiction to the choice of p_{η_0}. □

A notion that appears in the definition of other ranks is:

DEFINITION 2.31. Let Φ be a set of formulas (perhaps with parameters) whose free variables are among $\mathbf{x}$. For a positive integer k we say that Φ *is k-contradictory* iff Φ_0 is inconsistent for every $\Phi_0 \subseteq \Phi$ of cardinality k.

Here is another rank:

DEFINITION 2.32. By induction on α we define when $\mathrm{D}[p, \Delta, \lambda] \geq \alpha$.

(1) $\mathrm{D}[p, \Delta, \lambda] \geq 0$ if p is consistent.
(2) $\mathrm{D}[p, \Delta, \lambda] \geq \alpha$, for α limit, if for every $\beta < \alpha$, $\mathrm{D}[p, \Delta, \lambda] \geq \beta$.
(3) $\mathrm{D}[p, \Delta, \lambda] \geq \alpha + 1$ if for every finite $q \subseteq p$, for every $\mu < \lambda$, there are a finite $r \supseteq q$, $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$, $n < \omega$, and $\{\mathbf{a}_i : i \leq \mu\}$ such that:
   (a) set $\{\varphi(\mathbf{x}; \mathbf{a}_i) : i \leq \mu\}$ is n-contradictory over r; namely for every $S \subseteq \mu + 1$
   $$|S| = n \implies \mathfrak{C} \models \neg\exists\mathbf{x}[\bigwedge r \wedge \bigwedge_{i \in S} \varphi(\mathbf{x}; \mathbf{a}_i)]$$
   (b) for every $i \leq \mu$, $\mathrm{D}[r \cup \varphi(\mathbf{x}; \mathbf{a}_i), \Delta, \lambda] \geq \alpha$.

LEMMA 2.33 (Finite character).

PROOF. □

LEMMA 2.34 (Monotonicity).

PROOF. □

LEMMA 2.35. (1)
(2) *Invariance*

PROOF. Exercises. □

A localized version of $\mathrm{D}[p, \Delta, \aleph_1]$ turns out to be another important rank function. The importance of $\mathrm{D}[p, \Delta, k]$ is that its boundness is a characterization of the important class of first-order theories that extends that of stable theories.

DEFINITION 2.36. A complete first-order theory T is *simple* iff $\mathrm{D}[\mathbf{x} = \mathbf{x}, \Delta, k] < \infty$ for every finite Δ and all $k < \omega$.

DEFINITION 2.37. Let $p(\mathbf{x})$ be a set of formulas, possibly with parameters. Let Δ be a set of formulas without parameters and let $k < \omega$. We define the *rank* $\mathrm{D}[p, \Delta, k]$. The rank $\mathrm{D}[p, \Delta, k]$ is either an ordinal, or -1, or ∞. The relation $\mathrm{D}[p, \Delta, k] \geq \alpha$, is defined by induction on α.

(1) $\mathrm{D}[p, \Delta, k] \geq 0$ if p is consistent;
(2) $\mathrm{D}[p, \Delta, k] \geq \delta$ when δ is a limit if $\mathrm{D}[p, \Delta, k] \geq \beta$ for every $\beta < \delta$;
(3) $\mathrm{D}[p, \Delta, k] \geq \alpha + 1$ if for every finite $r \subseteq p$, there exist a formula $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ and a set $\{\mathbf{a}_i \mid i < \omega\}$ with $\ell(\mathbf{a}_i) = \ell(\mathbf{y})$ such that:
   (a) $\mathrm{D}[r \cup \varphi(\mathbf{x}; \mathbf{a}_i), \Delta, k] \geq \alpha$ for every $i < \omega$;
   (b) The set $\{\varphi(\mathbf{x}; \mathbf{a}_i) \mid i < \omega\}$ is k-contradictory.

We write

$\mathrm{D}[p, \Delta, k] = -1$ if p is not consistent;
$\mathrm{D}[p, \Delta, k] = \alpha$ when $\mathrm{D}[p, \Delta, k] \geq \alpha$ but $\mathrm{D}[p, \Delta, k] \not\geq \alpha + 1$;
$\mathrm{D}[p, \Delta, k] = \infty$ when $\mathrm{D}[p, \Delta, k] \geq \alpha$ for every ordinal α.

The next propositions establish some of the most basic properties of the rank. The proofs are rather easy exercises. They are included for completeness.

PROPOSITION 2.38.

(1) (Monotonicity) *If* $p_1 \vdash p_2$, $\Delta_1 \subseteq \Delta_2$, *and* $k_1 \leq k_2$, *then* $\mathrm{D}[p_1, \Delta_1, k_1] \leq \mathrm{D}[p_2, \Delta_2, k_2]$;
(2) (Finite Character) *For every type* p *there exists a finite* $r \subseteq p$ *such that* $\mathrm{D}[p, \Delta, k] = \mathrm{D}[r, \Delta, k]$
(3) (Invariance) *If* f *is an automorphism, then* $\mathrm{D}[p, \Delta, k] = \mathrm{D}[f(p), \Delta, k]$.

PROOF. (1) We prove by induction on the ordinal α, that

$$\mathrm{D}[p_1, \Delta_1, k_1] \geq \alpha \quad \text{implies} \quad \mathrm{D}[p_2, \Delta_2, k_2] \geq \alpha.$$

If $\mathrm{D}[p_1, \Delta_1, k_1] \geq 0$, then p_1 is consistent, so p_2 is consistent since $p_1 \vdash p_2$, and hence $\mathrm{D}[p_2, \Delta_2, k_2] \geq 0$.

When α is a limit ordinal, the implication is immediate from the induction hypothesis.

Suppose $\mathrm{D}[p_1, \Delta_1, k_1] \geq \alpha + 1$ and let $r_2 \subseteq p_2$ be finite. Since $p_1 \vdash p_2$, there is a finite $r_1 \subseteq p_1$ such that $r_1 \vdash r_2$. By the definition of the rank, there exist $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta_1$ and $\{\mathbf{a}_i \mid i < \omega\}$ with $\ell(\mathbf{a}_i) = \ell(\mathbf{y})$ such that $\mathrm{D}[r_1 \cup \varphi(\mathbf{x}; \mathbf{a}_i)\, \Delta_1, k_1] \geq \alpha$ for every $i < \omega$ and $\{\varphi(\mathbf{x}; \mathbf{a}_i) \mid i < \omega\}$
is k_1-contradictory. Now, $r_1 \cup \varphi(\mathbf{x}; \mathbf{a}_i) \vdash r_2 \cup \varphi(\mathbf{x}; \mathbf{a}_i)$,
so $\mathrm{D}[r_2 \cup \varphi(\mathbf{x}; \mathbf{a}_i), \Delta_2, k_2] \geq \alpha$ for every $i < \omega$ by induction hypothesis. But $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta_2$ (since $\Delta_1 \subseteq \Delta_2$), and $\{\varphi(\mathbf{x}; \mathbf{a}_i) \mid i < \omega\}$ is k_2-contradictory (since $k_2 \geq k_1$). Hence, $\mathrm{D}[p_2, \Delta_2, k_2] \geq \alpha + 1$, by definition of the rank.

(2) If $\mathrm{D}[p, \Delta, k] = -1$, then p is inconsistent and by the compactness theorem there is an inconsistent finite $r \subseteq p$. Then, $\mathrm{D}[r, \Delta, k] = -1$.

If $\mathrm{D}[p, \Delta, k] = \infty$, then for every finite $r \subseteq p$, $\mathrm{D}[r, \Delta, k] = \infty$ by Monotonicity.

If $\mathrm{D}[p, \Delta, k] = \alpha$, then $\mathrm{D}[p, \Delta, k] \geq \alpha$ and $\mathrm{D}[p, \Delta, k] \not\geq \alpha + 1$, so there exists a finite $r \subseteq p$ with $\mathrm{D}[r, \Delta, k] \geq \alpha$ such that there are no $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ and $\{\mathbf{a}_i \mid i < \omega\}$ such that $\{\varphi(\mathbf{x}; \mathbf{a}_i) \mid i < \omega\}$ is k-contradictory and
$\mathrm{D}[r \cup \varphi(\mathbf{x}; \mathbf{a}_i)\, \Delta, k] \geq \alpha$ for every $i < \omega$. But this demonstrates that
$\mathrm{D}[r, \Delta, k] \not\geq \alpha + 1$. Thus, $\mathrm{D}[r, \Delta, k] = \alpha$.

(3) Immediate. □

LEMMA 2.39 (Ultrametric Property). *For every p, Δ, $k, n < \omega$ and formulas $\{\psi_l(\mathbf{x}; \mathbf{b}_l) \mid l < n\}$, we have*

$$\mathrm{D}[p \cup \bigvee_{l<n} \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k] = \max_{l<n} \mathrm{D}[p \cup \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k].$$

PROOF. By Monotonicity, for every $l < n$ we have

$$\mathrm{D}[p \cup \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k] \leq \mathrm{D}[p \cup \bigvee_{l<n} \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k].$$

Hence,

$$\max_{l<n} \mathrm{D}[p \cup \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k] \leq \mathrm{D}[p \cup \bigvee_{l<n} \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k].$$

To prove the reverse inequality, we show by induction that for every ordinal α and every type p,

$$\mathrm{D}[p \cup \bigvee_{l<n} \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k] \geq \alpha \quad \text{implies} \quad \max_{l<n} \mathrm{D}[p \cup \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k] \geq \alpha.$$

When $\alpha = 0$ or α is a limit ordinal, the implication is easy. For the successor stage suppose, for contradiction, that

(*)

$$\mathrm{D}[p \cup \bigvee_{l<n} \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k] \geq \alpha + 1, \quad \text{but} \quad \max_{l<n} \mathrm{D}[p \cup \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k] \not\geq \alpha + 1.$$

Then, for every $l < n$ we have $\mathrm{D}[p \cup \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k] \leq \alpha$. Choose a finite $r_l \subseteq p$ such that

$$\mathrm{D}[r_l \cup \psi_l(\mathbf{x}; \mathbf{b}_l), \Delta, k] \leq \alpha$$

and let $r := \bigcup_{l<n} r_l$. Then $r \subseteq p$ is finite, so, by (*) and the definition of the rank, there exist $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ and $\{\mathbf{a}_i \mid i < \omega\}$ with $\ell(\mathbf{a}_i) = \ell(\mathbf{y})$ such that $\{\varphi(\mathbf{x}; \mathbf{a}_i) \mid i < \omega\}$ is k-contradictory and for every $i < \omega$

$$\mathrm{D}[r \cup \bigvee_{l<n} \psi(\mathbf{x}; \mathbf{b}_l) \cup \varphi(\mathbf{x}; \mathbf{a}_i), \Delta, k] \geq \alpha.$$

By induction hypothesis, for every $i < \omega$

$$\max_{\ell<n} \mathrm{D}[r \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell) \cup \varphi(\mathbf{x}; \mathbf{a}_i)\Delta, k] \geq \alpha,$$

so there exists $\ell_i < n$ such that

$$\mathrm{D}[r \cup \psi_{\ell_i}(\mathbf{x}; \mathbf{b}_{\ell_i}) \cup \varphi(\mathbf{x}; \mathbf{a}_i), \Delta, k] \geq \alpha.$$

By the pigeonhole principle, we may assume that $\ell_i = \ell^* < n$ is fixed and

$$\mathrm{D}[r \cup \psi_{\ell^*}(\mathbf{x}; \mathbf{b}_{\ell^*}) \cup \varphi(\mathbf{x}; \mathbf{a}_i), \Delta, k] \geq \alpha,$$

for every $i < \omega$. By definition of the rank,

$$\mathrm{D}[r \cup \psi_{\ell^*}(\mathbf{x}; \mathbf{b}_{\ell^*}), \Delta, k] \geq \alpha + 1,$$

and by Monotonicity,

$$\mathrm{D}[r_{\ell^*} \cup \psi_{\ell^*}(\mathbf{x}; \mathbf{b}_{\ell^*}), \Delta, k] \geq \alpha + 1.$$

But this contradicts the choice of r_{ℓ^*}. Therefore,

$$\max_{\ell<n} \mathrm{D}[p \cup \psi_\ell(\mathbf{x}; \mathbf{b}_\ell), \Delta, k] \geq \alpha + 1.$$

□

The proof of the following is similar to that of Theorem 1.11.

THEOREM 2.40 (Extension theorem for the D rank). *Let p be an $\mathbf{x}$-type, Δ and Φ be sets of formulas and $k < \omega$. Suppose that $\mathrm{D}^{\mathbf{x}}[p, \Delta, k] < \infty$. Then, for every set B there exists a type $q \in S^{\mathbf{x}}_{\Delta}(B)$ such that*

$$\mathrm{D}^{\mathbf{x}}[p, \Delta, k] = \mathrm{D}^{\mathbf{x}}[p \cup q, \Delta, k].$$

PROOF. We may assume that Φ is closed under conjunction. Suppose that $\mathrm{D}[p, \Delta, k] = \alpha$. Consider

$$\Gamma := \{\neg\psi(\mathbf{x}; \mathbf{b}) \mid \mathbf{b} \in B, \psi(\mathbf{x}; \mathbf{y}) \in \Phi, \mathrm{D}[p \cup \psi(\mathbf{x}; \mathbf{b}), \Delta, k] < \alpha\}.$$

Let us show that $p \cup \Gamma$ is consistent. If $p \cup \Gamma$ were inconsistent, there would be $\{\neg\psi_i(\mathbf{x}; \mathbf{b}_i) \mid i < n\} \subseteq \Gamma$ such that $p \cup \{\neg\psi_i(\mathbf{x}; b_i) \mid i < n\}$ is inconsistent. But then, $p \vdash \bigvee_{i<n} \psi_i(\mathbf{x}; \mathbf{b}_i)$. By Monotonicity and Lemma 2.39, we have

$$\alpha = \mathrm{D}[p, \Delta, k] \leq \mathrm{D}[p \cup \bigvee_{i<n} \psi_i(\mathbf{x}; \mathbf{b}_i), \Delta, k] = \max_{i<n} \mathrm{D}[p \cup \psi_i(\mathbf{x}; \mathbf{b}_i), \Delta, k] < \alpha,$$

which is, of course, a contradiction.

Choose $q \in S_\Phi(B)$ extending Γ. If $\mathrm{D}[p \cup q, \Delta, k] < \alpha$, then by Finite Character, there exists $\psi(\mathbf{x}; \mathbf{b}) \in q$ such that $\mathrm{D}[p \cup \psi(\mathbf{x}; \mathbf{b}), \Delta, k] < \alpha$. Hence, by definition of Γ, we must have $\neg\psi(\mathbf{x}; \mathbf{b}) \in \Gamma \subseteq q$, which is a contradiction. □

The following corollary is a generalization of Proposition 2.29 and $(3) \implies (2)$ from Theorem 3.3.2:

COROLLARY 2.41. *If $\mathrm{D}[\mathbf{x} = \mathbf{x}, \Delta, \lambda] < \infty$ ($\mathrm{R}[\mathbf{x} = \mathbf{x}, \Delta, \lambda] < \infty$) then $\mathrm{D}[\mathbf{x} = \mathbf{x}, \Delta, \lambda] < (2^{|T|})^+$ ($\mathrm{R}[\mathbf{x} = \mathbf{x}, \Delta, \lambda] < (2^{|T|})^+$),*

PROOF. Use theorem 2.25, finite character and the invariance together with the fact that $|\mathrm{D}(T)| \leq 2^{|T|}$. □

In the following section we will show several improvements of Corollary 2.41. E.g. $\mathrm{R}[\mathbf{x} = \mathbf{x}, L, \infty] < \infty \implies \mathrm{R}[\mathbf{x} = \mathbf{x}, L, \infty] < |T|^+$.

PROPOSITION 2.42. *For every type p and set of formulas Δ there exists a cardinal $\lambda(p, \Delta)$ such that for every $\mu \geq \lambda(p, \Delta)$ we have that $\mathrm{R}[p, \Delta, \mu] = \mathrm{R}[p, \Delta, \infty]$. Moreover, there exists a cardinal $\lambda_R(T)$ such that for every p and Δ*

$$\mathrm{R}[p, \Delta, \mu] = \mathrm{R}[p, \Delta, \infty] \quad \textit{for all } \mu \geq \lambda_{R(T)}.$$

Similarly for the D rank, there exists $\lambda_D(T)$ such that

$$\mathrm{D}[p, \Delta, \mu] = \mathrm{D}[p, \Delta, \infty] \quad \textit{for all } \mu \geq \lambda_{R(T)}.$$

PROOF. Since $\mu > \lambda$ implies that $\mathrm{R}[p, \Delta, \mu] \leq \mathrm{R}[p, \Delta, \lambda]$. There exists $\lambda = \lambda_{\Delta,p}$ such that

$$\mathrm{R}[p, \Delta, \mu] = \mathrm{R}[p, \Delta, \lambda] \quad \text{for every } \mu > \lambda.$$

Use finite character and apply invariance to the fact that $|D(T)| \leq 2^{|T|}$.

The argument for $\mathrm{D}[p, \Delta, \mu]$ is identical. □

An interesting (and difficult) problem is finding the least $\lambda_D(T)$ we will prove below the surprising result that $\lambda_D(T) = \aleph_1$. The bound on $\lambda_R(T)$ turn out to be $(2^{|T|})^{++}$ which is much harder to prove and will be delayed for another section.

There is another rank-related function that Shelah called *multiplicity* associated with $R[p, \Delta, \lambda]$, the special case when $\Delta = L$ and $\lambda = \aleph_1$, i.e. we are dealing with Morley's rank that Multiplicity is called by everybody (besides Shelah *Morley's degree*). .

DEFINITION 2.43. Let p, Δ and λ be as in Definition [**?**]. Suppose $\alpha := \mathrm{R}[p, \Delta, \lambda]$ is an ordinal.

Let μ_0 be the minimal μ such that there exists $q \subseteq p$ and the negation of the following statement: there exists $\{q_i \mid i \leq \mu\}$ e.c. Δ-types such that $\mathrm{R}[q \cup q_i, \Delta, \lambda] = \alpha$ for all $i \leq \mu$.

Denote that μ_0 by $\mathrm{Mlt}[p, \Delta, \lambda]$.

LEMMA 2.44 (Finite Character). *If $\alpha := \mathrm{R}[p, \Delta, \lambda]$ is an ordinal and $mu := \mathrm{Mlt}[p, \Delta, \lambda]$ then there exists a finite $p_0 \subseteq p$ such that $\mathrm{R}[p_0, \Delta, \lambda] = \alpha$ and $\mathrm{Mlt}[p_0, \Delta, \lambda] = \mu_0$.*

PROOF. For every finite $q \subseteq p$ with $\mathrm{R}[q, \Delta, \lambda] = \alpha$ let $\mu_q := \mathrm{Mlt}[q, \Delta, \lambda]$.

Notice that

$$\mathrm{Mlt}[p, \Delta, \lambda] = \min\{\mu_q \mid q \subseteq p \text{ satisfying the above requirements }\}.$$

Let p_0 be a finite sub type of p witnessing that minimum. □

LEMMA 2.45 (Uniqueness of extension). *Suppose $\mathrm{R}^{\mathbf{x}}[p, \Delta, \lambda] = \alpha$ and $\mathrm{Mlt}[p, \Delta, \lambda] = 1$. If $q_1, q_2 \in \mathrm{S}^{\mathbf{x}}(A)$ are such that $\mathrm{R}[p \cup q_1, \Delta, \lambda] = \mathrm{R}[p \cup q_2, \Delta, \lambda] = \alpha$ then $q_1 = q_2$.*

PROOF. Immediate. □

3. Characterizations of stability by rank and φ-types

THEOREM 3.1 (Basic characterization of stability). *Let T be a complete first-order theory, the following are equivalent:*

(1) *T is stable.*
(2) *T does not have the order property.*
(3) *T is stable in λ for every λ satisfying $\lambda^{|T|} = \lambda$.*
(4) *$\mathrm{R}[\mathbf{x} = \mathbf{x}, \varphi] < \omega$ for every $\varphi(\mathbf{x}; \mathbf{y})$.*
(5) *$\mathrm{R}[\mathbf{x} = \mathbf{x}, \varphi] < \infty$ for every $\varphi(\mathbf{x}; \mathbf{y})$.*
(6) *for every formula $\varphi(\mathbf{x}; \mathbf{y})$ we have that $|\mathrm{S}_{\varphi(\mathbf{x};\mathbf{y})}(A)| \leq |A|$ holds for every infinite set A.*
(7) *for every formula $\varphi(\mathbf{x}; \mathbf{y})$ we have that $\mathrm{S}^{\mathbf{x}}_{\varphi}(A)$ is countable for very countable set A.*
(8) *For every formula $\varphi(\mathbf{x}; \mathbf{y})$ there exists a formula $\psi_\varphi(\mathbf{y}; \mathbf{z}_\varphi)$ such that for all A and $p \in \mathrm{S}^{\mathbf{x}}_{\varphi}(A)$ there exists $\mathbf{b}_{\varphi,p} \in {}^{\ell(\mathbf{z})}\mathrm{A}$ satisfying*
$$\varphi(\mathbf{x}; \mathbf{a}) \in p \iff \mathfrak{C} \models \psi_\varphi[\mathbf{a}; \mathbf{b}_{\varphi,p}] \text{ for all } \mathbf{a} \in A.$$
(9) *$|\mathrm{S}(|M|)| \leq \|M\|^{|T|}$ holds for every $M \models T$.*
(10) *$\kappa_s(T) \leq |T|^+$.*
(11) *$\kappa_s(T) < \infty$.*
(12) *$\kappa_{\mathrm{def}}(T) \leq |T|^{+}$*[2].
(13) *$\kappa_{\mathrm{def}}(T) < \infty$.*

PROPOSITION 3.2. *If T is stable then $\kappa_s(T) \leq |T|^+$.*

PROOF. Given A and $p \in \mathrm{S}^n(A)$, by stability $\mathrm{R}[p, \varphi] < \omega$, denote this by k. By finite character there exists a finite $q_\varphi \subseteq p$ such that $\mathrm{R}[q_\varphi, \varphi] = \mathrm{R}[p, \varphi] = k$.

Let $B := \{\mathrm{dom}\, q_\varphi \mid \varphi \in \mathrm{Fml}(\mathrm{L}(T))\}$. Clearly $|B| \leq |T|$. Enough to verify that p does not split over B.

Given a formula $\varphi(\mathbf{x}; \mathbf{y})$ and $\mathbf{a}, \mathbf{b} \in {}^{\ell(\mathbf{y})}\mathrm{A}$ such that $\mathrm{tp}(\mathbf{a}/B) = \mathrm{tp}(\mathbf{b}/B)$, we want to show that
$$\varphi(\mathbf{x}; \mathbf{a}) \in p \iff \varphi(\mathbf{x}; \mathbf{b}) \in p.$$
Suppose $\varphi(\mathbf{x}; \mathbf{a}) \in p$ as
$$q_\varphi \subseteq q_p \cup \{\varphi(\mathbf{x}; \mathbf{a})\} \subseteq p$$
by Monotonicity of the rank function we have
$$(1) \quad k \leq \mathrm{R}[q_\varphi \cup \{\varphi(\mathbf{x}; \mathbf{a})\}, \varphi] \leq k.$$
If $\varphi(\mathbf{x}; \mathbf{b}) \notin p$ as p is a complete type we get that $\neg\varphi(\mathbf{x}; \mathbf{b}) \in p$, using monotonicity of the rank function once more we get as above that
$$(2) \quad \mathrm{R}[q_\varphi \cup \{\neg\varphi(\mathbf{x}; \mathbf{b})\}, \varphi] = k.$$

[2]The function $\kappa_{\mathrm{def}}(T)$ will be introduced in the next section.

Since $\mathrm{tp}(\mathbf{a}/B) = \mathrm{tp}(\mathbf{b}/B)$ there is $f \in \mathrm{Aut}_B(\mathfrak{C})$ such that $f(\mathbf{a}) = \mathbf{b}$. Using invariance we get from (1) that

$$(3) \quad \mathrm{R}[f(q_\varphi \cup \{\varphi(\mathbf{x}; \mathbf{a})\}), \varphi] = k.$$

Remembering that $f(q_\varphi) = q_\varphi$ and $f(\mathbf{a}) = \mathbf{b}$ from (3) we get

$$(4) \quad \mathrm{R}[q_\varphi \cup \{\varphi(\mathbf{x}; \mathbf{b})\}, \varphi] = k.$$

But the definition of the rank applied to (4) and (2) gives that $\mathrm{R}[q_\varphi, \varphi] \geq k + 1$ which is a contradiction to the assumption on q_φ. □

In Theorems x.yz & xx.yy.zz we will see many additional equivalent formulations of stability.

As an application we prove

THEOREM 3.3. *Suppose T is stable and Δ is a finite set of formulas.*

For any $A, \mathbf{I}$ such that there is an uncountable regular cardinal λ satisfying $|A| < \lambda \leq |\mathbf{I}|$ there exists $\mathbf{J} \subseteq \mathbf{I}$ of cardinality λ which is a sequence of Δ-indiscernibles over the set A.

PROOF. By an application of the pigeon hole principle we may assume that there exists $n < \omega$ such that $\mathbf{a} \in \mathbf{I} \implies \ell(\mathbf{a}) = n$.

Consider

$$S := \big\{R[p, \Delta, 2] \ : \ B \subseteq |M| \text{ of cardinality } < \lambda, B \supseteq A \text{ and } p \in \mathrm{S}^n_\Delta(B) \text{ is realized by } \geq \lambda \text{ elements of } \mathbf{I}\big\}.$$

CLAIM 3.4. *S is not empty.*

PROOF. Given any $B \subseteq |M|$ of cardinality $< \lambda$ consider $\mathrm{S}^n_\Delta(B)$. For every $\mathbf{a} \in \mathbf{I}$ let $p_{\mathbf{a}} := \mathrm{tp}_\Delta(\mathbf{a}/A) \in \mathrm{S}^n_\Delta(B)$. By stability, using the previous theorem we have that $|\mathrm{S}^n_\Delta(B)| < \lambda$. Since λ is regular by the pigeon hole principle there exists $p \in \mathrm{S}^n_\Delta(B)$ and $\mathbf{I}' \subseteq \mathbf{I}$ of cardinality λ such that $p_{\mathbf{a}} = p$ for every $\mathbf{a} \in \mathbf{I}'$. □

****

Let $\alpha := \min S$ and let $p \in \mathrm{S}^n_\Delta(B)$ with minimal rank. Given $\mathbf{c} \in |M|$ and $\psi(\mathbf{x}; \mathbf{c})$ either $p \cup \{\psi(\mathbf{x}; \mathbf{c})\}$ or $p \cup \{\neg\psi(\mathbf{x}; \mathbf{c})\}$ is realized by λ elements from $\mathbf{I}$. Iterating this, one can show that for every $C \subseteq |M|$ of cardinality $< \lambda$ there exists a type $q \in \mathrm{S}^n(C) \supseteq p$ realized by λ-many elements of $\mathbf{I}$. As q is realized by λ-many elements of $\mathbf{I}$ by minimality of α we have that $\mathrm{R}[q] \geq \alpha$, by Monotonicity as $q \supseteq p$ we get $\mathrm{R}[q] = \alpha$.

Define by induction on $i < \lambda$ an element $\mathbf{a}_i \in \mathbf{I}$ such that $p_i := \mathrm{tp}(\mathbf{a}_i / B \cup \{\mathbf{a}_j \mid j < i\}) \supseteq p$ and realized by $\geq \lambda$-many elements of $\mathbf{I}$. Notice that as α is the

minimal element of S we have that $R[p_i] = \alpha$ for all $i < \lambda$. Using Lemma 2.13 and Proposition 4.6 verify that $\mathbf{J} := \{\mathbf{a}_i \mid i < \lambda\}$ is as required as it satisfy the requirements in Lemma 4.9. $\square$

4. Definability of types is equivalent to stablity

One of the useful facts about stable theories is that for every type $p \in \mathrm{S}^{\mathbf{x}}(A)$ and for all formulas $\varphi(\mathbf{x};\mathbf{y})$ the property (of $\mathbf{a}$) "$\varphi(\mathbf{x};\mathbf{a})$ is a member of p" is first-order definable in $\mathfrak{C}$. I will make this notion of definability more precise and establish it equivalence to stability.

DEFINITION 4.1. Let $B \subseteq A$ be subsets of the monster model.

(1) Let $\varphi(\mathbf{x};\mathbf{y})$, $p \in \mathrm{S}^{\mathbf{x}}_{\varphi(\mathbf{x};\mathbf{y})}(A)$ and $\psi(\mathbf{y};\mathbf{z})$ be given. *The type p is ψ-definable over B* iff there exist $\mathbf{b} \in B$ such that
$$\varphi(\mathbf{x};\mathbf{a}) \in p \iff \mathfrak{C} \models \psi[\mathbf{a};\mathbf{b}] \quad \text{for all } \mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A}\,.$$
(2) Let $p \in \mathrm{S}^{\mathbf{x}}(A)$. The type p *is weakly B-definable* or p *is weakly definable over B* iff for every $\varphi(\mathbf{x};\mathbf{y})$ there exists a formula $\psi_{p,\varphi}(\mathbf{y};\mathbf{z}_{p,\varphi})$ such that $p \restriction \{\varphi\}$ is $\psi_{p,\varphi}$-definable over B
(3) If there exists a function d from formulas to formulas such that for every $\varphi(\mathbf{x};\mathbf{y})$ its value $d_\varphi(\mathbf{y};\mathbf{z})$ is a formula satisfying
$$\varphi(\mathbf{x};\mathbf{a}) \in p \iff \mathfrak{C} \models d_\varphi[\mathbf{a};\mathbf{b}] \quad \text{for all } \mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A} \text{ and all } \varphi(\mathbf{x};\mathbf{y})$$
for $p \in \mathrm{S}^{\mathbf{x}}(A)$, we say that the type p *is B-definable* or *definable over B*.

(4) Let $p \in \mathrm{S}^{\mathbf{x}}(A)$. The type p *is definable* iff there exists $B \subseteq A$ such that p is B-definable. Similarly: p *is weakly definable* iff there exists $B \subseteq A$ such that p is weakly B-definable.

PROPOSITION 4.2. *Let $p \in \mathrm{S}^{\mathbf{x}}(A)$ be definable (weakly definable) over some $B_1 \subseteq A$. If $B_2 \subseteq A$ satisfies $B_2 \supseteq B_1$ then p is definable (weakly definable) over B_2*

PROOF. Exercise. □

The following is yet another characterization of stability:

THEOREM 4.3. *Let T be a first-order complete theory. The following are equivalent*

(1) *T is stable*
(2) *for every A and every $p \in \mathrm{S}(A)$ the type p is weakly definable over A*
(3) *for every $\varphi(\mathbf{x};\mathbf{y})$ there exists a formula $\psi_\varphi(\mathbf{y};\mathbf{z}_\varphi)$ such that for every infinite A and every $p \in \mathrm{S}^{\mathbf{x}}(A)$ the type $p \restriction \varphi$ is ψ_φ-definable over A*

REMARK 4.4. Notice that in the notion of weakly definable type the defining formula ψ depends both on p and φ, however in the third part of the theorem we have a uniform ψ for all p that depends only on φ but not p.

First we prove the easy direction.

PROPOSITION 4.5 (weak definability implies stability). *If for every A and every $p \in \mathrm{S}(A)$ the type p is weakly definable over A then T is stable.*

PROOF. For the sake of contradiction suppose T is not stable in $2^{|T|}$. Let A be of cardinality $2^{|T|}$ such that $|\mathrm{S}(A)| > 2^{|T|}$. By Claim 5.13 there exists $\varphi(x;\mathbf{y})$ such that $|\mathrm{S}_{\varphi(x;\mathbf{y})}(A)| > |T|$. By the hypothesis for every $p \in \mathrm{S}_{\varphi(x;\mathbf{y})}(A)$ there are $\psi_\varphi(\mathbf{y};\mathbf{z})$ and $\mathbf{b} \in A$ such that

$$\varphi(x;\mathbf{a}) \in p \iff \mathfrak{C} \models \psi_\varphi[\mathbf{a};\mathbf{b}] \quad \text{for all } \mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A}\,.$$

Since $|\mathrm{S}_{\varphi(x;\mathbf{y})}(A)| > |T|$ there are $S \subseteq \mathrm{S}_{\varphi(x;\mathbf{y})}(A)$ of cardinality $|T|^+$, a sequence $\mathbf{a} \in A$ and a formula $\psi(\mathbf{y};\mathbf{z})$ such that

$$(*) \quad \varphi(x;\mathbf{a}) \in p \iff \mathfrak{C} \models \psi_\varphi[\mathbf{a};\mathbf{b}] \quad \text{for all } \mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A} \text{ and every } p \in S.$$

Pick $p_1 \neq p_2 \in S$. Let $\varphi(\mathbf{x};\mathbf{a}) \in p_1$ such that $\neg\varphi(\mathbf{x};\mathbf{a}) \in p_2$. Applying $(*)$ twice to φ produces the contradiction: $\mathfrak{C} \models \psi[\mathbf{a};\mathbf{b}]$ and $\mathfrak{C} \models \neg\psi[\mathbf{a};\mathbf{b}]$. □

The harder direction of Theorem 4.3 follows from the following two propositions:

PROPOSITION 4.6 (stability implies weak definability). *Let A, $\varphi(\mathbf{x};\mathbf{y})$ and $p \in \mathrm{S}^{\ell(\mathbf{x})}_\varphi(A)$ be given. If T is stable then there exists $\psi(\mathbf{x};\mathbf{b})$ with $\mathbf{b} \in A$ such that*

$$\varphi(\mathbf{x};\mathbf{a}) \in p \iff \mathfrak{C} \models \psi[\mathbf{a};\mathbf{b}] \quad \textit{for all } \mathbf{a} \in {}^{\ell(\mathbf{y})}\mathrm{A}\,.$$

PROOF. Since T is stable $n := \mathrm{R}[p,\varphi,2] < \omega$. Let q be finite of the same rank such that $q \subseteq p$.

For every $\mathbf{a} \in A$ define

$$q(\mathbf{a}) := q \cup \{\varphi(\mathbf{x};\mathbf{a})\}$$

CLAIM 4.7. $\varphi(\mathbf{x};\mathbf{a}) \in p \iff \mathrm{R}[q(\mathbf{a}),\varphi,2] \geq n$

PROOF. Notice that if $\varphi(\mathbf{x};\mathbf{a}) \in p$ then $q \subseteq q(\mathbf{a}) \subseteq p$ and application of monotonicity of the rank gives that $\mathrm{R}[q(\mathbf{a}),\varphi,2] = n$. By the uniqueness property of the rank in this case we have that $\mathrm{R}[q \cup \{\neg\varphi(\mathbf{x};\mathbf{a})\},\varphi,2] < n$.

On the other hand if $\varphi(\mathbf{x};\mathbf{a}) \notin p$ then as $\neg\varphi(\mathbf{x};\mathbf{a}) \in p$ we by monotonicity $\mathrm{R}[q \cup \{\neg\varphi(\mathbf{x};\mathbf{a})\},\varphi,2] = n$, thus by uniqueness $\mathrm{R}[q(\mathbf{a}),\varphi,2] < n$.

□

By the tree characterization (Lemma 2.28) we get that

$$\Gamma_{q(\mathbf{a})}(\varphi,n) \text{ is consistent} \iff \mathrm{R}[q(\mathbf{a}),\varphi,2] \geq n.$$

Now let

$$\psi_\varphi[\mathbf{a};\mathbf{b}] := \Big[\bigwedge_{\eta \in {}^n 2} [\exists \mathbf{x}_\eta \exists \mathbf{y}_{\eta\restriction 0} \exists \mathbf{y}_{\eta\restriction 1} \cdots \exists \mathbf{y}_{\eta\restriction (n-1)} \bigwedge \Gamma_{q(\mathbf{a})}(\varphi, n)] \Big].$$

Where $\mathbf{b}$ is an enumeration of the finite set $\operatorname{Dom} q$.

By the choice of q and the definition of $q(\mathbf{a})$ we have that

$$\varphi(\mathbf{x};\mathbf{a}) \in p \iff \mathfrak{C} \models \psi_\varphi[\mathbf{a};\mathbf{b}].$$

□

PROPOSITION 4.8 (Definability). *Suppose that T is stable.*
For every $\varphi(\mathbf{x};\mathbf{y})$ there exists a finite $\Delta_\varphi \subseteq \operatorname{Fml}(\mathrm{L}(T))$ such that for every A and every $p \in \mathrm{S}^{\mathbf{x}}(A)$ the type $p \restriction \{\varphi\}$ is definable by some $\psi \in \Delta$.

PROOF. Suppose for the sake of contradiction that there are a stable T and a formula φ such that for all finite Δ there exists A and $p \in \mathrm{S}^{\mathbf{x}}(A)$ such that $p \restriction \{\varphi\}$ is not definable with a formula from Δ.

Let $\langle b_1, \ldots, b_n \rangle \in {}^{\ell(\mathbf{x})}|\mathfrak{C}|$ be a realization of p. Let P be a new unary predicate symbol and $c_1, \ldots, c_n$ new constant symbols (not in $\mathrm{L}(T)$). Consider

$$T_\Delta := T \cup \Big\{ \neg \exists z_1, \ldots, z_n \Big[\bigwedge_{k=1}^n P(z_k) \to \forall y_1 \cdots \forall y_n \Big[\bigwedge_{k=1}^n P(y_k) \to [\varphi(\mathbf{c};\mathbf{y}) \leftrightarrow \psi(\mathbf{y}; z_1, \ldots, z_n)] \Big] \Big] \ : \ \psi \in \Delta \Big\}.$$

T_Δ is consistent for every finite Δ, since by the assumption $\langle \mathfrak{C}, A, \mathbf{b} \rangle \models T_\Delta$. By the compactness theorem there exists $M \models T$ with $A' \subseteq |M|$ and $\mathbf{b}' \in |M|$ such that $\langle M, A', \mathbf{b}' \rangle \models T_{\mathrm{L}(T)}$. Now by stability of T the type $\operatorname{tp}_\varphi(\mathbf{b}'/A')$ is weakly definable (Proposition 4.5), contradiction to the choice of $\mathbf{b}'$ and A'. □

PROPOSITION 4.9. *The last proposition establishes the implication* (1) $\implies$ (3) *from Theorem 4.3.*

PROOF. Given $\varphi(\mathbf{x};\mathbf{y})$ let Δ_φ be a finite set of formulas as in Proposition 4.8. Suppose $\psi(\mathbf{y};\mathbf{z}_i) \mid i < n\} = \Delta$. Now consider

$$\psi_\Delta(\mathbf{y};\mathbf{z}_0,\ldots,\mathbf{z}_{n-1};w,w_0,\ldots,w_{n-1}) := \bigwedge_{i<j<n} w_i \neq w_j \wedge \bigvee_{i<n} w = w_i \wedge \bigwedge_{i<n} [w = w_i \to \psi_i(\mathbf{y};\mathbf{z}_i)].$$

Fix $\mathbf{c} := \langle c_0, \ldots c_{n-1} \rangle$ be a sequence of distinct elements of A. By the previous proposition we have that for a given $p \in \mathrm{S}_\varphi(A)$ there exists $i < n$ such that p is ψ_i-definable and this witnessed by some $\mathbf{b} = \mathbf{b}_p$, but by our definition of ψ_Δ we have

$$\psi_i[\mathfrak{C};\mathbf{b}] = \psi_\Delta[\mathfrak{C};\mathbf{b};c_i,\mathbf{c}].$$

□

COROLLARY 4.10. *Suppose that $p \in \mathrm{S}^{\mathbf{x}}(M)$ is definable over $|M|$. If $A \supseteq |M|$ then there exists a unique $q \in \mathrm{S}^{\mathbf{x}}(A) \supseteq p$ also definable over $|M|$.*

PROOF. *Uniqueness:* Suppose FSOC that there are $q_1 \neq q_2 \in \mathrm{S}^{\mathbf{x}}(A)$ both extensions of p definable over $|M|$. Pick $\varphi(\mathbf{x};\mathbf{a}) \in q_1$ such that $\neg\varphi(\mathbf{x};\mathbf{a}) \in q_2$. Since p is definable over M there exists $\mathbf{b} \in M$ such that

$$\varphi(\mathbf{x};\mathbf{a}) \in p \iff \mathfrak{C} \models \psi_\varphi[\mathbf{a};\mathbf{b}]$$

As q_ℓ is also definable over M there exists $\mathbf{b}_\ell \in |M|$ such that

$$\varphi(\mathbf{x};\mathbf{a}) \in q_\ell \iff \mathfrak{C} \models \psi_\varphi[\mathbf{a};\mathbf{b}_\ell]$$

Since $p \subseteq q_1$ we have that $M \models \forall\mathbf{x}[\psi_\varphi(\mathbf{x};\mathbf{b}_1) \leftrightarrow \psi_\varphi(\mathbf{x};\mathbf{b})]$, similarly we get also $M \models \forall\mathbf{x}[\psi_\varphi(\mathbf{x};\mathbf{b}_2) \leftrightarrow \psi_\varphi(\mathbf{x};\mathbf{b})]$. Thus $M \models \forall\mathbf{x}[\psi_\varphi(\mathbf{x};\mathbf{b}_1) \leftrightarrow \psi_\varphi(\mathbf{x};\mathbf{b}_2)]$ and $\mathfrak{C} \models \forall\mathbf{x}[\psi_\varphi(\mathbf{x};\mathbf{b}_1) \leftrightarrow \psi_\varphi(\mathbf{x};b_2)]$ which is a contradiction to the assumptions that $\mathfrak{C} \models \psi_\varphi[\mathbf{a};\mathbf{b}_1]$ and $\mathfrak{C} \models \neg\psi_\varphi[\mathbf{a};\mathbf{b}_2]$.

Existence: As p is definable over M for every $\varphi(\mathbf{x};\mathbf{y})$ suppose $\mathbf{b} := \mathbf{b}_{p,\varphi} \in M$ witness that $p \restriction \varphi$ is definable. Let

$$q := \{\varphi(\mathbf{x};\mathbf{a}) \mid \varphi(\mathbf{x};\mathbf{y}) \in \mathrm{L}(T), \mathbf{a} \in A, \mathfrak{C} \models \psi_\varphi[\mathbf{a};\mathbf{b}]\}.$$

Clearly $q \supseteq p$. We need to show that q is consistent. If not there exists $\{\varphi_i(\mathbf{x};\mathbf{a}_i) \mid i < n\} \subseteq q$ such that

$$\mathfrak{C} \models \neg\exists\mathbf{x} \bigwedge_{i<n} \varphi(\mathbf{x};\mathbf{a}_i).$$

As we have also

$$\mathfrak{C} \models \bigwedge_{i<n} \psi_{\varphi_i}[\mathbf{a}_i;\mathbf{b}].$$

Quantifying out the $\mathbf{a}_i$'s we get

$$\mathfrak{C} \models \neg\exists\mathbf{x}\exists\mathbf{y}_0 \cdots \mathbf{y}_{n-1}[\bigwedge_{i<n} \varphi(\mathbf{x};\mathbf{y}_i) \wedge \bigwedge_{i<n} \psi_{\varphi_i}[\mathbf{y}_i;\mathbf{b}]].$$

As $M \prec \mathfrak{C}$ there are $\{\mathbf{a}_i^* \mid i < n\} \subseteq |M|$ such that

$$M \models \neg\exists\mathbf{x}[\bigwedge_{i<n} \varphi(\mathbf{x};\mathbf{a}_i^*) \wedge \bigwedge_{i<n} \psi_{\varphi_i}[\mathbf{a}_i^*;\mathbf{b}]].$$

By the second part of the above formula we get that $\{\varphi(\mathbf{x};\mathbf{a}_i^*) \mid i < n\} \subseteq p$, by the first part this set is inconsistent, this is a contradiction to the assumption that p is a type.

□

DEFINITION 4.11. Let $n < \omega$.

$\kappa^n_{\mathrm{def}}(T) := \min\{\kappa \mid$ for every $A, p \in \mathrm{S}^n(A)$ there exists $B \subseteq A$ s.t. p is definable over $B\}$.

If a κ as above does not exists we write $\kappa^n_{\mathrm{def}}(T) := \infty$.

$$\kappa_{\mathrm{def}}(T) := \sup_{n<\omega} \kappa^n_{\mathrm{def}}(T).$$

LEMMA 4.12. *If T is stable then $\kappa_{\rm def}(T) \leq |T|^+$.*

PROOF. Let $p \in \mathrm{S}^{\mathbf{x}}(A)$ be given. As T is stable, for every $\varphi(\mathbf{x};\mathbf{y})$ there are $\psi_\varphi(\mathbf{y};\mathbf{z}_\varphi)$ and $\mathbf{b}_{\varphi,p} \in {}^{\ell(\mathbf{y})}\mathrm{A}$ such that

$$\varphi(\mathbf{x};\mathbf{a}) \in p \iff \mathfrak{C} \models \psi_\varphi(\mathbf{a};\mathbf{b}_{\varphi,p}) \text{ for all } \mathbf{a} \in A.$$

Take B to be $\{\mathbf{b}_{\varphi,p} \mid \varphi(\mathbf{x};\mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T))\}$. Clearly $|B| \leq |T|$ and p is definable over B. □

COROLLARY 4.13. *If T is stable then $\kappa_s(T) \leq |T|^+$. Moreover, without any stability assumption of T we have that $\kappa_s(T) \leq \kappa_{\rm def}(T)$.*

The Corollary follows from the following Lemma (that does not assume stability):

LEMMA 4.14. *If $p \in \mathrm{S}^{\mathbf{x}}(A)$ is definable over B then p does not split over B.*

PROOF. Given $p \in \mathrm{S}^{\mathbf{x}}(A)$, suppose that $B \subseteq A$ is such that p is definable over B.

Let $\mathbf{a},\mathbf{c} \in A$ be such that $\mathrm{tp}(\mathbf{a}/B) = \mathrm{tp}(\mathbf{c}/B)$, and $\varphi(\mathbf{x};\mathbf{y})$ with its defining formula $\psi(\mathbf{y};\mathbf{b})$ (where $\mathbf{b} \in B$).

We get

$$\varphi(\mathbf{x};\mathbf{a}) \in p \iff \mathfrak{C} \models \psi[\mathbf{a};\mathbf{b}] \iff \mathfrak{C} \models \psi[\mathbf{c};\mathbf{b}] \iff \varphi(\mathbf{x};\mathbf{c}) \in p.$$

□

5. The order dichotomy

The following property will be shown to be equivalent to instability.

DEFINITION 5.1. We say that a complete first order theory has the *order-property* iff there are a positive integer ℓ, a formula $\varphi(\mathbf{x};\mathbf{y})$ and a set $\{\mathbf{a}_n \ : \ n<\omega\}$ such that for every $n<\omega$ we have that $\ell(\mathbf{a}_n)=\ell(\mathbf{x})=\ell(\mathbf{y})=\ell$ and for every $n,m<\omega$ we have that

$$n<m \iff \mathfrak{C} \models \varphi[\mathbf{a}_n;\mathbf{a}_m].$$

We say that a formula $\varphi(\mathbf{x};\mathbf{y})$ with the above property *witnesses the order-property* .

Clearly the theories T_{ord}, Zermelo-Fraenkel and $\mathrm{Th}(\langle \mathbb{R},+,\cdot\rangle)$ have the order property. There are many more examples of theories with the order property. The main theorem of this section implies that the order property is equivalent to instability.

In Theorem 5.19 we will generalize Exercise 2.2.**??**, by showing that if T has the order property then T does not have saturated models in any singular cardinal greater than $|T|$.

A related concept is

DEFINITION 5.2. Let $\varphi(\mathbf{x};\mathbf{y})\in \mathrm{L}(T)$. $\varphi(\mathbf{x};\mathbf{y})$ *has the order property* if there is a set $\{\mathbf{a}_n \mid n<\omega\}$ with $\ell(\mathbf{a}_n)=\ell(\mathbf{y})$ such that for every $k<\omega$ the type $\{\varphi(\mathbf{x};\mathbf{a}_n)^{\text{if } k\leq n} \mid n<\omega\}$ is consistent.

We introduce a *local* version of the rank function $\mathrm{R}[p]$ from Definition 2.14 and study some of its properties:

DEFINITION 5.3. Let p be a set of formulas in $\mathbf{x}$ and $\varphi(\mathbf{x};\mathbf{y})\in \mathrm{Fml}(\mathrm{L}(T))$.

(1) $\mathrm{R}[p,\varphi]\geq\alpha$ is defined by induction on α:
   (a) $\mathrm{R}[p,\varphi]\geq 0$, if p is consistent;
   (b) for α limit, $\mathrm{R}[p,\varphi]\geq\alpha$ if $\mathrm{R}[p,\varphi]\geq\beta$ for every $\beta<\alpha$
   (c) $\mathrm{R}[p,\varphi]\geq\alpha+1$ if for every finite $q\subseteq p$ there exists $\mathbf{a}$ such that $\mathrm{R}[q\cup\varphi(\mathbf{x};\mathbf{a}),\varphi]\geq\alpha$ and $\mathrm{R}[q\cup\neg\varphi(\mathbf{x};\mathbf{a}),\varphi]\geq\alpha$.
(2) For an ordinal α denote by $\mathrm{R}[p,\varphi]=\alpha$ the statement
$$\mathrm{R}[p,\varphi]\geq\alpha \quad \text{and} \quad \mathrm{R}[p,\varphi]\not\geq\alpha+1.$$
(3) We write $\mathrm{R}[p,\varphi]=\infty$ if $\mathrm{R}[p,\varphi]\geq\alpha$ for every ordinal α.

Similarly to the way we proved the basic properties of the function $\mathrm{R}[\cdot]$ one can prove

LEMMA 5.4.

(1) *(Monotonicity) If* $p \vdash q$ *then* $\mathrm{R}[p,\varphi] \leq \mathrm{R}[q,\varphi]$.
(2) *(Finite character) for every* p *there exists a finite* $q \subseteq p$ *such that* $\mathrm{R}[p,\varphi] = \mathrm{R}[q,\varphi]$.
(3) *(Uniqueness) Suppose* $A \subseteq B$ *and* $p \in \mathrm{S}_\varphi(A) \subseteq q_\ell \in \mathrm{S}_\varphi(B)$*, for* $\ell = 1, 2$*. If* $\mathrm{R}[p,\varphi] = \mathrm{R}[q_\ell,\varphi] < \infty$ *for both* $\ell = 1$ *and* $\ell = 2$ *then* $q_1 = q_2$.
(4) *(Invariance) For any* $f \in \mathrm{Aut}(\mathfrak{C})$ *we have that* $\mathrm{R}[p,\varphi] = \mathrm{R}[f(p),\varphi]$

PROOF. Exercise. □

Like [] use finite character to establish

PROPOSITION 5.5. *Let* p *be an* $\mathbf{x}$*-type and* $\varphi(\mathbf{x};\mathbf{y})$ *be given. If* $\mathrm{R}[p,\varphi] < \infty$ *then there exists a finite* $A \subseteq \mathrm{dom}\, p$ *such that* p *does not* (L,φ)*-split over* A.

Using an argument similar to the one used in Theorem [], the reader should verify

THEOREM 5.6 (Bounded rank implies local stability). *If* $\mathbf{x}$ *is a finite sequence and* $\mathrm{R}[\mathbf{x} = \mathbf{x},\varphi] < \infty$ *for some* $\varphi(\mathbf{x};\mathbf{y})$ *then for every infinite* A *we have that* $|\,\mathrm{S}^{\ell(\mathbf{x})}_{\varphi(\mathbf{x};\mathbf{y})}(A)| \leq |A|$.

Later we will introduce few more rank function whose definition will mimic Definitions 2.14 and 5.3. We often for a certain rank function R will associate a tree of formulas $\Gamma^{\mathrm{R}}(p,\alpha)$ such that $\mathrm{R}[p] \geq \alpha \iff$ the set $\Gamma^{\mathrm{R}}(p,\alpha)$ is consistent this kind of equivalence is called a *tree characterization lemma*. Now we will encounter our first such lemma.

In order to state the tree-characterization lemma we introduce the following device:

DEFINITION 5.7. Let p be a type in $\mathbf{x}$, let $\varphi(\mathbf{x};\mathbf{y})$ be a formula, and let α be an ordinal. Denote

$$\Gamma_p(\varphi,\alpha) := \{\varphi(\mathbf{x}_\eta;\mathbf{y}_{\eta\restriction\beta})^{\eta[\beta]} \mid \beta < \alpha,\ \eta \in {}^{\alpha}2\} \cup \bigcup_{\eta\in{}^{\alpha}2} p(\mathbf{x}_\eta).$$

LEMMA 5.8 (Tree characterization lemma). *Let* p *be a finite type in* $\mathbf{x}$ *and let* $\varphi(\mathbf{x};\mathbf{y})$ *be a formula. Then, for every* $n < \omega$*, the following are equivalent:*

(1) $\Gamma_p(\varphi,n)$ *is consistent;*
(2) $\mathrm{R}[p,\varphi] \geq n$.

PROOF. By the compactness theorem and finite character of the rank we may assume that p is finite.

Suppose $\mathrm{R}[p,\varphi] \geq n$.

By induction of $k \leq n$ define $\{\mathbf{a}_\nu \mid \nu \in {}^{k\geq}2\}$ such that for every $\eta \in {}^k2$ we have

$$(*) \quad \mathrm{R}[p_\eta, \varphi] \geq n - k$$

where

$$p_\eta := p \cup \{\varphi(\mathbf{x}; \mathbf{a}_{\eta\restriction i})^{\eta[i]} \mid i < k\}.$$

For $k = 0$ take $p_{\langle\rangle} := p$. Since $\mathrm{R}[p, \varphi] \geq n$ $(*)$ holds.

Suppose that $k < n$ and $\{p_\eta \mid \eta \in {}^{k\geq}2\}$ are defined.

Given $\eta \in {}^k2$ as $\mathrm{R}[p_\eta, \varphi] \geq n - k$ there exists $\mathbf{a}_\eta$ such that

$$\mathrm{R}[p_\eta \cup \{\varphi(\mathbf{x}; \mathbf{a}_\eta)\}, \varphi] \geq n - k - 1$$

and

$$\mathrm{R}[p_\eta \cup \{\neg\varphi(\mathbf{x}; \mathbf{a}_\eta)\}, \varphi] \geq n - k - 1.$$

For $t \in \{0, 1\}$ let $p_{\eta^\frown t} := p_\eta \cup \{\varphi(\mathbf{x}; \mathbf{a}_\eta)^t\}$.

When $k = n$ for every $\eta \in {}^n2$ we have $\mathrm{R}[p_\eta, \varphi] \geq n - n$, hence p_η is consistent. Let $\mathbf{b}_\eta \models p_\eta$. Now

$$\{\mathbf{b}_\eta, \mathbf{a}_{\eta\restriction\beta} \mid \eta \in {}^n2, \beta < n\} \models \Gamma_p(\varphi, n).$$

Now the converse that consistency of $\Gamma_p(\varphi, n)$ implies $\mathrm{R}[p, \varphi] \geq n$ left as an exercise. □

The following is a corollary of the compactness theorem:

PROPOSITION 5.9. *Let $\varphi(\mathbf{x}; \mathbf{y})$ be a given formula and let p be a set of formulas (perhaps with parameters from $\mathfrak{C}$ whose free variables are in $\mathbf{x}$. TFAE*

(1) *$\Gamma_p(\varphi, n)$ is consistent for every $n < \omega$,*
(2) *$\Gamma_p(\varphi, \omega)$ is consistent,*
(3) *$\Gamma_p(\varphi, \alpha)$ is consistent for every ordinal α*

PROOF. Exercise. □

In the next theorem we use the tree characterization lemma to establish

THEOREM 5.10. *Let $\varphi(\mathbf{x}; \mathbf{y})$ be a given formula and let p be a set of formulas (perhaps with parameters from $\mathfrak{C}$ whose free variables are in $\mathbf{x}$. TFAE*

(1) *$\mathrm{R}[p, \varphi] < \omega$,*
(2) *$\mathrm{R}[p, \varphi] < \infty$.*

PROOF. Clearly $(1) \implies (2)$.

(2) $\implies$ (1) : For the sake of contradiction suppose that (2) holds and $\mathrm{R}[p,\varphi] \geq \omega$. By the tree characterization lemma and Proposition 5.9 we get that for every limit α the set $\Gamma_p(\varphi,\alpha)$ is consistent. Let

$$\{\mathbf{b}_\eta, \mathbf{a}_{\eta\restriction\beta} \mid \eta \in {}^{\alpha}2, \beta < \alpha\} \models \Gamma_p(\varphi,\alpha).$$

For every $\eta \in {}^{\alpha>}2$ consider

$$p_\eta := p \cup \{\varphi(\mathbf{x};\mathbf{y}_{\eta\restriction\beta})^{\eta[\beta]} \mid \beta < \ell(\eta)\}.$$

By the assumption (2) the set $S := \{\mathrm{R}[p_\eta,\varphi] \mid \eta \in {}^{\alpha>}2\}$ is a set of ordinals. Let $\alpha := \min S$ and let $\eta \in {}^{\alpha>}2$ be such that $\alpha = \mathrm{R}[p_\eta,\varphi]$.

For $t \in \{0,1\}$ let $q_t := p_{\eta^\frown t}$. By minimality of α we get

$$\mathrm{R}[q_0,\varphi] \geq \alpha \text{ and } R[q_1,\varphi] \geq \alpha.$$

Since $q_t = p_\eta \cup \{\varphi(\mathbf{x};\mathbf{a})^t\}$, by the definition of R we get $\mathrm{R}[p_\eta,\varphi] \geq \alpha+1$ which is a contradiction to the choice of α and p_η. □

The main result of this section is the following characterization of stable theories:

THEOREM 5.11 (Characterization of stable theories). *Let T be a complete first-order theory. The following are equivalent*

(1) *T is unstable.*
(2) *T has the order-property.*
(3) *There exists $\varphi(\mathbf{x};\mathbf{y})$ has the order property.*
(4) *There exist a formula $\varphi(\mathbf{x};\mathbf{y})$ and an infinite set A such that $|A| < |\mathrm{S}_\varphi(A)|$.*
(5) *There exist a formula $\varphi(\mathbf{x};\mathbf{y})$ such that for every $\lambda \geq \aleph_0$ there A of cardinality λ such that $\lambda < |\mathrm{S}_\varphi(A)|$.*
(6) *$\mathrm{R}[\mathbf{x}=\mathbf{x},\varphi] \geq \omega$ for some $\varphi(\mathbf{x};\mathbf{y})$.*
(7) *$\mathrm{R}[\mathbf{x}=\mathbf{x},\varphi] \geq \infty$ for some $\varphi(\mathbf{x};\mathbf{y})$.*

Before starting the proof, I present an elementary fact about order linearly-ordered sets that generalizes the relationship between the real and the rational numbers:

LEMMA 5.12 (Existence of certain linearly-ordered sets). *For every infinite cardinal λ there exists a linearly-ordered set I of cardinality $> \lambda$ such that there exists $J \subseteq I$ of cardinality $\leq \lambda$ dense in I (i.e. for every $a,b \in I$ if $a <_I b$ then there exists $c \in J$ satisfying $a <_I c <_I b$) such that $|J| \leq \lambda$.*

PROOF. Let $\mu := \min\{\mu \mid 2^\mu > \lambda\}$. Notice that since $2^\lambda > \lambda$ always $\mu \leq \lambda$. Let I to be the set $\bigcup_{\alpha\leq\mu} {}^{\alpha}2$ ordered lexicographically, let $J := \bigcup_{\alpha<\mu} {}^{\alpha}2$. Since μ is the minimal such that $2^\mu > \lambda$ we get $|J| = 2^{<\mu} \leq \lambda$. □

PROOF OF THEOREM 5.11. $(1) \implies (2),(3)$ (*The hard direction*): We first show that it is enough to show a local version of the theorem:

Let A that witness the instability. By Proposition 1.3 we may assume that A is a set of cardinality λ such that $|\operatorname{S}^1(A)| > \lambda$.

CLAIM 5.13. *If T is not stable in some λ satisfying $\lambda = \lambda^{|T|}$ then there exists a formula $\varphi(x;\mathbf{y})$ and a set A such that $|\operatorname{S}^1_\varphi(A)| > \lambda = |A|$.*

PROOF. Fix $\{\varphi_i \ : \ i < |T|\}$ an enumeration of all the formulas in $\operatorname{L}(T)$. Suppose for the sake of contradiction that for every $i < |T|$ we have $|\operatorname{S}^1_\varphi(A)| \leq \lambda$. For every $p \in \operatorname{S}^1(A)$ define $\bar{p} \in \Pi_{i<|T|} S_{\varphi_i}(A)$ such that $\bar{p}_i = p \restriction \{\varphi_i, \neg\varphi_i\}$. Now consider the mapping

$$p \mapsto \bar{p}.$$

Observe that the mapping is injective.

The range of the mapping is $\Pi_{i<|T|} \operatorname{S}_{\varphi_i}(A)$, the following is an estimate of its cardinality

$$|\Pi_{i<|T|} \operatorname{S}_{\varphi_i}(A)| = \Pi_{i<|T|} |\operatorname{S}_{\varphi_i}(A)| \leq \Pi_{i<|T|} \lambda = \lambda^{|T|} = \lambda.$$

Since the domain of the mapping has cardinality $\geq \lambda^+$ there exists a $S \subseteq \operatorname{S}^1(A)$ of cardinality λ^+ such that for every $p, q \in S$ we have that $\bar{p} = \bar{q}$. Picking $p \neq q \in S$ contradicts the injectivity of the mapping. □

COROLLARY 5.14. *T is stable iff T is stable in every λ satisfying $\lambda = \lambda^{|T|}$. In particular, T is stable iff it is stable in $2^{|T|}$.*

PROOF. We need to prove that stability in a cardinal implies stability in $2^{|T|}$. We will show that stability is a cardinal implies stability in any λ satisfying $\lambda = \lambda^{|T|}$, notice that $(2^{|T|})^{|T|} = 2^{|T|}$.

Let λ be a cardinal satisfying $\lambda = \lambda^{|T|}$. For the sake of contradiction suppose that T is not stable in λ. By Claim 5.13 there exists a formula and an infinite set satisfying statement (5) in Theorem 5.11. Since $(5) \implies (1)$ we get that T is unstable. □

It suffices to prove:

LEMMA 5.15. *(Main Lemma) Let $\varphi(x;\mathbf{y})$ be a formula and suppose A is an infinite set. If $|\operatorname{S}^1_\varphi(A)| > |A|$ then T has the order-property, moreover the formula*
$\rho(x_1,\mathbf{y}_1,\mathbf{z}_1;x_2,\mathbf{y}_2,\mathbf{z}_2)$ witness the order-property where the formula ρ is defined as $[\varphi(x_1,\mathbf{y}_2) \leftrightarrow \varphi(x_1,\mathbf{z}_2)]$

PROOF. Denote by $\psi(\mathbf{y};x)$ the formula $\varphi(x;\mathbf{y})$, the difference is that we think of φ as formula in the free variable x and substitute parameters for $\mathbf{y}$, while in ψ the role of parameters and free variables is reversed. Let $\lambda := |A|$.

SUB CLAIM 5.16. *There exists* $\{A_n \ : \ n < \omega\}$ *such that*

(1) $A \subseteq A_0$,
(2) *for all* $n < \omega$ *the cardinality of* A_n *is* λ,
(3) $n < \omega \implies A_n \subseteq A_{n+1}$,
(4) *for all* $n < \omega$, *every* $p \in S^1_\varphi(A_n) \cup \mathrm{S}^{\ell(\mathbf{y})}_\psi(A_n)$ *and every finite* $B \subseteq A_n$ *the type* $p \restriction B$ *is realized in* A_{n+1}.

PROOF. By induction on n define A_n: Take $A_0 := A$.

Observe that for a finite $B \subseteq A_n$ the cardinality of

$$\{q \restriction B \ : \ q \in \mathrm{S}^{\ell(\mathbf{y})}_\psi(A_n)\} \quad \text{is at most} \quad 2^{|B|},$$

while the cardinality of $\{q \restriction B \ : \ q \in \mathrm{S}^1_\varphi(A_n)\}$ is no more than $2^{|B|^{\ell(\mathbf{y})}}$.

Thus in particular we have that

$$(*) \quad |B| < \aleph_0 \implies |\{p \restriction B \ : \ p \in \mathrm{S}^1_\varphi(A_n) \cup \mathrm{S}^{\ell(\mathbf{y})}_\psi(A_n)\}| < \aleph_0.$$

Since $|A_n| = \lambda$ is infinite, we can enumerate by $\{B_i \ : \ i < \lambda\}$ all the finite subsets of A_n, using $(*)$ we can enumerate by $\{q_j \ : \ j < \lambda\}$ the set $\{p \restriction B_i \ : \ p \in \mathrm{S}^1_\varphi(A_n) \cup \mathrm{S}^{\ell(\mathbf{y})}_\psi(A_n), \ i < \lambda\}$. Now for every $j < \lambda$ pick $\mathbf{a}_j \models q_j$ and define A_{n+1} to be the set $A_n \cup \{\mathbf{a}_j \ : \ j < \lambda\}$. □

By the hypothesis we can fix $\{a_i \ : \ i < \lambda^+\}$ such that $i \neq j < \lambda$ implies that $\mathrm{tp}_\varphi(a_i/A) \neq \mathrm{tp}_\varphi(a_j/A)$.

SUB CLAIM 5.17. *There exists* $i < \lambda^+$ *such that for every* $n < \omega$ *the type* $\mathrm{tp}_\varphi(a_i/A_{n+1})$ (ψ, φ)*-splits over* B *for every finite* $B \subseteq A_n$.

PROOF. Suppose not. Thus for every $i \in \lambda^+$ there are $n(i) < \omega$ and a finite $B_i \subseteq A_{n(i)}$ such that $\mathrm{tp}_\varphi(a_i/A_{n(i)+1})$ does not (ψ, φ)-split over B_i.
Consider $i \mapsto n(i)$. Since the range has cardinality $\aleph_0$ by the pigeonhole principle there are $S_1 \subseteq \lambda^+$ of cardinality λ^+ and an integer n such that

$$i \in S_1 \implies n(i) = n.$$

Consider $i \mapsto B_i$. Since the cardinality of the range of the mapping is λ and its domain has cardinality λ^+ by the pigeonhole principle there are $S_2 \subseteq S_1$ of cardinality λ^+ and B such that $i \in S_2 \implies B_i = B$.

For $i \in S_2$ since $\mathrm{tp}_\varphi(a_i/A_{n+1})$ does not (ψ, φ)-split over B for all $\mathbf{c}$ and $\mathbf{d}$ in A_{n+1} we have that

$$(*) \quad \text{if} \quad \mathrm{tp}_\psi(\mathbf{c}/B) = \mathrm{tp}_\psi(\mathbf{d}/B) \quad \text{then} \quad \models \varphi[a_i; \mathbf{c}] \leftrightarrow \varphi[a_i; \mathbf{d}].$$

By requirement (4) of Claim 5.16 there exists a finite $C \subseteq A_{n+1}$ such that every type $q \in S^{\ell(\mathbf{y})}_{\psi}(B)$ is realized in C.

Consider the mapping $i \mapsto \mathrm{tp}_{\varphi}(a_i/C)$, it has a range of cardinality $\leq 2^{|C|^{\ell(\mathbf{y})}} < \aleph_0 < \lambda^+$ there is a $S_3 \subseteq S_2$ of cardinality λ^+ such that

$$\mathrm{tp}_{\varphi}(a_i/C) = \mathrm{tp}_{\varphi}(a_j/C) \text{ for every } i, j \in S_3.$$

Pick $i \neq j \in S_3$. By the choice of the a_i's there exists $\mathbf{a} \in A$ such that $\models \varphi[a_i; \mathbf{a}]$ and $\models \neg\varphi[a_j; \mathbf{a}]$. By the choice of C there exists $\mathbf{b} \in C$ such that $\mathrm{tp}_{\psi}(\mathbf{b}/B) = \mathrm{tp}_{\psi}(\mathbf{a}/B)$. Using $(*)$ we get

$$(a) \quad \models \varphi[a_i; \mathbf{a}] \leftrightarrow \varphi[a_i; \mathbf{b}].$$

Using that $i, j \in S_3$ we get

$$\models \varphi[a_i; \mathbf{b}] \leftrightarrow \varphi[a_j; \mathbf{b}]$$

Another application of $(*)$ gives

$$(b) \quad \models \varphi[a_j; \mathbf{b}] \leftrightarrow \varphi[a_j; \mathbf{a}].$$

(a) and (b) give

$$\models \varphi[a_i; \mathbf{a}] \leftrightarrow \varphi[a_j; \mathbf{a}].$$

This is a contradiction to the choice of $\mathbf{a}$. $\square$

Using Sub Claim 5.17 there exists $c \in \{a_i \mid i < \lambda^+\}$ such that $p_n := \mathrm{tp}(c/A_{2n+1})$ has the following property: The type p_n splits over every finite subset of A_{2n} for every $n < \omega$.

By induction on $n < \omega$ define $\mathbf{a}_n, \mathbf{b}_n \in A_{2n+1}$ and $c_n \in A_{2n+2}$ such that $\mathbf{a}_n$ and $\mathbf{b}_n$ witness that p_n splits over B_n, namely $\varphi(x; \mathbf{a}_n) \in p_n$, $\neg\varphi(x; \mathbf{b}_n) \in p_n$ and $\mathrm{tp}_{\psi}(\mathbf{a}_n/B_n) = \mathrm{tp}_{\psi}(\mathbf{b}_n/B_n)$, where

$$B_n := \{\mathbf{a}_k, \mathbf{b}_k, c_k \ : \ k < n\}.$$

Using Sub Claim 5.16 pick $c_n \in A_{2n+2}$ a realization of the type

$$p_n \restriction (B_n \cup \{\mathbf{a}_n, \mathbf{b}_n\}).$$

In order to complete the proof of the Main Lemma (Lemma 5.15) it is enough to show that

$$\models \rho[\mathbf{d}_n; \mathbf{d}_k] \iff n < k < \omega \quad \text{where}$$

$\mathbf{d}_n := c_n{}^\frown \mathbf{a}_n{}^\frown \mathbf{b}_n$:

By the choice of $\mathbf{a}_n$ and $\mathbf{b}_n$ w we have that

$$\models \varphi[c; \mathbf{a}_n] \wedge \neg\varphi[c; \mathbf{b}_n] \text{ and } \mathrm{tp}_{\psi}(\mathbf{a}_n/B_n) = \mathrm{tp}_{\psi}(\mathbf{b}_n/B_n).$$

Since $c_k \models p_{2k} \restriction (B_n \cup \{\mathbf{a}_k, \mathbf{b}_k\})$ for $k \geq n$ we get that

$$(*) \quad \models \varphi[c_k; \mathbf{a}_n] \wedge \neg\varphi[c_k; \mathbf{b}_n] \quad \text{for every} \quad k \geq n.$$

The definition of ρ gives

$$n \leq k \implies \models \neg\rho[\mathbf{d}_n; \mathbf{d}_k].$$

When $n > k$ from

$$\mathrm{tp}_\psi(\mathbf{a}_n/B_n) = \mathrm{tp}_\psi(\mathbf{b}_n/B_n) \quad \text{and} \quad c_k \in B_n$$

we get that

$$\models \psi[\mathbf{a}_n; c_k] \iff \models \psi[\mathbf{b}_n; c_k]$$

which by the definition of ψ gives that

$$(**) \quad \models \varphi[c_k; \mathbf{a}_n] \leftrightarrow \varphi[c_k; \mathbf{b}_n] \quad \text{for every} \quad k < n.$$

The definition of ρ gives

$$n > k \implies \models \rho[\mathbf{d}_n; \mathbf{d}_k].$$

It is enough to prove

(6) $\implies$ (5) *The "easy" direction*: Since $\mathrm{R}[p, \psi] \geq \omega$, then, by Lemma 5.8, $\Gamma_{\mathbf{x}=\mathbf{x}}(\psi, \omega)$ is consistent and gives rise to many ψ-types.

Given a cardinal $\lambda \geq |\mathrm{L}(T)|$ we show that T is not stable in λ.
Let $\mu := \min\{\mu \ : \ 2^\mu > \lambda\}$. By Cantor's theorem we have that $\mu \leq \lambda$ and from the minimality of μ we get that for every cardinality $\kappa < \mu$ we have that $2^\kappa \leq \lambda$. By the Compactness Theorem, implies that $\Gamma_{\mathbf{x}=\mathbf{x}}(\varphi, \mu)$ is consistent.

Let $\{\mathbf{a}_\eta \mid \eta \in {}^{\mu>}2\}$ be realizations of $\mathbf{y}_\eta$'s from the tree. Thus, for every $\eta \in {}^{\mu}2$, $p_\eta := p \cup \{\psi(\mathbf{x}; \mathbf{a}_{\eta\restriction\beta})^{\eta[\beta]} \mid \beta < \mu\}$ is consistent.

Take $A := \{\mathbf{a}_\eta \mid \eta \in {}^{\mu>}2\}$. Clearly $|A| = 2^{\leq\mu} \leq \lambda < 2^\mu \leq |\mathrm{S}_\psi(A)|$.

□

The non-trivial direction follows from the following important analog of the Main Lemma above (Lemma 5.15):

LEMMA 5.18. *Let $\varphi(\mathbf{x}; \mathbf{y})$ be a formula. If there exist an infinite set A such that $|\mathrm{S}^{\ell(\mathbf{x})}_\varphi(A)| > |A|$ then either $\varphi(\mathbf{x}; \mathbf{y})$ has the order-property or $\neg\varphi(\mathbf{x}; \mathbf{y})$ has the order-property.*

PROOF. Suppose that p, $\mathbf{c}$ and $\{\mathbf{a}_n, \mathbf{b}_n, \mathbf{c}_n \ : \ n < \omega\}$ are as in the proof of Lemma 5.15. Thus we have that

$$(*) \quad \models \varphi[\mathbf{c}_k; \mathbf{a}_n] \wedge \neg\varphi[\mathbf{c}_k; \mathbf{b}_n] \quad \text{for every} \quad k \geq n.$$

and

$$(**) \quad \models \varphi[\mathbf{c}_k; \mathbf{a}_n] \leftrightarrow \varphi[\mathbf{c}_k; \mathbf{b}_n] \quad \text{for every} \quad k < n.$$

Now define for $k < n$

$$F(k,n) := \begin{cases} 1 & \text{if } \models \varphi[\mathbf{c}_k; \mathbf{a}_n] \\ \\ 0 & \text{Otherwise.} \end{cases}$$

By Ramsey's theorem there are an infinite $S \subseteq \omega$ and $\ell \in \{0,1\}$ such that for all $k < n \in S$ we have that $F(k,n) = \ell$.

Suppose that $\ell = 1$; in this case we show that $\{\mathbf{a}_n \ : \ n \in S\}$ witness that $\varphi(\mathbf{x};\mathbf{y})$ has the order property: Clearly for $k < n$ □

We illustrate now an application:

THEOREM 5.19. *Suppose T is unstable. If $\lambda > |T|$ is a singular cardinal then T does not have a locally saturated model of cardinality λ.*

PROOF. Suppose that $M \models T$ is a saturated model of cardinality λ and $\varphi(\mathbf{x};\mathbf{y})$ witness the order property . Let $\{\lambda_\alpha \mid \alpha < \operatorname{cf}\lambda\}$ be increasing sequence and $\{\mathbf{b}_\alpha \in M_{\alpha+1} \mid \alpha < \operatorname{cf}\lambda\}$ such that $\lambda > |T| + \aleph_0$ and $\sum_{\alpha<\operatorname{cf}\lambda} \lambda_\alpha = \lambda$. Pick $\{M_\alpha \mid \alpha < \operatorname{cf}\lambda\}$ an increasing elementary chain such that: $M_\alpha \prec M$, $|M_\alpha| = \lambda_\alpha$ and the following condition holds: Let $S \subseteq |M_\alpha|$ be a maximal subset (with respect to inclusion) such that the set

$$p_\alpha(\mathbf{y}) := \{\varphi(\mathbf{a};\mathbf{y}) \mid \mathbf{a} \in {}^{\ell(\mathbf{x})}\mathrm{S}\} \text{ is consistent.}$$

Using λ_α^+-saturation of $M_{\alpha+1}$ let $\mathbf{b}_\alpha \in |M_{\alpha+1}|$ such that

$$M \models \varphi[\mathbf{a};\mathbf{b}_\alpha] \text{ for every } \mathbf{a} \in \operatorname{dom} p_\alpha.$$

Now, consider

$$p := \{\varphi(\mathbf{b}_\alpha;\mathbf{y}) \mid \alpha < \operatorname{cf}\lambda\}.$$

Verify that p is omitted in M. □

Exercises 8.5.

(1) Suppose that $\varphi(\mathbf{x};\mathbf{y})$ witness the order property. Show that any linearly-ordered set I there exists $\mathbf{I} := \{\mathbf{a}_i \mid i \in I\}$ an indiscernible sequence such that

$$\mathfrak{C} \models \varphi[\mathbf{a}_i;\mathbf{a}_j] \iff i < j \quad \text{for all } i,j \in I.$$

(2) Suppose that $\varphi(\mathbf{x};\mathbf{y})$ has the order property. Show that for any linearly-ordered set I there exists $\mathbf{I} := \{\mathbf{a}_i \mid i \in I\}$ an indiscernible sequence with $\ell(\mathbf{y}) = \ell(\mathbf{a}_i)$ for all $i \in I$ such that

$$\{\varphi(\mathbf{x};\mathbf{a}_j)^{i\leq j} \mid j \in I\} \quad \text{is consistent for all } i \in I.$$

(3) Prove Lemma 5.4
(4) Prove proposition 5.5.
(5) Prove Theorem 5.6.
(6) Using Lemma 5.12 give an alternative proof to $(2) \implies (1)$ of Theorem 5.11.
(7)
(8)
(9)
(10)

6. Sequences and sets of indiscernibles

Recall from section 3 of Chapter 2 the notions of sequence of indiscernibles and set. Here for we will show that under weak assumptions on the theory T (stability) one can find rather large sets of indiscernibles inside arbitrary sets.

DEFINITION 6.1. Let A be a subset of the monster model. Let $\langle I, <_I \rangle$ be a linearly ordered set and let $\mathbf{I} = \{\mathbf{a}_i \ : \ i \in I\}$

(1) $\mathbf{I}$ *is a set of indiscernibles over* A iff for every $n < \omega$ and every $\{i_1, \ldots, i_n\} \subseteq I$ and every $\{j_1, \ldots, j_n\} \subseteq I$ we have
$$\mathrm{tp}(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}/A) = \mathrm{tp}(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}/A).$$
(2) $\mathbf{I}$ *is a sequence of indiscernilbles over* A iff for every $n < \omega$ and every $i_1 <_I \cdots <_I i_n \in I$ and every $j_1 <_I \cdots <_I j_n \in I$ we have
$$\mathrm{tp}(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}/A) = \mathrm{tp}(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}/A).$$

EXAMPLE 6.2. Any set of linearly independent elements in a vector space, or algebraically independent elements in a field are examples for sets of indiscernible.

We will start with another characterization of stability and concludes with the main result of this section is a generalization of Theorem 7.**??**.

THEOREM 6.3 (Characterizing stability by sets of indiscernible sequences). *Let T be a theory. The theory T is stable iff every infinite sequence of indiscenibles is a set of indiscernible.*

PROOF. If T is unstable then it has the order property. I.e. there are $\varphi(\mathbf{x}; \mathbf{y})$ and $\mathbf{I} := \{\mathbf{a}_n \mid n < \omega\}$ such that
$$\mathfrak{C} \models \varphi[\mathbf{a}_k; \mathbf{a}_n] \iff k < n \text{ for all } n, k < \omega.$$
By Exercise 4.3.20 we may assume that $\mathbf{I}$ is a sequence of indiscernible. Since $\mathfrak{C} \models \varphi[\mathbf{a}_0; \mathbf{a}_1] \wedge \neg\varphi[\mathbf{a}_1; \mathbf{a}_0]$ we get that $\mathrm{tp}(\mathbf{a}_0, \mathbf{a}_1/\emptyset) \neq \mathrm{tp}(\mathbf{a}_1, \mathbf{a}_0/\emptyset)$.

Suppose $\mathbf{I} = \{\mathbf{a}_i \mid i \in I\}$ is an infinite sequence of indiscernible which is not a set of iniscenibles. We will find a formula ψ and a countable set A such that $\mathrm{S}_\psi(A)$ is uncountable, this by Theorem 5.11 implies that T is unstable. We will be using an elementary property of permutations of finite sets: Every $\sigma \in S_n$ is a product of transpositions (see Corollary I.6.5 of [**Hu**]) of the form $(k, k+1)$ when $1 \leq k \leq n$.

Since $\mathbf{I}$ is not an indiscernible set there are $i_1 < \cdots < i_n \in I$, a permutation $\sigma \in S_n$ and a formula $\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ such that
$$\mathfrak{C} \models \varphi[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}] \wedge \neg\varphi[\mathbf{a}_{\sigma(i_1)}, \ldots, \mathbf{a}_{\sigma(i_n)}].$$

Since σ is a product of transpositions there exists $k \in \{1, \ldots, n\}$ such that

$$(*) \quad \mathfrak{C} \models \varphi[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}] \wedge \neg\varphi[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{k-1}}, \mathbf{a}_{i_{k+1}}, \mathbf{a}_{i_k}, \ldots, \mathbf{a}_{i_n}].$$

Since $\mathbf{I}$ is a sequence of indiscernibles $(*)$, holds for all increasing sequences from the infinite I. By the compactness theorem we may assume that I is the set of real numbers with the usual order.

Let $\psi(\mathbf{x}; \mathbf{y})$ be the formula $\varphi(\mathbf{y}_0, \ldots, \mathbf{y}_{k-1}, \mathbf{x}, \mathbf{y}_{k+1}, \ldots, \mathbf{y}_n)$ and $A := \{\mathbf{a}_q \mid q \in \mathbf{Q}\}$.

For any real r let $p_r := \mathrm{tp}^{\mathbf{x}}_{\psi}(\mathbf{a}_r/A)$. I claim that $r < s \implies p_r \neq p_s$: Given $r < s$ by density of the rationals, fix an increasing sequence of rationals $q_1 < \cdots < q_n$ such that $q_{k-1} < r < q_{k+1}$ and $q_{k+1} < s < q_{k+2}$. By $(*)$ we have

$$\mathfrak{C} \models \psi[\mathbf{a}_r; \mathbf{a}^*] \text{ and } \mathfrak{C} \models \neg\psi[\mathbf{a}_s; \mathbf{a}^*]$$

When $\mathbf{a}^*$ is the concatenation of sequence of all the $\mathbf{a}_i$'s with the exception of the k-th coordinate. This establishes that $\mathrm{S}^{\mathbf{x}}_{\psi}(A)$ has cardinality $2^{\aleph_0}$ a contradiction to the assumption that T is stable. □

The local stability characterization of stability in Theorem 5.11, has another important application. The following is fundamental

COROLLARY 6.4. *Suppose T is stable. For every $\varphi(\mathbf{x}; \mathbf{y})$ there exists $n_\varphi < \omega$ such that for every infinite indiscernible set $\mathbf{I} \subseteq {}^{\ell(\mathbf{y})}|\mathfrak{C}|$ and every $\mathbf{a} \in {}^{\ell(\mathbf{x})}|\mathfrak{C}|$ either*

$$|\varphi(\mathbf{a}; \mathbf{I})| < n_\varphi \text{ or } |\neg\varphi(\mathbf{a}; \mathbf{I})| < n_\varphi.$$

PROOF. Suppose we have a counterexample. We may assume that $\mathbf{I} = \{\mathbf{b}_i \mid i < \omega\}$. Let $S := \{i < \omega : \mathfrak{C} \models \varphi[\mathbf{a}; \mathbf{b}_i]\}$, it is an infinite, co-infinite set. For any infinite $X \subseteq \omega$ such that both $S - X$ and $X - S$ are infinite. Let f_X be a permutation of ω mapping $X - S$ onto $S - X$. Since $\mathbf{I}$ is a set of indiscernibles there is $\bar{f}_X \in \mathrm{Aut}(\mathfrak{C})$ extending $\bar{f}_X(\mathbf{a}_i) = \mathbf{a}_{f_X(i)}$. Let $\mathbf{a}_X := \bar{f}_X(\mathbf{a})$.

Given X and Y be as above we have $\mathrm{tp}_\varphi(\mathbf{a}_X/\mathbf{I}) \neq \mathrm{tp}_\varphi(\mathbf{a}_Y/\mathbf{I})$, thus $|\mathrm{S}^{\mathbf{x}}_{\varphi}(\mathbf{I})| = 2^{\aleph_0} > \aleph_0 = |\mathbf{I}|$ a contradiction to local stability. □

THEOREM 6.5. *If T is stable in λ then for every A and every $\mathbf{I} \subseteq {}^{<\omega}\mathfrak{C}$ such that $|A| \leq \lambda < |\mathbf{I}|$ there exists $\mathbf{J} \subseteq \mathbf{I}$ a sequence of indiscernibles over A such that $\mathbf{J} = \{\mathbf{a}_i \mid i < \lambda^+\}$.*

Before proving it we generalize a result (Corollary 7.4.7) that was established for $\aleph_0$-stable theories.

THEOREM 6.6 ($|T|$-character of non-splitting). *If T is stable then for every A and every type p over A there exists $B \subseteq A$ of cardinality ? $|T|$ at most such that p does not split over B.* ?!

PROOF. Suppose for the sake of contradiction that there exists A, a finite sequence of variables $\mathbf{x}$ and $p \in \mathrm{S}^{\mathbf{x}}(A)$ such that for every $B \subseteq A$ of cardinality $|T|$-at most the type p splits over B.

By induction on $i < |T|^+$ define $\{A_i \mid i < |T|^+\}$ such that

(1) $A_i \subseteq A$ of cardinality $|T|$,
(2) $i < j < |T|^+ \implies A_i \subseteq A_j$ and for limit $i < |T|^+$ we have that $A_i = \bigcup_{j<i} A_j$,
(3) $p \restriction A_{i+1}$ splits over A_i for all $i < |T|^+$ and
(4) for every $\varphi(x;\mathbf{y}) \in L(T)$ denote by $\psi(\mathbf{y},x)$ the formula $\varphi(x;\mathbf{y})$, for every $q \in \mathrm{S}^1_\varphi(A_i) \cup \mathrm{S}^{\ell(\mathbf{y})}_\psi(A_i)$ and every finite $B \subseteq A_i$ the type $q \restriction B$ is realized in A_{i+1}.

The construction can be carried out using the above assumption together with Theorem 5.11 using the stability hypothesis gives $|A| \leq |T| \implies |\mathrm{S}_\varphi(A)| \leq |T|$ for all $\varphi \in \mathrm{L}(T)$.

For every $i < |T|^+$ let $\mathbf{a}_i, \mathbf{b}_i \in A_{i+1}$ and $\varphi_i(\mathbf{x};\mathbf{y}_i) \in \mathrm{L}(T)$ be such that they witness the splitting of $p \restriction A_{i+1}$ over A_i. Namely

$$\mathrm{tp}(\mathbf{a}_i/A_i) = \mathrm{tp}(\mathbf{b}_i/A_i) \quad \text{and} \quad \{\varphi_i(\mathbf{x};\mathbf{a}_i), \neg\varphi_i(\mathbf{x};\mathbf{b}_i)\} \subseteq p.$$

Since the mapping $i \mapsto \varphi_i$ is from a set of cardinality $|T|^+$ into a set of cardinality $|T|$ there are $\varphi(\mathbf{x};\mathbf{y})$ and a set $S \subseteq |T|^+$ of cardinality $|T|^+$ such that

$$i \in S \implies \varphi_i = \varphi.$$

Pick an increasing $\{i_n \mid n < \omega\} \subseteq S$ such that $i_n + 1 < i_{n+1}$ for all $n < \omega$.

Using an argument similar to that in the proof of Lemma 5.15 we show that $\rho(\mathbf{x}_1,\mathbf{y}_1,\mathbf{z}_1;\mathbf{x}_2,\mathbf{y}_2,\mathbf{z}_2) := [\varphi(\mathbf{x}_1,\mathbf{y}_2) \leftrightarrow \varphi(\mathbf{x}_1,\mathbf{z}_2)]$ has the order property which by Theorem 5.11 contradicts stability.

Fix $\mathbf{c} \models p$. Pick an increasing $\{i_n \mid n < \omega\} \subseteq S$. By induction of $n < \omega$ pick $\mathbf{c}_n \in A_{i_{n+1}}$ such that $\mathbf{c}_n \models \mathrm{tp}_\varphi(\mathbf{c}/B_n \cup \{\mathbf{a}_{i_n}, \mathbf{b}_{i_n}\})$ where $B_n := \{\mathbf{a}_{i_k}, \mathbf{b}_{i_k}, \mathbf{c}_k \mid k < n\}$.

Let $\mathbf{d}_n := \mathbf{c}_n \hat{} \mathbf{a}_{i_n} \hat{} \mathbf{b}_{i_n}$ for all $n < \omega$.

Very that $\mathfrak{C} \models \rho[\mathbf{d}_n;\mathbf{d}_k] \iff n > k$ for all $n,k < \omega$.

□

Inspired by the last theorem it is natural to introduce:

DEFINITION 6.7. Let $n < \omega$

$$\kappa^n_s(T) := \min\{\kappa \mid \text{ for every } A, \text{ and every } p \in S^n(A) \text{ there exists } B \subseteq A \text{ of cardinality } < \kappa \text{ such that } p \text{ does not split over } B\}.$$

$$\kappa_s(T) := \sup_{n<\omega} \kappa_s^n(T).$$

By Corollary 6.4.7 when T is $\aleph_0$-stable then $\kappa_s(T) = \aleph_0$. Notice that we use only the assumption that T is totally transcendental.

The statement of Theorem 6.6 is that $\kappa_s(T) \leq |T|^+$.

DEFINITION 6.8. Let δ be an ordinal. We say that T *has a splitting sequence of length* δ provided there exists an increasing sequence of sets $\{A_i \mid i \leq \delta\}$ of the monster model and a type $p \in \mathrm{S}^n(A_\delta)$ such that

$$p \restriction A_{i+1} \text{ splits over } A_i \text{ for every } i < \delta.$$

REMARK 6.9. If there are a set A of cardinality λ and a type $p \in \mathrm{S}^n(A)$ such that p splits over every $B \subseteq A$ of cardinality $< \lambda$ then there exists $\{A_i \subseteq A \mid i \leq \lambda\}$ increasing and continuous such that $p \restriction A_{i+1}$ splits over A_i for every $i < \lambda$.

PROPOSITION 6.10 (A continuous splitting sequence). *If T has a splitting sequence of length δ then there exists an increasing and continuous sequence of sets $\{B_i \mid i \leq \delta\}$ and $p^* \in \mathrm{S}^{\mathbf{x}}(B_\delta)$, $\mathbf{a}_i, \mathbf{b}_i \in B_{i+1}$ and formulas $\varphi_i(\mathbf{x}; \mathbf{y}_i)$ such that*

$$\mathrm{tp}(\mathbf{a}_i/B_i) = \mathrm{tp}(\mathbf{b}_i/B_i) \text{ and } \varphi_i(\mathbf{x}; \mathbf{a}_i), \neg\varphi_i(\mathbf{x}; \mathbf{b}_i) \in p \text{ for all } i < \delta$$

PROOF. Define $\{B_i \subseteq A_i \mid i < \delta\}$ by induction on $i < \delta$ when $B_0 := A_0$, at limit i take union and at successor use splitting. The type p^* is $p \restriction (\bigcup_{i<\delta} B_i)$. $\square$

It turns out that in case T is stable in μ and $\mu \leq |T|$ then the previous theorem can be improved.

THEOREM 6.11. *If T has a splitting sequence of length λ then T is not λ-stable.*

PROOF. Using the Proposition we have $\{A_i \mid i < \lambda\}$ and $p \in \mathrm{S}_n(A)$ for $A := \bigcup_{i<\lambda} A_i$ such that

(1) $i < j < \lambda \implies A_i \subseteq A_j$ and
    for limit $i < \lambda$ we have that $A_i = \bigcup_{j<i} A_j$,
(2) $p \restriction A_{i+1}$ splits over A_i for all $i < \lambda$
(3) $i < \lambda \implies |A_i| = |i| + \aleph_0$.

Let $\mathbf{a}_i, \mathbf{b}_i \in A_{i+1}$ and $\varphi_i(\mathbf{x}; \mathbf{y}_i) \in \mathrm{L}(T)$ witness that $p \restriction A_{i+1}$ splits over A_i. Thus we have

$$\mathrm{tp}(\mathbf{a}_i/A_i) = \mathrm{tp}(\mathbf{b}_i/A_i), \quad \varphi_i(\mathbf{x}; \mathbf{a}_i) \in p \text{ and } \neg\varphi_i(\mathbf{x}; \mathbf{b}_i) \in p.$$

Fix $\mathbf{c} \models p$. Let $\chi := \min\{\chi \mid 2^\chi > \lambda\}$. Notice that (by Cantor's theorem) $\chi \leq \lambda$, thus $2^{<\chi} \leq \lambda$. We will contradict the assumption of λ-stability by constructing a set A^* of cardinality $2^{<\chi}$ with 2^χ many types over it.

For every $\eta \in {}^{\chi>}2$ define $(\mathfrak{C}, \mathfrak{C})$-mapping F_η such that

(1) $F_{\langle\rangle} = \mathrm{id}_{A_0}$,
(2) $\mathrm{dom}\, F_\eta = A_{\ell(\eta)}$,
(3) if $\ell(\eta)$ is limit then $F_\eta = \bigcup_{\alpha<\ell(\eta)} F_{\eta\restriction\alpha}$
(4) Denote by i the length of η. Let $F_{\eta\hat{}0}$ be a $(\mathfrak{C}, \mathfrak{C})$-mapping extending F_η to A_{i+1}. Since $\mathrm{tp}(\mathbf{a}_i/A_i) = \mathrm{tp}(\mathbf{b}_i/A_i)$ there exists a $(\mathfrak{C}, \mathfrak{C})$-mapping G extending F_η to $A_i \cup \mathbf{b}_i$ such that $G(\mathbf{b}_i) = F_{\eta\hat{}0}(\mathbf{a}_i)$. Let $F_{\eta\hat{}1}$ be a $(\mathfrak{C}, \mathfrak{C})$-mapping extending G to A_{i+1}.

Clearly the construction can be carried out by induction on $\ell(\eta)$.

Let $A^* := \bigcup\{F_\eta(\mathbf{a}_i, \mathbf{b}_i) \mid i < \chi,\ \eta \in {}^{i+1}2\}$.

For $\eta \in {}^{\chi}2$ let $F_\eta := \bigcup_{\alpha<\chi} F_{\eta\restriction\alpha}$. Now let $\bar{F}_\eta$ be an extension of F_η to the set $\bigcup_{i<\chi} A_i \cup \mathbf{c}$.

For $\eta \in {}^{\chi}2$ let $p_\eta := \mathrm{tp}(\bar{F}_\eta(\mathbf{c})/A^*)$.
It suffices to show that $\eta \neq \nu \in {}^{\chi}2 \implies p_\eta \neq p_\nu$.

Let $i := \min\{i < \chi \mid \eta[i] \neq \nu[i]\}$. Without loss of generality we may assume that $\eta[i] = 0$ and $\nu[i] = 1$. Let $\rho := \eta \restriction i$. Since $\varphi_i(\mathbf{x}; \mathbf{a}_i) \in p$ we have that

$$\models \varphi_i[\bar{F}_\eta(\mathbf{c}); \bar{F}_\eta(\mathbf{a}_i)].$$

As $F_{\rho\hat{}0} \subseteq \bar{F}_\eta$ and $\mathbf{a}_i \in A_{i+1} = \mathrm{dom}\, F_{\rho\hat{}0}$ we have

$$\varphi_i(\mathbf{x}; F_{\rho\hat{}0}(\mathbf{a}_i)) \in p_\eta.$$

By $\neg\varphi_i(\mathbf{x}; \mathbf{b}_i) \in p$ we get

$$\neg\varphi_i(\mathbf{x}; \bar{F}_\nu(\mathbf{b}_i)) \in p_\nu.$$

By the definition of $F_{\rho\hat{}1}$ we have that

$$\bar{F}_\nu(\mathbf{b}_i) = F_{\rho\hat{}1}(\mathbf{b}_i) = G(\mathbf{b}_i) = F_{\rho\hat{}0}(\mathbf{a}_i)$$

so

$$\neg\varphi_i(\mathbf{x}; F_{\rho\hat{}0}(\mathbf{a}_i)) \in p_\nu.$$

Thus $p_\eta \neq p_\nu$. □

We can now obtain an improvement of Theorem 6.6. For the case of uncountable T.

COROLLARY 6.12. *It T is λ-stable then $\kappa_s(T) \leq \lambda$.*

PROOF. If $\kappa_s(T) > \lambda$ there exists are a set $A, n < \omega$ and $p \in \mathrm{S}^n(A)$ such that p splits over B for every $B \subseteq A$ of cardinality $< \lambda$.

By induction on $i < \lambda$ construct a nice witness of splitting sequence like in Proposition 6.10.

The last Theorem gives us a set A such that $|A| \leq \lambda < |\mathrm{S}^n(A)|$. $\square$

Now to the proof of Theorem 6.5: At first we show we make an effort to use the method used in the proof of Theorem 7.**??**. We prove a special case of Theorem 6.5:

THEOREM 6.13. *Let λ be a cardinal satisfying $\lambda^{|T|} = \lambda$. If T is stable in λ then for every A and every $\mathbf{I} := \{\mathbf{a}_i : i < \alpha\}$ such that $|A| \leq \lambda < |\mathbf{I}|$ there exists $\mathbf{J} \subseteq \mathbf{I}$ of cardinality λ^+ a sequence of indiscernibles over A.*

PROOF. By induction on $i < \lambda^+$ define $\{M_i \mid i < |T|^+\}$ such that

(1) $|M_0| \supseteq A$,
(2) $i < \lambda^+ \implies \|M_i\| = \lambda$,
(3) $i < j < \lambda^+ \implies M_i \prec M_j$, and
    if $i < \lambda^+$ is limit then $M_i = \bigcup_{j<i} M_j$,
(4) for every $p \in S(|M_i|)$ the type p is realized in M_{i+1} and
(5) $i < \lambda^+ \implies \mathbf{a}_i \in M_{i+1}$.

The construction is possible by the DLS theorem and condition (4) is possible by λ-stability. Let $p_i := \mathrm{tp}(\mathbf{a}_i/|M_i|)$. Now consider

$$f(i) := \min\{j < \lambda^+ \mid \text{ the type } p_i \text{ does not split over } |M_j|\}.$$

Let $S^{\lambda^+}_{|T|^+} := \{\delta < \lambda^+ \mid \mathrm{cf}\,\delta = |T|^+\}$.

Using Theorem 6.6 note that the function f is pressing down on the set $S^{\lambda^+}_{|T|^+}$. Since $S^{\lambda^+}_{|T|^+}$ is stationary an application of Fodor's theorem gives a stationary set $S \subseteq S^{\lambda^+}_{|T|^+}$ and $j_0 < \lambda^+$ such that $i \in S \implies p_i$ does not split over $|M_{j_0}|$. For every $i \in S$ let $B_i \subseteq |M_{j_0}|$ be of cardinality $\leq |T|$ such that p_i does not split over B_i.

Now consider

$$i \mapsto B_i \quad \text{for } i \in S.$$

It is a function from S into the set $\{B \subseteq |M_{j_0}| : |B| \leq |T|\}$. The cardinality of the latter set is $\leq \lambda^{|T|}$ which by Corollary 5.14 is equal to λ. Thus by the pigeon hole principle there are $S' \subseteq S$ of cardinality λ^+ and a set $B \subseteq |M_{j_0}|$ such that $i \in S' \implies B_i = B$.

Consider the mapping $i \mapsto p_i \restriction |M_{j_0+1}|$. For arguments from the set S' it is into $\mathrm{S}(|M_{j_0+1}|)$ now by λ-stability the cardinality of the range is at most λ.

Another application of the pigeon hole principle gives $S'' \subseteq S'$ of cardinality λ^+ and a type $p \in \mathrm{S}(|M_{j_0+1}|)$ such that $i \in S'' \implies p_i \restriction |M_{j_0+1}| = p$.

Since $|S''| = \lambda^+$ also $S^* := \{i \in S'' \mid i > j_0\}$ has cardinality λ^+.

CLAIM 6.14. *The sequence* $\mathbf{J} := \langle \mathbf{a}_i \mid i \in S^* \rangle$ *is a sequence of indiscernibels over* $|M_{j_0}|$.

PROOF. Using Lemma 7.4.9 since $i \in S^* \implies p_i$ does not split over $|M_{j_0}|$ all we need to show is to verify that $i < j \in S^* \implies p_i \subseteq p_j$.

For the sake of contradiction suppose that there are $i < j \in S^*$ such that there exists $\varphi(\mathbf{x}; \mathbf{a}) \in p_i$ such that $\varphi(\mathbf{x}; \mathbf{a}) \notin p_j$.

Now consider $\mathrm{tp}(\mathbf{a}/B)$, by requirement (4) there exists $\mathbf{a}' \in M_{i+1}$ such that $\mathbf{a}' \models \mathrm{tp}(\mathbf{a}/B)$. Since p_i does not split over B and $\varphi(\mathbf{x}; \mathbf{a}) \in p_i$ we get that also $\varphi(\mathbf{x}; \mathbf{a}') \in p_i$. Since $\mathbf{a}' \in \mathrm{dom}\, p$ and $p_i \restriction |M_{j_0}| = p_j \restriction |M_{j_0}| = p$ we get that $\varphi(\mathbf{x}; \mathbf{a}') \in p_j$. Now using the fact that p_j does not split over B necessarily $\varphi(\mathbf{x}; \mathbf{a}) \in p_j$. This contradicts that $\varphi(\mathbf{x}; \mathbf{a}) \notin p_j$. □

□

What about the hypothesis $\lambda = \lambda^{|T|}$? It was needed to carry out a pigeon hole argument and also to conclude that $\lambda > |T|$. An attempt to remove the assumption focuses attention on several interesting questions: Does stability in λ implies that $\lambda = \lambda^{|T|}$ holds? Does stability in λ implies that $\lambda > |T|$? It will be shown in **??** that for stable but not superstable theories the answer to the first question is yes and for superstable theories there is a separate argument. It is easy to see that the answer to the second question is no. Later in **??** by replacing "non-splitting" by "non-forking" we will show that essentially the above argument can be used to prove Theorem 6.5.

Fortunately there is another simple argument to prove Theorem 6.5. The idea is to consider the *extension property for non-splitting* (see Exercise 7.4.7c) as a dichotomy property – if it holds than the conclusion of the theorem is obtained from Lemma 7.4.9 like in the previous argument and if the extension property fails we get a contradiction to Theorem **??**. Compare with the second proof of Theorem 4.1.

THE IMPORTANT PROOF OF THEOREM 6.5. . By an applications of the pigeon hole principle to $\mathbf{a} \mapsto \ell(\mathbf{a})$ (a function from $\mathbf{I}$ to ω) we may assume that $\mathbf{I}$ has cardinality λ^+ and there exists $n < \omega$ such that $\mathrm{S} \in \mathbf{I} \implies \ell(\mathbf{a}) = n$.

Suppose that:

There are sets $B \subseteq C \subseteq |M|$ and a type $p \in \mathrm{S}^n(C, M)$ such that

(1) $A \subseteq B$, $|C| \leq \lambda$,
(2) for every $C' \subseteq |M|$ of cardinality λ such that $C' \supseteq C$ there exists $\mathbf{a} \in \mathbf{I} - {}^n\mathrm{C}'$ such that $\mathrm{tp}(\mathbf{a}/C') \supseteq p$ which does not split over B,
(3) for every $\mathbf{c}' \in |M|$ there exists $\mathbf{c} \in C$ such that $\mathbf{c} \models \mathrm{tp}(\mathbf{c}'/B)$.

This is enough: Suppose that there are B, C and p as above. By induction on i define $\langle \mathbf{c}_i \mid i < \lambda^+ \rangle$ from $\mathbf{I}$ such that $C^i := \{\mathbf{c}_j \mid j < i\} \cup C$ and $p_i \in \mathrm{S}^n(C^i, M) \supseteq p$ does not split over B and $\mathbf{c}_i \in \mathbf{I} - C^i \models p_i$. By Lemma 7.4.9 it suffices to show that $i < j < \lambda^+ \implies p_i \subseteq p_j$. Let $\varphi(\mathbf{x}; \mathbf{c}') \in p_i$ be given. Let $\mathbf{c} \in C$ be such that $\mathbf{c} \models \mathrm{tp}(\mathbf{c}'/B)$. Since p_i does not split over B we get that $\varphi(\mathbf{x}; \mathbf{c}) \in p_i$. Since $p_i \restriction C = p = p_j \restriction C$ we get $\varphi(\mathbf{x}; \mathbf{c}) \in p_j$. Now by the non-splitting of p_j over B this implies that $\varphi(\mathbf{x}; \mathbf{c}') \in p_j$.

This is possible: For the sake of contradiction suppose that B, C and p as above do not exist. By induction on $i \leq \lambda$ define $B_i \subseteq |M|$ such that

(1) $B_0 = A$, $i < j \leq \lambda \implies B_i \subseteq B_j$,
(2) $|B_i| \leq \lambda$, if i is limit then $B_i = \bigcup_{j<i} B_j$,
(3) By the assumption for every B_i there exists $C^i \supseteq B_i$ a subset of $|M|$ of cardinality $\leq \lambda$ such that
   (a) for every $\mathbf{c}' \in |M|$ there exists $\mathbf{c} \in C^i$ such that $\mathbf{c} \models \mathrm{tp}(\mathbf{c}'/B_i)$ and
   (b) for every $p \in \mathrm{S}(C^i, M)$ that does not split over B_i there exists $C_p \subseteq |M|$ of cardinality $\leq \lambda$ containing C^i such that no non-splitting extension of p over B_i is realized in $\mathbf{I} - {}^n\mathrm{C}_p$.

Let

$$B_{i+1} := \bigcup_{p \in \mathrm{S}(C^i, M),\ p \text{ does not split over } B_i} C_p \cup C^i.$$

Stability in λ is used to show that $|B_{i+1}| \leq \lambda$.

Since $|B_\lambda| \leq \lambda$ and $|\mathbf{I}| > \lambda$ there exists $\mathbf{a} \in \mathbf{I} - B_\lambda$. Let $p := \mathrm{tp}(\mathbf{a}/B_\lambda)$. Notice that p contradicts that $\kappa_s(T) \leq \lambda$ (Theorem 6.11). It is enough to show that $p \restriction B_{i+1}$ splits over B_i for all $i < \lambda$. Otherwise $p \restriction C^i$ does not split over B_i for some $i < \lambda$. Let $q := p \restriction C^i$ thus $p \restriction C_q$ is an extension of q to C_q that does not split over B_i but by the definition of p it is realized by an element of $\mathbf{I} - C_q$ which is a contradiction to the construction of $\{B_i \mid i \leq \lambda\}$. □

As an application of the fact that $\aleph_1$ is a regular cardinal we will establish existence of Δ-systems.

DEFINITION 6.15. Let $\mathcal{F}$ be a family of sets. We say that $\mathcal{F}$ is a Δ-*System* if there exists a set K such that

$$A \cap B = K \text{ for all } A \neq B \in K.$$

THEOREM 6.16. *Let $\mathcal{F}$ be a family of finite sets. If $\mathcal{F}$ is uncountable then there exists an uncountable $\mathcal{F}^* \subseteq \mathcal{F}$ Δ-System*

PROOF. Without loss of generality we may assume that $|\mathcal{F}| = \aleph_1$. □

THEOREM 6.17.

PROOF. □

Exercises 8.6.

(1) Let T be a complete theory. Prove
Theorem T is stable iff every countable $T' \subseteq T$ is stable.
(2) Let $F := \langle \mathbb{R}, +, \cdot \rangle$ be the field of real numbers show that $\mathrm{Th}(F)$ is unstable.
(3) The theory of atomless boolean algebras is unstable.
(4) T_{ind} and T_{ord} are unstable theories.
(5) Is every model complete theory unstable?
(6) Is there a stable completion of ZFC?
(7) Do you know an example of a stable theory which is not $\aleph_1$-categorical?
(8) Do you know an example of a stable theory which is not superstable?
(9) Do you know a field F such that $\mathrm{Th}(F)$ is unstable?
(10) Prove the following

LEMMA 6.18. *Let μ be a cardinal number. $\lambda(T) \geq \mu$ iff there are $\{B_i \mid i < \mu\}$ increasing chain of sets and a type $p \in \mathrm{S}(\bigcup_{i<\mu} B_i)$ such that $p \restriction B_{i+1}$ splits over B_i for all $i < \mu$.*

(11) Suppose that T is a stable theory.
(12)
(13)
(14)
(15)

7. Towards Los conjecture for uncountable first-order theories

Morley concluded his seminal paper [**Mo1**] with a question whether his theorem holds also for uncountable theories. Namely given a first-order T, does categoricity in some $\lambda > |T|$ implies that T is categorical in all $\mu > |T|$?

This quickly was recognized as an interesting question and several people in the sixties obtained important partial results among Frank Rowbottom [], [] and Jean Pierre Ressayr []. Eventually the problem was solved by Shelah in 1970 who proved

THEOREM 7.1. *Let T be a first-order theory. If T is categorical in some cardinal $\lambda > |T|$ then T is categorical in μ for every $\mu > |T|$.*

Shela's proof required several new concepts. The first step will be described in this section. One of the obstacle in generalizing the proof of Theorem **??** to uncountable languages was the observation that unlike for countable theories which are $\aleph_0$-stable every set can be closed to a primary model (which is also prime), see **??** this is not true in uncountable languages even if we assume that T is categorical in some $\lambda > |T|$. Thus we need a replacement.

In this section I will present that replacement as well as will establish that categoricity implies that T has no Vaughtian pairs. However as in the proof of this fact for countable theories we used Vaught's two cardinal theorem that is not available we need to work harder to get that same result.

One of the important ideas introduced by Shelah was that in order to understand a class of models $\mathcal{K}$ (in our case it is $\mathrm{Mod}(T)$, but this idea appears in other situations as well) it is better to identify a natural subclass $\mathcal{K}^*$ that is well behaved and use it to obtain facts on members of $\mathcal{K}$. The very first instance of that process will be taking $\mathcal{K}^*$ to be the class of $|T|^+$-saturated models of T.

As always since categoricity of T implies that T is complete, we work in a monster model $\mathfrak{C}$ of T.

DEFINITION 7.2. Let $\mathbf{x}$ be a finite set of variables and let λ be an infinite cardinal. A type p in the variables $\mathbf{x}$ is called λ-*isolated* provided there exists an $\mathbf{x}$-type q of cardinality $< \lambda$ such that $q \vdash p$.

Thus, the old notion of isolated type from Definition **??** means $\aleph_0$-isolated.

Here is an analogue of Lemma 5.5 for stable theories:

THEOREM 7.3 (Density of λ-isolated types). *Let $\mathbf{x}$ be a finite sequence of variables and $\lambda > |T|$. Suppose T is a stable theory. If p is an $\mathbf{x}$-type of cardinality $< \lambda$ and A a subset of the monster model such that $A \supseteq \mathrm{dom}\, p$ then there exists a λ-isolated type $q \in \mathrm{S}^{\mathbf{x}}(A)$ extending p.*

PROOF. Fix $\{\varphi_i(\mathbf{x};\mathbf{y}_i) \mid i < |T|\}$ be an enumeration of all formulas with free variables containing $\mathbf{x}$ and let $\Delta_i := \{\varphi_j \mid j < i\}$. .

By induction on $i < |T|$ we define $\{q_i \mid i < |T|\}$ such that $\{q_i \mid i < \lambda\}$, q_i is a φ_i-type and $\operatorname{dom} q_i \subseteq A$ such that for every $i < |T|$ there exists a unique $q \in \mathrm{S}_{\Delta_i}(A)$ containing $p \cup \bigcup_{j<i} q_j$.

Given $i < |T|$ consider $p_i := p \cup \bigcup_{j<i} q_j$. Let

$$S := \{\mathrm{R}[q, \varphi_i, 2] \mid q \in \mathrm{S}^{\mathbf{x}}(A) \supseteq p_i\}.$$

By stability S is a set of ordinals (Theorem 3.1), let $\alpha := \min S$ and q witnessing this. By the finite character of R there exists a finite $q_i \subseteq q$ such that $\mathrm{R}[q_i, \varphi_i, 2] = \alpha$.

Let $q^* := p \cup \bigcup_{i<|T|} q_i$ as q^* is consistent of cardinality $< \lambda$, there exists $\mathbf{b} \models q^*$, let $q := \operatorname{tp}(\mathbf{b}/A)$. Verify that $q \supseteq p$ and $q^* \vdash q$. □

DEFINITION 7.4. Let M be a model and $A \subseteq |M|$. We say that M *is λ-primary over A* provided there exists $\{\mathbf{a}_i \mid i < \alpha\}$ such that $\operatorname{tp}(\mathbf{a}_i/A \cup \{\mathbf{a}_j \mid j < i\})$ is λ-isolated for all $i < \alpha$ and $|M| = A \cup \{\mathbf{a}_i \mid i < \alpha\}$.

The analogue of Theorem 5.9 is

THEOREM 7.5 (Existence of a λ-primary model).

8. The independence and strict-order properties

DEFINITION 8.1. We say that $\varphi(\mathbf{x};\mathbf{y})$ has the *independence property* IP if for every $M \models T$ and every $n < \omega$ there exists $\{\mathbf{a}_i \mid i < n\} \subseteq {}^{\ell(\mathbf{y})}|M|$ such that

$$M \models \exists \mathbf{x} \bigwedge_{i<n} \varphi(\mathbf{x};\mathbf{a}_i)^{i\in w} \text{ for every } w \subseteq n.$$

When $\mathbf{b} \in {}^{\mathbf{x}}|M|$ is satifying

$$M \models \bigwedge_{i<n} \varphi(\mathbf{b};\mathbf{a}_i)^{i\in w}.$$

we say that $\mathbf{b}$ *codes the set* w.

If a formula φ does not have the independence property we say φ *is dependent*. We say that T has the *independence property* if there exists a formula $\varphi(\mathbf{x};\mathbf{y})$ with independence property.

If T does not have the independence property we we say T *is a dependent theory*, some authors name such theories as *NIP*.

PROPOSITION 8.2. *If $\varphi(\mathbf{x};\mathbf{y})$ has the independence property then for every I there exists a set $\{\mathbf{a}_i \mid i \in I\}$ such that for every $w \subseteq I$ there exists $\mathbf{b}_w \in {}^{\ell(x)}|\mathfrak{C}|$ such that*

$$\mathfrak{C} \models \varphi[\mathbf{b}_w;\mathbf{a}_i] \iff i \in w.$$

Thus for every λ there is a set A of cardinality λ such that $|\mathrm{S}^{\mathbf{x}}_{\varphi}(A)| = 2^{\lambda}$.

PROOF. Given new constants $\{\mathbf{d}_w \mid w \subseteq I\}$ and $\{\mathbf{c}_i \mid i \in I\}$. Using the assumption that φ has the independence property show that the following theory is consistent:

$$T \cup \{\mathbf{c}_i \neq \mathbf{c}_j \mid i \neq j \in I\} \cup \{\varphi(\mathbf{d}_w;\mathbf{c}_i) \mid i \in w\} \cup \{\neg\varphi(\mathbf{d}_w;\mathbf{c}_i) \mid i \notin w\}.$$

□

EXAMPLE 8.3. The following theories have the independence property: The theory of the random graph, T_{ind} has the independence property, since the edge relation has the IP. Peano's arithmetic and ZFC also have the IP. Notice that in PA the formula $x|y$ that stands for x is a divisor of y has the IP: For every n let $\{p_0,\ldots,p_{n-1}\}$ be a set of distinct primes then for every $w \subseteq n$ the element $b_w := \prod_{i\in w} p_i$ codes w.

If T is stable then it is dependent. If T is a theory with the binary relation $<$ in its language such that $T \vdash$ " $<$ is a linear order of the entire universe" then the formula $<$ does not have the independence property: take $a < b$ in a model M of T, they witness that $M \models \neg\exists x[\neg[x < b] \wedge x < a]$.

PROPOSITION 8.4. *If $\varphi(\mathbf{x};\mathbf{y})$ is dependent then there exists $n<\omega$ such that there*

$$(*)_{\varphi,n} \quad \mathfrak{C} \models \neg\exists \mathbf{y}_0 \cdots \exists \mathbf{y}_{n-1} \bigwedge_{w\subseteq n} [\exists \mathbf{x} \bigwedge_{i<n} \varphi(\mathbf{x};\mathbf{y}_i)^{i\in w}].$$

PROOF. Exercise. □

DEFINITION 8.5. The *Vapnik-Chervonenkis-dimension of* $\varphi(\mathbf{x};\mathbf{y})$ is

$$VC(\varphi) := \min\{n<\omega \mid (*)_{\varphi,n}\}$$

when φ is NIP, if the formula has the independence property we write $VC(\varphi)=\infty$.

LEMMA 8.6 (Poizat). *Let $\varphi(\mathbf{x};\mathbf{y})$ be given and let $\psi(\mathbf{y};\mathbf{x})$ be the formula φ. Then φ IP iff ψ is IP.*

PROOF. As φ is IP using the compactness theorem there exists $A=\{\mathbf{a}_X \mid X\subseteq\omega\}$ such that for every $w\subseteq A$ there exists $\mathbf{b}_w$ such that

$$X\in w \iff \mathfrak{C}\models\varphi[\mathbf{b}_w;\mathbf{a}_X]$$

For $n<\omega$ let $w_n:=\{X\subseteq\omega \mid n\in X\}$.

To show that ψ has the independence property enough to show that $\psi(\mathbf{y};\mathbf{x})$ allows us to code all the subsets of ω; Given $w\subseteq\omega$ we have

$$n\in w \iff \mathfrak{C}\models\psi[\mathbf{a}_w;\mathbf{b}_{w_n}]$$

□

PROPOSITION 8.7. *Suppose that $\varphi(\mathbf{x};\mathbf{y})$ has the independence property if and only if for every limit ordinal α there are $\mathbf{a}\in|\mathfrak{C}|$ and an indiscernible sequence $\{\mathbf{b}_i \mid i<\alpha\}$ such that*

$$(*) \quad \models\varphi[\mathbf{a};\mathbf{b}_i] \iff i \text{ is even}.$$

PROOF. $\Leftarrow$: Suppose $\{\mathbf{b}_i \mid i<\omega\}$ is an indiscernible sequence and $\mathbf{a}$ is an element such that $(*)$ holds. Enough to show that for every $w\subseteq\omega$ there exists $\mathbf{a}_w$ such that $i\in w \iff \mathfrak{C}\models\varphi[\mathbf{a}_w;\mathbf{b}_i]$.

Define an increasing $f:\omega\to\omega$ such that $i\in w \iff f(i)$ is even. Using $(*)$ we get

$$(**) \quad \models\varphi[\mathbf{a};\mathbf{b}_{f(i)}] \iff i\in w.$$

Since $\{\mathbf{b}_i \mid i<\omega\}$ is an indiscernible sequence the mapping F_w that is defined by $F(\mathbf{b}_{f(i)}):=\mathbf{b}_i$ is an $(\mathfrak{C},\mathfrak{C})$-mapping thus there is $F_w^*\in \mathrm{Aut}(\mathfrak{C})$ extending it. An application of F_w^* to $(**)$ gives

$$\models\varphi[F_w^*(\mathbf{a});\mathbf{b}_i] \iff i\in w.$$

Namely, $\mathbf{a}_w:=F_w^*(\mathbf{a})$ codes the set w.

$\Rightarrow$: Let $\mathbf{c}$ and $\{\mathbf{d}_i \mid i < \alpha\}$ be new constants, using the Erdős-Rado theorem like in the proof of Theorem **??** show that T^* is consistent, where

$$T^* := T \cup \{\varphi(\mathbf{c};\mathbf{b}_i) \mid i < \alpha \text{ is even } \} \cup \{\neg\varphi(\mathbf{c};\mathbf{d}_i) \mid i < \alpha \text{ is odd}\}$$
$$\{\psi(\mathbf{d}_0,\dots,\mathbf{d}_{m-1}) \leftrightarrow \psi(\mathbf{d}_{i_0},\dots,\mathbf{d}_{i_{m-1}}) \mid i_0 < \cdots i_{m-1} < \alpha\}.$$

□

The following is of fundamental importance[3]:

LEMMA 8.8 (Shelah). *If $\varphi(\mathbf{x};\mathbf{y})$ does not have the independence property then there exists $n_\varphi < \omega$ such that for every limit α and indiscernible sequence $\mathbf{I} := \{\mathbf{b}_i \mid i < \alpha\}$ we have that for every $\mathbf{a} \in \mathfrak{C}$ either*

$$|\varphi(\mathbf{a};\mathbf{I})| < n_\varphi \text{ or } |\neg\varphi(\mathbf{a};\mathbf{I})| < n_\varphi$$

PROOF. Otherwise, for every $n < \omega$ there are $\mathbf{a}_n$ and infinite indiscernible sequences $\mathbf{I}_n$ such that

$$|\varphi(\mathbf{a}_n;\mathbf{I}_n)| \geq n \text{ and } |\neg\varphi(\mathbf{a}_n;\mathbf{I}_n)| \geq n.$$

Using the Ehrenfeucht Mostowski argument we get that there are $\mathbf{a}$ and an infinite indiscernible sequence $\mathbf{I} := \{\mathbf{b}_i \mid i < \omega\}$ such that both $\varphi(\mathbf{a};\mathbf{I})$ and $\neg\varphi(\mathbf{a};\mathbf{I})$ are infinite. Now define an increasing $f : \omega \to \omega$ such that

$$\mathfrak{C} \models \varphi[\mathbf{a};\mathbf{b}_{f(2n)}] \wedge \neg\varphi[\mathbf{a};\mathbf{b}_{f(2n+1)}] \text{ for all } n < \omega.$$

This by the previous lemma implies that $\{\mathbf{b}_{f(n)} \mid n < \omega\}$ is an indiscernible sequence which together with $\mathbf{a}$ witness that φ has the independence property. □

THEOREM 8.9. *If there is no formula $\varphi(x;\mathbf{y})$ with the independence property then T is a dependent theory.*

Notice that in the statement we require φ be a formula in a single valuable x unlike in the definition when we deal with $\mathbf{x}$. This theorem appeared in Shelah's 1971 paper [**Sh 10**]. The original proof used forcing and absoluteness together with Theorem 8.16 below. Several model theorists worked hard to eliminate the usage of non-trivial set theory in the proof. I will present a super elegant argument that avoids non-trivial set theory. This beautiful and incredibly elegant argument was published by Hans Adler in [**Ad**], the key lemma is a simplified and readable version of Shelah's Claim 3.1 from [**Sh 715**].

First we establish the following lemma.

In the next lemma we will generalize the previous lemma. Suppose that $\mathbf{I} := \{\mathbf{b}_i \mid i < \beta\}$, where β is a limit ordinal. Given $m < \omega$ denote by $\mathbf{I}^m$ increasing m-sequences from $\mathbf{I}$. Namely if $\mathbf{b} \in \mathbf{I}^m$ then $\ell(\mathbf{b}) = m$ and there are $i_0 < \cdots i_{m-1} < \beta$ such that $\mathbf{b} = \langle \mathbf{b}_{i_0},\dots,\mathbf{b}_{i_{m-1}}\rangle$.

[3]Compare with Theorem 9.4.

LEMMA 8.10. *Suppose $\varphi(\mathbf{x};\mathbf{y}_0,\ldots,\mathbf{y}_{m-1})$ is dependent, $\mathbf{a}\in\mathfrak{C}$ and $\mathbf{I}:=\{\mathbf{b}_i \mid i<\beta\}$, is an indiscernible sequence, such that $\ell(\mathbf{b}_i)=\ell(\mathbf{y}_k)$ for all $i<\beta$. There exists $\alpha_\varphi<\beta$ such that*

$$\models \varphi[\mathbf{a};\mathbf{b}_{i_0},\ldots,\mathbf{b}_{i_{m-1}}] \iff \models \varphi[\mathbf{a};\mathbf{b}_{j_0},\ldots,\mathbf{b}_{j_{m-1}}]$$

for all $\alpha_\varphi<i_0<\cdots<i_{m-1}<\beta$ and every $\alpha_\varphi<j_0<\cdots<j_{m-1}<\beta$.

PROOF. Otherwise, define $\mathbf{J}:=\{\mathbf{d}_n\in\mathbf{I}^m \mid n<\omega\}$ such that both

$$(*)\quad \varphi(\mathbf{a};\mathbf{J}) \text{ and } \neg\varphi(\mathbf{a};\mathbf{J}) \text{ are infinite.}$$

By induction on n: Pick any $i_0<\cdots<i_{m-1}<\beta$, let $\mathbf{d}_0:=\langle \mathbf{b}_{i_0},\ldots,\mathbf{b}_{i_{m-1}}\rangle$.

Suppose $\mathbf{d}_n=\langle \mathbf{b}_{i_0},\ldots,\mathbf{b}_{i_{m-1}}\rangle$. Since we assume that the assertion of the Lemma fails there are $j_0<\ldots<j_{m-1}<\beta$ such that $j_0>i_{m-1}$ and

$$\models \varphi[\mathbf{a};\mathbf{b}_{i_0},\ldots,\mathbf{b}_{i_{m-1}}] \iff \models \neg\varphi[\mathbf{a};\mathbf{b}_{j_0},\ldots,\mathbf{b}_{j_{m-1}}].$$

Verify that $\mathbf{d}_{n+1}:=\langle \mathbf{b}_{j_0},\ldots,\mathbf{b}_{j_{m-1}}\rangle$ is as required.

Notice that $\mathbf{J}$ is an indiscernible sequence, this together with (*) contradicts the previous lemma. □

LEMMA 8.11. *Let a be any element of the monster model and $\mathbf{I}:=\{\mathbf{b}_i \mid i<\lambda\}$ be a sequence of indiscernibles, such that λ is regular $>|T|$. If T has the property that every formula $\varphi(x;\mathbf{y})$ is dependent then for every element a there exists $\alpha<\lambda$ such that $\{\mathbf{b}_i \mid \alpha<i<\lambda\}$ is an indiscernible sequence over a.*

PROOF. Enough to show that there exists $\alpha<\lambda$ such that

$$\mathrm{tp}(\mathbf{b}_{i_0},\ldots,\mathbf{b}_{i_{m-1}}/a)=\mathrm{tp}(\mathbf{b}_{j_0},\ldots,\mathbf{b}_{j_{m-1}}/a) \text{ for all increasing sequences.}$$

For every $\varphi(x;\mathbf{y}_0,\ldots,\mathbf{y}_{m-1})$ there exists n_φ as in the previous lemma (applied to $\mathbf{I}^m$), thus there is $\alpha_\varphi<\lambda$ such that for all $\alpha_\varphi<i_0<\cdots<i_{m-1}<\lambda$ and every $\alpha_\varphi<j_0<\cdots<j_{m-1}<\lambda$ we have that

$$\models \varphi[a;\mathbf{b}_{i_0},\ldots,\mathbf{b}_{i_{m-1}}] \iff \models \varphi[a;\mathbf{b}_{j_0},\ldots,\mathbf{b}_{j_{m-1}}]$$

Let

$$\alpha:=\sup\{\alpha_\varphi \mid \varphi(x;\mathbf{z})\in \mathrm{L}(T)\}.$$

Since λ is regular and $>|T|$ we have that $\alpha<\lambda$ and $\{\mathbf{b}_i \mid \alpha<i<\lambda\}$ is an indiscernible sequence over a. □

PROOF OF THEOREM 8.9. For sake of contradiction suppose that $\varphi(\mathbf{x};\mathbf{y})$ has the independence property. By Proposition 8.7 fix $\mathbf{I}:=\{\mathbf{b}_i \mid i<|T|^+\}\subseteq {}^{\ell(\mathbf{y})}|\mathfrak{C}|$ an indiscernible sequence and $\mathbf{a}$ such that

$$(*)\quad \models \varphi[\mathbf{a};\mathbf{b}_i] \iff i \text{ is even.}$$

Suppose that $\mathbf{a} = \langle a_0, \dots, a_{n-1} \rangle$. Using Lemma 8.11 find $\alpha_0 < \lambda$ such that $\{\mathbf{b}_i \mid \alpha_0 < i < |T|^+\}$ is indiscernible sequence over a_0. Similarly by induction on $1 \leq k < n$ find $\alpha_{k-1} < \alpha_k < |T|^+$ such that $\{\langle a_0, \dots, a_{k-1} \rangle^\frown \mathbf{b}_i \mid \alpha_k < i < |T|^+\}$ is an indiscernible sequence over a_k.

The fact that $\{\mathbf{b}_i \mid \alpha_{n-1} < i < |T|^+\}$ is an indiscernible sequence over $\mathbf{a}$ contradicts (*). □

COROLLARY 8.12. *Suppose T is dependent. If $\mathbf{I} = \{\mathbf{a}_i \mid i < \lambda\}$ is an indiscernible sequence over A then for any finite $\mathbf{b}$ there exists $\mathbf{J}_\mathbf{b} \subseteq \mathbf{I}$ of cardinality $< \lambda$ such that $\mathbf{I} - \mathbf{J}_\mathbf{b}$ is an insidscernible sequence over $A \cup \mathbf{b}$.*

PROOF. Exercise. □

EXAMPLE 8.13. $T_{\rm ind}$ has the independence property, but the theories $T_{\rm ord}$ and $(\langle \mathbb{R}, +, \cdot \rangle)$ are examples of NIP theories.

DEFINITION 8.14. Let μ be an infinite cardinal. We denote by

$$\lambda \leq \operatorname{Ded} \mu \quad \text{the statement:}$$

There exists a linearly ordered set I of cardinality λ such that I has a dense subset J of cardinality μ.

REMARK 8.15. An alternative definition for $\operatorname{Ded} \mu$ is

$$\sup\{|I| : \text{ there exists a linearly ordered } I \text{ with a dense subset } J \text{ of cardinality } \mu\}.$$

Using the Erdős Rado theorem observe that $\operatorname{Ded} \mu \leq 2^\mu$ for all μ. By Lemma 5.12 always $\operatorname{Ded} \mu \geq \mu^+$.

While clearly we have that $\operatorname{Ded} \aleph_0 = 2^{\aleph_0}$. Mitchell [**Mit**] and Baumgartner [] proved that it is consistent that $\operatorname{Ded} \mu < 2^\mu$ holds for all uncountable μ.

The following result is in my opinion one of the most amazing theorems of model theory as it connects a behavior in an uncountable cardinal with the existence of a first order formula with a finitary property (IP). This theorem appeared in section 4 of Shelah's 1971 paper [**Sh 10**].

THEOREM 8.16. *If there exists an infinite A and a formula φ such that $|\operatorname{S}^{\ell(\mathbf{x})}_\varphi(A)| > \operatorname{Ded} |A|$ then $\varphi(\mathbf{x}; \mathbf{y})$ has the independence property.*

PROOF. Let μ be the minimal infinite cardinal such that there exists A of cardinality μ satisfying $|\operatorname{S}_\varphi(A)| > \operatorname{Ded} \mu$. Let $\lambda := \operatorname{Ded} \mu$. Fix $\{\mathbf{a}_i \mid i < \mu\} = {}^{\ell(\mathbf{y})}\mathrm{A}$ and let $A_i := \{\mathbf{a}_j \mid j < i\}$, here by A_i I mean exactly what is written and not $\bigcup_{j<i} \cup \mathbf{a}_j$. Namely

$$p \restriction A_i := \{\varphi(\mathbf{x}; \mathbf{a}_j) \mid \varphi(\mathbf{x}; \mathbf{a}_j) \in p, j < i\} \cup \{\neg\varphi(\mathbf{x}; \mathbf{a}_j) \mid \neg\varphi(\mathbf{x}; \mathbf{a}_j) \in p, j < i\}.$$

By minimality of μ we have that

$$(*) \quad i < \mu \implies |\,\mathrm{S}_\varphi(A_i)| \leq \lambda.$$

Consider

$$S_i := \{p \in \mathrm{S}_\varphi(A_i) \mid p \text{ has } \lambda^+ \text{ many extensions in } \mathrm{S}_\varphi(A)\}.$$
$$S_\mu := \{p \in \mathrm{S}_\varphi(A) \mid \forall i < \mu\ p \restriction A_i \in S_i\}.$$

Let $S := \bigcup_{i \leq \mu} S_i$.

For $p \neq q \in S$ define a binary relation $q < p$ as follows:

If $\exists i < \mu$ such that $p \restriction A_i = q \restriction A_i$, $\neg\varphi(\mathbf{x}; \mathbf{a}_i) \in q$ and $\varphi(\mathbf{x}; \mathbf{a}_i) \in p$.

If $\exists i < \mu$ such that $p \in S_i, q \restriction A_i = p$ and $\neg\varphi(\mathbf{x}; \mathbf{a}_i) \in q$.

If $\exists i < \mu$ such that $q \in S_i, p \restriction A_i = q$ and $\varphi(\mathbf{x}; \mathbf{a}_i) \in p$.

Verify that $<$ is a linear order on S and that $S^* := \bigcup_{i<\mu} S_i$ is a dense subset.

Using $|\,\mathrm{S}_\varphi(A)| \geq \lambda^+$ we get that $S^* \neq \emptyset$.

For $q \in S^*$, let $S_q := \{p \in S \ : \ p \supseteq q\}$. The set S_q is linearly ordered (by $<$) and $S^*_q := \{p \in S^* \mid p \supseteq q\}$ is a dense subset of S_q.

CLAIM 8.17. *For every $q \in S^*$ we that $|S_q| \geq \lambda^+$.*

PROOF. Given $q \in S^*$, let $i < \mu$ be such that $q \in S_i$.

Since $q \in S_i$ it has λ^+-many extensions in $\mathrm{S}_\varphi(A)$. It is enough to show that

$$|\,\mathrm{S}_\varphi(A) - S_\mu| \leq \lambda.$$

Using (*) we have that

$$|S_i| \leq |\,\mathrm{S}_\varphi(A_i)| \leq \lambda.$$

If $p \in \mathrm{S}_\varphi(A) - S_\mu$ then there exists $i < \mu$ such that $p \restriction A_i \notin S_i$. From the definition of S_i if $p \restriction A_i \notin S_i$ then the type $p \restriction A_i$ has $\leq \lambda$-many extensions to $\mathrm{S}_\varphi(A)$ we get that

$$\{p \in \mathrm{S}_\varphi(A) \ : \ p \restriction A_i \notin S_i\} \text{ has cardinality } \leq \lambda \text{ for all } i < \mu.$$

Thus, using that $|S_i| \leq \lambda$ we get

$$\{p \in \mathrm{S}_\varphi(A) \ : \ p \notin S_\mu\} \text{ has cardinality } \leq \lambda.$$

□

To complete the proof of the theorem it suffices to establish the following:

CLAIM 8.18. *For every $q \in S^*$ and every $n < \omega$ there are $\mathbf{b}^q_0, \dots, \mathbf{b}^q_{n-1} \in A$ such that*

$$(*)_n \quad q(\mathbf{x}) \cup \{\varphi(\mathbf{x}; \mathbf{b}^q_j)^{j \in w} \mid j < n\} \text{ is consistent for all } w \subseteq n.$$

PROOF. We argue by induction on $n < \omega$: For $n = 0$ there is nothing to prove.

Suppose that the statement of the claim holds for n and we'll establish it for $n + 1$:

Since S^*_q is a dense subset of S_q the above Claim $|S_q| \geq \lambda^+ \geq \operatorname{Ded} |S^*_q|$. As $\lambda = \operatorname{Ded} \mu$ we get that $|S^*_q| \geq \mu^+$ must hold.

As S^* is a union of μ-many sets there exists $i_0 < \mu$ such that $S^*_q \cap S_{i_0}$ has cardinality $\geq \mu^+$.

By the induction hypothesis for every $p \in S^*_q \cap S_{i_0}$ there are witnesses $\mathbf{b}^p_0, \dots, \mathbf{b}^p_{n-1} \in A$ satisfying $(*)_n$ for p replacing q.

Consider the mapping

$$p \mapsto \langle \mathbf{b}^p_0, \dots, \mathbf{b}^p_{n-1} \rangle \text{ for all } p \in S^*_q \cap S_{i_0}$$

As $|S^*_q \cap S_{i_0}| \geq \mu^+$ and there are only μ-many finite sequences of elements of A, there are $p_0 \neq p_1 \in S^*_q \cap S_{i_0}$ such that

$$\langle \mathbf{b}_0, \dots, \mathbf{b}_{n-1} \rangle := \langle \mathbf{b}^{p_0}_0, \dots, \mathbf{b}^{p_0}_{n-1} \rangle = \langle \mathbf{b}^{p_1}_0, \dots, \mathbf{b}^{p_1}_{n-1} \rangle.$$

and

$$p_\ell(\mathbf{x}) \cup \{\varphi(\mathbf{x}; \mathbf{b}_j)^{j \in w} \mid j < n\} \text{ is consistent for all } w \subseteq n \text{ and } \ell = 0, 1.$$

Take any $\mathbf{b}_n \in A_{i_0}$ such that $\varphi(\mathbf{x}; \mathbf{b}_n) \in p_0$ and $\neg\varphi(\mathbf{x}; \mathbf{b}_n) \in p_1$.

Since

$q(\mathbf{x}) \cup \{\varphi(\mathbf{x}; \mathbf{b}_n)^\ell\} \cup \{\varphi(\mathbf{x}; \mathbf{b}_j)^{j \in w} \mid j < n\}$ are consistent for $w \subseteq n$ and $\ell = 0, 1$.

We have that $\{\mathbf{b}_j \mid j \leq n\}$ satisfy $(*)_{n+1}$. □

□

COROLLARY 8.19. *If T is NIP then $f_T(\mu) \leq (\operatorname{Ded} \mu)^{|T|}$.*

If in every model of set theory there exist an uncountable μ such that $f_\varphi(\mu) = 2^\mu$ for some formula then φ has the independence property.

If T has the independence property then $f_T(\mu) = 2^\mu$ for all $\mu \geq |T|$.

PROOF. For the first part use the previous theorem together with the idea of proof of Theorem **??**. For the second part use the compactness theorem. □

Thus in a model of set theory where $(\mathrm{Ded}\,\mu)^{|T|} < 2^\mu$ we can identify a dependent unstable theory from a theory with the independence property by computing $f_T(\mu)$. Namely under the assumption of $(\mathrm{Ded}\,\mu)^{|T|} < 2^\mu$, the theory T is dependent iff $f_T(\mu) < 2^\mu$.

DEFINITION 8.20. Let $\mathbf{I}$ be a sequence of elements such that $n < \omega$ such that $\mathbf{b} \in \mathbf{I} \implies \ell(\mathbf{b}) = n$ and let be A be a given set.

For a sequence $\mathbf{x} = \langle x_1, \dots, x_n \rangle$ let

$$\mathrm{Av}(\mathbf{I}/A) := \Big\{ \varphi^t(\mathbf{x}; \mathbf{a}) \ : \ \varphi(\mathbf{x}; \mathbf{y}) \in \mathrm{Fml}(\mathrm{L}(T)), \mathbf{a} \in A, t \in \{0,1\} \text{ for all except finitely many elements of } \mathbf{I} \text{ we have } \mathbf{b} \in \mathbf{I} \implies \mathfrak{C} \models \varphi^t[\mathbf{b}; \mathbf{a}] \Big\}.$$

Recall that $\varphi^0(\mathbf{x})$ is $\varphi(\mathbf{x})$ and $\varphi^1(\mathbf{x})$ stands for $\neg\varphi(\mathbf{x})$.

$\mathrm{Av}(\mathbf{I}/A)$ is called *the average of* $\mathbf{I}$ *over* A.

PROPOSITION 8.21. *Let* $\mathbf{I}$ *be an indiscernible sequence without a last element and* A *be a set. For any* $\mathbf{J} \subseteq \mathbf{I}$ *without a last element we have*

$$\mathrm{Av}(\mathbf{I}/A) \subseteq \mathrm{Av}(\mathbf{J}/A).$$

PROOF. Trivial. □

THEOREM 8.22 (Average is a complete type). *Let* $\mathbf{I}$, A *and* n *be as above. Suppose* T *is a dependent theory. If* $\mathbf{I}$ *is an indiscernible sequence without a last element then* $\mathrm{Av}(\mathbf{I}/A) \in \mathrm{S}^n(A)$.

PROOF. It is enough to prove: Given an infinite indiscernible sequence I, a formula $\varphi(\mathbf{x}; \mathbf{y})$ and any $\mathbf{c} \in {}^{\ell(\mathbf{x})}|\,\mathfrak{C}\,|$ we have

$$|\varphi(\mathbf{I}; \mathbf{c})| < \aleph_0 \text{ or } |\neg\varphi(\mathbf{I}; \mathbf{c})| < \aleph_0.$$

Suppose there exists $\mathbf{I} = \{\mathbf{a}_n \mid n < \omega\}$, $\varphi(\mathbf{x}; \mathbf{c})$ and infinite sets $A, B \subseteq \omega$ such that

$$\models \varphi[\mathbf{a}_i; \mathbf{c}] \text{ for all } i \in A \text{ and } \models \neg\varphi[\mathbf{a}_j; \mathbf{c}] \text{ for all } j \in B.$$

Now we show that $\varphi(\mathbf{x}; \mathbf{y})$ has the independence property. Given $n < \omega$ and $w \subseteq n$. Since both A and B are infinite subsets of ω we can fix an increasing $\{j_i \mid i \in n\} \subseteq A \cup B$ such that

$$i \in w \iff j_i \in A.$$

By the assumption of A and B we have

$$\models \exists \mathbf{y} [\bigwedge_{i \in w} \varphi(\mathbf{a}_{j_i}; \mathbf{y}) \wedge \bigwedge_{i \in n - w} \neg\varphi(\mathbf{a}_{j_i}; \mathbf{y})].$$

By indiscernibelity of I

$$\mathrm{tp}(\mathbf{a}_0, \dots \mathbf{a}_{n-1}/\emptyset) = \mathrm{tp}(\mathbf{a}_{j_0}, \dots, \mathbf{a}_{j_{n-1}}/\emptyset)$$

we have

$$\models \exists \mathbf{y}[\bigwedge_{i \in w} \varphi(\mathbf{a}_i; \mathbf{y}) \wedge \bigwedge_{i \in n-w} \neg\varphi(\mathbf{a}_i; \mathbf{y})].$$

□

A simple application of the compactness theorem applied to the previous proof gives us the following important

LEMMA 8.23. *Suppose T is a dependent theory. For every $\varphi(\mathbf{x}; \mathbf{y})$ there exists $n_\varphi < \omega$ such that for every $\mathbf{I}$ infinite indiscernible sequence we have*

$$|\varphi(\mathbf{c}; \mathbf{I})| < n_\varphi \text{ or } |\neg\varphi(\mathbf{c}; \mathbf{I})| < n_\varphi \text{ for every } \mathbf{c} \in {}^{\ell(\mathbf{x})}\mathfrak{C}.$$

PROOF. If the assertion fails then there exists an infinite indiscernible sequence $\mathbf{J}$ and a sequence $\mathbf{d}$ such that both

$$\varphi(\mathbf{d}; \mathbf{J}) \text{ and } \varphi(\mathbf{d}; \mathbf{J}) \text{ are infinite.}$$

The argument used in the previous proof gives that $\varphi(\mathbf{x}; \mathbf{y})$ has the independence property. □

An immediate application of Theorem 8.22 gives

PROPOSITION 8.24. *Suppose T is a dependent theory. Let $\mathbf{I}$ be an indiscernible sequence without a last element over A and $\mathbf{b}$ be a finite sequence of the same arity as the elements of $\mathbf{I}$. If $\mathbf{b} \models \mathrm{Av}(\mathbf{I}/A \cup \mathbf{I})$ then $\mathbf{I}\{\mathbf{b}\}$ is an indiscernible sequence over A.*

PROOF. Exercise. □

THEOREM 8.25. *T has the independence property iff for every $m < \omega$ there exists $\varphi_m(\mathbf{x}; \mathbf{y})$ with $\ell(\mathbf{x}) = m$ such that φ_m has the independence property.*

Notice that the statement of Theorem 8.29 is non-trivial as the property of a theory is for $\ell(x) = 1$.

DEFINITION 8.26. A formula $\varphi(\mathbf{x}; \mathbf{y})$ is said to have the *strict order property* if for every M and every $n < \omega$ there exists $\{\mathbf{a}_i \mid i < n\} \subseteq |M|$ such that

$$M \models \exists \mathbf{x}(\neg\varphi(\mathbf{x}; \mathbf{a}_i) \wedge \varphi(\mathbf{x}; \mathbf{a}_j)) \iff i < j < n.$$

We say that T has the strict order property if there exists a formula $\varphi(x; \mathbf{y})$ with the strict order property.

strict-order property

EXAMPLE 8.27. $T_{\rm ord}$ and $\mathrm{Th}(\langle \mathbf{R}, +, \cdot, 0, 1\rangle)$ have the strict-order property but not the independence property. Peano's Arithmetic and Zermelo Fraenkel set theory have both the strict-order property but the independence property.

Using the compactness theorem and Ramsey's theorem (or Erős-Rado) we can prove:

PROPOSITION 8.28. (1) *Suppose $\varphi(\mathbf{x};\mathbf{y})$ has the independence property then for any infinite linearly ordered set I there exists $\mathbf{I} = \{\mathbf{a}_i \mid i \in I\}$ indiscernible sequence such that*

$$\{\varphi(\mathbf{x};\mathbf{a}_i)^{i\in w} \mid i \in I\} \quad \text{is consistent for every } w \subseteq I.$$

(2) *Suppose $\varphi(\mathbf{x};\mathbf{y})$ has the strict-order property then for any infinite linearly ordered set I there exists $\mathbf{I} = \{\mathbf{a}_i \mid i \in I\}$ indiscernible sequence such that*

$$\mathfrak{C} \models \exists \mathbf{x}(\neg\varphi(\mathbf{x};\mathbf{a}_i) \wedge \varphi(\mathbf{x};\mathbf{a}_j)) \iff i < j.$$

PROOF. Exercise. □

THEOREM 8.29. *? T has the strict-order property iff there exists $\varphi(\mathbf{x};\mathbf{y})$ for some finite $\mathbf{x}$ with the strict-order property.* ?!

THEOREM 8.30. *T unstable iff T st. order prop or T has indep prop.*

The easier direction follows from the following:

PROPOSITION 8.31. *T is unstable, provided it has the independence property or has the strict-order property.*

PROOF. Suppose T has the independence property witnessed by $\varphi(\mathbf{x};\mathbf{y})$. Let A be a a given set of cardinality λ, fix $\{\mathbf{a}_i \mid i < \lambda\} = A$ when $\ell(\mathbf{y}) = \ell(\mathbf{a}_i)$ for all $i < \lambda$. For $S \subseteq \lambda$ let

$$q_S := \{\varphi(\mathbf{x};\mathbf{a}_i)^{i\in S} \mid i < \lambda\} \quad \text{and let } p_S \text{ be a completion of } p_S \text{ to } A.$$

As $S_1 \neq S_2 \subseteq \lambda \implies p_{S_1} \neq p_{S_2}$ we have that $\mathrm{S}^{\mathbf{x}}(A) = 2^\lambda$ while A has cardinality λ.

Now suppose that T has the strict-order property. Clearly this implies that T has the order property, an application of Theorem 3.1 establishes that T is unstable. □

Using that instability implies the order property (Theorem 3.1) the non-trivial direction is:

THEOREM 8.32. *Let $\varphi(\mathbf{x};\mathbf{y})$ be a formula with the order property. Then, either $\varphi(\mathbf{x};\mathbf{y})$ has the independence property, or there exists a formula $\chi(\mathbf{y})$ perhaps with parameters from $\mathfrak{C}$ such that $\chi(\mathbf{x}) \wedge \varphi(\mathbf{x};\mathbf{y})$ has the strict-order property.*

PROOF. Suppose $\varphi(\mathbf{x};\mathbf{y})$ has the order-property but does not have the independence property. By Exercise 5.1 let $\{\mathbf{a}_q \mid q \in \mathbf{Q}\}$ be an indiscernible sequence satisfying

$$\mathfrak{C} \models \varphi[\mathbf{a}_q;\mathbf{a}_r] \iff r < q \quad \text{for all } q, r \in \mathbf{Q}.$$

Since $\varphi(\mathbf{x};\mathbf{y})$ does not have the independence property there are $n < \omega$ and $w \subseteq n$ such that

$$(*) \quad \mathfrak{C} \models \neg\exists\mathbf{y}\psi \text{ where } \psi := \bigwedge_{i<n} \varphi(\mathbf{a};\mathbf{y})^{i\in w}.$$

Let $\eta_0 \in {}^n2$ be defined by $\eta_0[i] = 1 \iff k \in w$. Recall our convention $\varphi^0 := \varphi$ and $\varphi^1 := \neg\varphi$. Using this we rewrite ψ as

$$\psi_\eta(\mathbf{y}) := \bigwedge_{i<n} \varphi(\mathbf{a};\mathbf{y})^{\eta_0[i]}$$

Using the assumption that $\varphi(\mathbf{x};\mathbf{y})$ has the order property we have that

$$(**) \quad \mathfrak{C} \models \exists\mathbf{y} \bigwedge_{i<i_0} \varphi(\mathbf{a}_i;\mathbf{y}) \wedge \bigwedge_{i_0\le i<n} \neg\varphi(\mathbf{a}_i;\mathbf{y}) \text{ for every } i_0 < n.$$

Note that the last formula is realized by $\mathbf{a}_q$ for all rationals such that $i_0 - 1 < q < i_0$.

The formula in $(**)$ is of the form $\psi_{\eta(i_0)}$ when

$$\eta(i_0) = \langle 0,\dots,0,1,\dots,1\rangle \text{ with the first 1 on the } i_0\text{-th place.}$$

Given $\eta \in {}^n2$, if there exists $i < n$ such that $\eta[i] = 1$ and $\eta[i+1] = 0$ let $i_0 < n$ be the smallest such that $\eta[i_0] = 1$ and $\eta[i_0+1] = 0$. Denote by η^* the sequence that obtained from η by switching the values at i_0 and i_0+1, namely $\eta^*[k] := \eta[k]$ for all $k \notin \{i_0, i_0+1\}$ while $\eta^*[i_0] := 0$ and $\eta^*[i_0+1] = 1$.

Now define $\{\eta_k \in {}^n2 \mid k < k_0 \le n\}$ be defined as follows $\eta_{k+1} := \eta_k^*$.

Since ψ_{η_0} is inconsistent while $\psi_{\eta(i_0)}$ is consistent for all $i_0 < n$, there exists $k_0 < n$ such that $\psi_{\eta_{k_0-1}}$ is inconsistent and $\psi_{\eta_{k_0}}$ is consistent. Denote by ν the sequence η_{k_0-1}. As ψ_ν is inconsistent and ψ_{ν^*} is consistent, writing this explicitly we have that ψ_ν is

$$\bigwedge_{i<i_0} \varphi(\mathbf{a}_i;\mathbf{y})^{\nu[i]} \wedge \neg\varphi(\mathbf{a}_{i_0};\mathbf{y}) \wedge \varphi(\mathbf{a}_{i_0+1};\mathbf{y}) \wedge \bigwedge_{i_0+1<i<n} \varphi(\mathbf{a}_i;\mathbf{y})^{\nu[i]},$$

while ψ_{ν^*} is

$$\bigwedge_{i<i_0} \varphi(\mathbf{a}_i;\mathbf{y})^{\nu[i]} \wedge \varphi(\mathbf{a}_{i_0};\mathbf{y}) \wedge \neg\varphi(\mathbf{a}_{i_0+1};\mathbf{y}) \wedge \bigwedge_{i_0+1<i<n} \varphi(\mathbf{a}_i;\mathbf{y})^{\nu[i]}.$$

Let $\chi(\mathbf{y})$ be the common subformula to ψ_ν and ψ_{nu^*}:

$$\bigwedge_{i<i_0} \varphi(\mathbf{a}_i;\mathbf{y})^{\nu[i]} \wedge \bigwedge_{i_0+1<i<n} \varphi(\mathbf{a}_i;\mathbf{y})^{\nu[i]}.$$

CLAIM 8.33. *The sequence* $\{\mathbf{a}_q \mid i_0 < q < i_0 + 1\}$ *witness that* $\chi(\mathbf{y}) \wedge \varphi(\mathbf{x}; \mathbf{y})$ *has the SOP.*

PROOF. Given rationals q and q' such that $i_0 < q < q' < i_0 + 1$, by the assumption that $\{\mathbf{a}_q \mid q \in \mathbf{Q}\}$ is an indiscernible sequence, from

$\mathfrak{C} \models \exists \mathbf{y}[\chi(\mathbf{y}) \wedge \varphi(\mathbf{a}_{i_0}; \mathbf{y}) \wedge \neg\varphi(\mathbf{a}_{i_0+1}; \mathbf{y})]$ and $\mathfrak{C} \models \neg\exists \mathbf{y}[\chi(\mathbf{y}) \wedge \neg\varphi(\mathbf{a}_{i_0}; \mathbf{y}) \wedge \varphi(\mathbf{a}_{i_0+1}; \mathbf{y})]$

we get that

$\mathfrak{C} \models \exists \mathbf{y}[\chi(\mathbf{y}) \wedge \varphi(\mathbf{a}_q; \mathbf{y}) \wedge \neg\varphi(\mathbf{a}_{q'}; \mathbf{y})]$ and $\mathfrak{C} \models \neg\exists \mathbf{y}[\chi(\mathbf{y}) \wedge \neg\psi(\mathbf{a}_q; \mathbf{y}) \wedge \varphi(\mathbf{a}_{q'}; \mathbf{y})]$

□

□

Exercises 8.8.

(1) The formula $\varphi(\mathbf{x};\mathbf{y})$ has the SOP if and only if for any linearly ordered set I there exists $\{\mathbf{a}_i \mid i \in I\}$ an indiscernible sequence such that

$$\mathfrak{C} \models \exists \mathbf{x}[\neg\varphi(\mathbf{x};\mathbf{a}_i) \wedge \varphi(\mathbf{x};\mathbf{a}_j)] \iff i < j \text{ for all } i,j \in I.$$

(2) Let T be a complete theory. Prove
Theorem T is NIP iff every countable $T' \subseteq T$ is NIP.
(3) Let T be a complete theory. Prove
Theorem T has the SOP iff every countable $T' \subseteq T$ has the SOP.
(4) Let $F := \langle \mathbb{R}, +, \cdot \rangle$ be the field of real numbers show that $\mathrm{Th}(F)$ is NIP.
(5) .
(6) T_{ind} has the independence property.
(7) Do you know a field F such that $\mathrm{Th}(F)$ is unstable?
(8)
(9) .
(10)
(11)
(12)
(13)

9. Superstable theories

DEFINITION 9.1. T is *superstable* iff there exists a cardinal μ such that T is λ-stable in all $\lambda \geq \mu$.

By Corollary 3.3.4 if T is $\aleph_0$-stable then it is superstable.

A theory is called *strictly stable* iff it is stable but not superstable.

EXAMPLE 9.2.

DEFINITION 9.3. T has the *λ-tree property* provided there are $\{\varphi_n(\mathbf{x};\mathbf{y}_n) \mid n < \omega\}$ and $\{\mathbf{a}_\eta \mid \eta \in {}^{\omega\geq}\lambda\} \subseteq \mathfrak{C}$ such that

$$\mathfrak{C} \models \varphi_n[\mathbf{a}_\eta;\mathbf{a}_\nu] \iff \nu \lhd \eta \quad \text{for all } \nu \in {}^n\lambda, \eta \in {}^\omega\lambda\,.$$

From the compactness theorem we get that

PROPOSITION 9.4. *If T has the μ-tree property for some infinite μ then T has the λ-tree property for all λ.*

DEFINITION 9.5. Deg

THEOREM 9.6. *T is superstable iff $Deg[\mathbf{x} = \mathbf{x}] < \infty$*

THEOREM 9.7. *T is not superstable iff T has the tree property*

THEOREM 9.8. *If T is not superstable then $I(\lambda, T) = 2^\lambda$ for all λ regular such that $\lambda > |T|$.*

10. Simple Theories

DEFINITION 10.1. D[]

DEFINITION 10.2. T is simple

DEFINITION 10.3. It T is simple then it does not have the strict-order property.

11. Noetherian topological spaces

An idea that appeared in commutative algebra is that of a Noetherian topological space. The motivation was Zariski's topology.

It turns out that several important ideas of model theory are best understood using this language. The terminology of Noetherian spaces is not necessary, for classification theory as most of the theory was developed by Shelah without knowing what Zariski's topology or Noetherian spaces are.

However since in contemporary model theory the concept of Noetherian topological space appeared explicitly I have decided to use this language.

DEFINITION 11.1. Let X be a topological space.

(1) A closed $V \subseteq X$ is called *irreducible* iff there are no closed $V_\ell \subsetneq V$ (for $\ell = 1, 2$) such that $V = V_1 \cup V_2$.
(2) X is called *Noetherian space* iff there is no infinite decreasing chain of closed subsets of X.

EXAMPLES 11.2.

(1) Let R be a commutative ring. The space $Spec(R)$ is Noetherian iff the ring R is Noetherian. A closed set $Y \subseteq Spec(R)$ is irreducible iff the ideal
$$\bigcap_{I \in Spec(R), I \supseteq Y} I$$
is prime.
(2) Let F be a field and n a positive integer. It is the contents of Hilbert's basis theorem that the Zariski topology on $\mathbb{A}^n(F)$ is a Noetherian topology. A closed $V \subseteq \mathbb{A}^n(F)$ is irreducible iff the ideal $I(V)$ is prime.
(3)

The fundamental theorem of arithmetic (-every integer greater than 1 has essentially a unique factorization) can be generalized to Noetherian spaces, with essentially the same proof.

THEOREM 11.3 (decomposition). *Let X be a Noetherian space. For every closed $V \subseteq X$ there are $d < \omega$ and $\{W_k \subseteq V \mid k < d\}$ irreducible such that*

(1) $V = \bigcup_{k<d} W_k$ *and*
(2) $W_{k_1} \not\subseteq W_{k_2} \wedge W_{k_2} \not\subseteq W_{k_1}$ *for all* $k_1 \neq k_2$.

If $\{W'_k \subseteq V \mid k < d'\}$ are irreducible sets satisfying both above conditions then $d = d'$ and there is a permutation σ of $\{0, \ldots, d-1\}$ such that $W_k = W'_{\sigma(k)}$ for all $k < d$.

PROOF. *Existence of decomposition:* Suppose there exists a closed $V \subseteq X$ that does not have a decomposition into irreducible subsets. Consider

$$F := \{V \subseteq X \mid V \text{ is closed and there is no finite } \{W_k \subseteq V \mid k < d\} \text{ satisfying } (1)\&(2)\}.$$

Since X is Noetherian there exists a minimal $V \in F$ i.e. if $V' \subsetneq V$ is closed than $V' \notin F$.

There two possibilities: If V is irreducible then we have contradicted the definition of the family F. So V is not irreducible. Let $V_\ell \subseteq V$ be closed such that $V = V_1 \cup V_2$. By minimality of V there are irreducible $\{W^1_k \subseteq V_1 \mid k < d_1\}$ and irreducible $\{W^2_k \subseteq V_2 \mid k < d_2\}$ such that $V = \bigcup_{k=1}^{d_1} W^1_k \cup \bigcup_{k=1}^{d_2} W^2_k$ which contradicts the hypothesis that $V \in F$.

Uniqueness of decomposition: Let $\{W_k \subseteq V \mid k < d\}$ and $\{W'_k \subseteq V \mid k < d'\}$ be irreducible sets satisfying $(1)\&(2)$. We argue by induction on $m := d + d'$: The initial case ($m = 2$) is clear.

Since $W_1 \subseteq \bigcup_{k=1}^{d'} W'_k$ there exists $0 \leq i \leq d'$ such that $W^* := W_1 \cap W'_i \neq \emptyset$. Thus

$$W_1 = W^* \cup \bigcup_{k=1}^{i-1} (W'_k \cap W_1) \cup \bigcup_{k=i+1}^{d'} (W'_k \cap W_1).$$

Since all the sets of the form $W'_k \cap W_1$ are closed and a finite union of closed sets is closed by the irreducibility assumption of W_1 necessarily $W_1 = W^*$ thus $W_1 \subseteq W'_i$. Now starting with W'_i instead of W_1 the same argument as before (using the irreducibility of W'_i) show that $W'_i \subseteq W_1$, namely $W_1 = W'_i$. Subtracting W_1 produces:

$$V - W_1 = \bigcup_{k=2}^{d} W_k = \bigcup_{k=1}^{i-1} W'_k \cup \bigcup_{k=i+1}^{d'} W'_k.$$

We are finished by applying the induction hypothesis to $V - W_1$ (we can do that since the set is a finite union of closed sets which is closed). □

REMARK 11.4. An inspection of the above argument show that we did not use the full power of the assumption that X is a topological space. All we have been using was that the family of closed sets is closed under finite unions and intersection. Thus Theorem 11.3 holds also for for any set X with a prescribed $\mathcal{F}$ family of subsets which is closed under finite intersections and unions. An important example is the following: Let X be the set ${}^n|M|$ for some model and $\mathcal{F}$ is a subfamily of the definable subsets of M by formulas of the form $\varphi(x_0, \ldots, x_{n-1}; \mathbf{a})$.

DEFINITION 11.5. Let M be an L-structure. An ordinal-valued function R from subsets of M is said to have the *ultrametric property* iff

$$R(D_1 \cup D_2) \leq \max\{R(D_1), R(D_2)\}.$$

for every $D_1, D_2 \in \operatorname{Dom} R$.

We will see later that Morley's rank and several other rank functions all have the ultrametric property. Notice that for ultrametric rank function which is also monotone (all rank functions in model theory are monotone) we have stronger identity:

$$R(D_1 \cup D_2) = \mathrm{Max}\{R(D_1), R(D_2)\} \quad \text{for all } D_1, D_2 \in \mathrm{Dom}\, R.$$

The reason that the proof of Theorem 11.3 applies for many model-theoretic situations is the following:

PROPOSITION 11.6 (decomposition of minimal rank). *Suppose that $R(\cdot)$ is an ordinal valued function from the definable subsets of a model M. If $R(\cdot)$ is*

(1) *monotone and*
(2) *has the ultrametric property*

Then for every definable set D in M

In addition to the decomposition theorem we have the following important concept:

DEFINITION 11.7. Let X be a Noetherian topological space. For $V \subseteq X$ let

$$\dim V := \mathrm{Max}\Big\{ n < \omega \mid \exists V_0 \cdots \exists V_{n-1} \big[[\bigwedge_{i<n-1} V_i \subsetneq V_{i+1}] \\ V_i \text{ is a closed subset of } X, V_{n-1} \subseteq V \big] \Big\}.$$

$\dim V$ is called the *geometric dimension of* V.

Exercises 8.11.

(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)

CHAPTER 9

Forking calculus

Introduction

There are several dependence relations in model theory. the most successful notion is called *forking*. In this chapter we develop the notion of forking and its close relative *dividing*.

Forking was discovered by Shelah and presented for the first time in the 1978 edition of his book. Already than Shelah realized that forking could be developed for some unstable theories, his development in his book focused on stable theories. Quickly several alternative approaches were presented most notably the one by Lascar and Poizat (later named the Parisian School of Forking) with often simplifying Shelah's original proofs. In the late nineties after Byunghan Kim's seminal PhD thesis people realized that forking makes lots of sense also for simple unstable theories. It turned out that the Parisian approach while made some of the basic results for stable theories easier it does not extend. Focus shifted back to Shelah's original definition. Later it was realized that Shelah's original definition of forking is also helpful to understand the class of unstable dependent theories (=NIP theories), in fact the exercises at the end of section 1 of chapter III (page 92 of [**Sh a**]) indicate that in the seventies he had some results in that direction.

In my development of forking I started by setting results that are true for all theories and in later stages considered more detailed structural results under specific assumptions on T like stable, super stable, simple and dependent.

Not unlike splitting and strong-splitting for a type p and a set B (usually $B \subseteq \operatorname{dom} p$) we say that p *does not split (strongly-split) over* B iff to some extent the type p is determined by its restriction to the set B. While notions considered perviously are nice in some respects the notion of forking is the nicest, this claim will be made precise (in Theorem **??**). Similarly to previous notions the really important concept is: Given type $p \subseteq q$ we want to describe when q is a *non-forking extension* of p. We will have a precise definition below. Immediately after the definition we show:

PROPOSITION 0.1. *Suppose $B \subseteq A$ and $p \in \mathrm{S}(A)$ does not fork over B then for every formula $\varphi(\mathbf{x};\mathbf{c})$ such that $\varphi(\mathbf{x};\mathbf{c})$ is almost over A. If $p \cup \{\varphi(\mathbf{x};\mathbf{c})\}$ is consistent then $p \cup \{\varphi(\mathbf{x};\mathbf{c})\}$ is a non-forking extension of p.*

Later in Theorem **??.??** we will see that when T is stable then the converse of the Proposition is also true.

In this chapter we make no assumptions on the theory T, except that it is complete (in order to be able to work in $\mathfrak{C}$ and avoid some trivial difficulties).

1. Basics of Forking

In this section we make no particular assumptions on the theory T (besides completeness), unless explicitly stated.

NOTATION 1.1 (first approximation for $*$-types). Let A, B be sets. Fix $\{b_i \mid i < |B|\}$ an enumeration of the set B. Let $\langle x_i \ : \ i < |B|\rangle$ be a sequence of distinct free variables.

$$\mathrm{tp}_*(B/A) := \{\varphi(x_{i_0}, \ldots, x_{i_{n-1}}; \mathbf{a}) \ : \ \mathfrak{C} \models \varphi[b_{i_0}, \ldots, b_{i_{n-1}}; \mathbf{a}]\}.$$

The problem with the above, is that the definition depends on the particular enumeration.

Recall that a sequence $\mathbf{I}$ in $\mathfrak{C}$ is indiscernible over a set A if any two finite subsequences of $\mathbf{I}$ of the same length have the same type over A. We always assume that the index set does not have a last element.

For $k < \omega$, we say that a set of formulas $q(\mathbf{x})$ is *k-contradictory* if every subset of q of k elements is inconsistent. Note that if $\langle \mathbf{a}_i \mid i \in I\rangle$ is an infinite sequence of indiscernibles and the set $\{\varphi(\mathbf{x}; \mathbf{a}_i) \mid i \in I\}$ is inconsistent, then it is k-contradictory for some $k < \omega$.

We begin by introducing the fundamental notion of *dividing*.

DEFINITION 1.2.

(1) A formula $\varphi(\mathbf{x}; \mathbf{a})$ *divides over B* if there exist infinite sequence $\mathbf{I}$ and $k < \omega$ such that
   (a) $\mathrm{tp}(\mathbf{c}/B) = \mathrm{tp}(\mathbf{a}/B)$ for every $\mathbf{c} \in \mathbf{I}$;
   (b) The set $\{\varphi(\mathbf{x}; \mathbf{c}) \mid \mathbf{c} \in \mathbf{I}\}$ is k-contradictory.
(2) A type p (possibly in infinitely many variables) *divides over B* if there exists a formula $\varphi(\mathbf{x}; \mathbf{a})$ such that $p \vdash \varphi(\mathbf{x}; \mathbf{a})$ and $\varphi(\mathbf{x}; \mathbf{a})$ divides over B.

LEMMA 1.3 (Monotonicity of dividing). (1) *Suppose $p \vdash q$ if q divides over B then also p divides over B. In particular, if a type does not divide over B that also any finite subtype also does not divide over B.*
(2) *If p does not divide over B then also p does not divide over C for every $C \supseteq B$.*

PROOF. Immediate from the definition. □

In Definition 1.2 I presented Shelah's original definition for dividing, it turns out that often an equivalent form is used. This is (2) of the next lemma.

LEMMA 1.4 (Equivalent statements to dividing). *Let B be a subset of the monster model and $\varphi(\mathbf{x};\mathbf{a})$ a formula with parameters.*

(1) *$\varphi(\mathbf{x};\mathbf{a})$ divides over B*
(2) *there exists an infinite B-indiscernible sequence $\mathbf{J}$ such that $\mathbf{a} \in \mathbf{J}$ and $\{\varphi(\mathbf{x};\mathbf{c}) \mid \mathbf{c} \in \mathbf{J}\}$ is inconsistent*
(3) *there are $k < \omega$ and $\{f_n \in \mathrm{Aut}_B(\mathfrak{C}) \mid n < \omega\}$ such that $\{\varphi(x; f_n(\mathbf{a})) \mid n < \omega\}$ is k-contradictory.*

PROOF. (2) $\implies$ (1): If $\{\varphi(\mathbf{x};\mathbf{c}) \mid \mathbf{c} \in \mathbf{J}\}$ is inconsistent then there exists $\mathbf{J}_0 \subseteq \mathbf{J}$ finite such that $\{\varphi(\mathbf{x};\mathbf{c}) \mid \mathbf{c} \in \mathbf{J}_0\}$ is inconsistent. Take $k := |\mathbf{J}_0|$. By indiscernibility of $\mathbf{J}$ for any k-many elements $\mathbf{J}_1 \subseteq \mathbf{J}$ we have $\mathrm{tp}(\mathbf{J}_0/B) = \mathrm{tp}(\mathbf{J}_1/B)$. In particular we get

$$\mathfrak{C} \models \exists \mathbf{x}[\bigwedge_{\mathbf{c}\in\mathbf{J}_1} \varphi(\mathbf{x};\mathbf{c})] \implies \mathfrak{C} \models \exists \mathbf{x} \bigwedge_{\mathbf{c}\in\mathbf{J}_0} \varphi(\mathbf{x};\mathbf{c})].$$

(1) $\implies$ (2): Suppose $\varphi(\mathbf{x};\mathbf{a})$, $\mathbf{I}$ and k are as in the definition of dividing.

Let $\{\mathbf{c}_n \mid n < \omega\}$ be new constants, consider the extension of T by the axioms

$$\{\neg\exists \mathbf{x}[\bigwedge_{n\in w} \varphi(\mathbf{x};\mathbf{c}_n)] \ : \ w \subseteq \omega, |w| = k\}$$

and the sentences stating that $\{\mathbf{c}_n \mid n < \omega\}$ is an indiscernible sequence over the set B with $\mathrm{tp}(\mathbf{c}_0/A) = \mathrm{tp}(\mathbf{b}/A)$.

By the argument we used to establish the Ehrenfeucht-Mostowski theorem (see the proof of Theorem 4.3.17) the new theory is consistent and taking $\mathbf{J}$ to be the set of interpretations of the set of new constants. Note that $\mathbf{J}$ as all the required properties with the possible exception of $\mathbf{a} \in \mathbf{J}$. Since $\mathrm{tp}(\mathbf{c}_0/B) = \mathrm{tp}(\mathbf{a}/B)$ there exists $f \in \mathrm{Aut}_B(\mathfrak{C})$ such that $f(\mathbf{c}_0) = \mathbf{a}$. Check that $\mathbf{I} := f[\mathbf{J}]$ is as required. □

The following is an immediate consequence of the definition:

PROPOSITION 1.5 (disjointness). *Suppose that the set B is a subset of the sets A, and C is any set. If $\mathrm{tp}(\mathbf{c}/A)$ does not divide over B for every $\mathbf{c} \in C$ then $C \cap A \subseteq \mathrm{acl}(B)$.*

PROOF. Suppose that $\mathbf{c} \in C \cap A - \mathrm{acl}(B)$. In that case

$$\mathbf{x} = \mathbf{c} \in \mathrm{tp}(\mathbf{c}/A)$$

must hold. If $\mathbf{I}$ is an infinite sequence of distinct elements over B such that $\mathbf{c} \in \mathbf{I}$ then $\{\mathbf{x} = \mathbf{d} \mid \mathbf{d} \in \mathbf{I}\}$ is 2-contradictory thus the type contains a formula that divides. □

Let us start by stating some immediate but fundamental properties of the concept of dividing.

REMARK 1.6 (Invariance). Let p be a type and B a set. The following conditions are equivalent:

(1) The type p does not divide over B;
(2) For every A-automorphism f, the type $f(p)$ does not divide over B;
(3) There exists an B-automorphism f such that the type $f(p)$ does not divide over B.

REMARK 1.7 (Monotonicity). Let B be a subset of $\mathfrak{C}$ and suppose p is a type that does not divide over B then p does not divide over A for every $A \supseteq B$.

We now introduce the most important concept of model theory introduced in the last 50 years.

DEFINITION 1.8.

(1) A formula $\varphi(\mathbf{x}; \mathbf{a})$ *forks over* B if there exist $n < \omega$ and $\{\psi_i(\mathbf{x}; \mathbf{a}^i) \mid i < n\}$, such that
   (a) $\varphi(\mathbf{x}; \mathbf{a}) \vdash \bigvee_{i<n} \psi_i(\mathbf{x}; \mathbf{a}^i)$ and
   (b) $\psi_i(\mathbf{x}; \mathbf{a}^i)$ divides over B for every $i < n$.
(2) A type p *forks over* B if there exists a formula $\varphi(\mathbf{x}; \mathbf{a})$ such that $p \vdash \varphi(\mathbf{x}; \mathbf{a})$ and $\varphi(\mathbf{x}; \mathbf{a})$ forks over B.
   We use the abbreviation p *f/*B for: the type p forks over the set B.
   The negation of forking is equally important, we write p *dnf over* B for: the type p does not fork over the set B.
(3) For a sequence $\mathbf{a}$ and sets A, B we denote "$\mathrm{tp}(\mathbf{a}/A)$ dnf over B" by $\mathbf{a} \underset{B}{\downarrow} A$.
(4) For sets A, B, C the symbol $C \underset{B}{\downarrow} A$ is short-hand for:

$$\mathbf{c} \underset{B}{\downarrow} A \text{ for every finite } \mathbf{c} \in C.$$

REMARK 1.9.

(1) If $\varphi(\mathbf{x}; \mathbf{a})$ divides over B, then $\varphi(\mathbf{x}; \mathbf{a})$ forks over B;
(2) If $\varphi(\mathbf{x}; \mathbf{a}) \vdash \bigvee_{i<n} \varphi_i(\mathbf{x}; \mathbf{b}^i)$, and $\varphi_i(\mathbf{x}; \mathbf{b}^i)$ forks over A for every $i < n$, then $\varphi(\mathbf{x}; \mathbf{a})$ forks over A;
(3) (Monotonicity of forking) Suppose $p \vdash q$ if q forks over B then also p forkes over B. In particular, if a type does not fork over B that also any finite subtype also does not fork over B.

(4) (Another monotonicity of forking) If p does not fork over B then also p does not fork over C for every $C \supseteq B$.
(5) (Finite character of forking) If p forks over B, then there exists a finite $p_0 \subseteq p$ such that p_0 forks over B. In the case p is a complete type then there exists $\varphi(\mathbf{x};\mathbf{a}) \in p$ such that $\varphi(\mathbf{x};\mathbf{a})$ forks over B.
(6) (Invariance of dividing) If $\varphi(\mathbf{x};\mathbf{a})$ divides over B then $\varphi(\mathbf{x};f(\mathbf{a}))$ divides over $f[B]$ for all $f \in \mathrm{Aut}(\mathfrak{C})$.
(7) (Invariance of forking) If $\varphi(\mathbf{x};\mathbf{a})$ forks over B then $\varphi(\mathbf{x};f(\mathbf{a}))$ forks over $f[B]$ for all $f \in \mathrm{Aut}(\mathfrak{C})$.
(8) If $\mathbf{a} \in B \subseteq A$ then $\mathrm{tp}(\mathbf{a}/A)$ does not fork over B.

The ideal of forking formulas over a set: From the definition of forking of a formula we have that when $\varphi(\mathbf{x};\mathbf{b})$ forks over A then $\psi(\mathfrak{C};\mathbf{d}) \subseteq \varphi(\mathfrak{C};\mathbf{b})$ implies that $\psi(\mathbf{x};\mathbf{d})$ forks over A, thus by (2) we get that

$$I^f_A := \{\varphi(\mathfrak{C};\mathbf{b}) \mid \varphi(\mathbf{x};\mathbf{b}) \text{ forks over } A\}$$

is an ideal of the Lindenbaum boolean algebra. Given a set A it is natural to ask is I^f_A a proper ideal? We will show later that when T is stable (or even Simple) then for every set A the ideal is non-trivial (this property will be named *existence property*). A related important key question: Is there a cardinal χ such that I^f_A satisfies the χ-chain condition?

A consequence of Proposition 1.5 that plays a central role in Chapter **??** is the following:

PROPOSITION 1.10. *Suppose A, B, C are sets satisfying $B \subseteq A \cap C$ then*

$$A \underset{B}{\downarrow} C \implies A \cap C = B.$$

The reader can observe that the advantage of forking over dividing is exactly that the argument below can be carried out.

THEOREM 1.11 (Extension). *If p does not fork over B and $\mathrm{dom}(p) \subseteq A$, then there exists $q \in \mathrm{S}(A)$ extending p such that q does not fork over B.*

PROOF. Consider

$$\Gamma := \{\neg\psi(\mathbf{x};\mathbf{a}) \mid \mathbf{a} \in A, \psi(\mathbf{x};\mathbf{a}) \text{ forks over } B\}.$$

Let us show that $p \cup \Gamma$ is consistent. If $p \cup \Gamma$ is inconsistent, then there is $\{\neg\psi_i(\mathbf{x};,\mathbf{a}_i) \mid i < n\} \subseteq \Gamma$ such that $p \cup \{\neg\psi_i(\mathbf{x};\mathbf{a}_i) \mid i < n\}$ is inconsistent. But then, $p \vdash \bigvee_{i<n} \psi_i(\mathbf{x};\mathbf{a}_i)$ and every $\psi_i(\mathbf{x};\mathbf{a}_i)$ forks over B. Hence, p forks over B, which is a contradiction.

Choose a complete extension $q \in \mathrm{S}(A)$ of $p \cup \Gamma$. If q forks over B, there exists $\psi(\mathbf{x};\mathbf{a}) \in q$ such that $\psi(\mathbf{x};\mathbf{a})$ forks over B. By the definition of Γ, we have $\neg\psi(\mathbf{x};\mathbf{a}) \in \Gamma \subseteq q$, which is a contradiction. Hence, q does not fork over B. □

DEFINITION 1.12. For a set B and a set of formulas p We say that p is *finitely satisfiable in* B iff for every finite $q \subseteq p$ there exists $\mathbf{a} \in B$ such that $\mathbf{a} \models q$.

Later (Theorem 1.66) we will show that a type dnf over over a universe of a model iff it is finitely satisfiable in that model.

We start with a couple of simple observations:

LEMMA 1.13. *Let B be a set and p a set of formulas. If p in finitely satisfiable in B then*

(1) p does not split over B and

(2) p does not fork over B.

PROOF. (1) Suppose that $\mathbf{a}$ and $\mathbf{c}$ and $\varphi(\mathbf{x}; \mathbf{y})$ are such that

$$\varphi(\mathbf{x}; \mathbf{a}),\ \neg\varphi(\mathbf{x}; \mathbf{c}) \in p.$$

By finite satisfiability there exists $\mathbf{b} \in B$ such that

$$(*) \quad \models \varphi[\mathbf{b}; \mathbf{a}] \wedge \neg\varphi[\mathbf{b}; \mathbf{c}].$$

Notice that $(*)$ implies that

$$\mathrm{tp}(\mathbf{a}/B) \neq \mathrm{tp}(\mathbf{c}/B).$$

(2) Suppose that p forks over B, there exists $\{\psi_i(\mathbf{x}; \mathbf{a}_i) \mid i < n\}$ such that

$$p \vdash \bigvee_{i<n} \psi_i(\mathbf{x}; \mathbf{a}_i) \text{ and } \psi_i(\mathbf{x}; \mathbf{a}_i) \text{ divides over } B.$$

Let $q \subseteq p$ be finite such that $q \vdash \bigvee_{i<n} \psi_i(\mathbf{x}; \mathbf{a}_i)$. By finite satisfiability there exists $\mathbf{b} \in B$ such that $\mathbf{b} \models q$. There exists $i < n$ such that $\models \psi_i[\mathbf{b}; \mathbf{a}_i]$. By dividing of $\psi_i(\mathbf{x}; \mathbf{a}_i)$ there exists $\{\mathbf{c}_n \mid n < \omega\}$ an indiscernible sequence over B such that $\mathbf{c}_0 = \mathbf{a}_i$ and $\{\psi_i(\mathbf{x}; \mathbf{c}_n) \mid n < \omega\}$ inconsistent. But by indiscernibility we have $\mathfrak{C} \models \psi_i[\mathbf{b}; \mathbf{a}_i] \iff \psi_i[\mathbf{b}; \mathbf{c}_n]$ for all $n < \omega$ thus

$$\mathbf{b} \models \{\psi_i(\mathbf{x}; \mathbf{c}_n) \mid n < \omega\},$$

is a contradiction. □

In light of the second part of the Lemma clearly if $p \in \mathrm{S}(M)$ then p does not fork over M. Since when p is an algebraic type over a model M then $p[\mathfrak{C}] = p[M]$, of much greater interest are the non-algebraic types. It is a natural question to ask can a universe of a model in the above replaced by an arbitrary set? Namely, given A and $p \in \mathrm{S}(A)$ does p fork over A? This question will be completely answered in **??** and **??**. This depends of certain properties of the theory T. As we will sone see for Simple theories the following even stronger property holds: For any A and $p \in \mathrm{S}(A)$ there exists $B \subseteq A$ such that p does not fork over B and $|B| \leq |T|$.

DEFINITION 1.14. For $n < \omega$ let

$$\kappa^n(T) := \min\{\kappa \mid \text{ for every } A \text{ and } p \in \mathrm{S}^n(A) \text{ there exists } B \subseteq A, \text{ such that } |B| < \kappa \text{ and } p \text{ dnf over } B\}.$$

If such a cardinal does not exists we write $\kappa^n(T) = \infty$.

$$\kappa(T) := \sup_{n<\omega} \kappa^n(T).$$

$\kappa_r(T)$ is defined similarly, as the first *regular* cardinal satisfying: For every $n < \omega$ and every A, $p \in \mathrm{S}^n(A)$ there exists $B \subseteq A$ of cardinality $< \kappa_r(T)$ such that p dnf over B.

Recall the notion of simple theory (Definition 2.36). Later we will prove that T is simple iff $\kappa(T) < \infty$ iff $\kappa(T) \leq |T|^+$. First we show that if T is stable then $\kappa(T) \leq |T|^+$ and also T is simple implies that $\kappa(T) \leq |T|^+$. This will be established by connecting forking with a drop in the approperiate rank functions (for stable and simple theories).

When T is not simple it is still possible that for a certain set A we have that all types over it dnf.

THEOREM 1.15. *Suppose T is stable. Let $p \in \mathrm{S}^{\mathbf{x}}(B)$ and $q \supseteq p$ be a type also in $\mathbf{x}$. If q forks over B then for every infinite λ there exists a finite set of formulas Δ_0 such that for every finite $\Delta \supseteq \Delta_0$ we have*

$$\mathrm{R}^{\mathbf{x}}[q, \Delta, \lambda] < \mathrm{R}^{\mathbf{x}}[p, \Delta, \lambda].$$

First we will establish the following special case:

LEMMA 1.16. *Let $p \in \mathrm{S}^{\mathbf{x}}(B)$, $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$, $\lambda \geq \aleph_0$ and suppose $\varphi(\mathbf{x}; \mathbf{a})$ divides over B. If $\alpha := \mathrm{R}^{\mathbf{x}}[p, \Delta, \lambda] < \infty$ then we have*

$$\mathrm{R}^{\mathbf{x}}[p \cup \{\varphi(\mathbf{x}; \mathbf{a})\}, \Delta, \lambda] < \alpha.$$

PROOF. Let $\{\mathbf{a}_i \mid i < \lambda\}$ be a sequence of indiscernibles over B such that $\mathbf{a}_0 = \mathbf{a}$ and $k < \omega$ satisfying that $\{\varphi(\mathbf{x}; \mathbf{a}_i) \mid i < \lambda\}$ is k-contradictory. Let $p_i := p \cup \{\varphi(\mathbf{x}; \mathbf{a}_i)\}$, take $A := B \cup \{\mathbf{a}_i \mid i < \lambda\}$. Suppose for a sake of contradiction that $\mathrm{R}^{\mathbf{x}}[p \cup \{\varphi(\mathbf{x}; \mathbf{a})\}, \Delta, \lambda] = \alpha$. By invariance we get that also $\mathrm{R}^{\mathbf{x}}[p_i, \Delta, \lambda] = \alpha$ for all $i < \lambda$.

By the extension property for R there exists $q_i \in \mathrm{S}^{\mathbf{x}}_{\Delta}(B) \supseteq \{\varphi(\mathbf{x}; \mathbf{a}_i)\}$ such that $\mathrm{R}^{\mathbf{x}}[p \cup q_i, \Delta, \lambda] = \alpha$ for all $i < \lambda$.

As the set $\{q_i \mid i < \lambda\}$ is k-contradictory there exists an $S \subseteq \lambda$ of cardinality λ such that $i \neq j \in S \implies q_i \neq q_j$. As the types $\{q_i \mid i \in S\}$ are complete distinct Δ-types over B, they must be explicitly contradictory thus $\mathrm{R}^{\mathbf{x}}[p, \Delta, \lambda] \geq \alpha + 1$ which is a contradiction. □

PROOF OF THEOREM 1.15. Suppose q forks over B. Let $\{\psi_i(\mathbf{x};\mathbf{a}_i) \mid i<n\}$ such that $q \vdash \bigvee_{i<n} \psi_i(\mathbf{x};\mathbf{a}_i)$ and $\psi_i(\mathbf{x};\mathbf{a}_i)$ divides over B for all $i<n$.

Let $\Delta_0 := \{\psi_i(\mathbf{x};\mathbf{y}_i) \mid i<n\}$. Given a finite $\Delta \supseteq \Delta_0$, by stability $\alpha := \mathrm{R}^{\mathbf{x}}[p,\Delta,\lambda] < \infty$. An application of the Lemma gives:

$$\mathrm{R}^{\mathbf{x}}[p \cup \{\psi_i(\mathbf{x};\mathbf{a}_i)\},\Delta,\lambda] < \alpha \text{ for all } i<n.$$

By the ultra-metric property we have

$$\mathrm{R}^{\mathbf{x}}[p \cup \{\bigvee_{i<n} \psi_i(\mathbf{x};\mathbf{a}_i)\},\Delta,\lambda] < \alpha.$$

Since $q \vdash p \cup \{\bigvee_{i<n} \psi(\mathbf{x};\mathbf{a}_i)\}$, monotonicity gives

$$\mathrm{R}^{\mathbf{x}}[q,\Delta,\aleph_0] < \alpha.$$

$$\mathrm{R}^n[p,\Delta,\lambda] < \infty \implies \mathrm{R}^n[q,\Delta,\lambda] < \mathrm{R}^n[p,\Delta,\lambda].$$

□

Basically the same argument as the last theorem can be used to establish one of the most fundamental properties of stable theories. It's called the *existence property*.

THEOREM 1.17 (Existence property of forking). *If T is stable then for every B and every $p \in \mathrm{S}^{\mathbf{x}}(B)$ the type p does not fork over B.*

theorem

PROOF. Otherwise, there are $\{\psi_i(\mathbf{x};\mathbf{a}_i) \mid i<n\}$ formulas such that

$$(*) \quad p \vdash \bigvee_{i<n} \psi_i(\mathbf{x};\mathbf{a}_i)$$

and for every $i<n$ the formula $\psi_i(\mathbf{x};\mathbf{a}_i)$ divides over B.

Let $\Delta := \{\psi_i(\mathbf{x};\mathbf{y}_i) \mid i<n\}$. Using stability $\alpha := \mathrm{R}^{\mathbf{x}}[p,\Delta,\aleph_0]$ is an ordinal. Using monotonicity of R from $(*)$ we get that

$$(1) \quad \alpha \leq \mathrm{R}^x[p \cup \{\bigvee_{i<n} \psi_i(\mathbf{x};\mathbf{a}_i)\},\Delta,\aleph_0].$$

Let $\gamma_i := \mathrm{R}^{\mathbf{x}}[p \cup \{\psi_i(\mathbf{x};\mathbf{a}_i)\},\Delta,\aleph_0]$ for $i<n$.

Using the ultra-metric property of R^1 we get that

$$(2) \quad \beta := \mathrm{R}^{\mathbf{x}}[p \cup \{\bigvee_{i<n} \psi_i(\mathbf{x};\mathbf{a}_i)\},\Delta,\aleph_0] = \mathrm{Max}\{\gamma_i \mid i<n\}.$$

But since p is a complete type over B and $\psi_i(\mathbf{x};\mathbf{a}_i) \in \Delta$ divides over B using Lemma 1.19 we have

$$(3) \quad \alpha = \mathrm{R}^{\mathbf{x}}[p,\Delta,\aleph_0] > \mathrm{R}^{\mathbf{x}}[p \cup \{\psi_i(\mathbf{x};\mathbf{a}_i)\},\Delta,\aleph_0] = \gamma_i.$$

[1]Theorem 2.15.

A combination of the last three statements gives $\alpha = \beta = \mathrm{Max}\{\gamma_i \mid i < n\} < \alpha$ which is a contradiction. □

Theorem 1.15 and its lemma also hold in many more cases (for other rank functions) and even without assuming stability:

THEOREM 1.18. *Let $p \in \mathrm{S}^n(B)$ and $q \supseteq p$ be a type in the same variables as p. If q forks over B then*

(1) *there exists a finite set of formulas Δ_0 such that for every $\Delta \supseteq \Delta_0$ and for all $\lambda \geq \aleph_0$*

$$\mathrm{D}^n[p, \Delta, \lambda] < \infty \implies \mathrm{D}^n[q, \Delta, \lambda] < \mathrm{D}^n[p, \Delta, \lambda].$$

(2) *Suppose that $\mathrm{D}^n[p, \Delta, k] < \infty$ for all finiteΔ and every $k < \omega$. There exists a finite Δ_0 and $k_0 < \omega$ such that*

$\mathrm{D}^n[q, \Delta, k] < \mathrm{D}^n[p, \Delta, k]$ *for all finite $\Delta \supseteq \Delta_0$ and very $k \geq k_0$.*

The proof of part (1) is similar to the previous theorem using the following:

LEMMA 1.19. *Let $p \in \mathrm{S}^{\mathbf{x}}(B)$, $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$, $n := \ell(\mathbf{x})$ and suppose $\varphi(\mathbf{x}; \mathbf{a})$ divides over B.*

$$\mathrm{D}^n[p, \Delta, \lambda] < \infty \implies \mathrm{D}^n[p \cup \{\varphi(\mathbf{x}; \mathbf{a})\}, \Delta, \lambda] < \mathrm{D}^n[p, \Delta, \lambda].$$

PROOF OF PART (2) OF THE THEOREM. By finite character of forking we may assume that $\varphi(\mathbf{x}; \mathbf{a}) \in q$ forks over B and by monotonicity of this rank it is enough to show that $\mathrm{D}^n[p \cup \{\varphi(\mathbf{x}; \mathbf{a})\}, \Delta, k] < \mathrm{D}^n[p, \Delta, k]$ for all Δ and k.

First suppose that $\varphi(\mathbf{x}; \mathbf{a})$ divides over B. Let $\{\mathbf{a}_n \mid n < \omega\}$ be an indiscernible sequences over B and $k_0 < \omega$ such that $\mathbf{a}_0 = \mathbf{a}$ and $\{\varphi(\mathbf{x}; \mathbf{a}_n) \mid n < \omega\}$ is k_0-contradictory. Take $\Delta_0 := \{\varphi(\mathbf{x}; \mathbf{y})\}$. If there are a finite $\Delta \supseteq \Delta_0$ and $k \geq k_0$ such that

$$\mathrm{D}^n[p \cup \{\varphi(\mathbf{x}; \mathbf{a})\}, \Delta, k] = \mathrm{D}^n[p, \Delta, k] = \alpha < \infty.$$

By invariance we get that

$$\mathrm{D}^n[p \cup \{\varphi(\mathbf{x}; \mathbf{a}_n)\}, \Delta, k] = \alpha \text{ for all } n < \omega.$$

Since $\varphi \in \Delta$ and $\{\varphi(\mathbf{x}; \mathbf{a}_n) \mid n < \omega\}$ is k-contradictory we get that $\mathrm{D}^n[p \cup \{\varphi(\mathbf{x}; \mathbf{a})\}, \Delta, k] \geq \alpha + 1$.

Now for the general case, i.e. $\varphi(\mathbf{x}; \mathbf{a})$ forks over B. Let $\{\psi_i(\mathbf{x}; \mathbf{a}_i) \mid i < m\}$ formulas dividing over B be such that $\varphi(\mathbf{x}; \mathbf{a}) \vdash \bigvee_{i<m} \psi_i(\mathbf{x}; \mathbf{a}_i)$. By the first part there finite Δ_i and $k(i) < \omega$ for $i < m$ such that

$$\mathrm{D}^n[p \cup \{\psi_i(\mathbf{x}; \mathbf{a}_i)\}, \Delta, k] < \mathrm{D}^n[p, \Delta, k] \text{ for all finite } \Delta \supseteq \Delta_i, k \geq k(i).$$

□

COROLLARY 1.20. *If T is stable or simple then for every $p \in \mathrm{S}^n(B)$ the type p does not fork over B.*

PROOF. If p forks over B then apply the theorem for $q = p$ to get that $\mathrm{R}[p, \Delta_0, \aleph_0] < \mathrm{R}[p, \Delta_0, \aleph_0]$ for some finite Δ_0, this is a contradiction to stability. For simple theory the proof is similar. □

THEOREM 1.21. *If T is stable then $\kappa(T) \leq |T|^+$.*

PROOF. Otherwise, there exists A and $p \in \mathrm{S}^n(A)$ such that p forks over every $B \subseteq A$ of cardinality $|T|$. Using the finite character of forking there exists an increasing and continuos sequence of subsets of A, $\{B_i \mid i \leq |T|^+\}$ such that $p \restriction B_{i+1}$ forks over B_i for every $i < |T|^+$. Let $q := p \restriction B_{|T|^+}$. By stability for every finite Δ we have $\mathrm{R}[q, \Delta, \aleph_0] < \infty$. By finite character for every finite Δ there exists $i_\Delta < |T|^+$ such that $\mathrm{R}[q, \Delta, \aleph_0] = \mathrm{R}[q \restriction B_{i_\Delta}, \Delta, \aleph_0]$. Let

$$\alpha := \sup\{i_\Delta \mid \Delta \text{ is a finite set of formulas }\}.$$

I claim that q dnf over B_α: Otherwise by Theorem 1.15 there exists a finite $\Delta = \Delta_0$ such that $\mathrm{R}[q, \Delta, \aleph_0] < \mathrm{R}[q \restriction B_\alpha, \Delta, \aleph_0]$. But by the choice of i_Δ we have $\mathrm{R}[q, \Delta, \aleph_0] = \mathrm{R}[q \restriction B_{i_\Delta}, \Delta, \aleph_0]$ but this is impossible since by monotonicity we have $\mathrm{R}[q \restriction B_\alpha, \Delta, \aleph_0] \geq \mathrm{R}[q \restriction B_{i_\Delta}, \Delta, \aleph_0]$. □

Similarly one can prove

THEOREM 1.22. *If T is simple then $\kappa(T) \leq |T|^+$.*

PROOF. Use the finite character of the rank $\mathrm{D}[p, \Delta, k]$ with Theorem 1.18(2), instead of Theorem 1.15. □

DEFINITION 1.23. T has *the tree property* provided there exists a formula $\varphi(\mathbf{x}; \mathbf{y})$ and integer k and $\{\mathbf{a}_\eta \mid \eta \in {}^{\omega>}\omega\}$ such that

(1) $\{\varphi(\mathbf{x}; \mathbf{a}_{\eta\restriction n}) \mid n < \omega\}$ is consistent for all $\eta \in {}^{\omega}\omega$ and
(2) $\{\varphi(\mathbf{x}; \mathbf{a}_{\eta\hat{}n}) \mid n < \omega\}$ is k-contradictory for all $\eta \in {}^{\omega>}\omega$.

Our near term goal is to establish

THEOREM 1.24. *For a complete first-order theory T the following are equivalent.*

(1) *T is simple.*
(2) *T does not have the tree property*
(3) *$\kappa(T) < \infty$.*
(4) *$\kappa(T) \leq |T|^+$.*
(5) *$\mathrm{D}[\mathbf{x} = \mathbf{x}, \Delta, k] < \omega$ for every $k < \omega$ and every finite Δ.*
(6) *$\mathrm{D}[\mathbf{x} = \mathbf{x}, \varphi, k] < \omega$ for every $k < \omega$ and every formula $\varphi(\mathbf{x}; \mathbf{y})$.*

DEFINITION 1.25. We say that *a set A has the existence property* provided p dnf over A for all $p \in \mathrm{S}(A)$.

While splitting may not imply forking, strong splitting does:

DEFINITION 1.26. A type p *strongly splits over* B iff there are $\varphi(\mathbf{x};\mathbf{a}) \in p$ and an indiscernible sequence $\{\mathbf{a}_n \mid n < \omega\}$ over B such that $\mathbf{a}_0 = \mathbf{a}$ and $\neg\varphi(\mathbf{x};\mathbf{a}_1) \in p$.

EXERCISE 1.27. Suppose p is a complete type that does not strongly splits over B and let $\mathbf{I}$ be an infinite indiscernible sequence over B. If $\mathbf{a} \models p$ then $\mathbf{I}$ is an indiscernible sequence also over $B \cup \{\mathbf{a}\}$

Forking and strong splitting are almost equivalent when the theory is NIP.

THEOREM 1.28. *Suppose T is dependent and p is a type. The following are equivalent*

(1) *p forks over B.*
(2) *There exists a finite A_0 such that for every $q \in \mathrm{S}(\mathrm{dom}\, p \cup A_0)$ extending p the type q strongly splits over B.*

Only (2) $\implies$ (1) uses the assumption on T, the other direction holds for all theories. The main step is the following:

LEMMA 1.29. *Suppose T is dependent. If p strongly splits over B then p divides over B.*

PROOF. Since p strongly splits over B there are B-indiscernibles $\mathbf{I} := \{\mathbf{a}_i \mid i < \omega\}$ and $\varphi(\mathbf{x};\mathbf{y})$ such that $\varphi(\mathbf{x};\mathbf{a}_0), \neg\varphi(\mathbf{x};\mathbf{a}_1) \in p$.

Since T does not have the independence property by Lemma 8.23 there exists $n := n_\varphi$ such that for every $\mathbf{c} \in {}^{\ell(\mathbf{y})}\mathfrak{C}$ we have

$$(*) \quad |\varphi(\mathbf{c};\mathbf{I})| < n \text{ or } |\neg\varphi(\mathbf{c};\mathbf{I})| < n.$$

Take $\psi(\mathbf{x};\mathbf{y}_0,\mathbf{y}_1) := \varphi(\mathbf{x};\mathbf{y}_0) \wedge \neg\varphi(\mathbf{x};\mathbf{y}_1)$ and $\mathbf{b}_i := \mathbf{a}_{2i}{}^\frown\mathbf{a}_{2i+1}$. Also $\{\mathbf{b}_i \mid i < \omega\}$ is an indiscernible sequence over B. Notice that by the choice of φ and $\{\mathbf{a}_0,\mathbf{a}_1\}$ we have that $p \vdash \psi(\mathbf{x};\mathbf{b}_0)$.

To show that $\psi(\mathbf{x};\mathbf{b}_0)$ divides over B it is enough to show that $\{\psi(\mathbf{x};\mathbf{b}_i) \mid i < \omega\}$ is inconsistent:

Take $w \subseteq \omega$ of cardinality n, if there exists $\mathbf{c} \models \{\psi(\mathbf{x};\mathbf{b}_i) \mid i \in w\}$ then we get that

$$\mathfrak{C} \models \varphi[\mathbf{c};\mathbf{a}_{2i}] \wedge \varphi[\mathbf{c};\mathbf{a}_{2i+1}] \text{ for all } i \in w.$$

Thus both the sets $\varphi(\mathbf{c};\mathbf{I})$ and $\neg\varphi(\mathbf{c};\mathbf{I})$ have cardinality n which is a contradiction to $(*)$. □

PROOF. (2) $\Rightarrow$ (1): Suppose for the sake of contradiction that p dnf over B. Let A_0 be a finite set as in (2). By the extension property there exist $q \in \mathrm{S}(\mathrm{dom}\, p \cup$

A_0) extending p such that q does not fork over B. However the assumption gives that q split strongly and by Lemma 1.29 this means that q divides (hence forks) over B.

(1) $\Rightarrow$ (2): Since p forks over B there are $\{\psi_i(\mathbf{x};\mathbf{a}^i_m), k(i) \mid i < n, m < \omega\}$ such that

$$p \vdash \bigvee_{i<n} \psi_i(\mathbf{x};\mathbf{a}^i_0)$$

$$(*) \quad \{\psi_i(\mathbf{x};\mathbf{a}^i_m) \mid m < \omega\} \text{ is } k(i)\text{-contradictory for all } i < n.$$

and for every $i < n$ the sequence $\{\mathbf{a}^i_m \mid m < \omega\}$ is an indiscernible sequence over B.

Let $A_0 := \{\mathbf{a}^i_m \mid i < n, m < k(i)\}$. Given any $q \in \mathrm{S}(\mathrm{dom}\, p \cup A_0)$ extending p, there exists $i < n$ such that

$$(a) \quad \varphi_i(\mathbf{x};\mathbf{a}^i_0) \in q.$$

Using $(*)$ there is $m_0 < k(i)$ such that

$$(b) \quad \neg\varphi_i(\mathbf{x};\mathbf{a}^i_{m_0}) \in q.$$

By $(a)\&(b)$ the type q strongly splits over A, this is witnessed by the sequence $\{\mathbf{a}^i_0\} \cup \{\mathbf{a}^i_k \mid m_0 \leq k < \omega\}$. □

The last Lemma is very useful. Here is a fundamental application to establish an important property of indiscernible sequence in a stable theory.

THEOREM 1.30. *Suppose T is stable. If $\mathbf{I}$ is an indicsernible sequence over A of cardinality $\geq \kappa_r(T)$ then for any finite $\mathbf{b}$ there exists $\mathbf{J}_\mathbf{b} \subseteq \mathbf{I}$ of cardinality $< \kappa(T)$ such that $\mathbf{I} - \mathbf{J}_\mathbf{b}$ is an indiscernible sequence over $A\mathbf{b}$.*

Compare this theorem to Corollary 7.8.12 where we obtained a weaker conclusion from a weaker assumption.

PROOF. Consider $p := \mathrm{tp}(\mathbf{b}/A \cup \mathbf{I})$. There exists $\mathbf{J}_\mathbf{b} \subseteq \mathbf{I}$ of cardinality $< \kappa(T)$ such that p does not fork over $A \cup \mathbf{J}_\mathbf{b}$. To verify the assertion it is enough to show that for every $\mathbf{c}_1 < \cdots < \mathbf{c}_n, \mathbf{c}'_1 < \cdots < \mathbf{c}'_n \in \mathbf{I}^* := \mathbf{I} - \mathbf{J}_\mathbf{b}$, any $\varphi(\mathbf{x};\mathbf{a};\mathbf{z}_1,\ldots,\mathbf{z}_n)$ with $\mathbf{a} \in A$ we have

$$\varphi(\mathbf{x};\mathbf{a};\mathbf{c}_1,\ldots,\mathbf{c}_n) \in p \iff \varphi(\mathbf{x};\mathbf{a};\mathbf{c}'_1,\ldots,\mathbf{c}'_n) \in p.$$

Otherwise, as $\{\mathbf{c}_1\hat{}\cdots\hat{}\mathbf{c}_n \mid \mathbf{c}_1 < \cdots < \mathbf{c}_n \in \mathbf{I}^*\}$ is an infinite indiscernible sequence over $A \cup \mathbf{J}_\mathbf{b}$ we have shown that p strongly splits over $A \cup \mathbf{J}_\mathbf{b}$ which by the previous lemma contradicts the choice of $\mathbf{J}_\mathbf{b}$. □

The following generalizes the uniqueness property (see Corollary 3.9.18) of strongly minimal types:

DEFINITION 1.31. A type p *is stationary over* B iff p dnf over B and for every $A \supseteq \operatorname{Dom} p$ and all $q_1, q_2 \in \mathrm{S}(A)$ extending p the following implication holds

$$q_1 \text{ and } q_2 \text{ dnf over } B \implies q_1 = q_2.$$

REMARK 1.32. Let p be a non-algebraic type. If there exists $\varphi(\mathbf{x}; \mathbf{a}) \in p$ such that φ is strongly minimal then the type p is stationary over $B := \bigcup \mathbf{a}$.

The following observations connects forking with splitting:

PROPOSITION 1.33. *Let p be a type such that p does not fork over B. If there exists $A \supseteq B$ such that $p \restriction A$ is stationary over B then p does not split over A.*

PROOF. Suppose p splits over A. Thus there are $\mathbf{a}, \mathbf{b} \in \operatorname{Dom} p$ and a formula $\varphi(\mathbf{x}; \mathbf{y})$ such that

$$\operatorname{tp}(\mathbf{a}/A) = \operatorname{tp}(\mathbf{b}/A) \quad \text{and} \quad \varphi(\mathbf{x}; \mathbf{a}), \neg\varphi(\mathbf{x}; \mathbf{b}) \in p.$$

Let $f \in \operatorname{Aut}_A(\mathfrak{C})$ be such that $f(\mathbf{b}) = \mathbf{a}$. Since $p \restriction A \cup \{\neg\varphi(\mathbf{x}; \mathbf{b})\} \subseteq p$, by monotonicity of nonforking:

$$p \restriction A \cup \{\neg\varphi(\mathbf{x}; \mathbf{b})\} \quad \text{does not fork over } B.$$

By invariance:

$$f(p \restriction A \cup \{\neg\varphi(\mathbf{x}; \mathbf{b})\}) \quad \text{does not fork over } f[B].$$

Since $f \restriction A = \mathrm{id}_A$ and $f(\mathbf{b}) = \mathbf{a}$.

$$q_1 := p \restriction A \cup \{\neg\varphi(\mathbf{x}; \mathbf{a})\} \quad \text{does not fork over } B.$$

Since $p \restriction A \cup \{\varphi(\mathbf{x}; \mathbf{a})\} \subseteq p$

$$q_2 := p \restriction A \cup \{\varphi(\mathbf{x}; \mathbf{a})\} \quad \text{does not fork over } B.$$

The types q_i contradicts the hypothesis that $p \restriction A$ is stationary. □

The next definition is central to the theory and should be viewed as a close generalization to variuos theorems in algebra that provides existence of independent sets of elements with variuos properties.

DEFINITION 1.34. Let p be a type and suppose that $B \subseteq A := \operatorname{dom} p$. Let I be an infinite linearly ordered set, a sequence $\mathbf{I} := \langle \mathbf{a}_i \mid i \in I \rangle$ is a *Morley sequence for p over B* if

(1) $\mathbf{I} \subseteq p[\mathfrak{C}]$ and is an indiscernible sequence over A;

(2) The type $\operatorname{tp}(\mathbf{a}_i / A \cup \{\mathbf{a}_j \mid j < i\})$ does not fork over B for every $i \in I$.

REMARK 1.35. When $\mathbf{a}$ and $\mathbf{b}$ are sequences we write $\mathbf{ab}$ for $\mathbf{a}^\frown\mathbf{b}$.

(1) Any subsequence of a Morley sequence is a Morley sequence (for the same type).

(2) (Invariance of Morley sequence) Suppose that $\mathbf{I}$ is a Morley sequence of p over B. Then, $f(\mathbf{I})$ is a Morley sequence of $f(p)$ over $f[B]$ for all $f \in Aut(\mathfrak{C})$.
(3) Let p be a type over a set A such that p does not fork over some $B \subseteq A$. If $\mathbf{I} = \{\mathbf{a}_i \mid i \in I\}$ is a Morley sequence for p over B then for every $\mathbf{J} := \{\mathbf{b}_i \mid i \in I\}$ we have

$$\mathbf{J} \models \mathrm{tp}_*(\mathbf{I}/A) \implies \mathbf{J} \text{ is a Morley sequence for p over } B.$$

(4) If $\langle \mathbf{a}_i \mathbf{b}_i \mid i \in I\rangle$ is a Morley sequence for $\mathrm{tp}(\mathbf{ab}/A)$ over B, then $\langle \mathbf{a}_i \mid i \in I\rangle$ is a Morley sequence for $\mathrm{tp}(\mathbf{a}/A)$ over B and $\langle \mathbf{b}_i \mid i \in I\rangle$ is a Morley sequence for $\mathrm{tp}(\mathbf{b}/A)$ over B.
(5) If $\langle \mathbf{a}_i \mid i \in I\rangle$ is a Morley sequence for $\mathrm{tp}(\mathbf{a}/A)$ over B and $\mathbf{b} \in A$ then $\langle \mathbf{ba}_i \mid i \in I\rangle$ is a Morley sequence for $\mathrm{tp}(\mathbf{ba}/A)$ over B

The following related notion (due to Shelah), is of utmost importance when studying stable theories. Recall that when T is dependent than for any $A \subseteq |\mathfrak{C}|$ and $\mathbf{I} \subseteq {}^m|\mathfrak{C}|$ indiscernible sequence without last element $\mathrm{Av}(\mathbf{I}/A) \in \mathrm{S}^m(A)$.

DEFINITION 1.36. Let $B \subseteq |\mathfrak{C}|$ and $\mathbf{I} \subseteq {}^m|\mathfrak{C}|$ be an indiscernible sequence without last element. We say that $\mathbf{I}$ *is based on* B provided for every $C \supseteq B$ the type $\mathrm{Av}(I/C)$ does not for over B.

LEMMA 1.37 (Invariance). *If* $\mathbf{I} \subseteq {}^m|\mathfrak{C}|$ *is based over* B *then for every* $f \in \mathrm{Aut}_B(\mathfrak{C})$ *also* $f(\mathbf{I})$ *is based over* B.

PROOF. Exercise. □

The following remark will be used in the proof of Theorem 3.5. Let I be an infinite linear order. For $J_1, J_2 \subseteq I$, we write $J_1 < J_2$ if each element of J_1 is less than all elements of J_2. If $\langle \mathbf{a}_i \mid i \in I\rangle$ is sequence indexed by I and $J \subseteq I$, we write $\mathbf{a}_J$ for $\langle \mathbf{a}_i \mid i \in J\rangle$.

PROPOSITION 1.38. *Let* $B \subseteq A$ *and* $p \in \mathrm{S}(A)$ *be given. If* $\langle \mathbf{a}_i \mid i \in I\rangle$ *is a Morley sequence for* p *over* B*, then for any* $J_1, J_2 \subseteq I$ *such that* $J_1 < J_2$*, the type* $\mathrm{tp}(\mathbf{a}_{J_2}/A \cup \mathbf{a}_{J_1})$ *does not divide over* A.

PROOF. By definition of dividing, we may assume that both J_1 and J_2 are finite. But then the follows easily by induction on $|J_2|$, using Lemma 3.2 for the induction step. □

The following theorem is used to produce Morley sequences much in the same way that Ramsey's Theorem is used to produce indiscernibles.

THEOREM 1.39 (Morley's method). *Let* J *be a linearly ordered set. For every sequence* $\langle \mathbf{a}_i \mid i < \beth_{(2^{|T|})^+}\rangle$ *there exists an sequence of indiscernibles* $\langle \mathbf{b}_j \mid j \in J\rangle$ *with the following property:*

(*) For every $n < \omega$ and every $j_0 <_J \cdots <_J j_{n-1} \in J$ there are ordinals $i_0 < \cdots < i_{n-1}$ such that

$$\mathrm{tp}(\mathbf{b}_{j_0}, \ldots, \mathbf{b}_{j_{n-1}}/\emptyset) = \mathrm{tp}(\mathbf{a}_{i_0}, \ldots, \mathbf{a}_{i_{n-1}}/\emptyset).$$

A proof of this theorem is a corollary of Hanf number computations (see Corollary 2.7.41). Here we present a direct alternative argument that does not depend on what was done in section 6 of chapter 2.

PROOF. Fix $\langle \mathbf{a}_i \mid i < \beth_{(2^{|T|})^+} \rangle$ in order to find an sequence of indiscernibles as in the statement of the theorem. For $n \geq 1$ let

$$\Gamma_n := \{p \in \mathrm{S}^n(\emptyset) \mid \text{There exist } i_1 < \cdots < i_n < \beth_{(2^{|T|})^+} \text{ such that } \models p(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n})\}.$$

Let $\{\mathbf{c}_j \mid j \in J\}$ be constants not in $\mathrm{L}(T)$. We will find a sequence of types

$$\langle\, p_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) \mid n < \omega \,\rangle, \qquad p_n \in \Gamma_n \tag{*}$$

such that the union of the following sets of sentences is consistent:

- T;
- $p_n(\mathbf{c}_{j_1}, \ldots, \mathbf{c}_{j_n})$, whenever $j_1 <_J \cdots <_J j_n \in J$.

This will clearly prove the proposition.

To prove the consistency of above sentences, we will construct, by induction on n, a sequence of subsets of $(2^{|T|})^+$,

$$\langle F_n \mid n < \omega \rangle, \text{ such that } |F_n| = (2^{|T|})^+ \text{ for all } n < \omega,$$

a family

$$\{\, X_{\xi,n} \subseteq \beth_{(2^{|T|})^+} \mid \xi \in F_n,\ n < \omega \,\},$$

and a sequence of types (*) such that the following conditions hold:

(1) $F_{n+1} \subseteq F_n$;
(2) $|X_{\xi,n}| > \beth_\beta(2^{|T|})$, where ξ is the β-th element of F_n;
(3) $\models p_n(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n})$, whenever $i_1 < \cdots < i_n$ are in $X_{\xi,n}$.

We let $F_0 = (2^{|T|})^+$ and $X_{\xi,0} = \beth_{(2^{|T|})^+}$ for every $\xi \in F_0$. Then, (1)–(3) are obvious. Assume, then, that F_n and the $X_{\xi,n}$'s have been defined, and let us define F_{n+1} and the $X_{\xi,n+1}$'s.

Let $g\colon (2^{|T|})^+ \to F_n$ be the increasing enumeration of F_n. Define

$$G_n = \{\, g(\beta + n + 1) \mid \beta < \alpha \,\}.$$

Note that $|G_n| = (2^{|T|})^+$. Furthermore, if $\xi = g(\beta + n)$,

$$|X_{\xi,n}| > \beth_{\beta+n}(2^{|T|}). \tag{**}$$

The map

$$(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_{n+1}}) \mapsto \mathrm{tp}(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_{n+1}}/\emptyset)$$

is a coloring of $[X_{\xi,n}]^{n+1}$ by $2^{|T|}$-many colors. Hence, by (**) and the Erdős-Rado theorem (see, []), there exist $X_{\xi,n+1} \subseteq X_{\xi,n}$ with

$$|X_{\xi,n+1}| > \beth_\beta(2^{|T|})$$

and a type $p_{\xi,n+1} \in \Gamma_{n+1}$ such that (3) holds with $p_{\xi,n+1}$ in place of p_{n+1}. Now the pigeonhole principle allows us to find a $F_{n+1} \subseteq G_n$ of cardinality $(2^{|T|})^+$ and a type p_{n+1} such that $p_{\xi,n+1} = p_{n+1}$ for every $\xi \in F_{n+1}$. Renumbering the $X_{\xi,n+1}$'s with respect to the ordering of F_{n+1} preserves (2). This concludes the construction. □

Before establishing that Morley sequences exist for non-forking types. I will show an application to the last theorem to infinitary logic.

THEOREM 1.40. *Suppose Ψ is a set of sentences of $L_{\lambda^+,\omega}$ in a language with $\leq \lambda$ many symbols. If Ψ has a model of cardinality $\beth_{(2^\lambda)^+}$ then for every $\chi \geq \lambda$ the theory Ψ has a model of cardinality χ.*

Moreover, Let T be a first-order theory, Γ is a set of T-types (without parameters). Suppose

$$\mathcal{K} = \{M \models T \mid M \text{ omits } p \text{ for every } p \in \Gamma\}.$$

If $\mathcal{K}$ has an element of cardinality $\beth_{(2^{|T|})^+}$ then $\mathcal{K}$ has an element of every cardinality $\geq |T|$.

PROOF. By Chang's presentation theorem (Theorem 1.8.57) the moreover part implies the statement about $L_{\lambda^+,\omega}$.

Let $\mu := \beth_{(2^{|T|})^+}$. By assumption there is $M \in \mathcal{K}_\mu$. As every model can be expanded by Skolem functions, we may assume that T is a complete theory with built in Skolem functions[2].

Let $\{a_i \mid i < \mu\}$ be an enumeration of $|M|$. An application to the last theorem to $J := \langle \chi, \in \rangle$ gives an indiscernible sequence $\{b_j \mid j < \chi\}$ satisfying also (*).

I claim that $N := \mathrm{EM}(\{b_j \mid j < \chi\}) \in \mathcal{K}$. Clearly the cardinality of N is χ and $N \models T$. We need to verify that N omits p for any $p \in \Gamma$. Otherwise, there exists $\mathbf{c} \in |N| \models p$ for some $p \in \Gamma$. The case $\ell(\mathbf{c}) > 1$ is little technical but essentially the same, so I will assume that p is a type in one variable (instead of a sequence) and $\mathbf{c} = c$.

[2]Use Exercise 1.14.9.

As N is the Skolem Hull of $\{b_j \mid j < \chi\}$ there exists a term $\tau(x_1, \mid x_n)$ and $j_1 < \cdots j_n < \chi$ such that $c = \tau(b_{j_1}, \ldots, b_{j_n})$. Using $(*)$ there are ordinals $i_1 < \cdots i_n$ such that

$$M \models \varphi[\tau(a_{i_1}, \ldots, a_{i_n})] \iff N \models \varphi[\tau(b_{ji_1}, \ldots, b_{j_n})] \text{ for every } \varphi(x).$$

As this holds also for every $\varphi \in p$ we get

$$c \models p \implies N \models \varphi[\tau(b_{j_1}, \ldots, b_{j_n})] \text{ for all } \varphi \in p \implies$$
$$M \models \varphi[\tau(a_{i_1}, \ldots, a_{i_n})] \text{ for all } \varphi \in p.$$

Namely the element $\tau^M[a_{i_1}, \ldots, a_{i_n}]$ realize p a contradiction to the assumption that M omits all types in Γ. □

Another application to illustrate the power of Theorem 1.39, is to the following two cardinal theorem.

THEOREM 1.41. *Let λ be an infinite cardinal. $\Psi \subseteq L_{\lambda^+,\omega}$ a theory in a language of cardinality $\leq \lambda$ containing a unary predicate P. If there exists $M \models \Psi$ such that $\|M\| \geq \beth_{(2^{|P^M|})^+}$ then for every $\mu \geq \lambda$ there exists $N \models \Psi$ such that $\|N\| = \mu$ while $|P^N| = |P^M|$.*

Moreover, suppose $P \in L$ is unary, T is a first-order theory such that $\mathrm{L}(T) \supseteq L$, Γ is a set of first-order types in $\mathrm{L}(T)$ and let

$$\mathcal{K} := \{M \restriction L : M \models T, \ \ M \text{ omits } p \text{ for all } p \in \Gamma\}.$$

If there exists $M \in \mathcal{K}$ such that $\|M\| \geq \beth_{(2^{|T|})^+}(|P^M|)$ then for every $\mu \geq |T| + |P^M|$ there exists $N \in \mathcal{K}_\mu$ such that $|P^M| = |P^N|$.

PROOF. Using Chang's presentation theorem, the moreover part implies the first part. So we assume that $M \models T$ (a first-order theory) such that $\|M\| \geq \beth_{(2^{|T|})^+}(|P^M|)$ and omitting all the types from Γ. Denote by $\{a_\alpha \mid \alpha < |P^M|\}$ the set P^M.

Expand M with new constant symbols $\{c_\alpha \mid \alpha < |P^M|\}$, so that they interpreted by the a_α's and the expanded theory will contain also the sentences

$$\{c_\alpha \neq c_\beta \mid \alpha \neq \beta < |P^M|\} \cup \{P(c_\alpha) \mid \alpha < |P^M|\}.$$

As in the previous proof we may assume that T has built in Skolem functions and is complete. Let $\{a_i \mid i < \beth_{(2^{|T|})^+}(|P^M|)\}$ be an enumeration of a subset of $|M| - P^M$. An application of Theorem 1.39 produces an indiscernible sequence $\{b_j \mid j < \mu\}$ satisfying $(*)$. Let $N := \mathrm{EM}(\{b_j \mid j < \mu\})$ enough to show that $|P^N| = |P^M|$.

Given $a \in P^N$ there is a Skolem term $\tau(x_1, \ldots, x_n)$ and $j_1 < \cdots < j_n < \mu$ such that $a = \tau(b_{j_1}, \ldots, b_{j_n})$, by $(*)$ there are $i_1 < \ldots i_n$ such that

$$(**) \quad \mathrm{tp}(b_{j_1}, \ldots, b_{j_n}/\emptyset, N) = \mathrm{tp}(a_{i_1}, \ldots, a_{i_n}/\emptyset, M).$$

Since $N \models P(\tau(b_{j_1}, \ldots, b_{j_n}))$ using $(**)$ we have $M \models P(\tau(a_{i_1}, \ldots, a_{i_n}))$, thus there is $\alpha < |P^M|$ such that $M \models \tau(a_{i_1}, \ldots, a_{i_n}) = c_\alpha$. Using $(**)$ once more we get that $N \models \tau(b_{j_1}, \ldots, b_{j_n}) = c_\alpha$. Namely we have $P^N = \{c^N_\alpha \mid \alpha < |P^M|\}$ thus $|P^N| = |P^M|$. The fact that N omits all the types from Γ is established exactly like in the previous theorem. □

THEOREM 1.42. *Let $B \subseteq A$. Suppose that $p \in \mathrm{S}^n(A)$ does not fork over B. Then for every linearly ordered set J there exists a Morley sequence $\langle \mathbf{b}_j \mid j \in J \rangle$ for p over B. Moreover, if $p = \mathrm{tp}(\mathbf{b}/A)$, the Morley sequence can be chosen such that $\mathbf{b} = \mathbf{b}_j$ for some $j \in J$.*

PROOF. Let us first expand the language with constants for the elements of A and call T^* the resulting expansion T. Now we use Theorem 1.11 to construct by induction a sequence $\langle \mathbf{a}_i \mid i < \beth_{(2^{|T^*|})^+} \rangle$ such that $\mathbf{a}_i \models p$ and $\mathrm{tp}(\mathbf{a}_i/A \cup \{\mathbf{a}_j \mid j < i\})$ does not fork over A.

By Theorem 1.39, there exists a sequence $\mathbf{J} = \langle \mathbf{b}_j \mid j \in J \rangle$ indiscernible over A (since $\mathrm{L}(T^*)$ has names for the elements of A) such that for every $n < \omega$ and any $j_0 <_I \cdots <_I j_n \in J$ there are $i_0 < \cdots < i_{n-1} < \beth_{(2^{|T^*|})^+}$ satisfying

$$\mathrm{tp}(\mathbf{b}_{j_0}, \ldots, \mathbf{b}_{j_{n-1}}/A) = \mathrm{tp}(\mathbf{a}_{i_0}, \ldots, \mathbf{a}_{i_{n-1}}/A).$$

We claim that $\mathbf{I}$ is a Morley sequence for p over B. First, $\mathbf{J}$ is a indiscernible over A and every $\mathbf{b}_i$ realizes p. Now suppose, by contradiction, that $\mathrm{tp}(\mathbf{b}_j/A \cup \{\mathbf{b}_i \mid i < j\})$ divides over A and fix a formula $\varphi(\mathbf{x}; \mathbf{c}, \mathbf{b}_{j_1}, \ldots, \mathbf{b}_{j_{n-1}}) \in \mathrm{tp}(\mathbf{b}_i/A \cup \{\mathbf{b}_j \mid j < i\})$ such that $\varphi(\mathbf{x}; \mathbf{c}, \mathbf{b}_{j_1}, \ldots, \mathbf{b}_{j_{n-1}})$ forks over B. Choose $i_0 < \cdots < i_n$ such that $\mathrm{tp}(\mathbf{b}_{j_1}, \ldots, \mathbf{b}_{j_n}, \mathbf{b}_i/A) = \mathrm{tp}(\mathbf{a}_{i_0}, \ldots, \mathbf{a}_{i_n}/A)$. Then $\varphi(\mathbf{x}; \mathbf{c}, \mathbf{a}_{i_0}, \ldots, \mathbf{a}_{i_{n-1}}) \in \mathrm{tp}(\mathbf{a}_{i_n}/A \cup \{\mathbf{a}_j \mid j < i_n\})$, and $\varphi(\mathbf{x}; \mathbf{c}, \mathbf{a}_{i_0}, \ldots, \mathbf{a}_{i_{n-1}})$ forks over B. Thus, $\mathrm{tp}(\mathbf{a}_{i_n}/B \cup \{\mathbf{a}_j \mid j < i_n\})$ forks over A, which is a contradiction.

Pick $\mathbf{c} \in \mathbf{J}$ since $\mathrm{tp}(\mathbf{c}/A) = \mathrm{tp}(\mathbf{b}/A)$ there is $f \in \mathrm{Aut}_A(\mathfrak{C})$ such that $f(\mathbf{c}) = \mathbf{b}$. It is now easy to check that $f(\mathbf{I})$ is as required. □

REMARK 1.43 (Existence of Morley Sequences for Infinitary Types). We leave to the reader the verification that. If C is an arbitrary set such that $\mathrm{tp}_*(C/A)$ does not fork over a subset B of A then for any linearly ordered set I there exists a Morley sequence $\langle C_i \mid i \in I \rangle$ for $\mathrm{tp}_*(C/A)$ over B.

COROLLARY 1.44. *Let A, B, J and p be as in Theorem 1.42. If M is a $(|A| + |J|)^+$-saturated model then a Morley sequence can be found inside M.*

For theories without the independence property, averages of Morley sequences are powerful enough to characterize non-forking extensions of stationary types.

THEOREM 1.45 (Stationary types are realized by indiscernible sequence based on a given set). *Let $B \subseteq A$ and p is a type with parameters from A which is stationary over B.*

(1) If $\mathbf{I} := \{\mathbf{a}_i \mid i \in I\}$ is an infinite sequence such that $p_i := \mathrm{tp}(\mathbf{a}_i/A \cup \{\mathbf{a}_j \mid j < i\})$ extends p and does not fork over B for all $i \in I$ then $\mathbf{I}$ is a Morley sequence for p over B.

(2) Suppose T is dependent. If $\mathbf{I}$ is a Morley sequence for p over B then for every $C \supseteq A$ the type $\mathrm{Av}(\mathbf{I}/C)$ is the non-forking extension of p in $\mathrm{S}(C)$.

(3) Any infinite $\mathbf{I} := \{\mathbf{a}_i \mid i \in I\} \subseteq p[\mathfrak{C}]$ such that $\mathrm{tp}(\mathbf{a}_i/A \cup \{\mathbf{a}_j \mid j < i\})$ dnf over B for all $i \in I$ is based over B.

PROOF. (1) By Proposition 1.33 p_i does not split over B and by stationarity and the construction we have that $p_i \subseteq p_j$ for all $i < j \in I$. Thus by Lemma 7.4.9 $\mathbf{I}$ is an indiscernible sequence over A.

(2) For any $C \supseteq B$ let $D := C \cup A \cup \mathbf{I}$, by induction define $\mathbf{J} := \{\mathbf{a}_j \mid j < \omega\}$ such that $\mathrm{tp}(\mathbf{a}_j/D \cup \{\mathbf{a}_i \mid i < j\})$ the non-forking extension of p. By (1) since $\mathbf{J}$ is an indiscernible sequence over D and the failure of the independence property we get that

$$\mathrm{Av}(\mathbf{I}/C) \subseteq \mathrm{Av}(\mathbf{I}/D) = \mathrm{Av}(\mathbf{I}\hat{}\mathbf{J}/D) = \mathrm{Av}(\mathbf{J}/D) = p_0.$$

Since p_0 dnf over A the type $\mathrm{Av}(\mathbf{I}/C)$ is as required. □

DEFINITION 1.46. Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be infinite sequences of indiscernibles. We say that $\mathbf{I}_1$ equivalent to $\mathbf{I}_2$ iff there exists an infinite linearly indexed sequence $\mathbf{J}$ such that both $\mathbf{I}_1\mathbf{J}$ and $\mathbf{I}_2\mathbf{J}$ are both indiscernible sequences.

LEMMA 1.47. *Suppose T is a dependent theory. $\mathbf{I}_\ell$ for $\ell = 1, 2$ are infinite indiscernible sequences. The following are equivalent.*

(1) $\mathbf{I}_1$ is equivalent to $\mathbf{I}_2$.

(2) $\mathrm{Av}(\mathbf{I}_1/A) = \mathrm{Av}(\mathbf{I}_2/A)$ for all A.

(3) There exists a model M containing $\mathbf{I}_1 \cup \mathbf{I}_2$ such that $\mathrm{Av}(\mathbf{I}_1/|M|) = \mathrm{Av}(\mathbf{I}_2/|M|)$.

PROOF. Recall that (by Theorem 8.22) since T is dependent for any infinite $\mathbf{I}$ indiscernible sequence of n-sequences we have that $\mathrm{Av}(\mathbf{I}/A) \in \mathrm{S}^n(A)$ for all A. Since (2) $\implies$ (3) is trivial, it is enough to prove (1) $\implies$ (2), (3) $\implies$ (2) and (2) $\implies$ (1).

(1) $\implies$ (2) Let $\mathbf{J}$ be infinite such that $\mathbf{I}_\ell\mathbf{J}$ indiscernible sequence. By Proposition 8.21 $\mathrm{Av}(\mathbf{I}_\ell\mathbf{J}/A) \subseteq \mathrm{Av}(\mathbf{J}/A)$. As both are complete types we get that

they must be equal. Applying this observation several times gives

$$\mathrm{Av}(\mathbf{I}_1/A) = \mathrm{Av}(\mathbf{I}_1\mathbf{J}/A) = \mathrm{Av}(\mathbf{J}/A) = \mathrm{Av}(\mathbf{I}_2\mathbf{J}/A) = \mathrm{Av}(\mathbf{I}_2/A).$$

(3) $\implies$ (2) Suppose there is A such that $\mathrm{Av}(\mathbf{I}_1/A) \neq \mathrm{Av}(\mathbf{I}_2/A)$. Let $\varphi(\mathbf{x};\mathbf{a}) \in \mathrm{Av}(\mathbf{I}_1/A)$ such that $\neg\varphi(\mathbf{x};\mathbf{a}) \in \mathrm{Av}(\mathbf{I}_2/A)$. By Lemma 8.23 there are $\{\mathbf{b}_i \mid i < n_\varphi\} \subseteq \mathbf{I}_1$ and $\{\mathbf{c}_i \mid i < n_\varphi\} \subseteq \mathbf{I}_2$ such that

$$\models \bigwedge_{i<n_\varphi} \varphi[\mathbf{b}_i;\mathbf{a}] \wedge \bigwedge_{i<n_\varphi} \neg\varphi[\mathbf{c}_i;\mathbf{a}].$$

This gives

$$\models \exists\mathbf{x}[\bigwedge_{i<n_\varphi} \varphi[\mathbf{b}_i;\mathbf{x}] \wedge \bigwedge_{i<n_\varphi} \neg\varphi[\mathbf{c}_i;\mathbf{x}]].$$

As M is an elementary sub model of $\mathfrak{C}$ we get

$$M \models \exists\mathbf{x}[\bigwedge_{i<n_\varphi} \varphi[\mathbf{b}_i;\mathbf{x}] \wedge \bigwedge_{i<n_\varphi} \neg\varphi[\mathbf{c}_i;\mathbf{x}]].$$

Thus there exists $\mathbf{a}^* \in |M|$ such that

$$\models \bigwedge_{i<n_\varphi} \varphi[\mathbf{b}_i;\mathbf{a}^*] \wedge \bigwedge_{i<n_\varphi} \neg\varphi[\mathbf{c}_i;\mathbf{a}^*].$$

Using Lemma 8.23 once more gives

$$|\varphi(\mathbf{I}_1;\mathbf{a}^*)| \geq n_\varphi \implies \varphi(\mathbf{x};\mathbf{a}^*) \in \mathrm{Av}(\mathbf{I}_1/|M|)$$

and

$$|\neg\varphi(\mathbf{I}_2;\mathbf{a}^*)| \geq n_\varphi \implies \neg\varphi(\mathbf{x};\mathbf{a}^*) \in \mathrm{Av}(\mathbf{I}_2/|M|).$$

(2) $\implies$ (1) By induction on $n < \omega$ define $\mathbf{J} := \{\mathbf{c}_n \mid n < \omega\}$ such that

$$\mathbf{c}_n \models \mathrm{Av}(\mathbf{I}_1/C_n) (= \mathrm{Av}(\mathbf{I}_2/C_n), \text{where}$$

$$C_n := \mathbf{I}_1 \cup \mathbf{I}_2 \cup \langle \mathbf{c}_i \mid i < n\rangle.$$

By Proposition 8.24 $\mathbf{I}_1\mathbf{J}$ and $\mathbf{I}_2\mathbf{J}$ are indiscernible sequences. □

In the following three theorems we will have a parameter B and the cardinal $\chi := (|B|+2)^{|T|}$ will play a role. Recall from Corollary 5.14 that if T is stable then it is stable in every λ that satisfy $\lambda = \lambda^{|T|}$, thus in particular, stability of T implies that T is stable in χ.

THEOREM 1.48. *Suppose T is stable and $p \in \mathrm{S}(M)$ does not fork over some $B \subseteq |M|$. If M is χ^+-saturated then p is stationary over B.*

PROOF. Otherwise, there exists $A \supseteq |M|$ and there are $q_0 \neq q_1 \in \mathrm{S}(A)$ both non-forking extensions of p. Thus there exists $\varphi(\mathbf{x};\mathbf{a}) \in q_0$ such that $\neg\varphi(\mathbf{x};\mathbf{a}) \in q_1$.

By induction on $i < \chi^+$ using χ^+-saturation define $\{\mathbf{a}_i \mid i < \chi^+\} \subseteq |M|$ such that

$$(*) \quad \mathrm{tp}(\mathbf{a}_i/A_i) = \mathrm{tp}(\mathbf{a}/A_i)$$

where $A_i := B \cup \{\mathbf{a}_j \mid j < i\}$.

As $\mathbf{a}_i \in \mathrm{dom}\, p$ either $\varphi(\mathbf{x};\mathbf{a}_i) \in p$ or $\neg\varphi(\mathbf{x};\mathbf{a}_i) \in p$. Let

$$F(i) := \begin{cases} 1 & \text{if } \varphi(\mathbf{x};\mathbf{a}_i) \in p \\ \\ 0 & \text{Otherwise.} \end{cases}$$

By the pigeon hole principle there exists $S \subseteq \chi^+$ of cardinality χ^+ such that F is constant on S.

Suppose $i \in S \implies \varphi(\mathbf{x};\mathbf{a}_i) \in p$.

By monotonicity of non-forking, as $p \restriction A_i \cup \{\neg\varphi(\mathbf{x};\mathbf{a})\} \subseteq q_1$, the type

$$p \restriction A_i \cup \{\neg\varphi(\mathbf{x};\mathbf{a})\} \text{ does not fork over } B.$$

Using invariance and $(*)$ we get that also

$$p \restriction A_i \cup \{\neg\varphi(\mathbf{x};\mathbf{a}_i)\} \text{ does not fork over } B.$$

By the extension property there exists

$$(*)_i \quad p_i \in \mathrm{S}(M) \supseteq p \restriction A_i \cup \{\neg\varphi(\mathbf{x};\mathbf{a}_i)\}$$

that does not fork over B.

Since T is stable in χ, by Theorem 7.6.5 there exists $S^* \subseteq S$ of cardinality χ^+ such that $\{\mathbf{a}_i \mid i \in S^*\}$ is an indiscernible sequence over B.

Take $i < j \in S^*$. As $i \in S$ we have $\varphi(\mathbf{x};\mathbf{a}_i) \in p \restriction A_j \subseteq p_j$. By the choice of p_j, using $(*)_j$ we get that $\neg\varphi(\mathbf{x};\mathbf{a}_j) \in p_j$. So we found an infinite indiscernible sequence over A with elements satisfying $\varphi(\mathbf{x};\mathbf{a}_i) \in p_j$ and $\neg\varphi(\mathbf{x};\mathbf{a}_j) \in p_j$, this means that p_j splits strongly over the set B. Using stability once more (in fact NIP is enough, see Lemma 1.29) we conclude that p_j divides over B which is a contadiction to the assumption that p_j dnf over B.

The case that $i \in S \implies \neg\varphi(\mathbf{x};\mathbf{a}_i) \in p$ is treated similarly. As $\varphi(\mathbf{x};\mathbf{a}) \in q_0$ by invariance we get that also $p \restriction A_i \cup \{\varphi(\mathbf{x};\mathbf{a}_i)\}$ dnf over A and select $p_i \in \mathrm{S}(M)$ as before where the only change is in $(*)_i$, require $\varphi(\mathbf{x};\mathbf{a}_i) \in p_i$. The rest is similar. □

THEOREM 1.49. *Suppose T is stable, for every $A \supseteq B$ we have*

$|\Gamma^f(A/B)| \leq |\Gamma^{stsp}(A/B)| \leq \chi$, *where*

$\Gamma^f(A/B) := \{p \in \mathrm{S}(A) \mid p \text{ dnf } /B\}$ *and*

$\Gamma^{stsp}(A/B) := \{p \in \mathrm{S}(A) \mid p \text{ does not split strongly } /B\}$.

PROOF. By Lemma 1.29 it is enough to prove that $|\Gamma^{stsp}(A/B)| \leq \chi$.

For every $\mathbf{b}, \mathbf{c} \in A$ (of the same arity) consider

$$\begin{aligned}\mathbf{b} \sim_B \mathbf{c} \iff & \exists n < \omega \exists \{\mathbf{I}_i \mid i < n\} \text{ infinite indiscernible sequences over } B \\ & \exists \{\mathbf{a}_i \mid i \leq n\} \text{ such that } \mathbf{a}_i, \mathbf{a}_{i+1} \in \mathbf{I}_i \text{ for all } i < n \\ & \text{with } \mathbf{b} = \mathbf{a}_0 \text{ and } \mathbf{c} = \mathbf{a}_n.\end{aligned}$$

Clearly $\sim_B$ is an equivalence relation of sequences of elements of A. Let $A^* \subseteq A$ be a complete set of representatives of all equivalence classes.

CLAIM 1.50. *The mapping $p \mapsto p \restriction A^*$ is an injective mapping from $\Gamma^{stsp}(A/B)$ into $\mathrm{S}(A^*)$.*

PROOF. Let $p_1 \neq p_2 \in \Gamma^{stsp}(A/B)$. Pick $\varphi(\mathbf{x}; \mathbf{a}) \in p_1$ such that $\neg\varphi(\mathbf{x}; \mathbf{a}) \in p_2$. Let $\mathbf{a}^* \in A^*$ be such that $\mathbf{a}^* \sim \mathbf{a}$. As p_1 does not split strongly over B also $\varphi(\mathbf{x}; \mathbf{a}^*) \in p_1$. Similarly we get $\neg\varphi(\mathbf{x}; \mathbf{a}^*) \in p_2$ Thus $p_1 \restriction A^* \neq p_2 \restriction A^*$. □

Since we have established that $|\Gamma^{stsp}(A/B)| \leq |\mathrm{S}(A^*)|$. By χ-stability it is enough to verify

CLAIM 1.51. $|A^*| \leq \chi$.

PROOF. Otherwise, there $\{\mathbf{a}_i \mid i < \chi^+\}$ such that $i \neq j \implies \mathbf{a}_i \not\sim_B \mathbf{a}_j$. By Theorem 6.5 we may assume that this an indiscernible sequence over B. □

□

THEOREM 1.52 (A bound on number of Morley sequences). *Let $\mathcal{F}$ be*

$$\{\mathbf{I} \mid \mathbf{I} \text{ is an indiscernible seq } /B \text{ and } \mathrm{Av}(\mathbf{I}/C) \text{ dnf } B \text{ for all } C \supseteq B\}$$

If T is stable then $|\mathcal{F}/\sim| \leq \chi$.

PROOF. Otherwise, there $\{\mathbf{I}_\alpha \mid \alpha < \chi^+\}$ such that $\alpha \neq \beta \implies \mathbf{I}_\alpha \not\sim \mathbf{I}_\beta$. □

In a stable theory for every formula $\varphi(\mathbf{x}; \mathbf{y})$ and a Morley sequence $\mathbf{I}$ we can construct a natural formula $\varphi_n(\mathbf{x}; \mathbf{a}^n)$ which is almost over B.

LEMMA 1.53. *Suppose T is stable. $\mathbf{I} = \{\mathbf{a}_\ell \mid \ell < \omega\}$ is a Morley sequence for a stationary type over B. For $n < \omega$ and a formula $\varphi(\mathbf{x}; \mathbf{y})$ let*

$$\varphi_n(\mathbf{x}; \mathbf{a}^n) := \bigvee_{\substack{w \subseteq 2n-1 \\ |w|=n}} \wedge \bigwedge_{\ell \in w} \varphi(\mathbf{x}; \mathbf{a}_\ell).$$

Let n_φ be from Lemma 6.8.23. Then for all $n \geq n_\varphi$ the formula $\varphi_n(\mathbf{x}; \mathbf{a}^n)$ is almost over B.

PROOF. Denote by $\psi(\mathbf{x}; \mathbf{b})$ the formula $\varphi_n(\mathbf{x}; \mathbf{a}^n)$.

CLAIM 1.54. $\mathfrak{C} \models \psi[\mathbf{c};\mathbf{b}] \iff \varphi(\mathbf{c};\mathbf{y}) \in \mathrm{Av}(\mathbf{I}/\mathbf{c})$ *for any* $\mathbf{c}$.

PROOF. Suppose $\mathfrak{C} \models \psi[\mathbf{c};\mathbf{b}]$, since $n \geq n_\varphi$ we get that $|\varphi(\mathbf{c};\mathbf{I})| \geq n_\varphi$, thus $|\neg\varphi(\mathbf{c};\mathbf{I})| < n_\varphi$, by the definition of Av this gives that $\varphi(\mathbf{c};\mathbf{y}) \in Av(\mathbf{I}/\mathbf{c})$ which implies that $\mathfrak{C} \models \psi[\mathbf{c};\mathbf{b}]$ holds. □

If $\psi(\mathbf{x};\mathbf{b})$ is not almost over B then by Theorem 6.1.25 $\Gamma_{\psi(\mathbf{x};\mathbf{b}),B}$ is unbounded. Let $\chi := (|B|+2)^{|T|}$. There are $\{F_i \in \mathrm{Aut}_B(\mathfrak{C}) \mid i < \chi^+\}$ such that

$$i \neq j \implies \psi(\mathbf{x};F_i(\mathbf{b})) \not\equiv \psi(\mathbf{x};F_j(\mathbf{b})).$$

Pick $\mathbf{c}_{i,j}$ such that

$$(*) \quad \mathfrak{C} \models \psi[\mathbf{c}_{i,j};F_i(\mathbf{b})] \wedge \neg\psi[\mathbf{c}_{i,j};F_j(\mathbf{b})].$$

Let M be a model containing the set $\{\mathbf{c}_{i,j} \mid i,j < \chi^+\}$.

An application of the Claim to both parts of (*) gives

$$\varphi(\mathbf{c}_{i,j};\mathbf{y}) \in \mathrm{Av}(F_i(\mathbf{I})/M) \text{ and } \neg\varphi(\mathbf{c}_{i,j};\mathbf{y}) \in \mathrm{Av}(F_j(\mathbf{I})/M).$$

We obtained that there are χ^+ many pairwise non equivalent Morley sequences over A, this is a contradiction to Theorem 1.52. □

The following is an important connection between formulas *almost over* (recall Definition 5.1.14 and Theorem 5.1.60) and non-forking.

THEOREM 1.55. *Let* $B \subseteq A$ *and suppose that* $p \in \mathrm{S}^m(A)$ *does not fork over* B.

(1) *Let* $E \in \mathrm{FE}^m(A)$. *If* $\mathbf{a}$ *is such that* $p \cup \{E(\mathbf{x};\mathbf{a})\}$ *is consistent then* $p \cup \{E(\mathbf{x};\mathbf{a})\}$ *dnf over* B.
(2) *If* $q \supseteq p$ *(in the same variables) and is almost over* A *then* q *dnf over* B.

PROOF. (1) Suppose $p \cup \{E(\mathbf{x};\mathbf{a})\}$ forks over B. Let $\{\mathbf{a}_i \mid i \leq k < n(E)\}$ a maximal set such that

$$\mathbf{a}_i \models \mathrm{tp}(\mathbf{a}/A) \quad \text{and} \quad \mathfrak{C} \models \bigwedge_{i<j\leq k} \neg E[\mathbf{a}_i;\mathbf{a}_j].$$

By maximality of k:

$$\mathrm{tp}(\mathbf{a}/A) \cup \{\neg E(\mathbf{x};\mathbf{a}_i) \mid i < k\} \text{ is inconsistent.}$$

Let $\varphi(\mathbf{x};\mathbf{c}) \in \mathrm{tp}(\mathbf{a}/A)$ be such that

$$\{\varphi(\mathbf{x};\mathbf{c})\} \cup \{\neg E(\mathbf{x};\mathbf{a}_i) \mid i < k\} \text{ is inconsistent.}$$

Thus

$$(*) \quad \models \forall\mathbf{x}[\varphi(\mathbf{x};\mathbf{c}) \to \bigvee_{i<k} E(\mathbf{x};\mathbf{a}_i)].$$

Since $\mathrm{tp}(\mathbf{a}_i/A) = \mathrm{tp}(\mathbf{a}/A)$ and $\mathrm{dom}\, p \subseteq A$, by invariance of forking we get from the assumption that $p \cup \{E(\mathbf{x};\mathbf{a})\}$ forks over B that

$$p \cup \{E(\mathbf{x};\mathbf{a}_i)\} \text{ forks over } B \text{ for all } i < k.$$

By property xx we get also that $p \cup \{\bigvee_{i<k} E(\mathbf{x};\mathbf{a}_i)\}$ forks over B.

CLAIM 1.56. $p \vdash \bigvee_{i<k} E(\mathbf{x};\mathbf{a}_i)$.

PROOF. Since $\varphi(\mathbf{x};\mathbf{c}) \in \mathrm{tp}(\mathbf{a}/A)$ and $p \cup \{E(\mathbf{x};\mathbf{a})\}$ is consistent

$$\exists \mathbf{y}[\varphi(\mathbf{y};\mathbf{c}) \wedge E(\mathbf{x};\mathbf{y})] \in p.$$

Take $\mathbf{b} \models p$. By the above existential formula there exists $\mathbf{a}^*$ such that $\models \varphi[\mathbf{a}^*;\mathbf{c}] \wedge E[\mathbf{b};\mathbf{a}^*]$. By $(*)$ we have that $\models E[\mathbf{b};\mathbf{a}^*]$ using transitivity of E there exists $i < k$ such that $\models E[\mathbf{b};\mathbf{a}_i]$. Since $\mathbf{b} \models p$ we have shown that $p \vdash \bigvee_{i<k} E(\mathbf{x};\mathbf{a}_i)$. □

The Claim contradicts the assumption that p dnf $/B$.

(2) Suppose $q \supseteq p$ almost over A. For the sake of contradiction suppose that q forks over B. By the finite character there exists a finite $q_0 \subseteq q$ such that q_0 forks over B.

Let $\varphi(\mathbf{x};\mathbf{c}) := \bigwedge q_0$. By assumption on q the formula $\varphi(\mathbf{x};\mathbf{c})$ is almost over A. By Theorem 5.1.25 there is $E \in \mathrm{FE}^{\ell(\mathbf{x})}(A)$ such that

$$(*) \quad E(\mathbf{x};\mathbf{y}) \vdash [\varphi(\mathbf{x};\mathbf{c}) \leftrightarrow \varphi(\mathbf{y};\mathbf{c})]$$

Pick $\mathbf{a} \models q$ since q_0 is a subtype of q we have $\models \varphi[\mathbf{a};\mathbf{c}]$. By $(*)$ we get that $\models \forall \mathbf{x}[E(\mathbf{x};\mathbf{a}) \rightarrow \varphi(\mathbf{x};\mathbf{c})]$. Since φ forks over B the formula $E(\mathbf{x};\mathbf{a})$ forks over B. Since $\mathbf{a} \models p$ we have that $p \cup \{E(\mathbf{x};\mathbf{a})\}$ is consistent, a contradiction to part (1). □

COROLLARY 1.57. *Suppose T is simple, in fact it suffices to assume that B has the existence property. If q is almost over B then q does not fork over B.*

PROOF. Let $A := B \cup \mathrm{dom}\, p$. Take $p \in \mathrm{S}(A) \supseteq q$. Since $p \restriction B \in \mathrm{S}(B)$ by the existence property it does not fork over B. By assumption on q we have that

$$r := p \restriction B \cup q \text{ is almost over } B.$$

By part (2) of the previous theorem r does not fork over B. □

THEOREM 1.58 (Finite equivalence relation theorem Ver 1). *Suppose T is stable, and $p \neq q \in \mathrm{S}^m(M)$ both dnf over some $B \subseteq |M|$.*

If M is a $((|B|+2)^{|T|})^+$-saturated model then there exists $E \in \mathrm{FE}^m(B)$ such that

$$p(\mathbf{x}) \cup q(\mathbf{y}) \vdash \neg E(\mathbf{x};\mathbf{y}).$$

PROOF. Let $\varphi(\mathbf{x};\mathbf{a}) \in p$ such that $\neg\varphi(\mathbf{x};\mathbf{a}) \in q$. As $\mathrm{tp}(\mathbf{a}/B)$ dnf over B there exists $p^* \in \mathrm{S}(M) \supseteq \mathrm{tp}(\mathbf{a}/B)$ that dnf over B. By saturation of M and Theorem 1.48 p^* is stationary over A. Pick $\mathbf{I} := \{\mathbf{a}^*_n \mid n < \omega\}$ a Morley sequence for p^* over B. By saturation of M find $\mathbf{J} := \{\mathbf{a}_\ell \mid \ell < \omega\} \subseteq |M|$ with $\mathbf{a}_0 = \mathbf{a}$ such that

$$\mathbf{J} \models \mathrm{tp}(\mathbf{I}/B).$$

Since p and q do not strongly split over B we have

$$\varphi(\mathbf{x};\mathbf{a}_\ell) \in p \text{ and } \neg\varphi(\mathbf{x};\mathbf{a}_\ell) \in q \text{ for all } \ell < \omega.$$

Let $n \geq n_\varphi$. As $\mathbf{J}$ is also a Morley sequence for p^*, an application of Lemma 1.53 gives that $\varphi_n(\mathbf{x};\mathbf{a}^n)$ is almost over B. By Theorem 5.1.25 there exists $E \in \mathrm{FE}^m(B)$ such that

$$E(\mathbf{x};\mathbf{y}) \vdash [\varphi_n(\mathbf{x};\mathbf{a}^n) \leftrightarrow \varphi_n(\mathbf{y};\mathbf{a}^n)]$$

Since $\varphi_n(\mathbf{x};\mathbf{a}^n) \in p$ and $\neg\varphi_n(\mathbf{y};\mathbf{a}^n) \in q$ we get that

$$p(\mathbf{x}) \cup q(\mathbf{y}) \vdash \neg E(\mathbf{x};\mathbf{y}).$$

□

THEOREM 1.59 (Finite equivalence relation theorem Ver 2). *Suppose T is stable.*

(1) $\mathrm{stp}(\mathbf{a}/B)$ is stationary over B for all $\mathbf{a}$.

(2) For any $A \supseteq B$. If $p \neq q \in \mathrm{S}^m(A)$ dnf over B then there exists $E \in \mathrm{FE}^m(B)$ such that

$$p(\mathbf{x}) \cup q(\mathbf{y}) \vdash \neg E(\mathbf{x};\mathbf{y}).$$

PROOF. (1) By the existence property[3] $\mathrm{tp}(\mathbf{a}/B)$ dnf over B. As $p := \mathrm{stp}(\mathbf{a}/B)$ is almost over B and extends $\mathrm{tp}(\mathbf{a}/B)$[4] which does not fork over B also p dnf over B.

If p is not stationary there are some $q_1 \neq q_1 \in \mathrm{S}^m(M) \supseteq p$ dnf over B and M is χ^+-saturated. Let $E \in \mathrm{FE}^m(B)$ be such that

$$q_1(\mathbf{x}) \cup q_2(\mathbf{y}) \vdash \neg E(\mathbf{x};\mathbf{y}).$$

Since $q_\ell \supseteq p$ we have $E(\mathbf{x};\mathbf{a}) \in q_\ell$ for $\ell = 1, 2$. So we have

$$q_1(\mathbf{x}) \cup q_2(\mathbf{y}) \vdash \neg E(\mathbf{x};\mathbf{y}) \wedge E(\mathbf{x};\mathbf{a}) \wedge E(\mathbf{y};\mathbf{a}).$$

A contradiction to the assumption that E is an equivalence relation.

(2) If there is no such E then

$\Gamma(\mathbf{x};\mathbf{y}) := p(\mathbf{x}) \cup q(\mathbf{y}) \cup \{E(\mathbf{x};\mathbf{y}) \mid E \in \mathrm{FE}^m(B)\}$ is consistent.

[3] Theorem 1.17.

[4] Theorem 1.55.

Take $\mathbf{ab} \models \Gamma$. Let $r := \mathrm{stp}(\mathbf{a}/B)$. By the third part of Γ we have that $r \equiv \mathrm{stp}(\mathbf{b}/B)$.

Now both $q_1(\mathbf{x}) := p(\mathbf{x}) \cup r(\mathbf{x})$ and $q_2(\mathbf{x}) := q(\mathbf{x}) \cup r(\mathbf{x})$ are distinct extensions of r that dnf over B. A contradiction to the fact that $\mathrm{stp}(\mathbf{a}/B)$ is stationary. □

COROLLARY 1.60 (types over models are stationary). *Suppose $p \in \mathrm{S}(M)$ dnf over some $B \subseteq |M|$. If T is stable then p is stationary.*

PROOF. For the sake of contradiction suppose that there exists $A \supseteq |M|$ and there are $q_1 \neq q_2 \in \mathrm{S}(A)$ both non forking extensions of p. By the Finite Equivalence relation theorem there exists $E \in \mathrm{FE}^m(B)$ such that

$$(*) \quad p(\mathbf{x}) \cup q(\mathbf{y}) \vdash \neg E(\mathbf{x}; \mathbf{y}).$$

Since there exist $n = n(E) < \omega$ such that

$$\mathfrak{C} \models \exists \mathbf{x}_0 \cdots \mathbf{x}_{n-1} [\bigwedge_{i<j<n} \neg E(\mathbf{x}_i; \mathbf{x}_j) \wedge \forall \mathbf{y} \bigvee_{j<n} E(\mathbf{y}; \mathbf{x}_j)].$$

Let $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1} \in |M|$ be representatives of al equivalence classes of E in M.

Pick $\mathbf{a} \models q_1$ and $\mathbf{b} \models q_2$. Let $\mathbf{a}_j$ be such that $M \models E(\mathbf{a}; \mathbf{a}_j)$. Since $q_1 \supseteq p \in \mathrm{S}(M)$ we get that $E(\mathbf{x}; \mathbf{a}_j) \in p$. Thus also $E(\mathbf{x}; \mathbf{a}_j) \in q_2$. Summerizing:

$$\models E(\mathbf{a}; \mathbf{a}_j) \wedge E(\mathbf{b}; \mathbf{a}_j) \wedge \neg E(\mathbf{a}; \mathbf{b}).$$

A contradiction to E being a transitive relation. □

The following is a central connection between saturation and stability:

THEOREM 1.61 (Harnik). *Suppose T is stable and δ an ordinal such that $\mathrm{cf}\, \delta \geq \kappa(T)$. If $\{M_i \mid i < \delta\}$ is an increasing elementary chain such that M_{i+1} is λ-saturated for all $i < \delta$ then $\bigcup_{i<\delta} M_i$ is also λ-saturated.*

PROOF. Let $N := \bigcup_{i<\lambda} M_i$. It is easy to see that if $\lambda \leq \kappa(T)$ then N is λ-saturated. Thus we assume that $\lambda > \kappa(T)$. Let A a subset of cardinality $< \lambda$ of N and $p \in \mathrm{S}(A)$ be given. Take any $p^* \in \mathrm{S}(N)$ extending p. By the definition of $\kappa(T)$ there exists $B \subseteq |N|$ of cardinality $< \kappa(T)$ such that p^* dnf over B. Since $\mathrm{cf}\, \delta \geq \kappa(T)$ there exists $j < \delta$ such that $B \subseteq |M_j|$.

By Corollary 1.60, $p^* \restriction |M_j|$ is stationary over B. By λ-saturation of M_{j+1} there exists $\mathbf{I} := \{\mathbf{a}_i \mid i < \lambda\} \subseteq |M_{j+1}|$ a Morley sequence for $p^* \restriction |M_j|$ over B. Recall (Theorem 1.45 (3)) that $\mathbf{I}$ is a sequence of indiscernibles based on B.

As p^* is a non-forking extension of $p^* \restriction |M_j|$ to N not forking over B and $\mathbf{I}$ is a based on B, by Theorem 1.45(2)

$$(*) \quad p^* = \mathrm{Av}(\mathbf{I}/N).$$

Now for any $a \in A$ there exists $\mathbf{J}_a \subseteq \mathbf{I}$ of cardinality $< \kappa(T)$ such that $\mathbf{I} - \mathbf{J}_a$ is an indiscernible sequence over Ba. Let

$$\mathbf{I}^* := \mathbf{I} - \bigcup_{a \in A} \mathbf{J}_a, \text{ it is an indiscernible sequence over } B \cup A.$$

Since $|\bigcup_{a\in A} \mathbf{J}_a| \leq \kappa(T) + |A| + \aleph_0 < \lambda$ and $|\mathbf{I}| = \lambda$, also $|\mathbf{I}^*| = \lambda$.

By choice of p^*, (*) and since $\mathbf{I}^*$ is infinite we have

$$p \subseteq \mathrm{Av}(\mathbf{I}^*/|N|).$$

For every $\varphi(\mathbf{x}; \mathbf{a}) \in p$ there exists a finite $S_\varphi \subseteq \mathbf{I}^*$ such that

$$\mathbf{b} \in \mathbf{I}^* - S_\varphi \implies \mathfrak{C} \models \varphi[\mathbf{b}; \mathbf{a}].$$

Since $\mathbf{I}^*$ is an indiscernible sequence over A, we have for every $\mathbf{b} \in \mathbf{I}^*$ that

$$\mathfrak{C} \models \varphi[\mathbf{b}; \mathbf{a}] \text{ for all } \varphi(\mathbf{x}; \mathbf{a}) \in p.$$

Thus, all elements of $\mathbf{I}^*$ realize the type p. □

The following is the non-trivial result generalizing Theorem 4.3.8.

COROLLARY 1.62 (Uniqueness of limit models). . *Suppose T is super stable. If $\lambda \geq |T|$ then for any $N \precneqq M_\ell \models T$ and $\alpha_1, \alpha_2 < \lambda^+$ both limit we have the implication that if M_ℓ are α_ℓ-limit over N then $M_1 \cong_N M_2$.*

PROOF. Similar to the proof of the last theorem. □

DEFINITION 1.63 (the first stability cardinal).

$$\lambda(T) := \min\{\lambda \geq |T| \ : \ T \text{ is stable in } \lambda\}.$$

By Corollary 5.14 if T is stable then $\lambda(T) \leq 2^{|T|}$.

Our next application of the finite equivalence relation theorem establishes that a stable theory with $\kappa(T) = \aleph_0$ must be superstable and more:

THEOREM 1.64. *Suppose T is stable. If $\kappa(T) = \aleph_0$ then T is stable in every $\lambda \geq \lambda(T)$.*

The theorem follows easily from:

THEOREM 1.65 (Stability Spectrum Theorem 1). *Suppose T is stable. If λ is a cardinal satisfying*

$$\lambda = \lambda^{<\kappa_1(T)} \geq \lambda(T)$$

then T is stable in λ.

PROOF. Otherwise, there exists a set A of cardinality λ such that $|\,\mathrm{S}^1(A)| > \lambda$. Pick distinct $\{p_i \in \mathrm{S}^1(A) \mid i < \lambda^+\}$. By definition of $\kappa_1(T)$ for every $i < \lambda^+$ there exists $B_i \subseteq A$ of cardinality $< \kappa_1(T)$ such that p_i dnf over B_i.

Since the number of subsets of A of cardinality $< \kappa_1(T)$ is $\lambda^{<\kappa_1(T)}$, by the assumption of λ and the pigeonhole principle we get there there exists $B \subseteq A$ and $S \subseteq \lambda^+$ of cardinality λ^+ such that $i \in S \implies B_i = B$. We may assume that $S = \lambda^+$.

By stability of T we get that $\kappa(T) \leq |T|^+$. Thus $|B| \leq |T|$. Since $|B| \leq |T|$ we can fix a model M of cardinality $\lambda(T)$ such that $B \subseteq |M|$. By the extension property for every $i < \lambda^+$ let $q_i \in \mathrm{S}^1(M \cup A) \supseteq p_i$ such that q_i dnf over B.

Using the finite equivalence relation theorem from stability of T we get that q_i is the unique non-forking extension (over B) of the stationary type $q_i \restriction M$. Namely $i \neq j \implies q_i \restriction M \neq q_j \restriction M$. Thus

$$\lambda^+ = |\{q_i \,:\, i < \lambda^+\}| = |\{q_i \restriction M \,:\, i < \lambda^+\}| \leq |\,\mathrm{S}(M)| \leq \lambda(T).$$

A contradiction to the assumption that $\lambda \geq \lambda(T)$. □

As a third application of the finite equivalence relation theorem, I present a useful equivalent property of "p dnf over B" when B is a universe of a model. Unfortunately this characterization fails for unstable theories (see comments on the Parisian approach to forking in the historical appendix).

THEOREM 1.66. *Suppose T is stable, $M \models T$ and p a set of formulas, perhaps with parameters outside M.*

$$p \textit{ dnf over } |M| \iff p \textit{ is finitely satisfiable in } M.$$

PROOF. By Lemma 1.13 it is enough to prove the implication $\implies$.

By the finite character of forking enough to show:

$$\varphi(\mathbf{x};\mathbf{a}) \text{ dnf over } M \implies M \models \exists \mathbf{x}[\varphi(\mathbf{x};\mathbf{a})].$$

By the extension property take $p^* \in \mathrm{S}^{\mathbf{x}}(M)$ such that $p^* \cup \varphi(\mathbf{x};\mathbf{a})$ dnf over M. Let $\mathbf{I} := \{\mathbf{b}_n \mid n < \omega\}$ be a Morley sequence for p^*. By Theorem 1.60 the type p^* is stationary. By Theorem 1.45 $\mathbf{I}$ is based over M.

CLAIM 1.67. *If $n := n_\varphi$ (from Lemma 8.8) and*

$$\varphi_n(\mathbf{x}^n;\mathbf{a}) := \bigvee_{\substack{w \subseteq 2n \\ |w|=n}} \wedge \bigwedge_{\ell \in w} \varphi(\mathbf{x}_\ell;\mathbf{a}).$$

then

$$(*) \quad \mathrm{tp}(\mathbf{b}_0,\ldots,\mathbf{b}_{2n-1}/|M|) \vdash \varphi_n(\mathbf{x}^n;\mathbf{a}).$$

PROOF OF CLAIM. Suppose that

$$\langle \mathbf{b}'_0, \dots \mathbf{b}'_{2n-1} \rangle \models \quad \mathrm{tp}(\mathbf{b}_0, \dots, \mathbf{b}_{2n-1}/|M|).$$

Let $f \in \mathrm{Aut}_M(\mathfrak{C})$ such that $f(\mathbf{b}_i) = \mathbf{b}'_i$ for all $i < 2n$. For $2n \leq i < \omega$ let $\mathbf{b}'_i$ be $f(\mathbf{b}_i)$.

Denote by $\mathbf{J}$ the set $\{\mathbf{b}'_n \mid n < \omega\}$, as $\mathbf{J} = f(\mathbf{I})$, using Lemma 1.37 also $\mathbf{J}$ is based over M.

Since both $\mathrm{Av}(\mathbf{I}/|M| \cup \mathbf{a})$ and $\mathrm{Av}(\mathbf{J}/|M| \cup \mathbf{a})$ are non forking extensions of the stationary type p^* we have that $\mathrm{Av}(\mathbf{I}/|M| \cup \mathbf{a}) = \mathrm{Av}(\mathbf{J}/|M| \cup \mathbf{a})$.

Since $p^* \cup \varphi(\mathbf{x}; \mathbf{a})$ dnf over M and p^* is stationary we have that $\varphi(\mathbf{x}; \mathbf{a}) \in \mathrm{Av}(\mathbf{I}/|M| \cup \mathbf{a})$. By the equality of the averages we have that

$$(**) \quad \varphi(\mathbf{x}; \mathbf{a}) \in \mathrm{Av}(\mathbf{J}/|M| \cup \mathbf{a}).$$

By the choice of n we have that $|\neg\varphi(\mathbf{J}; \mathbf{a})| < n$ thus from $(**)$ we get that $\mathfrak{C} \models \varphi_n[\mathbf{b}'_0, \dots, \mathbf{b}'_{2n-1}; \mathbf{a}]$. □

Now using the claim we will find $\mathbf{b}' \in |M|$ such that $M \models \varphi[\mathbf{b}'; \mathbf{a}]$. Using $(*)$ there exists $\psi(\mathbf{x}^n; \mathbf{c}) := \psi(\mathbf{x}_0, \dots, \mathbf{x}_{2n-1}; \mathbf{c}) \in \mathrm{tp}(\mathbf{b}_0, \dots, \mathbf{b}_{2n-1}//|M|)$ such that

$$(***) \quad \psi(\mathbf{x}^n; \mathbf{c}) \vdash \varphi_n(\mathbf{x}^n; \mathbf{a}).$$

Since $\mathfrak{C} \models \exists \mathbf{x}^n \psi[\mathbf{x}^n; \mathbf{c}]$, for some $\mathbf{b}^n \in {}^{2n}|M|$ we have that $M \models \psi[\mathbf{b}^n; \mathbf{c}]$. By $(***)$ we have $M \models \varphi_n[\mathbf{b}^n; \mathbf{a}]$. By the definition of φ_n this means that there exists $\mathbf{b}'$ (in fact there are at least $n+1$-many such) in the sequence $\mathbf{b}^n$ that satisfies $\varphi(\mathbf{x}; \mathbf{a})$. □

Now we turn to a characterization of dividing that will be invoked several times (Lemma 3.1).

LEMMA 1.68. *A formula $\varphi(\mathbf{x}; \mathbf{b})$ divides over A if and only if there exist $k < \omega$ and an indiscernible over A $\langle \mathbf{b}_i \mid i < \omega \rangle$ such that $\mathbf{b}_0 = \mathbf{b}$ and $\{\varphi(\mathbf{x}; \mathbf{b}_i) \mid i < \omega\}$ is k-contradictory.*

PROOF. Necessity is clear. We prove sufficiency. Assume that $\varphi(\mathbf{x}; \mathbf{b})$ divides over A and take $k < \omega$ and $I = \{\mathbf{b}_i \mid i < \omega\}$ such that $\mathrm{tp}(\mathbf{b}_i/A) = \mathrm{tp}(\mathbf{b}/A)$ for every $i < \omega$ and $\{\varphi(\mathbf{x}; \mathbf{b}) \mid i < \omega\}$ is k-contradictory. Expand the language with names for the elements of A and let $\{\mathbf{c}_n \mid n < \omega\}$ be constants not in $\mathrm{L}(T)$. Consider the union of the following sets of sentences:

- T;
- $\neg\exists \mathbf{x}[\varphi(\mathbf{x}; \mathbf{c}_{i_0}) \wedge \cdots \wedge \varphi(\mathbf{x}; \mathbf{c}_{i_{k-1}})]$, whenever $i_0 < \cdots < i_{k-1} < \omega$;
- $\psi(\mathbf{c}_0, \dots; \mathbf{c}_n; \mathbf{d}) \leftrightarrow \psi(\mathbf{c}_{i_0}, \dots, \mathbf{c}_{i_n}, \mathbf{d})$, whenever $i_0 < \cdots < i_n < \omega$, $\mathbf{d} \in A$, and ψ is in $\mathrm{L}(T)$;

- $\psi(\mathbf{c}_0; \mathbf{d})$, for every $\psi(\mathbf{x}; \mathbf{d}) \in \mathrm{tp}(\mathbf{b}_0/A)$.

An application of Ramsey's Theorem shows that this set of sentences is consistent.

Let N be a model for it and let $\mathbf{d}_n$ be the interpretation of $\mathbf{c}_n$ in the model N. Then, $\langle \mathbf{b}_n \mid n < \omega \rangle$ is a sequence indiscernible over A and the set $\{\varphi(\mathbf{x}; \mathbf{b}_n) \mid n < \omega\}$ is k-contradictory. Furthermore, there exists an A-automorphism f such that $f(\mathbf{d}_0) = \mathbf{b}$. Therefore, $\langle f(\mathbf{d}_n) \mid n < \omega \rangle$ satisfies the requirements of the lemma. □

Exercises 9.1.

(1) For $n < \omega$ let

$$\kappa_d^n(T) := \min\{\kappa \mid \text{ for every } A \text{ and } p \in \mathrm{S}^n(A) \text{ there exists } B \subseteq A, \text{ such that } |B| < \kappa \text{ and } p \text{ does not divide over } B\}.$$

If such a cardinal does not exists we write $\kappa_d^n(T) = \infty$.

$$\kappa_d(T) := \sup_{n<\omega} \kappa_d^n(T).$$

Prove that $\kappa_d(T) \leq |T|^+ \iff \kappa(T) \leq |T|^+$.

(2)

(3) Suppose that p is a strongly minimal type in $\mathbf{x}$ with parameters from the set A. Let $I = \{\mathbf{b}_n \ : \ n < \omega\} \subseteq {}^{\mathbf{x}}\mathfrak{C}$. If $\mathrm{tp}(\mathbf{b}_n/A \cup \{\mathbf{b}_k : k < n\})$ is a non-algebraic extension of the type p for every $n < \omega$ then I is a Morely Sequence for p over B.

(4)

(5) Show for any complete first-order T, if $\kappa(T) \leq |T|^+$ and "$M \models T$, $p \in \mathrm{S}(|M|)$ implies that p is stationary" then T must be stable.

(6)

(7)

(8)

(9)

2. Stability spectrum theorem

Stability spectrum theorem and Lachlan's theorem.

3. Forking in Simple theories is symmetric and transitive

The next lemma appeared in Shelah's original article [**?**] and is crucial to analyze forking and dividing. It will be used in the proof Theorem 3.3, the Independence Theorem (Theorem **??**), and the characterization of forking through the rank (Theorem **??**).

LEMMA 3.1 (Shelah's Characterization of Dividing). *Let A be a set, $\mathbf{a}$ be a possibly infinite sequence and $\mathbf{b}$ be a finite sequence. The following conditions are equivalent.*

(1) $\mathrm{tp}(\mathbf{a}/A\mathbf{b})$ *does not divide over A;*
(2) *For every infinite sequence of indiscernibles $\mathbf{I}$ over A with $\mathbf{b} \in \mathbf{I}$ there exists $\mathbf{a}'$ realizing $\mathrm{tp}(\mathbf{a}/A\mathbf{b})$ such that $\mathbf{I}$ is indiscernible over $A\mathbf{a}'$;*
(3) *For every infinite sequence of indiscernibels $\mathbf{I}$ over A with $\mathbf{b} \in \mathbf{I}$ there exists an $A\mathbf{b}$-automorphism f such that $\mathbf{I}$ is indiscernible over $Af(\mathbf{a})$.*

PROOF. The equivalence between (2) and (3) is a consequence of the homogeneity of $\mathfrak{C}$.

(2)⇒(1): By contradiction, suppose $\mathrm{tp}(\mathbf{a}/A\mathbf{b})$ divides over A and take $\varphi(\mathbf{x}; \mathbf{c}, \mathbf{b}) \in \mathrm{tp}(\mathbf{a}/A\mathbf{b})$ with $\mathbf{c} \in A$ such that $\varphi(\mathbf{x}; \mathbf{c}, \mathbf{b})$ divides over A. Lemma 1.68 provides a sequence $\mathbf{I} = \langle \mathbf{b}_i \mid i < \omega \rangle$ indiscernible over A such that $\mathbf{b} = \mathbf{b}_0$ and $\{\varphi(\mathbf{x}; \mathbf{c}, \mathbf{b}_i) \mid i < \omega\}$ is k-contradictory. By (2) there exists $\mathbf{a}'$ realizing $\mathrm{tp}(\mathbf{a}/A\mathbf{b})$ such that I is indiscernible over $A\mathbf{a}'$. But then $\models \varphi[\mathbf{a}', \mathbf{c}, \mathbf{b}_0]$, and $\models \varphi[\mathbf{a}', \mathbf{c}, \mathbf{b}_i]$ for every $i < \omega$ by indiscernibility. This contradicts the fact that $\{ \varphi(\mathbf{x}; \mathbf{c}, \mathbf{b}_i) \mid i < \omega \}$ is k-contradictory.

(1)⇒(2): Let $\mathbf{I} = \langle \mathbf{b}_i \mid i < \omega \rangle$ be a sequence of indiscernibles over A and $\mathbf{b} \in \mathbf{I}$. Denote $p(\mathbf{x}; \mathbf{b}) := \mathrm{tp}(\mathbf{a}/A\mathbf{b})$ and let $q := \bigcup_{\mathbf{b}_i \in I} p(\mathbf{x}; \mathbf{b}_i)$. We claim that q is consistent.

If q is inconsistent, there exist a finite $\mathbf{I}^* \subseteq \mathbf{I}$, $\mathbf{c} \in A$ and a formula $\varphi(\mathbf{x}; \mathbf{c}, \mathbf{b}) \in \mathrm{tp}(\mathbf{a}/A\mathbf{b})$ such that $\{\varphi(\mathbf{x}; \mathbf{c}, \mathbf{b}_i) \mid \mathbf{b}_i \in \mathbf{I}^*\}$ is inconsistent. By the indiscernibility of $\mathbf{I}$ over A, $\{\varphi(\mathbf{x}; \mathbf{c}, \mathbf{b}_i) \mid \mathbf{b}_i \in \mathbf{I}\}$ is $|\mathbf{I}^*|$-inconsistent, so $\mathrm{tp}(\mathbf{a}/A\mathbf{b})$ divides over A. But this is a contradiction.

Now let $\Gamma(\mathbf{x})$ be the union of the following formulas:

- $q(\mathbf{x})$;
- $\psi(\mathbf{x}; \mathbf{b}_0, \ldots, \mathbf{b}_{n-1}, \mathbf{d}) \leftrightarrow \psi(\mathbf{x}; b_{i_0}, \ldots, \mathbf{b}_{i_{n-1}}, \mathbf{d})$, whenever $\psi \in \mathrm{L}(T)$, $n < \omega$, $i_0 < \cdots < i_{n-1} < \omega$, and $\mathbf{d} \in A$.

We show that $\Gamma(\mathbf{x})$ is consistent, which implies (2). The proof is by induction on the cardinality of the finite subsets of $\Gamma(\mathbf{x})$. For the induction step, it is sufficient to show that for any $\mathbf{d} \in A$, $i_0 < \cdots < i_n$ and $\varphi(\mathbf{x}; \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{i_{n-1}}, \mathbf{d}) \in q(\mathbf{x})$

we have

$$(**) \qquad \models \exists \mathbf{x} \left[\varphi(\mathbf{x}; \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{i_{n-1}}, \mathbf{d}) \wedge \left[\psi(\mathbf{x}; b_0, \ldots, \mathbf{b}_{n-1}, \mathbf{d}) \leftrightarrow \psi(\mathbf{x}; \mathbf{b}_{i_0}, \ldots, \mathbf{b}_{i_{n-1}}, \mathbf{d}) \right] \right].$$

Fix $\mathbf{a}'$ realizing q. By Ramsey's Theorem there is an infinite subsequence $\mathbf{I}'$ of $\mathbf{I}$ which is ψ-indiscernible over $\mathbf{d}\mathbf{a}'$. Take $\mathbf{b}'_0, \ldots, \mathbf{b}'_{n-1}, \mathbf{b}'_{i_0}, \ldots, \mathbf{b}'_{i_{n-1}} \in \mathbf{I}'$. Then,

$$\models \varphi[\mathbf{a}', \mathbf{b}'_0, \ldots, \mathbf{b}'_{i_{n-1}}, \mathbf{d}] \wedge \left[\psi[\mathbf{a}', \mathbf{b}'_0, \ldots, \mathbf{b}'_{n-1}, \mathbf{d}] \leftrightarrow \psi[\mathbf{a}', \mathbf{b}'_{i_0}, \ldots, \mathbf{b}'_{i_{n-1}}, \mathbf{d}] \right].$$

Therefore,

$$\models \exists \mathbf{x} [\varphi(\mathbf{x}; \mathbf{b}'_0, \ldots, \mathbf{b}'_{i_{n-1}}, \mathbf{d}) \wedge [\psi(\mathbf{x}; \mathbf{b}'_0, \ldots, \mathbf{b}'_{n-1}, \mathbf{d}) \leftrightarrow \psi(\mathbf{x}; \mathbf{b}'_{i_0}, \ldots, \mathbf{b}'_{i_{n-1}}, \mathbf{d})]],$$

which implies (**) by the indiscernibility of I over A. □

The next lemma is sometimes called the Pairs Lemma, or Left Transitivity. It is also from [**?**]:

LEMMA 3.2 (Left Transitivity). *Let* $\mathbf{a}_0$, $\mathbf{a}_1$, *and* $\mathbf{b}$ *be possibly infinite sequences. If* $\mathrm{tp}(\mathbf{a}_0/A\mathbf{b})$ *does not divide over* A *and* $\mathrm{tp}(\mathbf{a}_1/A\mathbf{b}\mathbf{a}_0)$ *does not divide over* $A\mathbf{a}_0$, *then* $\mathrm{tp}(\mathbf{a}_0\mathbf{a}_1/A\mathbf{b})$ *does not divide over* A.

PROOF. By Finite Character, we may assume that $\mathbf{b}$ is finite. Let $\mathbf{I}$ be a sequence indiscernible over A such that $\mathbf{b} \in \mathbf{I}$. By Lemma 3.1, showing that $\mathrm{tp}(\mathbf{a}_0\mathbf{a}_1/A\mathbf{b})$ does not divide over A is equivalent to finding $\mathbf{c}_0\mathbf{c}_1$ realizing $\mathrm{tp}(\mathbf{a}_0\mathbf{a}_1/A\mathbf{b})$ such that $\mathbf{I}$ is indiscernible over $A\mathbf{c}_0\mathbf{c}_1$. By Lemma 3.1, since $\mathrm{tp}(\mathbf{a}_0/A\mathbf{b})$ does not divide over A, we can find $\mathbf{c}_0$ realizing $\mathrm{tp}(\mathbf{a}_0/A\mathbf{b})$ such that $\mathbf{I}$ is indiscernible over $A\mathbf{c}_0$. Take an $A\mathbf{b}$-automorphism f such that $f(\mathbf{a}_0) = \mathbf{c}_0$. Since $\mathrm{tp}(\mathbf{a}_1/A\mathbf{b}\mathbf{a}_0)$ does not divide over $A\mathbf{a}_0$, the type $\mathrm{tp}(f(\mathbf{a}_1)/A\mathbf{b}\mathbf{c}_0)$ does not divide over $A\mathbf{c}_0$. Since $\mathbf{I}$ is indiscernible over $A\mathbf{c}_0$, by Lemma 3.1 we can choose $\mathbf{c}_1$ realizing $\mathrm{tp}(f(\mathbf{a}_1)/A\mathbf{b}\mathbf{c}_0)$ such that I is indiscernible over $A\mathbf{c}_0\mathbf{c}_1$. We have $\mathrm{tp}(\mathbf{c}_0\mathbf{c}_1/A\mathbf{b}) = \mathrm{tp}(f(\mathbf{a}_0)f(\mathbf{a}_1)/A\mathbf{b}) = \mathrm{tp}(\mathbf{a}_0\mathbf{a}_1/A\mathbf{b})$, so we are done. □

THEOREM 3.3 (Kim's Characterization of Dividing in Simple Theories). *Let* T *be simple. The following conditions are equivalent:*

1. *The formula* $\varphi(\mathbf{x}; \mathbf{b})$ *divides over* A;
2. *For every Morley sequence* $\langle \mathbf{b}_i \mid i \in I \rangle$ *for* $\mathrm{tp}(\mathbf{b}/A)$, *the set* $\{\varphi(\mathbf{x}; \mathbf{b}_i) \mid i \in I\}$ *is inconsistent;*
3. *For some Morley sequence* $\langle \mathbf{b}_i \mid i \in I \rangle$ *for* $\mathrm{tp}(\mathbf{b}/A)$, *the set* $\{\varphi(\mathbf{x}; \mathbf{b}_i) \mid i \in I\}$ *is inconsistent.*

PROOF. (3) ⇒ (1) is obvious and (2) ⇒ (3) is given by Corollary **??**. Let us prove (1) ⇒ (2).

CLAIM 3.4. *Let* $\mathbf{I} = \langle \mathbf{b}_i \mid i \in I \rangle$ *be a Morley sequence for* $\mathrm{tp}(\mathbf{b}/A)$. *For each* $I_0 \subseteq I$ *such that* $\inf(I_0)$ *exists and each* $i \in I$ *such that* $i < \inf(I_0)$, *the formula* $\varphi(\mathbf{x}; \mathbf{b}_i)$ *divides over* $A \cup \bigcup_{j \in I_0} \mathbf{b}_j$.

PROOF. Recall that $\mathbf{b}_{I_0}$ stands for $\langle \mathbf{b}_i \mid i \in I_0 \rangle$. Let $q(\mathbf{x}, \mathbf{y}) = \mathrm{tp}(\mathbf{b}_i, \mathbf{b}_{I_0}/A)$. First, by using an automorphism we have that $\varphi(\mathbf{x}; \mathbf{b}_i)$ divides over A. Let $\mathbf{J}$ be an infinite sequence of indiscernibles over A such that $\{\varphi(\mathbf{x}; \mathbf{c}) \mid \mathbf{c} \in \mathbf{J}\}$ is inconsistent and $\mathbf{b}_i \in J$. By Proposition 1.38, the type $q(\mathbf{b}_i; \mathbf{y})$ does not divide over A. Hence, by Lemma 3.1, there exists $\mathbf{b}' \models q(\mathbf{b}_i, \mathbf{x}/A)$ such that J is an sequence of indiscernibles over $A\mathbf{b}'$. Since $\mathrm{tp}(\mathbf{b}'/A\mathbf{b}_i) = \mathrm{tp}(\mathbf{b}_{I_0}/A\mathbf{b}_i)$ there is $f \in \mathrm{Aut}_{A\mathbf{b}_i}(\mathfrak{C})$ such that $f(\mathbf{b}') = \mathbf{b}_{I_0}$. The sequence $f(\mathbf{J})$ witnesses that $\varphi(\mathbf{x}; \mathbf{b}_i)$ divides over $A\mathbf{b}_{I_0}$. □

Let $\mathbf{I}$ be a Morley sequence for $\mathrm{tp}(\mathbf{b}/A)$. Assume, for a contradiction, that $\{\varphi(\mathbf{x}; \mathbf{c}) \mid \mathbf{c} \in \mathbf{I}\}$ is consistent. Let J be the reverse order of $(\kappa(T) + \aleph_0)^+$. Use Theorem 1.39 to fix an sequence of indiscernibles $\mathbf{J} = \langle \mathbf{b}'_i \mid i \in J \rangle$ such that for any finite increasing sequence from $\mathbf{J}$ there is a finite sequence of elements of $\mathbf{I}$ that realize the same type over A. Then $\mathbf{J}$ is a Morley sequence for $\mathrm{tp}(\mathbf{b}/A)$. Furthermore, by compactness, since $\{\varphi(\mathbf{x}; \mathbf{c}) \mid \mathbf{c} \in \mathbf{I}\}$ is consistent, then so is $r := \{\varphi(\mathbf{x}; \mathbf{b}_j) \mid j \in \mathbf{J}\}$. Choose $\mathbf{c}$ realizing r. By simplicity, there exists a set Y of cardinality less than $\kappa(T)$ such that $\mathrm{tp}(\bar{c}/AJ)$ does not fork over $A\bar{b}_Y$. Choose $i \in X$ such that $i < \min(Y)$. Then $\varphi(\bar{x}, \bar{b}_i)$ divides over $A\bar{b}_Y$ by the Claim applied to J. But $\bar{c}$ realizes $\varphi(\bar{x}, \bar{b}_i)$, so $\mathrm{tp}(\bar{c}/AJ)$ forks over $A\bar{b}_Y$, which contradicts the choice of Y. □

We can now prove the equivalence of forking and dividing under simplicity.

THEOREM 3.5. *Let T be simple. The following conditions are equivalent:*

(1) $\varphi(\bar{x}, \bar{b})$ *divides over A;*
(2) $\varphi(\bar{x}, \bar{b})$ *forks over A.*

PROOF. (1) ⇒ (2) is obvious. We prove (2) ⇒ (1): Since $\varphi(\mathbf{x}; \mathbf{b})$ forks over A, there exist $\{\psi_i(\mathbf{x}; \mathbf{a}^i) \mid m < \omega\}$ such that $\psi_i(\mathbf{x}; \mathbf{a}^i)$ divides over A for $i < m$ and

$$(*) \qquad \varphi(\mathbf{x}; \mathbf{b}) \vdash \bigvee_{i<m} \psi_i(\mathbf{x}; \mathbf{a}^i).$$

Let $\mathbf{a} := \langle \mathbf{a}^0, \ldots, \mathbf{a}^{m-1} \rangle$. As T is simple, $\mathrm{tp}(\mathbf{ab}/A)$ dnf over A. By Theorem 1.42 we can fix a Morley sequence for $\mathrm{tp}(\mathbf{ab}/A)$,

$$\langle \mathbf{a}_n \mathbf{b}_n \mid n < \omega \rangle, \quad \text{with} \quad \mathbf{a}_0 = \mathbf{a}, \mathbf{b}_0 = \mathbf{b}.$$

Let us write $\bar{a}_n = \langle \mathbf{a}^0_n, \ldots, \mathbf{a}^{m-1}_n \rangle$ for every $n < \omega$.

Since $\langle \mathbf{b}_n \mid n < \omega \rangle$ is indiscernible over A, to show that $\varphi(\mathbf{x}; \mathbf{b})$ divides over A, it suffices to show that the set $\{\varphi(\mathbf{x}, \mathbf{b}_n) \mid n < \omega\}$ is inconsistent. Assume that it is consistent and let $\mathbf{c}$ realize it. By the definition of Morley sequence,

$$\mathrm{tp}(\mathbf{ab}/A) = \mathrm{tp}(\mathbf{a}_n \mathbf{b}_n/A), \quad \text{for every } n < \omega.$$

Hence, using an A-automorphism and (*), we conclude that

$$(**) \qquad \varphi(\mathbf{x};\mathbf{b}_n) \vdash \bigvee_{i<m} \psi_i(\mathbf{x};\mathbf{a}_n^i), \quad \text{for every } n<\omega.$$

By the choice of $\mathbf{c}$, we have $\models \varphi[\mathbf{c};\mathbf{b}_n]$ for every $n<\omega$. Therefore, (**) implies that for every $n<\omega$ there exists $i(n)<m$ such that $\models \psi_{i(n)}[\mathbf{c};\mathbf{a}_n^{i(n)}]$. By the pigeonhole principle, there exist an infinite $S\subseteq\omega$ and a fixed $k<m$ such that

$$(\dagger) \qquad \models \psi_k[\,\mathbf{c},\mathbf{a}_n^k], \quad \text{for every } n\in S.$$

But $\langle \mathbf{a}_n^k \mid n\in S\rangle$ is a Morley sequence for $\mathrm{tp}(\mathbf{a}_0^k/A)$. Furthermore, (†) shows that the set $\{\psi_k(\mathbf{x};\mathbf{a}_n^k) \mid n\in S\}$ is consistent. Thus, $\psi_k(\mathbf{x};\mathbf{a}^k)$ does not divide over A by Theorem 3.3. This contradicts the choice of $\psi_k(\mathbf{x};\mathbf{a}^k)$. □

THEOREM 3.6 (Symmetry). *Let T be simple. Then $\mathrm{tp}(\mathbf{a}/A\mathbf{b})$ forks over A if and only if $\mathrm{tp}(\mathbf{b}/A\mathbf{a})$ forks over A.*

PROOF. It is, of course, sufficient to prove one direction. Suppose $\mathrm{tp}(\mathbf{a}/A\mathbf{b})$ forks over A and take $\varphi(\mathbf{x};\mathbf{c};\mathbf{b})\in \mathrm{tp}(\mathbf{a}/A\mathbf{b})$ with $\mathbf{c}\in A$ such that $\varphi(\mathbf{x};\mathbf{c};\mathbf{b})$ forks over A. By Theorem 3.5, the formula $\varphi(\mathbf{x};\mathbf{c};\mathbf{b})$ divides over A. If $\mathrm{tp}(\mathbf{b}/A\mathbf{a})$ does not fork over A, we can choose $\mathbf{I}=\langle \mathbf{b}_n \mid n<\omega\rangle$, a Morley sequence for $\mathrm{tp}(\mathbf{b}/A\mathbf{a})$ over A such that $\mathbf{b}_0=\mathbf{b}$. We have $\models\varphi[\mathbf{a};\mathbf{c};\mathbf{b}]$, so, by the indiscernibility of $\mathbf{I}$ over $A\mathbf{a}$, we also have $\models\varphi[\mathbf{a};\mathbf{c};\mathbf{b}_n]$ for every $n<\omega$. Thus, $\{\varphi(\mathbf{x};\mathbf{c};\mathbf{b}_n) \mid n<\omega\}$ is consistent (as it is realized by $\mathbf{a}$). Notice that $\langle \mathbf{c}\mathbf{b}_n \mid n<\omega\rangle$ is a Morley sequence for $\mathrm{tp}(\mathbf{c}\mathbf{b}/A)$. But then the set $\{\varphi(\mathbf{x};\mathbf{c};\mathbf{b}_n) \mid n<\omega\}$ must be inconsistent, by Theorem 3.3. We have a contradiction, so $\mathrm{tp}(\mathbf{b}/A\mathbf{a})$ forks over A. □

THEOREM 3.7 (Transitivity). *Let T be simple and $A\subseteq B\subseteq C$. If $\mathrm{tp}(\mathbf{a}/C)$ does not fork over B and $\mathrm{tp}(\mathbf{a}/B)$ does not fork over A, then $\mathrm{tp}(\mathbf{a}/C)$ does not fork over A.*

PROOF. We use the fact that forking is equivalent to dividing. Let $\mathbf{b}$ enumerate B and $\mathbf{c}$ enumerate $C\setminus B$. By Symmetry and Finite Character, we have that $\mathrm{tp}(\mathbf{b}/A\mathbf{a})$ does not divide over A and that $\mathrm{tp}(\mathbf{c}/A\mathbf{b}\mathbf{a})$ does not divide over $A\mathbf{b}$. Hence, by Lemma 3.2, $\mathrm{tp}(\mathbf{c}\mathbf{b}/A\mathbf{a})$ does not divide over A. Another use of Symmetry finishes the proof. □

THEOREM 3.8. *T is simple iff $\kappa(T)\leq |T|^+$ iff $\kappa(T)<\infty$*

Kim's theorem:

THEOREM 3.9. *Suppose T is simple. p divides over B iff p forkes over B.*

Kim's theorem:

THEOREM 3.10. *T is simple iff forking is symmetric.*

THEOREM 3.11. *Let T be stable. T is superstable iff $\kappa(T)=\aleph_0$.*

4. Applications of forking

THEOREM 4.1. *Let T be stable in a countable language. For every A if M and N are prime over A then $M \cong_A N$.*

5. Forking is canonical

Various axioms and theorems.

CHAPTER 10

Applications

Introduction

The purpose of this chapter is to establish several results of fundamental importance that are supposed to convince the reader that the tools we developed are indeed powerful.

1. Harnik's theorem

2. Uniqueness and characterization of prime models

THEOREM 2.1. *Suppose T is stable and countable or superstable. If M is primary over a subset A then there does not exists an uncountable $\mathbf{I} \subseteq |M|$ indiscernible sequence over A.*

COROLLARY 2.2. *Suppose T totally transcendental. If M is prime over A then there is no uncountable $\mathbf{I} \subseteq |M|$ indiscernible sequence over A.*

PROOF. □

PROOF. Notice that the assumptions on T imply that $\kappa(T) \leq \aleph_1$. For the sake of contradiction suppose that here exists an uncountable sequence of indiscernibles $\mathbf{I}$ in M. By stability we may assume that $\mathbf{I} = \{\mathbf{b}_i \mid i < \omega_1\}$.

Define $\{B_\alpha \subseteq |M| \ : \ \alpha < \omega_1\}$ increasing and continuous such that

(1) $|B_\alpha| = \aleph_0$,
(2) B_α/A is constructible and
(3) $\mathbf{b}_\alpha \in B_{\alpha+1}$.

Consider

$$F(\delta) := \min\{j \mid \mathrm{tp}(\mathbf{b}_\delta/A \cup B_\delta) \text{ is isolated over } A \cup B_j\}.$$

CLAIM 2.3. $\mathrm{cf}\,\delta = \aleph_0 \implies F(\delta) < \delta$.

PROOF. As B_δ/A is constructible, by Lemma **??** M is primary over $A \cup B_\delta$. By Lemma **??** we get that M is atomic over $A \cup B_\delta$, thus there are $\mathbf{a} \in A, \mathbf{b} \in B_\delta$ and a formula $\varphi(\mathbf{x}; \mathbf{y}; \mathbf{z})$ such that

$$\varphi(x; \mathbf{a}; \mathbf{b}) \vdash \mathrm{tp}(\mathbf{b}_\delta/A \cup B_\delta).$$

Since $\mathrm{cf}\,\delta = \aleph_0$ and $B_\delta = \bigcup_{j<\delta} B_j$, there exists $j < \delta$ such that $\mathbf{b} \in B_j$. □

By stability for every finite $\mathbf{b}$ there exists $\mathbf{J}_\mathbf{b} \subseteq \mathbf{I}$ of cardinality $< \kappa(T)$ such that $\mathbf{I} - \mathbf{J}_\mathbf{b}$ is an indiscernible sequence over $A\mathbf{b}$. (this is Theorem **??**). By repeated application of this fact we get that for every $\alpha < \omega_1$ there exists $\beta(\alpha) < \omega_1$ such that

$$\{\mathbf{b}_i \mid \beta(\alpha) \leq i < \omega_1\} \text{ is an indiscernible sequence over } A \cup B_\alpha.$$

We may assume that $\alpha_< \alpha_2 \implies \beta(\alpha_1) < \beta(\alpha_2)$.

The set

$$C := \{\delta < \omega_1 \mid \alpha < |delta \implies \beta(\alpha) < \delta\}$$

is closed and unbounded. The set $S := \mathfrak{C} \cap \{\delta < \omega_1 \mid \operatorname{cf} \delta = \aleph_0\}$ is stationary. By the Claim F is a pressing down function on S thus by Fodor there are $S^* \subseteq S$ stationary and j_0 such that

$$\delta \in S^* \implies F(\delta) = j_0.$$

□

THEOREM 2.4. *T is NIP*

3. Uniqueness of prime models

4. Stability spectrum

CHAPTER 11

Survey

Introduction

The aim of this chapter is to overview several important topics that are not covered by this book but are in my opinion important. Some of the topics need (and deserve) entire monographs bigger than the current one to explore the main results and the interesting open problems.

I hope that the reader will find the few pages dedicated to each section not a complete waste of time. Some of the topics are central to current research in contemporary model theory, others were central a decade ago and may become fashionable again. Since I expect the field of classification theory for non elementary classes (and perhaps its merger with geometric model theory) to become the central branch of pure model theory I have included a rather lengthy section about non elementary classes.

1. The main gap (Shelah's great theorem)

Since Shelah begun his research in model theory around 1968 much of his wrok was aimed at discovering the main "dichotomies" in first-order model theory and studying thier properties.

This was guided by several test-case problems. In the begining Los conjecture for uncountable theories and later several brilliant far reaching conjectures concerning the spectrum function that were proposed by Shelah himself in his 1971 paper [**Sh16**] (and restated again in his 1974 paper [**Sh31**])..

Around 1982 Shelah announced ([**Sh200**]) that he proved a theorem that to some extent is the crown of his theory, all the old conjectures are true! This follows from :

THEOREM 1.1. *Let T be a complete first-order theory in a countable language. If there exists a $\lambda > |T| + \aleph_0$ such that $\mathrm{I}(\lambda, T) < 2^\lambda$ then for every $\mu > \aleph_0$ and every $M \models T$ of cardinality μ there exists*
$\{M_\eta \prec M \mid \eta \in {}^{\omega>}\mu\}$ countable models such that

(1) *M is prime and minimal over the set $\bigcup_{\eta \in {}^{\omega>}\mu} |M_\eta|$*

(2) *For every* $\eta \in {}^{\omega>}\mu$ *let* $\eta^- := \eta \restriction (\ell(\eta) - 1)$ *and the following requirement holds*

$$M_\eta \underset{M_{\eta^-}}{\perp} \bigcup_{\substack{\nu \in {}^{\omega>}\mu \\ \neg[\eta \lhd \nu]}} M_\nu \quad \textit{for all} \quad \eta \in {}^{\omega>}\mu\,.$$

It is relatively easy to derive from it the following corollary:

THEOREM 1.2 (The Main Gap). *Let* T *be a complete first-order theory in a countable language. There are two mutually exclusive possible behavior for the spectrum function* $\lambda \mapsto \mathrm{I}(\lambda, T)$ *for uncountable* λ*:*

(1) $\mathrm{I}(\lambda, T) = 2^\lambda$ *for every uncountable* λ*, or*
(2) $\mathrm{I}(\aleph_\alpha, T) \leq \beth_{\omega_1}(|\alpha| + \aleph_0)$ *for every* α

The theorem is called the main gap due to the following observation: Notice that in models of ZFC where cardinal arithmetic is "reasonable" there are cardinal numbers $\aleph_\alpha$ such that $\beth_{\omega_1}(|\alpha| + \aleph_0)$ is much smaller than $2^{\aleph_\alpha}$.

In 1982 Shelah identified the following as a major conjecture in model theory:

CONJECTURE 1.3 (Shelah). *The satetement of Theorem 1.1 holds also for uncountable theories.*

A crucial new property needed for the main gap is the following

DEFINITION 1.4. T has *omitting types order-property* iff there exists a type $p(\mathbf{x}; \mathbf{y}; \mathbf{z})$ with $\ell(\mathbf{x}) = \ell(\mathbf{y}) < \omega$ such that for every λ there exists $\{\mathbf{a}_i \mid i < \lambda\}$ such that

$$i < j \iff \mathfrak{C} \models \exists \mathbf{z} \bigwedge p(\mathbf{a}_i; \mathbf{a}_j; \mathbf{z}).$$

Using Theorem **??** we get

THEOREM 1.5. *Let* T *be a complete first-order theory. If there exists* $\lambda > |T| + \aleph_0$ *such that* $\mathrm{I}(\lambda, T) < 2^\lambda$ *then* T *does not have the omitting types order-property.*

DEFINE DOP.

2. Classification theory for non-elementary classes

2.1. independence.

2.2. global behaviour.

3. Geometric stability (or the fine structure theory)

[**Pi2**]

4. Lang-Mordell

5. Ax and Kochen

6. o-minimal theories

A book by Lou van den dries [**Dr**].

7. Abstract model theory

In abstract model theory the objects of study are logics and their model theoretic properties rather than theories or models for a given theory.

First we have to define the notion of a logic and we will introduce few important properties of logics.

In chapter 1 we followed the common tradition in much of the current literature in model theory and denoted by L similarity type and in later parts of the book abused notation and also used the same symbol to denote all first-order formulas. In this section we have to be more careful.

In what follows we will limit ourself to *single sorted logics*.

DEFINITION 7.1. Let $\mathbf{R}, \mathbf{F}, \mathbf{n}, \mathbf{C}$ be like in Definition 1.1.1. Denote by the letter σ the quadruple $\langle \mathbf{R}, \mathbf{F}, \mathbf{n}, \mathbf{C} \rangle$ we call σ a *signature*.

NOTATION 7.2. Let σ be a signature, denote by Struc_σ the class

$$\{M \mid M \text{ is a structure for } \sigma\}.$$

DEFINITION 7.3. A *logic* is a mapping from similarity types $\sigma \mapsto \langle \mathcal{L}_\sigma, \models_{\mathcal{L}_\sigma} \rangle$ satisfying

(1) *existence of occurance number:* For every $\varphi \in \mathcal{L}_\sigma$ there exists a set $\sigma_\varphi \subseteq \sigma$ such that $\varphi \in \mathcal{L}_{\sigma_\varphi}$. For every $\sigma' \supseteq \sigma_\varphi$ and any $M \in \mathrm{Struc}_{\sigma'}$ the relation $M \models_{\mathcal{L}_{\sigma'}} \varphi$ is well defined.
(2) *expansion property:*
(3) *isomorphism property:*
(4) *renaming:*
(5) *popositional closure:*
(6) *quantifier closure:*
(7) *relativization:*

DEFINITION 7.4. $\mathcal{L}$ *occurance number*

DEFINITION 7.5. Let $\kappa \geq \lambda$, $\mathcal{L}$ is (κ, λ)*-compact*

Note that first-order logic is $(\kappa, \aleph_0)$-compact for every κ.

THEOREM 7.6 (William Hanf, 1964). *Let κ be an uncountable cardinal. The logic $L_{\kappa,\kappa}$ is (κ, κ)-compact iff κ is weakly compact cardinal.*

This is probably the right place to mention that Tarski's original definition of weakly compact cardinal was not the one we gave in Chapter 1. It was the statement of the previous theorem. Only about 20 years later Hanf managed to show that the partition relation definition is equivalent to the one involving infinitary logics.

The following related concept was introduced by Tarski:

DEFINITION 7.7. An uncountable cardinal κ is *strongly compact* iff the logic $L_{\kappa,\kappa}$ is $(\kappa, \aleph_0)$-compact.

THEOREM 7.8 (Lindström, 1969). *Let $\mathcal{L}$ be a logic extending first-order. If $\mathcal{L}$ satisfies the dounward Löwenhiem Skolem theorem for sentences as well as the $(\aleph_0, \aleph_0)$-compactness property then $\mathcal{L}$ is equivalent to first-order.*

PROOF. □

Similarly to the proof of uniqueness of model-companion of a theory one can prove:

THEOREM 7.9 (Makowsky, ?). *Let $\mathcal{L}$ be a logic extending first-order. If $\mathcal{L}$ satisfies the elementary chain property as well as the $(\aleph_0, \aleph_0)$-compactness property then $\mathcal{L}$ is equivalent to first-order.*

DEFINITION 7.10. $\mathcal{L}$ has the amalgamation property.

THEOREM 7.11 (Makowsky Shelah 1983). *Let $\mathcal{L}$ be a logic extending first-order. If $Oc(\mathcal{L})$ is less than the first measurable cardinal then $\mathcal{L}$ satisfies the compactness theorem iff the logic has the amalgamation property.*

Recently Xavier Caicedo improved the above theorem (See [**Cai**]):

8. Finite model theory

9. Non standard analysis

CHAPTER 12

A miniguide to the literature

This guide had three parts. In the first I present my favorite basic sources for elementary logic and algebra. In the second part I describe sources for for additional information for topics described in the book, and extensions. In the last part I discuss articles and books for further reading about topics discussed in the book and some that are not even mentioned.

This guide as the rest of the book are supposed to be viewed as reflecting my personal taste (while trying to be honest with the reader).

In case your favorite articles or books don't get the attention you think they deserve I hope you will not be too angry with me.

The basic material

(1) naive set theory – Hrbacek & Jech or Levi
(2) predicate calculus – Goldstern and Ihoda [**GJ**] maybe Monk [**M**]
(3) Undergraduate algebra (Herstein, Dummit and Foote), basic graduate algebra – Hungerford.

Further recommended reading

(1) Shelah's bible [**Sh c**]
(2) Pillay's [**Pi2**]
(3) Non elementary classes, Rami's new book [**Gr1**]
(4) Shelah's [**Sh 300**] Universal classes
(5) Shelah's nonstructure theory and black boxes - new book
(6) classification over a predicate Shelah - another new book.
(7) Keisler's infinitary logic [**Ke71**]
(8) van den Dries o-minimal book [**Dr**]
(9) Marker-Messmer-Pillay field theory book [**MMP**]
(10) Frank Wagner's [**Wa1**] and [**Wa2**]
(11) Lang-Mordel Elisabeth [**Bou**] and Anand's [**Pi2**]

CHAPTER 13

Open Problems

Introduction

INCOMPLETE!

1. Classification theory for non-elementary classes

Shelah's categoricity conjecture.

The simplest conjecture of this family is Conjecture **??**.**??** stated for $L_{\omega_1,\omega}$ in a countable language.

There are several natural extensions of this for abstract elementary classes and uncountable lanbguages:

CONJECTURE 1.1.

CONJECTURE 1.2.

Some more accessible special cases:

CONJECTURE 1.3. *Eliminate the successor assumption*

CONJECTURE 1.4.

CONJECTURE 1.5.

CONJECTURE 1.6. *Hanf number for amalgamation*

2. Shelah's categoricity conjecture

3. Main Gap for uncountable theories

Namely is a generalization of Theorem **??**.**??** true?

CONJECTURE 3.1. *For every first order T in a language of cardinality λ either*

(1) $\mathrm{I}(\aleph_\alpha, T) = 2^{\aleph_\alpha}$ *for all* $\aleph_\alpha > \lambda$ *or*

(2) $\mathrm{I}(\aleph_\alpha, T) < \beth_{\delta(\lambda)}(|\alpha| + \aleph_0)$.

It is reasonable to expect that Conjecture 3.1 will be deduced from the generalization of Theorem **??**:

CONJECTURE 3.2.

CONJECTURE 3.3. *nice categories without CH.*

CONJECTURE 3.4.

CONJECTURE 3.5.

4. Other problems

PROBLEM 4.1. *Let $T \subseteq L_{\omega_1,\omega}$ be a countable in a countable language. If T has a model of cardinality $\aleph_{\omega_1}$ then it has a model of cadinality $2^{\aleph_0}$.*

Explain connection to Borel hirarchy of subsetets of $\mathbb{R}$. The intuition and the reduction to a partition theorem on trees.

Include the two card theorem?

PROBLEM 4.2. *Gap-2 without morass?*

The next problem concerns uniqueness of prime models for simple (unstable) first-order theoreis:

PROBLEM 4.3. *Let T be simple and unstable. Is the following true: Let $A \subseteq \mathfrak{C}$, if there exists a prime model M containing A then for any prime model N over A we have $M \cong_A N$.*

Since forking calculus is well beahved in simple theories. If one attempts to follow the "standard approach" then in order to solve Problem **??** affirmatively it suffices to prove the following instance of the *open mapping theorem*:

PROBLEM 4.4. *For T simple. Suppose $A \subseteq B$ Suppose that $p \in \mathrm{S}(B)$ does not fork over A. If there exists $\varphi(\mathbf{x}; \mathbf{b})$ with parameters from B such that*

$$\varphi(\mathbf{x}; \mathbf{b}) \vdash p$$

then there exists $\psi(\mathbf{x}; \mathbf{a})$ with parameters from A such that

$$\psi(\mathbf{x}; \mathbf{a}) \vdash p \restriction A.$$

"Reverse mathematics"– simplicity requires ω_1-many iterations of powerset.

PROBLEM 4.5. *Is $\langle H(\chi), \in <^* \rangle$ sufficient for stable theories? When $\chi := (2^{2^{|T|}})^+$ and $<^*$ is a well-ordering of the universe.*

Is there perhaps a nice axiomatization of the sufficient proper fargment of ZFC *sufficient to develop the model theory of simplicity?*

What is a sufficient fragment of ZFC sufficient to develop classification theory?

Parhaps

PROBLEM 4.6. *$\langle H(\chi), \in <^* \rangle$ is enough when χ is regular and little bigger than the Hanf number? I.e. is $\beth_{(2^{2^{|T|}})^+}$ enough?*

Is it possible that the bound from Problem 4.6 is the "correct" framework to do mathematics? I.e. for non set-theoretic mathematics provable in ZFC theorems are deriveable already in the above fragment of ZFC.

The philosophical ramifications of this are dremendous, this seems to imply that classification theory (including non-elemntary classes) captures all of mathematics! I personally don't think that this is true but there is increasing evidence to believe in this.

PROBLEM 4.7 (Silver's dream). *In "mathematics" there is no need to use large cardinals above $\beth_{(2^{2^{|T|}})^+}$?*

DEFINITION 4.8. Let T be a complete first-order theory. For cardinaly $\lambda \geq \mu$ let

$$\lambda \xrightarrow{T} (\mu)^{<\omega}$$

stand for the statement for for every sequence $\mathbf{I} \subseteq |\mathfrak{C}|$ of cardinality λ ther exists $\mathbf{J} \subseteq \mathbf{I}$ an indiscernible sequence of cardianlity μ.

$$\lambda \xrightarrow{T} (\mu)^{<\omega}_{\kappa}$$

stands for for every $A, \mathbf{I} \subseteq \mathfrak{C}$ such that $|A| \leq \kappa$ and $|\mathbf{I}| \geq \lambda$ there exists $\mathbf{J} \subseteq \mathbf{I}$ an A-indiscernible sequence of cardinality μ.

Mention Rowbottom, Ramsey cardinals, and stability. Negative results

PROBLEM 4.9. *Find a reasonable class of unstable first-order theories such that*

$$\forall \mu \exists \lambda_\mu [\lambda_\mu \xrightarrow{T} (\mu)^{<\omega}].$$

This question is interesting since it provides a framework to discover a notion generalizing forking (or non-splitting) that is nicely behaved for some unstable thoeries.

Perhaps theories without the independence property is such a class. Mention Shelah's result concerning theories such that no monadic expansion has the indep prop.

PROBLEM 4.10. *Suppose T is without the independence property then*

$$\forall \mu \exists \lambda_\mu [\lambda_\mu \xrightarrow{T} (\mu)^{<\omega}].$$

CHAPTER 14

Historical comments

VERY INCOMPLETE!

Chapter I.

The subject of model theory was created by several people in the late forties and earliy fifthies. The main figures were Anatolii Mal'cev, Abraham Robinson and Alfred Tarski.

Bourbaki **??** - multi sorted structure, Bjarni Jonsson in his review (MR0097335) of Chapter IV (Structures) from 1957 wrote: "Due to the extreme generality, the definitions are rather cumbersome, and all the results derived are of a very trivial nature.". Overall Bourbaki is quite hostile and ignorant of mathematical logic, Adrian Mathias published a wonderful article titled "On Bourbaki's ignorance of Mathematical Logic" **??**.

Most experts agree that the very first result of model theory appeared in 1915:

THEOREM 0.1 (Löwenheim 1915). *Let T be a set of sentences in a countable language. If T has an infinite model then T has a countable model.*

Five years later another approximation to Theorem 1.3.20 appeared:

THEOREM 0.2 (Skolem 1920). *Let T be a set of first-order sentences in a countable language. If T has an infinite model M then T has a countable model N such that N is a submodel of M.*

It turns out that the above was not well known outside logic. Evidence to this can be found in [**Bae**].

The completeness theorem for propositional calculus goes back to 1920 E. L. Post, [**?**] contains a proof.

Gödel's completeness theorem (for theories in a finite or countable language) was proved in his 1929 PhD thesis and appeared in print in 1930 [**?**]. Gödel also discovered the compactness theorem.

Lindenbaum's theorem 2.22 (according to Tarski 1930) was proved by Lindenbaum for propositional calculus, but was never published by him. Already in 1930 Tarski used the same argument to show that every ultrafilter can be extended to an ultrafilter. It turns out that (perhaps independently) the notion of ultrafilter was introduced by Birkhoff and Cartan in the mid thirties. Often people attribute the fact that filter can be extended to Bourbaki, however as Tarski published that result already in 1930 it is clear that Cartan and Bourbaki were not the first.

Note that the modern proof of Gödel's completeness theorem uses Lindebaum's theorem, however as the original version dealt with countable theories the proof didn't use the Axiom of Choice as an inductive construction gives a ZF proof.

A non-standard model for true arithmetic was constructed by Skolem in 1934 ([**Sk33**], [**Sk34**]) without use of the compactness theorem. His construction did not use the compactness theorem but instead used a *definable* form of the ultrapower construction that depended on the rich definability features of arithmetic.

In an article by William Aspray [**Asp**] I found the following remraks: According to Veblen the term "categorical" was introduced by John Dewey and David Hilbert (1899) and Huntington (1902) had previously investigted categoricity of systems. Corcoran (1980, 1981) points out that Veblen infers completetness from categoricity without an argument and apparently does not distinguish between the two concepts.The notion of categoricity was introduced by Huntington at the beginning of the 20th century.

The compactness theorem for propositional calculus with uncountable many symbols appears in a paper by Anatolii Mal'cev [**?**] in 1936. It seems that Mal'cev did not know of Gödel's work. While in that paper he did not state the compactness for first-order logic, he had the following

THEOREM 0.3 (Mal'cev 1936). *Let T be a set of first-order sentences. If T has an infinite model M than M has a proper extension which is also a model of T.*

Mal'cev also introduced the method of elementary diagrams to construct extensions of given models.

In 1941 Mal'cev published [**?**] containing the modern formulation of the compactness theorem for first-order logic for languages containing arbitrarily many symbols. That paper contained several applications of the compactness theorem to algebra, primarily to group theory.

Unfortunately due to world war II this paper was not widely known until Henkin [**He**] and Robinson [**?**] rediscovered and reorganized some of Mal'cev's results. In 1959 Henkin and Mostowski[**He Mo**] published a review of Mal'cev's important 1941 paper.

During the academic years 1926-1928 Tarski held a seminar at Warsaw University on elimination of quantifiers. The method originated by Löwenheim [**Lo15**] and fully developed by Skolem [**?**], and C. H. Lanford [**La26**] who proved elimination of quantifiers for T_{ord}. In 1928 in his seminar Tarksi presented a proof of

THEOREM 0.4. *Let T be a set of first-order sentences in a countable language. If T has an infinite model M then T has an uncountable model*

This was couple of years before Gödel's compactness theorem was discovered. The first to publish the above result was Mal'cev (1936). Unfortunately when asked many years later Tarski was not able to reconstruct his original proof (Vaught **??**).

Tarski was not happy with the precision of various results obtained in his seminar since there was not a precise notion of truth. The modern notion of satisfaction was introduced by Tarski in his paper from 1936 [**Ta36**].

The fact that every consistent set of propositional sentences can be extended to a complete set of sentences is a result of Adolf Lindembaum who did not publish it, the result appeared in paper by Tarski's from 1930 who in another paper from the same year [**Ta30**] stated and proved the related fact that every filter can be extended to an ultrafilter.

INCLUDE LETTER of recommendation FROM ALONZO CHURCH?

Sections:Structures and languages: Example 1.2.2.64 is due to Morley (unpublished) from the early sixties.

Example 1.2.2.65 is also unpublished found by Ralph Robinson in the early sixties. Both examples were introduced in the analysis of two-cardinal transfer theorems.

The recognition that the characterizations of the Cantor's set and a Polish space via equivalence relations in Examples 1.2.2.58 and 1.2.2.59 are due to Shelah.

Sections:On existence of models and elementary submodels: The history of the compactness theorem (Theorem 3.3) is complicated. The theorem for countable theories was proved by Gödel in 1929, for uncountable theories by Mal'cev in 1936 and the modern proof is due to Leo Henkin and is dated 1949.

Henkin, Rasiowa ... See pages 163–167 in [**Dau**].

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Section 4 (The Erdős-Rado Theorem): Ramsey's theorem appeared in his 1930 paper [**Ra30**] titled "On a problem of formal logic" while it was the begining of the branch of combinatorics called Ramsey's theory it was also one of the first important papers of model theory. While Ramsey's paper had the word

"logic" in its title, the value of Ramsey's theorem to logicians was recognized not before the publication of Andrzej Ehrenfeucht and Andrzej Mostowski paper only a quarter of century later [**EM**].

In the thirties several people continued and extended Ramsey's work. Seirpiński's example is from [**Se33**]. Kurepa published several papers that did not get much attention at the time. The most influencial papers in the field were papers coauthored by Paul Erdős and Richard Rado and some other papers also with András Hajnal. The paper containing the theorem known as the Erdő-Rado theorem appared in [**ErRa56**]. It turns out that Kurepa published the statement and a proof for the ER theorem before ER (stated under GCH and he used alephs instead of beths. In the definitive exposition of partition calculus [**EHR65**], the authors in the first page stated this fact and gave full credit to Kurepa. Logicians to this day continue referring to the result as the Erdos-Rado theorem.

The first time the Erdős-Rado theorem was used in model theory was when Michael Morley proved his omitting-types theorem. In May 2004 Morley told me that he did not know of the Erdős-Rado theorem but proved a necessary combinatorial lemma, he was told later that he rediscovered the Erdős-RadoTheorem.

Erdős and Rado for their proof used a generalization of one of the standard arguments to prove the infinite Ramsey's theorem (based on an application of König's Lemma to find an infinite branch in a tree). They introduced a device called *partition trees* and under certain cardinal arithmetic assumptions derived that the tree has an infinite branch.

The elegant model theoretic proof the section was discovered by Steve Simpson [**Si70**]. This technique was exploited later mainly by Shelah to obtain several new results in combinatorial set theory including the solution of several hard problems of Erdős.

Arhangelski's solution used the original tree-ramification Lemma, in the exercise I have outlined an argument patterned after Simpson's proof, which is essentially a downward Löwenhweim-Skolem theorem for the logic $L_{\aleph_1,\aleph_1}$.

The idea of relative saturation, Shelah's improvements (see also Section 10)

Section 9 12:

The important idea (Example 12.4(3))of coding the power-set of a predicate via an $\in$-relation satisfying the extensionality axiom (which was later used by Morley to establish lower bounds for Hanf number and Shelah, Grossberg & VanDIeren to get independence results in AECs), is attributed by [**?**] to Ralph Robinson.

Vaught's two cardinal theorem **??** was announced by Vaught in 1961 (see **??**), the proof appeared in the important paper Morley Vaught paper **??** 1962, Chang's conjecture was mentioned first in Vaught 1963. Keisler's two cardinal theorem if from Keisler 1966.

Sections:A class: Amalgamation , JMP

Sections:AEC: Theorem 2.7 in its present form is from Shelah [**?**].

The earliest version of this theorem is a special case for lattices was established by Tsurane Iwamura in 1944 (see [**Iw**]). Very similar statements to Theorem 2.7 appear as Exercises in Borbaki;s Set Theory (Exercise) and also an Exercise in R. Cohn's book on universal algebra and as Theorem 21.5 from Gätzer's book. In October 2008 I asked Gätzer if he knows the precise history of this argument and he answered that it was "folklore" among universal algebraists when e wrote his book in 1968.

Sections: Some extensions of first-order logic: Ernest Zermelo (see [**Ze**]), Scott, Carol Karp 1965, Keisler, Hanf-Morley. Basic reference: M. A. Dickmann's [**Di**].

Sections:Ultraproducts: The fact that every filter can be extended to an ultrafilter – Lemma 2.10 is due to Tarski. Earlier Lindenbaum (a student of Tarski) proved that in what is called now the Lindenbaum Boolean Algebra of a theory every filter has a maximal extension. Often Lemma 2.10 is called the Lindenbaum-Tarski theorem.

Theorem 2.27 and the related lemmas appeared in [**EK**]

Sections:Models of weak set theory: Jensen, large cardinals, Shelah non structure and proper forcing

Sections:Ehernfeucht-Fraisse games: Fra¨issé introduced in 1954 [**Fre**] and Ehernefecht independently later [**Eh61**].

Internal sequence - Solovay's proof of existence of Kurepa trees in **L**.

Theorem **??** is due to several people: The equivalence of the compactness theorem to the Boolean Prime Ideal theorem is due to Henkin [**?**]. It was John Kelly [**?**] who realized that the full Tychonov theorem implies the axiom of choice. Ł⁄oś and Ryll-Nardzewski (1955) independenlty from Also Rubin and Scott proved the equivalence of BPI and Tychonov's theorem for Hausdorff spaces.

Conjecture 2.4.1 was introduced by me in my MSc thesis and published in [**?**]. In October 1994 I managed to convince Shelah of its importance, After I rejected his couple of set-theoretic attempts (using Kurepa trees) he realized its value and his [**?**], [**?**] and [**?**] are partial progress towards the original problem (eventually published in volume I of his AEC book)

Chapter II.

It was Jerzy Loś who used the perfect set argument to prove Theorem 2.7.24. A precursor of this result was Mostowski's theorem [**?**] using results of

Tarski, Stone and earlier results in topology to show that the number of completion of a countable theory if is uncountable then it has cardinality $2^{\aleph_0}$. Loś also begun investigating the connection between the set of types and the spectrum function and categoricity in particular.

Theorem **??**.**??**.**??** is from Morley and Vaught's paper 1961, based on Jónsson 1960.

Some of the early work with types was motivated by the quest to understand countable models of a first-order theory.

The most fundamental fact concerning countable models is Henkin's omitting types theorem. This theorem found several important applications for uncountable models and still influence current research.

One of the early applications past the characterization of $\aleph_0$-categoricity Theorem 2.9.38, was Vaught's two cardinal Theorem 12.9 appeared in [**Va61**]. The same volume contains also MacDowel and Specker's theorem [**MacSp61**]

Theorem 3.45 is due to Erdős and Rado dated 1956, the proof is taken from the argument for Theorem 12.1 from [**EHMR**].

Aronszjan's existence theorem (Theorem 2.2.68) appeared in Kurepa's paper [**?**] the modern proof is due to Ernest Specker [**Sp49**].

THEOREM 0.5. *Every countable $M \models \mathrm{PA}$ has a proper elementary N end-extension. [i.e. for every $a \in |M|$ and $b \in |N|$ if $N \models [b < a]$ then $b \in |M|$.]*

This theorem was a precursor for several much stronger theorems. One of the strongest theorems in this family is Theorem **??**. Instead of going over each of this theorems separately I made an effort to emphasize this uniformity and I have derived vaghut's theorem as well as the much stronger Keisler's two cardinal theorem from Theorem **??**. I find it surprising that people did not realize that the proof of 0.5 actually implies almost all theorems.

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:
Sections:More on omitting types: Morley ...

The most basic theorem that supplies models of arbitrary large cardinality omitting types was discovered by Morley [**Mo4**] (who also discovered the Erdoős-Rado theorem on the fly). Corollary 7.42 is a generalization of Morley's original result. Our treatment follows that of C.C. Chang ([**?**] 1968), Barwise and Kunen ([**BaKu**]) that emphasizes EC classes instead of infinitary logics. Note that via Chang's trick this is more general (see **??**.**??**).

Theorem **??** was discovered by Schmerl and Shelah and it was known to Morley already in the sixties who did not publish it.

Chapter III.

Morley's theorem appeared in the University of Chicago 1962 Ph.D. thesis of Michael Morely. It is interesting to note that Morley was not in Chicago when he did his important work. But was visit Vaught in Berkeley for an extended visit (which also produced their paper [**?**]). It turns out that when Saunders MacLane heard of Morley's theorem he did not think that it warrants' a Ph.D. only after pressure from Vaught who threatened him that if Chicago will not Morley a PhD for this work Berkeley will MacLane accepted the dissertation. The concepts of totally transcendental theory, ordinal-valued rank function to analyze types and the concept of prime model over a set were originated there.

Exercise 3.4.6 is credited by Chang and Keisler to Erhrenfeucht and Mostowski 1956 [**EM**].

:

:

Indiscernibles and Ehrenfeucht-Mostowski models: According to Erhrenfeucht and Mostowski the original motivation was the following special case of Question 3.3:

> QUESTION 0.6 (Gisbert Hasenjaeger[1]). Do models of arithmetic admit non-trivial automorphisms?

Chapter IV.

The notion of stability was introduced in the sixties by Frank Rowbottom in an unpublished work. To the best of my knowledge the first publication dealing with stability is Saharon Shelah's article [] based on his 1969 Ph.D. thesis.

Shelah told me that already around 1968 he suspected that the order property follows from instability (the converse is trivial). Shelah spent considerable amount of time searching the literature of combinatorial set theory for a device he could use for the proof. After several years of search he discovered the paper "Some remarks on set theory part X" by Erdős and Makkai from 1965 (see [**EM**]) a proof using this combinatorial theorem appeared in [**Sh16**]. Our proof of Lemma **??**.5.5.15 is different than Shelah's original, it is a simplification of his second proof from [**Sh a**].

Chapter V.

[1]While some logician know that Gisbert Hasenjaeger (1919-2006) spent most of his professional career at the University of Bonn; less well known is that during World War II he worked for the coding department of the german armed forces (OKW/Chi), his main accomplishment was to identify flaws and break the code of the Enigma machine, only 35 years later he found out that Alan Turing did the same in England [**Schmeh 2009**].

Chapter VI.

Differentially closed fields where originally defined as follows: F is a differentially closed field iff for any finite system of ODEs Γ if threr is a differential field E extending F containing a solution for Γ then F has a solution for Γ. Originally Abraham Robinson proposed a first-order axiomatization for DCF, Lenore Blum found a flaw in that axiomatization and corrected it. Definition **??** and Theorem **??** are from her 1968 PhD thesis.

Theorem 7.6 is due to Shelah was included in his 1969 PhD thesis [**Sh 0**].

Chapter VIII.

In his ground-breaking paper [**Sh3**], Shelah gave the first description to the stability spectrum of T. He characterized the class of cardinals $\lambda \geq 2^{|T|}$ such that T is stable in λ. For the combinatorial analysis of models involved, he devised an intricate tool which he called *strong splitting*. Later, in order to describe the full stability spectrum (*i.e.*, include the cardinals $\lambda < 2^{|T|}$ such that T is stable in λ), he refined the concept of strong splitting, and introduced the fundamental concept of *forking*.

Between the early 1970's and 1978, Shelah concentrated his efforts in model theory to the completion of his first book [**Sh c**]. The complete description of the stability spectrum of T is given in Section III-5. Shelah, however, realized quickly that the range of applicability of the concept of forking extends well beyond the realm of the spectrum problem.

In 1977, Daniel Lascar and Bruno Poizat published [**LaPo**] an alternative approach to forking which appeared to be more understandable than Shelah's. They replaced the original "combinatorial" definition with one closely related to Shelah's notion of *semidefinability* from Chapter VII of [**Sh c**]. The approach of Lascar and Poizat had a remarkable impact on the dissemination of the concept of forking in the logical community. Several influential publications, such as the books of Anand Pillay [**Pi1**] and Daniel Lascar [**La3**] and the papers of Victor Harnik and Leo Harrington [**HH**] and Michael Makkai [**?**], adopted the French approach and avoided Shelah's definition of forking. Both of these approaches were presented in John Baldwin's book [**Ba88**].

Parallel to these events, Shelah isolated a class of first-order theories which extends that of stable theories, the class of *simple theories*. Originally he named simple theories as *theories without the tree property* (echoing Theorem **??**), unfortunately he changed is mind and used the non descriptive term simple theories. This concept originated in the study of yet another property of theories motivated by combinatorial set theory, namely,

$(\lambda, \kappa) \in \mathrm{SP}(T)$: Every model of T of cardinality λ has a κ-saturated elementary extension of cardinality λ.

For stable T, the class of pairs λ, κ such that $(\lambda, \kappa) \in \mathrm{SP}(T)$ had been completely identified in Chapter VIII-4 of [**?**] (using the stability spectrum theorem and some combinatorial set theory). Notice that the question when $\kappa = \lambda$ is equivalent to the existence of saturated model of cardinality λ^2. Some of the basic facts about existence of saturated elementary extensions are stated as Fact **??**.

Chapter XIV.

The terminology of *quasminimal* is from Zilber's [**Zi**]. It is interesting to notice that the same concept appeared 27 years earlier in Shelah's [**Sh48**].

Chapter XIV.

Shelah in section 4 of Chapter XII of [**?**] used the notion (λ, n) *Existence property* to denote essentially what in [**Sh87b**] he called (λ, n) *Goodness property*. Due to the level of current renewed interest in Excellent classes I decided to use the older (and clearer) terminology.

[2]Suppose $N \models T$ is saturated of cardinality λ and let $M \models T$ be a given model of cardinality λ, since saturated models are universal there exists $N' \succ M$ saturated and isomorphic to N.

APPENDIX

CHAPTER 15

Some set theory

Introduction

This chapter was not written for an independent reading. It is here for the reader to consult while reading the book. However the reader is encouraged to complete reading the first 4 sections.

While the first two sections are quit easy, I state some basic facts from set theory (often without proof) that usually are covered in undergraduate courses in set theory or topology. Section 3 introduces Martin's Axiom and includes some simple comments that are used only at one stop in the book (see Theorem **??**). The contents of section four (dedicated to several weak Diamond principles) is quite difficult, it contains facts not previously published and I advice you to postpone its reading until the principles are used. In section five I introduce a little known but important theorem of combinatorial set theory, namely Hajnal's free subset theorem that used only at one place in the text. I don't recommend reading this before the actual application.

1. Sets, functions and relations

When A is a set the *power set of* A is the collection of all subsets of A denoted by $\mathcal{P}(A)$. Namely,

$$\mathcal{P}(A) := \{S \mid S \subseteq A\}.$$

When $A := \{1, 2, 3\}$ then $\mathcal{P}(A)$ is the following 8-element set:

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Denote by $\mathcal{P}^-(A)$ the set $\mathcal{P}(A) - \{A\}$. In the above example $\mathcal{P}^-(A)$ is $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$

When A and B are sets, for $a \in A$ and $b \in B$ the symbol $\langle a, b\rangle$ stands for the set $\{\{a\}, \{a, b\}\}$. It is easy to check that $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$ implies that $a_1 = a_2$ and $b_1 = b_2$. The set $\langle a, b\rangle$ is called the *ordered pair of* a *and* b.

By induction on $2 \leq n < \omega$ we define $\langle a_1, \ldots, a_{n+1}\rangle := \langle\langle a_1, \ldots, a_n\rangle, a_{n+1}\rangle$.

Let $A \times B := \{\langle a, b\rangle \mid a \in A, b \in B\}$.

REMARK 1.1. Notice that the notion of cartesian product allows us to perform the following "trick": Given sets A and B we can replace them by pairwise disjoint A' and B' such that $|A'| = |A|$ and $|B'| = |B|$.

Define A' to be $\{1\} \times A$ and let B' to be $\{2\} \times B$. Check that the sets A' and B' are as required.

When A and B are sets a function f from A to B is denoted by $f : A \to B$ which means that f is a subset of $A \times B$ with the following properties: *the domain of* f is A, namely for every $a \in A$ there exists $b \in B$ such that $(a, b) \in f$ and f *is single valued* which means that for all $a \in A$ and every $b, b' \in B$ we have that $(a, b) \in f \wedge (a, b') \in f$ implies that $b = b'$.

When $f : A \to B$ and $Y \subseteq B$ then $f^{-1}[Y] := \{a \in A \mid f(a) \in Y\}$.

When $R \subseteq A \times B$ often instead of writing $(a, b) \in R$ we will write aRb.

DEFINITION 1.2 (Transitive closure of a relation). Let A be a set, suppose $\sim \subseteq A \times A$ is a reflexive (for every $a \in A$ we have that $a \sim s$) and symmetric relation (for every $a \sim b$ we have that $b \sim a$). The relation $\sim^* \subseteq A \times A$ is called *the transitive closure of the relation* $\sim$ provided

$$a \sim^* b \iff \text{there are } 1 \leq n < \omega, \{a_i \mid i \leq n\} \subseteq A \text{ such that } a = a_0, b = a_n \text{ and } a_k \sim a_{k+1} \text{ for all } k < n.$$

It is not difficult to see that given a symmetric and reflexive relation R on a set A the relation $\sim^*$ is an equivalence relation on A and is the minimal (with respect to set theoretic inclusion) such that $\sim^* \supseteq \sim$.

2. Cardinal numbers

[1]

Cantor in his monumental 1874 paper introduced the following definition:

DEFINITION 2.1. The sets A and B are said to to be *equinumeouos* (or *have the same cardinality* iff there exists a bijection $f : A \to B$. The statement

$$\exists f[f : A \overset{1-1}{\underset{\text{onto}}{\to}} B]$$

is denoted by $|A| = |B|$. The statement "there exists an injective function from A to B" is denoted by $|A| \leq |B|$.

We write $|A| < |B|$ for the statement:

There is an injective mapping $f : A \to B$ and $|A| \neq |B|$.

REMARK 2.2. Note that the above definitions (unlike in some modern texts) we do not depend or require the axiom of choice.

A set S is *finite* iff there exists a natural number n such that $|S| = |\{1, 2, \ldots, n\}|$. If a set is not finite we say it is infinite. A set S is *countable* iff there is a bijection from S onto the natural numbers. This is denoted by $|S| = \aleph_0$.

DEFINITION 2.3. Set A and B have *the same cardinality* and called *equinomoruos* provided there exists a bijection $f : A \to B$. We denote this by writing $|A| = |B|$.

For a set S the *power set of* S is $\{X \mid X \subseteq S\}$ and is denoted by $\mathcal{P}(S)$.

THEOREM 2.4 (Cantor's theorem). *For any set A we have that $|A| < |\mathcal{P}(A)|$. Moreover, there is no surjective function from A onto $\mathcal{P}(A)$.*

PROOF. For the sake of contradiction suppose that there exists a surjective $f : A \to \mathcal{P}(A)$. Consider

$$D := \{a \in A \mid a \notin f(a)\}.$$

As $D \in \mathcal{P}(A)$ and f is surjective there exists $a_0 \in A$ such that $f(a_0) = D$.

Check that $a_0 \in D \iff a_0 \notin D$ which is a contradiction. □

THEOREM 2.5 (Cantor Bernstein[2]). *For any pair os sets A and B , if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.*

PROOF. By the assumption there are injective functions $f : A \to B$ and $g : B \to A$.

[1]The results of this section are proved in ZF (the Axiom of Choice is not used here).

[2]Cantor's original proof used the Axiom of Choice. Bernestein was the first to find a proof that avoids usage of the AC.

Let $B' := g[B]$ and $A_1 := f[B']$. Note that $A_1 \subseteq B' \subseteq A$ and that $h := g \circ f$ is a bijection from A onto A_1. Since $|B'| = |B|$ it is enough to verify the following

LEMMA 2.6. *Suppose $A_1 \subseteq B \subseteq A$. If there exists a bijection $f : A \to A_1$ then there exist a bijection $g : A \to B$.*

PROOF. By induction on $n < \omega$ define $\{A_n \mid n < \omega\}$ and $\{B_n \mid n < \omega\}$:

$A_0 := A$ and $B_0 := B$.

$A_{n+1} := f[A_n]$ and $B_{n+1} := f[B_n]$.

Consider

$$g(x) := \begin{cases} f(x) & \text{if there exists } n < \omega \text{ such that } x \in A_n - B_n \\ \\ x & \text{Otherwise.} \end{cases}$$

Verify that g is a bijection as required. □

Denote by the letters λ, μ, κ cardinal numbers. There are three natural operations one can define on cardinal numbers:

DEFINITION 2.7. Let $\lambda,\ \mu$ and κ be cardinal numbers and let A, B and C be sets such that $\lambda = |A|, \mu = |B|$ and $\kappa = |C|$.

(1) $\lambda + \mu := |\{0\} \times A \cup \{1\} \times B|$.
(2) $\lambda \cdot \mu := |A \times B|$.
(3) $\lambda^\mu := |A^B|$, i.e. it is the cardinality of the set of functions from B into A.

It is easy to verify that the above definition does not depend on choice of representatives. Direct application of the definition (by constructing the appropriate bijections) gives:

PROPOSITION 2.8 (Basic cardinal arithmetic rules). *Let $\lambda,\ \mu$ and κ be cardinal numbers.*

(1) $\lambda + \mu = \mu + \lambda$.
(2) $\lambda \cdot \mu = \mu \cdot \lambda$.
(3) $\lambda + (\mu + \kappa) = (\lambda + \mu) + \kappa$.
(4) $\lambda \cdot (\mu \cdot \kappa) = (\lambda \cdot \mu) \cdot \kappa$.
(5) $\lambda \cdot (\mu + \kappa) = \lambda \cdot \mu + \lambda \cdot \kappa$.
(6) $\lambda^\mu \cdot \lambda^\kappa = \lambda^{\mu+\kappa}$.
(7) $(\lambda^\mu)^\kappa = \lambda^{\mu \cdot \kappa}$.
(8) $(\lambda + \mu)^2 = \lambda^2 + 2 \cdot \lambda \cdot \mu + \mu^2$.

PROOF. The assertions (1)-(7) can be verified by an explicit construction of a bijection for the appropriate sets using Definition 2.2.

(5) Let A be a set of cardinality λ, B and C pairwise disjoint sets of cardinality (respectively) μ and κ. Consider

$$f(\langle x, y\rangle) := \langle x, y\rangle \quad \text{for } x \in A \text{ and } y \in B \cup C.$$

Verify that f is a bijection from $A \times (B \cup C)$ onto $A \times B \cup A \times C$.

(6) Suppose A, B, C and λ, μ, κ are as above. Verify that $g(\langle f_1, f_2\rangle) := h$ is a bijection from $A^B \times A^C$ onto $A^{B\cup C}$, when

$$h(x) := \begin{cases} f_1(x) & \text{if } x \in B \\ \\ f_2(x) & x \in C. \end{cases}$$

(8) Use the previous algebraic identities. □

By constructing the appropriate injections one can establish the following

PROPOSITION 2.9 (Monotonicity). *Suppose $\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$ then*

(1) $\lambda_1 + \mu_1 \leq \lambda_2 + \mu_2$.
(2) $\lambda_1 \cdot \mu_1 \leq \lambda_2 \cdot \mu_2$.
(3) $\lambda_1^{\mu_1} \leq \lambda_2^{\mu_2}$.
(4) *if we assume in addition that $\lambda_1 + \lambda_1 = \lambda_1 \geq \mu_1$ and $\mu_2 \geq 2$ (to avoid trivial counter examples) then we have $\lambda_1 + \mu_1 \leq \lambda_2 \cdot \mu_2$.*
(5) *If $\lambda \cdot \lambda = \lambda$ then $\lambda + \lambda = \lambda$.*

PROOF. (1),(2) and (3) by constructing the obvious injections, similarly to the previous Proposition.

(4) $\lambda_1 + \mu_1 \leq \lambda_1 + \lambda_1 \leq \lambda_1 \cdot 2 \leq \lambda_2 \cdot \mu_2$.

(5) Apply Cantor Bernestein to $\lambda \leq \lambda + \lambda \leq \lambda \cdot \lambda = \lambda$. □

In section 4 using the AC we will show that $\lambda = \lambda \cdot \lambda$ for all infinite λ. Here we verify that identity holds for many cardinals without using AC.

PROPOSITION 2.10. (1) *There is a bijection from $\omega \times \omega$ onto ω.*
(2) $\aleph_0 + \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0$.
(3) $\Sigma_{n<\omega} \aleph_0^n = \aleph_0$.

PROOF. (1) Verify that $f(n, k) := 2^n \cdot (2k + 1) - 1$ is a bijection from $\omega \times \omega$ onto ω.

(2) Monotonicity and usage of (1) gives:

$$\aleph_0 \leq \aleph_0 + \aleph_0 \leq \aleph_0 \cdot \aleph_0 = \aleph_0.$$

An application of the Cantor-Berenstein Theorem to the last inequality gives the assertion.

(3) Using (2) to verify:

$$\aleph_0 \leq \Sigma_{n<\omega}\aleph_0^n \leq \aleph_0 \cdot \aleph_0 = \aleph_0.$$

An application of the Cantor-Berenstein Theorem to the last inequality gives the assertion.

□

COROLLARY 2.11. *For λ satisfying $\lambda = \lambda \cdot \lambda$ we have*

$2^\lambda = \lambda^\lambda$. *In particular $2^{\aleph_0} = \aleph_0^{\aleph_0}$ holds.*

PROOF. By monotonicity and basic cardinal arithmetic rules and the assumption on λ we have

$$2^\lambda \leq \lambda^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda.$$

An application of he Cantor Bernstein theorem to

$2^\lambda \leq \lambda^\lambda \leq 2^\lambda$ gives the desired equality.

□

PROPOSITION 2.12. *The cardinality of the set of real numbers $\mathbb{R}$ is $2^{\aleph_0}$.*

PROOF. Apply the Cantor Bernestein Theorem to

$$2^{\aleph_0} = |\,{}^\omega 2\,| \leq |\mathbb{R}| \leq |\,{}^\omega\omega\,| = \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$

□

PROPOSITION 2.13. *Let A be a set and λ its cardinality. The cardinality of $\mathcal{P}(A)$ is 2^λ.*

PROOF. For any $S \subseteq A$ consider $\chi_S : A \to \{0,1\}$ defined by:

$$\chi_S(x) := \begin{cases} 1 & \text{if } x \in A \\ \\ 0 & \text{Otherwise.} \end{cases}$$

Verify that the mapping $S \mapsto \chi_S$ is a bijection from $\mathcal{P}(A)$ onto $\{0,1\}^A$. □

Combining the last proposition with Cantor's Theorem (2.4) gives

COROLLARY 2.14. $2^\lambda > \lambda$ *for all* λ.

From the last Corollary and Proposition 2.12 we get by substituting $\lambda := \aleph_0$:

COROLLARY 2.15. *The set of real numbers is uncountable.*

DEFINITION 2.16. Let A be a set.

A is called *finite* if there exists an injection $f : A \to \omega$ such that there is $n \in \omega$ satisying $k > n \implies k \notin f[A]$ for every $k < \omega$.

If there exist a bijection between the set A and the set of natural numbers we say that A is *countable*, denote this by writing $|A| = \aleph_0$.

If there is an injective mapping from A into the integers we say that A is *at most countable*, this is denoted by writing $|A| \leq \aleph_0$.

An infinite set that is not countable is called *uncountable set*

For reasons that will become clear soon logicians like to denote the set of natural numbers (it includes zero) by the greek letter ω.

FACT 2.17 (Elementary facts).

(1) *Thus the set of integers $\mathbb{Z}$ and the set rational numbers $\mathbb{Q}$ are both countable. Thus by induction on $n < \omega$ one can show that $\aleph_0^n = \aleph_0$.*
(2) $|\mathbb{Q}[x]| = \aleph_0$.
(3) *Let A be a set of cardinality λ.*
$$2^\lambda = |\mathcal{P}(A)|.$$
(4) *There is a bijection between the set of real numbers $\mathbb{R}$ and $\mathcal{P}(\omega)$ the cardinality of $\mathbb{R}$ is $2^{\aleph_0}$ and called the* continuum.
(5) *For every cardinal number λ satisfying $\lambda \cdot \lambda = \lambda$ we have that $2^\lambda \cdot 2^\lambda \cdot = 2^\lambda \cdot$ in particular, $2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$. Thus for every positive integer n $|\mathbb{R}^n| = 2^{\aleph_0}$.*
(6) *Suppose A is a set of cardinality λ and λ is satisfying $\lambda \cdot \lambda = \lambda$, denote by ${}^{\omega>}\mathrm{A}$ the set of finite sequences of elements of A. Then $|\,{}^{\omega>}\mathrm{A}\,| = \lambda$.*

PROOF. (1) The function $f(n, k) := 2^n(2m + 1) - 1$ is a bijection.
(2)
(3)
(4)
(5) $2^\lambda \cdot 2^\lambda = 2^{\lambda \cdot \lambda} = 2^\lambda$.
(6)

□

One of the basic results of elementary set theory is is intuitively a natural statement that after some thinking needs a proof.

FACT 2.18 (Cantor-Bernestien's theorem). *For sets A and B*

$$|A| \leq |B| \text{ and } |B| \leq |A| \implies |A| = |B|.$$

An application:

PROPOSITION 2.19. *Let λ be an infinite cardinal number such that $\lambda \cdot \lambda = \lambda$. Then $\lambda^\lambda = 2^\lambda$. In particular $\aleph_0^{\aleph_0} = 2^{\aleph_0}$.*

PROOF. By the repeated application of different monotonicity properties of cardinal arithmetic we get:

$$2^\lambda \leq \lambda^\lambda \leq (2^\lambda)^\lambda = 2^{(\lambda \cdot \lambda)} = 2^\lambda.$$

An application of the Cantor-Bernstein theorem to the previous inequality proves the theorem. □

3. The Axiom of Choice, Zorn's Lemma and the well-ordering theorem

The *Axiom of Choice* is denoted by AC , is the statement that for any family of sets F there exists a fuction $g : F \to \cup F$ satisfying $g(s) \in s$ for every $s \in F$. A function g satisfying the above is called a *choice function for* F.

A set P with a binary relation $\leq$ on P is called a *partially ordered set* or poset for short. Provided we have

$$\langle P, \leq \rangle \models \forall x \forall y [x \leq y \wedge y \leq x \to x = y]$$

and

$$\langle P, \leq \rangle \models \forall x \forall y \forall z [x \leq y \wedge y \leq z \to x \leq z].$$

A poset $\langle P, \leq \rangle$ is *linearly ordered* provided

$$\langle P, \leq \rangle \models \forall x \forall y [x \leq y \vee y \leq x].$$

Let $\langle P, \leq \rangle$ be a poset. A subset C of P is a *chain* iff $\langle C, \leq \rangle$ is linearly ordered (i.e. for any $x, y \in C$ we have either $x \leq y$ or $y \leq x$.

Let $\langle P, \leq \rangle$ be a poset $S \subseteq P$ and $b \in P$. The element b is an *upper bound*of S provided $\langle P, \leq \rangle \models [a \leq b]$ for every $a \in S$.

Let $\langle P, \leq \rangle$ be a poset. An element $m \in P$ is *maximal* provided there is no $a \in P$ such that $\langle P, \leq \rangle \models [a \geq m] \wedge [a \neq m]$.

Let $\langle P, \leq \rangle$ be a poset. An element $m \in P$ is *minimal* provided $\langle P, \leq \rangle \models [m \leq a]$ for every $a \in P$. Alternatively, $m \in S$ is a minimal element of S provided $\forall x [x \in P \wedge x < m \to x \notin S]$.

A poset $\langle P, \leq \rangle$ is *well ordered* provided it is linearly ordered and for every non empty $S \subseteq P$ there exists $m \in S$ minimal in S.

Denote by *WOP* the well-ordering principle. Namely the statement: For every non-empty set P there exists a binary relation $\leq$ on P such that $\langle P, \leq \rangle$ is well-ordered.

THEOREM 3.1 (The well-ordering theorem, Zermelo 1908). $AC \implies WOP$.

REMARK 3.2. It is relatively easy to show the converse, namely $WOP \implies AC$.

The statement known by the name Zorn's Lemma, denoted by ZL is the following: Let $\langle , P, \leq \rangle$ be a partially order set. If every chain has an upper bound in P then P has a maximal element.

Kuratowski (around 1922) was the first to prove

THEOREM 3.3. $AC \iff ZL$.

Thus we have that

$$ZF \vdash AC \iff WOP \iff ZL.$$

4. Ordinals

DEFINITION 4.1. Let λ and κ be cardinal numbers.

$$\lambda^{<\kappa} := \sum_{\mu<\kappa} \lambda^{\mu}.$$

REMARK 4.2. Without using the axiom of choice it is not clear that all cardinals satisfy $\lambda \cdot \lambda = \lambda$ in addition to what stated above. TARSKI's THEOREM and remark on GERSHON SAGEEV

DEFINITION 4.3.

(1) AÊset P with a binary relation $\leq \subseteq P \times P$ is called *partially ordered set* POSET for short iff $x \leq y \wedge y \leq z \implies x \leq z$ for all $x, y, z \in P$.
(2) A POSET $\langle P, \leq \rangle$ is *linearly ordered* iff $\forall x \forall y [\mathbf{x} \leq y \vee y \leq x]$.
(3) Let $\langle P, \leq \rangle$ be a POSET. $C \subseteq P$ is a *chain* iff $\langle C, \leq \rangle$ is linearly ordered.

DEFINITION 4.4. A poset $\langle P, < \rangle$ is *well ordered* iff for every non empty $S \subseteq P$ there exists $m \in S$ which is *minimal* i.e.

$$p < m \implies p \notin S \text{ for every } p \in P.$$

DEFINITION 4.5. An *ordinal number* is a set α such that $\langle \alpha, \in \rangle$ is well ordered and for every $\beta \in \alpha$ we have that $\gamma \in \beta \implies \gamma \in \alpha$

The following are examples of ordinals: $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}, \emptyset\}$, $\{\{\{\emptyset\}, \emptyset\}, \emptyset\}$, $\{\{\{\{\emptyset\}, \emptyset\}, \emptyset\}, \emptyset\}$.

By induction on natural numbers define: $\underline{0} := \{\emptyset\}$ and $\underline{n+1} := \underline{n} \cup \{\underline{n}\}$. Verify that $\omega := \{\underline{n} \mid n \text{ is a natural number}\}$ is an ordinal. It is not difficult to see that

PROPOSITION 4.6.

(1) *If α is an ordinal then* its successor

$$\alpha + 1 := \alpha \cup \{\alpha\}$$

is also an ordinal.
(2) *If S is a set of ordinals then $\bigcup S$ is also an ordinal.*
(3) *Let* On *be the class $\{\alpha \mid \alpha$ is an ordinal$\}$.*
(4) On *is a proper class i.e. not a set.*
(5) *If S is a class of ordinals then $\langle S, \in \rangle$ is well-ordered.*

DEFINITION 4.7. For an ordinal number α the *cofinality of* α is

$$\mathrm{cf}\, \alpha := \min\{\beta \in \mathrm{On} \mid \text{ there exists an increasing } f : \beta \to \alpha\}.$$

DEFINITION 4.8. An ordinal number α is a *successor ordinal* iff there exists an ordinal β such that $\alpha = \beta + 1$. If $\alpha \neq 0$ and is not successor then it is caller a *limit ordinal*.

The class On is an extension of the natural numbers, its importance is that we can prove theorems by transfinite induction.

THEOREM 4.9. *Let $P(\cdot)$ be a property of ordinals. If $P(0)$ is true and*

$$\forall \beta < \alpha [P(\beta) \to P(\alpha)] \text{ for every } \alpha \in \mathrm{On}$$

Then $\forall \alpha \in \mathrm{On} P(\alpha)$

Alternatively

THEOREM 4.10. *Let $P(\cdot)$ be a property of ordinals. If*

(1) *$P(0)$ is true,*
(2) *for every ordinal α such that $\beta + 1 = \alpha$ we have $P(\beta) \to P(\alpha)$ and*
(3) *for α limit we have $\forall \beta < \alpha P(\beta) \to P(\alpha)$.*

Then $\forall \alpha \in \mathrm{On} P(\alpha)$.

DEFINITION 4.11. The *well-ordering principle* is:

For every set P there exists $R \subseteq P \times P$ such that $\langle P, R \rangle$ is a well ordered set. We denote this principle by WO.

It is not difficult to see (by using induction on natural numbers) that every WO holds for every finite set.

THEOREM 4.12. *For every well-ordered set $\langle P, < \rangle$ there exists a unique ordinal α such that $\langle P, < \rangle \cong \langle \alpha, \in \rangle$*

ordinals, Zorn's lemma,

THEOREM 4.13. *The follwing are equivalent*

(1) *The axiom of choice.*
(2) *The well ordering principle.*
(3) *Zorn's lemma.*

The the direction $(1) \implies (2)$ is due to Zermelo and often is refered to as *Zermelo's theorem* . The direction $(2) \implies (1)$ is easier. The equivalence of the Axiom of choice and Zorn's lemma is due to Kuratowski []. In the mid thirties Zorn published an influential paper [**?**] containing several applications of Kuratowski's principle and the general mathematical public renamed Kuratowski's principle as Zorn's lemma.

DEFINITION 4.14. $\alpha + \beta$ by transfinite induction.

THEOREM 4.15. *$\alpha + \beta$ is the order type of $\{1\} \times \alpha \cup \{2\} \times \beta$ when ....*

DEFINITION 4.16. $\alpha \cdot \beta$ by transfinite induction.

THEOREM 4.17. *$\alpha \cdot \beta$ is the order type of $\alpha \times \beta$ with reverse lexicographic ordering.*

DEFINITION 4.18. Let λ be an infinite cardinal number. By induction on ordinals α define $\aleph_\alpha(\lambda)$ as follows:

$\aleph_0(\lambda) := \lambda$,

$\aleph_{\alpha+1}(\lambda) := (\aleph_\alpha(\lambda))^+$,

$\aleph_\delta(\lambda) := \sum_{\alpha<\delta} \aleph_\alpha(\lambda)$ when δ is a limit ordinal.

Recall that for a cardinal λ we denote by λ^+ its successor, namely

$$\lambda^+ := \min\{\mu \in Card \mid \mu > \lambda\}.$$

DEFINITION 4.19. Let λ be an infinite cardinal number. By induction on ordinals α define $\beth_\alpha(\lambda)$ as follows:

$\beth_\alpha(\lambda)$ as follows:

$\beth_0(\lambda) := \lambda$,

$\beth_{\alpha+1}(\lambda) := 2^{\beth_\alpha(\lambda)}$,

$\beth_\delta(\lambda) := \sum_{\alpha<\delta} \beth_\alpha(\lambda)$ when δ is a limit ordinal.

When $\lambda = \aleph_0$ in above definitions. Thus $\beth_n$ stands for $\beth_n(\aleph_0)$ and $\aleph_n$ stands for $\aleph_n(\aleph_0)$.

The following special case is of great importnace in the *construction of primary models*, in 5.9:

FACT 4.20. *Let λ be an infinite cardinal and suppse $\alpha = \lambda \cdot \omega$. If $i < \alpha$ then there are unique $n_i < \omega$ and $\zeta_i < \lambda$ such that $i = \lambda \cdot n_i + \zeta_i$.*

DEFINITION 4.21 (cofinality). Let α be a limit ordinal. The *cofinality of* α is

$$\operatorname{cf} \alpha = \min\{\beta \leq \alpha \mid \exists f : \beta \to \alpha \text{ increasing}\}.$$

FACT 4.22. *Let α be a limit ordinal.*

(1) $\operatorname{cf} \alpha$ *is a cardinal.*
(2) *Suppose α is an infinite cardinal, then α is regular iff* $\operatorname{cf} \alpha = \alpha$.

DEFINITION 4.23 (closed unbounded set).

FACT 4.24. *Let λ be an uncountable regular and M an L-structure with $|L| < \lambda$ such that L contains a unary predicate P with $P^M = \lambda$. If $\{M_\alpha \prec M \mid \alpha < \lambda\}$ is an increasing continuous elementary chain such that $\|M_\alpha\| < \lambda$ for all $\alpha < \lambda$ and $\bigcup_{\alpha<\lambda} P^{M_\alpha} = \lambda$ then*

$$\{\alpha < \lambda \mid P^{M_\alpha} = \alpha\} \quad \textit{contains a club.}$$

.

DEFINITION 4.25 (stationary set).

NOTATION 4.26. Let $\kappa < \mu$ be infinite cardinals such that $\kappa = \aleph_0$ or κ is regular.

$$S^\mu_\kappa := \{\alpha < \mu \mid \operatorname{cf} \alpha = \kappa\}.$$

Note that for λ uncountable regular, always S^λ_κ is a stationary subset of λ.

Of particular interest is the set $S^{\lambda^+}_\kappa$ (meaningful when κ is $\aleph_0$ or is regular).

transfinite induction (def and proof) , cardinal arithmetic, Zermelo-Konig(?), trees , posets,

5. The commulative hierarchy and the reflection principle

By induction on ordinals α define V_α as follows:

For $\alpha = 0$: Let V_0 be the empty set.

For $\alpha = \beta + 1$ let V_α be the power set of V_β.

For α limit let $V_\alpha := \bigcup_{\beta \in On} V_\beta$.

DEFINITION 5.1. The sequence $\{V_\alpha \mid \alpha \in On\}$ is called *the commulative hierarchy*.

6. Martin's Axiom

DEFINITION 6.1. Let $\langle P, \leq_P \rangle$ be a partially ordered set. A subset G of P is called *directed* iff

$$\langle G, \leq_P \rangle \models \forall x \forall y \exists z [x \leq z \wedge y \leq z].$$

DEFINITION 6.2. Let $\langle P, \leq \rangle$ be a partially ordered set. $D \subseteq P$ is called *dense* iff

For every $x \in P$ there exists $y \in G$ such that $P \models x \leq y$.

DEFINITION 6.3. A partially ordered set satisfies the *countable chain condition* of *ccc* for short provided for every $\{p_\alpha \mid \alpha < \omega_1\} \subseteq P$ then there are $\alpha \neq \beta < \omega_1$ such that there exists $r \in P$ satisfying

$$P \models [p_\alpha \leq r \wedge p_\beta \leq r].$$

DEFINITION 6.4. Martin's axiom.

MA_λ stands for the assertion: If a poset $\langle P, \leq \rangle$ satisfies the ccc the for any family $\mathcal{F} := \{D_i \mid i < \lambda\}$ of dense subsets of P there exists a directed $G \subseteq P$ such that $G \cap D_i \neq \emptyset$ for all $i < \lambda$. Such a set G is called $\mathcal{F}$*-generic*.

FACT 6.5. *It is consistent that* $2^{\aleph_0} > \aleph_1$ *and* MA_λ *holds for every* $\lambda < 2^{\aleph_0}$.

7. On weak diamonds

The following is one of the equivalent forms of Jensen's diamond.

DEFINITION 7.1. Let μ be an uncountable regular cardinal and S a stationary subset of μ. $\diamondsuit_\mu(S)$ holds if and only if there exists a family of functions, $\{g_\delta : \delta \to \delta \mid \delta < \mu\}$ such that for all functions, $f : \mu \to \mu$ the set $\{\delta \in S : f \restriction \delta = g_\delta\}$ is stationary.

When $S = \mu$ instead of $\diamondsuit_\mu(S)$ we write $\diamondsuit_\mu$.

FACT 7.2.

(1) *(Ronald Jensen)*

$\langle \mathbf{L}, \in\rangle \models \diamondsuit_\mu(S)$ *for every regular uncountable* μ *and every stationary* $S \subseteq \mu$.

(2) *(Dodd and Jensen) The principle* $\diamondsuit_{\aleph_1}$ *is strictly stronger than the continuum hypothesis, namely*

$$\mathrm{Con(ZF)} \text{ implies } \mathrm{Con(ZFC} + 2^{\aleph_0} = \aleph_1 + \neg\diamondsuit_{\aleph_1}).$$

(3) *(John Gregory) For* $\lambda > \aleph_0$, $2^\lambda = \lambda^+$ *implies* $\diamondsuit_{\lambda^+}$.

DEFINITION 7.3 (Devlin-Shelah's weak diamond). Let $k < \omega$ and $S \subseteq \lambda^+$ be a stationary set. The principle $\Phi^k_{\lambda^+}(S)$, holds if and only if for all functions $F : {}^{\lambda^+>}2 \to k$ there exists a function $g : \lambda^+ \to k$ such that for all $f : \lambda^+ \to 2$ the set $\{\delta \in S : F(f \restriction \delta) = g(\delta)\}$ is stationary.

When $S = \lambda^+$ we write $\Phi^k_{\lambda^+}$.

The set ${}^{\lambda^+>}2$ stands for the set $\{f : \alpha \to 2 \mid \alpha < \lambda^+\}$.

The function F is called *coloring* and k is the *number of colors*. Instead of attempting gussing the function $f \restriction \delta$ as in $\diamondsuit_{\lambda^+}$ we guess its color.

REMARK 7.4. It is easy to see that $\diamondsuit_{\lambda^+}$ implies $\Phi^2_{\lambda^+}$ and that $\diamondsuit_{\lambda^+}$ implies $2^\lambda = \lambda^+$.

The main result of this section is

THEOREM 7.5 (Devlin-Shelah). *The statement* $2^\lambda < 2^{\lambda^+}$ *is equivalent to* $\Phi^2_{\lambda^+}$.

The implication left to right is the important one.

Notice that by Cantor's theorem we have that $2^\lambda = \lambda^+$ implies $2^\lambda < 2^{\lambda^+}$.

The assumption that $2^{\aleph_0} < 2^{\aleph_1}$ is called the *Weak Continuum Hypothesis* WCH.

The assumption $\forall\lambda[2^\lambda < 2^{\lambda^+}]$ is called the *Weak Generalized Continuum Hypothesis* or WGCH.

One of the early applications of Martin's Axiom, was Solovaly's

$$(*) \quad \mathrm{ZFC} + \mathrm{MA}_{\aleph_1} \vdash 2^{\aleph_0} = 2^{\aleph_1}.$$

Solovay's proof involved what is called maximally almost disjoint sets (see **??**), I will not reproduce it here. Notice that Theorems **??** and **??** gives an alternative proof to $(*)$.

DEFINITION 7.6.

(1) Θ_{λ^+} is said to hold if and only if for all $\{f_\eta : \lambda^+ \to \lambda^+ \mid \eta \in {}^{\lambda^+}2\}$ there exists $\eta \in {}^{\lambda^+}2$ such that the set
$\{\delta < \lambda^+ : \exists \rho \in {}^{\lambda^+}2[f_\eta \restriction \delta = f_\rho \restriction \delta \wedge \eta \restriction \delta = \rho \restriction \delta \wedge \eta[\delta] \neq \rho[\delta]\}$ is stationary.
(2) Θ'_{λ^+} is said to hold if and only if for all
$\{f_\eta : \lambda^+ \to 2^\lambda \mid \eta \in {}^{\lambda^+ x}2\}$ and for every club $C \subseteq \lambda^+$, there are $\eta \neq \nu \in {}^{\lambda^+}2$ and there exists a $\delta \in C$ such that $\eta \restriction \delta = \nu \restriction \delta$, $f_\eta \restriction \delta = f_\nu \restriction \delta$ and $\eta[\delta] \neq \nu[\delta]$.

Since the only difference between the last two principles is that in the first the functions f_η's are into λ^+ and in the second into 2^λ it is trivial to see that Θ_{λ^+} follows from Θ'_{λ^+}. The second form is what we use in model theory, the first form is the original version from the Devlin-Shelah paper that was introduced in the context of Whitehead problem.

THEOREM 7.7 (The weak diamond theorem). *For all infinite cardinals λ we have that*

$$2^\lambda < 2^{\lambda^+} \iff \Phi^2_{\lambda^+} \iff \Theta'_{\lambda^+} \iff \Theta_{\lambda^+}.$$

The main step is proving $2^\lambda < 2^{\lambda^+} \implies \Phi^2_{\lambda^+}$, i.e. Theorem 7.5. The implication $\Phi^2_{\lambda^+} \implies \Theta'_{\lambda^+}$ is simpler and is presented as Theorem 7.10 below. $\Theta'_{\lambda^+} \implies \Theta_{\lambda^+}$ is trivial. The next proposition completes the sequence of implicatios.

PROPOSITION 7.8. Θ_{λ^+} *implies* $2^\lambda < 2^{\lambda^+}$.

PROOF. For the sake of contradiction suppose that there exists a injective function

$$G : {}^{\lambda^+}2 \to {}^{\lambda}2 .$$

For every $\eta \in {}^{\lambda^+}2$ and every $\alpha < \lambda^+$ let

$$f_\eta(\alpha) := \begin{cases} G(\eta)[\alpha] & \text{if } \alpha < \lambda \\ \\ \eta[\alpha] & \text{Otherwise.} \end{cases}$$

As $\{f_\eta : \lambda^+ \to 2 \mid \eta \in {}^{\lambda^+}2\}$, by Θ_{λ^+} since $\{\delta < \lambda^+ \mid \delta \geq \lambda\}$ is a club there exists δ such that $\lambda \leq \delta < \lambda^+$ and there are $\eta, \nu \in {}^{\lambda^+}2$ such that

$$(*) \quad f_\eta \restriction \delta = f_\nu \restriction \delta \wedge \eta[\delta] \neq \nu[\delta].$$

Since $\delta \geq \lambda$ from reading out the definition of f_η and the first part of $(*)$ we get that $G(\eta)[\alpha] = G(\nu)[\alpha]$ for every $\alpha < \lambda$. Thus $G(\eta) = G(\nu)$, this together with the second part of $(*)$ is a contradiction to the assumption that G is injective. □

We show below that $\Phi^k_{\lambda^+}$ implies a stronger version to be used in applications:

LEMMA 7.9 (Coding lemma). *Let $S \subseteq \lambda^+$ be a stationary set $1 < k < \omega$ and suppose that $\Phi^k_{\lambda^+}(S)$ holds.*

For every positive integer n and every $F : \underbrace{{}^{<\lambda^+}2^\lambda \times \cdots \times {}^{<\lambda^+}2^\lambda}_{n\text{-times}} \to k$ there exists a guess $g : \lambda^+ \to k$ such that for all $f_1, \ldots, f_n : \lambda^+ \to 2^\lambda$ the set $\{\delta \in S \mid F(f_1 \restriction \delta, \ldots, f_n \restriction \delta) = g(\delta)\}$ is stationary; Namely it is possible to guess also the colors of several functions from λ^+ to 2^λ and not only colors of a single function from λ^+ to 2.

PROOF. Argue by induction on n. For $n = 1$:

Let $F : {}^{\lambda^+>}(2^\lambda) \to k$ be given. Since $|{}^{\lambda^+>}(2^\lambda)| = 2^\lambda = |{}^{\lambda^+>}2|$ there is a bijection G from ${}^{\lambda^+>}(2^\lambda)$ onto ${}^{\lambda^+>}2$.

Now define $F^* : {}^{\lambda^+>}2 \to k$ by: $F^*(h) := F(G^{-1}(h))$. Given $f : \lambda^+ \to 2^\lambda$ consider

$$C_f := \{\delta < \lambda^+ \mid \forall \gamma[\gamma < \delta \to \operatorname{dom} G(f \restriction \gamma)) < \delta]\}.$$

Since C_f is a club, by $\Phi^k_{\lambda^+}(S)$ there exists $g : \lambda^+ \to k$ a guess for F^*.
Now for any $h : \lambda^+ \to 2$ we have that

$$S_h := \{\delta \in S \mid F^*(h \restriction \delta) = g(\delta)\} \text{ is stationary.}$$

By the definition of F^* we have that for every $\delta \in S_{G(f)} \cap C_f$

$$F(f \restriction \delta) = F^*(G(f) \restriction \delta)$$

thus $\{\delta \in S \ : \ F(f \restriction \delta) = g(\delta)\} \supseteq S_{G(f)} \cap C_f$ and is stationary.

For $n > 1$, fix a bijection $H : \underbrace{{}^{\lambda^+>}(2^\lambda) \times \cdots \times {}^{\lambda^+>}(2^\lambda)}_{n\text{-times}} \to {}^{\lambda^+>}2$.

Suppose is $F : \underbrace{{}^{\lambda^+>}(2^\lambda) \times \cdots \times {}^{\lambda^+>}(2^\lambda)}_{n\text{-times}} \to k$ a given coloring. For $\alpha < \lambda^+$ and functions $f_1, \ldots, f_n$ from α into 2^λ, let $h_{\bar{f}} := H(f_1, \ldots, f_n)$

Define now $F^* : {}^{\lambda^+>}2 \to k$ by

$$F^*(h) := F(H^{-1}(h)) \text{ for all } h \in {}^{\lambda^+>}2\,.$$

Now given $\bar{f} = \langle f_1, \ldots, f_n \rangle$ functions from λ^+ into 2^λ Let

$$C_{\bar{f}} := \{\delta < \lambda^+ \mid \forall \beta_1 \cdots \forall \beta_n [[\bigwedge_{1 \le k \le n} \beta_k < \delta \to \operatorname{dom} H(f_1 \restriction \beta_1, \ldots, f_n \restriction \beta_n) < \delta]\}.$$

Verify that $C_{\bar{f}}$ is a club.

By $\Phi^k_{\lambda^+}(S)$ there is a guess g for $\bar{f}$ and a stationary set witnessing this. The rest of the argument is like in the case $n = 1$. □

THEOREM 7.10. *If* $\Phi^2_{\lambda^+}$ *then* Θ'_{λ^+}.

PROOF. In order to prove Θ'_{λ^+} it is enough to show that given $\{f_\eta : \lambda^+ \to 2^\lambda \mid \eta \in {}^{\lambda^+}2\}$ and a club $C \subseteq \lambda^+$, there exists $\eta, \rho \in {}^{\lambda^+}2$ and $\delta \in C$ such that $f_\eta \restriction \delta = f_\rho \restriction \delta, \rho \restriction \delta = \eta \restriction \delta$, and $\eta[\delta] \neq \rho[\delta]$.

For $\alpha < \lambda^+$, $\eta \in^\alpha 2$, and $h : \alpha \to 2^\lambda$, define

$$F(\eta, h) := \begin{cases} 1 & \text{if } \exists \nu \in {}^{\lambda^+}2 [\nu \succ \eta^\frown 0 \wedge f_\nu \supseteq h] \\ \\ 0 & \text{Otherwise.} \end{cases}$$

By $\Phi^2_{\lambda^+}$ and Lemma 7.9 there exists a guess g for F: for all $\eta \in {}^{\lambda^+}2$ and for all $h : \lambda^+ \to 2^\lambda$, the set $\{\delta < \lambda^+ \mid F(\eta \restriction \delta, h \restriction \delta) = g(\delta)\}$ is stationary.

Define $\eta[\alpha] = g(\alpha)$ for all $\alpha < \lambda^+$. Consider the case $h = f_\eta$. Pick $\delta \in C$ such that

$$(*) \quad F(\eta \restriction \delta, h \restriction \delta) = g(\delta).$$

We consider the two possible values for $\eta[\delta]$:

Case 1: $\eta[\delta] = 1$. By $(*)$ and the definition of F we have that $F(\eta \restriction \delta, h \restriction \delta) = 1$. There exists $\nu \in^{\lambda^+} 2$ such that $\nu \succ \eta \restriction \delta, f_\nu \restriction \delta = f_\eta \restriction \delta$, and $\nu[\delta] = 0$.

Case 2: $\eta[\delta] = 0$. By the definition of F we have that $F(\eta \restriction \delta, f_\eta \restriction \delta) = 1$. By the choices of δ and g we get $g(\delta) = 1$, contradicting $\eta[\delta] = g(\delta)$.

Notice that the same proof also establishes Θ_{λ^+}. □

Devlin and Shelah in their paper [**DSh**] proved that $2^{\aleph_0} < 2^{\aleph_1}$ implies $\Phi^2_{\omega_1}$. They ended the paper with a remark that a similar proof can be used to obtain a stronger weak diamond for higher cardinals. Unfortunately they did not provide a proof and the hint they provided is incorrect.

Below I introduce the stronger and more general result with a complete proof.

DEFINITION 7.11. Let μ be a regular uncountable cardinal, λ another cardinal number ≥ 2 and $k \geq 2,\ n \geq 1$ integers.

By $\bar{\Phi}^k_{\mu,\lambda,n}(S)$ denote

For every

$$F : \underbrace{{}^{\mu>}\lambda \times \cdots \times {}^{\mu>}\lambda}_{n\text{-times}} \to 2$$

there exists $g : \mu \to 2$ such that for every $f^1, \ldots, f^n : \mu \to \lambda$ the set

$$\{\delta \in S \ :\ F(f^1 \restriction \delta, \ldots, f^n \restriction \delta) = g(\delta)\} \text{ is a stationary subset of } \mu.$$

When $S = \mu$ we omit S. Namely $\bar{\Phi}^k_{\mu,\lambda,n}$ stands for $\bar{\Phi}^k_{\mu,\lambda,n}(\mu)$.

Notice that $\neg\,\mathrm{Unif}(\mu,\lambda,\lambda,k) \iff \bar{\Phi}^k_{\mu,\lambda,1}(\mu)$.

DEFINITION 7.12. Let μ be a regular uncountable cardinal and λ another cardinal number ≥ 2.

By $\neg\,\mathrm{Unif}(\mu,\lambda,\lambda,2)$ denote:

For every $F : {}^{\mu>}\lambda \to 2$ there exists $g : \mu \to 2$ such that for every $f : \mu \to \lambda$

$$\{\delta < \mu \ :\ F(f \restriction \delta) = g(\delta)\} \text{ is a stationary subset of } \mu.$$

Notice that $\neg\,\mathrm{Unif}(\mu,2,2,2) \iff \Phi^2_\mu$.

There are applications of $\Phi^3_{\lambda^+}$ in Abelian group theory. It is known that $2^\lambda < 2^{\lambda^+}$ does not imply $\Phi^3_{\lambda^+}$.

THEOREM 7.13. *Let $\mu > \aleph_0$ and $\lambda \geq 2$ such that*

$$\lambda^{<\mu} < 2^\mu \text{ and } \exists \theta < \mu[2^\theta = \lambda^{<\mu}].$$

Then $\bar{\Phi}^k_{\mu,\lambda,n}(S)$ holds for al $n < \omega$.

COROLLARY 7.14. $2^\lambda < 2^{\lambda^+}$ *implies* $\Phi^2_{\lambda^+}$.

PROOF. For the sake of contradiction suppose that $\mathrm{Unif}(\mu, \lambda, \lambda, 2)$ holds. Then, there exists $F\colon {}^{\mu>}\lambda \to 2$ such that for all $g\colon \mu \to k$ there is $f\colon \mu \to \lambda$ for which the set

$$\{\delta < \mu \ : \ F(f \restriction \delta) = g(\delta)\}$$

is not stationary. Then, by definition, since $k = 2$, the set

$$\{\delta < \mu \ : \ F(f \restriction \delta) = 1 - g(\delta)\}$$

contains a club. Thus, by replacing g by $1 - g$, we conclude that

> There exists $F\colon {}^{\lambda^+>}2 \to 2$ such that for all $g\colon \mu \to 2$ there is a function $f^g\colon \mu \to \lambda$ and a club $C^g \subseteq \mu$ such that
>
> $$C^g \subseteq \{\delta < \mu \ : \ F(f^g \restriction \delta) = g(\delta)\}.$$

We will define an equivalence relation on the set of functions ${}^{\mu}2$. In order to do this, we consider the following set of models.

$$\mathcal{K} := \{\langle \alpha, f_i, g_i\rangle_{i<\theta n} \mid n < \omega, \alpha < \mu, g_i\colon \alpha \to 2, f_i\colon \alpha \to \lambda\}$$

There are μ choices for α, since $2^{|\alpha|} \leq 2^{<\mu}$ the number of possible choices for each f_i, g_i is $\leq \lambda^{<\mu}$. Namely,

$$|K| \leq \lambda^{<\mu} = 2^{\theta}.$$

Therefore, we can fix an injective function

$$H\colon \mathcal{K} \to {}^{\theta}2\,.$$

Now for each function $g\colon \mu \to 2$, we construct two sequences of functions

$$\langle g_i : \mu \to 2 \mid i < \theta\omega\rangle, \langle f_i : \mu \to \lambda \mid i < \theta\omega\rangle$$

and a sequence of clubs

$$\langle C_n \subseteq \mu \mid n < \omega\rangle.$$

For $n = 0$, we just let $C_0 = \mu$. For $n = 1$, we let

$$g_i := g, \ \ f_i := f^g, \ \text{ and } \ C_1 = C^g, \ \text{ for every } i < \theta.$$

Having constructed $f_i, g_i\colon \mu \to 2$ and $f_i\colon \mu \to \lambda$ for $i < \theta n$ and C_n, we proceed to define $f_{\theta n+j},\ g_{\theta n+j}\colon \mu \to 2$ and $f_{\theta n+j}\colon \mu \to \lambda$ for $j < \lambda$ and C_{n+1} as follows.

Fix $\alpha < \mu$ and let

$$\beta_{\alpha,n} = \min\{\beta \in C_n \mid \beta > \alpha\}.$$

Then $\beta_{\alpha,n} < \mu$ and we can form the model

$$M_{\alpha,n} = \langle \beta_{\alpha,n},\ f_i \restriction \beta_{\alpha,n},\ g_i \restriction \beta_{\alpha,n}\rangle_{i<\theta n}.$$

We define $g_{\theta n+j}(\alpha)$ to be the j^{th} member of the sequence $H(M_{\alpha,n})$, i.e.

$$\langle g_{\theta n+j}(\alpha) \mid j < \theta\rangle = H(M_{\alpha,n}).$$

We let $f_{\theta n+j} := f^{g_{\theta n+j}}$, for $j < \theta$ and

$$C_{n+1} := C_n \cap \bigcap_{j<\theta} C^{g_{\theta n+j}}.$$

Since C_n and $C^{g_{\theta n+j}}$ are clubs in μ, also C_{n+1} is a club.

We now turn to the equivalence relation. For $g\colon \mu \to 2$ and $g'\colon \mu \to 2$, the previous construction has provided us with sequences

$$\langle f_i, g_i \mid i < \theta\omega\rangle \text{ and } \langle C_n \mid n < \omega\rangle \text{ for } g,$$

and sequences

$$\langle f'_i, g'_i \mid i < \theta\omega\rangle \text{ and } \langle C'_n \mid n < \omega\rangle \text{ for } g'.$$

We say that gEg' if

(1) $\min \bigcap_{n<\omega} C_n = \min \bigcap_{n<\omega} C'_n$, denote this ordinal by γ;
(2) $g_i \restriction \gamma = g'_i \restriction \gamma$, for every $i < \theta\omega$;
(3) $f_i \restriction \gamma = f'_i \restriction \gamma$, for every $i < \theta\omega$.

Clearly E is an equivalence relation on ${}^{\mu}2$. Now, since there are only μ ways for (1) to fail and only $\lambda^{<\mu}$ ways for (2) or (3) to fail, there are at most $\lambda^{<\mu}$ equivalence classes.

CLAIM 7.15. *$gEg' \implies g = g'$, for all $g, g'\colon \mu \to 2$.*

PROOF OF CLAIM. Let $C := \bigcap_{n<\omega} C_n$ and $C' := \bigcap_{n<\omega} C'_n$.
Let $\langle \gamma_j \mid j < \lambda^+\rangle$ and $\langle \gamma'_j \mid j < \mu\rangle$ be the normal enumerations of the clubs C and C'.

We will prove by induction on $j < \mu$ that

$$\gamma_j = \gamma'_j,\ \ g_i \restriction \gamma_j = g'_i \restriction \gamma_j,\ \text{ and }\ f_i \restriction \gamma_j = f'_i \restriction \gamma_j,\quad \text{for every } i < \theta\omega.$$

This will clearly imply the conclusion of the claim: If $\alpha < \mu$, there is $\gamma_j > \alpha$, so $g_0 \restriction \gamma_j = g'_0 \restriction \gamma_j$. But $g = g_0$ and $g' = g'_0$ by construction, so $g(\alpha) = g'(\alpha)$.

For $j = 0$, this is given by (1), (2), and (3).

For j a limit ordinal, we have by normality of each enumeration that $\gamma_j = \sup_{k<j} \gamma_k$ and $\gamma'_j = \sup_{k<j} \gamma'_k$. Hence $\gamma_j = \gamma'_j$ by induction hypothesis, and the rest follows similarly.

For $j + 1$, notice that for each $n < \omega$, the following series of equality holds:

$$\begin{aligned} H(M_{\gamma_j,n}) &= \langle g_{\lambda n+i}(\gamma_j) \mid i<\theta\rangle && \text{(by definition)}\\ &= \langle F(f_{\lambda n+i} \restriction \gamma_j) \mid i<\theta\rangle && (\text{since } \gamma_j \in C_n)\\ &= \langle F(f'_{\lambda n+i} \restriction \gamma'_j) \mid i<\theta\rangle && \text{(by induction)}\\ &= \langle g'_{\lambda n+i}(\gamma'_j) \mid i<\theta\rangle && (\text{since } \gamma'_j \in C'_n)\\ &= H(M'_{\gamma'_j,n}) && \text{(by definition)}. \end{aligned}$$

But H is one-to-one, so this implies that for every $n<\omega$

$$M_{\gamma_j,n} = M'_{\gamma'_j,n}.$$

By definition, we have for every $n<\omega$:

(*) $\quad \langle \beta_{\gamma_j,n},\, f_i \restriction \beta_{\gamma_j,n},\, g_i \restriction \beta_{\gamma_j,n}\rangle_{i<\theta n} = \langle \beta'_{\gamma'_j,n},\, f'_i \restriction \beta'_{\gamma'_j,n},\, g'_i \restriction \beta'_{\gamma'_j,n}\rangle_{i<\theta n}.$

Let us first show that this implies that

(**) $$\gamma_{j+1} = \gamma'_{j+1}.$$

We claim that

$$\gamma_{j+1} = \sup_{n<\omega} \beta_{\gamma_j,n} = \sup_{n<\omega} \beta'_{\gamma'_j,n} = \gamma'_{j+1}.$$

First we show that $\gamma_{j+1} = \sup_{n<\omega} \beta_{\gamma_j,n}$. Fix $m<\omega$. Then, for every $n \geq m$, we have $\beta_{\gamma_j,n} \in C_n \subseteq C_m$. Thus, since C_m is a club, also $\sup_{n<\omega} \beta_{\gamma_j,n} \in C_m$. Since this is true for every $m<\omega$, then $\sup_{n<\omega} \beta_{\gamma_j,n} \in C$. But $\sup_{n<\omega} \beta_{\gamma_j,n} > \gamma_j$ by definition of $\beta_{\gamma_j,n}$, so $\sup_{n<\omega} \beta_{\gamma_j,n} \geq \gamma_{j+1}$, by definition of γ_{j+1}. The fact that $\sup_{n<\omega} \beta_{\gamma_j,n} \leq \gamma_{j+1}$ follows from the fact that $\beta_{\gamma_j,n} \leq \gamma_{j+1}$ for every $n<\omega$. Hence, the equality holds. Similarly

$$\gamma'_{j+1} = \sup_{n<\omega} \beta'_{\gamma'_j,n}.$$

But for every $n<\omega$ we have $\beta'_{\gamma'_j,n} = \beta_{\gamma_j,n}$ by (*), so $\gamma_{j+1} = \gamma'_{j+1}$.

The fact that $g_i \restriction \gamma_{j+1} = g'_i \restriction \gamma_{j+1}$ and $f_i \restriction \gamma_{j+1} = f'_i \restriction \gamma_{j+1}$, for every $i<\theta\omega$ follows now easily from (*) and (**). This proves the final claim. □

Now, since there are only $\lambda^{<\mu}$ E-equivalence classes and 2^μ functions in ${}^\mu 2$, the previous claim implies that $\lambda^{<\mu} = 2^\mu$, which contradicts the assumption of the theorem. □

The following generalization of Φ^k_μ is useful in the model theory of AECs.

THEOREM 7.16. *Let $\mu > \aleph_0$ and $\lambda \geq 2$ such that*

$$\lambda^{<\mu} < 2^\mu \text{ and } \exists \theta < \mu [2^\theta = \lambda^{<\mu}].$$

Then $\bar{\Phi}^2_{\mu,\lambda,n}(\mu)$ holds for every $n<\omega$.

PROOF. Almost literally repeat the argument we had in the proof of Theorem 7.13:

Suppose $n \geq 1$ such that $\neg\bar{\Phi}^2_{\mu,\lambda,n}(\mu)$. Thus there exists F such that for every $g : \mu \to 2$ there are $f^\ell : \mu \to \lambda$ for $\ell = 1, n$ and a club $C \subseteq \mu$ such that

$$\{\delta < \mu \ : \ F(f^1 \restriction \delta, \dots, f^n \restriction \delta) = g(\delta)\} \supseteq C.$$

Work with

$$\mathcal{K} := \{\langle \alpha, f^\ell_i, g_i\rangle_{i<\theta\omega} \mid g_i : \alpha \to 2, \ f^\ell_i : \alpha \to \lambda, \ 1 \leq \ell \leq n\}.$$

Fix an injection $H : \mathcal{K} \to {}^\theta 2$. Given g define sequences $\langle g_i \mid i < \theta\omega\rangle$, $\langle f^\ell_i : \mu \to \lambda \mid i < \theta\omega, \ell \leq n\rangle$ and $\{C_n \mid n < \omega\}$. Use them to define E^* on ${}^\mu 2$ as before, replacing (3) by

$$(3)' \quad f^\ell_i \restriction \gamma = f'^\ell_i \restriction \gamma, \text{ for every } i < \theta\omega \text{ and every } 1 \leq \ell \leq n.$$

The proof that $gE^*g' \implies g = g'$ is also similar. □

The principle we we need is the following special case:

COROLLARY 7.17. *Let λ be an infinite cardinal.*

$$2^\lambda < 2^{\lambda^+} \implies \bar{\Phi}^2_{\lambda^+,\lambda^+,3}.$$

PROOF. Take $\mu \leftarrow \lambda^+$. $\lambda \leftarrow \lambda^+$, $\theta \leftarrow \lambda$. □

Devlin and Shelah in [**DSh**] (see 7.1) state

> Generalizing 4.1 (2) we can prove that if μ is regular, and for some $\theta < \mu$ we have $2^\theta = \lambda^{<\mu} < 2^\mu$, then the conclusion of 4.1 (2) holds with μ in place of ω_1 (in the proof, θ takes the place of ω).

Namely, they asserted that under the above cardinal arithmetic assumptions on μ and λ the following strengthening of what is called the Devlin Shelah weak diamond holds:

DEFINITION 7.18. The principle $\neg\,\mathrm{Unif}(\mu, \lambda, \lambda, 2)$ means:

For every $F : {}^{\mu>}\lambda \to 2$ there exists $g : \mu \to 2$ such that for every $f : \mu \to \lambda$

$$\{\delta < \mu \ : \ F(f \restriction \delta) = g(\delta)\} \text{ is a stationary subset of } \mu.$$

In 1981 I got very interested in weak diamonds due to their role in the classification theory for Abstract Elementary Classes and used this to solve and old question of Macintyre and Shelah concerning universal locally finite groups, in [**Gr1**], an improved version included in see [**?**]. While reading [**DSh**] I was intrigued by 7.1 but did not manage to see how to prove this, nor to follow the hint.

I discussed this with Menachem Magidor and we realized that all we can prove was the case

$$\mu = \omega_1, \lambda < \aleph_{\omega_1} \text{ and } \lambda^{\aleph_0} < 2^{\aleph_1}.$$

Magidor communicated this to Shelah who also admitted not to be able to prove the assertion as stated in [**DSh**]. Shelah introduced a family of principles generalizing the original weak diamond.

8. The small subsets of λ^+ is a normal ideal

Recall that an ideal $I \subseteq \mathcal{P}(\lambda^+)$ is said to be *normal* if for every family $\{X_\alpha \mid \alpha < \lambda^+\} \subseteq I$, the set

$$\nabla_{\alpha<\lambda^+} X_\alpha := \{\beta < \lambda^+ \mid \beta \in X_\alpha \text{ for some } \alpha < \beta\} \in I.$$

Denote by I_{NS} the ideal of non-stationary sets of λ^+. The following facts are well-known.

FACT 8.1 (Fodor). *The ideal I_{NS} is normal.*

An ideal J is λ^+-complete if for every family $\mathfrak{F} \subseteq J$ of cardinality at most λ, the set $\bigcup \mathfrak{F} \in J$.

FACT 8.2. *Let $J \subseteq \mathcal{P}(\lambda^+)$ be a normal ideal such that $\{\alpha\} \in J$ for any $\alpha < \lambda^+$. Then, J is λ^+-complete and $J \supseteq I_{NS}$.*

Recall the following combinatorial principle.

$\Phi^k_{\lambda^+}(S)$: For all $F\colon {}^{\lambda^+>}2 \to k$ there exists $g\colon \lambda^+ \to k$ such that for all $f\colon \lambda^+ \to k$ the set $\{\delta \in S \mid F(f \restriction \delta) = g(\delta)\}$ is stationary.

Until the rest of this lecture, we let

$$J = \{X \subseteq \lambda^+ \mid \neg\Phi^k_{\lambda^+}(X)\}.$$

Clearly, if $X \subseteq \lambda^+$ is not stationary, then $\neg\Phi^k_{\lambda^+}(X)$. Hence, $I_{NS} \subseteq J$.

We first prove the following theorem.

THEOREM 8.3 (Devlin-Shelah). *The ideal J is normal.*

PROOF. It is easy to see that J is an ideal, so we will only show that it is normal.

Let $\{X_\alpha \mid \alpha < \lambda^+\} \subseteq J$ and let $X := \nabla_{\alpha<\lambda^+} X_\alpha$. We will show that $\neg\Phi^k_{\lambda^+}(X)$. To this end, it is enough to produce a coloring $F\colon {}^{\lambda^+>}2 \to k$ such that for any $g\colon \lambda^+ \to k$ there is $f\colon \lambda^+ \to 2$ such that the set

$$\{\delta \in X \mid F(f \restriction \delta) = g(\delta)\} \in I_{NS}.$$

In order to do this, we fix for each $\alpha < \lambda^+$

$$F_\alpha\colon {}^{\lambda^+>}2 \to k,$$

witnessing $\neg\Phi^k_{\lambda^+}(X_\alpha)$. Now fix a bijection

$$h\colon (\lambda^+ + 1) \times \lambda^+ \to \lambda^+.$$

We first claim that the set

$$C = \{\delta < \lambda^+ \mid h\colon (\delta+1)\times\delta \to \delta \text{ is a bijection }\}$$

is a club.

To see this, define $M = \langle \lambda^+ + 1, \in, h, \lambda^+\rangle$, where $\in$ is the usual epsilon relation, h the above bijection and λ^+ a unary predicate whose interpretation is λ^+. Notice that M knows that its universe is the successor of $(\lambda^+)^M$, since this is definable using $\in$ and $h^M\colon |M|\times(\lambda^+)^M \to (\lambda^+)^M$ is a bijection, as this is expressible in first order. Take $\langle M_\alpha \mid \alpha < \lambda^+\rangle$ a resolution of M. Then a standard result shows that the set

$$C' = \{\delta < \lambda^+ \mid (\lambda^+)^{M_\delta} = \delta\}$$

is a club. Since C' is contained in C, we are done with the first claim.

We now define $F\colon {}^{\lambda^+>}2 \to k$. Given $\beta < \lambda^+$ and $f\colon \beta \to 2$, we let $F(f) := 0$, unless $\beta \in C$, $\beta \in X_\gamma$ for some $\gamma < \beta$ and then we let

$$F(f) := F_{\gamma_0}(f^*),$$

where

$$\gamma_0 = \min\{\gamma < \beta \mid \beta \in X_\gamma\}$$

and $f^*\colon \beta \to 2$ is the unique function defined by

$$f^*(\xi) = f(h(\beta,\xi)),$$

for every $\xi < \beta$.

The conclusion of the theorem will follow from the final claim.

CLAIM 8.4. *For any $g\colon \lambda^+ \to k$ there exists $f\colon \lambda^+ \to 2$ such that the set*

$$\{\alpha \in X \mid F(f\restriction\alpha) = g(\alpha)\} \in I_{NS}.$$

PROOF OF THE CLAIM: Let $g\colon \lambda^+ \to k$ be given. By definition of X_α, we can find $f_\alpha\colon \lambda^+ \to 2$, such that

$$Y_\alpha := \{\xi \in X_\alpha \mid F_\alpha(f_\alpha\restriction\xi) = g(\xi)\} \in I_{NS}.$$

By normality of the ideal of non-stationary sets, we know that

(†) $$Y := \nabla_{\alpha<\lambda^+} Y_\alpha \in I_{NS}.$$

Let $f\colon \lambda^+ \to 2$ be the unique function satisfying

$$f(h(\alpha,\xi)) = f_\alpha(\xi), \quad \text{for every } \xi < \alpha.$$

We sill show that f satisfies the conclusion of the claim.

Suppose not. Then, the set

$$S := \{\alpha \in X \mid F(f\restriction\alpha) = g(\alpha)\} \text{ is stationary..}$$

Hence, the set $S \cap C$ is also stationary. We will show that $S \cap C \subseteq Y$, which contradicts (†). Let $\beta \in S \cap C$. Then $\beta \in C \cap X$ and

(*) $$F(f \restriction \beta) = g(\beta).$$

Since $\beta \in X$, there exists $\gamma < \beta$ such that $\beta \in X_\gamma$. Let

$$\gamma_0 = \min\{\gamma < \beta \mid \beta \in X_\gamma\}.$$

Thus, the following equalities hold:

$$\begin{aligned} g(\beta) &= F(f \restriction \beta) && (\text{ by (*) }) \\ &= F_{\gamma_0}((f \restriction \beta)^*) && (\text{ by definition of } F) \end{aligned}$$

Moreover, if $\xi < \beta$, then

$$\begin{aligned} (f \restriction \beta)^*(\xi) &= f \restriction \beta(h(\beta, \xi)) && (\text{ by definition of } (f \restriction \beta)^*) \\ &= f(h(\beta, \xi)) && (\ \beta \in C, \text{ so } h(\beta, \xi) \in \beta) \\ &= f_\beta(\xi) && (\text{ by definition of } f). \end{aligned}$$

This shows that for every $\beta \in C \cap S$, there exists $\gamma_0 < \beta$ such that

$$F_{\gamma_0}(f_\beta) = g(\beta).$$

We want $F_{\gamma_0}(f_{\gamma_0} \restriction \beta) = g(\beta)$.

We will show that □

□

9. Kuratowski's and Hajnal's free subset theorems

DEFINITION 9.1. Let A be a set and g be a set-valued function with domain A. $F \subseteq A$ is called *free with respect to* g provided

$$x \neq y \in F \implies x \notin g(y).$$

THEOREM 9.2 (Kuratowski). *For every* $1 \leq n < \omega$ *and function* $g : [\omega_n]^n \to [\omega_n]^{<\aleph_0}$ *then there exists* $F \subseteq \omega_n$ *of cardinality* $n+1$ *free with respect to* g.

PROOF. By induction on n:

For $n = 1$: Given $g : \omega_1 \to [\omega_1]^{<\aleph_0}$ the set

$$C := \{\delta < \omega_1 \mid g : \delta \to [\delta]^{<\aleph_0}\}$$

Contains a club.

For any $\alpha < \omega_1$ define an increasing sequence $\{\alpha_k \mid k < \omega\}$ by induction on k: Let $\alpha_0 := \alpha$ and

$$\alpha_{k+1} := \min\{\delta < \omega_1 \mid \delta > \sup g[\alpha_k]\}.$$

Let $\alpha^* := \sup_{k<\omega} \alpha_k$. Notice that $\alpha^* \in C$. Pick $\alpha < \omega_1$. Notice that for any $\beta < \alpha^*$ there is $k < \omega$ such that $\beta < \alpha_{k+1}$, as $\alpha^* \notin g[\alpha_{k+1}]$ also $\alpha^* \notin f[\beta]$. So we have shown that $\alpha^* \notin f[\beta]$ for all $\beta < \alpha^*$. As α^* is infinite we can find $\beta_0 < \alpha^*$ such that $\beta_0 \notin f(\alpha^*)$. Thus we have shown that $F := \{\beta_0, \alpha^*\}$ is a free set for g.

For $n+1$: Given $g : [\omega_{n+1}]^{n+1} \to [\omega_{n+1}]^{<\aleph_0}$. the set

$$C := \{\delta < \omega_{n+1} \mid g : [\delta]^{n+1} \to [\delta]^{<\aleph_0}\}$$

Contains a club. For $\omega_n \leq \alpha < \omega_{n+1}$ define an increasing and continuous sequence $\{\alpha_i \mid i < \omega_n\}$ such that $\alpha_0 := \alpha$ and

$$\alpha_{i+1} := \min\{\delta < \omega_{n+1} \mid \delta > \sup g[\alpha_i]\}.$$

Let $\alpha^* := \sup_{i<\omega_n} \alpha_i$. Clearly $\alpha^* \in C$ and $\alpha^* \notin g[\alpha^*]$.

As $g(\alpha^*)$ is a finite set and α^* is infinite there exists $\beta_0 < \alpha^*$ such that $\beta_0 \notin g(\alpha^*)$.

Consider

$$g^*(\beta_1, \ldots, \beta_n) := g(\beta_0, \beta_1, \ldots, \beta_n).$$

As $\alpha^* \in C$ we have that $g^* : [\alpha^*]^n \to [\alpha^*]^{<\aleph_0}$. Since $|\alpha^*| = \aleph_n$ an application of the induction hypothesis gives $F^* \subseteq \alpha^*$ free for g^* with $n+1$-many elements.

Fix $\delta \in C$ such that $\delta > \omega_n$ and a $\delta' \in C$ such that $\delta' > \delta$. Consider

$$g^*(\alpha_1, \ldots, \alpha_n) := g(\alpha_1, \ldots, \alpha_n, \delta).$$

Clearly $g* : [\delta']^n \to [\delta']^{<\aleph_0}$. As $|\delta'| = \aleph_n$ by the induction hypothesis there exists $F^* \subseteq \delta'$ of cardinality $n+1$ free for g'.

□

THEOREM 9.3 (Hajnal 1961). *Suppose $\mu > \lambda$ are infinite cardinal numbers. For every A of cardinality μ and any set valued function g with domain A such that $x \in A \implies |g(x)| < \lambda$ then there exists $F \subseteq A$ of cardinality μ free with respect to g.*

The statement of this theorem for the case μ is regular was established by Sierepinski (Lazard 1936?). In 1936 by Ruziewicz [**Ruz**] conjectured that this also holds for μ singular. The theorem is Hajnal's proof of the conjecture [**Haj**] (for a more modern general exposition see the book by Erdős, Hajnal, Mate and Rado [**EHMR**]).

PROOF. Without loss of generality we may assume that $A = \mu$ and $x \in A \implies g(x) \subseteq \mu$.

Distinguish between two cases:

Suppose μ is regular. By induction on $i < \lambda$ define $F_i \subseteq \mu$ and ordinals $\alpha_i^1 < \alpha_i^2 < \mu$ such that

F_i is a maximal free subset of $\mu - \alpha_i^2$ with respect to g; $\alpha_i^1 := \sup \bigcup_{j<i} F_j$; and $\alpha_i^2 := \sup\{g(x) \cup \{x\} \mid x < \alpha_i^1\}$.

If there exists $i < \lambda$ such that $|F_i| = \mu$ we are done. Otherwise, suppose $|F_i| < \mu$ for every $i < \lambda$. In this case by regularity of μ we get that $\alpha := \sup_{i<\lambda} \alpha_i^2 < \mu$. Pick $x \in \mu - \alpha$.

Notice that

$$(*) \quad g(x) \cap F_i \neq \emptyset \text{ for every } i < \lambda.$$

Otherwise, there exists $i < \lambda$ such that $g(x) \cap F_i = \emptyset$, thus $\{x\} \cup F_i$ is free which is a contradiction to maximality of F_i.

By the definitions of α_i^1 and α_i^2 we have that

$$i < j < \lambda \implies S_i \cup \bigcup g[F_i] \subseteq \alpha_j^2 \text{ and } F_j \cup g[F_j] \subseteq (\alpha_j^2, \mu)$$

thus $i < j < \lambda \implies (F_i \cup g[F_i]) \cap (F_j \cup g[F_j]) = \emptyset$.

Together with $(*)$ we get that $|g(x)| \geq \lambda$ which is a contradiction to our assumption on g.

Suppose μ is singular, let $\kappa := \operatorname{cf} \mu$. As $\lambda < \mu$ by replacing λ by a bigger cardinal we may assume that λ is regular and $\lambda > \kappa$. Fix $\{\mu_i \mid i < \kappa\}$ increasing sequence of regular cardinals such that $\mu = \sum_{i<\kappa} \mu_i$ and $\mu_0 > \lambda$.

Since $\lambda < \mu_i$ by the previous case for every $i < \kappa$ there exists $F_i \subseteq \mu$ free of cardinality μ_i. For every $i < \kappa$ consider

$$G_i := F_i - \bigcup_{j<i}(A_j \cup \bigcup g[A_j]).$$

As $x \in \mu \implies |g(x)| < \lambda$, μ_i regular and $\mu_i > \mu_j \geq \lambda$ we have that $|G_i| = \mu_i$ for all $i < \kappa$.

Now define $\{S_i^\zeta \subseteq G_i \mid i < \kappa, \zeta < \lambda\}$ and $\{\zeta_i < \lambda \mid i < \kappa\}$ such that

$$|S_i^\zeta| = \mu_i, \quad \zeta \neq \zeta' \implies S_i^\zeta \cap S_i^{\zeta'} = \emptyset \quad \text{for all } i < \kappa.$$

and

$$(*) \quad x \in \bigcup_{\zeta<\lambda} S_i^\zeta \implies g(x) \cap (\bigcup_{j<i} \bigcup_{\zeta_i \leq \zeta < \lambda} S_j^\zeta) = \emptyset.$$

By induction on $i < \kappa$ define $\{S_\zeta^i \subseteq G_i \mid \zeta < \lambda\}$ and ζ_i.

Suppose $\{S_j^\zeta \subseteq G_i \mid j < i, \zeta < \lambda\}$ and $\{\zeta_j < \lambda \mid j < i\}$ already satisfy the inductive hypothesis.

For every $\xi < \lambda$ let

$$G_i^\xi := \{x \in G_i \mid f(x) \cap (\bigcup_{\xi \leq \zeta < \lambda} \bigcup_{j<i} S_j^\zeta) = \emptyset\}.$$

CLAIM 9.4. *for every $x \in G_i$ there exists $\xi(x) < \lambda$ such that $x \in G_i^\xi$.*

PROOF. Otherwise, there exists $x \in G_i$ such that for every $\xi < \lambda$ there exists $\zeta \geq \xi$ and there exists $j(\xi) < i$ such that $S_j^\zeta \cap g(x) \neq \emptyset$. Since λ is regular and $\lambda > \kappa$ there exists j such that for λ-many ζ's $S_j^\zeta \cap g(x) \neq \emptyset$. As $|g(x)| < \lambda$ and $\zeta_1 \neq \zeta_2 \implies S_j^{\zeta_1} \cap S_j^{\zeta_2} = \emptyset$ we get that $|g(x)| \geq \lambda$ a contradiction. □

Since $G_i = \bigcup_{\xi<\lambda} G_i^\xi$ and $\lambda < \mu_i = |G_i|$ is a regular cardinal there exists $\xi_0 < \lambda$ such that $|G_i^{\xi_0}| = \mu_i$. Define ζ_i as ξ_0 and pick $\{S_i^\zeta \mid \zeta < \lambda\}$ a partition of $G_i^{\xi_0}$ such that $\zeta < \lambda \implies |S_i^\zeta| = \mu_i$.

Using that $\lambda > \kappa$ and is regular there exists $\zeta^* < \lambda$ such that $\zeta^* \geq \zeta_i$ for all $i < \kappa$.

CLAIM 9.5. $F := \bigcup_{i<\kappa} S_i^{\zeta^*}$ *is free of cardinality μ*

PROOF. Let $x \neq y \in F$ be given.

If there exists $i < \kappa$ such that $x, y \in S_i^{\zeta^*}$ then we are done since $x, y \in F_i$. Suppose $i \neq j$ such that $x \in S_i$ and $y \in S_j$.

Suppose $j < i$ then since $x \in G_i$ we get that $x \notin f(y)$; We get $y \notin f(x)$ using $(*)$. $\square$

10. The building-stones of many models

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10.1. A theorem of Ulam. The following theorem due to Ulam in combination with last lecture's result that the collection of sets for which the Devlin-Shelah weak-diamond fails is a normal ideal completes the proof that we can split up a stationary set for which $\Phi^k_{\lambda^+}(S)$ holds into λ^+ many disjoint stationary sets S_α, so that $\Phi^k_{\lambda^+}(S_\alpha)$ holds for each piece.

The following notion was introduced by Tarski

DEFINITION 10.1. Let κ be an uncountable regular cardinal, an ideal I of subsetes of κ is *saturated* iff for every $\{S_\alpha \mid \alpha < \kappa\}$ such that $S_\alpha \notin I$ for all $\alpha < \kappa$ there exists $\alpha \neq \beta$ such that $S_\alpha \cap S_\beta = \emptyset$.

The contents of the following theorem of Ulam is that any complete ideal on a successor cardinal that contains all bounded sets is not saturated.

THEOREM 10.2 (Ulam). *Let I be a complete ideal on λ^+ so that $\forall \alpha < \lambda^+$, $\{\alpha\} \in I$. If $S \subseteq \lambda^+$ is a collection of limit ordinals, $S \notin I$, then there is a pairwise disjoint family $\{S_\alpha \subseteq S \mid \alpha < \lambda^+\}$ so that $\forall \alpha < \lambda^+$, $S_\alpha \notin I$.*

PROOF. Let I and S be as specified.

First, recall that a normal ideal in λ^+ which contains all the singletons $\{\alpha\}$, $\alpha < \lambda^+$, must be λ^+-complete, and therefore I must contain every set bounded in λ^+.

There must exist $\mu \leq \lambda$ be so that $S_\mu = \{\delta \in S \mid \mathrm{cf}(\delta) = \mu\} \notin I$. Otherwise, $\forall \mu \leq \lambda,\ S_\mu \in I$. By λ^+-completeness, $S = \bigcup_{\mu \leq \lambda} S_\mu \in I$, a contradiction.

Fix an appropriate $\mu \leq \lambda$ so that $S_\mu \notin I$.

For every $\delta \in S_\mu$, fix a continuous, increasing enumeration $\eta_\delta : \mu \to \delta$.

For each $\beta < \lambda^+$, and each $i < \mu$, let $S(\beta, i) := \{\delta \in S_\mu \mid \eta_\delta[i] = \beta\}$.

Notice that for every $i < \mu$ and all $\beta_1 \neq \beta_2$, $S(\beta_1, i) \cap S(\beta_2, i) = \emptyset$. Therefore, it will suffice to show that there exists $i < \mu$ so that $|\{\beta < \lambda^+ \mid S(\beta, i) \notin I\}| = \lambda^+$.

Suppose there is no such $i < \mu$. Then, for every $i < \mu$, $|\{\beta < \lambda^+ \mid S(\beta, i) \notin I\}| < \lambda^+$. Since λ^+ is regular, $\forall i < \mu$,

$\{\beta < \lambda^+ \mid S(\beta, i) \notin I\}$ is bounded, and for each $i < \mu$ we may choose $\beta_j < \lambda^+$ so that $\forall \beta \geq \beta_j,\ S(\beta, i) \in I$. By the regularity of λ^+, $\beta^* := \sup_{i<\mu} \beta_i < \lambda^+$. This β^* works as a uniform lower bound: $\forall i < \mu,\ \forall \beta \geq \beta^*,\ S(\beta, i) \in I$.

Let

$$S^* := \underset{\beta^* \leq \beta < \lambda^+}{\bigtriangledown} \left(\bigcup_{i<\mu} S(\beta, i) \right).$$

By the completeness of I, each $\bigcup_{i<\mu} S(\beta, i) \in I$, and by its normalcy, their diagonal union is in I as well.

By assumption, $S_\mu \notin I$, so it must be unbounded in λ^+. Moreover, because ideals are closed down, $S_\mu \not\subseteq S^*$. So we may take $\delta \in S_\mu - S^*$, $\delta > \beta^*$. Because $\delta \notin S^*, \forall \beta \in [\beta^*, \delta),\ \delta \notin \bigcup_{i<\mu} S(\beta, i)$.
Which is really just saying $\forall \beta \in [\beta^*, \delta),\ \forall i < \mu,\ \eta_\delta[i] \neq \beta$. A contradiction to η_δ approaching δ.

□

Since the ideal of nonstationary sets is normal we get the following

COROLLARY 10.3. *For every stationary $S \subseteq \lambda^+$ there exists $\{S_\alpha \subseteq S \ :\ \alpha < \lambda^+\}$ such that for every $\alpha < \lambda^+$ the set S_α is stationary and $\alpha \neq \beta \Longrightarrow S_\alpha \cap S_\beta = \emptyset$.*

COROLLARY 10.4. *Let $S \subseteq \lambda^+$ be such that $\Phi_{\lambda^+}(S)$ holds. Then there exists $\{S_\alpha \subseteq S \ :\ \alpha < \lambda^+\}$ such that for every $\alpha < \lambda^+$ the set $\Phi_{\lambda^+}(S_\alpha)$ holds and $\alpha \neq \beta \Longrightarrow S_\alpha \cap S_\beta = \emptyset$.*

Exercises ??.

(1) Prove that $\diamondsuit_{\lambda^+}$ implies $2^\lambda = \lambda^+$.
(2) Prove that $\diamondsuit_{\lambda^+}$ implies $\Phi^k_{\lambda^+}$ for all k.
(3) $\diamondsuit_{\lambda^+}$ holds iff there exists $\{g_\delta : \delta \to 2 \mid \delta < \lambda^+\}$ such that for all functions $f : \lambda^+ \to 2$, the set $\{\delta < \lambda^+ : f \restriction \delta = g_\delta\}$ is stationary.
(4) $\diamondsuit_{\lambda^+}$ holds iff there exists $\{S_\delta \subseteq \delta : \delta < \lambda^+\}$ such that for all $S \subseteq \lambda^+$, $\{\delta < \lambda^+ : S \cap \delta = S_\delta\}$ is stationary.
(5) Show that Θ_{λ^+} implies Θ'_{λ^+}.
(6)
(7)

CHAPTER 16

Combinatorial geometry

Introduction

Whitney, Van Der Wearden, not in the first edittion... Matroids, R. Baer and E. Artin. Aigner

1. Pregeometries (or Matroids)

In order to motivate what will come we present an example which is a direct generalization of the notion of linear dependence in vector spaces.

DEFINITION 1.1. Let R be a ring, and let M be a left $R-$module. For $X \subseteq M$ let $Sp(X) := \{\sum_k r_k \cdot a_k \quad : \quad r_k \in R, a_k \in X\}$. The submodule of M generated by X is the minimal (with respect to inclusion) submodule of M containing the set X.

It is easy to check the following:

LEMMA 1.2. *Let M be R-module, for every $X \subseteq M$, $Sp(X)$ is the $R-$submodule of M generated by the set X.*

DEFINITION 1.3. Let M be an R- module, and suppose $X \subseteq M$.

(1) we say that X *generates* M iff $M = Sp(X)$.
(2) X is *linearly independent* iff for every $a \in X$ we have that $a \in Sp(X - \{a\})$
(3) X is *a basis of* M iff X is linearly independent and generates M.

THEOREM 1.4. *Let R be a division ring, and let M be an R-module, suppose that X and Y are subsets of M.*

(1) *for every $X \subseteq M$ we have $Sp(X) = Sp(Sp(X))$*
(2) *for every $X \subseteq M$ and we have that $X \subseteq Sp(X)$*
(3) *if $a \in Sp(X)$ then there exists a finite $X' \subseteq X$ such that $a \in Sp(X')$.*
(4) *if $Y \subseteq Sp(X)$ and $b \in Sp(Y)$ then $b \in Sp(X)$.*
(5) *if $a \in Sp(X \bigcup \{b\})$ and $a \notin Sp(X)$ then $b \in Sp(X \bigcup \{a\})$.*

PROOF.

(1) Using (1) it is enough to show that $Sp(Sp(X)) \subseteq Sp(X)$.
(2) Since $1 \in R$.
(3) Suppose $a \in Sp(X)$, then by the definition of $Sp(X)$ there are $r_1, \ldots, , r_n \in R$ and $a_1, \ldots, a_n \in X$ such that $a = r_1 \cdot a_1 + \cdots + r_n \cdot a_n$. Clearly $X' := \{a_1, \ldots, a_n\}$ is as required.
(4) Let $b \in Sp(Y)$ be given, by (3) let Y' be a finite subset of Y such that $b \in Sp(Y')$. From the definition of $Sp(\cdot)$ it is clear that $Sp(Y') \subseteq Sp(X)$.
(5) If $a \in Sp(X \bigcup \{b\})$ there are $r, r_1, \ldots, r_n \in R$ and $a_1, \ldots, a_n \in X$ such that $a = r_1 \cdot a_1 + \cdots + r_n \cdot a_n + r \cdot b$. Since $a \notin Sp(X)$ necessarily $r \neq 0$. Since R is a division ring we have in M that
$$b = r^{-1} \cdot (a - \{r_1 \cdot a_1 + \cdots + r_n \cdot a_n\}).$$
Namely $b \in Sp(X \bigcup \{a\})$.

□

DEFINITION 1.5 (Pre-Geometry). Let W be a non empty set.
A function $\mathrm{cl} : \mathcal{P}(W) \to \mathcal{P}(W)$ is called a *pre-geometry* if the following conditions holds:

(1) For every $X \subseteq W$ we have that $X \subseteq \mathrm{cl}(X)$.
(2) *Finite character*: If $a \in \mathrm{cl}(X)$ for some $X \in W$, then there exists a finite $X' \subseteq X$ such that $a \in \mathrm{cl}(X')$.
(3) *Transitivity* let $X, Y \subseteq W$ and suppose $a \in W$. if $a \in \mathrm{cl}(X)$ and $X \subseteq \mathrm{cl}(Y)$ then $a \in \mathrm{cl}(Y)$.
(4) *Exchange principle*: if $a \in \mathrm{cl}(X \bigcup \{b\}) - \mathrm{cl}(X)$ then $b \in \mathrm{cl}(X \bigcup \{a\})$.
(5) We will assume that cl is non-vacuous namely $\mathrm{cl}(\emptyset)$ is a proper subset of W.
(6) Let (W, cl) be a pre-geometry, and let A be a fixed subset of W.
The *localization of* (W, cl) *at* A is the pre-geometry (W, cl_A) when for every $X \subseteq W$ $\mathrm{cl}_A(X) := \mathrm{cl}(A \bigcup X)$.

DEFINITION 1.6 (Matroid). Let W be a nonempty set and let $S : \mathcal{P}(W) \to \mathcal{P}(W)$ be a pre-geometry (W, S) is called a *matroid* iff there exists a finite $B \subseteq W$ such that $\mathrm{cl}(B) = W$. A matroid (W, S) is called *simple* if for every $a \in W$ we have that $\mathrm{cl}(\{a\}) = \{a\}$, and $\mathrm{cl}(\emptyset) = \emptyset$.

Matroids play a central role in combinatorics and graph theory.

EXAMPLE 1.7 (The trivial matroid). Let W be a finite set. For every $X \subseteq W$ let $\mathrm{cl}(X) := X$. Clearly (W, cl) is a matroid.

EXAMPLE 1.8 (The trivial pre-geometry). Let W be a set. For every $X \subseteq W$ let $\mathrm{cl}(X) := X$. Clearly (W, cl) is a pre-geometry.

EXAMPLE 1.9 (Modules over division rings). Let R be a division ring, and let M be a left R-module. By Theorem 1.4 the function $Sp : \mathcal{P}(M) \to \mathcal{P}(M)$ is a pre-geometry.

EXAMPLE 1.10 (Fields). Let F be a subfield of the field E. An element $a \in E$ is called *algebraic over* F iff there exists a nonzero polynomial $p(x) \in F[x]$ such that $p(a) = 0$. An element $a \in E$ which is not algebraic over F is called *transcendental over* F. For a set $X \subseteq E$ let $F(X)$ be the subfield of E generated by $F \bigcup X$. Let $\mathrm{cl} : \mathcal{P}(E) \to \mathcal{P}(E)$ be defined as follows, for $X \subseteq E$ let

$$\mathrm{cl}(X) := \{a \in E \ : \ a \text{ is algebraic over } F(X)\}.$$

Namely, $\mathrm{cl}(X)$ is the algebraic closure of $F(X)$ in E. Similarly to Theorem 1.4 one can prove that (W, cl) is a pre-geometry. While the finite character and exchange are easy to verify, notice that the argument to get transitivity is little more involved than in linear algebra we use the following elementary facts from field theory:

(1) If $F \leq E$ are fields and $\alpha \in E$ then α is algebraic over F iff $F(\alpha)$ as an F-vector space is finite dimensional.
(2) If $F \leq K \leq E$ are fields then $[E : F] = [E : K] \cdot [K : F]$.

The following is a natural way matroids occur in graph theory:

EXAMPLE 1.11 (Connected components in graphs). We describe here an important matroid that can be associated with any finite graph it is called *the polygon matroid of a graph* .

Recall that a connected component of a graph $\langle V, E\rangle$ is $C \subseteq V$ such that $a \neq b \in V \implies \{a, b\} \in E$ and for every $a \in C$ and every $b \in V$ $\{a, b\} \in E \implies b \in C$.

Clearly connected components are equivalence classes. I.e. for $a, b \in V$ the relation $a \sim b$ defined by:

$$\exists n < \omega \exists e_0 \cdots \exists e_n \in E[\bigwedge_{i<n} e_i \cap e_{i+1} \neq \emptyset] \wedge a \in e_0 \wedge b \in e_n.$$

is an equivalence relation and the equivalence class of a is the connected component containing a.

Thus if C_1 and C_2 are connected components then $C_1 \neq C_2 \implies C_1 \cap C_2 = \emptyset$.

Let $\langle V, E\rangle$ be a finite graph. For $X \subseteq E$ let

$$\mathrm{cl}(X) := \Big\{ \{a, b\} \in E \mid \text{ there exists a connected component } C \text{ of } \langle V, X\rangle \text{ such that } a \in C \wedge b \in C \Big\}.$$

Let's verify the exchange principle: Suppose $X \subseteq E$ and $e_1, e_2 \in E$ are such that $e_1 \in \mathrm{cl}(X \cup e_2) - \mathrm{cl}(X)$. As $e_1 \notin \mathrm{cl}(X)$ there are distinct connected components C_1 and C_2 of $\langle V, X\rangle$ such that $e_\ell \in C_\ell$. By $e_2 \in \mathrm{cl}(X \cup e_1)$ the elements of $e_1 \cup e_2$

are in the same connected component of $\langle V, X \cup e_2 \rangle$, namely e_2 connects C_1 and C_2, thus $e_1 \in \mathrm{cl}(X \cup e_1)$.

EXAMPLE 1.12.

DEFINITION 1.13. Let (W, cl) be a pre-geometry, and $A \subset W$ be given. (W_A, cl_A) is called a *localization at* A iff $W_A := \{B \subseteq W \mid B \supseteq A\}$ and $\mathrm{cl}_A(B) := \mathrm{cl}(A \cup B)$.

EXAMPLE 1.14. AFFINE GEOMETRY,

EXAMPLE 1.15. There are several examples in model theory .....

LEMMA 1.16. *Let W be a nonempty set, and let X, Y be subsets of W. Suppose $S : \mathcal{P}(W) \to \mathcal{P}(W)$ is a pre-geometry*

(1) $X \subseteq Y \ \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$ *(monotonicity)*
(2) $\mathrm{cl}(Y) = \mathrm{cl}(\mathrm{cl}(Y))$ *(idempotance).*

PROOF.

(1) Let $a \in \mathrm{cl}(X)$ be given. By Definition 1.5 (1) $Y \subseteq \mathrm{cl}(Y)$ so in particular $X \subseteq \mathrm{cl}(Y)$. By transitivity we have that $a \in \mathrm{cl}(Y)$.
(2) By(1) $\mathrm{cl}(Y) \subseteq \mathrm{cl}(\mathrm{cl}(Y))$. Suppose $a \in \mathrm{cl}(\mathrm{cl}(Y))$ is given. By the finite character there exists $Z \subseteq \mathrm{cl}(Y)$ finite such that $a \in \mathrm{cl}(Z)$. An application of the transitivity property gives $a \in \mathrm{cl}(Y)$.

□

DEFINITION 1.17. Let (W, cl) be a pre-geometry. Suppose X is a nonempty subset of W.

(1) X is called cl-*independent* if for every $a \in X$ we have that $a \notin \mathrm{cl}(X - \{a\})$.
(2) when the identity of cl is clear from the context we will omit a mention of it.
(3) X *generates* W if $W = \mathrm{cl}(X)$.
(4) X *is a basis of* W if X is independent and X generates W.
(5) we say that W is *finitely generated* if there exists a finite $X \subseteq W$ such that $\mathrm{cl}(X) = W$. When (W, cl) is finitely generated in the literature quite often (W, cl) is called a *matroid*.
(6) When (W, cl) is finitely generated and for every $a \in W$ we have $\mathrm{cl}(\{a\}) = \{a\}$ and $\mathrm{cl}(\emptyset) = \emptyset$, (W, cl) is called *simple matroid* or also *combinatorial geometry*.

EXAMPLE 1.18. Notice that in a module over a division ring D (with the notion of $Sp_D(\cdot)$) our terminology coincides with that of linear algebra.

EXAMPLE 1.19. Suppose $F \leq E$ are fields, for $x \subseteq E$ let

$$\mathrm{cl}(X) = \{a \in E | a \text{ is algebraic over } F(X)\}$$

then a base for (E, cl) is called a *transcendence base*.

DEFINE RELATIVEZATION (LOCALIZATION)

THEOREM 1.20. *Let (W, S) be a pre-geometry. Suppose X is a nonempty subset of W. The following are equivalent*

(1) *X is a basis*
(2) *X is a minimal set of generators of W (i.e. $W = \mathrm{cl}(X)$ and if Y is a proper subset of X then $\mathrm{cl}(Y) \neq W$).*
(3) *X is a maximal independent (i.e. for every $a \in X$ we have that $a \notin \mathrm{cl}(X - \{a\})$ and for every proper extension Y of X, Y is not independent).*

PROOF. (1)⇒(2): Suppose X is a basis. The fact that X generates W is part of the definition of basis. Let Y be a proper subset of X. any $a \in X - Y$ exemplify that Y is not a set of generators since by independence of X and Lemma 1.16(1) $a \notin \mathrm{cl}(Y)$.

(2)⇒(3): Suppose X is a minimal set of generators. It is enough to show that X is independent. Let $a \in X$, define $Y := X - \{a\}$. Since X is a minimal set of generators $\mathrm{cl}(Y) \neq W$. In fact from Lemma 1.16 it follows that $X \subseteq \mathrm{cl}(Y)$. But clearly $Y \subseteq \mathrm{cl}(Y)$ so $X \subseteq \mathrm{cl}(Y)$ since $a \in \mathrm{cl}(Y)$ $(= \mathrm{cl}(X - \{a\})$.

(3)⇒(1): Suppose X is maximal independent subset of W. It is enough to show that X generates W. Let $a \in W$ be given, by Definition 1.5 if $a \in X$ then $a \in \mathrm{cl}(X)$, suppose $a \in W - X$, by the maximality of X we have that $a \in \mathrm{cl}(X)$. Namely we have shown that $W \subseteq \mathrm{cl}(X)$. □

THEOREM 1.21. *Let (W, cl) be a pre-geometry. Suppose X and Y are subsets of W, such that X is independent and Y generates W. There exists $Y' \subseteq Y$ such that $X \bigcup Y'$ is a basis and $X \cap Y' = \emptyset$.*

PROOF. Consider $P := \{Y^* | Y^* \subseteq Y, X \bigcup Y^* \text{is independent}, X \cap Y^* = \emptyset\}$ it is partially ordered by $\subseteq$. Suppose Y^* is a maximal element of P.
Define $B := X \cup Y^*$. In order to show that B is a basis it is enough to show that B generates W. Let $a \in W$ be given, since $\mathrm{cl}(Y) = W$ we have that $a \in \mathrm{cl}(Y)$. By the finite character there exists $Y_0 \subseteq Y$ finite such that $a \in \mathrm{cl}(Y_0)$. By transitivity it is enough to show that $Y_0 \subseteq \mathrm{cl}(B)$. For the sake of contradiction choose $b \in Y_0 - \mathrm{cl}(B)$, by Definition 1.5 and Lemma 1.16 $b \notin B$. Since $b \notin X \bigcup Y^*$ by the maximality of Y^* clearly $X \bigcup Y^* \bigcup \{b\}$ is dependent. Namely $b \in \mathrm{cl}(B)$ contradiction.

By the above it is enough to prove that P has a maximal element. Clearly if C is finite or just have a last element there is nothing to prove. So we may assume that C have no last element. By Zorn's lemma it is enough to verify that every chain C has an upper bound in P. Let $Y' := \bigcup C$, in order to show that Y' is an upper bound of C it suffices to show that $X \bigcup Y'$ is independent. For the sake of contradiction suppose that there exists $a \in X \bigcup Y'$ such that $a \in \mathrm{cl}(X \bigcup Y' - \{a\})$. By the finite character there exists a finite $Z \subseteq X \bigcup Y' - \{a\}$ such that $a \in \mathrm{cl}(Z)$. Let $Y'' \subseteq Y'$, and $X'' \subseteq X$ both finite such that $Z = X'' \bigcup Y'' - \{a\}$. Since C does not have a last element, the fact that Y'' is finite and the definition of Y'' we have that there exists $Y^* \in C$ such that $Y'' \subseteq Y^* \subseteq Y'$. By monotonicity property of $\mathrm{cl}(\cdot)$ we have that $\mathrm{cl}(Z) \subseteq \mathrm{cl}(X \bigcup Y^* - \{a\})$.
Thus $a \in \mathrm{cl}(X \bigcup Y^* - \{a\})$. Which is a contradiction to the hypothesis that $X \bigcup Y^*$ is independent. □

which

COROLLARY 1.22. *Every pre-geometry has a basis.*

PROOF. Let (W, cl) be a pre-geometry. Let $Y \subseteq W$ be a set of generators for W (we can take $Y = W$), since cl is non trivial there exists $a \in W - \mathrm{cl}(\emptyset)$. Clearly $X := \{a\}$ is independent. Now apply Theorem 1.21 to X and Y. □

THEOREM 1.23. *Let (W, cl) be a pre-geometry. Suppose X and B are nonempty subsets of W, such that X is independent. If B is a basis for W then $|X| \leq |B|$.*

PROOF. Suppose for the sake of contradiction that $|B| < |X|$. First we will show that we may assume that B is finite and later prove the theorem by induction on $|B|$. Suppose that B is infinite. Since $\mathrm{cl}(B) \supseteq X$, by the finite character for every $a \in X$ there exists a finite $B_a \subseteq B$ such that $a \in \mathrm{cl}(B_a)$. By the pigeonhole principle there exists an infinite $X' \subseteq X$ and a finite $B_0 \subseteq B$ such that $a \in X' \Rightarrow B_a = B_0$. Let $V := B_0 \bigcup X'$, and define $\mathrm{cl}^* : \mathcal{P}(V) \to \mathcal{P}(V)$ as follows: for $Y \subseteq V$ let $\mathrm{cl}^*(Y) := \mathrm{cl}(Y) \cap V$. It is easy to verify that cl^* is a pre-geometry on the set V, and B_0 is a cl^* - basis for (V, cl^*), and X' is cl^* - independent. Since X' is infinite and B_0 is finite it is clear that the following lemma contradict our original assumption that $|X| > |B|$. □

LEMMA 1.24. *Let (V, cl) be a pre-geometry. Suppose X and B are nonempty subsets of V, such that X is independent. If B is a finite basis for V then $|X| \leq |B|$.*

PROOF. For the sake of contradiction suppose that $|B| < |X|$. Since X is independent, and B is a maximally independent set B is not a subset of X. We will prove

CLAIM 1.25. *Let B be a basis for V with n elements, and let $m := |X \cap B|$, V has a basis B_1 with n elements such that $|X \cap B_1| = m + 1$.*

Apply the Claim $n - m$ times to obtain $B_2, B_3, \ldots, B_{n-m}$ all V-bases containing n elements when B_2 is produced from B_1 by the Claim having the property that $|X \cap B_2| = m + 2$, so eventually B_{n-m} will be a basis which is a subset of X in contradiction to the discussion above the Claim.

PROOF. Without loss of generality we may assume that X is a basis (by Theorem 1.21 we may replace X by a super-set which is a basis). Let $|B| = n$, there exists $m < n$ and we may choose an enumeration $\{a_1, \ldots, a_n\}$ of B such that $B \cap X = \{a_1, \ldots, a_m\}$. By Theorem 1.20 $B - \{a_{m+1}\}$ does not generate V. Since X generates V by monotonicity $X \not\subseteq \mathrm{cl}(B - \{a_{m+1}\})$. Let $c \in X - \mathrm{cl}(B - \{a_{m+1}\})$, and define $B_1 := (B - \{a_{m+1}\}) \bigcup \{c\}$. Since $|B_1| = n$ and $|B_1 \cap X| = m + 1$ it is enough to show B_1 is a basis. We need (using Theorem 1.20) to show that:

(1) B_1 is independent, and
(2) B_1 generates V.

B_1 is independent: Suppose there exists an element $a \in B_1$ such that $a \in \mathrm{cl}(B_1 - \{a\})$. Notice that since $c \notin X - \mathrm{cl}(B - \{a_{m+1}\})$ necessarily $a \neq c$. Since $a \neq c$ and $a \in B_1$ we have $a \in B - \{a_{m+1}\}$. By the independence of B we have $a \notin \mathrm{cl}(B - \{a_{m+1}\} - \{a\})$. Denote by S the set $B - \{a_{m+1}\} - \{a\}$. We have shown that $a \notin \mathrm{cl}(S)$ and $a \in \mathrm{cl}(S \cup \{c\})$. By the exchange principle we get $c \in \mathrm{cl}(S \cup \{a\})$. Which stands for $c \in \mathrm{cl}(B - \{a_{m+1}\})$, this contradicts the choice for c.

B_1 generates V: Since $\mathrm{cl}(B) = W$ by monotonicity it is enough to show that $a_{m+1} \in \mathrm{cl}(B_1)$. By the choice of c we have $c \notin \mathrm{cl}(B - \{a_{m+1}\})$. Since $c \in \mathrm{cl}(B)$ we have $c \in \mathrm{cl}(B - \{a_{m+1}\} \cup \{a_{m+1}\})$, an application of the exchange principle gives $a_{m+1} \in \mathrm{cl}(B - \{a_{m+1}\} \cup \{c\})$. □

□

COROLLARY 1.26. *Let W be a nonempty set, and let S be a pre-geometry. If B_1 and B_2 are bases of W then $|B_1| = |B_2|$.*

PROOF. Since B_1 is independent we have by Theorem 1.23 that $|B_1| \leq |B_2|$, another application of Theorem 1.22 to B_1 and B_2 gives $|B_1| \geq |B_2|$. By Cantor-Bernstein's theorem $|B_1| = |B_2|$. □

EXAMPLE 1.27. Let R be a division ring, and M left R-module.
In (M, Sp_R) by Corollary 1.22 M has a basis, and by Corollary 1.26 the cardinality of a basis is determined by the module. The cardinality of a basis for M is called the cardinality of M over R and is denoted by $\dim_R M$ another commonly used notation is $[M : R]$.

EXAMPLE 1.28. When $F \leq E$ are fields and cl_F is algebraic closure (see Example 1.9 (2)), a basis for (E, cl_F) is called a *transcendence basis for E over*

F, its cardinality (which by Corollary 1.26 is determined by E and F) is called *the transcendence degree of E over F* also denoted as $tr.d.E/F$.

By $tr.d.F$ denote $tr.d.E/F$ when F is the prime field of E.

As an application we present an alternative proof to

THEOREM 1.29 (Steinitz's theorem). *Let F and E be algebraically closed fields. If $\mathrm{Char}\, F = \mathrm{Char}\, E$ and $tr.d.F = tr.d.E$ then $F \cong E$. In particular, if both F and E are algebraically closed fields with $\mathrm{Char}\, F = \mathrm{Char}\, E$ then $|F| = |E| > \aleph_0$ implies $F \cong E$.*

PROOF. Let P be the prime field of F, we may assume that it is also the prime field of E. Let $\lambda := tr.d.F$. Fix $I := \{a_i \mid i < \lambda\} \subseteq F$ and $J := \{b_j \mid j < \lambda\} \subseteq E$ to be both transcendental bases.

Let $f : P[I] \cong P[J]$ (it is a ring isomorphism) defined by $f(q(a_{i_1}, \ldots, a_{i_n})) = q(b_{i_1}, \ldots, b_{i_n})$ for every $q(x_1, \ldots, x_n) \in P[x_1, \ldots, x_n]$. Let f_1 be the extension of f to $P(I)$ the field of fractions of $P[I]$. Clearly f_1 is a field isomorphism from $P(I)$ onto $P(J)$. Let F_1 be the algebraic closure of $P(I)$ and E_1 the algebraic closure of $P(J)$. As the original fields were algebraically closed we may assume that $F_1 \leq F$ and $E_1 \leq E$. By the uniqueness of algebraic closure there exists f_2 extending f_1 such that $f_2 : F_1 \cong E_1$

As I is a basis of F, $F = \mathrm{cl}(I)$. Namely every element of F is algebraic over F_1, as F_1 is algebraically closed we get that $F_1 = F$. Similarly $E_1 = E$. Thus $f_2 : F \cong E$.

For the last statement notice that if P is the prime field of F and as $\{a \in F \mid a \text{ is algebraic } \}$ has cardinality $\leq |P| + \aleph_0$, thus if F is uncountable then $tr.d.F = |F|$.

□

EXAMPLE 1.30. In model theory - the contents of Corollaries 10 and 12 is that regular types have true dimension.

REMARK 1.31. There exists a model for ZF (-Zermelo Fraenkel set theory "the usual set theory without the axiom of choice") where the real numbers have two bases B_1 and B_2 (as a vector space over $\mathbb{Q}$) such that $|B_1| \neq |B_2|$. This does not contradict Corollary 1.26 since in Theorem 1.23 which was used for the proof of Corollary 1.26 we have made a use of the axiom of choice! In the use of the pigeonhole principle when both sets involved are infinite (i.e. we had more pigeons (-elements of X) than holes but the set of infinitely many holes (- finite subsets of the set B)).

DEFINITION 1.32. A pre-geometry (W, cl) is a *geometry* iff

$$\mathrm{cl}(\{a\}) = \{a\} \quad \text{for all } a \in W.$$

It is easy to get a geometry from a given pre-geometry:

LEMMA 1.33. *Let* (W, cl) *be a pre-geometry. For* $a, b \in W - \mathrm{cl}(\emptyset)$ *the relation* $a \sim b$ *defined as* $\mathrm{cl}(\{a\}) = \mathrm{cl}(\{b\})$*, is an equivalence relation on the set* $W - \mathrm{cl}(\emptyset)$.

LEMMA 1.34. *If* (W, cl) *is a pre-geometry then* $(\bar{W}, \bar{\mathrm{cl}})$ *is geomerty where* $\bar{W} := W - \mathrm{cl}(\emptyset)/\sim$ *and* $\bar{\mathrm{cl}}(S) := \mathrm{cl}(\{a\} - \mathrm{cl}(\emptyset)$.

DEFINITION 1.35. (W, cl) be a pre-geometry.

(1) Let The *dimension* of W is the cardinality of its basis and is denoted by $\dim W$. If $A \subseteq W$ by $\dim A$ we denote $\dim \mathrm{cl}(A)$.
(2) For $B \subseteq A \subseteq W$ $\dim(A/B)$ is the dimension of the localization of $\mathrm{cl}(A)$ to the set B.
(3) Projective geometry
(4) The pre-geometry is *modular* iff for all finite dimensional A and B we have that
$$\dim(A \cup B) + \dim(A \cap B) = \dim(A) + \dim(B).$$
(5) The pre-geometry is *locally modular*

LEMMA 1.36 (additivity). (W, cl) *be a pre-geometry. Suppose* $B \subseteq A \subseteq W$. *we have*
$$\dim(A/B) + \dim(B) = \dim(A).$$

PROOF. □

LEMMA 1.37. *Let* (W, cl) *be a pre-geometry. Suppose* X *and* Y *are closed subsetes of* W. *we have*
$$\dim(X \cup Y) + \dim(X \cap Y) \leq \dim X + \dim Y.$$

PROOF. □

DEFINITION 1.38 (Flats). Let $\langle W, \mathrm{cl}\rangle$ be a pre-geometry. $X \subseteq W$ is called a *flat* provided $X = \mathrm{cl}(X)$.

DEFINITION 1.39 (Lattice).

PROPOSITION 1.40. *Let* $\langle W, \mathrm{cl}\rangle$ *be a pre-geometry. The set of flats of the pre-geometry is a lattice.*
$$L(W) := \{X \subseteq W \mid \mathrm{cl}(X) = X\}.$$

DEFINITION 1.41 (Linear). A pre-geometry $\langle W, \mathrm{cl}\rangle$ is called *linear pre-geometry* or *coordinatizable pre-geometry* provided there exists a division ring D and a D-module M such that there is also $f : W \to M$ satisfying
$$X \text{ is cl independent} \iff \{f(a) \mid a \in X\} \text{ is linearly independent in } M \text{ for all } X \subseteq W.$$

DEFINITION 1.42 (Graphic). A matroid isomorphic to a polygon matroid of a graph (see Example 1.11 is called *graphic matroid*.

DEFINITION 1.43 (Modular lattice). Semi-modular

modular

DEFINITION 1.44 (Geometric lattice). Semi-modular

modular

THEOREM 1.45 (Birkhoff-Whitney).

PROOF. □

THEOREM 1.46. *Let* (W, cl) *be a pre-geometry, The following are equivalent:*

(1) $\langle W, \mathrm{cl}\rangle$ *is modular*
(2) $\forall X \subseteq W$ *and for every* $a, c \in W$

$$c \in \mathrm{cl}(X \cup \{a\}) \to \exists b \in \mathrm{cl}(X) \text{ such that } c \in \mathrm{cl}(\{b, a\}).$$

(3) *For any closed* $X, Y \subseteq W$ *and all* $c \in W$ *if* $c \in \mathrm{cl}(A \cup B)$ *then there are* $a \in A$ *and* $b \in B$ *such that* $c \in \mathrm{cl}(a, b)$.

PROOF. □

EXAMPLE 1.47. Modules over division rings

EXAMPLE 1.48. Affine

EXAMPLE 1.49. Algebraically closed fields are not locally modular.

THEOREM 1.50. *Every modular pre-geometry is projective*

PROOF. □

EXAMPLE 1.51. Definition Homogeneous pre-geometry

1.1. Exercises.

(1) Let (W, cl) be a pre-geometry. For $a, b \in W$ define

$$a \sim b \iff \mathrm{cl}(a) = \mathrm{cl}(b).$$

Veirfy that $\sim$ is an equivalence relation on $W - \mathrm{cl}(\emptyset)$.
(2) Let $W' := W/sim$ and $\mathrm{cl}'(A) := \{\mathrm{cl}(a) \mid a/\sim \in A\}$ for all non-empty $A \subseteq W'$. Show that (W', cl') is a geometry whenever (W, cl) is a pre-geometry.
(3)

2. Abstract dependence

3. Projective geometries

CHAPTER 17

Plato: The Allegory of the Cave, from book VII The Republic

And now, I said, let me show in a figure how far our nature is enlightened or unenlightened:–Behold! human beings living in an underground den, which has a mouth open towards the light and reaching all along the den; here they have been from their childhood, and have their legs and necks chained so that they cannot move, and can only see before them, being prevented by the chains from turning round their heads. Above and behind them a fire is blazing at a distance, and between the fire and the prisoners there is a raised way; and you will see, if you look, a low wall built along the way, like the screen which marionette players have in front of them, over which they show the puppets.

I see.

And do you see, I said, men passing along the wall carrying all sorts of vessels, and statues and figures of animals made of wood and stone and various materials, which appear over the wall? Some of them are talking, others silent.

You have shown me a strange image, and they are strange prisoners.

Like ourselves, I replied; and they see only their own shadows, or the shadows of one another, which the fire throws on the opposite wall of the cave?

True, he said; how could they see anything but the shadows if they were never allowed to move their heads?

And of the objects which are being carried in like manner they would only see the shadows?

Yes, he said.

And if they were able to converse with one another, would they not suppose that they were naming what was actually before them?

Very true.

And suppose further that the prison had an echo which came from the other side, would they not be sure to fancy when one of the passers-by spoke that the voice which they heard came from the passing shadow?

No question, he replied.

To them, I said, the truth would be literally nothing but the shadows of the images.

That is certain.

And now look again, and see what will naturally follow if the prisoners are released and disabused of their error. At first, when any of them is liberated and compelled suddenly to stand up and turn his neck round and walk and look towards the light, he will suffer sharp pains; the glare will distress him, and he will be unable to see the realities of which in his former state he had seen the shadows; and then conceive some one saying to him, that what he saw before was an illusion, but that now, when he is approaching nearer to being and his eye is turned towards more real existence, he has a clearer vision,–what will be his reply? And you may further imagine that his instructor is pointing to the objects as they pass and requiring him to name them,–will he not be perplexed? Will he not fancy that the shadows which he formerly saw are truer than the objects which are now shown to him?

Far truer.

And if he is compelled to look straight at the light, will he not have a pain in his eyes which will make him turn away to take refuge in the objects of vision which he can see, and which he will conceive to be in reality clearer than the things which are now being shown to him?

True, he said.

And suppose once more, that he is reluctantly dragged up a steep and rugged ascent, and held fast until he is forced into the presence of the sun himself, is he not likely to be pained and irritated? When he approaches the light his eyes will be dazzled, and he will not be able to see anything at all of what are now called realities.

Not all in a moment, he said.

He will require to grow accustomed to the sight of the upper world. And first he will see the shadows best, next the reflections of men and other objects in the water, and then the objects themselves; then he will gaze upon the light of the moon and the stars and the spangled heaven; and he will see the sky and the stars by night better than the sun or the light of the sun by day?

Certainly.

Last of all he will be able to see the sun, and not mere reflections of him in the water, but he will see him in his own proper place, and not in another; and he will contemplate him as he is.

Certainly.

He will then proceed to argue that this is he who gives the season and the years, and is the guardian of all that is in the visible world, and in a certain way the cause of all things which he and his fellows have been accustomed to behold?

Clearly, he said, he would first see the sun and then reason about him.

And when he remembered his old habitation, and the wisdom of the den and his fellow-prisoners, do you not suppose that he would felicitate himself on the change, and pity them?

Certainly, he would.

And if they were in the habit of conferring honors among themselves on those who were quickest to observe the passing shadows and to remark which of them went before, and which followed after, and which were together; and who were therefore best able to draw conclusions as to the future, do you think that he would care for such honors and glories, or envy the possessors of them? Would he not say with Homer,

Better to be the poor servant of a poor master, and to endure anything, rather than think as they do and live after their manner?

Yes, he said, I think that he would rather suffer anything than entertain these false notions and live in this miserable manner.

Imagine once more, I said, such a one coming suddenly out of the sun to be replaced in his old situation; would he not be certain to have his eyes full of darkness?

To be sure, he said.

And if there were a contest, and he had to compete in measuring the shadows with the prisoners who had never moved out of the den, while his sight was still weak, and before his eyes had become steady (and the time which would be needed to acquire this new habit of sight might be very considerable), would he not be ridiculous? Men would say of him that up he went and down he came without his eyes; and that it was better not even to think of ascending; and if any one tried to loose another and lead him up to the light, let them only catch the offender, and they would put him to death.

No question, he said.

This entire allegory, I said, you may now append, dear Glaucon, to the previous argument; the prison-house is the world of sight, the light of the fire is the sun, and you will not misapprehend me if you interpret the journey upwards to be the ascent of the soul into the intellectual world according to my poor belief, which, at your desire, I have expressed– whether rightly or wrongly God knows. But whether true or false, my opinion is that in the world of knowledge the idea of good appears last of all, and is seen only with an effort; and, when seen, is also inferred to be the universal author of all things beautiful and right, parent of light and of the lord of light in this visible world, Here Plato describes his notion of God in a way that was influence profoundly Christian theologians. and the immediate source of reason and truth in the intellectual; and that this is the power upon which he would act rationally either in public or private life must have his eye fixed.

I agree, he said, as far as I am able to understand you.

Moreover, I said, you must not wonder that those who attain to this beatific vision are unwilling to descend to human affairs; for their souls are ever hastening into the upper world where they desire to dwell; which desire of theirs is very natural, if our allegory may be trusted.

Yes, very natural.

And is there anything surprising in one who passes from divine contemplations to the evil state of man, misbehaving himself in a ridiculous manner; if, while his eyes are blinking and before he has become accustomed to the surrounding darkness, he is compelled to fight in courts of law, or in other places, about the images or the shadows of images of justice, and is endeavoring to meet the conception of those who have never yet seen absolute justice?

Anything but surprising, he replied.

Any one who has common sense will remember that the bewilderments of the eyes are of two kinds, and arise from two causes, either from coming out of the light or from going into the light, which is true of the mind's eye; and he who remembers this when he sees any one whose vision is perplexed and weak, will not be too ready to laugh; he will first ask whether that soul of man has come out of the brighter life, and is unable to see because unaccustomed to the dark, or having turned from darkness to the day is dazzled by excess of light. And he will count the one happy in his condition and state of being, and he will pity the other; or, if he have a mind to laugh at the soul which comes from below into the light, there will be more reason in this than in the laugh which greets him who returns from above out of the light into the den.

That, he said, is a very just distinction.

But then, if I am right, certain professors of education must be wrong when they say that they can put a knowledge into the soul which was not there before, like sight into blind eyes.

They undoubtedly say this, he replied.

Whereas our argument shows that the power and capacity of learning exists in the soul already; and that just as the eye was unable to turn from darkness to light without the whole body, so too the instrument of knowledge can only by the movement of the whole soul be turned from the world of becoming into that of being, and learn by degrees to endure the sight of being and of the brightest and best of being, or in other words, of the good.

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