Notions of amalgamation for abstract elementary classes

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Introduction and Outline

Motivated by the free products of groups, the direct sums of modules, and Shelah's $(\lambda, 2)$-goodness, we study strong amalgamation properties in Abstract Elementary Classes. Such a notion of amalgamation consists of a selection of certain amalgams for every triple $M_0 \leq M_1, M_2$, and we show that if a weak AEC K designates a unique strong amalgam to every triple $M_0 \leq M_1, M_2$, then K satisfies categoricity transfer at cardinals $\geq \theta(K) + 2^{\mathrm{LS}(K)}$, where $\theta(K)$ is a cardinal associated with the notion of amalgamation. We also show that if such a unique choice does not exist, then there is some model $M \in K$ having $2^{|M|}$ many extensions which cannot be embedded in each other over M. Thus, for AECs which admit a notion of amalgamation, the property of having unique amalgams is a dichotomy property in the sense of Shelah's classification theory.

We present a framework of a "notion of amalgamation" for a given abstract elementary class. Abstracting from the examples of free amalgamation of groups and direct sum of modules, we isolate the axiomatic properties of absolute minimality, regularity, continuity, and admitting decomposition (Definition 2.6), which we assume throughout the paper. We also define the uniqueness property of amalgams, which intuitively states that for any triple of models there is a unique amalgam (up to isomorphism) which is "nice". We refer to a notion satisfying all of the above as a notion of free amalgamation, and establish that when a class K has a notion of free amalgamation and is categorical in a sufficiently large cardinal, then it behaves analogously to the models of a unidimensional first order theory. This allows us to prove a categoricity transfer theorem (Theorem 5.6):

Theorem. *Suppose $\mathcal{A}$ is a notion of free amalgamation in K, and K has a prime and minimal model. If K is λ^*-categorical in some $\lambda^* \geq \theta(K)$, then K is λ-categorical in every cardinal $\lambda \geq \theta(K) + (2^{LS(K)})^+$.*

In this formulation, the cardinal $\theta(K)$ is defined from the given notion of free amalgamation, and is analogous to $\kappa(T)$ for T a simple theory.

Of course, this begs the question of how strong the assumptions above are. In particular, we have mentioned previously that the assumption of unique "nice" amalgams implies that types have unique nonforking extensions. In fact, like stability, the uniqueness property delineates between structural results on one hand and anti-structural results on the other. This can be seen by combining the above theorem and Theorem 6.8:

Theorem. *Suppose $\mathcal{A}$ is regular, continuous, absolutely minimal and has weak 3-existence. If (M_b, M^*, M) is a non-uniqueness triple and $p = gtp(M^*/M_b, M^*)$, then there is $N \geq M$ such that p has $2^{|N|}$-many extensions to N.*

As an application of the categoricity transfer result, we consider a type p with $U(p) = 1$ and K_p the class of realizations of p: this class (under some assumptions) is naturally associated with corresponding pregeometries, which allows us to conclude (Theorem 7.13):

Theorem. *Suppose K admits finite intersections and has a stable independence relation with the $(< \aleph_0)$-witness property. If $U(p) = 1$, then K_p is λ-categorical in all $\lambda > |dom\ p| + LS(K)$*

Notably, this is analogous to the case of an uncountably categorical countable theory, where the sets $\phi(M)$ for a strongly minimal $\phi(x)$ are also uncountably categorical. This is, of course, a crucial component of the Baldwin-Lachlan proof of Morley's categoricity theorem.

The outline of this paper is as follows: in Chapter 2, we formally define notions of amalgamation for an abstract class, and establish some basic properties which follow from the definition. We then introduce some axiomatic properties for notions of amalgamation, and also explore both examples and counter-examples to these properties.

Chapter 3 introduces sequential amalgamation, and most of the section is dedicated to proving Theorem 3.14, which roughly states that when $\mathcal{A}$ is well-behaved, then the ordering of the sequence of amalgamation does not affect the $\mathcal{A}$-amalgam. In Chapter 4, we introduce some notation for amalgams and the cardinal invariant $\mu(K)$, and use them to show that for an independence relation defined from a given notion of amalgamation, this independence relation behaves similarly to forking in a simple, stable, or even superstable theory.

Chapter 5 uses the additional assumption that $\mathcal{A}$ has uniqueness (as well as some other axiomatic properties introduced in Chapter 2) to show that the class K admits categoricity transfer at cardinals $> \theta(K) + 2^{\mathrm{LS}(K)}$, where $\theta(K)$ is a cardinal characteristic derived from the notion of amalgamation. On the other hand, in Chapter 6 we show that failing to have uniqueness implies that there are arbitrarily large models with the maximal number of non-isomorphic (in fact non-biembeddable) extensions. Finally, in Chapter 7 we apply the technology developed to the class K_p, which are the realizations of some type p with $U(p) = 1$, and show that K_p is necessarily categorical in a tail of cardinals. Some open questions and possible directions of further work are addressed in Chapter 8.

Contents

Chapter 1

Background

1.1 A Primer on Abstract Elementary Classes

One of the first facts that modern students of model theory can come to realize is that first-order logic is quite weak with regards to defining classes of models, such that even simple ideas such as the class of locally finite groups is not firstorderizable. This was, of course, well known to model theorists of earlier decades, and much work was done exploring stronger logics which allow infinitary conjunctions, infinitary quantifiers, cardinality quantifiers, and many other constructions. On the other hand, since the compactness theorem fails for all of these stronger logics (without assuming the existence of e.g. strongly compact cardinals), model theory for such infinitary logics has to progress without one of the most widely used tools of first-order model theory.

Even in prior to the modern framework of non-elementary classes, it was recognized that to effectively work in such infinitary logics, a notion of being a "strong" submodel that is more nuanced than the notion of elementary submodels used in first-order logic was needed: for example, in [Kei71], a classical textbook for model theory in $\mathcal{L}_{\omega_1,\omega}$, Keisler makes extensive use of the "end extension" relation between models with a specified definable ordering. It is in this context in which Shelah first introduced abstract elementary classes (AEC) in [She87]: its key idea being that one focuses on some notion of a "strong submodel" relation within a class of models and study the class of structures with an ordering relation, rather than focusing on the specific syntactic properties of different logics and how this affects the classes of models it can define.

Given a fixed language τ, an AEC in τ consists of a pair $(K, \leq_K)$, where K is a class of τ-structures and $\leq_K$ is a partial ordering on K. We further require the following properties to hold of the pair:

- K is closed under τ-isomorphisms
- Given models $M, N \in K$, $M \leq_K N$ implies that M is a τ-substructure of N, and moreover $\leq_K$ is preserved under τ-isomorphisms
- (The Löwenheim-Skolem property) There is a cardinal λ such that for any model $N \in K$ and a set $A \subseteq N$, there is a submodel $M \leq_K N$ with $A \subseteq M$ and such that $|M| \leq |A| + \lambda$; we denote the minimal such λ to be $\mathrm{LS}(K)$
- (The Tarski-Vaught property) If α is a limit ordinal and $(M_i : i < \alpha)$ is an $\leq_K$-increasing and continuous chain of models in K such that each $M_i \leq_K M^*$, then $N := \bigcup_{i<\alpha} M_i$ is also a model in K with each $M_i \leq_K N$. Moreover, if M^* is a model in K such that each $M_i \leq_K M^*$, then $N \leq_K M^*$ as well
- (The coherence property) Given models $M_1, M_2, N \in K$, if $M_1 \leq_K N$, $M_2 \leq_K N$, and $M_1 \subseteq M_2$, then $M_1 \leq_K M_2$

It is straightforward to see that the class of models which satisfy some first-order theory T (i.e. the elementary class $\mathrm{Mod}(T)$) with the elementary submodel ordering satisfies the above properties, and in this sense the axiomatic definition above guarantees enough structure in the class of models to apply some arguments analogous to those used in first-order model theory. As an example, whereas in first-order model theory one can present arguments in terms of elementary embeddings instead of elementary substructures, for an AEC K we define a K-embedding as a τ-embedding $f : M \to N$ where $f[M] \leq_K N$, and we can proceed in proofs using K-embeddings instead of the $\leq_K$ ordering.

As is the case of first-order model theory, the above axiomatic setting merely lays the foundation necessary for model-theoretic arguments, and further assumptions are necessary if we wish to prove stronger results. One such assumption is the amalgamation property, which had been studied extensively by Fraïssé, Robinson, and Jónnson for its importance in constructing Fraïsse limits and also in the context of algebraic logic due to the connection between interpolation theorems for various logics and the amalgamation property of the corresponding algebraizations (for example, as in Chapter 12 of [Mon76]). In the current context, an AEC K has the amalgamation property (abbreviated to AP) if for any triple of models M_0, M_1, M_2 and K-embeddings $f : M_0 \to M_1$, $g : M_0 \to M_2$, there exists a model N with K-embeddings f', g' such that the following diagram commutes:

$$\begin{tikzcd} M_2 \arrow[r, "g'"] & N \\ M_0 \arrow[u, "g"] \arrow[r, "f"] & M_1 \arrow[u, "f'"'] \end{tikzcd}$$

It should be noted that despite the use of commutative squares above, AP does not refer to any universal property, and thus an AEC having AP does not imply that K has pushouts (when considering the class of models and K-embeddings as a category).

1.2 Historical context and motivating works

In [She75], Shelah extended his solution to the Whitehead problem (under $V = L$) to show that Whitehead groups of arbitrary cardinality are free, and a key result within this paper became what is now referred to as Shelah's singular compactness theorem. Whilst the precise statement of the singular compactness theorem is beyond the scope of this introduction, it was in this context that Shelah, and later Hodges and Eklof introduces an axiomatic notion of "bases" and "freeness", abstracted from the relation between free groups and free bases which generate them (for a concise and modern presentation, see [Ekl08]). This axiomatic notion of "freeness" has seen application to other structures such as modules, graph colorings, and set transversals, but one weakness of this particular formulation is the reliance on the idea that an object is "generated" by some abstract "basis" of elements, which does not have a direct analog in more general classes of models such as an AEC. Further investigation, however, hinted that this notion of basis and generating elements can be replaced in an AEC by choosing certain well-behaved amalgams of models; in particular, the free amalgam of groups and the direct sum of modules are the guiding example for how this replacement might proceed.

This use of some "nice" amalgamation in lieu of objects generated by bases brings up interesting possibilities: as an example, both the class of vector spaces (over $\mathbb{Q}$, for example) and the class of algebraically closed fields over a prime p are classes where forking can be equivalently defined by reference to dimensions (specifically, $\mathbb{Q}$-linear dimension for vector spaces over $\mathbb{Q}$ and transcendental dimension for ACF_p). In fact, in [GL00] Grossberg and Lessmann derived a forking-like independence relation on an arbitrary pregeometry (X, cl), and showed that many of the defining properties of forking in a stable theory are also satisfied in this setting. It seems plausible, therefore, that for a suitably well-behaved AEC with a suitably "nice" amalgamation property, a notion of forking independence can be defined.

Forking independence is, of course, one of the most important concepts in first-order model theory, and much work has been devoted to finding analogous results for AECs. Of the many related work, two are especially important in the scope of their applicability:

- In [She09], Shelah introduced the notion of (λ-) good frames, an axiomatic framework for (non)forking of types over models.
- In [Bon+16], the authors showed sufficient conditions under which a notion of subsets inside a model in an AEC being independent is canonical, analogous to how forking independence is canonical for

models of a stable theory: in other words, for any AEC which is sufficiently well-behaved, there is a unique notion of forking independence that satisfies all the axiomatic properties.

In particular, if we are able to define an independence relation within for our context of a (still hypothetical) suitably well-behaved AEC with a "nice" amalgamation property, then the canonicity of nonforking independence implies that these notions of independence must be equivalent.

Assuming that a class of models has some "nice" amalgamation property to derive more macroscopic structural results about the class is not unheard of; this is, in fact, reminiscent of the "stable amalgams" first introduced by Shelah in [She83a] and [She83b], where the fact that stable amalgams can be extended and are preserved under continuous chains is used to construct a model in a higher cardinality. In this case (and, as we will detail later, for the current context as well), we are only interested in certain "nice" amalgams, but the collection of nice amalgams have certain extendibility, continuity, and/or uniqueness properties which is needed for the analysis. It should be noted that this idea (and Shelah's earlier work in [She82] on the Dimensional Order Property) marks the beginning of the body of literature concerning amalgamation of independent sets/types/diagrams, which has since developed in multiple directions such as study of the homology of such diagrams by Goodrick, Kim, and Kolesnikov in [GKA13] and the quasiminimal excellent classes studied by Zilber in [Zil05].

In summary, from the above historical context we can outline the goal for this project as follows:

1. Define some reasonable form of amalgamation on AECs that shares characteristics with e.g. the free amalgamation of nonabelian groups

2. Use the "nice" amalgam to derive properties similar to that of a class with pregeometries; in particular, try to derive an independence relation analogous to nonforking independence

3. Derive further structural properties about AECs which admit such a "nice" amalgamation

1.3 Selected contemporary work

Before beginning the main body of this work, we present here some specific results from within the last decade that the author finds to be of particular relevance. We will also make further direct comparisons between these works and the present work in later sections. Note that all of the following research is independent of this paper.

In [BR16], Beke and Rosický extends the singular compactness theorem to a category-theoretical setting:

Theorem (from [BR16], Proposition 4.4 and the following remark)**.** *Let $\mathcal{A}$ be an accessible category with filtered colimits, $\mathcal{B}$ an AEC and $F : \mathcal{A} \to \mathcal{B}$ a functor preserving filtered colimits. Assume that F-structures extend along morphisms (see Definition 1.1 in [BR16]). Let $X \in \mathcal{B}$ be a model of size μ, a singular cardinal. If all subobjects of X of size less than μ are in the image of F, then X itself is in the image of F.*

In the previous section on the history of the singular compactness hypothesis, we mentioned that a key difficulty we encountered in attempting to translate the earlier work of Shelah, Hodges, and Eklof into the context of AECs was to identify a suitable notion of models being "generated"; Beke and Rosický resolved this issue with the introduction of the category $\mathcal{A}$ and the functor F as in the stated result. Not only is this work (in the author's opinion) significant in its own right in connecting AECs, category theory and concepts from homology (see the section on cellular structures in [BR16]), but it is also an important source of examples for the current work: translating the (subclass of) AECs for which singular compactness theorem holds into some notion of "basis" and generation often immediately leads to finding the relevant notion of amalgamation within the subclass.

In [SV18], Shelah and Vasey extend the notion of excellence, first applied to classes defined by a $\mathcal{L}_{\omega_1,\omega}$ sentence in [She83b], to the context of AECs, and proved:

Theorem (Theorem 14.2 of [SV18])**.** *If K is excellent and categorical in some $\mu > LS(K)$, then K is categorical in all $\mu' \geq \mu + \beth_{(2^{\aleph_{LS(K)^+}})^+}$*

Besides the clear significance of an eventual categoricity theorem for excellent AECs, this paper is of particular relevance to the current work as excellence relies on the existence of nonforking amalgams with structural properties similar to those discussed later in section Chapter 2 and extending such amalgams to multidimensional diagrams. Given that the amalgamation of 3-dimensional diagrams plays a significant role in Chapter 5 and 6, this work, together with the canonicity of nonforking in AECs, helps to provide examples where 3-dimensional amalgams do indeed exist.

In [Vas18a], Vasey extended Zilber's quasiminimal pregeometry classes and defined what was named *quasiminimal AECs* using the following semantic properties

Definition 1.1 (Definition 4.1, [Vas18a])**.** An AEC K is quasiminimal if:

1. $\mathrm{LS}(K) = \aleph_0$
2. K has a prime model
3. $K_{\leq\aleph_0}$ admits intersections (see Definition 3.2 of [Vas18a] for the precise definition)
4. For every $M \in K_{\leq\aleph_0}$, there is a unique 1-type over M that is not realized inside M

Vasey then showed that there exists a direct correspondence between such quasiminimal AECs and Zilber's quasiminimal pregeometry classes, which implies that such AECs also have the full uncountable categoricity structure of pregeometry classes. Moreover, it was shown (see section 4, in particular Corrollary 4.10, of [Vas18a]) that the exchange property in Zilber's definition of quasiminimal pregeometry classes was in fact redundant. For the purpose of this paper, Vasey's quasiminimal AECs is also a major inspiration for the categoricity transfer results of Chapter 5.

In [LRV19], Lieberman, Rosický, and Vasey introduces a notion of *stable independence* for accessible categories (and thus AECs) in a category-theoretical language. Importantly, stable independence is defined in terms of the class having sufficiently many and sufficiently nice amalgams, and in particular the required properties of such amalgams are morally equivalent to several properties we explore in this paper; compare, specifically, section 3 of [LRV19] and Chapter 2 of this paper. On the other hand, whilst [LRV19] focuses on proving forking-like properties of stable independence (see section 8 and Theorem 9.1 of their paper) and showing when stable independence can exist, in the current work we use some stronger assumptions about amalgams (specifically, that "nice" amalgamation should be absolutely minimal and admit decomposition) to obtain the structural theorems of sections Chapters 5 and 6.

Finally, in [Kam20], Kamsma extends the above work by Lieberman et al. to obtain a more general notion of *simple independence*; as in the case of first-order model theory, the key dividing line between stable and simple independence relations is the stationarity or uniqueness of nonforking types (after some suitable translation into the applicable frameworks). Importantly, it is shown that:

Theorem (Corollary 1.2, [Kam20])**.** *An abstract elementary category (a generalization of AECs) with the amalgamation property admits at most one simple independence relation.*

This is particularly relevant to the current work, as we will show that an AEC admitting a "nice" amalgamation will also admit an independence relation that is a simple independence relation in section Chapter 4, which the above result implies is indeed the canonical independence relation even without the uniqueness property.

1.4 Preliminaries

We first recall some basic definitions regarding abstract elementary classes (AECs), and the more general notion of abstract classes introduced by Grossberg in [Gro]. A more detailed overview of basic concepts and results can be found in [Bal09].

Definition 1.2. Let τ be a language.

1. $(K, \leq_K)$ is an **abstract class** (in τ) iff:
   - K is a class of τ-structures which are closed under τ-isomorphisms

- $\leq_K$ is a partial order on K, and $M \leq N$ implies that M is a τ-substructure of N
- The partial order is invariant under isomorphisms: if $M \leq_K N$, $M' \subseteq N'$, $f : M \simeq M'$ and $g : N \simeq N'$ are isomorphisms, and $f \subseteq g$, then $M' \leq_K N'$

2. $(K, \leq_K)$ is a **very weak abstract elementary class** if it is an abstract class that satisfies:

   - The **Löwenheim-Skolem property**: there is a cardinal $\mathrm{LS}(K)$ such that for any model $N \in K$ and any set $A \subseteq N$, there is $M \leq_K N$ such that $A \subseteq M$ and $|M| \leq |A| + \mathrm{LS}(K)$, and $\mathrm{LS}(K)$ is the minimal cardinal to satisfy this property.
   - The **(weak) Tarski-Vaught chain property**: if α is a limit ordinal and $(M_i)_{i<\alpha}$ is an $\leq_K$-increasing continuous chain of models in K, then $N := \bigcup_{i<\alpha} M_i$ is also a model in K, and each $M_i \leq_K N$

3. $(K, \leq_K)$ is a **weak abstract elementary class** if it is a very weak AEC which additionally satisfies the **Coherence property**: if $M_1 \leq_K N$, $M_2 \leq_K N$, and $M_1 \subseteq M_2$, then $M_1 \leq_K M_2$

4. $(K, \leq_K)$ is an **abstract elementary class** if it is a weak AEC which additionally satisfies the **Smoothness property**: if α is a limit ordinal, $(M_i)_{i<\alpha}$ is an $\leq_K$-increasing continuous chain, and for each $i < \alpha$ we have that $M_i \leq_K N$, then $M_\alpha := \bigcup_{i<\alpha} M_i \in K$ and $M_\alpha \leq_K N$

For $(K, \leq_K)$ an abstract class, we denote by $\tau(K)$ the language of the models in K. We drop the subscript in $\leq_K$ when it is clear from context.

Definition 1.3. Let $(K, \leq)$ be an abstract class.

1. Given $M, N \in K$, a τ-homomorphism $f : M \longrightarrow N$ is a K**-embedding** iff f is a τ-isomorphism between M and $f[M]$, and $f[M] \leq N$

2. $(K, \leq)$ has the **Amalgamation Property** (AP) if for models M_0, M_1, M_2 with K-embeddings $f_1 : M_0 \longrightarrow M_1, f_2 : M_0 \longrightarrow M_2$, there is a model $N \in K$ with K-embeddings $g_1 : M_1 \longrightarrow N, g_2 : M_2 \longrightarrow N$ such that the following diagram commutes:

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f_2"] \arrow[r, "f_1"] & M_1 \arrow[u, "g_1"'] \end{tikzcd}$$

3. We define the class $K^3 := \{(\bar{a}, M, N) : M \leq N, \bar{a} \in N\}$

4. Given $(\bar{a}_1, M, N_1), (\bar{a}_2, M, N_2) \in K^3$, we define the relation $\sim$ such that $(\bar{a}_1, M, N_1) \sim (\bar{a}_2, M, N_2)$ iff there is a model $N' \geq N_2$ and a K-embedding $f : N_1 \longrightarrow N'$ such that $f \restriction M = \mathrm{id}_M$ and $f(\bar{a}_1) = \bar{a}_2$

Fact 1.4. *If $(K, \leq)$ has AP, then $\sim$ is an equivalence relation.*

Definition 1.5. Given $(\bar{a}, M, N) \in K^3$, the **Galois type** $\mathrm{gtp}(\bar{a}/M, N)$ is the equivalence class of $(\bar{a}, M, N)$ under $\sim$. We say that $\bar{a}$ realizes the Galois type p if $\mathrm{gtp}(\bar{a}/M, N) = p$. Given an ordered set I, we let $S^I(M)$ denote the collection of Galois types of the form $\mathrm{gtp}((a_i)_{i\in I}/M, N)$

Chapter 2

Notions of Amalgamation

2.1 The basic idea

Let $(K, \leq)$ be an abstract class. We would like to capture the idea of selecting certain amalgams of triples $M_0 \leq M_1, M_2$ and designating them as the "nice" amalgams that we will focus on; this is formalized in the following definition.

Definition 2.1. Let the tuple (M_0, M_1, M_2, f) be given such that $M_0, M_1, M_2 \in K$, $M_0 \leq M_1$ and $f : M_0 \longrightarrow M_2$ is a K-embedding. A triple (N, g_1, g_2) is an **amalgam of** M_1 **and** M_2 **over** M_0 **via** f if $N \in K$, $g_1 : M_1 \longrightarrow N$ and $g_2 : M_2 \longrightarrow N$ are K-embeddings, and the following diagram commutes (where ι denotes the inclusion embedding):

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] & M_1 \arrow[u, "g_1"'] \end{tikzcd}$$

For simplicity, we will also refer to the above diagram as an amalgam (of M_1 and M_2 over M_0 via f). We denote the collection of such amalgams by $\mathrm{Amal}(M_0, M_1, M_2, f)$.

A (class) function $\mathcal{A}$ is a **pre-notion of amalgamation** if:

- Its domain is the class of tuples (M_0, M_1, M_2, f) such that $M_0 \leq M_1$ and $f : M_0 \longrightarrow M_2$ is a K-embedding; and
- For each such tuple, $\mathcal{A}(M_0, M_1, M_2, f) \subseteq \mathrm{Amal}(M_0, M_1, M_2, f)$

For a triplet $(N, g_1, g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$, we say that (N, g_1, g_2) is an $\mathcal{A}$**-amalgam of** M_1 **and** M_2 **over** M_0 **via** f, which we will also denote by the annotated diagram:

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"'] \end{tikzcd}$$

We say that $\mathcal{A}$ is a **notion of amalgamation** if in addition to being a pre-notion, the following properties hold of $\mathcal{A}$:

- (Completeness) For every tuple (M_0, M_1, M_2, f) as above, $\mathcal{A}(M_0, M_1, M_2, f)$ is nonempty.
- $\mathcal{A}$ contains trivial amalgams: For any $M_0 \leq M_1$, $(M_1, \iota, \mathrm{id}) \in \mathcal{A}(M_0, M_0, M_1, \iota)$. Diagrammatically,

$$\begin{tikzcd} M_1 \arrow[r, "\mathrm{id}"] & M_1 \\ M_0 \arrow[u, "\iota"] \arrow[r, "\mathrm{id}"] \arrow[ur, phantom, "\mathcal{A}"] & M_0 \arrow[u, "\iota"'] \end{tikzcd}$$

- (Top Invariance) For every $(N, g_1, g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$ and every K-isomorphism $h : N \simeq N'$, $(N', h \circ g_1, h \circ g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$. Diagrammatically,

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \arrow[r, "h"] & N' \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"'] & \end{tikzcd} \implies \begin{tikzcd} M_2 \arrow[r, "h \circ g_2"] & N' \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "h \circ g_1"'] \end{tikzcd}$$

- (Side Invariance 1) For every $(N, g_1, g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$ and K-isomorphism $h : M_1 \simeq M'$, $(N, g_1 \circ h^{-1}, g_2) \in \mathcal{A}(h[M_0], M', M_2, f \circ (h \restriction M_0)^{-1})$. Diagrammatically,

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"'] \arrow[d, "h"] \\ & M' \end{tikzcd} \implies \begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ h[M_0] \arrow[u, "f \circ (h \restriction M_0)^{-1}"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "g_1 \circ h^{-1}"'] \end{tikzcd}$$

- (Side Invariance 2) For every $(N, g_1, g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$ and K-isomorphism $h : M_2 \simeq M'$, $(N, g_1, g_2 \circ h^{-1}) \in \mathcal{A}(M_0, M_1, M', h \circ f)$. Diagrammatically,

$$\begin{tikzcd} M' & \\ M_2 \arrow[u, "h"] \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"'] \end{tikzcd} \implies \begin{tikzcd} M' \arrow[r, "g_2 \circ h^{-1}"] & N \\ M_0 \arrow[u, "h \circ f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"'] \end{tikzcd}$$

- (Symmetry) If $(N, g_1, g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$, then $(N, g_2, g_1) \in \mathcal{A}(f[M_0], M_2, M_1, f^{-1})$. Diagrammatically,

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"'] \end{tikzcd} \implies \begin{tikzcd} M_1 \arrow[r, "g_1"] & N \\ f[M_0] \arrow[u, "f^{-1}"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "g_2"'] \end{tikzcd}$$

Remark. Technically, $\mathcal{A}$ fails to even be a class function in the strictest sense, as $\mathcal{A}(M_0, M_1, M_2, f)$ is a proper class because of the invariance properties and also because there is no bound on the cardinality of the amalgams. This can of course be resolved by the assumption of a strongly inaccessible cardinal κ such that every model of K has cardinality $< \kappa$; in any case, this is inconsequential to this paper.

Clearly, if $\mathcal{A}$ is a notion of amalgamation for K, then K must have the Amalgamation Property as $\mathcal{A}$ is complete. Generally, we are interested in notions of amalgamation which specify certain well-behaved amalgams: for example, if K has the Disjoint Amalgamation Property, we may define $\mathcal{A}_d$ as only the amalgams

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}_d"] & M_1 \arrow[u, "g_1"'] \end{tikzcd}$$

where $g_1[M_1] \cap g_2[M_2] = g_1[M_0]$. Since we would like to work in K while ignoring the other amalgams which are not well-behaved, the properties defined above are designed such that some basic results which hold for amalgamation in general also hold for $\mathcal{A}$. For example:

Lemma 2.2. *Suppose $\mathcal{A}$ is a notion of amalgamation, and:*

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"'] \end{tikzcd}$$

1. *There is some $N' \geq M_1$ and $g_2' : M_2 \longrightarrow N'$ such that*

$$\begin{tikzcd} M_2 \arrow[r, "g_2'"] & N' \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"'] \end{tikzcd}$$

2. *There is some $N'' \geq M_2$ and $g_1' : M_1 \longrightarrow N''$ such that*

$$\begin{tikzcd} M_2 \arrow[r, "\iota"] & N'' \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1'"'] \end{tikzcd}$$

Proof. 1. Let N' be a copy of N such that $M_1 \leq N'$, and $h : N \simeq N'$ be such that $h \circ g_1 = \iota : M_1 \hookrightarrow N'$. Letting $g_2' = h \circ g_2$, the desired result follows from Top Invariance.

2. Similar to (1), using N'' a copy of N such that $M_2 \leq N''$.

$\square$

Lemma 2.3. *Suppose $\mathcal{A}$ is a notion of amalgamation, and:*

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"'] \end{tikzcd}$$

Then

$$\begin{tikzcd} g_2[M_2] \arrow[r, "\iota"] & N \\ g_1[M_0] \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & g_1[M_1] \arrow[u, "\iota"'] \end{tikzcd}$$

Proof. Firstly, note that as the diagram is commutative, indeed $g_1[M_0] = (g_2 \circ f)[M_0] \leq g_2[M_2]$. By Side Invariance 1 (via the isomorphism $g_1 : M_1 \simeq g_1[M_1]$),

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ g_1[M_0] \arrow[u, "f \circ (g_1 \restriction M_0)^{-1}"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & g_1[M_1] \arrow[u, "\iota"'] \end{tikzcd}$$

Then, by Side Invariance 2 (via the isomorphism $g_2 : M_2 \simeq g_2[M_2]$),

$$\begin{tikzcd} g_2[M_2] \arrow[r, "\iota"] & N \\ g_1[M_0] \arrow[u, "g_2 \circ f \circ (g_1 \restriction M_0)^{-1}"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & g_1[M_1] \arrow[u, "\iota"'] \end{tikzcd}$$

Finally, as $g_1 \restriction M_0 = g_2 \circ f$, hence $g_2 \circ f \circ (g_1 \restriction M_0)^{-1} = \iota : g_1[M_1] \hookrightarrow g_2[M_2]$ as desired. $\square$

Given the above lemmas, we see that to specify a notion of amalgamation $\mathcal{A}$, it suffices to specify when a commutative square of the form

$$\begin{tikzcd} M_2 \arrow[r, "\iota"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] & M_1 \arrow[u, "\iota"'] \end{tikzcd}$$

is in fact a $\mathcal{A}$-amalgam. Similarly, for most results of $\mathcal{A}$, it suffices to prove the statement only for commutative diagrams as above.

Remark. Within the model theory literature, it is customary to say that N is an amalgam of M_1, M_2 over M_0 if there is a K-embedding f such that

$$\begin{tikzcd} M_2 \arrow[r, "f"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] & M_1 \arrow[u, "\iota"'] \end{tikzcd}$$

Hence, if N is an amalgam of M_1, M_2 over M_0, then for any $N' \geq N$, in this customary language it is also true that N' is an amalgam of M_1, M_2 over M_0. On the other hand, in this paper the phrase "N (with g_1, g_2) is an $\mathcal{A}$-amalgam of M_1, M_2 over M_0 (via f)" refers specifically to the statement "$(N, g_1, g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$". In particular, since we do not assume that $\mathcal{A}$ has any upward-closure property, it is not necessarily true that for every $N' \geq N$, $(N', g_1, g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$. It is, however, a relevant concept for the current investigation, and so we introduce a slight variant of the phrase to differentiate this interesting case:

Definition 2.4. We say that N is an **$\mathcal{A}$-amalgam by inclusion** of M_1 and M_2 over M_0 if the following diagram is an $\mathcal{A}$-amalgam:

$$\begin{tikzcd} M_2 \arrow[r, "\iota"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"'] \end{tikzcd}$$

For $M_0 \leq M_1, M_2 \leq N$, we say that M_1 **and** M_2 **are $\mathcal{A}$-subamalgamated over** M_0 **inside** N if there is some $N' \leq N$ such that N' is an $\mathcal{A}$-amalgam by inclusion of M_1, M_2 over M_0.

Remark. It is important to note that we are <u>not</u> asserting that every triple $M_0 \leq M_1, M_2$ can be amalgamated by inclusions; nor will we be assuming that such a property holds for any notion of amalgamation we consider. This definition simply allows us to refer specifically to $\mathcal{A}$-amalgams of the above form.

Example 2.5. Let K be the class of vector spaces over a fixed field F, with $\leq_K$ the subspace relation. We can define $\mathcal{A}$ such that for $V \leq W_1, W_2 \leq U$, U is an $\mathcal{A}$-amalgam of W_1, W_2 over V (by inclusion) iff $W_1 \cap W_2 = V$ and $\mathrm{span}(W_1 \cup W_2) = U$. In this example, if $U' \gneq U$, then U' is <u>not</u> an $\mathcal{A}$-amalgam of W_1, W_2 over V. However, W_1, W_2 are $\mathcal{A}$-subamalgamated over V inside U'. More generally, if T is (for example) a first order stable theory, we can define $\mathcal{A}$ such that for models $M_0 \preccurlyeq M_1, M_2$ with $M_1 \underset{M_0}{\perp} M_2$, N is an $\mathcal{A}$ amalgam iff N is $(a, \kappa_r(T))$-prime over $M_1 \cup M_2$.

Some other examples of notions of amalgamation which we are interested in include:

1. Consider the class of groups with the subgroup ordering. Given $G \leq H, K$, the free amalgamated product $H *_G K$ is formed by taking the free product of H, K and identifying the two copies of G together. This defines a notion of amalgamation on the class.

2. Similarly, consider the class of (left-) modules over a fixed ring R with the submodule ordering. As in the case for vector spaces, we can define $\mathcal{A}$ such that given $M_0 \leq M_1, M_2 \leq N$, N is an $\mathcal{A}$-amalgam of M_1, M_2 over M_0 iff $M_1 \cap M_2 = M_0$ and $\mathrm{span}(M_1 \cup M_2) = N$. Note that this is equivalent to defining $\mathcal{A}$-amalgams by taking direct sums and quotienting to identify the copies of the amalgamation base.

3. More generally, if $\mathcal{V}$ is a variety of algebras (in the sense of an equational class in universal algebra) and the category $\mathcal{C}$ consisting of algebras in $\mathcal{V}$ with embeddings has pushouts, then the pushout construction is a notion of amalgamation for $\mathcal{C}$. This example will be developed in more detail below (see Example 2.8).

4. Consider the class of algebraically closed fields with characteristic p: Given $K_0 \leq K_1, K_2 \leq L$, we define $\mathcal{A}$ such that L is an $\mathcal{A}$-amalgam of K_1, K_2 over K_0 iff $K_1 \cap K_2 = K_0$, K_1 and K_2 are algebraically independent over K_0, and $\mathrm{acl}(K_1 \cap K_2) = L$. More generally, this construction holds for any AEC K where each model has a pregeometry which is "coherent" with K; we will develop this idea further in section 7.

5. In a different vein, let K be a class of algebras which is an expansion of Boolean algebras, for example the class of cylindric algebras or polyadic algebras (of some fixed dimension α). K is said to have the *super amalgamation property* if any span A_0, A_1, A_2 can be amalgamated by some (B, f_1, f_2) satisfying: for every $x \in A_1$ and $y \in A_2$, if $f_1(x) \leq f_2(y)$ then there is $z \in A_0$ such that $x \leq z$ and $z \leq y$, and vice versa. If K has the super amalgamation property, then the super amalgams define a notion of amalgamation.

6. In [SV18], the notion of ϕ-amalgamation is defined over an *AEC* for a quantifier-free formula ϕ (assuming for simplicity that the language τ is relational): the diagram

$$\begin{tikzcd} M_2 \arrow[r, "f_2"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] & M_1 \arrow[u, "f_1"'] \end{tikzcd}$$

is a ϕ-amalgam iff $\phi(M_1), \phi(M_2)$ are equal as τ-structures and $f_1 \upharpoonright \phi(M_1) = f_2 \upharpoonright \phi(M_2)$. This is clearly also a notion of amalgamation in the current sense.

7. Adapting [Ekl08] to the present context, there a notion of amalgamation for $\mathcal{Q}$-filtered modules which respects the "free factor" ordering. We explore this example in detail in Appendix B.

8. On the other hand, in the class of groups with the subgroup ordering, we can define another notion $\mathcal{A}$ such that for $G_0 \leq G_1, G_2 \leq H$, H is an $\mathcal{A}$-amalgam of G_1, G_2 over G_0 iff $H = \langle G_1 \cup G_2 \rangle$. This is an example where $\mathcal{A}$ gives very little structural information about the class.

2.2 Some structural properties

Some of the examples above show that even with $\mathcal{A}$ a specifically defined notion of amalgamation, $\mathcal{A}$ might not provide any structural information on the underlying class besides having the amalgamation property. As we are interested in stronger results which do not follow simply from the fact that K has AP, we are interested in notions which satisfy some extra properties.

Definition 2.6. Let K be an abstract class, and let $\mathcal{A}$ be a notion of amalgamation in K.

- $\mathcal{A}$ is **minimal** if for every $(N, g_1, g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$, N is minimal over $g_1[M_1] \cup g_2[M_2]$ i.e. if $N' \leq N$ and $g_1[M_1] \cup g_2[M_2] \subseteq N'$, then $N' = N$.
- $\mathcal{A}$ is **absolutely minimal** if for every $(N, g_1, g_2) \in \mathcal{A}(M_0, M_1, M_2, f)$ and for any $N^* \geq N$, if $N' \leq N^*$ is such that $g_1[M_1] \cup g_2[M_2] \subseteq N'$, then $N \leq N'$.
- $\mathcal{A}$ is **regular** if for every commutative square in K, the following conditions are equivalent:
  1. The commutative square is an $\mathcal{A}$-amalgam i.e.

$$\begin{tikzcd} M_2 \arrow[r, "g_2"] & N \\ M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"'] \end{tikzcd}$$

2. There is some M', N', g' such that $M_0 \leq M' \leq M_1$, $g_2[M_2] \leq N' \leq N$, and $g' = g_1 \restriction M'$, with both of the following commutative squares being $\mathcal{A}$-amalgams:

$$\begin{tikzcd}
M_2 \arrow[r, "g_2"] & N' \arrow[r, "\iota"] & N \\
M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "g'"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"']
\end{tikzcd}$$

3. For every M' such that $M_0 \leq M' \leq M_1$, there exists $N' \leq N$ such that the following commutative square is an $\mathcal{A}$-amalgam:

$$\begin{tikzcd}
M_2 \arrow[r, "g_2"] & N' \arrow[r, "\iota"] & N \\
M_0 \arrow[u, "f"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "g_1 \restriction M'"'] &
\end{tikzcd}$$

   Moreover, for any such choice of N', the following commutative square is also an $\mathcal{A}$-amalgam:

$$\begin{tikzcd}
N' \arrow[r, "\iota"] & N \\
M' \arrow[u, "g_1 \restriction M'"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "g_1"]
\end{tikzcd}$$

- $\mathcal{A}$ is **continuous** if for any limit δ and increasing continuous chains $(M_i)_{i<\delta}, (N_i)_{i<\delta}$ with K-embeddings $(f_i : M_i \longrightarrow N_i)_{i<\delta}$ such that:

$$\begin{tikzcd}
N_0 \arrow[r, "\iota"] & N_1 \arrow[r, "\iota"] & N_2 \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & N_i \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & \cdots \\
M_0 \arrow[u, "f_0"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "f_1"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "f_2"] \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M_i \arrow[u, "f_i"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1} \arrow[u, "f_{i+1}"'] \arrow[r, "\iota"] & \cdots
\end{tikzcd}$$

  The commutative square of the respective unions is also an $\mathcal{A}$-amalgam:

$$\begin{tikzcd}
N_0 \arrow[r, "\iota"] & \bigcup_{i<\delta} N_i \\
M_0 \arrow[u, "f_0"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & \bigcup_{i<\delta} M_i \arrow[u, "\bigcup_{i<\delta} f_i"']
\end{tikzcd}$$

- $\mathcal{A}$ **admits decompositions** if for every $M_0 \lneq M_1 \lneq N$, there is a M_2 such that $M_0 \lneq M_2 \lneq N$ and N is an $\mathcal{A}$-amalgam of M_1, M_2 over M_0 (via the inclusion maps).

- $\mathcal{A}$ has **uniqueness** if for any two amalgams $(N, g_1, g_2), (N', g_1', g_2') \in \mathcal{A}(M_0, M_1, M_2, f)$, there exists a K-isomorphism $h : N \cong N'$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & N' \\
M_2 \arrow[r, "g_2"] \arrow[urr, bend left, "g_2'"] & N \arrow[ur, "h"] & \\
M_0 \arrow[u, "f"] \arrow[r, "\iota"] & M_1 \arrow[u, "g_1"] \arrow[uur, bend right, "g_1'"'] &
\end{tikzcd}$$

Remark.

- The "absolute" in "absolutely minimal" refers to the fact that the amalgam N of M_1, M_2 over M_0 is minimal over $M_1 \cup M_2$ only relative to models N' which can be jointly embedded with N; this is only an issue in the current framework since we do not assume the existence of monster models.

- The literature is unfortunately split over the nomenclature for what is defined as uniqueness above: this property is sometimes known as "strong uniqueness", whereas (using the language of [SV18]) "uniqueness" would refer to the property that two amalgamation diagrams can be amalgamated as indexed system of models. However, it is our opinion that within the current presentation the unqualified name "uniqueness" is more natural in terms of the existence of isomorphisms.

- Furthermore, the uniqueness property is substantially different from the other properties defined above. This is because the properties such as minimality, continuity, and regularity are necessary for $\mathcal{A}$ to resemble taking direct sums enough to motivate any further work (as we will discuss in Section 3). On the other hand, both the uniqueness property and its failure have significant model-theoretical consequences; we will explore the consequences of the positive case in Section 5, and the consequences of the negative case in Section 6.

With these properties, we can start differentiating between various notions of amalgamation and the implications on the structure of the underlying class. A simple but illustrative example comes from abelian groups, and more specifically the torsion divisible groups:

Example 2.7. Fix S a family of abelian groups such that for $G, H \in S$ with $G \neq H$, for any abelian group K and group embeddings $f : G \longrightarrow K$, $g : H \longrightarrow K$, $f[G] \cap g[H] = 0$, where 0 is the trivial group (for example, the Prüfer p-groups $S := \{\mathbb{Z}(p^\infty) : p \text{ a prime}\}$). Define the class K such that $M \in K$ iff M is a direct sum $\bigoplus_{i<\alpha} G'_i$, where each G'_i is isomorphic to some $G_i \in S$, and let the ordering $\leq_K$ be the subgroup ordering. Note that the condition on S implies that if $G, H \in K$ and $G \leq_K H$, then $H = G \oplus (\bigoplus_{i<\alpha} H'i)$ for some sequence of subgroups H'_i which are isomorphic to groups in S.

In this case, K has an obvious notion of amalgamation $\mathcal{A}$, where H is a $\mathcal{A}$-amalgam of G^1, G^2 over G^0 (by inclusion) iff $H = \bigoplus_{i<\alpha} H_i$, and there are sets $S_0, S_1, S_2 \subseteq \alpha$ such that:

- For $l = 0, 1, 2$, $G^l = \bigoplus_{i \in S_l} H_i$
- $S_1 \cap S_2 = S_0$ and $S_1 \cup S_2 = \alpha$

It is straightforward to see that $\mathcal{A}$ is minimal, absolutely minimal, regular, continuous, admits decomposition, and has uniqueness.

It is interesting to note that S as defined above cannot contain $\mathbb{Q}$ since $\mathbb{Q}$ can be embedded as a proper subgroup of itself. Of course, in the case where K is the class of divisible groups, since any divisible group admits a unique decomposition into copies of $Z(p^\infty)$ and $\mathbb{Q}$, $\mathcal{A}$ can be naturally extended to a notion of amalgamation in the class of divisible groups. In particular, this extension of $\mathcal{A}$ formally relies on the natural notion of amalgamation in the class of vector spaces over $\mathbb{Q}$, which obviously satisfies all of the above properties. In this case, the notion $\mathcal{A}$ on the class of divisible groups also satisfies all of these properties.

Generalizing the above construction from abelian groups to varieties of algebra, we get:

Example 2.8. Let $\mathcal{V}$ be a (finitary) variety of algebra (in the sense of universal algebra), and suppose $\mathcal{C}$ is a subcategory of $\mathcal{V}$ such that:

- $\mathcal{C}$ is closed under V-isomorphisms
- Every morphism of $\mathcal{C}$ is an embedding
- Defining $M \leq_C N$ iff the inclusion map $\iota : M \hookrightarrow N$ is a morphism in $\mathcal{C}$, then the abstract class $C = (\text{obj}(\mathcal{C}), \leq_C)$ is a weak AEC.
- $\mathcal{C}$ is closed under pushouts

We can then define a notion of amalgamation $\mathcal{A}$ on C by defining pushout diagrams to be $\mathcal{A}$-amalgams. Note that the fact that all morphisms in $\mathcal{C}$ are embeddings implies that this pushout amalgamation is absolutely minimal, regular, continuous, and has uniqueness. In fact, one can see that the fact that $\mathcal{V}$ is a variety of algebra was not particularly important in this example, although often the fact that $\mathcal{C}$ has pushouts stems from the fact that the variety $\mathcal{V}$ has pushouts (even though $\mathcal{C}$ itself might not be a variety).

The free product over groups also gives rise to more complicated examples of amalgamation, for example using small cancellation theory:

Example 2.9. Let S be a class function on triples of groups, such that for $G_0 \leq G_1, G_2$, $S(G_0, G_1, G_2) \subseteq \mathcal{P}(G_1 *_{G_0} G_2)$ is a nonempty family of sets such that each $R \in S(G_0, G_1, G_2)$ is symmetrized and satisfies $C'(1/6)$, where $C'(\lambda)$ is the metric small cancellation condition (see [LS01], Chapter 5 for discussion related to small cancellation theory, including the relevant definitions).

Now, let K be the class of groups ordered by the subgroup relation, and define $\mathcal{A}$ such that given $G_0 \leq G_1, G_2 \leq H$, H is an $\mathcal{A}$-amalgam of G_1, G_2 over G_0 (by inclusion) iff $H \cong G_1 *_{G_0} G_2 / \langle R \rangle_N$, where $R \in S(G_0, G_1, G_2)$ and $\langle R \rangle_N$ is the normal closure of R in H.

In particular, we note that if $S(G_0, G_1, G_2)$ contains (for example) both the empty set and a set not contained inside G_0, then there are two $\mathcal{A}$-amalgams of G_1, G_2 over G_0 which are not isomorphic over $G_1 \cup G_2$, and hence $\mathcal{A}$ does not have uniqueness. Similarly, whether or not $\mathcal{A}$ satisfies regularity, continuity, and admission of decomposition depends on the function S. On the other hand, $\mathcal{A}$ is necessarily absolutely minimal as H is generated by $G_1 \cup G_2$.

There also exist examples outside the realm of algebra, although in such cases we may lose absolute minimality:

Example 2.10. Let T be a countable, totally trascendental, superstable theory that has NDOP, and let K be the class of $\aleph_0$-saturated models of K with the elementary substructure relation. Note that as T is superstable, K is closed under union of continuous chains, and thus K is an AEC with $\mathrm{LS}(K) = \aleph_0$. Working within a monster model of T, define $\mathcal{A}$ such that N is an $\mathcal{A}$-amalgam of M_1, M_2 over M_0 by inclusion iff $M_1 \underset{M_0}{\perp} M_2$ and N is a minimal a-prime model over $M_1 \cup M_2$, which exists as T is NDOP by assumption. Then $\mathcal{A}$ is minimal, regular, continuous, and has uniqueness.

For the rest of this paper, we will restrict our attention to very weak AECS:

Hypothesis 2.11. *$(K, \leq)$ is a very weak AEC.*

Despite not requiring the class K to satisfy Smoothness and Coherence (see Definition 1.2), the properties defined in Definition 2.6 for a notion of amalgamation puts additional constraints on the class, and the example below shows that even a very "natural" notion of amalgamation in a very weak AEC can fail to have the above properties:

Example 2.12. Consider the class $(K_{ACFp}, \leq_K)$, where K_{ACFp} is the class of algebraically closed fields of characteristic p but $L_1 \leq_K L_2$ iff $|L_1| < |L_2|$ or L_2 is a limit model over L_1. It is straightforward to check that $(K_{ACFp}, \leq_K)$ is a very weak AEC. Note that L_2 is a limit model over L_1 iff $\mathrm{td}(L_2/L_1) = |L_2|$, where $\mathrm{td}(K/F)$ is the transcendental degree of K over F.

We define a notion of amalgamation $\mathcal{A}$ in the following manner: given $L_0 \leq_K L_1, L_2 \leq_K M$, M is an $\mathcal{A}$-amalgam of L_1, L_2 over L_0 (by inclusion) iff

1. $L_1 \cap L_2 = L_0$
2. L_1 and L_2 are algebraically independent over L_0
3. Assuming WLOG $|L_1| \leq |L_2|$, $\mathrm{td}(M/L_3) = 0$ if $|L_1| = |L_2|$ and $\mathrm{td}(M/L_3) = |L_2|$ otherwise, where $L_3 := \mathrm{acl}(L_1 \cup L_2)$

The third condition is necessary (for example) in the case where $|L_1| < |L_2|$, since in this case

$$\mathrm{td}(L_3/L_2) = \mathrm{td}(L_1/L_0) = |L_1| < |L_2|$$

which implies that L_3 is not a limit model over L_2. On the other hand, $\mathcal{A}$ does not satisfy some of the above properties:

- $\mathcal{A}$ is not minimal: given models L_0, L_1, L_2, L_3, M as above, there is some model $M' \lneq_K M$ such that $|M'| = |M|$ and $L_3 \leq_K M'$, so in particular $\mathrm{td}(M'/L_3) = \mathrm{td}(M/L_3)$. Hence M' is also an $\mathcal{A}$-amalgam of L_1, L_2 over L_0.

- $\mathcal{A}$ is not continuous: Suppose the set $\{a_i : i < \omega\} \cup \{b_j : j < \omega_1\} \subseteq M$ are algebraically independent, and define:
  1. $M_0 = \bar{\mathbb{Q}}$
  2. $N_0 = M_0(a_i : i < \omega)$
  3. For $\alpha \geq 1$, $M_i = M_0(b_j : j < \omega \cdot \alpha)$
  4. For $\alpha \geq 1$, $N_i = M_0(\{a_i : i < \omega\} \cup \{b_j : j < \omega \cdot \alpha\})$

  Note then this gives $\mathcal{A}$-amalgams:

$$\begin{tikzcd}
N_0 \arrow[r, "\iota"] & N_1 \arrow[r, "\iota"] & N_2 \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & N_i \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & \cdots \\
M_0 \arrow[r, "\iota"] \arrow[u, "f_0"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[r, "\iota"] \arrow[u, "f_1"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[r, "\iota"] \arrow[u, "f_2"] & \cdots \arrow[r, "\iota"] & M_i \arrow[r, "\iota"] \arrow[u, "f_i"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1} \arrow[r, "\iota"] \arrow[u, "f_{i+1}"'] & \cdots
\end{tikzcd}$$

  On the other hand, $N_{\omega_1} = M_0(\{a_i : i < \omega\} \cup \{b_j : j < \omega_1\})$ is not a limit model over $M_{\omega_1} = M_0(b_j : j < \omega_1)$ as $\mathrm{td}(N_{\omega_1}/M_{\omega_1}) = \aleph_0$, and so in particular N_{ω_1} is not an $\mathcal{A}$-amalgam of N_0, M_{ω_1} over M_0.

By assuming that $\mathcal{A}$ satisfies some of the properties from Definition 2.6, a few basic results can be deduced. In particular, these results are analogous to basic properties of the direct sum on vector spaces.

Lemma 2.13. *Suppose $\mathcal{A}$ is a notion of amalgamation that is regular. If M_0, M_1, M_2, M_3, M', N are models such that:*

1. *M' is a $\mathcal{A}$-amalgam of M_1, M_2 over M_0 by inclusion, i.e.*

$$\begin{tikzcd}
M_1 \arrow[r, "\iota"] & M' \\
M_0 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"]
\end{tikzcd}$$

2. *N is a $\mathcal{A}$-amalgam of M_3, M' over M_0 by inclusion, i.e.*

$$\begin{tikzcd}
M_3 \arrow[r, "\iota"] & N \\
M_0 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "\iota"]
\end{tikzcd}$$

Then there is $N' \leq N$ such that:

1. *N' is an $\mathcal{A}$-amalgam of M_2, M_3 over M_0 by inclusion; and*
2. *N is an $\mathcal{A}$-amalgam of M_1, N' over M_0 by inclusion*

Proof. Note that by the regularity, since $M_0 \leq M_2 \leq M'$, there exists $N' \leq N$ such that:

$$\begin{tikzcd}
M_3 \arrow[r, "\iota"] & N' \arrow[r, "\iota"] & N \\
M_0 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "\iota"]
\end{tikzcd}$$

In particular, we have the following diagram:

$$\begin{tikzcd}
M_3 \arrow[r, "\iota"] & N' \arrow[r, "\iota"] & N \\
M_0 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] \arrow[dr, "\mathrm{id}"] & M_2 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "\iota"] \\
& M_0 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"]
\end{tikzcd}$$

Applying regularity to the two commutative squares on the right, this shows that N is indeed a $\mathcal{A}$-amalgam of N', M_1 over M_0 by inclusion. $\square$

Lemma 2.14. *Suppose $\mathcal{A}$ is a notion of amalgamation and is absolutely minimal. If M_1, M_2 are $\mathcal{A}$-subamalgamated over M_0 inside N, then there is a unique $N' \leq N$ such that N' is the $\mathcal{A}$-amalgam of M_1, M_2 over M_0 by inclusion.*

Proof. Let $N' \leq N$ be an $\mathcal{A}$-amalgam of M_1, M_2 over M_0 by inclusion, and suppose $N^* \leq N$ is also an $\mathcal{A}$-amalgam of M_1, M_2 over M_0 by inclusion. In particular, hence $M_1 \cup M_2 \subseteq N^*$. As $\mathcal{A}$ is absolutely minimal and $N', N^* \leq N$, hence $N' \leq N^*$. The symmetric argument also shows that $N^* \leq N'$, and hence $N' = N^*$. $\square$

Notation 2.15. If $\mathcal{A}$ is absolutely minimal, and the models $M_0 \leq M_1, M_2 \leq N$ are such that M_1, M_2 are $\mathcal{A}$-subamalgamated over M_0 inside N, then we denote the unique $\mathcal{A}$-amalgam inside N by $M_1 \oplus^N_{M_0} M_2$.

Lemma 2.16. *Suppose $\mathcal{A}$ is absolutely minimal and regular. Then for any $M \leq N$, the operation $\oplus^N_M$ is commutative and associative where defined.*

Proof. That $\oplus^N_M$ is commutative is from $\mathcal{A}$ being symmetric. Associativity follows from Lemma 2.13. $\square$

Definition 2.17. We say an abstract class K **admits finite intersections** (abbreviated to has **FI**) if whenever M_1, M_2 are such that there exists M_0, N with $M_0 \leq M_1, M_2 \leq N$, then the intersection $M_1 \cap M_2$ is a model in K.

Lemma 2.18. *Let $\mathcal{A}$ be a notion of amalgamation.*

1. *If $\mathcal{A}$ is absolutely minimal, then it is minimal.*
2. *If K admits finite intersections and $\mathcal{A}$ is minimal, then $\mathcal{A}$ is absolutely minimal.*

Proof. Note that by the Invariance properties of $\mathcal{A}$, it suffices to show that the above statements hold for any M_0, M_1, M_2, N such that:

$$\begin{tikzcd} M_1 \arrow[r, "\iota"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"'] \end{tikzcd}$$

1. Assume that $\mathcal{A}$ is absolutely minimal. If $N' \leq N$ is such that $M_1 \cup M_2 \subseteq N'$, then $N \leq N'$ by absolute minimality, and hence $N' = N$. This shows that $\mathcal{A}$ is minimal.
2. Assume that K admits finite intersections and $\mathcal{A}$ is minimal. Then, if $N^* \geq N$ and $N' \leq N^*$ is such that $M_1 \cup M_2 \subseteq N'$, since K admits finite intersection, $N'' = N \cap N'$ is also a model of K, and furthermore $M_1 \cup M_2 \subseteq N''$. But then by minimality, $N'' = N$, and hence $N \leq N'$ as desired.

$\square$

Lemma 2.19. *Suppose $\mathcal{A}$ is minimal. If N is an $\mathcal{A}$-amalgam of M_1, M_2 over M_0 by inclusion and $|N| \geq LS(K)$, then $|N| = |M_1| + |M_2| + LS(K)$.*

Proof. Since $M_1, M_2 \leq N$, by the Löwenheim-Skolem axiom there is some $N' \leq N$ such that $|N'| \leq \mathrm{LS}(K) + |M_1 \cup M_2|$. Since $\mathcal{A}$ is minimal, hence $N' = N$, giving the desired result. $\square$

Lemma 2.20. *Suppose $\mathcal{A}$ is a notion of amalgamation that is regular and continuous. Let N be an $\mathcal{A}$-amalgam of M^*, M over M_b by inclusion, δ be a limit ordinal, and $(M_i)_{i<\delta}$ be a continuous resolution of M such that $M_b \leq M_0$. Then there is a continuous resolution $(N_i)_{i<\delta}$ of N such that for each $i < \delta$, N_i is an $\mathcal{A}$-amalgam of M^*, M_i over M_b by inclusion.*

Proof. We will construct N_i by induction:

1. Since N is an $\mathcal{A}$-amalgam of M^*, M over M_b by inclusion, and M_0 is such that $M_b \leq M_0 \leq M$, by regularity there is $N_0 \leq N$ such that:

$$\begin{tikzcd}
M^* \arrow[r, "\iota"] & N_0 \arrow[r, "\iota"] & N \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"]
\end{tikzcd}$$

2. If N_i is already defined, by construction

$$\begin{tikzcd}
M^* \arrow[r, "\iota"] & N_i \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_i \arrow[u, "\iota"]
\end{tikzcd}$$

   As $\mathcal{A}$ is regular, it is also the case that

$$\begin{tikzcd}
N_i \arrow[r, "\iota"] & N \\
M_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"]
\end{tikzcd}$$

   Since $M_i \leq M_{i+1} \leq M$, again by regularity, there is $N_{i+1} \leq N$ such that

$$\begin{tikzcd}
N_i \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & N \\
M_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"]
\end{tikzcd}$$

3. At limit stage α, we have

$$\begin{tikzcd}
M^* \arrow[r, "\iota"] & N_0 \arrow[r, "\iota"] & N_1 \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & N_i \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & \cdots \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots
\end{tikzcd}$$

   As $\mathcal{A}$ is continuous and $(M_i)_{i<\alpha}$ is an increasing continuous chain, letting $N_\alpha = \bigcup_{i<\alpha} N_i$, we get that

$$\begin{tikzcd}
M^* \arrow[r, "\iota"] & N_\alpha \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_\alpha \arrow[u, "\iota"]
\end{tikzcd}$$

$\square$

Chapter 3

Sequential amalgamation

3.1 Associativity of amalgamation

From a model-theoretic perspective, that the class of vector spaces over a fixed (countable) field is uncountably categorical stems from the exchange property of vectors and the fact that all vector spaces are direct sums of 1-dimensional spaces. In order to mimic this structure (or equivalently, the structure of models with a pregeometry), we must first define the amalgam of not only two models but of a possibly infinite sequence of models. We thus devote this section to showing that under the assumptions of $\mathcal{A}$ being absolutely minimal, regular, and continuous, then sequential amalgamation under $\mathcal{A}$ behaves as one would expect from the example of direct sums.

Notation 3.1. For an ordinal α, we define the ordinal $s(\alpha)$ by:

- $s(\alpha) = \alpha$ for limit α
- $s(\alpha) = \alpha + 1$ otherwise

Definition 3.2. Let $M_b \in K$, and let $(M_i)_{i<\alpha}$ be a sequence of models such for each $i < \alpha$, $M_b \leq M_i$. We say that N is an **$\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b** if there exists a sequence of models $(N_i)_{i<s(\alpha)}$ and K-embeddings $(f_i : M_i \longrightarrow N_{i+1})_{i<\alpha}$ such that:

1. $N_0 = M_b$ and $N_1 = f_0[M_0]$
2. For each $i < \alpha$, $f_i[M_b] = M_b$ and $f_i[M_i] \leq N_{i+1}$
3. $(N_i)_{i<s(\alpha)}$ is a continuous resolution of N i.e. it is an increasing continuous chain with $N = \bigcup_{i<s(\alpha)} N_i$.
4. For every $i \geq 1$, the following diagram is an $\mathcal{A}$-amalgam:

$$\begin{tikzcd} N_i \arrow[r, "\iota"] & N_{i+1} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_i \arrow[u, "f_i"'] \end{tikzcd}$$

Paralleling the two-model case, we say that N is an **$\mathcal{A}$-amalgam by inclusion** of $(M_i)_{i<\alpha}$ over M_b if N is an $\mathcal{A}$-amalgam as above with each f_i being an inclusion map $\iota_i : M_i \hookrightarrow N_{i+1}$. When each $M_i \leq N$, we say that $(M_i)_{i<\alpha}$ **is $\mathcal{A}$-subamalgamated over** M_b **inside** N if there is some $N' \leq N$ such that N' is an $\mathcal{A}$-amalgam by inclusion.

In order to understand what properties of sequential amalgams are desirable for our analysis, recall that any divisible group can be uniquely decomposed as a direct sum of (copies of) the rationals and Prüfer p-groups. Using this as a guiding example, ideally the amalgamation of a sequence of models should be independent from the order of amalgamation, and moreover it should be possible to take subsets of a "basis" to construct smaller models. In order to prove this claim (Theorem 3.14), we proceed by a number of lemmas:

Lemma 3.3. *If N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b, then for any $\beta \leq \alpha$, there exists some $L \leq N$ such that:*

1. *L is an $\mathcal{A}$-amalgam of $(M_i)_{i<\beta}$ over M_b; and*
2. *N is an $\mathcal{A}$-amalgam of the sequence $(L)^\frown(M_i)_{\beta\leq i<\alpha}$ over M_b*

Proof. Let $(N_i)_{i<s(\alpha)}$ be a continuous resolution of N witnessing that N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b via the maps $(f_i : M_i \longrightarrow N_{i+1})_{i<\alpha}$, and let $L = \bigcup_{i<\beta} N_i$. Then the resolution $(N_i)_{i<s(\beta)}$ witnesses that L is the desired $\mathcal{A}$-amalgam, and moreover the sequence $(L)^\frown(N_i)_{\beta\leq i<s(\alpha)}$ witnesses that N is also an $\mathcal{A}$-amalgam of $(L)^\frown(M_i)_{\beta\leq i<\alpha}$ over M_b (via the maps $(\iota : L \longrightarrow N)^\frown(f_i)_{\beta\leq i<\alpha}$). $\square$

Lemma 3.4. *Suppose that $\mathcal{A}$ is a notion of amalgamation which is regular and continuous. If N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b via the maps $(f_i : M_i \longrightarrow N)_{i<\alpha}$, then there is some $L \leq N$ such that:*

- *L is an $\mathcal{A}$-amalgam of $(M_i)_{1\leq i<\alpha}$ over M_b via the same maps; and*
- *N is an $\mathcal{A}$-amalgam of L and M_0 over M_b in the following diagram:*

$$\begin{tikzcd} M_0 \arrow[r, "f_0"] & N \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & L \arrow[u, "\iota"'] \end{tikzcd}$$

Proof. Fix $(N_i)_{i<s(\alpha)}$ a continuous resolution of N witnessing that N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b via $(f_i)_{i<\alpha}$. Let us first construct the model L as the union of an increasing continuous chain $(L_i)_{1\leq i<s(\alpha)}$, with the following conditions:

1. $L_1 = f_1[M_1]$, and each $L_i \leq N_i$
2. For limit δ, $L_\delta = \bigcup_{1\leq i<\delta} L_i$
3. For $i \geq 1$, the following diagram is an $\mathcal{A}$-amalgam:

$$\begin{tikzcd} M_{i+1} \arrow[r, "f_{i+1}"] & L_{i+1} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & L_i \arrow[u, "\iota"'] \end{tikzcd}$$

4. For $i \geq 1$, the following diagram is an $\mathcal{A}$-amalgam:

$$\begin{tikzcd} L_i \arrow[r, "\iota"] & N_i \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M_0 \arrow[u, "f_0"'] \end{tikzcd}$$

For the successor step, recall that as $(N_i)_{i<s(\alpha)}$ witnesses that N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b, in particular for each $i < \alpha$, the following diagram is an $\mathcal{A}$-amalgam:

$$\begin{tikzcd} M_i \arrow[r, "f_i"] & N_{i+1} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & N_i \arrow[u, "\iota"'] \end{tikzcd}$$

Hence, as $L_i \leq N_i$ by assumption, by regularity there exists some $L_{i+1} \leq N_{i+1}$ such that:

$$\begin{tikzcd} M_i \arrow[r, "f_i"] & L_{i+1} \arrow[r, "\iota"] & N_{i+1} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & L_i \arrow[u, "\iota"'] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & N_i \arrow[u, "\iota"'] \end{tikzcd}$$

It remains to show that (4) is satisfied. We note that combining the above diagram and assuming (4) holds for L_i, we get the following diagram:

$$\begin{tikzcd}
M_{i+1} \arrow[r, "f_i"] & L_{i+1} \arrow[r, "\iota"] & N_{i+1} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] \arrow[dr, "\mathrm{id}"] & L_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_i \arrow[u, "\iota"] \\
 & M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_0 \arrow[u, "f_0"]
\end{tikzcd}$$

Applying regularity to the two commutative squares on the right, we see that (4) is satisfied at the $i+1$ step:

$$\begin{tikzcd}
L_{i+1} \arrow[r, "\iota"] & N_{i+1} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_0 \arrow[u, "f_0"]
\end{tikzcd}$$

For the limit step, it suffices to check again that L_δ satisfies (4). Since L_i satisfies (4) by assumption for $i < \delta$, we have the diagram:

$$\begin{tikzcd}
f_0[M_0] \arrow[r, "\iota"] & N_1 \arrow[r, "\iota"] & N_2 \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & N_i \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & \cdots \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & L_1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & L_2 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & L_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & L_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots
\end{tikzcd}$$

Hence by continuity (and invariance), we get that

$$\begin{tikzcd}
M_0 \arrow[r, "f_0"] & \bigcup_{i<\delta} N_i \arrow[r, "\mathrm{id}"] & N_\delta \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & \bigcup_{i<\delta} L_i \arrow[u, "\iota"] \arrow[r, "\mathrm{id}"] & L_\delta \arrow[u, "\iota"]
\end{tikzcd}$$

This completes the definition of $(L_i)_{1 \leq i < s(\alpha)}$. Note then that this resolution of $L = \bigcup_{i<s(\alpha)} L_i$ is a witness to the fact that L is an $\mathcal{A}$-amalgam of $(M_i)_{1 \leq i < \alpha}$ over M_b, and moreover the proof for (4) in the limit case also shows that $N = \bigcup_{i<s(\alpha)} N_i$ is an $\mathcal{A}$-amalgam of M_0, L over M_b, as desired. $\square$

Corollary 3.5. *Suppose $\mathcal{A}$ is regular and continuous. If N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion, then for any $0 < j < \alpha$, there are $L_1, L_2 \leq N$ such that:*

- *L_1 is an $\mathcal{A}$-amalgam of $(M_i)_{i<j}$ over M_b by inclusion*
- *L_2 is an $\mathcal{A}$-amalgam of $(M_i)_{j<i<\alpha}$ over M_b by inclusion; and*
- *N is an $\mathcal{A}$-amalgam of (L_1, M_j, L_2) over M_b by inclusion*

Proof. That L_1 exists by Lemma 3.3 and L_2 exists by Lemma 3.4. $\square$

Lemma 3.6. *Suppose $\mathcal{A}$ is a notion of amalgamation that is regular and continuous. Let N be an $\mathcal{A}$-amalgam of M^*, M' over M_b by the following diagram:*

$$\begin{tikzcd}
M^* \arrow[r, "g"] & N \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "\iota"]
\end{tikzcd}$$

If M' is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b (via the K-embeddings $(f_i : M_i \longrightarrow M)_{i<\alpha}$), then the sequence $(M^*)^\frown(M_i)_{i<\alpha}$ is $\mathcal{A}$-subamalgamated over M_b inside N.

Proof. Since M' is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b via the maps $(f_i)_{i<\alpha}$, there is a continuous resolution $(M'_i)_{i<s(\alpha)}$ of M' such that $M'_0 = M_b, M'_1 = f_0[M_0]$ and for each $1 \leq i < \alpha$,

$$\begin{tikzcd}
M'_i \arrow[r, "\iota"] & M'_{i+1} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M_i \arrow[u, "f_i"']
\end{tikzcd}$$

So let us define an increasing continuous chain $(N_i)_{i<\beta}$ such that

1. $\beta = \alpha + 2$ iff $\alpha < \omega$; otherwise $\beta = s(\alpha)$
2. $N_0 = M_b$ and $N_1 = g[M^*]$
3. For limit δ, $N_\delta = \bigcup_{i<\delta} N_i$
4. For each $i < \omega$, $M'_i \leq N_{i+1} \leq N$ and the commutative squares in the following diagram are $\mathcal{A}$-amalgams:

$$\begin{tikzcd}
N_1 \arrow[r, "\iota"] & N_{i+2} \arrow[r, "\iota"] & N \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] \arrow[dr, "\iota"] & M'_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"] \\
& M_i \arrow[u, "f_i"] &
\end{tikzcd}$$

5. For each i such that $\omega \leq i < \alpha$, $M'_i \leq N_i \leq N$ and the commutative squares in the following diagram are $\mathcal{A}$-amalgams:

$$\begin{tikzcd}
N_1 \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & N \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] \arrow[dr, "\iota"] & M'_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"] \\
& M_i \arrow[u, "f_i"] &
\end{tikzcd}$$

We will define N_i inductively to satisfy the above conditions:

- For $i = 0$ and $i = 1$, the construction of N_i is specfied as above.
- For $i = 2$, note that since $N_1 = g[M^*]$ by definition, we have (by Side Invariance) that

$$\begin{tikzcd}
g[M^*] \arrow[r, "\mathrm{id}"] & N_1 \arrow[r, "\iota"] & N \\
M'_0 \arrow[r, "\mathrm{id}"] & M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"]
\end{tikzcd}$$

  As $M_b \leq M'_1 \leq M$, by regularity there exists some $N_2 \leq N$ such that

$$\begin{tikzcd}
N_1 \arrow[r, "\iota"] & N_2 \arrow[r, "\iota"] & N \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"]
\end{tikzcd}$$

- If $1 \leq i < \omega$, then by the inductive hypothesis, we have:

$$\begin{tikzcd} N_1 \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & N \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"] \end{tikzcd}$$

As $M'_i \leq M'_{i+1} \leq M$, again by regularity there is some $N_{i+1} \leq N$ such that

$$\begin{tikzcd} N_1 \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & N_{i+2} \arrow[r, "\iota"] & N \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"] \end{tikzcd}$$

Furthermore, apply regularity to the two commutative squares on the left, we also get:

$$\begin{tikzcd} N_1 \arrow[r, "\iota"] & N_{i+2} \arrow[r, "\iota"] & N \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"] \end{tikzcd}$$

- If $i = \omega$, then by the inductive hypothesis we have

$$\begin{tikzcd} N_1 \arrow[r, "\iota"] & N_2 \arrow[r, "\iota"] & N_3 \arrow[r, "\iota"] & \cdots \arrow[r] & N_{i+1} \arrow[r, "\iota"] & N_{i+2} \arrow[r, "\iota"] & \cdots \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_2 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \arrow[r] & M'_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \end{tikzcd}$$

Defining $N_\omega = \bigcup_{i<\omega} N_i$, by continuity we have that

$$\begin{tikzcd} N_1 \arrow[r, "\iota"] & N_\omega \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_\omega \arrow[u, "\iota"] \end{tikzcd}$$

Now, since $N_\omega \leq N$, by regularity (specifically, the "moreover" part of condition (3), see Definition 2.6), it is also true that

$$\begin{tikzcd} N_\omega \arrow[r, "\iota"] & N \\ M'_\omega \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"] \end{tikzcd}$$

- For successor and limit i's beyond ω, the construction is the same as above except for the shifted indices.

Letting $N' = \bigcup_{i<\beta} N_i$, it remains to show that N' is an $\mathcal{A}$-amalgam of $(M^*)^\frown(M_i)_{i<\alpha}$ over M_b (via the maps $(g)^\frown(f_i)_{i<\alpha}$) i.e. that for each $i < \omega$ and j such that $\omega \leq j < \alpha$,

$$\begin{tikzcd} N_{i+1} \arrow[r, "\iota"] & N_{i+2} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_i \arrow[u, "f_i"] \end{tikzcd} \qquad \begin{tikzcd} N_j \arrow[r, "\iota"] & N_{j+1} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_j \arrow[u, "f_j"] \end{tikzcd}$$

For the $i < \omega$ case, recall that $(M'_i)_{i<\alpha}$ witnesses that M is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b, and hence for each i,

$$\begin{tikzcd}
M'_i \arrow[r, "\iota"] & M'_{i+1} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_i \arrow[u, "f_i"']
\end{tikzcd}$$

Combining this with condition (4) above and the construction of N_i, we get the diagram

$$\begin{tikzcd}
N_1 \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & N_{i+2} \arrow[r, "\iota"] & N \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] \arrow[dr, "\mathrm{id}"] & M'_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"] \\
& M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_i \arrow[u, "f_i"'] &
\end{tikzcd}$$

Note that apply regularity to the two commutative squares in the middle column gives us the desired result. As the same argument applies to the case of $\omega \le j < \alpha$ with shifted indices, this completes the proof. $\square$

Lemma 3.7. *Suppose N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b via the maps $(f_i : M_i \longrightarrow N)_{i<\alpha}$. If additionally M_0 is an $\mathcal{A}$-amalgam of $(L_j)_{j<\beta}$ over M_b via the maps $(g_j : L_j \longrightarrow M_0)_{j<\beta}$, then N is an $\mathcal{A}$-amalgam of the concatenated sequence $(L_j : j < \beta)^\frown(M_i : i < \alpha)$ over M_b.*

Proof. Fix $(M'_j)_{j<s(\beta)}$ a continuous resolution of M_0 witnessing that it is an $\mathcal{A}$-amalgam of $(L_j)_{j<\alpha}$ over M_b, and also fix $(N_i)_{i<s(\alpha)}$ a continuous resolution of N witnessing that it is an $\mathcal{A}$-amalgam of $(M_i)_{i<s(\alpha)}$ over M_b. Consider then the concatenated sequence $S = (f_0[M'_j] : j < s(\beta))^\frown(N'_i : 1 \le i < s(\alpha))$: it is a continuous resolution of N since $N_0 = f_0[M_0] = \bigcup_{j<s(\beta)} f_0[M'_j]$. Since $f_0 \restriction M'_j$ is a K-isomorphism between M'_j and $f_0[M'_j]$, Invariance of $\mathcal{A}$ implies the desired result. $\square$

3.2 Commutativity of amalgamation

Having shown that sequential amalgamation is associative, let us now turn to the question of commutativity. Not surprisingly given the results of the previous section, this is in fact directly connected to the ability to amalgamate along a subsequence of models. First, however, we need to establish:

Lemma 3.8. *Suppose $\mathcal{A}$ is absolutely minimal. Let N be an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion, and suppose that $N' \ge N$, $N^* \le N'$ is also an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion. Then $N = N^*$.*

Proof. By induction on α:

- If $\alpha = 2$, then this is true by Lemma 2.14.
- Assuming the statement is true for α. Given N, N^* both $\mathcal{A}$-amalgams of $(M_i)_{i<\alpha+1}$ over M_b (by inclusion) as above, let $N_\alpha \le N$ be such that N_α is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b, and similarly define $N^*_\alpha \le N^*$. By induction $N_\alpha = N^*\alpha$, and hence both N and N^* are $\mathcal{A}$-amalgams of N_α, M_α over M_b. Hence by Lemma 2.14, $N = N^*$.
- For limit δ, if N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\delta}$ over M_b, then fix $(N_\alpha)_{\alpha<\delta}$ a continuous resolution of N witnessing that N is an $\mathcal{A}$-amalgam. Similarly fix $(N^*_\alpha)_{\alpha<\delta}$. In particular, each N_α and N^*_α are $\mathcal{A}$-amalgams of $(M_i)_{i<\alpha}$ over M_b, and hence by induction each $N_\alpha = N^*_\alpha$. Then

$$N = \bigcup_{\alpha<\delta} N_\alpha = \bigcup_{\alpha<\delta} N^*_\alpha = N^*$$

This completes the proof.

$\square$

Corollary 3.9. *Suppose $\mathcal{A}$ is absolutely minimal. If each $M_i \leq N$ and the sequence $(M_i)_{i<\alpha}$ is $\mathcal{A}$-subamalgamated over M_b inside N, then there is a unique $N' \leq N$ which is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b.*

Notation 3.10. If $(M_i)_{i<\alpha}$ are such that each $M_b \leq M_i \leq N$ and the sequence $(M_i)_{i<\alpha}$ is $\mathcal{A}$-subamalgamated inside N via inclusion, then we denote the unique $\mathcal{A}$-amalgam inside N by $\bigoplus^N_{M_b, i<\alpha} M_i$.

Lemma 3.11. *Suppose $\mathcal{A}$ is regular, continuous, and absolutely minimal. Let α be a limit ordinal, and $(M_i^1)_{i\leq\alpha}, (M_i^2)_{i\leq\alpha}, (N_i)_{i\leq\alpha}$ be increasing continuous chains such that for each $i < \alpha$, N_i is an $\mathcal{A}$-amalgam of M_i^1, M_i^2 over M_b by inclusion. Then for any $i, j < \alpha$, there is a unique $M_{ij} \leq N_{max(i,j)}$ which is an $\mathcal{A}$-amalgam of M_i^1, M_j^2 over M_b by inclusion. Moreover, the M_{ij}'s are such that if i is a limit ordinal, then $M_{ij} = \bigcup_{k<i} M_{kj}$, and similarly if j is a limit ordinal.*

Proof. Let M_i^1, M_i^2, N_i be as above, so that we have the diagram

$$\begin{tikzcd}
\vdots & & & \iddots \\
M_1^2 \arrow[u] \arrow[rr] & & N_1 \arrow[ur] & \\
M_0^2 \arrow[u] \arrow[r] & N_0 \arrow[ur] & & \\
M_b \arrow[u] \arrow[r] & M_0^1 \arrow[u] \arrow[r] & M_1^1 \arrow[uu] \arrow[r] & \cdots
\end{tikzcd}$$

where all the arrows are inclusions and all the commutative squares with a vertex at M_b are $\mathcal{A}$-amalgams. Letting M_{ii} be defined as N_i, we will define M_{ij} by induction on $\max(i,j) < \alpha$ such that in addition to the requirements above, we have additionally that the condition $(A(ij))$ holds when i, j are not limits:

$$\begin{tikzcd}
M_{i-1,j} \arrow[r, "\iota"] & M_{i,j} \\
M_{i-1,j-1} \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i,j-1} \arrow[u, "\iota"']
\end{tikzcd} \tag{A(i,j)}$$

(where $M_{-1,-1} = M_b, M_{-1,j} = M_j^2, M_{i,-1} = M_i^1$)

- For M_{01}, note that N_1 is an $\mathcal{A}$-amalgam of M_1^1, M_1^2 over M_b (by inclusion). As $M_b \leq M_0^1 \leq M_1^1$, by regularity there is $M_{01} \leq N_1$ such that

$$\begin{tikzcd}
M_1^2 \arrow[r, "\iota"] & M_{01} \arrow[r, "\iota"] & N_1 \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_0^1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1^1 \arrow[u, "\iota"]
\end{tikzcd}$$

  M_{10} is defined symmetrically.

- If M_{ij} is defined for all $i, j \leq \alpha$, then for any $i \leq \alpha$, by regularity there is $M_{i,\alpha+1}$ such that

$$\begin{tikzcd}
M_{\alpha+1}^2 \arrow[r, "\iota"] & M_{i,\alpha+1} \arrow[r, "\iota"] & N_{\alpha+1} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_i^1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{\alpha+1}^1 \arrow[u, "\iota"]
\end{tikzcd}$$

It is straightforward to see that condition ($A(i, \alpha+1)$) by induction on i (and using regularity for the base case). We define $M_{\alpha+1,j}$ which satisfies ($A(\alpha+1, j)$) by the symmetric argument. Finally, to see that condition ($A(\alpha+1, \alpha+1)$) holds, note that by definition of $M_{\alpha,\alpha+1}$, we have

$$\begin{tikzcd} M^2_{\alpha+1} \arrow[r, "\iota"] & M_{\alpha,\alpha+1} \arrow[r, "\iota"] & N_{\alpha+1} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^1_\alpha \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^1_{\alpha+1} \arrow[u, "\iota"] \end{tikzcd}$$

Apply regularity to the commutative square on the right (and symmetry), we get that

$$\begin{tikzcd} M^1_{\alpha+1} \arrow[r, "\iota"] & M_{\alpha+1,\alpha} \arrow[r, "\iota"] & N_{\alpha+1} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_\alpha \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{\alpha,\alpha+1} \arrow[u, "\iota"] \end{tikzcd}$$

The $\mathcal{A}$-amalgam on the right shows that ($A(\alpha+1, \alpha+1)$) is indeed satisfied.

- If δ is a limit and M_{ij} are defined for $i, j < \delta$, then by regularity let $M_{i,\delta} \leq N_\delta$ be an $\mathcal{A}$-amalgam of M^1_i, M^2_δ over M_b. We need to show that:

  Claim. $M_{i,\delta} = \bigcup_{j<\delta} M_{ij}$

  To prove the claim, note that since $(M^2_j)_{j<\delta}$ is a continuous resolution of M^2_δ, by Lemma 2.20 there is a continuous resolution $(M'_{ij})_{j<\delta}$ of $M_{i,\delta}$ such that each M'_{ij} is an $\mathcal{A}$-amalgam of M^1_i, M^2_j over M_b. But then each $M'_{ij}, M_{ij} \leq N_\delta$, and as $\mathcal{A}$ is absolutely minimal, by Lemma 2.14 we have that $M_{ij} = M'_{ij}$. This proves the claim. Additionally, this construction implies that when γ is also a limit, then $M_{\gamma,\delta} = \bigcup_{i<\gamma} M_{i,\delta}$.

  Symmetrically, we define $M_{\delta,j}$. To finish the construction, we need to show that:

$$\bigcup_{i<\delta} M_{i,\delta} = N_\delta = \bigcup_{j<\delta} M_{\delta,j}$$

  But this is true since each $N_\alpha \leq M_{\alpha,\delta}, M_{\delta,\alpha} \leq N_\delta$, and $\bigcup_{\alpha<\delta} N_\alpha = N_\delta$.

□

Corollary 3.12. *Suppose $\mathcal{A}$ is absolutely minimal, regular, and continuous. If α is a limit ordinal, and $(M^1_i)_{i\leq\alpha}, (M^2_i)_{i\leq\alpha}, (N_i)_{i\leq\alpha}$ are increasing continuous chains such that for each $i < \alpha$, N_i is an $\mathcal{A}$-amalgam of M^1_i, M^2_i over M_b by inclusion, then N_α is an $\mathcal{A}$-amalgam of M^1_α, M^2_α over M_b.*

Proof. For $i, j < \alpha$, let $M_{ij} \leq N_\alpha$ be constructed as in the above Lemma, and for each $i < \alpha$, let $M_{i,\alpha} = \bigcup_{j<\alpha} M_{ij}$.
Claim.

$$\begin{tikzcd} M^2_\alpha \arrow[r, "\iota"] & M_{0,\alpha} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^1_0 \arrow[u, "\iota"] \end{tikzcd}$$

Proof. Note that by condition ($A(0, j)$) for each $j < \alpha$, we have that

$$\begin{tikzcd} M^1_0 \arrow[r, "\iota"] & M_{0,0} \arrow[r, "\iota"] & M_{0,1} \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M_{0,i} \arrow[r, "\iota"] & M_{0,i+1} \arrow[r, "\iota"] & \cdots \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^2_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^2_1 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M^2_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^2_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \end{tikzcd}$$

Hence the claim holds as $\mathcal{A}$ is continuous. □

Claim. For each $i < \alpha$,

$$\begin{tikzcd} M_{i,\alpha} \arrow[r, "\iota"] & M_{i+1,\alpha} \\ M_i^1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1}^1 \arrow[u, "\iota"'] \end{tikzcd}$$

This holds by the same argument.

Claim. For limit $\delta < \alpha$, $M_{\delta,\alpha} = \bigcup_{i<\delta} M_{i,\alpha}$, and moreover

$$\begin{tikzcd} M_\alpha^2 \arrow[r, "\iota"] & M_{\delta,\alpha} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_\delta^1 \arrow[u, "\iota"'] \end{tikzcd}$$

Proof. Note that

$$\bigcup_{i<\delta} M_{i,\alpha} = \bigcup_{i<\delta} \bigcup_{j<\alpha} M_{ij} = \bigcup_{j<\alpha} M_{\delta,j} = M_{\delta,\alpha}$$

For the moreover part, combining the above claims and induction along δ, we get that

$$\begin{tikzcd} M_\alpha^2 \arrow[r, "\iota"] & M_{0,\alpha} \arrow[r, "\iota"] & M_{1,\alpha} \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M_{i,\alpha} \arrow[r, "\iota"] & M_{i+1,\alpha} \arrow[r, "\iota"] & \cdots \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_0^1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1^1 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M_i^1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1}^1 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \end{tikzcd}$$

As $\mathcal{A}$ is continuous, hence $\bigcup_{i<\delta} M_{i,\alpha} = M_{\delta,\alpha}$ is an $\mathcal{A}$-amalgam of M_α^2, M_δ^1 over M_b. □

Combining the above claims, we get the diagram (for all $i < \alpha$)

$$\begin{tikzcd} M_\alpha^2 \arrow[r, "\iota"] & M_{0,\alpha} \arrow[r, "\iota"] & M_{1,\alpha} \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M_{i,\alpha} \arrow[r, "\iota"] & M_{i+1,\alpha} \arrow[r, "\iota"] & \cdots \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_0^1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1^1 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M_i^1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1}^1 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \end{tikzcd}$$

As $\mathcal{A}$ is continuous, hence $\bigcup_{i<\alpha} M_{i,\alpha}$ is an $\mathcal{A}$-amalgam of M_α^1, M_α^2 over M_b. But since for any $i < j < \alpha$, $N_i \leq M_{ij}, M_{ji} \leq N_j$, we have that $N_\alpha = \bigcup_{i<\alpha} M_{i,\alpha}$. This completes the proof. □

Theorem 3.13. *Suppose $\mathcal{A}$ is absolutely minimal, regular, and continuous. Let N be an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion. Then for any subsequence $S \subseteq \alpha$, there is some $M_S \leq N$ which is an $\mathcal{A}$-amalgam of $(M_{S(j)})_{j<|S|}$ over M_b by inclusion. Moreover, if S^c is the complement of S in α (and considered as an increasing sequence), then there is M_{S^c} such that additionally, N is an $\mathcal{A}$-amalgam of M_S, M_{S^c} over M_b by inclusion.*

Proof. We will proceed by induction on the length of α:

- When $\alpha = 2$, this is trivial.
- Assume the claim holds for α. Given N an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha+1}$ over M_b, suppose that S is a subsequence of $\alpha + 1$. This breaks down into three cases:
  1. If $S = \{\alpha\}$, then the case is trivial.
  2. If $S \subseteq \alpha$, then consider $N_\alpha \leq N$ which is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b (as guaranteed by Lemma 3.3): by the inductive hypothesis, $M_S \leq N_\alpha$ exists, and so does $M_{S_\alpha^c}$, where S_α^c is the

complement of S w.r.t. α. Now, since N is an $\mathcal{A}$-amalgam of N_α and M_α over M_b, we get the $\mathcal{A}$-amalgams

$$\begin{tikzcd} M_S \arrow[r, "\iota"] & N_\alpha \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{S^c_\alpha} \arrow[u, "\iota"'] \end{tikzcd} \qquad \begin{tikzcd} N_\alpha \arrow[r, "\iota"] & N \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_\alpha \arrow[u, "\iota"'] \end{tikzcd}$$

By Lemma 2.13, hence there is some M_{S^c} such that:

- M_{S^c} is an $\mathcal{A}$-amalgam of $M_{S^c_\alpha}, M_\alpha$ over M_b; and
- N is an $\mathcal{A}$-amalgam of M_S, M_{S^c} over M_b

Furthermore, by Lemma 3.7, M_{S^c} is also an $\mathcal{A}$-amalgam of $(M_j : j < |S^c_\alpha|)^\frown(M_\alpha)$ over M_b.

3. If $S \ni \alpha$ and $S \cap \alpha \neq \varnothing$, then S^c satisfies the above case (2), so the same construction gives the required submodels.

- Let δ be a limit, and suppose the claim holds for all $\alpha < \delta$. Given N an $\mathcal{A}$-amalgam of $(M_i)_{i<\delta}$ over M_b by inclusion, let $(N_i)_{i<\delta}$ be a continuous resolution of N such that each N_i is an $\mathcal{A}$-amalgam of $(M_j)_{j<i}$ over M_b. Now, if S is a subsequence of δ, denote $S_\alpha := S \restriction \alpha$ and $S^c_\alpha := S^c \restriction \alpha$. Note then that for each α, S^c_α is the complement of S_α relative to α, and hence the inductive hypothesis implies that there are models $M^\alpha_{S_\alpha}, M^\alpha_{S^c_\alpha} \leq N_\alpha$ such that:

  - $M^\alpha_{S_\alpha}$ is an $\mathcal{A}$-amalgam of $(M_{S(j)})_{j<|S_\alpha|}$ over M_b
  - $M^\alpha_{S^c_\alpha}$ is an $\mathcal{A}$-amalgam of $(M_{S^c(j)})_{j<|S^c_\alpha|}$ over M_b
  - N_α is an $\mathcal{A}$-amalgam of $M^\alpha_{S_\alpha}, M^\alpha_{S^c_\alpha}$ over M_b

  Moreover, by Lemma 3.8, $M^\alpha_{S_\alpha}$ is the unique $\mathcal{A}$-amalgam of $(M_{S(j)})_{j<|S|}$ over M_b inside N_δ, and similarly for $M^\alpha_{S^c_\alpha}$. Hence we will drop the superscript, and define $M_S := \bigcup_{\alpha<\delta} M_{S_\alpha}, M_{S^c} := \bigcup_{\alpha<\delta} M_{S^c_\alpha}$. Note then that the chains $(M_{S_\alpha})_{\alpha\leq\delta}, (M_{S^c_\alpha})_{\alpha\leq\delta}, (N_\alpha)_{\alpha\leq\delta}$ satisfies the hypothesis of Corollary 3.12 above, and hence N_δ is an $\mathcal{A}$-amalgam of M_S, M_{S^c} over M_b. Moreover, the continuous resolution $(M_{S_\alpha})_{\alpha<\delta}$ witnesses that M_S is an $\mathcal{A}$-amalgam of $(M_{S(j)})_{j<|S|}$ over M_b as desired.

□

Remark. It should be noted that if K is assumed to satisfy Smoothness (for example, if K is an AEC), then the proof of the above theorem can be simplified considerably: if N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion and S is a subsequence of α, then the $\mathcal{A}$-amalgam of $(M_{S(i)})_{i<\mathrm{otp}(S)}$ over M_b can be easily defined by induction. This works even at limit stages when K is assumed to have Smoothness; otherwise, the above argument seems to be necessary.

Theorem 3.14. *Suppose $\mathcal{A}$ is absolutely minimal, regular, and continuous. Let $\alpha \geq 2$ be an ordinal, and $\sigma : |\alpha| \longrightarrow \alpha$ be any enumeration of α. Then N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion iff N is also an $\mathcal{A}$-amalgam of $(M_{\sigma(j)})_{j<|\alpha|}$.*

Proof. Let N be an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion. We proceed by induction on α:

- When $\alpha = 2$, this is just Lemma 2.16.

- Suppose the claim holds for n, which is finite. If σ is an enumeration of $n+1$, then σ is a permutation of $n+1$. There are two cases to consider:

  - If $\sigma(n) = n$, then $\sigma \restriction n$ is a permutation of n, and the claim follows from the inductive hypothesis.
  - Otherwise, let $m = \sigma(n) < n$. By Corollary 3.5, there are models $N_1, N_2 \leq N$ such that:
    * N_1 is an $\mathcal{A}$-amalgam of $(M_i)_{i<m}$ over M_b
    * N_2 is an $\mathcal{A}$-amalgam of $(M_i)_{m<i\leq n}$ over M_b
    * N is an $\mathcal{A}$-amalgam of (N_1, M_m, N_2) over M_b

But then by Lemma 2.16, N is also an $\mathcal{A}$-amalgam of (N_1, N_2, M_m) over M_b. Now, if $N' \leq N$ is an $\mathcal{A}$-amalgam of N_1, N_2 over M_b, then by Lemma 3.7 and 3.6, N' is an $\mathcal{A}$-amalgam of $(M_j)_{j \neq m}$ over M_b. Since $\sigma(n) = m$ and σ is a permutation, (by re-indexing) the inductive hypothesis implies that N' is also an $\mathcal{A}$-amalgam of $(M_{\sigma(i)})_{i<n}$ over M_b, and hence N is an $\mathcal{A}$-amalgam of $(M_{\sigma(i)})_{i<n+1}$ over M_b.

- Suppose the claim holds for an infinite α, and so $|\alpha| = |\alpha+1|$. Given $\sigma : |\alpha| \longrightarrow \alpha+1$ an enumeration, there is some $\beta < |\alpha|$ such that $\sigma(\beta) = \alpha$. Let S be the subsequence of α such that ran S = ran $\sigma \restriction \beta$, and let S^c be its complement in α, so in particular $S^c = S'^\frown\alpha$ for some subsequence S' of α. Now, since N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha+1}$ over M_b, there is an $N^* \leq N$ such that N^* is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b, and N is an $\mathcal{A}$-amalgam of N^*, M_α over M_b. But since S is a subsequence of α and S' is its complement w.r.t. α, by Theorem 3.13 there are models $N_S, N_{S'} \leq N^*$ such that

  - N_S is an $\mathcal{A}$-amalgam of $(M_{S(i)})_{i<|S|}$ over M_b
  - $N_{S'}$ is an $\mathcal{A}$-amalgam of $(M_{S'(i)})_{i<|S'|}$ over M_b
  - N^* is an $\mathcal{A}$-amalgam of $N_S, N_{S'}$ over M_b

  Furthermore, since S, S' are subsequences of α, $\mathrm{otp}(S), \mathrm{otp}(S') \leq \alpha$, and so by the inductive hypothesis N_S is also an $\mathcal{A}$-amalgam of $(M_{\sigma(i)})_{i<\beta}$ over M_b. Similarly, $N_{S'}$ is an $\mathcal{A}$-amalgam of $(M_{\sigma(i)})_{\beta<i<|\alpha|}$. Moreover, N is also an $\mathcal{A}$-amalgam of (N_S, M_α, N'_S) over M_b by Lemma 2.16, and so by Lemma 3.7 and 3.6, N is indeed an $\mathcal{A}$-amalgam of $(M_{\sigma(i)})_{i<|\alpha|}$ over M_b. We also need to show that if N is an $\mathcal{A}$-amalgam of $(M_i)_{i<|\alpha|}$ over M_b, then N is also an $\mathcal{A}$-amalgam of $(M_{\sigma^{-1}(j)})_{j<\alpha+1}$. Again letting β be such that $\sigma(\alpha) = \beta$, by Lemma 3.5 there are models $N^1, N^2 \leq N$ such that

  - N^1 is an $\mathcal{A}$-amalgam of $(M_i)_{i<\beta}$ over M_b
  - N^2 is an $\mathcal{A}$-amalgam of $(M_i)_{\beta<i<|\alpha|}$ over M_b
  - N is an $\mathcal{A}$-amalgam of (N^1, M_β, N^2) over M_b

  By Lemma 2.16 again, we see that N is also an $\mathcal{A}$-amalgam of (N^1, N^2, M_β) over M_b. If $N' \leq N$ is such that N' is an $\mathcal{A}$-amalgam of N^1, N^2 over M_b, then by the inductive hypothesis N' is also an $\mathcal{A}$-amalgam of $(M_{\sigma^{-1}(j)})_{j<\alpha}$ over M_b. By Lemma 3.7, hence N is an $\mathcal{A}$-amalgam of $(M_{\sigma^{-1}(j)})_{j<\alpha+1}$ over M_b.

- Suppose α is a limit ordinal, and that the claim holds for all $\beta < \alpha$. As N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b, let $(N_\beta)_{\beta<\alpha}$ be a continuous resolution of N such that each N_β is an $\mathcal{A}$-amalgam of $(M_i)_{i<\beta}$ over M_b. Now, given $\sigma : |\alpha| \longrightarrow \alpha$ an enumeration, for $j < |\alpha|$ let $\sigma_j := \sigma \restriction j$, and let S_j be a subsequence of α such that ran S_j = ran σ_j i.e. S_j is the set enumerated by σ_j but re-indexed by the ordinal ordering. Note that since each S_j is a subsequence of α, by Theorem 3.13 there is $N_{S_j} \leq N$ which is an $\mathcal{A}$-amalgam of $(M_{S_j(i)})_{i<\mathrm{otp}(S_j)}$ over M_b. Furthermore, since each $|S_j| = j < |\alpha|$, $\mathrm{otp}(S_j) < \alpha$, and hence by the inductive hypothesis N_{S_j} is also an $\mathcal{A}$-amalgam of $(M_{\sigma_j(i)})_{i<j}$ over M_b. Letting $N' = \bigcup_{j<|\alpha|} N_{S_j}$, this implies that N' is an $\mathcal{A}$-amalgam of $(M_{\sigma(i)})_{i<|\alpha|}$ over M_b.

  Claim. $N' = N$

  Proof. Since $(N_i)_{i<\alpha}$ is a continuous resolution of N, $N' = \bigcup_{j<|\alpha|} N_{S_j}$, and each $N_{S_j} \leq N$, it suffices to show that each $N_i \subseteq N'$. Now, for each $i < \alpha$, let ζ_i be a subsequence of σ such that ran $\zeta_i = i$, and so by the inductive hypothesis N_i is an $\mathcal{A}$-amalgam of $(M_{\zeta_i(j)})_{j<\mathrm{otp}(\zeta_i)}$ over M_b. But by Theorem 3.13 there is $N'_i \leq N'$ which is an $\mathcal{A}$-amalgam of $(M_{\zeta_i(j)})_{j<\mathrm{otp}(\zeta_i)}$ over M_b, and as $\mathcal{A}$ is absolutely minimal, by Corollary 3.9 $N'_i = N_i$. This proves the claim. □

  It remains to show, that when α is <u>not</u> an initial ordinal, that if N is an $\mathcal{A}$-amalgam of $(M_i)_{i<|\alpha|}$ over M_b, then it is also an $\mathcal{A}$-amalgam of $(M_{\sigma^{-1}(j)})_{j<\alpha}$. However, we note that the argument analogous to the one given above also works here, and hence the claim is proven for α.

□

Given Theorems 3.13 and 3.14, we see that when $\mathcal{A}$ is absolutely minimal, regular, and continuous, then $\mathcal{A}$-amalgamation of models indexed by a sequence is independent of the ordering, and hence can be considered as being indexed by a set. Moreover, if N is an $\mathcal{A}$-amalgam of $(M_i)_{i\in Y}$ over M_b by inclusion, then for any $X \subseteq Y$, there is $N_X \leq N$ which is an $\mathcal{A}$-amalgam of $(M_i)_{i\in X}$ over M_b.

Chapter 4

An Independence Relation

4.1 Decomposition and the cardinal $\mu(K)$

As we have mentioned previously, a guiding example to our study of notion of amalgamation is the case of direct sums of vector spaces and divisible groups. We are still missing one key ingredient in this analogy: the ability to decompose larger models into smaller ones.

Lemma 4.1. *Suppose $\mathcal{A}$ is a notion of amalgamation which is minimal, regular, and admits decomposition. Then for any $M_b \lneq N$, there exists an ordinal $\alpha < |N|^+$ and a sequence of models $(M_i)_{i<\alpha}$ such that:*

- *For every $i < \alpha$, $M_b \lneq M_i \leq N$ and $|M_i| = LS(K) + |M_b|$*
- *N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion.*

Proof. Let $\lambda = |N|^+$ and $\mu = |M_b| + \mathrm{LS}(K)$. We will try to define two sequences of models, $(M_i)_{i<\lambda}$ and $(N_i)_{i<\lambda}$, such that:

1. For each i, $M_b \leq M_i \leq N$, $N_i \leq N$, and $|M_i| = \mu$
2. $(N_i)_{i<\lambda}$ is an increasing continuous chain with $N_0 = M_b$ and $N_1 = M_0$
3. For every $i \geq 1$, the following diagram is an $\mathcal{A}$-diagram:

$$\begin{tikzcd} N_i \arrow[r, "\iota"] & N_{i+1} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_i \arrow[u, "\iota"'] \end{tikzcd}$$

Proceeding inductively:

- For $i = 0$, let M_0 be any model such that $M_b \lneq M_0 \leq N$ and $|M_0| = \mu$, and let $N_0 = M_b, N_1 = M_0$.
- Suppose inductively that M_i, N_{i+1} has been defined to satisfy (3). Since $N_{i+1} \leq N$, either $N_{i+1} = N$ or $N_{i+1} \lneq N$. In the former case, we terminate the inductive construction; otherwise, since $\mathcal{A}$ admits decomposition, there is some M'_{i+1} such that

$$\begin{tikzcd} N_{i+1} \arrow[r, "\iota"] & N \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_{i+1} \arrow[u, "\iota"'] \end{tikzcd}$$

  Note that as $\mathcal{A}$ is minimal, $M'_{i+1} - N_{i+1}$ must be nonempty as otherwise $N_{i+1} = N$. So let M_{i+1} be any model of cardinality μ such that $M_b \lneq M_{i+1} \leq M'_{i+1}$ and $M_{i+1} - N_{i+1}$ is nonempty. Then, as

$M_b \leq M_{i+1} \leq M'_{i+1}$, by regularity there exists some N_{i+2} such that

$$\begin{tikzcd} N_{i+1} \arrow[r, "\iota"] & N_{i+2} \arrow[r, "\iota"] & N \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M'_{i+1} \arrow[u, "\iota"] \end{tikzcd}$$

- For limit δ, let $N_\delta = \bigcup_{i<\delta} N_i$. If $N = N_\delta$, then the construction terminates; otherwise, M_δ and $N_{\delta+1}$ can be defined by the same procedure as in the successor case.

Note that by construction, each $N_i \lneq N_{i+1} \leq N$, and as $\lambda = |N|^+$, hence the above procedure must terminate at some ordinal $\alpha < \lambda$. In that case, $(N_i)_{i<s(\alpha)}$ witnesses the fact that N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion. $\square$

One other important property of the direct sum in vector spaces and divisible groups is that under any "basis" decomposition, any element is contained within the "span" (or amalgam) of finitely many basis elements. Whilst this is clearly true in the two examples because such algebraic objects are finitary, in the present context we are also interested in classes which are infinitary but not unboundedly; analogously, there are interesting classes which are $\mathcal{L}_{\kappa,\omega}$ classes rather than just a $\mathcal{L}_{\infty,\omega}$ class. To this end, we will define a cardinal $\mu(K)$ by:

Definition 4.2. Suppose that $\mathcal{A}$ is a notion of amalgamation which is regular, continuous, absolutely minimal, and admits decomposition.

1. For $M \in K$ and $a \in M$, we define $\mu(a, M)$ to be the least cardinal μ such that: for any $M_b \leq M$ and any sequence $(M_i)_{i<\alpha}$ such that M is the $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion, there is a subsequence $S \subseteq \alpha$ with $|S| < \mu$ such that $a \in \bigoplus^{M}_{M_b, j\in S} M_j$.
2. We define $\mu(M) := \sup\{\mu(a, M) : a \in M\}$
3. We define $\mu(K) := \sup\{\mu(M) : M \in K\}$ if it exists, or $\mu(K) = \infty$ otherwise.
4. If $\mu(K) < \infty$, then we define $\mu_r(K)$ to be the least regular cardinal $\geq \mu(K)$.

Remark. Strictly speaking, $\mu(K)$ should be considered as $\mu^{\mathcal{A}}(K)$ since the definition depends on $\mathcal{A}$ and different notions of amalgamation might give rise to different values of $\mu(K)$. However, since in this paper we will always be considering a class K with a fixed notion of amalgamation $\mathcal{A}$, we have chosen to suppress the extra notation.

In the next sections, we will see how $\mu(K)$ acts similar to the cardinal $\kappa(T)$ for a first order theory T.

4.2 Simple independence

Before proceeding, let us make a quick observation connecting strong notions of amalgamation with stability:

Proposition 4.3. *Suppose K is a very weak AEC, $\mathcal{A}$ is a notion of amalgamation in K that is absolutely minimal, continuous, regular, has uniqueness, and admits decomposition with $\mu(K) < \infty$. Define an equivalence relation $\sim$ on pairs of models such that $(M_1, N_1) \sim (M_2, N_2)$ iff $M_1 \leq N_1, M_2 \leq N_2$ and there exists an isomorphism $f : N_1 \to N_2$ with $f \restriction M_1$ also an isomorphism between M_1 and M_2, and let*

$$\Gamma := \{(M, N)/\sim : |M| = |N| = LS(K)\}$$

Then K is λ-stable for any $\lambda \geq LS(K) + |\Gamma|^{<\mu(K)}$.

Proof. Let λ be as above, $M \in K_\lambda$ and fix a $M_b \leq M$ with $|M_b = \mathrm{LS}(K)$. Given $p \in S^1(M)$, if (a, M, N) realizes p, then as $\mathcal{A}$ admits decomposition and has uniqueness let $N' \leq N$ be such that $N = M \oplus_{M_b} N'$. Then by Lemma 4.1 and Theorem 3.14, we can decompose N' as the (unique) $\mathcal{A}$-amalgam $\bigoplus_{M_b, i<\lambda} N_i$ with each $|N_i| = \mathrm{LS}(K)$. Now, by Lemma 3.6 N is also the $\mathcal{A}$-amalgam $M \oplus_{M_b} \left(\bigoplus_{M_b, i<\lambda} N_i\right)$, and hence by the definition of $\mu(K)$, there is an $I \subseteq \lambda$ with $|I| < \mu(K)$ and such that $a \in M \oplus_{M_b} \left(\bigoplus_{M_b, i \in I} N_i\right)$.

Since every type $p \in S^1(M)$ can be realized in a model of the above form (with M_b fixed), this implies that

$$|S^1(M)| \leq \lambda \cdot |\Gamma|^{<\mu(K)} = \lambda$$

□

In the elementary class of algebraically closed fields with characteristic 0, the forking relationship can be easily understood in terms of transcendence degree: $\mathrm{gtp}(\bar{a}/F_1)$ does not fork over F_0 iff $\mathrm{td}(\bar{a}/F_1) = \mathrm{td}(\bar{a}/F_0)$. Since this is essentially a characterization of forking using the concept of bases, we would expect that a suitably well-behaved notion of amalgamation would also give rise to a forking-like independence relation. To that end, we define:

Definition 4.4. Suppose K is a very weak AEC, $\mathcal{A}$ is a notion of amalgamation in K. We define a notion of $\mathcal{A}$-independence, denoted by $\perp$, as follows: if $M \leq N$ and $A, B \subseteq N$, then $A \overset{N}{\underset{M}{\perp}} B$ if there exists models M_1, M_2 with $M \leq M_1, M_2 \leq N$ such that $A \subseteq M_1, B \subseteq M_2$, and M_1, M_2 are $\mathcal{A}$-subamalgamated inside N over M i.e. there is some $N' \leq N$ such that

$$\begin{tikzcd} M_2 \arrow[r, "\iota"] & N' \arrow[r, "\iota"] & N \\ M \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] & \end{tikzcd}$$

In such a case, we say that the pair (M_1, M_2) is a **witness** to $A \overset{N}{\underset{M}{\perp}} B$.

Our goal here is to establish the conditions necessary for $\perp$ to behave as forking for stationary types in a simple first order theories: To that end, we need to establish that $\perp$ has the defining properties of forking:

- Invariance
- Top monotonicity (i.e. forking does not depend on the ambient model)
- Right monotonicity
- Base monotonicity
- Symmetry
- Transitivity
- Existence of nonforking extensions
- Continuity
- κ-ary character for some cardinal κ
- Uniqueness of nonforking extensions

We will first show when all but the last of these properties hold for $\perp$, which in the first order case is the dividing line between nonforking for simple theories and stable theories.

Proposition 4.5 (Top Monotonicity)**.** *Let $\mathcal{A}$ be a notion of amalgamation.*

1. *If $A \overset{N}{\underset{M}{\perp}} B$ and $N' \geq N$, then $A \overset{N'}{\underset{M}{\perp}} B$*

2. *Suppose that K admits finite intersection and $\mathcal{A}$ is regular, minimal. If $A \underset{M}{\overset{N}{\downarrow}} B$ and $N' \leq N$ is such that $A, B, M \subseteq N'$, then $A \underset{M}{\overset{N'}{\downarrow}} B$*

Proof. That (1) is true is straightforward from the definition of $\downarrow$. For (2), let (M_1, M_2) witness that $A \underset{M}{\overset{N}{\downarrow}} B$; as K has FI and $M \leq M_1, M_2, N' \leq N$, both $M_1 \cap N'$ and $M_2 \cap N'$ are models of K, and by regularity $M_1 \cap N', M_2 \cap N'$ are $\mathcal{A}$-subamalgamated over M inside N. Since K admits finite intersection and $\mathcal{A}$ is minimal, hence $\mathcal{A}$ is absolutely minimal by Lemma 2.18, and so in particular $(M_1 \cap N') \oplus^N_M (M_2 \cap N') \leq N'$. Hence $(M_1 \cap N', M_2 \cap N')$ is a witness to $A \underset{M}{\overset{N'}{\downarrow}} B$. □

Some straightforward observations which follow from the definition of $\downarrow$ are:

Proposition 4.6. *Let $\mathcal{A}$ be a notion of amalgamation*

1. *(Existence) For any $M \leq N$ and $A \subseteq N$, $A \underset{M}{\overset{N}{\downarrow}} M$.*

2. *(Symmetry) $A \underset{M}{\overset{N}{\downarrow}} B$ implies $B \underset{M}{\overset{N}{\downarrow}} A$.*

3. *(Right Monotonicity) If $A \underset{M}{\overset{N}{\downarrow}} B$ and $B' \subseteq B$, then $A \underset{M}{\overset{N}{\downarrow}} B'$.*

4. *(Right Normality) $A \underset{M}{\overset{N}{\downarrow}} B$ iff $A \underset{M}{\overset{N}{\downarrow}} (B \cup M)$*

Proposition 4.7 (Base Monotonicity)**.** *Suppose $\mathcal{A}$ is regular. If M' is such that $M \leq M' \subseteq B$ and $A \underset{M}{\overset{N}{\downarrow}} B$, then $A \underset{M'}{\overset{N}{\downarrow}} B$.*

Proof. Let (M_1, M_2) witness that $A \underset{M}{\overset{N}{\downarrow}} B$. In particular, this implies that there is some $N' \leq N$ such that

$$\begin{tikzcd} M_1 \arrow[r, "\iota"] & N' \\ M \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"'] \end{tikzcd}$$

Since $M \leq M' \leq M_2$ (as $M' \subseteq B \subseteq M_2$), by regularity there exists some $N'' \leq N'$ with

$$\begin{tikzcd} M_1 \arrow[r, "\iota"] & N'' \arrow[r, "\iota"] & N' \\ M \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"] \end{tikzcd}$$

Since $A \subseteq M_1 \leq N''$, hence (N'', M_2) is a witness to $A \underset{M'}{\overset{N}{\downarrow}} B$. □

Lemma 4.8. *If $\mathcal{A}$ is regular and $M_0 \leq M_1$, then $A \underset{M_0}{\overset{N}{\downarrow}} M_1$ iff there is some $M_2 \leq N$ such that $M_0 \leq M_2$, $A \subseteq M_2$ and M_1, M_2 are $\mathcal{A}$-subamalgamated over M_0 inside N.*

Proof. For the reverse direction, note that (M_2, M_1) is a witness to $A \underset{M_0}{\overset{N}{\mathop{\smile\hskip-0.9em\mid\;}}} M_1$. For the forward direction, let (M_2, M') witness that $A \underset{M_0}{\overset{N}{\mathop{\smile\hskip-0.9em\mid\;}}} M_1$, and so in particular $M_0 \leq M_1 \leq M'$. Hence by regularity, M_1, M_2 are also $\mathcal{A}$-subamalgamated over M_0 inside N. $\square$

Proposition 4.9 (Transitivity)**.** *Suppose K admits finite intersection and $\mathcal{A}$ is regular, absolutely minimal. If $M_0 \leq M_1 \leq M_2 \leq N$ and $A \subseteq N$ is such that $A \underset{M_0}{\overset{N}{\mathop{\smile\hskip-0.9em\mid\;}}} M_1$ and $A \underset{M_1}{\overset{N}{\mathop{\smile\hskip-0.9em\mid\;}}} M_2$, then $A \underset{M_0}{\overset{N}{\mathop{\smile\hskip-0.9em\mid\;}}} M_2$.*

Proof. By the above lemma, there exists $M', M'' \leq N$ such that (M', M_1) witnesses $A \underset{M_0}{\overset{N}{\mathop{\smile\hskip-0.9em\mid\;}}} M_1$ and (M'', M_2) witnesses $A \underset{M_1}{\overset{N}{\mathop{\smile\hskip-0.9em\mid\;}}} M_2$. Hence, there are also models $M' \oplus^N_{M_0} M_1, M'' \oplus^N_{M_1} M_2 \leq N$ such that:

$$
\begin{tikzcd}
M' \arrow[r, "\iota"] & M' \oplus^N_{M_0} M_1 \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M'' \arrow[r, "\iota"] & M'' \oplus^N_{M_1} M_2 \\
M_1 \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"']
\end{tikzcd}
$$

Since K has FI and $M_0 \leq M', M'' \leq N$, there is a model $M^* := M' \cap M''$, and in particular $M_0 \leq M^*, A \subseteq M^*$. So by regularity, M^*, M_1 are also $\mathcal{A}$-subamalgamated over M_0 inside N i.e.

$$
\begin{tikzcd}
M_1 \arrow[r, "\iota"] & M_1 \oplus^N_{M_0} M^* \arrow[r, "\iota"] & M_1 \oplus^N_{M_0} M' \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M^* \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "\iota"]
\end{tikzcd}
$$

Note that since $M_1, M^* \leq M''$, $M_1 \oplus^N_{M_0} M^* \leq M''$ as $\mathcal{A}$ is absolutely minimal. Therefore, again by regularity, $(M_1 \oplus^N_{M_0} M^*), M_2$ are $\mathcal{A}$-subamalgamated over M_1 inside N, so there is some M^{**} such that:

$$
\begin{tikzcd}
M_2 \arrow[r, "\iota"] & M^{**} \arrow[r, "\iota"] & M_2 \oplus^N_{M_0} M'' \\
M_1 \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \oplus^N_{M_0} M^* \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M'' \arrow[u, "\iota"]
\end{tikzcd}
$$

Combining the commutative squares on the left of the two diagrams, we get that

$$
\begin{tikzcd}
M^* \arrow[r, "\iota"] & M_1 \oplus^N_{M_0} M^* \arrow[r, "\iota"] & M^{**} \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"]
\end{tikzcd}
$$

Applying regularity once more, hence (M^*, M_2) witness that $A \underset{M_0}{\overset{N}{\mathop{\smile\hskip-0.9em\mid\;}}} M_2$. $\square$

Proposition 4.10 (Invariance)**.** *If $\mathcal{A}$ is a notion of amalgamation, then $\mathop{\smile\hskip-0.9em\mid\;}$ is invariant under K-embeddings: if $A \underset{M}{\overset{N}{\mathop{\smile\hskip-0.9em\mid\;}}} B$ and $f : N \longrightarrow N'$ is a K-embedding, then $f(A) \underset{f[M]}{\overset{N'}{\mathop{\smile\hskip-0.9em\mid\;}}} f(B)$.*

Proof. First, for the case where $f : N \cong N'$ is a K-isomorphism, the statement above holds due to the Invariance properties of $\mathcal{A}$. Then Proposition 4.5 shows that this is true for the general case where f is a K-embedding. $\square$

Corollary 4.11. *If* $\bar{a} \underset{M_0}{\overset{N}{\smile}} M_1$ *and* $gtp(\bar{a}/M_1, N) = gtp(\bar{b}/M_1, N')$*, then* $\bar{b} \underset{M_0}{\overset{N'}{\smile}} M_1$*.*

The above corollary shows that when $\mathcal{A}$ is a notion of amalgamation and $\smile$ is derived from $\mathcal{A}$, then in fact $\smile$ can be extended to a form of nonforking notion for Galois types.

Notation 4.12. Let $p \in S(M_1)$. We say that p **does not fork** over M_0 if $M_0 \leq M_1$ and there is some $\bar{a}$ and a model $N \geq M_1$ such that $(\bar{a}, M_1, N)$ realize p, and $\bar{a} \underset{M_0}{\overset{N}{\smile}} M_1$.

We say that $q \geq p$ is a **nonforking extension** if q does not fork over dom p

Corollary 4.13. *Suppose K admits finite intersection, and $\mathcal{A}$ is regular, absolutely minimal. If p does not fork over M and q is a nonforking extension of p, then q does not fork over M.*

Proposition 4.14 (Extension)**.** *Let $p \in S(M_0)$. If $M_1 \geq M_0$, then there is $q \in S(M_1)$ such that $q \geq p$ and q does not fork over M_0.*

Proof. Let $(\bar{a}, M_0, M_2)$ realize p, and let N be an $\mathcal{A}$-amalgam of M_2, M_1 over M_0 via

$$\begin{tikzcd} M_2 \arrow[r, "f"] \arrow[dr, phantom, "\mathcal{A}"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] & M_1 \arrow[u, "\iota"] \end{tikzcd}$$

Then $f(\bar{a}) \underset{M_0}{\overset{N}{\smile}} M_1$ (as witnessed by $(f[M_2], M_1)$), and $\text{gtp}(f(\bar{a})/M_0, N) = p$. Hence $\text{gtp}(f(\bar{a})/M_1, N)$ is the desired nonforking extension. $\square$

Proposition 4.15 (Locality, version 1)**.** *Suppose $\mathcal{A}$ is regular, continuous, absolutely minimal and admits decomposition. Assume further that $\mu(K) < \infty$. If $(M_i)_{i<\alpha}$ is a strictly increasing continuous chain of models and $cf(\alpha) \geq \mu_r(K)$, then for every $p \in S(\bigcup_{i<\alpha} M_i)$ a Galois type of length $< \mu(K)$, there is some $i < \alpha$ such that p does not fork over M_i.*

Proof. Let $M_b := M_0, M := \bigcup_{i<\alpha} M_i$, and let us define a sequence of models $(M'_i)_{i<\alpha}$ such that:

1. For each $i < \alpha$, $M_b \leq M'_i \leq M_{i+1}$.
2. $M'_0 = M_1$
3. For each $i < \alpha$, M'_i is such that $M_i \oplus^M_{M_b} M'_i = M_{i+1}$.

Note that $\mathcal{A}$ admitting decomposition implies that such a sequence exists. Furthermore, by construction we have that $\bigoplus^M_{M_b, i<\alpha} M'_i = M$ (as witnessed by the resolution $(M_i)_{i<\alpha}$).

Given $p \in S(M)$ a Galois type of length $< \mu(K)$, let $(\bar{a}, M, N)$ realize p, and let $N^* \leq N$ be such that $N = M \oplus^N_{M_b} N^*$ (again, N^* exists as $\mathcal{A}$ admits decomposition). Hence we also have that N is the $\mathcal{A}$-amalgam of $\{N^*\} \cup \{M'_i\}_{i<\alpha}$ over M_b (by inclusion). Since $|\bar{a}| < \mu_r(K) \leq \text{cf}(\alpha)$, there is some $i_0 < \alpha$ such that

$$\bar{a} \in N^* \oplus^N_{M_b} \left(\bigoplus^N_{M_b, i<i_0} M'_i \right) = N^* \oplus^N_{M_b} M_{i_0+1}$$

Let $N' := N^* \oplus^N_{M_b} M_{i_0+1}$. Since N is the $\mathcal{A}$-amalgam of N^*, M over M_b by inclusion, by regularity we also have that N is the $\mathcal{A}$-amalgam of N', M over M_{i_0+1}. Diagrammatically,

$$\begin{tikzcd} N^* \arrow[r, "\iota"] \arrow[dr, phantom, "\mathcal{A}"] & N \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] & M \arrow[u, "\iota"] \end{tikzcd} \Longrightarrow \begin{tikzcd} N^* \arrow[r, "\iota"] \arrow[dr, phantom, "\mathcal{A}"] & N' \arrow[r, "\iota"] \arrow[dr, phantom, "\mathcal{A}"] & N \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] & M_{i_0+1} \arrow[u, "\iota"] \arrow[r, "\iota"] & M \arrow[u, "\iota"] \end{tikzcd}$$

Note then that (N', M) is a witness to $\bar{a} \underset{M_{i_0+1}}{\overset{N}{\smile}} M$, and therefore p does not fork over M_{i_0+1}. $\square$

In fact, a related formulation of the locality property can be shown to be true using the same proof:

Proposition 4.16 (Locality, version 2)**.** *Suppose $\mathcal{A}$ is regular, continuous, absolutely minimal, admits decomposition and is such that $\mu(K) < \infty$. If $|M| > \mu_r(K) + LS(K)$ and p is a Galois type over M of length $< \mu(K)$, then there is some $M^* \leq M$ such that $|M^*| < \mu_r(K) + LS(K)^+$ and p does not fork over M^*.*

Proof. Let $\lambda = |M|$, and take some $M_b \leq M$ with $|M_b| = \mathrm{LS}(K)$. As $\mathcal{A}$ admits decomposition and is absolutely minimal, by Lemma 4.1 there is a sequence $(M_i)_{i<\alpha}$ such that:

1. $\alpha < \lambda^+$
2. For each $i < \alpha$, $M_b \leq M_i \leq M$ and $|M_i| = \mathrm{LS}(K)$
3. $M = \bigoplus^M_{M_b, i<\alpha} M_i$

Further, as $\mathcal{A}$ is regular and continuous, by Theorem 3.14 we may assume that $\alpha = \lambda$. Letting $(\bar{a}, M, N)$ be a realization of p, as in the proof for the above proposition there exists some N^* such that $N = N^* \oplus^N_{M_b} M$. Now, as $|A| < \mu(K) < \infty$, there is some subset $S \subseteq \lambda$ such that $|S| < \mu_r(K)$ and $A \subseteq N^* \oplus^N_{M_b} \left(\bigoplus^N_{M_b, i\in S} M_i\right)$. Letting $N' = N^* \oplus^N_{M_b} \left(\bigoplus^N_{M_b, i\in S} M_i\right)$, hence (as in the above proof) N is the $\mathcal{A}$-amalgam of N', M over $\bigoplus^N_{M_b, i\in S} M_i$ by regularity. Therefore, letting $M^* = \bigoplus^N_{M_b, i\in S} M_i$, we have $A \overset{N}{\underset{M^*}{\mathop{\smile}\limits^{\vert}}} M$. Furthermore, as $|S| < \mu_r(K)$ and each $|M_i| = \mathrm{LS}(K)$, since $\mathcal{A}$ is minimal, $|M^*| < \mu_r(K) + \mathrm{LS}(K)^+$ as desired. □

Corollary 4.17. *Suppose K admits finite intersection, and $\mathcal{A}$ is regular, continuous, absolutely minimal, admits decomposition, and is such that $\mu(K) < \infty$. If $M \in K$, $(M_i)_{i<\alpha}$ is continuous resolution of M with $cf(\alpha) \geq \mu_r(K)$, and $p \in S(M)$ is a type of length $< \mu(K)$ such that each $p \restriction M_i$ does not fork over M_0, then p does not fork over M_0.*

Proof. By the previous proposition, there is some $i < \alpha$ such that p does not fork over M_i. But $p \restriction M_i$ does not fork over M_0 by assumption, and so by Proposition 4.9 p does not fork over M_0. □

4.3 The two notions of uniqueness

Let us now proceed onto the last property: the uniquenss of nonforking extensions. Unsurprisingly, this is directly connected to $\mathcal{A}$ having Uniqueness.

Proposition 4.18 (Uniqueness)**.** *Suppose K admits finite intersection, $\mathcal{A}$ has uniqueness and is regular. Then for any Galois type $p \in S(M)$ and any $N \geq M$, there is a unique $q \in S(N)$ such that q is a nonforking extension of p.*

Proof. Let $q_1, q_2 \in S(N)$ be nonforking extensions of p, and let $(\bar{a}, N, N_1), (\bar{b}, N, N_2)$ be realizations of the types respectively. Since $q_1 \restriction M = q_2 \restriction M = p$ and K has AP (since $\mathcal{A}$ is a notion of amalgamation), we may assume that there is a K-isomorphism $f : N_1 \longrightarrow_M N_2$ such that $f(\bar{a}) = \bar{b}$. Now, as each q_i is a nonforking extension of p, there exists $M_1 \leq N_1$ such that (M_1, N) is a witness to $\bar{a} \overset{N_1}{\underset{M}{\mathop{\smile}\limits^{\vert}}} N$, and similarly $M_2 \leq N_2$. Letting $M'' = f[M_1] \cap M_2$, note then that $M \leq M'', \bar{b} \in M''$. Hence, by regularity, (M'', N) is also a witness for $\bar{b} \overset{N_2}{\underset{M}{\mathop{\smile}\limits^{\vert}}} N$. Further, let $M' \leq N_1$ be such that $f[M'] = M''$, and similarly (M', N) is a witness for $\bar{a} \overset{N_1}{\underset{M}{\mathop{\smile}\limits^{\vert}}} N$. But as f is also an isomorphism between M' and M'' over M, by uniqueness of $\mathcal{A}$ there is an isomorphism

g satisfying the following commutative diagram (where all the unlabelled maps are inclusions):

$$\begin{tikzcd}
& N \oplus_M^{N_1} M' \arrow[dr, "g"] & \\
N \arrow[ur] \arrow[rr] & & N \oplus_M^{N_2} M'' \\
& M' \arrow[uu] \arrow[dr, "f"] & \\
M \arrow[uu] \arrow[ur] \arrow[rr] & & M'' \arrow[uu]
\end{tikzcd}$$

In particular, $g(\bar{a}) = f(\bar{a}) = f(\bar{b}) = g(\bar{b})$ and $g[N] = N$, and hence $\mathrm{gtp}(\bar{a}, N, N_1) = \mathrm{gtp}(\bar{b}, N, N_2)$. This completes the proof. $\square$

Corollary 4.19. *Suppose K admits finite intersection, $\mathcal{A}$ has uniqueness and is regular. If $(M_i)_{i<\alpha}$ is an increasing continuous chain, and $(p_i)_{i<\alpha}$ is an increasing chain of types (with each $p_i \in S(M_i)$) such that each p_{i+1} is a nonforking extension of p_i, then there is $p \in S\big(\bigcup_{i<\alpha} M_i\big)$ such that $p \restriction M_i = M_i$ and p does not fork over M_0.*

Proof. Denote $M_\alpha := \bigcup_{i<\alpha} M_i$. By Proposition 4.14, let $p \in S(M_\alpha)$ be a nonforking extension of p_0. Note then that for each $i < \alpha$, $p \restriction M_i$ also does not fork over M_0, and hence is a nonforking extension of p_0. By the above proposition, hence $p \restriction M_i = p_i$. $\square$

Definition 4.20. We say that $\mathcal{A}$ is a notion of **geometric amalgamation** if $\mathcal{A}$ is regular, continuous, absolutely minimal, and admits decomposition with $\mu(K) < \infty$.

Conclusion 4.21. *Suppose K is a very weak AEC admitting finite intersections, $\mathcal{A}$ is a notion of geometric amalgamation on K withb uniqueness. Then the relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfies:*

- *Invariance (Proposition 4.10)*
- *Existence (Proposition 4.6.1)*
- *Symmetry (Proposition 4.6.2)*
- *Top monotonicity (Proposition 4.5)*
- *Right (and left) monotonicity (Proposition 4.6.3 and symmetry)*
- *Base monotonicity (Proposition 4.7)*
- *Normality (Proposition 4.6.4)*
- *Transitivity (Proposition 4.9)*
- *Extension (Proposition 4.14)*
- *$\mu(K)$-local character for types of length $< \mu(K)$ (Proposition 4.15 and 4.16)*
- *Continuity (Proposition 4.17)*
- *Uniqueness of nonforking extensions (Proposition 4.18)*

This completes the propositions needed to prove that $\mathop{\smile\!\!\!\!\!\!|}$ has the desired properties (under certain assumptions on K and $\mathcal{A}$). A nontrivial example of such an independence relation comes from the class of free groups:

Example 4.22. Let K be the class of free groups, with an ordering $\leq_f$ such that $G \leq_f H$ iff G is a free factor in H i.e. there is some set Y such that we can consider $H = F(Y)$ (the free group with Y as the set of generators), and moreover there is some $X \subseteq Y$ such that $G = F(X)$.

Note then that K is a weak AEC which admits finite intersections (see the Appendix for details), and taking $\mathcal{A}$ to be the notion of free amalgamation gives us that $\mathcal{A}$ is minimal (hence absolutely minimal), regular, continuous, admits decomposition, and has uniqueness. It is also clear that $\mu(K) = \aleph_0$. In this case, $\bar{a} \underset{F}{\overset{G}{\downarrow}} \bar{b}$ iff there is a free basis X of G (so $F(X) = G$) along with subsets $X_0, X_1, X_2 \subseteq X$ such that:

- $F_0 = F(X_0)$
- $X_1 \cap X_2 = X_0$
- $\bar{a} \in F(X_1)$ and $\bar{b} \in F(X_2)$

Moreover, the above lemmas show that $\downarrow$ for the class of free groups behaves as if for a superstable first order theory; this is not surprising since by defining superstability in terms of uniqueness of limit models, the uncountable categoricity of the class implies that it is indeed superstable.

On the other hand, this example is notable for two reasons:

1. The free factors of a free group are <u>not</u> closed under infinite intersections (for an example, see [BCS77]), and in particular the class of free groups do not admit arbitrary intersection. This is in contrast to classes such as vector spaces and algebraically closed fields, where the pregeometry is used to define independence but implies that the class admits intersections.

2. The first order theory of free groups is known to be not superstable (see, for example, [Poi83]), whereas $(K, \leq_f)$ is indeed superstable. Furthermore, since $G \leq_f H$ implies that G is an elementary substructure of H (see the Appendix), this implies that the free groups lies within the superstable part of the theory of free groups. This fact is, of course, trivial given that the free groups are uncountably categorical, but does show how different the class of free groups is from the class of elementarily free groups.

Before ending this section, let us demonstrate the known fact that the existence of a superstable-like independence notion implies that the class is tame, a structural notion first isolated by Grossberg and VanDieren in [GV06b] which has since been been employed extensively in structural analysis of AECs:

Definition 4.23. Let I be a linear order. We say that K is $(< \lambda)$**-tame for** I**-types** if for any model M and $p, q \in S^I(M)$, $p \neq q$ iff there exists some $N \leq M$ such that $|N| < \lambda$ and $p \upharpoonright N \neq q \upharpoonright N$. We say that K is λ-tame if it is $(< \lambda^+)$-tame.

Corollary 4.24. *If K admits finite intersection and $\mathcal{A}$ is a notion of free amalgamation, then K is $(\mu_r(K) + LS(K))^+$-tame for types of length $< \mu(K)$.*

Proof. Let $M \in K$ with $|M| > \mu_r(K) + \mathrm{LS}(K)$, $p, q \in S(M)$ be types of length $l < \mu(K)$, and let $(\bar{a}, M, N_1), (\bar{b}, M, N_2)$ realize p, q respectively. By Proposition 4.16, there is $M_a \leq M$ such that $\bar{a} \underset{M_a}{\overset{N_1}{\downarrow}} M$ and $|M_a| = \mu_r(K) + \mathrm{LS}(K)$. Define $M_b \leq M$ similarly, and (by the Löwenheim-Skolem property) let $M_0 \leq M$ be such that $M_a, M_b \leq M_0$ and $|M_0| = \mu_r(K) + \mathrm{LS}(K)$. Then by Proposition 4.7, $\bar{a} \underset{M_0}{\overset{N_1}{\downarrow}} M$ and $\bar{b} \underset{M_0}{\overset{N_2}{\downarrow}} M$. Now, if p, q are such that $p \upharpoonright M' = q \upharpoonright M'$ for every $M' \leq M$ with $|M'| \leq \mu_r(K) + \mathrm{LS}(K)$, then in particular $p \upharpoonright M_0 = q \upharpoonright M_0$. But p is a nonforking extension of $p \upharpoonright M_0$ and similarly q, so by Proposition 4.18, $p = q$. ☐

Remark. The statement of the the above Corollary begs comparison to a result of Boney that appears as Theorem 3.7 of [Vas17], stating that a pseudouniversal AEC is $(\aleph_0)$-tame.[1] Since pseudouniversality

[1] The actual result is slightly stronger, but difficult to state here precisely due to small conflicts of notation.

is a strengthening[2] of admitting intersections with $\mu(K) = \aleph_0$ (when a suitable notion of amalgamation is defined), at first glance our result appears to be comparable. Besides the slightly different cardinal arithmetic, the main difference is that our result here relaxes the intersection requirement to only finite intersections, but at the expense of requiring $\mathcal{A}$ to have uniqueness (which, as Section 6 explores, has strong implications regarding the structure of K and is not a trivial assumption).

4.4 A pathological case and strictness

Although the properties of an independence relation explored in the previous section are standard for analysing the canonicity of forking independence in a simple or stable setting (for the latter case, both in first order model theory and in the study of AECs), in a even more general setting (such as thorn-independence in o-minimal theories) another property that is important strictness:

Definition 4.25. $\mathop{\smile\!\!\!\!\!\!|}$ is **strict** if for any $M \leq N$ and $A \subseteq N$, $A \mathop{\smile\!\!\!\!\!\!|}\limits^{N}_{M}$ iff $A \subseteq M$.

We would like to consider the case where K has finite intersections and $\mathcal{A}$ is a notion of geometric amalgamation with uniqueness. In this case, strictness of $\mathop{\smile\!\!\!\!\!\!|}$ corresponds to a property of $\mathcal{A}$:

Lemma 4.26.

1. *If $\mathcal{A}$ is minimal and regular, and $M_0 \leq M_1 \leq N$, then $M_1 \mathop{\smile\!\!\!\!\!\!|}\limits^{N}_{M_0} M_1$ iff*

$$\begin{tikzcd} M_1 \arrow[r, "\iota"] & M_1 \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"'] \end{tikzcd}$$

2. *If in addition K has finite intersections, and $A \subseteq N$ is such that $A - M_0$ is nonempty, then $A \mathop{\smile\!\!\!\!\!\!|}\limits^{N}_{M_0}$ iff there exists M_1 as above with $A \subseteq M_1$*

Proof. (1) is straightforward from Lemma 4.8. For (2), the reverse direction is also straightforward, and in the forward direction, if (M', M'') is a witness for $A \mathop{\smile\!\!\!\!\!\!|}\limits^{N}_{M_0} A$, then taking $M_1 = M' \cap M''$ is sufficient. □

This particular property of $\mathcal{A}$ is symptomatic of a more general pathological case of amalgamation:

Lemma 4.27. *Assume $\mathcal{A}$ is minimal and regular. If $M_0 \leq M_1 \leq N$, then*

$$\begin{tikzcd} N \arrow[r, "\iota"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"'] \end{tikzcd} \Rightarrow \begin{tikzcd} M_1 \arrow[r, "\iota"] & M_1 \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"'] \end{tikzcd}$$

Proof. Since $M_1 \leq N$, by regularity we have that M_1, M_1 is $\mathcal{A}$-subamalgamated over M_0 inside N. As $\mathcal{A}$ is minimal, hence the $\mathcal{A}$-amalgam must be M_1 itself. □

For a general notion of amalgamation $\mathcal{A}$, the fact that M_1 is an $\mathcal{A}$-amalgam of M_1, M_1 over M_0 may appear to be a triviality. However, if $\mathcal{A}$ is assumed to have uniqueness, then this has very strong implications:

[2] To quote [Vas17], the extra requirement is that "the isomorphism characterizing equality of Galois types is unique".

Lemma 4.28. *Suppose that $\mathcal{A}$ is regular and has uniqueness. Assume $M \leq N$ is such that*

$$\begin{tikzcd} N \arrow[r, "\iota"] & N \\ M \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N \arrow[u, "\iota"'] \end{tikzcd}$$

If N_1, N_2 are such that $M \leq N_1, N_2 \leq N$ and $N_1 \cong_M N_2$, then in fact $N_1 = N_2$

Proof. By regularity, we have that

$$\begin{tikzcd} N \arrow[r, "\iota"] & N \\ M \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_1 \arrow[u, "\iota"'] \end{tikzcd} \quad \begin{tikzcd} N \arrow[r, "\iota"] & N \\ M \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_2 \arrow[u, "\iota"'] \end{tikzcd}$$

Let $f : N_1 \to_M N_2$ be a K-isomorphism. By uniqueness of $\mathcal{A}$, we have an h such that

$$\begin{tikzcd} & & N \\ N \arrow[r, "\iota"] \arrow[urr, bend left, "\iota"] & N \arrow[ur, "h"] & \\ M \arrow[u, "\iota"] \arrow[r, "\iota"] & N_1 \arrow[u, "\iota"'] \arrow[uur, bend right, "f"'] & \end{tikzcd}$$

But this implies that $h = \mathrm{id}_N$, and as $N_1 \leq N$ and $f = h \upharpoonright N$, thus $f = \mathrm{id}_{N_1}$, and hence $N_1 = N_2$. $\square$

Corollary 4.29. *Suppose $\mathcal{A}$ is regular and has uniqueness, and $M \leq N$ is such that*

$$\begin{tikzcd} N \arrow[r, "\iota"] & N \\ M \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N \arrow[u, "\iota"'] \end{tikzcd}$$

Then for any $N' \geq M$, if there exists $f : N' \to_M N$, then f is unique. In particular, $Aut(N/M) = \{id\}$

This form of highly rigid pairs of models where the only automorphisms are trivial is found in some pathological AECs, for example the class of well-orderings of order type $\leq \alpha$ for some ordinal α and ordered by initial segments, where even the independence relation of non-splitting would fail to be strict. [3] As an example, in this case taking $\mathcal{A}$-amalgamation to simply be the greater of two ordinals would be a geometric notion of amalgamation with uniqueness that nonetheless has all the pathological properties mentioned above. In light of this, we will add a hypothesis to $\mathcal{A}$ to rule out such cases:

Definition 4.30. A notion of amalgamation $\mathcal{A}$ is **strict** if whenever $M \leq N$ is such that

$$\begin{tikzcd} N \arrow[r, "\iota"] & N \\ M \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N \arrow[u, "\iota"'] \end{tikzcd}$$

Then in fact $M = N$.

A notion of **free amalgamation** is a notion of geometric amalgamation that additionally is strict and has uniqueness.

Remark. Whilst ruling out pathological cases by an additional assumption may appear to be a dishonest method, we should note that this is true for our guiding examples of direct sums of abelian groups and free amalgamation of nonabelian groups. In fact, that a "free" amalgamation of N, N over M (not by inclusion) should be more than N itself makes $\mathcal{A}$ more akin to pushouts in algebra.

[3] We thank Samson Leung for bringing our attention to this illustrative example.

Strict amalgamation also allows us to give lower bounds on the cardinality of amalgams, which is necessary for the next chapter:

Lemma 4.31. *Suppose $\mathcal{A}$ is minimal, regular, and strict. If N is an $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b where each $M_i \gneq M_b$, then $|N| \geq |\alpha|$*

Proof. By Lemma 4.27, if $M_1 \gneq M_0$ and N_1 is an $\mathcal{A}$-amalgam of M_1, N_0 over M_0, then $N_1 \gneq N_0$. Then $|N| \gneq |\alpha|$ by induction. $\square$

Chapter 5

Categoricity Transfer with Free Amalgamation

5.1 With prime and minimal models

Up until this point, we have three primary examples of classes with a notion of free amalgamation which have guided our exploration:

- The class of vector spaces over a fixed field with the subspace (equivalently, elementary submodel) ordering
- The class of (torsion) divisible groups with the subgroup ordering
- The class of free groups with the "free factor" ordering (see Example 4.22)

The key characteristic shared, and indeed the driving intuition for this study, is that such classes have some notion of "basis" which generates each model. Now, in the case of the class of vector spaces, the eventual categoricity of the class can be derived from the fact that any bijection between two bases extends to an isomorphism between the spanned spaces. An analogous principle clearly holds also for the free groups, and the same argument can be applied more generally to the cases of strongly minimal first order theories and quasiminimal excellent classes with the countable closure property. On the other hand, this does not apply to the class of divisible groups, and the torsion divisible groups are not categorical in any cardinal whereas the class of free groups are uncountably categorical. In this sense, we will formalize the intuitive argument above to establish sufficient conditions for a categoricity transfer theorem.

One aspect of the argument above for vector spaces is that two superspaces V, W of U are isomorphic over U if $V/U, W/U$ have the same dimension. Although there is no notion of dimensionality within the current context, we note that the dimension of a vector space only differs from its cardinality for spaces of small dimension. This allows us to formalize the notion of two extensions being "isomorphic" when they are of sufficiently large cardinality:

Definition 5.1. Suppose $\mathcal{A}$ is a notion of free amalgamation in K.

1. We define $\theta(K) = \mu(K) + \mathrm{LS}(K)$
2. Given $M_b \leq M_t$ and α an ordinal, for a model $N \geq M_b$ we write "$N \cong M_t^\alpha / M_b$" to indicate that $N = \bigoplus_{M_b, i<\alpha}^N M_i$, where each $M_i \cong_{M_b} M_t$.
3. We define an equivalence relation $\sim$ on pairs of models of K by: given $M_1 \leq N_1$ and $M_2 \leq N_2$, $(N_1, M_1) \sim (N_2, M_2)$ iff there is a K-isomorphism $f : N_1^{\theta(K)}/M_1 \cong N_2^{\theta(K)}/M_2$ with $f[M_1] = M_2$.

Remark. Note that the above definition does not construct M_t^α / M_b as a particular model, but if N_1, N_2 are such that both $N_1 \cong M_t^\alpha / M_b$ and $N_2 \cong M_t^\alpha / M_b$, then in fact $N_1 \cong_{M_b} N_2$ by uniqueness of $\mathcal{A}$, and hence we may consider M_t^α / M_b as a particular choice of representative inside K. In this sense, for any ordinal

$0 < \beta < \alpha$ we may consider $M_b \leq M_t \leq M_t^\beta/M_b \leq M_t^\alpha/M_b$. In this sense, we extend the notation by defining $M_t^0/M_b = M_b$. Furthermore, note that by Theorem 3.13, $M_t^\alpha/M_b \cong M_t^{|\alpha|}/M_b$, and as $\mathcal{A}$ is strict and minimal, by Lemma 4.31, $|M_t^\alpha/M_b| = |\alpha| + |M_t| + \mathrm{LS}(K)$.

Also, although the notation employed here might be suggestive of some form of ultrapower, we should recall that this is not an ultrapower as M_b is a model in K rather than an ultrafilter over the index set α. More importantly, it is not guaranteed that K is closed under taking ultrapowers, as K is not necessarily an elementary class. On the other hand, we have chosen this notation to emphasize that M_t^α/M_b consists of many copies of M_t, but where the type of any given element is "calculated" based on its relation with M_b.

Lemma 5.2. *Suppose $\mathcal{A}$ is a notion of free amalgamation. If $(N_1, M) \sim (N_2, M)$, then for any $\lambda \geq \theta(K)$, there is a K-isomorphism $f : N_1^\lambda/M \cong_M N_2^\lambda/M$.*

Proof. Decompose $\lambda = \bigsqcup_{j<\lambda} S_j$ such that each $|S_j| = \theta(K)$. Defining models $N_1^*, N_1^i, N_2^*, N_2^i$ such that $N_l^* = \bigoplus_{M,i<\lambda}^{N_l^*} N_l^i$ for $l = 1, 2$, note then that for each $j, j' < \lambda$,

$$\bigoplus_{M,i\in S_j}^{N_1^*} N_1^i \cong N_1^{\theta(K)}/M \cong N_2^{\theta(K)}/M \cong \bigoplus_{M,i\in S_{j'}}^{N_2^*} N_2^i$$

So let us define $N_l^{S_j} = \bigoplus_{M,i\in S_j}^{N_l^*} N_1^i$. Then, by applying Theorem 3.13, we get that $N_l^* = \bigoplus_{M,j<\lambda}^{N_l^*} N_l^{S_j}$. Hence, as $\mathcal{A}$ has uniqueness, we get that N_1^*, N_2^* are isomorphic over M. □

Definition 5.3. Given K an abstract class, we say that $M \in K$ is a **prime and minimal model** of K if:

1. For every $N \in K$, there is a K-embedding $\iota_N : M \longrightarrow N$; and
2. For every K-embedding $f : N_1 \longrightarrow N_2$, $f \circ \iota_{N_1} = \iota_{N_2}$

If K has a prime and minimal model, we fix such a model and denote it by 0_K.

Theorem 5.4. *Suppose $\mathcal{A}$ is a notion of free amalgamation in K, and 0_K is a prime and minimal model. If K is λ-categorical in some $\lambda \geq \theta(K)$, then for any M_1, M_2 in K with $|M_1| = |M_2| = LS(K)$, $(M_1, 0_K) \sim (M_2, 0_K)$.*

Proof. Given M_1, M_2 and λ as above, $|M_1^\lambda/0_K| = |M_2^\lambda/0_K| = \lambda$. Hence, by λ-categoricity, there is some K-isomorphism $f : M_1^\lambda/0_K \cong M_2^\lambda/0_K$, and moreover $f[0_K] = 0_K$ as 0_K is prime and minimal. Thus, WLOG we may assume that $N = M_1^\lambda/0_K = M_2^\lambda/0_K$, and in fact that there exists sequence $(M_1^i)_{i<\lambda}, (M_2^i)_{i<\lambda}$ such that:

1. For each $i < \lambda$, M_1^i is isomorphic to M_1 and M_2^i is isomorphic to M_2 (over 0_K).
2. Each $0_K \leq M_1^i, M_2^i \leq N$; and
3. $N = \bigoplus_{0_K,i<\lambda}^N M_1^i = \bigoplus_{0_K,i<\lambda}^N M_2^i$

We will construct two sequences of sets $(S_j)_{j<\omega}, (T_j)_{j<\omega}$, satisfying:

1. Each $S_j \subseteq \lambda$ with $|S_j| = \theta(K)$, and similarly for T_j
2. $S_0 = T_0 = \theta(K)$
3. $S_j \subseteq S_{j+1}$ and $T_j \subseteq T_{j+1}$
4. For each $j < \omega$, $\bigoplus_{0_K,i\in T_j}^N M_2^i \leq \bigoplus_{0_K,i\in S_{j+1}}^N M_1^i$; and
5. For each $j < \omega$, $\bigoplus_{0_K,i\in S_j}^N M_1^i \leq \bigoplus_{0_K,i\in T_{j+1}}^N M_2^i$

Let us first show that such a construction is sufficient: defining $S := \bigcup_{j<\omega} S_j$ and $T := \bigcup_{j<\omega} T_j$, note that by Lemma 3.14.

$$\bigoplus_{0_K, i\in S}^{N} M_1^i = \bigcup_{j<\omega} \left(\bigoplus_{0_K, i\in S_j}^{N} M_1^i \right)$$

The same statement holds for T and M_2^i. Hence, by (3) and (4) of the construction above, we have that $\bigoplus_{0_K,i\in S}^N M_1^i = \bigoplus_{0_K,i\in T}^N M_2^i$. But since $|S| = |T| = \theta(K)$, hence we can take $\bigoplus_{0_K,i\in S} M_1^i \cong M_1^{\theta(K)}/0_K$, and therefore $(M_1, 0_K) \sim (M_2, 0_K)$.

Let us complete the proof by constructing S_j, T_j. Given S_j, T_j already defined, consider $M' = \bigoplus_{0_K,i\in T_j}^N M_2^i$: we have that $|M'| = \mathrm{LS}(K) + |T_j| = \theta(K)$, and hence there is $S_{j+1} \subseteq \lambda$ such that $|S_{j+1}| = \theta(K) \times \mu(K) = \theta(K)$ and $M' \subseteq \bigoplus_{0_K,i\in S_{j+1}}^N M_1^i$. Similarly we can define T_{j+1}, and this completes the proof. $\square$

Note that the conclusion of the above theorem holds for the classes of vector spaces and free groups, but not for divisible groups: for example, letting 0_G denote the trivial group, it is clear that if $p \neq q$ are primes, then $(\mathbb{Z}(p^\infty), 0_G), (\mathbb{Z}(q^\infty), 0_G)$ are not $\sim$ equivalent.

Lemma 5.5. *Suppose $\mathcal{A}$ is a notion of free amalgamation in K. Given models $M_0 \leq M_1, M_2$, if $(M_1, M_0) \sim (M_2, M_0)$, then for any ordinal β,*

$$(M_1^\beta/M_0) \oplus_{M_0} (M_2^{\theta(K)}/M_0) \cong_{M_0} M_1^{|\beta|+\theta(K)}/M_0 \cong_{M_0} M_2^{|\beta|+\theta(K)}/M_0$$

Proof. As $(M_1, M_0) \sim (M_2, M_0)$, $M_1^{\theta(K)}/M_0 \cong_{M_0} M_2^{\theta(K)}/M_0$, and hence

$$(M_1^\beta/M_0) \oplus_{M_0} (M_2^{\theta(K)}/M_0) \cong_{M_0} M_1^{\beta+\theta(K)}/M_0$$

Furthermore, by Theorem 3.14 and Lemma 5.2, we have that

$$M_1^{\beta+\theta(K)}/M_0 \cong_{M_0} M_1^{|\beta|+\theta(K)}/M_0 \cong_{M_0} M_2^{|\beta|+\theta(K)}/M_0$$

$\square$

Theorem 5.6. *Suppose $\mathcal{A}$ is a notion of free amalgamation in K, and K has a prime and minimal model. If K is λ-categorical in some $\lambda^* \geq \theta(K)$, then K is λ-categorical in every cardinal $\lambda \geq \theta(K) + (2^{LS(K)})^+$.*

Proof. By the previous theorem, for any $M_1, M_2 \in K_{\mathrm{LS}(K)}$, $(M_1, 0_K) \sim (M_2, 0_K)$. Hence by Lemma 5.2, it suffices to show that if $M \in K$ and $|M| = \lambda \geq \theta(K) + (2^{\mathrm{LS}(K)})^+$, then $M \cong M'^\lambda/0_K$ for some $M' \in K_{\mathrm{LS}(K)}$.

So given $M \in K$ and $|M| = \lambda$, by Lemma 4.1 we can decompose $M = \bigoplus_{0_K,i<\lambda}^M M_i$ such that each $|M_i| = \mathrm{LS}(K)$. Letting $\Gamma := \{M_i/\cong : i < \lambda\}$ (where $\cong$ is the equivalence relation of K-isomorphism), note that since $|\Gamma| \leq 2^{\mathrm{LS}(K)}$, there is some $P \in \Gamma$ which is realized $\geq \theta(K) + (2^{\mathrm{LS}(K)})^+$ times in the sequence $(M_i)_{i<\lambda}$. For each $Q \in \Gamma$, let us also fix some $M_Q \in \{M_i : i < \lambda\}$ such that $M_Q \vDash Q$. Note that by the previous theorem, for any $Q_1, Q_2 \in \Gamma$, $(M_{Q_1}, 0_K) \sim (M_{Q_2}, 0_K)$.

Defining $S := \{i \in \lambda : M_i \vDash P\}$, we can decompose S as a disjoint union $S = \bigsqcup_{Q\in\Gamma} S_Q$ which is indexed by Γ and such that each $|S_Q| \geq \theta(K) + (2^{\mathrm{LS}(K)})^+$ and is a regular cardinal (possibly except for S_P). Now, for each $Q \in \Gamma$, we have that $\bigoplus_{0_K,i\in S_Q}^M M_i \cong M_P^{|S_Q|}/0_K$ as each $i \in S_Q \subseteq S$. Defining $N_{S_Q} = \bigoplus_{0_K,i\in S_Q}^M M_i$, note that as $|S_Q| \geq \theta(K)$ and $(M_P, 0_K) \sim (M_Q, 0_K)$, by the above Lemma we have that $N_{S_Q} \cong M_Q^{|S_Q|}/0_K$. Hence there is a sequence $(N_Q^i)_{i<|S_Q|}$ such that $N_{S_Q} = \bigoplus_{0_K,i<|S_Q|}^M N_Q^i$ and such that each $N_Q^i \vDash Q$.

Now, for each $Q \in \Gamma$ such that $Q \neq P$, let $T_Q := \{i \in \lambda : M_i \vDash Q\}$, and define $N_Q^* := \bigoplus_{0_K,i\in T_Q}^M M_i$. By Theorems 3.13 and 3.14, each N_{S_Q}, N_Q^* are $\mathcal{A}$-subamalgamated inside M over 0_K, and so we have that

$$N_{S_Q} \oplus_{0_K}^M N_Q^* = \left(\bigoplus_{0_K, i<|S_Q|}^{M} N_Q^i \right) \oplus_{0_K}^M \left(\bigoplus_{0_K, i\in T_Q}^{M} M_i \right)$$

In other words, $N_{S_Q} \oplus_{0_K}^M N_Q^* \cong M_Q^{|S_Q|+|T_Q|}/0_K$ by Lemma 5.2. In particular, as $(M_Q, 0_K) \sim (M_P, 0_K)$, we also have that $N_{S_Q} \oplus_{0_K}^M N_Q^* \cong M_P^{|S_Q|+|T_Q|}/0_K$.

This implies that

$$M = \bigoplus_{0_K, i<\lambda}^{M} M_i = \left(\bigoplus_{0_K, i\in S}^{M} M_i \right) \oplus_{0_K}^{M} \left(\bigoplus_{0_K, Q\neq P}^{M} \left(\bigoplus_{0_K, i\in T_Q}^{M} M_i \right) \right)$$
$$= N_{S_P} \oplus_{0_K}^{M} \left(\bigoplus_{0_K, Q\neq P}^{M} N_{S_Q} \oplus_{0_K}^{M} N_Q^* \right)$$

Since $N_{S_P} \cong M_P^{|S_P|}/0_K$ and $N_{S_Q} \oplus_{0_K}^{M} N_Q^* \cong M_P^{|S_Q|+|T_Q|}/0_K$, thus we get that $M \cong M_P^{\lambda}/0_K$. This completes the proof. $\square$

Note that in the above argument, the fact that $\lambda > 2^{\mathrm{LS}(K)}$ was used to ensure that $|\Gamma| < \lambda$, and hence some $P \in \Gamma$ is realized by many M_i's. In particular, since each $|M_i| = \mathrm{LS}(K)$, in fact we can bound $|\Gamma| \leq I(K, \mathrm{LS}(K))$, where $I(K, \theta)$ is the number of non-isomorphic models in K_θ. This gives the following strengthening:

Theorem 5.7. *Suppose $\mathcal{A}$ is a notion of free amalgamation in K, and K has a prime and minimal model. If K is λ-categorical in some $\lambda^* \geq \theta(K)$, then K is λ-categorical in every cardinal $\lambda \geq \theta(K) + I(K, LS(K))^+$.*

5.2 Without prime and minimal models

The above case assumed the existence of a prime and minimal model, which for most algebraic examples is the trivial object inside the class. On the other hand, this is a very strong assumption from a model-theoretic point of view; for example, intuitively the class of saturated algebraically closed fields (equivalently, the algebreically closed fields of infinite transcendental degree) should also allow the same argument for categoricity transfer, but the class lacks a prime and minimal model. In order to modify the above argument to work in this case, we need to strengthen the notion of amalgamation with an additional property:

Definition 5.8. Let $\mathcal{A}$ be a notion of amalgamation that is regular and absolutely minimal. We say that $\mathcal{A}$ is **3-monotonic** if the following condition is satisfied: Given models $M_0 \leq M_1, M_2, M_3 \leq N$ such that

1. M_1, M_2 are $\mathcal{A}$-subamalgamated inside N over M_0; and
2. N is the $\mathcal{A}$-amalgam of $M_3, M_1 \oplus_{M_0}^{N} M_2$ over M_0 via inclusion

Then N is the $\mathcal{A}$-amalgam of $M_1 \oplus_{M_0}^{N} M_3, M_2 \oplus_{M_0}^{N} M_3$ over M_3.

Diagrammatically, if the following commutative squares are $\mathcal{A}$-amalgams:

$$\begin{tikzcd} M_1 \arrow[r, "\iota"] & M_1 \oplus_{M_0}^{N} M_2 \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"'] \end{tikzcd} \qquad \begin{tikzcd} M_3 \arrow[r, "\iota"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \oplus_{M_0}^{N} M_2 \arrow[u, "\iota"'] \end{tikzcd}$$

Then we also have the $\mathcal{A}$-amalgam

$$\begin{tikzcd} M_1 \oplus_{M_0}^{N} M_3 \arrow[r, "\iota"] & N \\ M_3 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \oplus_{M_0}^{N} M_3 \arrow[u, "\iota"'] \end{tikzcd}$$

In particular, these models also form the following commutative diagram (simplifying $M_{ij} := M_i \oplus_{M_0}^{N} M_j$

and where all the arrows are inclusion maps), where each face of the cube is an $\mathcal{A}$-amalgam:

$$
\begin{tikzcd}
 & M_{13} \arrow[rr] & & N \\
M_3 \arrow[ur] \arrow[rr] & & M_{23} \arrow[ur] & \\
 & M_1 \arrow[uu] \arrow[rr] & & M_{12} \arrow[uu] \\
M_0 \arrow[uu] \arrow[ur] \arrow[rr] & & M_2 \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

Lemma 5.9. *Suppose $\mathcal{A}$ is regular, continuous, absolutely minimal and 3-monotonic. If $M = \bigoplus^{M}_{M_b, i<\alpha} M_i$ and $N = N^* \oplus^N_{M_b} M$, then*

$$N = \bigoplus^{N}_{N^*, i<\alpha} (N^* \oplus^N_{M_b} M_i)$$

Proof. Let $(M'_i)_{i<s(\alpha)}$ be a continuous resolution of M witnessing that M is the $\mathcal{A}$-amalgam of $(M_i)_{i<\alpha}$ over M_b by inclusion, and so by Corollary 3.9 each $M'_i = \bigoplus^{N}_{M_b, i<\alpha} M_i$. We will prove the statement by induction on α:

1. When $\alpha = 1$, the statement is trivially true.
2. If α is a limit ordinal and the statement is true for all $i < \alpha$, then for each $i < \alpha$, we have

$$N'_i := N^* \oplus^N_{M_b} M'_i = N^* \oplus^{N'_i}_{M_b} \left(\bigoplus^{N'_i}_{M_b, j<i} M_j \right) = \bigoplus^{N'_i}_{N^*, j<i} (N^* \oplus^{N'_i}_{M_b} M_j)$$

   Note that as $\mathcal{A}$ is absolutely minimal, we can replace all the superscript N'_i by N. As a result, we thus have:

   (a) $N'_0 = N^* \oplus^N_{M_b} M'_0 = N^* \oplus^N_{M_b} M_b = N^*$

   (b) $N'_1 = N^* \oplus^N_{M_b} M_1$

   (c) For $1 < i < \alpha$, $N'_i = \bigoplus^{N}_{N^*, j<i} (N^* \oplus^N_{M_b} M_j)$

   Hence, letting $N'_\alpha := \bigcup_{i<\alpha} N'_i$, the sequence $(N'_i)_{i<\alpha}$ is a witness to

$$N'_\alpha = \bigoplus^{N}_{N^*, i<\alpha} (N^* \oplus^N_{M_b} M_i)$$

   But as $(M'_i)_{i<\alpha}$ is a continuous resolution of M, and each $N'_i = N^* \oplus^N_{M_b} M'_i$, by continuity of $\mathcal{A}$

$$N'_\alpha = \bigcup_{i<\alpha} (N^* \oplus^N_{M_b} M'_i) = N^* \oplus^N_{M_b} \left(\bigcup_{i<\alpha} M_i \right) = N$$

   This completes the proof for the limit step.

3. If the inductive hypothesis is true for α, then we have

$$N^* \oplus^N_{M_b} \left(\bigoplus^{N}_{M_b, i<\alpha} M_i \right) = \bigoplus^{N}_{N^*, i<\alpha} (N^* \oplus^N_{M_b} M_i)$$

Since $\mathcal{A}$ is 3-monotonic, we therefore get the following diagram where each face of the cube is an $\mathcal{A}$-amalgam:

$$\begin{tikzcd}
 & N^* \oplus^N_{M_b} M_\alpha \arrow[rr] & & N \\
N^* \arrow[ur] \arrow[rr] & & \bigoplus^N_{N^*, i<\alpha}(N^* \oplus^N_{M_b} M_i) \arrow[ur] & \\
 & M_\alpha \arrow[uu] \arrow[rr] & & M \arrow[uu] \\
M_b \arrow[uu] \arrow[ur] \arrow[rr] & & \bigoplus^N_{M_b, i<\alpha} M_i \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

In particular, the top face thus guarantees that

$$N = (N^* \oplus^N_{M_b} M_\alpha) \oplus^N_{N^*} \left(\bigoplus^N_{N^*, i<\alpha} (N^* \oplus^N_{M_b} M_i) \right) = \bigoplus^N_{N^*, i<\alpha+1} (N^* \oplus^N_{M_b} M_i)$$

This completes the successor step of the proof.

$\square$

Corollary 5.10. *Suppose $\mathcal{A}$ is a notion of free amalgamation and is 3-monotonic. For any $M_t \geq M_b$ and ordinals $0 < \beta < \alpha$, let $N_1 = M_t^\beta / M_b$ and $N_2 = M_t^{\beta+1}/M_b$. Then $M_t^\alpha / M_b \cong N_2^{\alpha-\beta}/N_1$.*

Proof. Let $M = \bigoplus^M_{M_b, i<\alpha} M'_i$ where each $M'_i \cong_{M_b} M_t$, and hence $M \cong M_t^\alpha/M_b$. Defining $M^* = \bigoplus^M_{M_b, i<\beta} M'_i$, note then that $M^* \cong M_t^\beta / M_b \cong_{M_b} N_1$. Moreover, therefore we have that for each i such that $\beta \leq i < \alpha$, $M^* \oplus^M_{M_b} M'_i \cong M_t^{\beta+1}/M_b \cong_{M_b} N_2$. Hence by the above lemma, we also have that

$$M = M^* \oplus^M_{M_b} \left(\bigoplus^M_{M_b, i<\alpha-\beta} M'_{\beta+i} \right) = \bigoplus^M_{M^*, i<\alpha-\beta} (M^* \oplus^M_{M_b} M'_{\beta+i}) \cong N_2^{\alpha-\beta}/N_1$$

$\square$

Theorem 5.11. *Suppose $\mathcal{A}$ is a notion of free amalgamation and is 3-monotonic. Given $M_1 \lneq M_2, N_1 \lneq N_2$ models of cardinality $LS(K)$, define $M_b = M_2^{\theta(K)}/M_1, M_t = M_2^{\theta(K)+1}/M_1$ and N_b, N_t likewise. If K is λ-categorical for some $\lambda > \theta(K)$, then $(M_t, M_b) \sim (N_t, N_b)$. In particular, $M_b \cong N_b$.*

Proof. As before, note that $|N_2^\lambda / N_1| = |M_2^\lambda/M_1| = \lambda$, and hence we can consider $M_2^\lambda/M_1 \cong N_2^\lambda/N_1$ by λ-categoricity. In other words, there is a model $N \in K_\lambda$ and models $(M'_i)_{i<\lambda}, (N'_i)_{i<\lambda}$ such that:

1. For each $i < \lambda$, $M_1 \leq M'_i \leq N$ and $M'_i \cong_{M_1} M_2$
2. For each $i < \lambda$, $N_1 \leq N'_i \leq N$ and $N'_i \cong_{N_1} N_2$
3. $N = \bigoplus^N_{M_1, i<\lambda} M'_i = \bigoplus^N_{N_1, i<\lambda} N'_i$

First, we will construct sequences of sets $(S_j)_{j<\omega}, (T_j)_{j<\omega}$ satisfying:

1. Each $S_j, T_j \subseteq \lambda$, and each $|S_j|, |T_j| = \theta(K)$
2. $(S_j)_{j<\omega}, (T_j)_{j<\omega}$ are increasing sequences of sets
3. For each $j < \omega$, $\bigoplus^N_{M_1, i \in S_j} M'_i \leq \bigoplus^N_{N_1, i \in T_{j+1}} N'_i$
4. For each $j < \omega$, $\bigoplus^N_{N_1, i \in T_j} N'_i \leq \bigoplus^N_{M_1, i \in S_{j+1}} M'_i$

We will construct these sets by induction:

- Since $|N_1| = \mathrm{LS}(K)$, there is $S_0 \subseteq \lambda$ such that $|S_0| = \theta(K)$ and $N_1 \leq \bigoplus_{M_1, i\in S_0}^N M_i'$. Similarly we can define T_0 such that $M_1 \leq \bigoplus_{N_1, i\in T_0}^N N_i'$.

- If T_j is defined and $|T_j| = \theta(K)$, then $\bigoplus_{N_1. i\in T_j}^N N_i'$ is of cardinality $\mu(K) + \mathrm{LS}(K)$, and hence there is $S_{j+1} \subseteq \lambda$ such that $|S_{j+1}| = \theta(K)$ and satisfying (3). Similarly we can define T_{j+1} such that (4) is satisfied.

Letting $S = \bigcup_{j<\omega} S_j$ and $T = \bigcup_{j<\omega}$, note then that $|S| = |T| = \theta(K)$, and therefore we have

$$M_b = M_2^{\theta(K)}/M_1 \cong \bigoplus_{M_1, i\in S}^N M_i' = \bigoplus_{N_1, i\in T}^N N_i' \cong N_2^{\theta(K)}/N_1 = N_b$$

Note that by Theorem 3.13, we also have that

$$N = \bigoplus_{M_1, i<\lambda}^N M_i' = \left(\bigoplus_{M_1, i\in S}^N M_i'\right) \oplus_{M_1}^N \left(\bigoplus_{M_1, i\notin S}^N M_i'\right)$$

So, letting $M^* = \bigoplus_{M_1, i\in S}^N M_i'$, we have by Lemma 5.9 that

$$N = \bigoplus_{M^*, i\notin S}^N (M^* \oplus_{M_1}^N M_i')$$

Furthermore, since $\bigoplus_{M_1, i\in S}^N M_i' = \bigoplus_{N_1, i\in T}^N N_i'$ by construction of S, T, we also have that

$$N = \bigoplus_{M^*, i\notin T}^N (M^* \oplus_{M_2}^N N_i')$$

Let us define $M_i'' = M^* \oplus_{M_1}^N M_i'$ for $i \notin S$, and note that we have $|M_i''| = |M^*| + |M_i'| = \theta(K)$. Also, by definition we have that $M_i'' \cong M_2^{\theta(K)+1}/M_1 = M_t$. Similarly defining N_i'' for $i \notin T$, we thus have

$$N = \bigoplus_{M^*, i\notin S}^N M_i'' = \bigoplus_{M^*, i\notin T}^N N_i''$$

Since $\lambda > \theta(K) = |S| = |T|$, by re-indexing the sequences we may consider

$$N = \bigoplus_{M^*, i<\lambda}^N M_i'' = \bigoplus_{M^*, i<\lambda}^N N_i''$$

Now, let us define new sequences of sets $(U_k)_{k<\omega}, (V_k)_{k<\omega}$ such that

1. For each $k < \omega$, $U_k, V_k \subseteq \lambda$ and $|U_k| = |V_k| = \theta(K)$
2. $(U_k)_{k<\omega}, (V_k)_{k<\omega}$ are increasing sequences of sets
3. $S_0 = T_0 = \theta(K)$
4. For each $k < \omega$, $\bigoplus_{M^*, i\in T_k}^N N_i'' \leq \bigoplus_{M^*, i\in S_{k+1}}^N M_i''$
5. For each $k < \omega$, $\bigoplus_{M^*, i\in S_k}^N M_i'' \leq \bigoplus_{M^*, i\in T_{k+1}}^N N_i''$

The construction is the same as in Theorem 5.4 and above, using the fact that since each $|U_k| = |V_k| = \theta(K)$, $\bigoplus^N_{M^*, i \in S_k} M''_i, \bigoplus^N_{M^*, i \in T_k} N''_i$ are also of cardinality $\theta(K)$. In particular, if $U = \bigcup_{k<\omega} U_k$ and $V = \bigcup_{k<\omega} V_k$, then we again have that

$$M^\theta_t(K)/M_b \cong \bigoplus^N_{M^*, i \in U} M''_i = \bigoplus^N_{M^*, i \in V} N''_i \cong N^\theta_t(K)/N_b$$

This completes the proof. $\square$

Definition 5.12. Let K be a very weak AEC. We say that K has **common small models** if for any models $N_1, N_2 \in K_{>\mathrm{LS}(K)}$, there is $M_1, M_2 \in K_{\mathrm{LS}(K)}$ such that $M_1 \leq N_1, M_2 \leq N_2$ and $M_1 \cong M_2$.

Remark. A quick discussion of the notion of having common small models is in order. Clearly:

1. If K is $\mathrm{LS}(K)$-categorical, then K has common small models.
2. If K is λ-categorical, then $K_{\geq \lambda}$ has common small models

Furthermore, consider the case of K being the elementary class of models of a first order theory T, and assume that T is λ-categorical for some $\lambda > \mathrm{LS}(K) = |T|$. Then by Morley's (and Shelah's, for the uncountable case) categoricity theorem, for any $M, N \in K_{>\mathrm{LS}(K)}$ either there is an elementary embedding from M into N or vice versa. In particular, K has common small models.

Alternatively, if N_1, N_2 are both $\mathrm{LS}(K)^+$-saturated, then in particular both models are $\mathrm{LS}(K)^+$-universal, and so there would be $M \leq N_1, N_2$ with $|M| = \mathrm{LS}(K)$. In the case where K is an AEC with JEP, AP, and no maximal models, using EM models it is known that λ-categoricity for a $\lambda > \mathrm{LS}(K)$ implies that K is $\mathrm{LS}(K)$-stable with respect to Galois types over models. Furthermore, letting $H(\lambda) := \beth_{(2^\lambda)^+}$, if K is λ-categorical in some regular $\lambda \geq H(\mathrm{LS}(K))$, then every $M \in K$ with $|M| \geq H(\mathrm{LS}(K))$ is $\mathrm{LS}(K)^+$-saturated (see e.g. [Bal09], Theorem 14.4). Whilst this is not exactly equivalent to K having common small models, this would be sufficient to prove λ-categoricity for all sufficiently large $\lambda \geq H(\mathrm{LS}(K))$.

However, within the current context where K is only assumed to be a very weak AEC, even if we assume JEP, Shelah's presentation theorem may fail, and there is no guarantee that there is a EM functor into K. In the absence of some Hanf number of omitting types for very weak AECs which might not be pseudoelementary, even if K is λ-categorical we cannot guarantee that there is $\mu < \lambda$ such that all $M \in K_\mu$ are $\mathrm{LS}(K)^+$ saturated, and thus require the additional assumption of K having common small models.

Lemma 5.13. *Suppose $\mathcal{A}$ is a notion of free amalgamation which is 3-monotonic and K is categorical in some $\lambda > \theta(K)$. Then for any $N \in K$ with $|N| > \theta(K) + 2^{LS(K)}$ and any $M_b \leq N$ with $|M_b| = LS(K)$, there is M_t such that $M_b \leq M_t \leq N$, $|M_t| = LS(K)$, and $N \cong N^{|N|}_t/N_b$, where $N_b \cong M^{\theta(K)}_t/M_b$ and $N_t \cong M^{\theta(K)+1}_t/M_b$.*

Proof. We will prove the lemma using a variation of the proof of Theorem 5.6. Let $\lambda := |N|$, and so by Lemma 4.1, we can decompose $N = \bigoplus^N_{M_b, i<\lambda} M_i$ where each $|M_i| = \mathrm{LS}(K)$. Letting $\Gamma := \{M_i/\cong_{M_b} : i < \lambda\}$, for each $P \in \Gamma$ let $S_P := \{i \in \lambda : M_i \vDash P\}$, and hence in particular $\lambda = \bigsqcup_{P\in\Gamma} S_P$. Note that $|\Gamma| \leq 2^{\mathrm{LS}(K)} < \lambda$ as $|M_b| = \mathrm{LS}(K)$, and hence there is some $Q \in \Gamma$ such that $|S_Q| > 2^{\mathrm{LS}(K)} + \theta(K)$. Additionally, for each $P \in \Gamma$, fix a $M_P \vDash P$.

Let us further decompose $S_Q = T^* \sqcup \bigsqcup_{P\in\Gamma} T_P$ such that $|T^*| = \theta(K)$, and whenever $P \neq Q$, $|T_P| > \theta(K)$ and is regular. Thus by Theorem 3.13 we have that

$$\begin{aligned} N &= \bigoplus^N_{M_b, i<\lambda} M_i = \bigoplus^N_{M_b, P\in\Gamma} \left(\bigoplus^N_{M_b, i\in S_P} M_i \right) \\ &= \left(\bigoplus^N_{M_b, i\in T^*} M_i \right) \oplus^N_{M_b} \left(\bigoplus^N_{M_b, i\in T_Q} M_i \right) \oplus^N_{M_b} \bigoplus^N_{M_b, P\neq Q} \left(\bigoplus^N_{M_b, i\in S_P \sqcup T_P} M_i \right) \end{aligned}$$

Letting $M^* = \bigoplus_{M_b, i\in T^*}^N M_i$, note that as $T^* \subseteq S_Q$, $M_i \vDash Q$ for each $i \in T^*$, and so $M^* \cong M_Q^{|T^*|}/M_b = M_Q^{\theta(K)}/M_b$. Now, as $\mathcal{A}$ is 3-monotonic, by Lemma 5.9, we have that

$$\begin{aligned} N &= \left(\bigoplus_{M_b, i\in T^*}^N M_i\right) \oplus_{M_b}^N \left(\bigoplus_{M_b, i\in T_Q}^N M_i\right) \oplus_{M_b}^N \bigoplus_{M_b, P\neq Q}^N \left(\bigoplus_{M_b, i\in S_P\sqcup T_P}^N M_i\right) \\ &= \left(\bigoplus_{M^*, i\in T_Q}^N (M_i \oplus_{M_b}^N M^*)\right) \oplus_{M^*}^N \bigoplus_{M^*, P\neq Q}^N \left(\bigoplus_{M^*, i\in S_P\sqcup T_p}^N (M_i \oplus_{M_b}^N M^*)\right) \end{aligned}$$

So for $i \notin T^*$, let $M_i' := M_i \oplus_{M_b}^N M^*$. In particular, for any $P \in \Gamma$ and $i \in T_P \subseteq S_Q$, $M_i' \cong M_Q^{\theta(K)+1}/M_b$. Furthermore, by Theorem 5.11, for any $P \in \Gamma$, $M^* \cong M_Q^{\theta(K)}/M_b \cong M_P^{\theta(K)}/M_b$, and so in fact for any $P \neq Q$ and $i \in S_P$, $M_i' = M_i \oplus_{M_b}^N M^* \cong M_P^{\theta(K)+1}/M_b$. Letting $N_P := M_P^{\theta(K)+1}/M_b$, hence by Theorem 5.11, for any $P \in \Gamma$, $(N_P, M^*) \sim (N_Q, M^*)$. So by Lemma 5.2, since for any $P \neq Q$, as $|T_P| > \theta(K)$, we have

$$\begin{aligned} \bigoplus_{M^*, i\in S_P\sqcup T_p}^N M_i' &= \left(\bigoplus_{M^*, i\in S_P}^N M_i'\right) \oplus_{M^*}^N \left(\bigoplus_{M^*, i\in T_P}^N M_i'\right) \\ &\cong (N_P^{|S_P|}/M^*) \oplus_{M^*} (N_Q^{|T_P|}/M^*) \\ &\cong N_Q^{|S_P|+|T_P|}/M^* \end{aligned}$$

Substituting this back, we get that

$$N \cong N_Q^{|\lambda - T^*|}/M^* = N_Q^{\lambda}/M^*$$

Since $N_Q = M_Q^{\theta(K)+1}/M_b$ and $M^* = M_Q^{\theta(K)}/M_b$, this completes the proof. $\square$

Theorem 5.14. *Suppose K has common small models, and $\mathcal{A}$ is a notion of free amalgamation which is 3-monotonic. If K is λ^*-categorical for some $\lambda^* > \theta(K)$, then K is λ-categorical for any $\lambda > \theta(K) + 2^{LS(K)}$.*

Proof. Given $M, N \in K_\lambda$ with $\lambda > \theta(K) + 2^{\mathrm{LS}(K)}$, since K has common small models, let $M_0 \leq M, N_0 \leq N$ such that $M_0 \cong N_0$. By the above lemma, there are models $M_1, M_b, M_t, N_1, N_b, N_t$ such that:

1. $M_0 \leq M_1 \leq M$ and $N_0 \leq N_1 \leq N$
2. $M_b \cong M_1^{\theta(K)}/M_0$ and $N_b \cong N_1^{\theta(K)}/N_0$
3. $M_t \cong M_1^{\theta(K)+1}/M_0$ and $N_t \cong N_1^{\theta(K)+1}/N_0$
4. $M \cong M_t^{\lambda}/M_b$ and $N \cong N_t^{\lambda}/N_b$

Since K is λ^*-categorical for some $\lambda^* > \theta(K)$, by Theorem 5.11 $(M_t, M_b) \sim (N_t, N_b)$. Hence by Lemma 5.2, $M \cong N$. $\square$

Before ending this section, let us compare our result with other results of categoricity transfer which are relevant to our case:

Fact 5.15 ([GV06a], Theorem 6.3)**.** *Suppose K is $LS(K)$-tame with the amalgamation property, joint embedding property, and arbitrary large models. If K is categorical in $LS(K)$ and $LS(K)^+$, then K is categorical in all $\lambda \geq LS(K)$*

Fact 5.16 ([Vas18b], Corollary 10.9)**.** *Suppose K is $LS(K)$-tame, has arbitrary large models, and has primes. If K is categorical in some $\lambda > LS(K)$, then K is categorical in all $\lambda' > \min(\lambda, \beth_{(2^{LS(K)})^+})$*

Fact 5.17 ([SV18], Theorem 14.2)**.** *Let K be an excellent AEC that is categorical in some $\mu > LS(K)$.*

1. *There is some $\chi < h(LS(K))$ such that K is categorical in all $\mu' \geq \min(\mu, \chi)$.*[1]
2. *If K is also categorical in $LS(K)$, then K is categorical in all $\mu' > LS(K)$.*

We note that classes with a notion of free amalgamation are $\mu_r(K) + \mathrm{LS}(K)$-tame (see Lemma 4.24), and hence Fact 5.15 is relevant here. On the other hand, many of the algebraic examples we have seen above are not $\mathrm{LS}(K)$-categorical, but we manage to prove categoricity transfer using the additional assumption of a notion of free amalgamation.

With regards to Fact 5.16, we recall from [Vas18b] that a class which admits (arbitrary) intersections over sets of the form $M \cup \{a\}$ does have primes, and so in particular the result applies to AECs which admit intersection. Now, if the closure operator additionally satisfies the exchange principle (or if a suitable notion of "independent sets" can be otherwise defined), then it admits a 3-monotonic notion of geometric amalgamation (see also section 7 below). However, this still does not guarantee that the notion of amalgamation has uniqueness, and in this sense the extra assumptions of the exchange principle and uniqueness significantly brings down the cardinality threshold in proving categoricity transfer. On the other hand, the present result is applicable even to classes which do <u>not</u> have primes: for example, the class of free groups with free factor ordering.

Finally, regarding Fact 5.17, there are two main points of comparison:

1. The relationship between K being excellent and K admitting a notion of free amalgamation is far from clear. Unlike the previous case, the greatest difference here is not regarding uniqueness but rather a sense of dimensionality:

   - For $\mathbb{I}$ to be an excellent multidimensional independence relation, it must have n-existence and n-uniqueness for amalgamation diagrams of all finite dimensions.
   - For $\mathcal{A}$ to be a notion of free amalgamation, it must admit decomposition and have bounded locality i.e. $\mu(K) < \infty$.

   Using first order model theory as an analogy, the proof of Theorem 5.6 and 5.14 shows that free amalgamation along with categoricity in a sufficiently large cardinal implies that the class is essentially "unidimensional", which implies that the class trivially has the NDOP (negation of the Dimensional Order Property). In contrast, the analysis of multidimensional amalgamation in excellence is a natural extension of analysing theories which have the NDOP but are not necessarily as simple as being unidimensional. On the other hand, our formulation in terms of free amalgamation has also allowed us to prove the anti-structural theorems in the negative case (see Section 6 below), whereas a full main gap theorem from a multidimensional approach has yet to be reached.

2. The other point of comparison is of course the cardinal bounds present; we believe that this is due much more to the machinery used, and is a reflection of the different level of generality given in the first point.

Before moving on to the case where $\mathcal{A}$ does not have uniqueness, we should note that $\mathcal{A}$ being a notion of free amalgamation implies that K has few models (in the sense of Shelah's main gap theorem), even without the assumption of categoricity:

Proposition 5.18. *Suppose $\mathcal{A}$ is a notion of free amalgamation. Then for any $\lambda \geq LS(K)$, $I(K, \lambda) \leq \lambda^{I_2(K, LS(K))}$*

Proof. By Lemma 3.14 and Lemma 4.1, any model M can be decomposed into the form

$$M = \bigoplus_{M_b, i<\alpha} (M_i^{\lambda_i}/M_b)$$

with

- $|M_b| = \mathrm{LS}(K)$

[1] Recall that $h(\lambda) := \beth_{(2^\lambda)^+}$.

- Each $|M_i| = \mathrm{LS}(K)$, and $\lambda_i \leq \lambda$ is a cardinal
- For $i \neq j$, $(M_i, M_b) \not\approx (M_j, M_b)$

Moreover, uniqueness of $\mathcal{A}$ guarantees that the isomorphism type of M is determined entirely by the choice of M_b, M_i's and λ_i's. This implies the stated bound. $\square$

In particular, if $K = \mathrm{Mod}(T)$ is an elementary class which has a notion of free amalgamation, then T has NDOP.

Chapter 6

Without Uniqueness, and having many Extensions

In the previous section, we proved arguably the strongest structural theorem which we could expect for classes with very "nice" notions of amalgamation. In particular, uniqueness of the notion of amalgamation was necessary to define the model M_t^λ/M_b, which was central to the argument above. On the other hand, having unique amalgams appears a priori to be a very strong assumption, and hence merits an investigation into when uniqueness can be derived.

The driving intuition here is that if a triple (M_0, M_1, M_2) has two $\mathcal{A}$-amalgams which cannot be embedded into each other (w.r.t. to the triple), then by taking λ-many copies of M_1 over M_0, we can construct 2^λ-many models which cannot be embedded into each other. However, before we can formalize this argument, we need an additional property to hold for $\mathcal{A}$:

Definition 6.1. Suppose $\mathcal{A}$ is a notion of amalgamation and is regular. We say that $\mathcal{A}$ has **weak 3-existence** if: given $M_0 \leq M_1, M_2, M_3$, if M_{ij} is a $\mathcal{A}$-amalgam of M_i, M_j over M_0, then there is a model N which is a $\mathcal{A}$-amalgam of M_3, M_{12} over M_0 and such that there are K-embeddings f_1, f_2 making the following diagram commute:

$$\begin{tikzcd}
& M_{13} \arrow[rr, "f_1"] & & N \\
M_3 \arrow[ur] \arrow[rr] & & M_{23} \arrow[ur, "f_2"] & \\
& M_1 \arrow[uu] \arrow[rr] & & M_{12} \arrow[uu] \\
M_0 \arrow[uu] \arrow[ur] \arrow[rr] & & M_2 \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Remark.

- The "weak" in "weak 3-existence" indicates that in the above diagram, the commutative square

$$\begin{tikzcd}
M_{12} \arrow[r, "f_1"] & N \\
M_3 \arrow[u, "\iota"] \arrow[r, "\iota"] & M_{23} \arrow[u, "f_2"']
\end{tikzcd}$$

  is not necessarily an $\mathcal{A}$-amalgam. Note that every other face of the cube is an $\mathcal{A}$-amalgam either by assumption or because $\mathcal{A}$ is regular. Furthermore, if $\mathcal{A}$ is 3-monotonic, then the above commutative square is also necessarily an $\mathcal{A}$-amalgam.

- There is an unfortunate clash in terminology within model theory, where a notion of n-amalgamation (and thus related notions such as n-existence and n-uniqueness) could either be referring to amalgamation of models or amalgamation of types over a system of independent sets. In general, the requirement of amalgamating models is stronger than the amalgamation of types: whereas (in the first order case) having n-amalgamation of types is a "measurement" of how close a simple theory is to being stable (as stable theories have n-amalgamation of types for all n), having n-amalgamation over models of size λ is related the class K having λ^{+n}-AP. In particular, in the context of K admitting finite intersections and having a notion of geometric amalgamation, being able to n-amalgamate $\perp\!\!\!\!\smile$-independent models implies that n-amalgamation of types over $\perp\!\!\!\!\smile$-independent models is also possible.

Lemma 6.2. *If $\mathcal{A}$ is regular and has uniqueness, then $\mathcal{A}$ has weak 3-existence.*

Proof. Given $M_0 \leq M_1, M_2, M_3$ and M_{ij} an $\mathcal{A}$-amalgam of M_i, M_j over M_0 by inclusion, let N be an $\mathcal{A}$-amalgam of M_3, M_{12} over M_0 by inclusion. Note that as $M_0 \leq M_1 \leq M_{12}$, by regularity there is $N_1 \leq N$ such that

$$\begin{tikzcd}
M_3 \arrow[r, "\iota"] & N_1 \arrow[r, "\iota"] & N \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{12} \arrow[u, "\iota"]
\end{tikzcd}$$

But then by uniqueness, there is a K-isomorphism $f_1 : M_{13} \longrightarrow N_1$ such that f_1 is the identity on $M_1 \cup M_3$. Defining f_2 analogously via M_2 and M_{23}, this proves the statement. $\square$

Definition 6.3. Given a triple (M_0, M_1, M_2), we say that it is a **non-uniqueness triple** if there are models N_1, N_2 and K-embeddings f_1, f_2 such that

$$\begin{tikzcd}
M_2 \arrow[r, "f_1"] & N_1 \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"]
\end{tikzcd}
\qquad
\begin{tikzcd}
M_2 \arrow[r, "f_2"] & N_2 \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"]
\end{tikzcd}$$

But there is no K-isomorphism g such that the following diagram commutes:

$$\begin{tikzcd}
 & & N_2 \\
M_2 \arrow[r, "f_1"] \arrow[urr, bend left, "f_2"] & N_1 \arrow[ur, "g"] & \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] & M_1 \arrow[u, "\iota"] \arrow[uur, bend right, "\iota"'] &
\end{tikzcd}$$

We say that the tuple (N_1, f_1, N_2, f_2) **witnesses** that (M_0, M_1, M_2) is a non-uniqueness triple.

We say that (M_0, M_1, M_2) is a **uniqueness triple** if it is not a non-uniqueness triple.

Lemma 6.4. *Suppose $\mathcal{A}$ is absolutely minimal. If (M_0, M_1, M_2) is a non-uniqueness triple as witnessed by (N_1, f_1, N_2, f_2), then for any $N' \geq N_2$, there is no K-embedding $g : N_1 \longrightarrow N'$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & & N' \\
M_2 \arrow[r, "f_1"] \arrow[urr, bend left, "f_2"] & N_1 \arrow[ur, "g"] & \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] & M_1 \arrow[u, "\iota"] \arrow[uur, bend right, "\iota"'] &
\end{tikzcd}$$

Proof. Let $M_0, M_1, M_2, N_1, N_2, f_1, f_2, N'$ be as above, and assume for a contradiction that there does exist a K-embedding g making the above diagram commute. Note then by Invariance of $\mathcal{A}$, $g[N_1] \leq N'$ is also an $\mathcal{A}$-amalgam of M_1 and $(g \circ f_1)[M_2] = f_2[M_2]$ over M_0. Since $\mathcal{A}$ is absolutely minimal, this implies that $g[N_1] = N_2$, contradicting that (N_1, f_1, N_2, f_2) is a witness to (M_0, M_1, M_2) being a non-uniqueness triple. ☐

Lemma 6.5. *Suppose $\mathcal{A}$ is absolutely minimal, regular, and continuous. Let δ be a limit and $(M_i)_{i\leq\delta}, (N_i)_{i<\delta}$ be increasing continuous chains such that*

$$\begin{tikzcd}
N_0 \arrow[r, "\iota"] & N_1 \arrow[r, "\iota"] & N_2 \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & N_i \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & \cdots \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots
\end{tikzcd}$$

If each (M_i, N_i, M_{i+1}) is a uniqueness triple, then (M_0, N_0, M_δ) is also a uniqueness triple.

Proof. Let N^1, f_1, N^2, f_2 be such that

$$\begin{tikzcd}
N_0 \arrow[r, "f_1"] & N^1 \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_\delta \arrow[u, "\iota"]
\end{tikzcd}
\qquad
\begin{tikzcd}
N_0 \arrow[r, "f_2"] & N^2 \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_\delta \arrow[u, "\iota"]
\end{tikzcd}$$

Inductively, define $N_i^1, g_i^1, N_i^2, g_i^2$ for $i < \delta$ such that:

1. $N_0^1 = f_1[N_0] \leq N^1, N_0^2 = f_2[N_0] \leq N^2$
2. $g_0^1 = f_1, g_0^2 = f_2$
3. For $k = 1, 2$, $(N_i^k)_{i<\delta}$ is a continuous resolution of N^k
4. For $k = 1, 2$ and $i < \delta$, the following is a $\mathcal{A}$-amalgam:

$$\begin{tikzcd}
N_0^k \arrow[r, "\iota"] & N_i^k \\
M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_i \arrow[u, "\iota"]
\end{tikzcd}$$

5. Each $g_i^k : N_i \cong N_i^k$ is an isomorphism such that

$$\begin{tikzcd}
N_i \arrow[rr] \arrow[dr, "g_i^k"] & & N_{i+1} \arrow[dr, "g_{i+1}^k"] & \\
& N_i^k \arrow[rr] & & N_{i+1}^k \\
M_i \arrow[uu] \arrow[ur] \arrow[rr] & & M_{i+1} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Note that taking $g^k = \bigcup_{i<\delta} g_i^k$ and $h = g^2 \circ (g^1)^{-1}$ shows that (N^1, f_1, N^2, f_2) is not a witness to non-uniqueness. Proceeding with the induction, note that only the successor step requires verification.

So given N_i^k, g_i^k for $k = 1, 2$, let $N_{i+1}^k := N_0^k \oplus_{M_0}^{N^k} M_{i+1}$. Note then that N_{i+1}^k is a $\mathcal{A}$-amalgam of N_i^k, M_{i+1} over M_i by inclusion. Furthermore, as $g_i^k : N_i \cong_{M_i} N_i^k$ is an isomorphism and (M_i, N_i, M_{i+1}) is a uniqueness triple, hence g_{i+1}^k satisfying (5) exists. ☐

Lemma 6.6. *Suppose $\mathcal{A}$ is regular and absolutely minimal. If (M_0, M_1, M_2) is a non-uniqueness triple and $N \geq M_2$, then (M_0, M_1, N) is also a non-uniqueness triple.*

Proof. We will show the contrapositive and assume that (M_0, M_1, N) is a uniqueness triple. Let M_1^*, f_1, M_2^*, f_2 be two $\mathcal{A}$-amalgams such that

$$\begin{tikzcd} M_1 \arrow[r, "\iota"] & M^1_* \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "f_1"'] \end{tikzcd} \qquad \begin{tikzcd} M_1 \arrow[r, "\iota"] & M^2_* \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "f_2"'] \end{tikzcd}$$

Further, let N^1, g_1, N^2, g_2 be $\mathcal{A}$-amalgams such that

$$\begin{tikzcd} M^1_* \arrow[r, "\iota"] & N^1 \\ M_2 \arrow[u, "f_1"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N \arrow[u, "g_1"'] \end{tikzcd} \qquad \begin{tikzcd} M^2_* \arrow[r, "\iota"] & N^2 \\ M_2 \arrow[u, "f_2"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N \arrow[u, "g_2"'] \end{tikzcd}$$

By regularity, each N^k, g_k is an $\mathcal{A}$-amalgam of (M_0, M_1, N), and hence by the assumption that this is a uniqueness triple there is an isomorphism $h : N^1 \cong N^2$ such that

$$\begin{tikzcd} & & N^2 \\ M_1 \arrow[urr, bend left, "\iota"] \arrow[r, "\iota"] & N^1 \arrow[ur, "h"] & \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] & N \arrow[u, "g_1"] \arrow[uur, bend right, "g_2"'] & \end{tikzcd}$$

In particular, since g_k extends f_k for both $k = 1, 2$,

$$h[f_1[M_2]] = (h \circ g_1)[M_2] = g_2[M_2] = f_2[M_2]$$

Now, as M^1_* is an $\mathcal{A}$-amalgam of $M_1, f_1[M_2]$ over M_0 by inclusion, by invariance $h[M^1_*]$ is an $\mathcal{A}$-amalgam of $M_1, f_2[M_2]$ over M_0 by inclusion. Hence, by absolute minimality, $h[M^1_*] = M^2_*$, and in particular $h \upharpoonright M^1_*$ is an isomorphism such that

$$\begin{tikzcd} & & M^2_* \\ M_1 \arrow[urr, bend left, "\iota"] \arrow[r, "\iota"] & M^1_* \arrow[ur, "h \upharpoonright M^1_*"] & \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] & N \arrow[u, "f_1"] \arrow[uur, bend right, "f_2"'] & \end{tikzcd}$$

In particular, since M^1_*, M^2_* are two arbitrary $\mathcal{A}$-amalgam of M_1, M_2 over M_0, hence (M_0, M_1, M_2) is also a uniqueness triple. □

Corollary 6.7. *Suppose $\mathcal{A}$ is absolutely minimal, regular, and continuous. If there is a non-uniqueness triple (M_0, M_1, M_2), then for any $\lambda \geq |M_1| + |M_2|$ there is a non-uniqueness triple (M_0', M_1', M_2') such that*

$$|M_0| \leq |M_0'| = |M_1'| \leq |M_1| \leq |M_2'| = \lambda$$

Proof. Let N, f be a $\mathcal{A}$-amalgam such that

$$\begin{tikzcd} M_2 \arrow[r, "\iota"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "f"'] \end{tikzcd}$$

Let $(M^{(i)})_{i<|M_1|}$ be a continuous resolution of M_1 with $M^{(0)} = M_0$, and by Lemma 2.20 let $(N^{(i)})_{i<|M_1|}$ be a continuous resolution of N such that each $N^{(i)}$ is a $\mathcal{A}$-amalgam by

$$\begin{tikzcd} M_2 \arrow[r, "\iota"] & N^{(i)} \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^{(i)} \arrow[u, "f\restriction M^{(i)}"'] \end{tikzcd}$$

By Lemma 6.5, there is some $i < |M_1|$ such that $(M^{(i)}, M^{(i+1)}, N^{(i)})$ is a non-uniqueness triple. Letting $M_0' = M^{(i)}, M_1' = M^{(i+1)}$, and taking $M_2' \geq N^{(i)}$ with $|M_2'| = \lambda$, then the above lemma shows that (M_0', M_1', M_2') is a non-uniqueness triple. $\square$

Theorem 6.8. *Suppose $\mathcal{A}$ is regular, continuous, absolutely minimal and has weak 3-existence. If (M_b, M^*, M) is a non-uniqueness triple and $p = gtp(M^*/M_b, M^*)$, then there is $N \geq M$ such that p has $2^{|N|}$-many extensions to N.*

Proof. Since (M_b, M^*, M) is a non-uniqueness triple, fix M^0, M^1 two $\mathcal{A}$-amalgams of M^*, M over M_b by inclusion such that there is no K-isomorphism from M^0 to M^1 fixing $M^* \cup M$ pointwise. Define $\lambda := |M| + \mathrm{LS}(K)$, and let N be such that N is a $\mathcal{A}$-amalgam of $(M_i)_{i<\lambda}$ over M_b by inclusion, with isomorphisms $g_i : M_i \cong_{M_b} M$. In particular, this means that there is a continuous resolution $(N_i)_{i<\lambda}$ such that:

1. $N_0 = M_b$ and $N_1 = M_0$
2. For each $i < \lambda$

$$\begin{tikzcd} N_i \arrow[r, "\iota"] & N_{i+1} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_i \arrow[u, "\iota"'] \end{tikzcd}$$

To prove the theorem, for every $\eta \in {}^{\lambda}2$ we will construct M_η a $\mathcal{A}$-amalgam of M^*, N over M_b, and such that for $\xi \neq \eta$, $\mathrm{gtp}(M^*/N, M_\xi) \neq \mathrm{gtp}(M^*/N, M_\eta)$. So given $\eta \in {}^{\lambda}2$, let us construct an increasing continuous chain $(M_{\eta,i})_{i<\lambda}$ and embeddings $(h_{\eta,i})_{i<\lambda}$ such that:

1. $M_{\eta,0} = M^*$ and $h_{\eta,0} = \iota : M_b \hookrightarrow M^*$
2. $(h_{\eta,i} : N_i \longrightarrow M_{\eta,i})_{i<\lambda}$ is an increasing sequence
3. For each $i < \lambda$

$$\begin{tikzcd} M^* \arrow[r, "\iota"] & M_{\eta,i} \\ M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_i \arrow[u, "h_{\eta,i}"] \end{tikzcd}$$

4. For each $i < \lambda$

$$\begin{tikzcd} M_{\eta,i} \arrow[r, "\iota"] & M_{\eta,i+1} \\ N_i \arrow[u, "h_{\eta,i}"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_{i+1} \arrow[u, "h_{\eta,i+1}"] \end{tikzcd}$$

5. For each $i < \lambda$, there is a K-embedding $f_{\eta,i}$ such that the following diagram commutes:

$$\begin{tikzcd}
& M^{\eta(i)} \arrow[dr, "f_{\eta,i}"] & \\
M^* \arrow[ur] \arrow[rr] & & M_{\eta,i+1} \\
& M \arrow[uu] \arrow[dr, "g_i"] & N_{i+1} \arrow[u, "h_{\eta,i+1}"'] \\
M_b \arrow[uu] \arrow[ur] \arrow[rr] & & M_i \arrow[u]
\end{tikzcd}$$

We proceed to construct by induction:

- For $i = 0$, define $M_{\eta,0} = M^*$ and $h_{\eta,0} = \iota$ as specified.
- For limit $\alpha < \lambda$, let $M_{\eta,\alpha} = \bigcup_{i<\alpha} M_{\eta,i}$, and similarly $h_{\eta,\alpha} = \bigcup_{i<\alpha} h_{\eta,\alpha}$. Note that by (4) and continuity, this implies that

$$\begin{tikzcd}
M^* \arrow[r, "\iota"] & M_{\eta,\alpha} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_i \arrow[u, "h_{\eta,\alpha}"']
\end{tikzcd}$$

- Given $M_{\eta,i}$ and $h_{\eta,i}$ defined, note that we have $\mathcal{A}$-amalgams:

$$\begin{tikzcd}
N_i \arrow[r, "\iota"] & N_{i+1} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "g_i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M^* \arrow[r, "\iota"] & M^{\eta(i)} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M \arrow[u, "\iota"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M^* \arrow[r, "\iota"] & M_{\eta,i} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_i \arrow[u, "h_{\eta,i}"']
\end{tikzcd}$$

  Hence, as $\mathcal{A}$ has weak 3-existence, there exists a model $M_{\eta,i+1}$ and maps $f_{\eta,i}, h_{\eta,i+1}$ such that $M_{\eta,i+1}$ is an $\mathcal{A}$-amalgam of M^*, N_{i+1} over M_b and the following diagram commutes:

$$\begin{tikzcd}
& M_{\eta,i} \arrow[rr] & & M_{\eta,i+1} \\
M^* \arrow[ur] \arrow[rr] & & M^{\eta(i)} \arrow[ur, "f_{\eta,i}"] & \\
& N_i \arrow[uu, "h_{\eta,i}"] \arrow[rr] & & N_{i+1} \arrow[uu, "h_{\eta,i+1}"'] \\
M_b \arrow[uu] \arrow[ur] \arrow[rr] & & M \arrow[uu] \arrow[ur, "g_i"] &
\end{tikzcd}$$

  In particular, by regularity the following commutative squares are also $\mathcal{A}$-amalgams:

$$\begin{tikzcd}
M^* \arrow[r, "\iota"] & M_{\eta,i} \arrow[r, "\iota"] & M_{\eta,i+1} \\
M_b \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_i \arrow[u, "h_{\eta,i}"'] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & N_{i+1} \arrow[u, "h_{\eta,i+1}"']
\end{tikzcd}$$

Letting $M_\eta := \bigcup_{i<\lambda} M_{\eta,i}$ and $h_\eta = \bigcup_{i<\lambda} h_{\eta,i}$, note then that h_η is a K-embedding from N to M_η which fixes M_b pointwise.

To complete the proof, it remains to show that for any $\xi \neq \eta$, there is no $M' \geq M_\eta$ and a K-embedding F such that the following diagram commutes:

$$\begin{tikzcd}
 & M_\xi \arrow[rr, "F"] & & M' \\
M^* \arrow[ur] \arrow[rr] & & M_\eta \arrow[ur] & \\
 & h_\xi[N] \arrow[uu] \arrow[dr, "h_\eta \circ h_\xi^{-1}"] & & \\
M_b \arrow[uu] \arrow[ur] \arrow[rr] & & h_\eta[N] \arrow[uu] &
\end{tikzcd}$$

So suppose for a contradiction that such a M', F exists. Since $\xi \neq \eta$, fix $i_0 < \lambda$ such that $\xi(i_0) \neq \eta(i_0)$. Assuming WLOG that $\eta(i_0) = 0$, by construction of M_η we have that

$$\begin{tikzcd}
 & M^0 \arrow[dr, "f_{\eta,i_0}"] & \\
M^* \arrow[ur] \arrow[rr] & & M^* \oplus_{M_b}^{M_\eta} h_\eta[M_i] \\
 & M \arrow[uu] \arrow[dr, "g_i"] & \\
M_b \arrow[uu] \arrow[ur] \arrow[rr] & & M_i \arrow[uu, "h_\eta"']
\end{tikzcd}$$

Similarly, since $\xi(i_0) = 1$, we have that

$$\begin{tikzcd}
 & M^1 \arrow[dr, "f_{\xi,i_0}"] & \\
M^* \arrow[ur] \arrow[rr] & & M^* \oplus_{M_b}^{M_\xi} h_\xi[M_i] \\
 & M \arrow[uu] \arrow[dr, "g_i"] & \\
M_b \arrow[uu] \arrow[ur] \arrow[rr] & & M_i \arrow[uu, "h_\xi"']
\end{tikzcd}$$

But note that as $\mathcal{A}$ is absolutely minimal and F is a K-embedding,

$$F[M^* \oplus_{M_b}^{M_\xi} h_\xi[M_{i_0}]] = M^* \oplus_{M_b}^{M'} h_\eta[M_{i_0}] = M^* \oplus_{M_b}^{M\eta} h_\eta[M_{i_0}]$$

This contradicts that (M^0, M^1) is a witness to (M_b, M^*, M) being a non-uniqueness triple. $\square$

Corollary 6.9. *Suppose $\mathcal{A}$ is regular, continuous, absolutely minimal, and has weak 3-existence. If $\mathcal{A}$ does not have uniqueness, then there is some $M \in K$ and a $p \in S^{LS(K)+|M|}(M)$ such that for every $\lambda \geq LS(K) + |M|$, there is $N \in K_\lambda$ such that $N \geq M$ and p has 2^λ (nonforking) extensions to N.*

In particular, if we assume that K is sufficiently type-short and $\mathcal{A}$ is a notion of geometric amalgamation with weak 3-existence, then $\mathcal{A}$ has uniqueness iff K is λ-stable on a tail of cardinals.

Chapter 7

Classes with Pregeometries and Regular Types

One last example which we would like to consider is the following: let T be the first order theory in a 2-sorted language, such that models of T are of the form (V, F), where F is a field and V is a vector space over F. Whilst T is clearly not categorical in any cardinal, the uncountable categoricity of vector spaces implies that categoricity transfer holds in the subclass where F is fixed. More generally, if we consider the vectors in a model of T to (essentially) realize a regular type, and define the class K where each model consists of the realization of the fixed regular type within a model in T, then K also has satisfies some categoricity transfer. In this sense, we wish to prove an analogous result for an AEC with some given notion of independence. This can be seen (essentially) as a case of Zilber's categoricity result for quasiminimal excellent classes from [Zil05] (see also [Kir10] and chapter 2 of [Bal09]), and more specifically as a quasiminimal AEC introduced by Vasey in [Vas18a]. However, we will be using the categoricity results of section 5 to provide an alternative proof.

Recall that if T is a stable first order theory, then the realizations of a regular type within a model form a pregeometry (where independence is forking independence). It is hence helpful for us to first investigate how an AEC where each model is a pregeometry admits a notion of amalgamation:

Definition 7.1. Let K be an AEC. A **system of pregeometries** for K consists of functions $(\mathrm{cl}_M)_{M \in K}$ such that:

1. For each $M \in K$, (M, cl_M) is a pregeometry i.e. $\mathrm{cl}_M : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ satisfies:
   (a) For each $X \subseteq M$, $X \subseteq \mathrm{cl}_M(X) = \mathrm{cl}_M(\mathrm{cl}_M(X))$
   (b) If $X \subseteq Y$, then $\mathrm{cl}_M(X) \subseteq \mathrm{cl}_M(Y)$
   (c) If $a \in \mathrm{cl}_M(X)$, then there exists $X_0 \subseteq X$ such that $|X| < \aleph_0$ and $a \in \mathrm{cl}_M(X_0)$
   (d) If $b \in \mathrm{cl}_M(A \cup \{a\}) - \mathrm{cl}_M(A)$, then $a \in \mathrm{cl}_M(A \cup \{b\})$
2. If $M \leq N$, then $\mathrm{cl}_M \subseteq \mathrm{cl}_N$. In particular, $M = \mathrm{cl}_M(M) = \mathrm{cl}_N(M)$
3. If $B \subseteq N$, $B = \mathrm{cl}_N(B)$, and there exists some $M_0 \leq N$ such that $M_0 \subseteq \mathrm{cl}_N(B)$, then there is some $M \leq N$ such that B is the universe of M.

Given $M \in K$ and $B \subseteq M$, we say that B is **closed** if B is a closed set relative to cl_M. We will similarly use terminology for pregeometries (independent sets, etc.) without explicit references to the ambient model.

Remark. The assumption that each closure operator is finitary is necessary for K to be an AEC: if cl_N is not finitary, the union of a chain of closed sets might not be closed, and thus K violates the Tarski-Vaught chain axioms. More generally, if each cl_N has $< \lambda$-character, then K is a λ-AEC.

Definition 7.2. Given $(\mathrm{cl}_M)_{M\in K}$ a system of pregeometries for K and AEC, we define $\mathcal{A}$ to be a notion of amalgamation on K by asserting that

$$\begin{tikzcd} M_1 \arrow[r, "\iota"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"'] \end{tikzcd}$$

if and only if there is $B_1, B_2 \subseteq N$ such that

1. $B_1 \cup B_2$ is an independent set and $\mathrm{cl}_N(B_1 \cup B_2) = N$
2. $\mathrm{cl}_N(B_1) = M_1$ and $\mathrm{cl}_N(B_2) = M_2$
3. $\mathrm{cl}_N(B_1 \cap B_2) = M_0$

Lemma 7.3. *$\mathcal{A}$ as defined above is 3-monotonic, absolutely minimal, continuous, and admits decomposition.*

Proof.

1. 3-monotonicity follows straightforwardly from the definition of $\mathcal{A}$
2. For absolute minimality, suppose N is a $\mathcal{A}$-amalgam of M_1, M_2 over M_0 by inclusion. Hence there are $B_1, B_2 \subseteq N$ such that $\mathrm{cl}_N(B_1 \cup B_2) = N$ and $\mathrm{cl}_N(B_i) = M_i$, and therefore $\mathrm{cl}_N(M_1 \cup M_2) = N$. Now, if $N' \geq N$ and $M' \leq N'$ is such that $M_1 \cup M_2 \subseteq M'$, then

   $$N = \mathrm{cl}_N(M_1 \cup M_2) = \mathrm{cl}_{N'}(M_1 \cup M_2) = \mathrm{cl}_{M'}(M_1 \cup M_2) \subseteq M'$$

   This shows that $\mathcal{A}$ is absolutely minimal.
3. For continuity, suppose δ is a limit ordinal and there are models $(M_i, N_i)_{i<\delta}$ such that

   $$\begin{tikzcd} N_0 \arrow[r, "\iota"] & N_1 \arrow[r, "\iota"] & N_2 \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & N_i \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & \cdots \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & M_i \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1} \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots \end{tikzcd}$$

   Inductively, we will define sets $B, (A_i)_{i<\delta}$ such that:

   (a) $B \subseteq N_0$ and $A_i \subseteq M_i$
   (b) $(A_i)_{i<\delta}$ is an increasing continuous sequence of sets
   (c) For each $i < \delta$, $B \cup A_i$ is independent, and $B \cap A_i = A_0$
   (d) $\mathrm{cl}_{N_0}(B) = N_0$
   (e) For each $i < \delta$, $\mathrm{cl}_{M_i}(A_i) = M_i$
   (f) For each $i < \delta$, $\mathrm{cl}_{N_i}(B \cup A_i) = N_i$

   Note that this is sufficient: letting $A_\delta = \bigcup_{i<\delta}$, A_δ is a basis for $\bigcup_{i<\delta} M_i$, $B \cup A_\delta$ is independent, and $B \cap A_\delta = A_0$. Moreover, since each $N_i = \mathrm{cl}_{N_i}(B \cup A_i)$, hence $B \cup A_\delta$ is a basis for $\bigcup_{i<\delta} N_i$. Thus the basis $B \cup A_\delta$ witnesses that

   $$\begin{tikzcd} N_0 \arrow[r, "\iota"] & \bigcup_{i<\delta} N_i \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & \bigcup_{i<\delta} M_i \arrow[u, "\iota"'] \end{tikzcd}$$

   So let us construct the sets $B, (A_i)_{i<\delta}$:

   - Since N_1 is an $\mathcal{A}$-amalgam of M_1, N_0 over M_0 by inclusion, fix B, A_1 a basis of N_0, M_1 respectively that witnesses the $\mathcal{A}$-amalgam, and let $A_0 = B \cap A_1$.
   - For limit α, let $A_\alpha = \bigcup_{i<\alpha} A_i$ as required.

- Given A_i, by induction A_i is a basis for M_i, $B \cup A_i$ is a basis for N_i, and N_{i+1} is an $\mathcal{A}$-amalgam of M_{i+1}, N_i over M_i. By the exchange property, thus there is A_{i+1} a basis of M_{i+1} which extends A_i and such that $B \cup A_{i+1}$ is independent. Moreover, thus $B \cap A_{i+1} \subseteq M_i$, and hence by induction $B \cap A_{i+1} = B \cap A_i = A_0$.

This completes the proof for continuity.

4. For decomposability, suppose $M_0 \leq M_1 \leq N$. Fix A_0 a basis of M_0. and extend to A_1 a basis of M_1. Extending further to B a basis for N, let $M_2 = \mathrm{cl}_N(A_0 \cup (B - A_1))$. Then N is an $\mathcal{A}$-amalgam of M_1, M_2 over M_0 by inclusion, as required.

□

Lemma 7.4. *$\mathcal{A}$ as defined above is regular.*

Proof. Recalling the definition of regularity (Definition 2.6), we shall prove the implications $2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 2$

- ($2 \Rightarrow 1$) Suppose that

$$\begin{tikzcd} N_0 \arrow[r, "\iota"] & N_1 \arrow[r, "\iota"] & N_2 \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"] \end{tikzcd}$$

Fix independent sets $A_0^1, A_1^1, A_1^2, A_2^2, B_0^1, B_1^1, B_1^2$ such that:

1. A_0^1 is a basis for M_0, B_0^1 is a basis for N_0
2. A_1^1, A_1^2 are bases for M_1, B_1^1, B_1^2 are bases for N_1
3. A_2^2 is a basis for M_2
4. $A_0^1 = B_0^1 \cap A_1^1$, $B_1^1 = A_1^1 \cup B_0^1$, and $A_1^2 = B_1^2 \cap A_2^2$
5. $A_2^2 \cup B_1^2$ is a basis for N_2

By applying the exchange property, we can find A_2^1 which extends A_1^1 and is a basis for M_2. Since $\mathrm{cl}_{N_1}(B_1^1) = \mathrm{cl}_{N_1}(B_1^2)$ and $A_2^2 \cup B_1^2$ is independent with $A_2^2 \cap B_1^2 \subseteq M_1$, hence $B_0^1 \cup A_2^1$ is also independent. Hence N_2 is an $\mathcal{A}$-amalgam of M_2, N_0 over M_0 by inclusion.

- ($1 \Rightarrow 3$) Suppose that

$$\begin{tikzcd} N_0 \arrow[r, "\iota"] & N_2 \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[u, "\iota"] \end{tikzcd}$$

Further, let M_1 be such that $M_0 \leq M_1 \leq M_2$. Now, as N_2 is an $\mathcal{A}$-amalgam of N_0, M_2 over M_0, there is a basis B of N_2 such that $B \cap N_0, B \cap M_2, B \cap M_0$ are all bases of the respective models. So extend $B \cap M_0$ to B_1, a basis of M_1, and note that $B_1 \cup (B \cap N_0)$ is still independent as $M_1 \leq M_2$. So taking $N_1 = \mathrm{cl}_{N_2}(B_1 \cup (B \cap N_0))$, we get

$$\begin{tikzcd} N_0 \arrow[r, "\iota"] & N_1 \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] \end{tikzcd}$$

Furthermore, we can extend B_1 to a basis B_2 of M_2, and still maintain that $B_2 \cup (B_1 \cup (B \cap N_0)) = B_2 \cup (B \cap N_0)$ is independent. Hence N_2 is also an $\mathcal{A}$-amalgam of M_2, N_1 over M_1. Note that this is sufficient to show $1 \Rightarrow 3$, since $\mathcal{A}$ being absolutely minimal implies that N_1 is the unique $\mathcal{A}$-amalgam of M_1, N_0 over M_0 inside N_2.

- ($3 \Rightarrow 2$) This is trivial.

□

Lemma 7.5. *For $\mathcal{A}$ as defined above, $\mu(K) = \aleph_0$*

Proof. This is straightforward from the fact that each closure operator has finite character. $\square$

Since we are interested in types which have U-rank 1, we require the class K to admit some suitable notion of nonforking. For this, we use the notions of stable and simple independence given in [GM21], which extends earlier work in [Bon+16] and [LRV19]. The reader is encouraged to consult [GM21] for the relevant definition.

Fact 7.6 ([GM21], Proposition 5.9)**.** *Suppose N is a monster model in K, and $\perp\!\!\!\perp$ is a simple independence relation on N. If $A \underset{M}{\perp\!\!\!\perp} B$, then there is a model $M' \geq M$ such that $B \subseteq M'$ and $A \underset{M}{\perp\!\!\!\perp} M'$.*

Lemma 7.7. *Suppose N is a monster model in K, $\perp\!\!\!\perp$ is a supersimple (in particular, simple) independence relation on N with the $(< \aleph_0)$-witness property for singletons, and $p \in S^1(M_0)$ is a Galois type with $U(p) = 1$. Define the operator cl_p on $p(N)$ by:*

$$cl_p(A) := \{x \in p(N) : x \underset{M_0}{\not\perp\!\!\!\perp} A\}$$

Then cl_p is a closure operator on $p(N)$, and $(p(N), cl_p)$ is a pregeometry.

Proof. We first need a claim:

Claim. If $M \geq M_0$ and $x \underset{M}{\not\perp\!\!\!\perp} A$, then for every model $M^* \geq M$ with $A \subseteq M^*$, $x \in M^*$ also.

Proof. Otherwise, if $M^* \geq M \geq M_0$ is such that $A \subseteq M^*$ but $x \notin M^*$, note then as $x \in p(N)$, $\mathrm{gtp}(x/M^*, N)$ must be the unique nonalgebraic extension of p to M^*, and hence is the nonforking extension of p to M^*. Thus by transitivity $x \underset{M}{\perp\!\!\!\perp} M^*$, contradicting $x \underset{M}{\not\perp\!\!\!\perp} A$. $\square$

We can now show the properties required of cl_p:

- cl_p is monotonic: for every $a \in A$, $a \underset{M_0}{\not\perp\!\!\!\perp} A$
- cl_p is idempotent: let X, y be such that $y \underset{M_0}{\not\perp\!\!\!\perp} A \cup X$ and for every $x \in X, x \underset{M_0}{\not\perp\!\!\!\perp} A$. Suppose for a contradiction that $y \underset{M_0}{\perp\!\!\!\perp} A$, so by Fact 7.6 there is some $M \geq M_0$ such that $A \subseteq M$ and $y \underset{M_0}{\perp\!\!\!\perp} M$. Note that since each $x \underset{M_0}{\not\perp\!\!\!\perp} A$, by the above claim $X \subseteq M$, and hence in particular $y \underset{M_0}{\perp\!\!\!\perp} A \cup X$, a contradiction.
- cl_p has finite character: If $x \underset{M_0}{\not\perp\!\!\!\perp} A$, then by the $(< \aleph_0)$-witness property there must be a finite $A_0 \subseteq A$ such that $x \underset{M_0}{\not\perp\!\!\!\perp} A_0$.
- cl_p satisfies the exchange property: Suppose that $x \in \mathrm{cl}_p(A \cup \{b\}) - \mathrm{cl}_p(A)$. Hence $x \underset{M_0}{\perp\!\!\!\perp} A$, and $b \underset{M_0}{\perp\!\!\!\perp} A$. Now, let M be a model such that $M_0 \leq M$ and $A \subseteq M$: note that since $x \in \mathrm{cl}_p(A \cup \{b\})$, by the above claim, for every model $M' \geq M_0$, if $b \in M'$ then $x \in M'$. In particular, this holds for any $M' \geq M$. But then by the previous facts this implies that $x \underset{M}{\not\perp\!\!\!\perp} b$, and hence by symmetry $b \underset{M}{\not\perp\!\!\!\perp} x$ for any such arbitrary M. So assume for a contradiction that $b \notin \mathrm{cl}_p(A \cup \{x\})$, and hence there must be some model $M \geq M_0$ such that $A \cup \{x\} \subseteq M$ but $b \notin M$. This contradicts that $b \underset{M}{\not\perp\!\!\!\perp} x$.

$\square$

Remark. The assumption that $\mathop{\smile\!\!\!\!\!\!\mid}$ has the $(<\aleph_0)$-witness property may appear at first to be very strong, but it was shown in [GM21] (Theorem 7.12 and Corollary 7.16) that having bounded U-rank is equivalent to $\mathop{\smile\!\!\!\!\!\!\mid}$ being supersimple, which for classes with (arbitrary) intersection implies that $\mathop{\smile\!\!\!\!\!\!\mid}$ does have the $(<\aleph_0)$-witness property. Since the assumption of $U(p)=1$ is necessary for the construction in consideration here, assuming that $\mathop{\smile\!\!\!\!\!\!\mid}$ does have the $(<\aleph_0)$-witness property does not significantly increase the strength of our assumptions in totality.

Definition 7.8. Suppose K is an AEC in a relational language τ and $\mathop{\smile\!\!\!\!\!\!\mid}$ is a supersimple independence relation with the $(<\aleph_0)$-witness property for singletons on a monster model N of K. Let $p\in S^1(M_0)$ be a Galois type such that $U(p)=1$. We define the abstract class $(K_p,\leq_p)$, where:

1. $\tau(K_p)=\tau_{M_0}=\tau\sqcup\{c_a\}_{a\in M_0}$, where each c_a is a new constant symbol.
2. A τ_{M_0} structure $\mathcal{M}$ is a model in K_p iff there is a τ_{M_0}-embedding f from $\mathcal{M}$ into a set $A\cup M_0\subseteq N$, such that:
   - $A\subseteq p(N)$ and A is closed with respect to $\mathop{\smile\!\!\!\!\!\!\mid}_{M_0}$ i.e. if $b\in p(N)$ and $b \mathop{\not\!\smile\!\!\!\!\!\!\mid}_{M_0} A$, then $b\in A$.
   - $f(c_a^{\mathcal{M}})=a$
3. $\mathcal{M}_1\leq_p\mathcal{M}_2$ iff there is a τ_{M_0}-isomorphism $f:\mathcal{M}\longrightarrow p(N)\cup M_0$ such that both f and $f\restriction\mathcal{M}_1$ satisfies the above conditions.

Remark. Of course, K_p as defined above is not strictly an AEC since all of its models are of bounded cardinality. However, by the lemma below, given some monster model $N'>N$ with a corresponding notion of independence, we can use N' to extend K_p, and so in particular K_p as already defined contains all "small" models.

Lemma 7.9. *K_p is an AEC with a system of pregeometries inherited from N, $LS(K_p)=|M_0|+LS(K)$, and M_0 as a τ_{M_0} structure is prime and minimal in K.*

Proof. Having fixed N a monster model of K and $\mathop{\smile\!\!\!\!\!\!\mid}$ a stable independence relation on N, let us first describe the system of pregeometries: for any $M\in K_p$, $M=(A,M_0)$ where there is a τ-embedding f such that $f[A]\subseteq p(N)$, $f\restriction M_0=\mathrm{id}_{M_0}$, and $f[A]$ is closed w.r.t. $\mathop{\smile\!\!\!\!\!\!\mid}$. We define cl_M by:

1. $\mathrm{cl}_M(\varnothing)=\mathrm{cl}_M(M_0)=M_0$
2. For any B, $\mathrm{cl}_M(B)=\mathrm{cl}_M(B\cup M_0)$
3. For $B\subseteq A$, $\mathrm{cl}_M(B)=M_0\cup\{x\in A: f(x)\mathop{\not\!\smile\!\!\!\!\!\!\mid}_{M_0} f[B]\}$

Note that as f is a τ-isomorphism from A to $f[A]$, cl_M as defined above is independent of the choice of f as $\mathop{\smile\!\!\!\!\!\!\mid}$ is invariant under τ-automorphisms of N. The other conditions for the closure operators to be a system of pregeometries for K_p follows straightforwardly. Moreover, since any τ_{M_0}-embedding must be the identity on M_0, M_0 is indeed prime and minimal in K_p. $\square$

Definition 7.10. Given (X,cl) a pregeometry and closed sets $A_0,A_1,A_2\subseteq X$, we say that A_1,A_2 **are independent over** A_0 if there are independent sets B_1,B_2 such that:

- $\mathrm{cl}(B_1)=A_1,\mathrm{cl}(B_2)=A_2$
- $\mathrm{cl}(B_1\cap B_2)=A_0$
- $B_1\cup B_2$ is an independent set.

We say that the pair (B_1,B_2) is a **witness** to A_1,A_2 being independent over A_0. Note that if A_1,A_2 are independent over A_0, then $A_1\cap A_2=A_0$.

Theorem 7.11. *Given K_p as defined above, if $\mathcal{A}$ is defined using the system of pregeometries inherited from N, then it has uniqueness.*

Proof. Since the system of pregeometries of K_p are inherited from the pregeometry (N, cl_p) and $\mathcal{A}$ is defined by independence w.r.t the system of pregeometries, it suffices to prove that:

Claim. Suppose A_1, A_2 are closed subsets of $p(N)$ and independent over A_0. If f, g are τ_{M_0}-automorphisms of N such that $f \restriction A_0 = g \restriction A_0$ and $f[A_1], g[A_2]$ are independent over $f[A_0]$, then there is h a τ_{M_0}-automorphism of N which is an isomorphism between $\mathrm{cl}(A_1 \cup A_2)$ and $\mathrm{cl}(f[A_1] \cup g[A_2])$.

So to prove the claim, fix (B_1, B_2) which witnesses that A_1, A_2 are independent over A_0, and let $B_0 := B_1 \cap B_2$. Letting $\lambda = |B_2 - B_0|$, fix also an enumeration $B_2 - B_0 = \{b_i : i < \lambda\}$, and we will construct a sequence $(h_i : i < \lambda)$ such that:

1. Each h_i is a restriction of a τ_{M_0}-automorphism of N, and the sequence is an increasing continuous chain
2. $h_0 = f \restriction A_1$
3. For each $i < \lambda$, $\mathrm{dom}\ h_i = \mathrm{cl}_p(A_1 \cup \{b_j : j < \lambda\}) =: A_1^i$
4. For each $i < \lambda$, $h_i \restriction \mathrm{cl}_p(b_j : j < i) = g \restriction \mathrm{cl}_p(b_j : j < i)$

This is sufficient, as letting $h = \bigcup_{i<\lambda} h_i$ gives the desired automorphism. So let us proceed inductively:

- For $i = 0$, take $h_0 = f \restriction A_1$ as required.
- At limit stages, we take the union as required.
- If h_i is constructed with $h_i = h^* \restriction A_1^i$ and $A_1^i = \mathrm{cl}_p(A_1 \cup \{b_j : j < i\})$ for some h^* a τ_{M_0}-automorphism of N, note that as B_2 is independent by assumption, b_i is independent from A_1^i, and so is $h^*(b_i)$ from $h_i[A_1^i]$. Hence there is some model M_1 such that $h_i[A_1^i] \subseteq M_1$ but $h^*(b_i) \notin M_1$. Similarly, $g(b_i)$ is independent from $f[A_1] \cup g[\mathrm{cl}_p(b_j : j < i)] = h_i[A_1^i]$, and we can find a model M_2 similarly with $g(b_i) \notin M_2$. Now, let $y \in p(N)$ be such that $y \notin M_1, M_2$: in particular, y is independent from M_1 over M_0, and as $U(p) = 1$ thus $\mathrm{gtp}(y/M_1, N) = \mathrm{gtp}(h^*(b_i)/M_1, N)$. Similarly, $\mathrm{gtp}(y/M_2, N) = \mathrm{gtp}(y/M_2, N)$. Note that since $A_1^i \subseteq M_1 \cap M_2$ by construction, this implies that there is some automorphism h' of N such that:
  - $h' \restriction A_1^i = h_i$: and
  - $(h' \circ h^*)(b_i) = g(b_i)$

  So we can take $h_{i+1} = h' \restriction \mathrm{cl}_p(A_1^i \cup \{b_i\})$ (possibly by composing with a suitable automorphism of N to ensure $h_{i+1} \restriction \mathrm{cl}_p(b_j : j < i + 1) = g \restriction \mathrm{cl}_p(b_j : j < i + 1)$

This completes the construction, and hence the proof. □

Lemma 7.12. *For any $\mathcal{M}_1, \mathcal{M}_2 \in K_p$ with $|\mathcal{M}_1| = |\mathcal{M}_2| = |M_0|$, $(\mathcal{M}_1, M_0) \sim (\mathcal{M}_2, M_0)$*

Proof. Note that if $|\mathcal{M}| = |M_0|$, then $\mathcal{M}^{\theta(K_p)}/M_0 = \mathcal{M}^{|M_0|}/M_0 = (A, M_0)$ where A has dimension $|M_0|$ as a pregeometry. Since $U(p) = 1$, if b_1, b_2 are both independent from A, then there is some τ_{M_0}-automorphism of N which fixes A pointwise but sends b_1 to b_2. This provides the desired τ_{M_0}-isomorphism between $\mathcal{M}_1^{\theta(K_p)}/M_0$ and $\mathcal{M}_2^{\theta(K_p)}/M_0$. □

Theorem 7.13. *Suppose K has a monster model and a supersimple independence relation with the $(< \aleph_0)$-witness property for singletons. If $U(p) = 1$, then K_p is λ-categorical in all $\lambda > |dom\ p| + LS(K)$*

Proof. We have shown that $\mathcal{A}$ is a notion of free amalgamation for K_p, and that M_0 is a prime and minimal model for K_p. Furthermore, the above lemma establishes that for there is a unique $\sim$ class for models of cardinality $|M_0| = \mathrm{LS}(K_p)$, so the proof of Theorem 5.6 also applies here. Furthermore, as stated in Theorem 5.6, we can improve the cardinality transfer bound to $\mathrm{LS}(K_p) + I(K_p, \mathrm{LS}(K_p))$; but the above lemma establishes that $I(K_p, \mathrm{LS}(K_p)) = \mathrm{LS}(K_p)$, which gives the desired bound. □

Corollary 7.14. *For any $M_1, M_2 \in K$, if $M_0 \leq M_1, M_2$ and $|p(M_1)| = |p(M_2)| > |M_0|$, then $p(M_1) \cong_{M_0} p(M_2)$ as τ-structures.*

Chapter 8

Open questions

There are some questions that arise immediately from our treatment of notions of amalgamation but which we have yet to answer. For example:

Question 8.1. How do we define an independence relation $\mathop{\smile\!\!\!\!\!\!\;|}$ from $\mathcal{A}$ (as was done in section 4) but without the assumption of finite intersections?

Note that the categoricity result of section 5 does not assume that the class admits finite intersections, and the canonicity of forking established in [Bon+16] implies that there should be a canonical notion of forking which is equivalent to $\mathop{\smile\!\!\!\!\!\!\;|}$ as defined in section 4 if the class does have FI. This suggest that there should be a "correct" definition of $\mathop{\smile\!\!\!\!\!\!\;|}$ using only properties of $\mathcal{A}$.

Question 8.2. Suppose that $\mathcal{A}$ is absolutely minimal, continuous, and regular. Is $\mathcal{A}$ having uniqueness equivalent to K being Galois stable in some λ?

Both directions require some further work beyond what we have presented here. In the forward direction, note that despite having defined a well-behaving notion of forking in section 4, we assumed that $\mathcal{A}$ admits decomposition and in particular this is necessary to establish $\mu(K)$ as the cardinal for local character. A more satisfying method would be to show some form of local character without assuming that $\mathcal{A}$ admits decomposition, and in particular without reference to $\mu(K)$.

In the reverse direction, note that Theorem 6.8 assumes weak 3-existence. On the other hand, there are simple first order theories for which 3-amalgamation of types is not possible; it therefore seems plausible that (assuming tameness and shortness) stability implies that there are no non-uniqueness triples.

Question 8.3. How can we weaken the property of admitting decomposition? Is admitting decomposition equivalent to being superstable?

A relevant result here is Theorem 4.26 by Mazari-Armida from [Maz21], which states (in particular) that for a class K of R-modules closed under direct sums, K (with the pure-submodule ordering) is superstable iff every module in K is pure-injective. Defining $\mathcal{A}$ to be amalgamation by direct sums and taking the relevant quotients, we note that every module of K being pure-injective implies that $\mathcal{A}$ admits decomposition. On the other hand, the result of [Maz21] depends heavily on corresponding Galois types with syntactic types, and hence it seems likely that any development in this direction would require at least tameness and shortness.

Question 8.4. If K is eventually categorical, must K admit a notion of free amalgamation? If K has uniqueness of limit models, does the subclass of limit models (with the "limit over" ordering) admit a notion of free amalgamation?

This is of course true when K is the elementary class of a strongly minimal theory as this is an AEC with a system of pregeometries. This then extends to all elementary classes, based on the Baldwin-Lachlan argument for Morley's categoricity theorem, by taking considering the free amalgamation of algebraically closed sets of a strongly minimal formula and appealing to the fact that all models are prime over its strongly minimal set; that this notion of amalgamation is absolutely minimal requires the fact that there are no Vaughtian pairs in such a theory. Hyttinen and Kangas also showed in [HK18] that universal classes which are eventually categorical are essentially either vector spaces or disintegrated, and in either case would admit a notion of free amalgamation. On the other hand, both arguments require the essential step of finding some "minimal" type on which a pregeometry can be defined, and showing that every model is prime and minimal (in K) over their realizations of the minimal type; in particular, for a general AEC K it is not yet clear to us whether or not there exists such a correspondence between models $M \in K$ and the (pre)geometric portions of such models. A simpler question would be whether the class of limit models admit free amalgamation under the assumption of uniqueness of limit models; this can be achieved if a universal resolution of N over M can be "copied" to any M' a limit model over M; essentially reducing back to Shelah's construction of $(\lambda, 2)$-good sets in [She83a].

Appendix A

The class of free groups as a weak AEC

For this appendix, let K be the class of free groups with the ordering $G \leq_f H$ iff G is a free factor of H. We will show in detail that $(K, \leq_f)$ is a weak AEC which admits finite intersection and has a notion of free amalgamation; this follows entirely from Perin's work in [Per11], which builds off a series of works by Sela, in particular [Sel06a] and [Sel06b].

Notation A.1. For any set X, we let $F(X)$ denote the free group with X as the set of generators. For any ordinal α, we let F_α denote the free group with α (as a set of ordinals) as the set of generators, so that if $\beta < \alpha$, then F_β is a subgroup of F_α.

We use $\preccurlyeq$ to indicate the relation of being an elementary submodel.

Fact A.2 ([Per11], Theorem 1.3)**.** *Let H be a proper subgroup of F_n, the free group on n-generators. Then H is an elementary submodel of F_n iff H is a free factor of F_n.*

In particular, if $X = \{x_0, \ldots, x_{n-1}\}$, $Y \subseteq X$, then $F(Y) \preccurlyeq F(X)$. Note that the result as stated only applies when X is finite; however, it is straightforward to see that this implies the same result for free groups of infinite rank:

Lemma A.3. *For any ordinal α, $F_\alpha \preccurlyeq F_{\alpha+1}$*

Proof. By induction on α:

- When α is finite, this follows from Fact A.2.
- Suppose the statement holds for α, and for $\beta \leq \alpha$ let $G_\beta := F(\beta \cup \{\alpha + 1\})$. By induction, we have that each $F_\beta \preccurlyeq G_\beta$, and hence

$$F_{\alpha+1} = \bigcup_{\beta\leq\alpha} F_\beta \preccurlyeq \bigcup_{\beta\leq\alpha} G_\beta = F((\alpha+1) \cup \{\alpha+1\}) = F_{\alpha+2}$$

□

Corollary A.4. *For ordinals $\alpha < \beta$, $F_\alpha \preccurlyeq F_\beta$*

Corollary A.5. *For any sets $X \subseteq Y$, $F(X) \preccurlyeq F(Y)$*

Fact A.6 (Corollary to Kurosh's Subgroup Theorem)**.** *If F, G are free factors of H, then $F \cap G$ is a free factor of both F and G. In particular, if $F \subseteq G$, then F is a free factor of G.*

Corollary A.7. *K admits finite intersection.*

Lemma A.8. *The class $(K, \leq_f)$ is a weak AEC.*

Proof. The only property which is not immediate is Coherence. So suppose that $F \leq_f H$, $G \leq_f H$, and $F \subseteq G$. Hence F, G are both free factors of H, and so $F \leq_f G$ by Fact A.6. $\square$

Remark. It should be noted that $(K, \leq_f)$ is <u>not</u> an AEC as it does not satisfy Smoothness, as exemplified by this example from [BCS77]: Let $X = \{x_i : i < \omega\}$, and define $y_i := x_i x_{i+1}^2$, $G_i := \langle y_j : j < i\rangle$. Note then that each G_i is a free factor of $F(X)$, but $\bigcup_{i<\omega} G_i = \langle x_i x_{i+1}^2 : i < \omega\rangle$ is not a free factor of $F(X)$.

In $(K, \leq_f)$, we define the notion of amalgamation $\mathcal{A}$ to be the group (nonabelian) free amalgamation: the commutative square

$$\begin{tikzcd}
G_1 \arrow[r, "\iota"] & H \\
G_0 \arrow[u, "\iota"] \arrow[r, "\iota"] & G_2 \arrow[u, "\iota"']
\end{tikzcd}$$

is an $\mathcal{A}$-amalgam iff there is a set Y with subsets $X_1, X_2 \leq Y$ such that $H = F(Y)$, $G_1 = F(X_1)$, $G_2 = F(X_2)$, and $G_0 = F(X_1 \cap X_2)$. Equivalently, there exists G_1', G_2' such that $G_1 = G_0 * G_1'$, $G_2 = G_0 * G_2'$, and $H = G_0 * G_1' * G_2'$.

Lemma A.9. *$\mathcal{A}$ is absolutely minimal.*

Proof. If H is a $\mathcal{A}$-amalgam of G_1, G_2 by inclusion over G_0, then $H = \langle G_1 \cup G_2\rangle$, which is the minimal subgroup containing G_1, G_2 in H (and every extension of H). $\square$

Lemma A.10. *If H_1 is an $\mathcal{A}$-amalgam of G_1, H_0 over G_0 by inclusion, and X_0, X_1 are free bases of G_0, G_1 respectively such that $X_0 \leq X_1$, then there is a set $Y_1 \supseteq X_0$ such that $Y_1, Y_1 \cup X_1$ are free bases of H_0, H_1 respectively.*

Proof. Translating to free products of groups, the assumption implies that there are groups G', H' such that:

- $G_1 = G_0 * G'$
- $H_0 = G_0 * H'$
- $H_1 = G_0 * G' * H'$

Hence, if Y' is any free basis of H', then letting $Y_1 = X_0 \cup Y'$ gives the desired result. $\square$

Lemma A.11. *$\mathcal{A}$ is continuous.*

Proof. Given the $\mathcal{A}$-amalgams

$$\begin{tikzcd}
H_0 \arrow[r, "\iota"] & H_1 \arrow[r, "\iota"] & \cdots \\
G_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & G_1 \arrow[u, "\iota"] \arrow[r, "\iota"] & \cdots
\end{tikzcd}$$

Fix a free basis X_0 of G_0, and let Y be such that $X_0 \cup Y$ is a basis for H_0. By the above lemma, we can find $X_1 \supseteq X_0$ such that $X_1, X_1 \cup Y$ are bases for G_1, H_1 respectively. Proceeding by induction, we get that $Y \cup \bigcup_i X_i$ is a basis for $\bigcup_i H_i$, and hence this is an $\mathcal{A}$-amalgam of $H_0, \bigcup_i G_i$ over G_0 by inclusion. $\square$

Lemma A.12. *$\mathcal{A}$ is regular.*

Proof. Recall the definition of regularity in Definition 2.6; We will prove that the three statements are equivalent for $\mathcal{A}$.

- $1 \Rightarrow 3$: If H is an $\mathcal{A}$-amalgam of G_1, G_2 over G_0 by inclusion, then $H = G_0 * G_1' * G_2'$. Now, if G_* is such that $G_0 \leq_f G^* \leq_f G_1$, then there is some $G_*' \leq_f G_1'$ such that $G_* = G_0 * G_*'$. Thus we have that

$$\begin{tikzcd}
G_2 \arrow[r, "\iota"] & G_0 * G_2' * G_*' \arrow[r, "\iota"] & H \\
G_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & G_* \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & G_1 \arrow[u, "\iota"]
\end{tikzcd}$$

- $2 \Rightarrow 1$: Assume that

$$\begin{tikzcd} H_0 \arrow[r, "\iota"] & H_1 \arrow[r, "\iota"] & H_2 \\ G_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & G_1 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & G_2 \arrow[u, "\iota"] \end{tikzcd}$$

  Hence we have that $H_1 = G_0 * G_1' * H'$ and $H_2 = G_1 * G_2' * H' = G_0 * G_1' * G_2' * H$. So H_2 is indeed an $\mathcal{A}$-amalgam of G_2, H_0 over G_0 by inclusion.

- $2 \Rightarrow 3$: This is straightforward.

$\square$

Lemma A.13. *$\mathcal{A}$ admits decomposition, and $\mu(K) = \aleph_0$*

Proof. If $G_0 \leq_f G_1 \leq_f G_2$, then there is some G' such that $G_2 = G_1 * G'$, and hence G_2 is the $\mathcal{A}$-amalgam of $G_1, G_0 * G'$ over G_0 by inclusion. That $\mu(K) = \aleph_0$ is equivalent to the fact that all words in a free group are of finite length. $\square$

Lemma A.14. *$\mathcal{A}$ has uniqueness.*

Proof. This is straightforward from the fact that free amalgamation is a pushout in the category of groups. $\square$

Corollary A.15. *$\mathcal{A}$ is a notion of free amalgamation on $(K, \leq_f)$*

Appendix B

The class of $\mathcal{Q}$-filtered modules and connections with singular compactness

Most of the material in this appendix is based on [Ekl08], albeit translated to the language of AECs. Let us recall from Chapter 2 the example of $\mathcal{Q}$-filtered modules:

Definition B.1. Fix a ring R, and let $\mathcal{Q}$ be a family of R modules with $0 \in \mathcal{Q}$. A R-module M is **$\mathcal{Q}$-filtered** if there is a continuous resolution $(M_i : i < \alpha)$ of M in the submodule ordering such that for each $i+1 < \alpha$, M_{i+1}/M_i is isomorphic to some $A \in \mathcal{Q}$. We call the sequence $(M_i : i < \alpha)$ a **$\mathcal{Q}$-filtration** of M.

Let K be the class of $\mathcal{Q}$-filtered modules. We define the partial ordering $\preccurlyeq$ on K as follows: $M \leq_K N$ iff there is a $\mathcal{Q}$-filtration $(N_i : i < \alpha)$ of N such that for every $i + q < \alpha$,

$$M \cap (N_{i+1} - N_i) \neq \emptyset \Rightarrow N_{i+1} \subseteq N_i + M$$

where $-$ is set difference and $+$ is the internal module sum within N.

Let $M \preccurlyeq N$ be $\mathcal{Q}$-filtered modules, $(N_i : i < \alpha)$ be a $\mathcal{Q}$-filtration of N, and denote $M_i := M \cap N_i$ for $i < \alpha$. By the above definition, $M_{i+1} = M_i$ or $N_{i+1} = N_i + M_{i+1}$, and hence $M_{i+1}/M_i \cong N_{i+1}/N_i \in \mathcal{Q}$. In particular, hence $(M_i : i < \alpha)$ is a $\mathcal{Q}$-filtration of M.

Proposition B.2 ([Ekl08], Section 2.III)**.** *Suppose $\mathcal{Q}$ is a family of $\leq \mu$-presented modules. Then $(K, \preccurlyeq)$ is a very weak AEC with $LS(K) = |R| + \mu + \aleph_0$*

Lemma B.3 ([Ekl08], Section 2.III)**.** *If $M \preccurlyeq N$ and $(A_i : i < \alpha)$ is a $\mathcal{Q}$-filtration of M, then there exists $(A_i : \alpha \leq i < \beta)$ such that $(A_i : i < \beta)$ is a $\mathcal{Q}$-filtration of N.*

Let us now define a notion of amalgamation $\mathcal{A}$ on the class K. Given $M^0 \preccurlyeq M^1, M^2, \preccurlyeq N$, N is an $\mathcal{A}$-amalgam of M^1, M^2 over M^0 by inclusion iff there is a $\mathcal{Q}$-filtration $(N_i : i < \alpha)$ of N and subsets $S_0, S_1, S_2 \subseteq \alpha$ such that:

- $S_1 \cup S_2 = \alpha$, and $S_0 = S_1 \cap S_2 \ni 0$
- Defining $M^l_i := N_i \cap M^l$ for $l = 0, 1, 2$, for each $0 < i < \alpha$ and $l = 0, 1, 2$:
  - If $i + 1 \in S_1 - S_2$, then $N_{i+1} = N_i + M^1_{i+1}$ and $M^2_{i+1} = M^2_i$
  - If $i + 1 \in S_2 - S_1$, then $N_{i+1} = N_i + M^2_{i+1}$ and $M^1_{i+1} = M^1_i$
  - If $i + 1 \in S_0 = S_1 \cap S_2$, then $N_{i+1} = N_i + M^0_{i+1}, M^1_{i+1} = M^1_i + M^0_{i+1}, M^2_{i+1} = M^2_i + M^0_{i+1}$

Lemma B.4. *Suppose N is an $\mathcal{A}$-amalgam of M^1, M^2 over M^0 by inclusion. Then $N = M^1 + M^2$ and $M^0 = M^1 \cap M^2$*

Proof. Let $(N_i : i < \alpha)$ be a $\mathcal{Q}$-filtration of N which witnesses that N is an $\mathcal{A}$-amalgam as supposed, and let S_l, M_i^l be as defined in the previous paragraph for $l = 0, 1, 2$ and $i < \alpha$. It suffices to show that each $N_i = M_i^1 + M_i^2$; we proceed by induction on $i < \alpha$:

- For $i = 0$, $i \in S_0$ and $N_0 = M_0^0 = M_0^1 = M_0^2 = 0$
- For i a limit ordinal, this is straightforward by taking unions.
- For $i + 1 < \alpha$, we break into cases:
  - Suppose $i + 1 \in S_0 = S_1 \cap S_2$. Then
  $$N_{i+1} = N_i + M_{i+1}^0 = M_i^1 + M_i^2 + M_{i+1}^0 = (M_i^1 + M_{i+1}^0) + (M_i^2 + M_{i+1}^0) = M_{i+1}^1 + M_{i+1}^2$$
  - Suppose $i + 1 \in S_1 - S_2$. Then
  $$N_{i+1} = N_i + M_{i+1}^1 = M_i^1 + M_i^2 + M_{i+1}^1 = M_{i+1}^1 + M_{i+1}^2$$
  - The case of $i + 1 \in S_2 - S_1$ is symmetric.

To see that $M^1 \cap M^2 \subseteq M^0$, suppose that $m \in M^1 \cap M^2$. Let $i + 1 < \alpha$ be minimal such that $m \in N_{i+1}$, so in particular $m \in M_{i+1}^1 \cap M_{i+1}^2$. Since $i + 1$ was chosen to be minimal, this implies that $M_{i+1}^1/M_i^1 \neq 0$ and $M_{i+1}^2/M_i^2 \neq 0$, and therefore $i + 1 \in S_1 \cap S_2 = S_0$. Thus $m \in M_{i+1}^0 \subseteq M^0$ □

Corollary B.5. *$\mathcal{A}$ is absolutely minimal.*

Lemma B.6. *Suppose N is an $\mathcal{A}$-amalgam of M^1, M^2 over M^0 by inclusion. Let $(A_i^0 : i < \alpha)$ be any $\mathcal{Q}$-filtration of M^0, and let $(A_i^1 : i < \beta), (A_i^2 : i < \gamma)$ be $\mathcal{Q}$-filtrations of M^1 and M^2 respectively such that for all $i < \alpha < \beta, \gamma$, $A_i^0 = A_i^1 = A_i^2$. Then there is a $\mathcal{Q}$-filtration $(N_i : i < \delta)$ of N such that:*

1. *$\delta = \beta + (\gamma - \alpha)$*
2. *For $i < \alpha$, $N_i = A_i^0$; and $N_\alpha = M^0$*
3. *For $\alpha < i < \beta$, $N_i = A_i^1$; and $N_\beta = M^1$*
4. *For $j < \gamma - \alpha$, $N_{\beta+j} = M^1 + A_{\alpha+j}^2$*

Moreover, this $\mathcal{Q}$-filtration of N witnesses the $\mathcal{A}$-amalgamation.

Proof. Let us first check that $(N_i : i < \delta)$ as defined above is indeed a $\mathcal{Q}$-filtration of N. Clearly $\bigcup_{i<\delta} N_i = M^1 + M^2 = N$, and hence we only need to check that each $N_{i+1}/N_i \in \mathcal{Q}$. For $i < \beta$, this is true by definition of $N_i = A_i^1$; For $i = \beta + j$ for some $j < \gamma - \alpha$, we defined $N_{\beta+j} = M^1 + A_{\alpha+j}^2$. Consider the map $f : M^1 + A_{\alpha+j+1}^2 \to A_{j+1}^2/A_j^2$ where for $x \in M^1$ and $y \in A_{\alpha+j+1}^2$, $f(x + y) = y + A_{\alpha+j}^2$. We check:

- f is well-defined: if $x_1, x_2 \in M^1$, $y_1, y_2 \in A_{\alpha+j+1}^2$ are such that $x_1 + y_1 = x_2 + y_2$, then $x_1 - x_2 = y_2 - y_1 \in M^1 \cap A_{\alpha+j+1}^2$. By the above lemma, since N is an $\mathcal{A}$-amalgam of M^1, M^2 over M^0 by inclusion, $M^1 \cap M_2 = M_0$, and hence
$$M^1 \cap A_{j+1}^2 \subseteq M^1 \cap M^2 = M^0 = A_\alpha^2 \subseteq A_{\alpha+j}^2$$
Hence, $y_1 + A_{\alpha+j}^2 = y_2 + A_{\alpha+j}^2$ as desired.
- For the kernel of f, note that $f(x + y) = A_{\alpha+j}^2$ iff $y \in A_{\alpha+j}^2$, and hence $\ker(f) = M^1 + A_{\alpha+j}^2$

Hence the map f demonstrates that
$$N_{i+1}/N_i = (M^1 + A_{\alpha+j+1}^2)/(M^1 + A_{\alpha+j}^2) \cong A_{\alpha+j+1}^2/A_{\alpha+j}^2 \in \mathcal{Q}$$

For the moreover part, note that $M^1 \cap (N_{i+1} - N_i)$ is nonempty iff $i + 1 < \beta$ by the above construction, and $M^2 \cap (N_{i+1} - N_i)$ is nonempty iff $i + 1 < \alpha$ or $\beta < i + 1 < \delta$, and hence the sets $S_0 = \alpha, S_1 = \beta, S_2 = \alpha \cup \{i < \delta : i \geq \beta\}$ demonstrates that $(N_i : i < \delta)$ indeed witnesses that N is an $\mathcal{A}$-amalgam of M^1, M^2 over M^0. □

Corollary B.7. *Suppose N is an $\mathcal{A}$-amalgam of M^1, M^2 over M^0 by inclusion. Let $(A_i : i < \alpha)$ be a $\mathcal{Q}$-filtration of M^0, with an end-extension $(A_i : i < \beta)$ a $\mathcal{Q}$-filtration of M^1. Then there is a further end-extension $(A_i : i < \delta)$ a $\mathcal{Q}$-filtration of N such that $(A_i \cap M^2 : i < \delta)$ is a $\mathcal{Q}$-filtration of M^2, and $M^2 \cap (Ai + 1 - A_i)$ is nonempty iff $i \in \alpha \cup (\delta - \beta)$*

Corollary B.8. *$\mathcal{A}$ is regular.*

Proof. 1. First, suppose that we have the following $\mathcal{A}$-amalgams:

$$\begin{tikzcd}
M^2 \arrow[r, "\iota"] & N' \arrow[r, "\iota"] & N \\
M^0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^1 \arrow[u, "\iota"]
\end{tikzcd}$$

By the previous corollary applied to the amalgam on the left, fix $(A_i : i < \alpha_2)$ a $\mathcal{Q}$-filtration of N' with $\alpha_0 < \alpha_1 < \alpha_2$ such that:

- $(A_i : i < \alpha_0)$ is a $\mathcal{Q}$-filtration of M^0
- $(A_i : i < \alpha_1)$ is a $\mathcal{Q}$-filtration of M'
- $(A_i \cap M_2 : i < \alpha_2)$ is a $\mathcal{Q}$-filtration of M^2

Applying the corollary again to the amalgam on the right, we can extend the filtration to $(A_i : i < \alpha_3)$ a $\mathcal{Q}$-filtration of N such that $(A_i \cap M^1 : i < \alpha_3)$ is also a $\mathcal{Q}$-filtration of M^1.

To check that this filtration witnesses that N is an $\mathcal{A}$-amalgam of M^1, M^2 over M^0, it suffices to check that $M^1 \cap (A_{i+1} - A_i)$ and $M^2 \cap (A_{i+1} - A_i)$ are both nonempty iff $i < \alpha_0$: for any such i, as $(A_i \cap M^2 : i < \alpha_2)$ is a $\mathcal{Q}$-filtration of M^2, this implies that $i < \alpha_2$, in which case $A_i \cap M^1 = A_i \cap M'$. Furthermore, by construction of $(A_i : i < \alpha_2)$, thus $i < \alpha_0$ as required.

2. Next, suppose that

$$\begin{tikzcd}
M^2 \arrow[r, "\iota"] & N \\
M^0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^1 \arrow[u, "\iota"]
\end{tikzcd}$$

Let M' be such that $M^0 \preccurlyeq M' \preccurlyeq M^1$, and by Lemma B.3 and fix a $\mathcal{Q}$-filtration $(A_i : i < \delta)$ of N with $\alpha < \beta < \gamma < \delta$ such that:

- $(A_i : i < \alpha)$ is a $\mathcal{Q}$-filtration of M^0
- $(A_i : i < \beta)$ is a $\mathcal{Q}$-filtration of M'
- $(A_i : i < \gamma)$ is a $\mathcal{Q}$-filtration of M^1
- $(A_i \cap M^2 : i < \delta)$ is a $\mathcal{Q}$-filtration of M^2, with $M^2 \cap (A_{i+1} - A_i)$ nonempty iff $i < \alpha \cup (\delta - \gamma)$

Let $N' = M^2 + M'$. We claim that

$$\begin{tikzcd}
M^2 \arrow[r, "\iota"] & N' \arrow[r, "\iota"] & N \\
M^0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^1 \arrow[u, "\iota"]
\end{tikzcd}$$

- For the amalgam on the left, consider the filtration $(A_i \cap N' : i < \delta)$:
  - For $i < \beta$, $A_i \subseteq M' \subseteq N'$, and hence $A_i \cap N' = A_i$
  - For $\beta \leq i < \gamma$, $A_i \cap N' = A_i \cap (M' + M^2) = (A_i \cap M') + (A_i \cap M^2) = M' = A_\beta$
  - For $\gamma \leq i < \delta$, $A_i \cap N' = A_i \cap (M' + M^2) = M' + (A_i \cap M^2)$. As before, the map $f : M' + (A_{i+1} \cap M^2) \to (A_{i+1} \cap M^2)/(A_i \cap M^2)$ with $f(x + y) = y + (A_i \cap M^2)$ for $x \in M'$ and $y \in A_{i+1} \cap M^2$ shows that $(A_{i+1} \cap N')/(A_i \cap N') \cong (A_{i+1} \cap M^2)/(A_i \cap M^2) \in \mathcal{Q}$; the maps is well-defined as $M' \cap (M^2 \cap A_{i+1}) = M^0 \cap A_{i+1} \subseteq A_i \cap M^2$ since $M' \cap M^2 \subseteq M^1 \cap M^2 = M^0$.

Thus this is a $\mathcal{Q}$-filtration which witnesses N' is an $\mathcal{A}$-amalgam of M^2, M' over M^0.

- For the amalgam on the right, the $\mathcal{Q}$-filtration $(A_i : i < \delta)$ is sufficient to witness that N is an $\mathcal{A}$-amalgam of N', M^1 over M'.

This shows that $\mathcal{A}$ is regular. ☐

Lemma B.9. *$\mathcal{A}$ is continuous.*

Proof. Let δ be a limit, and suppose

$$\begin{tikzcd}
N^0 \arrow[r, "\iota"] & N^1 \arrow[r, "\iota"] & N^2 \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & N^i \arrow[r, "\iota"] & N^{i+1} \arrow[r, "\iota"] & \cdots \\
M^0 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^1 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^2 \arrow[r, "\iota"] \arrow[u, "\iota"] & \cdots \arrow[r, "\iota"] & M^i \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M^{i+1} \arrow[r, "\iota"] \arrow[u, "\iota"] & \cdots
\end{tikzcd}$$

Let $M^\delta = \bigcup_{i<\delta} M^i, N^\delta = \bigcup_{i<\delta} N^i$, and by Lemma B.3 fix $(A_j : j < \alpha_\delta)$ a $\mathcal{Q}$-filtration of M^δ with $(\alpha_i : i \leq \delta)$ an increasing continuous chain such that each $(A_j : j < \alpha_i)$ is a $\mathcal{Q}$-filtration of M^i. In particular, $(A_j : j < \alpha_0)$ is a $\mathcal{Q}$-filtration of M^0, which can be end-extended to $(A_j : j < \alpha_0)^\frown(B_k : k < \beta)$, a $\mathcal{Q}$-filtration of N_0. It is then straightforward to check that the filtration $(A_j : j < \alpha_0)^\frown(B_k : k < \beta)^\frown(A_j + N^0 : \alpha_0 \leq j < \alpha_\delta)$ is a $\mathcal{Q}$-filtration of N^δ which witnesses that N^δ is an $\mathcal{A}$-amalgam of M^δ, N^0 over M^0. ☐

On the other hand, depending on $\mathcal{Q}$, $\mathcal{A}$ might not admit decomposition:

Example B.10. Consider $\mathcal{Q} = \{0, \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$ as $\mathbb{Z}$-modules. Then the sequence $0 \preccurlyeq 2\mathbb{Z} \preccurlyeq \mathbb{Z}$ is a $\mathcal{Q}$-filtration of $\mathbb{Z}$, but there is no proper submodule $M \lneq \mathbb{Z}$ such that $M + 2\mathbb{Z} = \mathbb{Z}$ and $\mathbb{Z}/M \in \mathcal{Q}$.

In the opposite direction, by taking $\mathcal{Q}$ to be sufficiently large (e.g. all modules with cardinality less than some strongly inaccessible λ), we guarantee that for any two modules M^1, M^2 with $M^0 = M^1 \cap M^2$, any $N \geq M^1, M^2$ with $N = M^1 + M^2$ is an $\mathcal{A}$-amalgam, thus easily giving examples where $\mathcal{A}$ has neither uniqueness nor weak 3-existence:

Example B.11. Let $A \cong B \cong R^2$, with $A = \langle a_1, a_2 \rangle$ and $B = \langle b_1, b_2 \rangle$. Identify $M_0 = 2a_1 \oplus 2a_2 = 3b_1 \oplus 3b_2$, so that $M_0 \leq A, B$, and consider two different amalgams:

- $N = A + B$ with the relations $2a_1 = 3b_1$ and $2a_2 = 3b_2$
- $N' = A + B$ with the relations $2a_2 = 3b_1$ and $2a_1 = 3b_2$

In this case, there cannot be an isomorphism between N and N' which fixes $A \cup B$, and hence uniqueness fails.

Continuing the example, let $C \cong R^2$ also with $C = \langle c_1, c_2 \rangle$ with $M_0 = 5c_1 \oplus 5c_2$. Consider three amalgams:

- $M_{AB} = A + B$ with the relations $2a_1 = 3b_1$ and $2a_2 = 3b_2$
- $M_{BC} = B + C$ with the relations $3b_1 = 5c_1$ and $3b_2 = 5c_2$
- $M_{AC} = A + C$ with the relations $2a_1 = 5c_2$ and $2a_2 = 5c_1$

If $\mathcal{A}$ has weak 3-existence, then there would be some $N \geq M_{AB}, M_{BC}, M_{AC}$ which forms a 3-amalgam, and in N we have $2a_1 = 3b_1 = 5c_1 = 2a_2$, contradicting that $A \leq N$.

Perhaps surprisingly, however, $\mathcal{A}$ is always 3-monotonic regardless of the choice of $\mathcal{Q}$:

Lemma B.12. *$\mathcal{A}$ is 3-monotonic.*

Proof. Suppose M^{12} is an $\mathcal{A}$-amalgam of M^1, M^2 over M^0, and N an $\mathcal{A}$-amalgam of M^3, M^{12} over M^0. Fix $(A_i : i < \alpha)$ a $\mathcal{Q}$-filtration of M^0, extend it to $(A_i : i < \beta)$ a $\mathcal{Q}$-filtration of M^{12} that witnesses M^{12} is an $\mathcal{A}$-amalgam, and let $(B_j < \gamma)$ be such that $(A_i : i < \alpha)^\frown(B_j : j < \gamma)$ is a $\mathcal{Q}$-filtration of M^3. It is straightforward to check that $(A_i : i < \alpha)^\frown(B_j : j < \gamma)^\frown(M^3 + A_i : \alpha \leq i < \beta)$ is a $\mathcal{Q}$-filtration of N which that witnesses N is an $\mathcal{A}$-amalgam of $M^1 + M^3, M^2 + M^3$ over M^3 ☐

Before ending this section, let us recall that the main purpose of [Ekl08] was to prove Shelah's singular compactness theorem in the context of a class of modules K with some abstract notion of "freeness" and "basis", of which the class of $\mathcal{Q}$-filtered modules is a specific example.

Definition B.13 (Adapted from [Ekl08], Hypothesis 1.2)**.** Let R be a ring, K a class of R-modules with a partial ordering $\preccurlyeq$ which is a refinement of the submodule relation, and such that K is closed under $\preccurlyeq$-increasing continuous chains. An abstract system of basis for K consists of a $B(M) \subseteq P(\{A \in K : A \preccurlyeq M\})$ for each $M \in K$, and for every $M \preccurlyeq N$ a relation $D(M.N) \subseteq B(M) \times B(N)$ satisfying:

1. For all $X \in B(M)$, $0 \in X$
2. Every $X \in B(M)$ is closed under union of chains
3. There exists a cardinal μ such that for every $N \in K$, $X \in B(N)$, $M \in X$, and $b \in N$, there is $M' \in X$ such that $M \leq M'$, $b \in M'$, and $|M'| \leq |M| + \mu$
4. For all $X \in B(N)$ and $M_0, M_1 \in X$ with $M_0 \leq M_1$, there is some $Y \in B(M_1)$ such that $M_0 \in Y$; and hence, in particular, $M_0 \preccurlyeq M_1$
5. If $M \preccurlyeq N$ and $X \in B(M)$, then there is a $Y \in B(N)$ such that $(X, Y) \in D(M, N)$. We denote the relation by $X = Y \restriction M$.
6. If $(M_i : i < \alpha)$ is a $\preccurlyeq$-chain, and each $X_i \in B(M_i)$ is such that $X_i = X_{i+1} \restriction M_i$, then there is $Y \in B(\bigcup_{i<\alpha} M_i)$ such that $\bigcup_{i<\alpha} X_i \subseteq Y$

As an example, for the class of $\mathcal{Q}$-filtered modules, $X \in B(M)$ iff there is a $\mathcal{Q}$-filtration $(M_i : i < \alpha)$ of M and for all $A \in X, i+1 < \alpha$, $A \cap (M_{i+1} - M_i) \neq \emptyset$ implies that $M_{i+1} \subseteq M_i + A$; note, in particular, that each $M_i \in X$. For $X \in B(M)$ and $Y \in B(N)$, $X = Y \restriction M$ iff there is a $\mathcal{Q}$-filtration $(N_i : i < \beta)$ of N with some $\alpha < \beta$ such that $M = N_\alpha$, Y is the basis determined by this $\mathcal{Q}$-filtration as above, and $X = \{A \in Y : S \leq M\}$. That this definition of $B(M)$ and $D(M, N)$ satisfies the conditions for an abstract system of basis is demonstrated in [Ekl08], Section 2.III.

So for a class $(K, \preccurlyeq)$ with such a system, how much of the above analysis for $\mathcal{Q}$-filtered modules carries over to the general case? The simplest notion of amalgamation $\mathcal{A}$ we can define given such a system is that if there is $X \in B(M)$ and $A_0, A_1, A_2 \in X$ such that $A_0 = A_1 \cap A_2$ and $M = A_1 + A_2$, then M is an $\mathcal{A}$-amalgam of A_1, A_2 over A_0 by inclusion.

It is clear that $\mathcal{A}$ is absolutely minimal. For regularity, consider the following hypotheses:

Hypothesis B.14.

1. *If $X \in B(M)$, then $M \in X$*
2. *For every $X \in B(M)$ and $A_1, A_2 \in X$, $A_1 + A_2 \in X$.*
3. *If $M \preccurlyeq N$ and $Y \in B(N)$ is such that $M \in Y$, then there is $X \in B(M)$ such that $X = \{A \in Y : A \leq M\}$, and X is unique in $B(M)$ satisfying $X = Y \restriction M$*
4. *Suppose $M_0 \preccurlyeq M_1, M_2 \preccurlyeq N$ are such that $M_0 = M_1 \cap M_2$, $N = M_1 + M_2$, and there is a $Y \in B(N)$ with $M_0, M_1, M_2 \in Y$ (i.e. N is an $\mathcal{A}$-amalgam of M_1, M_2 over M_0). If $X_1 \in B(M_1)$ and $X_2 \in B(M_2)$ are such that $X_1 \restriction M_0 = X_2 \restriction M_0$, then there exists a $Y' \in B(N)$ such that $X_1 = Y' \restriction M_1$ and $X_2 = Y' \restriction M_2$*

We first need to show that

$$\begin{tikzcd} M_2 \arrow[r, "\iota"] & N' \arrow[r, "\iota"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M' \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] \end{tikzcd} \Rightarrow \begin{tikzcd} M_2 \arrow[r, "\iota"] & N \\ M_0 \arrow[u, "\iota"] \arrow[r, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[u, "\iota"] \end{tikzcd}$$

Let $Y' \in B(N')$ be such that $M_0, M_2, M' \in Y'$, as guaranteed to exist by the $\mathcal{A}$-amalgam on the left. Consider the basis $X' = Y' \restriction M'$, and extend to a basis $X_1 \in B(M_1)$. Since N is an $\mathcal{A}$-amalgam of N', M_1

over M' and $X_1 \upharpoonright M' = X'$, there is a basis $Y \in B(N)$ such that $Y \upharpoonright N' = Y'$ and $Y \upharpoonright M_1 = X_1$. In particular, $M_0, M_2 \in Y' \subseteq Y$, $M_1 \cap M_2 = M' \cap M_2 = M_0$, and $N = M_1 + N' = M_1 + (M' + M_2) = M_1 + M_2$. Hence N is indeed an $\mathcal{A}$-amalgam of M_1, M_2 over M_0 by inclusion.

Next, suppose that $M_0 \preccurlyeq M' \preccurlyeq M_1$, and N is an $\mathcal{A}$-amalgam of M_1, M_2 over M_0. Let $Y \in B(N)$ witness this fact (so $M_0, M_1, M_2 \in Y$), and let $X_0 = Y \upharpoonright M_0$. Extend X_0 to a basis $X' \in B(M')$ i.e. $X_0 = X' \upharpoonright M_0$, and then extend X' to $X_1 \in B(M_1)$. Note then $X_0 = X_1 \upharpoonright M_0 = Y \upharpoonright M_0 = (Y \upharpoonright M_2) \upharpoonright M_0$, so there is a $Y' \in B(N)$ such that $Y' \upharpoonright M_2 = Y \upharpoonright M_2$ and $Y' \upharpoonright M_1 = X_1$. In particular, $M' \in X' \subseteq X_1 \subseteq Y'$, and so $M' + M_2 \in Y'$. This demonstrates that N is an $\mathcal{A}$-amalgam of $M_1, M' + M_2$ over M', and similarly $Y' \upharpoonright (M' + M_2)$ demonstrates that $M' + M_2$ is an $\mathcal{A}$-amalgam of M', M_2 over M_0. This shows that $\mathcal{A}$ is regular under the above additional hypotheses.

For continuity, the above hypotheses are also sufficient: suppose that

$$\begin{tikzcd}
N_0 \arrow[r, "\iota"] & N_1 \arrow[r, "\iota"] & N_2 \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & N_i \arrow[r, "\iota"] & N_{i+1} \arrow[r, "\iota"] & \cdots \\
M_0 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_1 \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_2 \arrow[r, "\iota"] \arrow[u, "\iota"] & \cdots \arrow[r, "\iota"] & M_i \arrow[r, "\iota"] \arrow[u, "\iota"] \arrow[ur, phantom, "\mathcal{A}"] & M_{i+1} \arrow[r, "\iota"] \arrow[u, "\iota"] & \cdots
\end{tikzcd}$$

Since N_1 is an $\mathcal{A}$-amalgam of M_1, N_0 over M_0, let $Y_1 \in B(N_1)$ witness this fact, and let $X_2 \in B(M_2)$ be an extension of $Y_1 \upharpoonright M_1$. Then, as N_2 is an $\mathcal{A}$-amalgam of M_2, N_1 over M_1, there is some $Y_2 \in B(N_2)$ such that $Y_2 \upharpoonright N_1 = Y_1$ and $Y_2 \upharpoonright M_2 = X_2$. We can thus construct $Y_i \in B(N_i)$ inductively, and hence there is $Y \in B(\bigcup_{i<\alpha} N_i)$ such that each $Y_i \subseteq Y$. Note that each $M_i \in Y$, and hence $\bigcup_{i<\alpha} M_i \in Y$, and thus Y witnesses that $\bigcup_{i<\alpha} N_i$ is an $\mathcal{A}$-amalgam of $N_0, \bigcup_{i<\alpha} M_i$ over M_0.

The case for 3-monotonicity is similar: suppose that M_{12} is an $\mathcal{A}$-amalgam of M_1, M_2 over M_0, and N is an $\mathcal{A}$-amalgam of M_3, M_{12} over M_0. Let $X \in B(M_{12})$ be such that $M_0, M_1, M_2 \in X$, and let $X_3 \in B(M_3)$ extend $X \upharpoonright M_0$. Then there is a $Y \in B(N)$ such that $Y \upharpoonright M_3 = X_3$ and $Y \upharpoonright M_{12} = X$, and hence $M_1 + M_3, M_2 + M_3 \in Y$. This shows that N is an $\mathcal{A}$-amalgam of $M_1 + M_3, M_2 + M_3$ over M_3.

As the example of $\mathfrak{Q}$-filtered modules demonstrates, it is not necessarily true that a system of basis satisfying the above hypotheses leads to a notion of amalgamation $\mathcal{A}$ which admits decomposition. In fact, with $\mathcal{A}$ as defined above, since $0 \preccurlyeq M$ for all $M \in K$, $\mathcal{A}$ admitting decomposition implies that for every $M \preccurlyeq N$, then there is $M' \preccurlyeq N$ such that $M \oplus M' = N$, and in particular that M is a direct summand of N. We note further that in this case, $\mu(K) = \aleph_0$ and $\mathcal{A}$ has uniqueness: if $M_0 \preccurlyeq M_1, M_2$, then there is N_1, N_2 such that $M_1 = M_0 \oplus N_1$ and $M_2 = M_0 \oplus N_2$, and hence the only $\mathcal{A}$-amalgam would be $N = M_0 \oplus N_1 \oplus N_2$. This is a case where Question 8.3 is partially answered in the positive: the assumption of admitting decomposition implies that K is superstable.

Although this analysis shows that a system of basis for K requires nontrivial hypotheses to be added to prove some fundamental properties of $\mathcal{A}$, this is not surprising when we consider that the axioms defining a system of basis were abstracted by Shelah (and later, Hodges and Eklof) for the purpose of proving the singular compactness theorem, which evidently requires much less structure on the class than the structural results we have shown for classes with free amalgamation. On the other hand, one should keep in mind that there may be other notions of amalgamation which can be defined on such a class; this demonstrates a fundamental weakness in the analysis of notions of amalgamation: the lack of canonicity of such notions makes it difficult to prove anti-structural results.

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