# EXCELLENT ABSTRACT ELEMENTARY CLASSES ARE TAME

RAMI GROSSBERG AND ALEXEI S. KOLESNIKOV

Abstract. The assumption that an AEC is tame is a powerful assumption permitting development of stability theory for AECs with the amalgamation property. Lately several upward categoricity theorems were discovered where tameness replaces strong set-theoretic assumptions.

We present in this article two sufficient conditions for tameness, both in form of strong amalgamation properties that occur in nature. One of them was used recently to prove that several Hrushovski classes are tame.

This is done by introducing the property of weak $(\mu, n)$-uniqueness which makes sense for all AECs (unlike Shelah's original property) and derive it from the assumption that weak $(\mathrm{LS}(\mathcal{K}), n)$-uniqueness, $(\mathrm{LS}(\mathcal{K}), n)$-symmetry and $(\mathrm{LS}(\mathcal{K}), n)$-existence properties hold for all $n < \omega$. The conjunction of these three properties we call *excellence*, unlike [Sh 87b] we do not require the very strong $(\mathrm{LS}(\mathcal{K}), n)$-uniqueness, nor we assume that the members of $\mathcal{K}$ are atomic models of a countable first order theory. We also work in a more general context than Shelah's good frames.

## Introduction

In 1977 Shelah influenced by earlier work of Jónsson ([Jo1] and [Jo2]) in [Sh 88] introduced a semantic generalization of Keisler's [Ke] treatment of $L_{\omega_1,\omega}(\mathbf{Q})$. It is the notion of *Abstract Elementary Class*:

**Definition 0.1.** Let $\mathcal{K}$ be a class of structures all in the same similarity type $\mathrm{L}(\mathcal{K})$, and let $\prec_{\mathcal{K}}$ be a partial order on $\mathcal{K}$. The ordered pair $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an *abstract elementary class, AEC for short* iff

A0 (Closure under isomorphism)
- (a) For every $M \in \mathcal{K}$ and every $\mathrm{L}(\mathcal{K})$-structure $N$ if $M \cong N$ then $N \in \mathcal{K}$.
- (b) Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.

---

*Date*: September 9, 2005.

1991 *Mathematics Subject Classification.* Primary: 03C45, 03C52. Secondary: 03C05, 03C95.

The research is part of the author's work towards his Ph.D. degree under direction of Prof. Rami Grossberg. I am deeply grateful to him for his guidance and support.

A1 For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.

A2 Let $M, N, M^*$ be $\mathrm{L}(\mathcal{K})$-structures. If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$.

A3 (Downward Löwenheim-Skolem) There exists a cardinal $\mathrm{LS}(\mathcal{K}) \geq \aleph_0 + |\mathrm{L}(\mathcal{K})|$ such that for every $M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \supseteq A$ and $\|N\| \leq |A| + \mathrm{LS}(\mathcal{K})$.

A4 (Tarski-Vaught Chain)

(a) For every regular cardinal $\mu$ and every $N \in \mathcal{K}$ if $\{M_i \prec_{\mathcal{K}} N \ : \ i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing (i.e. $i < j \Longrightarrow M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $\bigcup_{i<\mu} M_i \prec_{\mathcal{K}} N$.

(b) For every regular $\mu$, if $\{M_i \ : \ i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i<\mu} M_i$.

For $M$ and $N \in \mathcal{K}$ a monomorphism $f : M \to N$ is called a $\mathcal{K}$-*embedding* iff $f[M] \prec_{\mathcal{K}} N$. Thus, $M \prec_{\mathcal{K}} N$ is equivalent to "$\mathrm{id}_M$ is a $\mathcal{K}$-embedding from $M$ into $N$".

Many of the fundamental facts on AECs are due to Saharon Shelah and were introduced in [Sh 88], [Sh 394] and [Sh 576]. For a survey of some of the basics see [Gr1] or [Gr3].

In the late seventies Shelah established the program of developing *Classification Theory for Abstract Elementary Classes*, namely that there exists a vastly more general theory than the one presented in [Sh c] that can be developed without any reference to the compactness theorem (that fails already in small fragments of $L_{\omega_1,\omega}$). As such a theory undoubtedly will require new concepts and techniques Shelah proposed the following as a test problem:

**Conjecture 0.2** (Shelah's conjecture)**.** *Let $\psi \in L_{\omega_1,\omega}$ be a sentence in a countable language. If $\psi$ is $\lambda$-categorical in some $\lambda > \beth_{\omega_1}$ then $\psi$ is $\mu$-categorical for every $\mu \geq \beth_{\omega_1}$.*

Several authors wrote many papers trying to approximate this conjecture (Shelah alone produced more than 1,000 pages), the conjecture at present seems to be not accessible.

In 1990 Shelah proposed a generalization for AECs:

**Conjecture 0.3** (see [Sh c])**.** *Let $\mathcal{K}$ be an AEC. If $\mathcal{K}$ is categorical in some $\lambda > \mathrm{Hanf}(\mathcal{K})$ then $\mathcal{K}$ is $\mu$-categorical for every $\mu \geq \mathrm{Hanf}(\mathcal{K})$.*

**Notation 0.4.** Let $\mu$ be a cardinal number and $\mathcal{K}$ a class of models. By $\mathcal{K}_\mu$ we denote the subclass $\{M \in \mathcal{K} \ : \ \|M\| = \mu\}$.

Two classical concepts that introduced in the fifties and studied extensively by Fraisee, Robinson and Jonsson play also an important role in AECs:

**Definition 0.5.** Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an AEC and suppose $\mu \geq \mathrm{LS}(\mathcal{K})$. We say that $\mathcal{K}$ *has the $\mu$-amalgamation property* iff for all $M_\ell \in \mathcal{K}_\mu$ (for $\ell = 0, 1, 2$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell = 1, 2$) there exists $N^* \in \mathcal{K}_\mu$ and $f_\ell : M_\ell \to N^*$

(for $\ell = 1, 2$) such that $f_1 \restriction M_0 = f_2 \restriction M_0$, i.e. the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & N^* \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle f_2} \\ M_0 & \xrightarrow[\mathrm{id}]{} & M_2 \end{array}$$

The model $N^*$ is called an *amalgam* of $M_1$ and $M_2$ over $M_0$.

$\mathcal{K}$ *has the* $\mu$*-joint mapping property* iff for any $M_\ell \in \mathcal{K}_\mu$ for $\ell = 1, 2$ there are $N^* \in \mathcal{K}_\mu$ and $\mathcal{K}$-embeddings $f_\ell : M_\ell \to N^*$.

We say that $\mathcal{K}$ has the *amalgamation property* iff it has the $\mu$-amalgamation property for all $\mu \geq \mathrm{LS}(\mathcal{K})$.

Using the axioms of AECs one can prove the following:

**Fact 0.6.** *If* $\mathcal{K}$ *has the* $\mu$*-AP for all* $\mu \geq \mathrm{LS}(\mathcal{K})$ *then for any triple* $M_\ell \in \mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$*. If* $M_0 \prec_{\mathcal{K}} M_1, M_2$ *then there exists an amalgam of* $M_1$ *and* $M_2$ *over* $M_0$*.*

Using Axiom A0 from the definition of AEC it follows that both a stronger-looking and a weaker-looking amalgamation properties are equivalent to what we call above the amalgamation property:

**Fact 0.7.** *Let* $\mathcal{K}$ *be an AEC. The following are equivalent*

(1) $\mathcal{K}$ *has the* $\mu$*-amalgamation property,*

(2) *for all* $M_\ell \in \mathcal{K}_\mu$ *(for* $\ell = 0, 1, 2$*) such that* $M_0 \prec_{\mathcal{K}} M_\ell$ *(for* $\ell = 1, 2$*) there exists* $N^* \in \mathcal{K}_\mu$ *such that* $N^* \succ_{\mathcal{K}} N_2$ *and there is* $f : M_1 \to N$ *satisfying* $f \restriction M_0 = \mathrm{id}_{M_0}$*, i.e. the following diagram commutes:*

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & N^* \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ M_0 & \xrightarrow[\mathrm{id}]{} & M_2 \end{array}$$

(3) *for all* $M_\ell \in \mathcal{K}_\mu$ *(for* $\ell = 0, 1, 2$*) such that* $g_\ell : M_0 \to M_\ell$ *(for* $\ell = 1, 2$*) are* $\mathcal{K}$*-embeddings there are* $N^* \in \mathcal{K}_\mu$ *and there is* $f_\ell : M_\ell \to N^*$ *satisfying* $f_1 \circ g_1 \restriction M_0 = f_2 \circ g_2 \restriction M_0$ *i.e. the next diagram commutes:*

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & N^* \\ {\scriptstyle g_1}\uparrow & & \uparrow{\scriptstyle f_2} \\ M_0 & \xrightarrow[g_2]{} & M_2 \end{array}$$

An important tool in proving the above lemma is the following basic property of AECs.

**Fact 0.8.** *Suppose* $f : M \to N$ *is a* $\mathcal{K}$*-embedding. There are a model* $\bar{M} \succ M$ *and* $\bar{f} : \bar{M} \cong N$ *extending* $f$*.*

Robinson's consistency property implies that if $T$ is a complete first-order theory then $\mathrm{Mod}(T)$ has both the amalgamation and the joint mapping properties. As there are natural examples of AECs where the $\mu$-AP is a property fails (see [GrSh]) we must deal with AP as a property.

*Acknowledgment:* John Baldwin provided us with detailed very helpful comments, remarks and questions on the 8/30/2005 version that improved very much the presentation of this paper.

## Galois types, amalgamation and tameness

In the theory of AECs the notion of complete first-order type is replaced by that of a *Galois type*:

**Definition 0.9.** Let $\beta > 0$ be an ordinal. For triples $(\bar{a}_\ell, M, N_\ell)$ where $\bar{a}_\ell \in {}^{\beta}\mathrm{N}_\ell$ and $M \prec_{\mathcal{K}} N_\ell \in \mathcal{K}$ for $\ell = 1, 2$, we define a binary relation $E$ as follows: $(\bar{a}_2, M, N_2) E (\bar{a}_1, M, N_1)$ iff and there exists $N \in \mathcal{K}$ and $\mathcal{K}$-mappings $f_1, f_2$ such that $f_\ell : N_l \to N$ and $f_\ell \restriction M = \mathrm{id}_M$ for $\ell = 1, 2$ and $f_2(\bar{a}_2) = f_1(\bar{a}_1)$:

$$\begin{array}{ccc} N_1 & \xrightarrow[f_1]{} & N \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle f_2} \\ M & \xrightarrow[\mathrm{id}]{} & N_2 \end{array}$$

**Remark 0.10.** When $\mathcal{K}$ has the amalgamation property then $E$ is an equivalence relation on the class of triples of the form $(\bar{a}, M, N)$. If $\mathcal{K}$ fails to have the amalgamation property, $E$ may fail to be transitive, but the transitive closure of $E$ could be used instead.

**Remark 0.11.** Using Ax0 one can show that in the previous definition we may assume that $f_2 = \mathrm{id}_{N_2}$, i.e. that $N \succ_{\mathcal{K}} N_2$ and the condition is that $f_1(\bar{a}_1) = \bar{a}_2$.

**Definition 0.12.** Let $\beta$ be a positive ordinal.

(1) For $M, N \in \mathcal{K}$ and $\bar{a} \in {}^{\beta}\mathrm{N}$. The *Galois type of $\bar{a}$ in $N$ over $M$*, written $\text{ga-tp}(\bar{a}/M, N)$, is defined to be $(\bar{a}, M, N)/E$.

(2) For $M \in \mathcal{K}$,

$$\text{ga-S}^{\beta}(M) := \{\text{ga-tp}(\bar{a}/M, N) \mid M \prec N \in \mathcal{K}_{\|M\|}, \bar{a} \in {}^{\beta}\mathrm{N}\}.$$

We write $\text{ga-S}(M)$ for $\text{ga-S}^1(M)$.

(3) Let $p := \text{ga-tp}(\bar{a}/M', N)$ for $M \prec_{\mathcal{K}} M'$ we denote by $p \restriction M$ the type $\text{ga-tp}(\bar{a}/M, N)$. The *domain of $p$* is denoted by $\mathrm{dom}\, p$ and it is by definition $M'$.

(4) Let $p = \text{ga-tp}(\bar{a}/M, N)$, suppose that $M \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} N$ and let $\bar{b} \in {}^{\beta}\mathrm{N}'$ we say that $\bar{b}$ *realizes* $p$ iff $\text{ga-tp}(\bar{b}/M, N') = p \restriction M$.

(5) For types $p$ and $q$, we write $p \leq q$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and there exists $\bar{a}$ realizing $p$ in some $N$ extending $\mathrm{dom}(p)$ such that $(\bar{a}, \mathrm{dom}(p), N) = q \restriction \mathrm{dom}(p)$.

An important notion in this paper is that of an *amalgamation base*. A model is an amalgamation base iff every pair of models extending it of the same cardinality can be amalgamated over it. Sometimes we will be interested to consider amalgamation bases some special sets which are not models. Please note that the assumption that every subset of a model (from $\mathcal{K}$) is an amalgamation basis a very strong assumption. Making this assumption brings us to the very special context of AECs called *homogeneous model theory*, see [GrLe] for an introduction. Since there are many interesting examples of AECs with amalgamation over models (but not over all sets) like in Zilber's theory of pseudo exponentiation we do not make the assumption that all sets are amalgamation bases. Our interest is limited for very special sets that originate from certain systems of models we describe now.

**Definition 0.13.** Let $I$ be a subset of $\mathcal{P}(n)$ for some $n < \omega$ that is downward closed (i.e. $t \in I$ and $s \subseteq t$ implies $s \in I$).

For an $\mathbf{S} = \langle M_s \in \mathcal{K} \mid s \in I \rangle$ is an *$I$-system* iff for all $s, t \in I$

(1) $s \subseteq t \implies M_s \prec_{\mathcal{K}} M_t$ and
(2) $M_{s \cap t} = M_s \cap M_t$

$\mathbf{S}$ is a *$(\lambda, I)$-system* iff in addition all the models are of cardinality $\lambda$.

Denote by

$$A^{\mathbf{S}}_t := \bigcup_{s \subsetneq t} M_s \text{ and } A^{\mathbf{S}}_I := \bigcup_{s \in I} M_s$$

Occasionally, we use the notation $M^{\mathbf{S}}_t$ for the models $M_t$ in the system $\mathbf{S}$.

**Definition 0.14.** Suppose $\mathbf{S} = \langle M_s \in \mathcal{K}_\mu \mid s \in I \rangle$ is an *$I$-system* for some $I \subseteq \mathcal{P}(n)$ We say that *a set $A^{\mathbf{S}}_I$ is a $\mu$-amalgamation base* iff for all $M_\ell \in \mathcal{K}_\mu$ (for $\ell = 1, 2$) such that $M_s \prec_{\mathcal{K}} M_\ell$ (for all $s \in I$ and $\ell = 1, 2$) there exists $N^* \in \mathcal{K}_\mu$ such that $N^* \succ_{\mathcal{K}} M_2$ and there is a $\mathcal{K}$-embedding $f : M_1 \to N^*$ satisfying $f \restriction A^{\mathbf{S}}_I = \mathrm{id}_{A^{\mathbf{S}}_I}$, i.e. the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & N^* \\ {\scriptstyle \mathrm{id}_{A^{\mathbf{S}}_I}} \uparrow & & \uparrow {\scriptstyle \mathrm{id}_{M_2}} \\ A^{\mathbf{S}}_I & \xrightarrow[\mathrm{id}_{A^{\mathbf{S}}_I}]{} & M_2 \end{array}$$

**Notation 0.15.** Denote by $\mathrm{Ab}_\mu(\mathcal{K})$ the class

$$\{A^{\mathbf{S}}_I \mid A^{\mathbf{S}}_I \text{ is a } \mu\text{-amalgamation base for some } I\text{-system from } \mathcal{K}_\mu\}.$$

Thus $\mathcal{K}$ has the $\lambda$-amalgamation property iff $\mathcal{K}_\lambda \subseteq \mathrm{Ab}_\lambda(\mathcal{K})$. Under the assumption that $\mathcal{K}_\mu$ has the AP the notion of a Galois-type can be extended to include also ga-tp$(\bar{a}/A, M)$ for $A \in \mathrm{Ab}_\mu(\mathcal{K})$.

**Definition 0.16.** Let $\mathcal{K}$ be an AEC with the amalgamation property and let $\chi \geq \text{LS}(\mathcal{K})$. The class $\mathcal{K}$ is called $\chi$-*tame* iff

$$p \neq q \implies \exists N \prec_{\mathcal{K}} M \text{ of cardinality } \leq \chi \text{ such that } p \restriction N \neq q \restriction N$$

for any $M \in \mathcal{K}_{>\chi}$ and every $p, q \in \text{ga-S}(M)$

$\mathcal{K}$ is *tame* iff it is $\chi$-tame for some $\chi < \text{Hanf}(\mathcal{K})$

Suppose $\mu > \chi$. The class is $(\chi, \mu)$-*tame* iff

$$p \neq q \implies \exists N \prec_{\mathcal{K}} M \text{ of cardinality } \leq \chi \text{ such that } p \restriction N \neq q \restriction N$$

for any $M \in \mathcal{K}_{\mu}$ and every $p, q \in \text{ga-S}(M)$

In [GrV1] Grossberg and VanDieren introduced the notion of *tameness* as a candidate for a further "reasonable" assumption an AEC that permits development of stability-like theory. It turns out that essentially the same property was introduced earlier by Shelah implicitly in the proof of his main theorem in [Sh 394].

One of the better approximations to Shelah's categoricity conjecture for AECs can be derived from a theorem due to Makkai and Shelah ([MaSh]):

**Theorem 0.17** (Makkai and Shelah 1990)**.** *Let $\mathcal{K}$ be an AEC, $\kappa$ a strongly compact cardinal such that* $\text{LS}(\mathcal{K}) < \kappa$. *Let $\mu_0 := \beth_{(2^{\kappa})^+}$. If $\mathcal{K}$ is categorical in some $\lambda^+ > \mu_0$ then $\mathcal{K}$ is categorical in every $\mu \geq \mu_0$.*

Proposition 1.13 of [MaSh] asserts (using the assumption that $\kappa$ is strongly compact) that any AEC $\mathcal{K}$ as above has the AP (for models of cardinality $\geq \kappa$). Since Galois types in this context are sets of $L_{\kappa,\kappa}$ formulas the class is trivially $\kappa$-tame.

In [GrV2] Grossberg and VanDieren proved (in ZFC) a case of Shelah's categoricity conjecture for tame AECs with the amalgamation property which implies the above theorem of Makkai and Shelah. Thus the tameness assumption enables upward categoricity argument (instead of the large cardinal assumption). This is also an extension (upward) of Shelah's main theorem from [Sh 394].

**Theorem 0.18** (Grossberg and VanDieren 2003)**.** *Let $\mathcal{K}$ be an AEC, $\kappa := \beth_{(2^{\text{LS}(\mathcal{K})})^+}$. Denote by $\mu_0 := \beth_{(2^{\kappa})^+}$. Suppose that $\mathcal{K}_{>\kappa}$ has the amalgamation property and is $\kappa$-tame. If $\mathcal{K}$ is categorical in some $\lambda^+ > \mu_0$ then $\mathcal{K}$ is categorical in every $\mu \geq \mu_0$.*

Later Lessmann obtained finer upward categoricity results by using much stronger assumptions to tameness ($\aleph_0$-tameness and $\text{LS}(\mathcal{K}) = \aleph_0$) and existence of arbitrary large models.

In [Sh 394] Shelah proved that for an AEC with the amalgamation property. If $\mathcal{K}$ is $\lambda$-categorical for some $\lambda > \beth_{(2^{\text{Hanf}(\mathcal{K})})^+}$ then it is $(\text{Hanf}(\mathcal{K}), \mu)$-tame for all $\text{Hanf}(\mathcal{K}) < \mu < \lambda$.

Throughout this paper we will be using Shelah's presentation theorem for AECs which states that every AEC can be viewed as a PC$\Gamma$-class (see

[Sh 88] or [Gr3]). We state it in a form that is more convenient for our purposes.

**Lemma 0.19.** *Let $\mathcal{K}$ be an AEC, let $\mu = \mathrm{LS}(\mathcal{K})$. Let $\chi_0$ e a large regular cardinal. There are $\mu$ functions $\{f_i \mid i < \mu\}$ such that whenever $M \in \mathcal{K}$, $M \subset H(\chi_0)$, and $\mathcal{B} \prec \langle H(\chi_0), \in, \mathcal{K}, M, \{f_i \mid i < \mu\}\rangle$, $\|\mathcal{B}\| \geq \mu$, for $N = M^{\mathcal{B}}$ we have $N \in \mathcal{K}$ and $N \prec_{\mathcal{K}} M$.*

This is simply saying that Skolem functions can be defined in an appropriate set-theoretic universe and whenever a subset $N$ of a model $M \in \mathcal{K}$ is closed under those functions, $N$ is a $\mathcal{K}$-model.

## 1. The basic framework and concepts

Shelah in [Sh 600] introduced the axiomatic framework for the notion of *good frame*; his goal was to axiomatize superstability. Below we offer a much simpler (and more general) axiomatic setting we call *weak forking* that in the first-order case corresponds to simplicity.

**Definition 1.1.** A pair $\langle \mathcal{K}, \mathop{\downarrow}\rangle$ is a *weak forking notion* iff $\mathcal{K}$ is an AEC and $\mathop{\downarrow}$ is a four-place relation called *non-forking* $A \overset{N}{\underset{C}{\downarrow}} B$ for $C \subset A, B \subset N$, $A, B, C \in \mathrm{Ab}(\mathcal{K})$ such that $\mathop{\downarrow}$ satisfies

(1) *Invariance*: If $f : N \to N'$ is a $\mathcal{K}$-embedding, then $A \overset{N}{\underset{C}{\downarrow}} B$ if and only if $f(A) \overset{f(N')}{\underset{f(C)}{\downarrow}} f(B)$.

(2) *Monotonicity:* If $C \subset C' \subset B' \subset B$ and $N \prec N'$, then

$$A \overset{N}{\underset{C}{\downarrow}} B \text{ if and only if } A \overset{N'}{\underset{C'}{\downarrow}} B'.$$

(3) *Disjointness:*

$$A \overset{N}{\underset{C}{\downarrow}} B \implies A \cap B \subseteq C.$$

(4) *Extension of independence:* If $M_0 \prec M$ and $N_0 \succeq M_0$, then there is a model $N \in \mathcal{K}$, $M \prec N$, and $f : N_0 \to N$ such that $f \restriction M_0 = \mathrm{id}_{M_0}$ and $M \overset{N}{\underset{M_0}{\downarrow}} f(N_0)$.

(5) *Continuity:* If $\delta$ is a limit ordinal, $\{N_i \mid i < \delta\}$ is an increasing continuous chain, and $M \overset{N}{\underset{M_0}{\downarrow}} N_i$ for $i < \delta$, then $M \underset{M_0}{\downarrow} N_\delta$.

(6) *Symmetry:* if $M \overset{N}{\underset{M_0}{\downarrow}} N_0$, then $N_0 \overset{N}{\underset{M_0}{\downarrow}} M$.

(7) *Transitivity:* if $M \overset{N}{\underset{M_0}{\downarrow}} M_1$ and $M \overset{N}{\underset{M_1}{\downarrow}} M_2$, then $M \overset{N}{\underset{M_0}{\downarrow}} M_2$.

(8) *Local character:* There is a cardinal $\kappa = \kappa(\mathcal{K})$ such that for any amalgamation base $A' \subset A \cup B$ there is an amalgamation base $B' \subset B$, $|B'| = \kappa + |A'|$, with $A' \underset{B'}{\overset{N}{\downarrow}} B$.

(9) *Definability:* There is a family $\mathcal{F}$ of $\kappa(\mathcal{K})$ functions, definable set-theoretically, such that if $A' \subset A \cup B$ is closed under $\mathcal{F}$, then $A' \underset{B \cap A'}{\overset{N}{\downarrow}} B$.

**Remark 1.2.** Axiom 9 is a very mild strengthening of the local character axiom. It hides a brute force construction similar to the one in Lemma 0.19 and possible in the known examples. Suppose local character holds, and that dependence relation makes sense for all sets. Fix a well-ordering of the universe of $N \in \mathcal{K}$. For $\mathbf{a} \in A$, $\ell(\mathbf{a}) = n$, define $\{f^n_i(\mathbf{a}) \mid i < \kappa(\mathcal{K})\}$ to be an enumeration of the set $B' \subset N$ such that $\mathbf{a} \underset{B'}{\overset{N}{\downarrow}} N$. Letting $\mathcal{F} := \bigcup\{f^n_i \mid i < \kappa, n < \omega\}$, we get the desired family.

The property stated in Axiom 9 was extracted from Section 4 of Shelah's [Sh 87b].

Of course, the local character property follows from definability of independence.

Axiom 9 and transitivity immediately give the following useful version of the definability property.

**Claim 1.3.** *There is a family $\mathcal{F}$ of $\kappa(\mathcal{K})$ functions, definable set-theoretically, such that if $A \supset C$, $A \underset{C}{\overset{N}{\downarrow}} B$, and $A' \subset A$ is closed under $\mathcal{F}$, then $A' \underset{C \cap A'}{\overset{N}{\downarrow}} B$.*

**Remark 1.4.** While we assume that the independence relation $\downarrow$ is defined over amalgamation bases, it is enough, for our purposes, to demand that the main properties of independence such as symmetry, transitivity, and extension holds only over models.

The extension property for the class follows from the amalgamation assumptions we are making on the class, see Section 2.

**Remark 1.5.** To see that Shelah's notion of good frame is much more stronger than our, imagine that $\mathcal{K} = \mathrm{Mod}(T)$ when $T$ is a complete first-order theory and $\downarrow$ is the usual first-order forking. $\mathcal{K}$ is a good frame iff $T$ is superstable, while $\langle \mathcal{K}, \downarrow \rangle$ is a weak forking notion iff $T$ is simple.

In the formulation of extension property, if $M_0 = N_0$, we obtain existence property of independence. Let us state a form of the extension of independence property that will be useful later:

**Lemma 1.6.** *If* $M \overset{N}{\underset{M_0}{\downarrow}} N_0$ *and* $\bar{N}_0 \succ N_0$*, then there is a model* $\bar{N} \in \mathcal{K}$*,* $N \prec \bar{N}$*, and* $f : \bar{N}_0 \to \bar{N}$ *such that* $f \restriction N_0 = \mathrm{id}_{N_0}$ *and* $M \overset{\bar{N}}{\underset{M_0}{\downarrow}} f(\bar{N}_0)$.

*Proof.* Applying extension of independence to $N$, $N_0$, and $\bar{N}_0$, we get a model $\bar{N} \succ N$ and $f : \bar{N}_0 \to \bar{N}$, identity over $N_0$, such that $N \overset{\bar{N}}{\underset{N_0}{\downarrow}} f(\bar{N}_0)$. Using symmetry and monotonicity we get $M \overset{\bar{N}}{\underset{N_0}{\downarrow}} f(\bar{N}_0)$, and now symmetry and transitivity give $M \overset{\bar{N}}{\underset{M_0}{\downarrow}} f(\bar{N}_0)$. ⊣

**Examples 1.7.**

(1) Let $\mathcal{K} := \mathrm{Mod}(T)$ when $T$ is a first-order complete theory, $\prec_{\mathcal{K}}$ is the usual elementary submodel relation and $\downarrow$ is the non-forking relation. Clearly $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is a weak forking notion iff $T$ is simple. $\kappa$ in this case is $\kappa(T)$.

It is not difficult to see that $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is a weak forking notion with $\kappa = \aleph_0$ iff $T$ is super-simple.

(2) Let $T$ be a countable first-order theory, and let

$\mathcal{K}^a := \{M \models T \mid \mathrm{tp}(\mathbf{a}/\emptyset, M)$ is an isolated type for every $\mathbf{a} \in |M|\}$.

A type $p \in S(A)$ is called *atomic* iff $A \cup \{\mathbf{a}\}$ is atomic subset of $\mathfrak{C}$ and $\mathbf{a} \models p$.

Suppose that $T$ is $\aleph_0$-*atomically stable*, i.e. for $\mathrm{R}[p] < \infty$ for every atomic type, where

**Definition 1.8.** For $M \in \mathcal{K}^a$ and $\mathbf{a} \in M$ define by induction of $\alpha$ when $\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a})] \geq \alpha$

$\alpha = 0$; $M \models \exists \mathbf{x} \varphi(\mathbf{x}; \mathbf{a})$

For $\alpha = \beta + 1$;

There are $\mathbf{b} \supseteq \mathbf{a}$ and $\psi(\mathbf{x}; \mathbf{b})$ such that

$$\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b})] \geq \beta$$

$$\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a}) \wedge \neg\psi(\mathbf{x}; \mathbf{b})] \geq \beta \quad \text{and for every } \mathbf{c} \supseteq \mathbf{a}$$

there is $\chi(\mathbf{x}; \mathbf{c})$ complete s.t.

$$\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a}) \wedge \chi(\mathbf{x}; \mathbf{c})] \geq \beta$$

An atomic set $A \subseteq \mathfrak{C}$ is *good* iff for every consistent $\varphi(\mathbf{x}; \mathbf{a})$ (with $\mathbf{a} \in A$) there is an isolated type $p \in S(A)$ containing $\varphi(x; \mathbf{a})$. In the atomic case the countable good sets are amalgamation bases (compare with Definition 0.14). This follows from:

**Fact 1.9** ([Sh 87a])**.** *Suppose $A$ is countable. Then $A$ is good if and only if there is a universal model over $A$.*

Suppose $A \cup B \cup C$ are inside $N \in \mathcal{K}^a$ and $C$ is good. We let $A \underset{C}{\downarrow} B$ if for each $\mathbf{a} \in A$, $\text{tp}(\mathbf{a}/B)$ does not split over some finite subset of $C$. Then $\langle \mathcal{K}^a, \underset{C}{\downarrow} \rangle$ is a weak forking notion.

(3) Let $\mathcal{K}$ be the class of elementary submodels of a totally transcendental sequentially homogeneous model. Let $M_1 \underset{M_0}{\downarrow} M_2$ stand for $\text{tp}(\mathbf{a}/M_2)$ does not strongly-split over $M_0$ for every $\mathbf{a} \in |M_1|$.

Then $\langle \mathcal{K}, \underset{M_0}{\downarrow} \rangle$ is a weak forking notion.

Compare the following with XII.2 of [Sh c].

**Definition 1.10** (Stable systems)**.** Let $\langle \mathcal{K}, \underset{M_0}{\downarrow} \rangle$ be weak forking notion. Suppose $I \subseteq \mathcal{P}^-(n)$, suppose $\mathbf{S} = \{M_s \mid s \in I \cup \{n\}\}$ is a $(\lambda, n)$-system. The system $\mathbf{S}$ is called $(\lambda, I)$-*stable in* $M_n^{\mathbf{S}}$ if and only if

(1) $A_s^{\mathbf{S}}$ is an amalgamation base for all $s \in I$,

(2) for all $s \in I$, for all $t \subseteq s$

$$M_t^{\mathbf{S}} \underset{A_t^{\mathbf{S}}}{\overset{M_n^{\mathbf{S}}}{\downarrow}} \bigcup_{\substack{w \subset s \\ w \not\supseteq t}} |M_w^{\mathbf{S}}|.$$

We make one more assumption on the $\langle \mathcal{K}, \underset{M_0}{\downarrow} \rangle$.

**Axiom 1.11** (Generalized Symmetry)**.** *Let $\langle \mathcal{K}, \underset{M_0}{\downarrow} \rangle$ be weak forking notion. We say that $\langle \mathcal{K}, \underset{M_0}{\downarrow} \rangle$ has the* $(\lambda, n)$-symmetry property *if a system $\mathbf{S} = \{M_s \mid s \in \mathcal{P}(n)\}$, $\mathbf{S} \subset \mathcal{K}_\lambda$, is stable inside $M_n$ whenever there exists an enumeration $\bar{s} := \langle s(i) \mid i < 2^n - 1 \rangle$ of $\mathcal{P}^-(n)$ (always without repetitions such that $s(i_1) \subset s(i_2) \implies i_1 < i_2$) such that*

(1) *$A_{s(i)}^{\mathbf{S}}$ is an amalgamation base for all $i$;*

(2)

$$M_{s(j)}^{\mathbf{S}} \underset{A_{s(j)}^{\mathbf{S}}}{\overset{M_n^{\mathbf{S}}}{\downarrow}} \bigcup_{i<j} |M_{s(i)}^{\mathbf{S}}|.$$

In other words, under the generalized symmetry to get stability of the $\mathcal{P}^-(n)$-system it is enough to check the independence of just one "face" from the rest of the $n$-dimensional cube, not all the faces as in the Definition 1.10.

We now state the generalized amalgamation properties, we omit the superscripts $\mathbf{S}$ when the identity of the system is clear.

**Definition 1.12** ($n$-existence)**.** Let $\langle \mathcal{K}, \underset{M_0}{\downarrow} \rangle$ be weak forking notion. $\mathcal{K}$ has the $(\lambda, n)$-*existence property* iff for every $(\lambda, \mathcal{P}^-(n))$-system $\mathbf{S} = \langle M_s \mid s \in \mathcal{P}^-(n) \rangle$ such that $\{M_t \mid t \subseteq s\}$ is a stable $(\lambda, |s|)$-system for all $s \in \mathcal{P}^-(n)$, there exists a model $M_n$ and $\mathcal{K}$-embeddings $\{f_s \mid s \in \mathcal{P}^-(n)\}$ such that

(1) $\{f_s(M_s) \mid s \in \mathcal{P}^-(n)\} \cup \{M_n\}$ is a stable system indexed by $\mathcal{P}(n)$.

(2) the embeddings $f_s$ are coherent: $f_t \restriction M_s = f_s$ for $s \subset t \in \mathcal{P}^-(n)$.

**Remark 1.13.** Let us clarify what is going on in the case $n = 3$. We are given the models $M_\emptyset$, $\{M_i \mid i < 3\}$ and $\{M_{ij} \mid i < j < 3\}$ Such that $M_i \underset{M_\emptyset}{\overset{M_{ij}}{\downarrow}} M_j$ for all $i < j < 3$.

The 3-existence property asserts that the three models can be embedded into $M_{012}$ in a coherent way so that the images form a stable system inside $M_{012}$. Note that this fails even in the first order case.

Failure of $(\aleph_0, 3)$-existence is witnessed by the example of a triangle-free random graph. Start with a triple of models $M_i$, $i < 3$ extending some $M_\emptyset$, and fix some elements $a_i \in M_i$. Choose models $M_{01}$, $M_{02}$, and $M_{12}$ so that $M_i \underset{M_\emptyset}{\overset{M_{ij}}{\downarrow}} M_j$ for all $i < j < 3$, and such that $M_{ij} \models R(a_i, a_j)$ for $i < j < 3$. The system cannot be completed since the model $M_{012}$ would witness a triangle.

This is an example of a non-simple first order theory. It can be generalized to a failure of $(\aleph_0, n+1)$-amalgamation by using $n$-dimensional tetrahedron-free graphs. Those examples are simple first order theories.

**Definition 1.14** (weak $n$-uniqueness)**.** Let $\mathbf{S} = \{M_s \mid s \in \mathcal{P}^-(n)\}$, $\mathbf{S}' = \{M'_s \mid s \in \mathcal{P}^-(n)\}$ be stable systems of models in $\mathcal{K}$, where without loss of generality we assume $M_\emptyset = M'_\emptyset$. We say that $\mathbf{S}$ and $\mathbf{S}'$ are *piecewise isomorphic* if there are $\{f_s : M_s \cong M'_s \mid s \in \mathcal{P}^-(n)\}$, where $f_\emptyset = \mathrm{id}_{M_\emptyset}$ and $f_t \restriction M_s = f_s$ for $s \subset t$.

Let $\langle \mathcal{K}, \mathop{\downarrow} \rangle$ be weak forking notion. We say $\mathcal{K}$ has the *weak* $(\lambda, n)$-*uniqueness property* if the following holds. For any two $(\lambda, n)$-stable systems $\mathbf{S} = \langle M_s \mid s \in \mathcal{P}(n) \rangle$ and $\mathbf{S}' = \langle M'_s \mid s \in \mathcal{P}(n) \rangle$ such that $S \setminus \{M_n\}$ and $\mathbf{S}' \setminus \{M'_n\}$ are piecewise isomorphic there are $M^* \in \mathcal{K}_\lambda$ and $\mathcal{K}$-embeddings $g : M_n \to M^*$ and $g' : M'_n \to M^*$ such that $g(M_s) = g'(f_s(M_s))$ for all $s \in \mathcal{P}^-(n)$.

**Remark 1.15.** In [Sh 87b], Shelah states a variant of weak $(\lambda, n)$-uniqueness property. Shelah calls the property *failure of* $(\lambda, n)$*-non-uniqueness*, it is stated in item (2) of Proposition 1.16. We show that weak $(\lambda, n)$-uniqueness condition is equivalent to the failure of $(\lambda, n)$-non-uniqueness.

**Proposition 1.16.** *Let $\langle \mathcal{K}, \mathop{\downarrow} \rangle$ be weak forking notion. Then the following are equivalent:*

1. *$\mathcal{K}$ has the weak $(\lambda, n)$-uniqueness property;*
2. *for every stable system $\mathbf{S} = \langle M_s \mid s \in \mathcal{P}^-(n) \rangle \subseteq \mathcal{K}_\lambda$ inside some $M_n$ we have that $A^{\mathbf{S}}_n \in \mathrm{Ab}_\lambda(\mathcal{K})$.*

*Proof.* If the weak uniqueness holds, then clearly the set $A_n$ is an amalgamation base; we can take the identity isomorphisms as the "piecewise" embeddings.

Now the converse. Let $\mathbf{S}^1$, $\mathbf{S}^2$ be piecewise isomorphic stable systems indexed by $\mathcal{P}^-(n)$, inside $M^1_n$ and $M^2_n$ respectively. To show the weak uniqueness, it is enough to construct a model $N^2_n$ and $g : M^1_n \cong N^2_n$ such that

$g \supset f_s$, $s \in \mathcal{P}^-(n)$ (it is enough to consider only the $(n-1)$-element subsets $s$). Indeed, by invariance the system $\mathbf{S}^2$ is stable inside $N_n^2$; by (2) then there are $M^*$ and $h_M : M_n^2 \to M^*$, $h_N : N_n^2 \to M^*$ over $\mathbf{S}^2$. Then $h_N \circ g : M_n^1 \to M^*$ and $h_M : M_n^2 \to M^*$ are the needed embeddings.

The construction of $N_n^2$ and $g$ is a slight generalization of the construction in the proof of Fact 0.8. As the universe of $N_n^2$ we take the following set:

$$|N_n^2| := \bigcup_{s \in \mathcal{P}^-(n)} |M_s^2| \cup \left( |M_n^1| \setminus \bigcup_{s \in \mathcal{P}^-(n)} |M_s^1| \right).$$

Define the structure on the $|N_n^2|$ by copying it from the structure $M_n^1$. Take a tuple $\mathbf{a} \in |N_n^2|$, it can be uniquely presented as $\mathbf{a} = \cup_{s \in \mathcal{P}^-(n)} f_s(\mathbf{a}_s) \cup \mathbf{b}$, where $\mathbf{a}_s \in M_s^1$ and $\mathbf{b} \in |M_n^1| \setminus \bigcup_{s \in \mathcal{P}^-(n)} |M_s^1|$. For a relation $R \in L(\mathcal{K})$ define

$$N_n^2 \models R(\mathbf{a}) \text{ if and only if } M_n^1 \models R(\cup_{s \in \mathcal{P}^-(n)}(\mathbf{a}_s) \cup \mathbf{b}).$$

By construction $M_n^1$ is isomorphic to $N_n^2$. $\dashv$

**Definition 1.17** (goodness)**.** Let $\langle \mathcal{K}, \mathop{\smile\!\!\!\!\!\!|} \rangle$ be weak forking notion, it has the $(\lambda, n)$-*goodness property* iff $\langle \mathcal{K}, \mathop{\smile\!\!\!\!\!\!|} \rangle$ has the $(\lambda, n)$-symmetry property and has the $(\lambda, n)$-existence property and the weak $(\lambda, n)$-uniqueness property.

**Theorem 1.18** (characterizing goodness for f.o.)**.** *Let $T$ be a complete countable first order theory. Suppose $T$ is superstable without dop If $\mathbf{S} = \langle M_s \mid s \in \mathcal{P}^-(n) \rangle$ is a stable system of models of cardinality $\aleph_0$ then the following are equivalent:*

1. *the set $A_n^{\mathbf{S}}$ is an amalgamation base*
2. *There is a prime and minimal model over $A_n^{\mathbf{S}}$.*

**Definition 1.19** (excellence)**.** Let $\langle \mathcal{K}, \mathop{\smile\!\!\!\!\!\!|} \rangle$ be weak forking notion and let $\lambda \geq \mathrm{LS}(\mathcal{K})$. $\langle \mathcal{K}, \mathop{\smile\!\!\!\!\!\!|} \rangle$ is $\lambda$-*excellent* iff $\langle \mathcal{K}, \mathop{\smile\!\!\!\!\!\!|} \rangle$ has the $(\lambda, n)$-goodness property for every $n < \omega$. When $\lambda = \mathrm{LS}(\mathcal{K})$ we say that $\mathcal{K}$ excellent instead of $\lambda$-excellent.

**Theorem 1.20** (Shelah 1982)**.** *Let $T$ be a complete countable first order theory. Suppose $T$ is superstable without DOP. Then the following are equivalent:*

1. *$\langle \mathrm{Mod}(T), \prec \rangle$ is excellent.*
2. *$\mathrm{Mod}(T)$ has the $(\aleph_0, 2)$-goodness property.*
3. *$T$ does not have the OTOP.*

For proof see [Sh c].

**Fact 1.21** (Hart and Shelah 1986)**.** *For every $n < \omega$ there is an $\aleph_0$-atomically stable class $\mathcal{K}_n$ of atomic models of a countable f.o. theory such that $\mathcal{K}$ is has the $(\aleph_0, k)$-goodness property for all $k < n$ but is not excellent.*

In section 3 we will prove that the existential quantifier in the definition of excellent class can be replaced with a universal quantifier:

**Theorem 1.22.** *If $\langle \mathcal{K}, \downarrow \rangle$ is excellent then it has the $(\lambda, n)$-goodness property for every $n < \omega$ and every $\lambda \geq \mathrm{LS}(\mathcal{K})$.*

*Proof.* Immediate from Theorems 3.1 and 3.2. ⊣

## 2. A sufficient condition for Tameness

We start by explaining the main idea for obtaining $(\lambda, \lambda^+)$-tameness from weak $(\lambda, 2)$-uniqueness and $(\lambda, 2)$-existence. We outline the general construction and the induction step by a picture and later give a completely formal argument.

Suppose $(a_1, M, N^1) \in p$ and $(a_2, M, N^2) \in q$ and their restriction on small submodels of $M$ are equal. Pick $\{N^\ell_\alpha \prec_{\mathcal{K}} N_\ell \mid \alpha < \lambda\} \subseteq \mathcal{K}_{<\lambda}$ increasing and continuous resolutions of $N_\ell$ and $\{M^\alpha \prec_{\mathcal{K}} M \mid \alpha < \lambda\} \subseteq \mathcal{K}_{<\lambda}$ increasing resolution of $M$ such that $M_\alpha \prec N^\ell_\alpha$, require that $a_\ell \in N^\ell_0$.

By the assumption there exist $N^*_0 \succ_{\mathcal{K}} N^2_0$ of cardinality less than $\lambda$ amalgam of $N^1_0$ and $N^2_0$ over $M_0$ mapping $a_1$ to $a_2$.

Our goal is to find models $\bar{N}^\ell \succ_{\mathcal{K}} N^\ell$ and $\bar{f}_\lambda : \bar{N}^1 \cong \bar{N}^2$ such that $\bar{f}_\lambda(a_1) = a_2$.

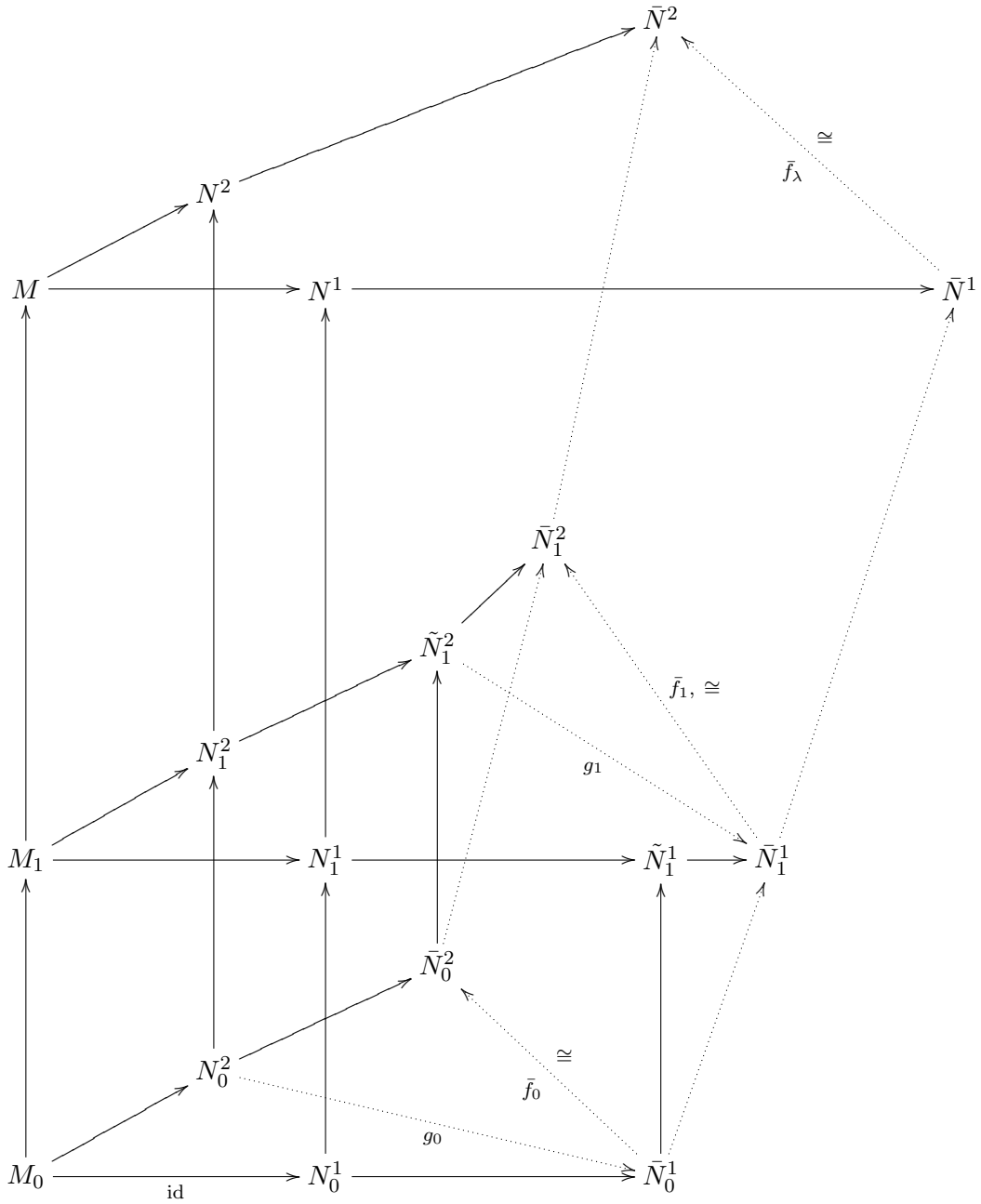
The construction of the models and the mapping will be by induction on $i < \lambda$ such that the following diagram commutes.

For $i = 0$; let $\bar{N}^1_0 \succ_{\mathcal{K}} N^1_0$ be an amalgam of $N^2_0$ and $N^1_0$ over $M_0$ such that $g_0 : N^2_0 \to \bar{N}^1_0$, $g_0 \restriction M_0 = \mathrm{id}_{M_0}$ and $g_0(a_2) = a_1$. By Fact 0.8 there are $\bar{N}^2_0 \succ_{\mathcal{K}} N^2_0$ and $\bar{f}_0 : \bar{N}^1_0 \cong \bar{N}^2_0$ such that $\bar{f}_0 \supseteq g_0^{-1}$. Using a strong form of the extension property (see Lemma 2.7 below) after renaming $\bar{N}^\ell_0$ we may assume that there exists $\hat{N}^\ell_0 \succ_{\mathcal{K}} \bar{N}^\ell_0$ of cardinality $\lambda$ such that $M \underset{M_0}{\overset{\hat{N}^\ell_0}{\downarrow}} \bar{N}^\ell_0$. Using Lemma 2.7 once more we find $\check{N}^\ell_1 \in \mathcal{K}_\lambda$ such that $\hat{N}^\ell_0 \underset{M_0}{\overset{\check{N}^\ell_1}{\downarrow}} N^\ell_1$. Since $\check{N}^\ell_0 \succ \hat{N}^\ell_0$, by monotonicity

$$(*) \qquad M \underset{M_0}{\overset{\check{N}^\ell_0}{\downarrow}} \bar{N}^\ell_0.$$

Now take $\tilde{N}^\ell_1 \prec \check{N}^\ell_1$ of cardinality $\lambda$ such that it contains $|\bar{N}^\ell_0| \cup |N^\ell_1|$.

Monotonicity applied to $(*)$ gives that $M_1 \underset{M_0}{\overset{\tilde{N}^\ell_1}{\downarrow}} \bar{N}^\ell_0$ holds for $\ell = 1, 2$. An application of the weak 2-uniqueness property produces a model $\bar{N}^1_1 \succ \tilde{N}^1_1$ and $g_1 : \tilde{N}^2_1 \to \bar{N}^1_1$ such that $g_1 \supseteq \bar{f}_0^{-1}$ and $g_1 \restriction M_1 = \mathrm{id}_{|M_1|}$. Now using Fact 0.8 there are $\bar{N}^2_1 \succ_{\mathcal{K}} \tilde{N}^2_1$ and $\bar{f}_1 : \bar{N}^2_1 \cong \bar{N}^1_1$ such that $\bar{f}_1 \supseteq g_1^{-1}$.


$\bar{N}^2$
$\cong$
$\bar{f}_\lambda$
$N^2$
$M$
$N^1$
$\bar{N}^1$
$\bar{N}_1^2$
$\tilde{N}_1^2$
$\bar{f}_1, \cong$
$N_1^2$
$g_1$
$M_1$
$N_1^1$
$\tilde{N}_1^1$
$\bar{N}_1^1$
$\bar{N}_0^2$
$\cong$
$N_0^2$
$\bar{f}_0$
$g_0$
$M_0$
id
$N_0^1$
$\bar{N}_0^1$

For the rest of this section we deal with $(\lambda, 2)$-existence and weak uniqueness properties. We make it explicit that $(\lambda, 2)$-existence is simply the $\lambda$-extension property for independence (see Definition 2.3); and weak $(\lambda, 2)$-uniqueness corresponds to the first-order stationarity. This makes transparent the argument showing, for example, $(\lambda^+, \lambda)$-tameness from $(\lambda, 2)$-existence and weak uniqueness. The first-order relativization of the proof goes along these lines: let $p$, $q$ be types over $M$ of size $\lambda^+$ that agree over all $\lambda$-submodels of $M$. With $\lambda \geq \kappa(\mathcal{K})$, by local character we can find $M_0 \prec M$, $\|M_0\| = \lambda$, such that $p$, $q$ do not fork over $M_0$. By assumption $p \restriction M_0 = q \restriction M_0$, so stationarity gives $p = q$. Of course this outline avoids several important issues; for example, we assume "stationarity" only in $\lambda$, and we used $\lambda^+$-stationarity in the argument above.

Let us restate the definitions of 2-existence and weak uniqueness here.

**Definition 2.1.** We say that $\mathcal{K}$ has $(\lambda, 2)$-*existence property* if for any triple of models $M_0 \prec M_1, M_2$, all in $\mathcal{K}_\lambda$, there is a model $M^* \in \mathcal{K}_\lambda$ and $\mathcal{K}$-embeddings $f_\ell : M_i \to M^*$, $i\ell = 1, 2$, such that $f_1 \restriction M_0 = f_2 \restriction M_0$ and $f_1(M_1) \underset{f_1(M_0)}{\overset{M^*}{\downarrow}} f_2(M_2)$.

**Remark 2.2.** Equivalently, $(\lambda, 2)$-existence property holds if there is $M^* \succ M_1$ and a map $f : M_2 \to M^*$ over $M_0$ such that $M_1 \underset{M_0}{\overset{M^*}{\downarrow}} f(M_2)$. So $(\lambda, 2)$-existence is really the $\lambda$-extension property:

**Definition 2.3.** If in the extension for independence property all the models are in $\mathcal{K}_\lambda$, we say that $\mathcal{K}$ has the $\lambda$-*extension property for independence*.

**Definition 2.4.** The class $\mathcal{K}$ has *weak* $(\lambda, 2)$-*uniqueness property* if for any two $\lambda$-systems $\mathbf{S}^\ell = \{M_0^\ell, M_1^\ell, M_2^\ell, M_3^\ell\}$, $\ell = 0, 1$, that are stable (i.e., $M_1^\ell \underset{M_0^\ell}{\overset{M_3^\ell}{\downarrow}} M_2^\ell$) and piecewise isomorphic (i.e., there are $f_i : M_i^0 \cong M_i^1$ for $i = 0, 1, 2$ with $f_1 \restriction M_0 = f_2 \restriction M_0$), let $f_0(M_0) := f_1(M_0)$ there is a model $M^* \in \mathcal{K}_\lambda$ and embeddings $g^\ell : M_3^\ell \to M^*$ such that $g^0(M_i) = g^1(f_i(M_i))$ for $i = 0, 1, 2$.

**Remark 2.5.** (1) Equivalently, in $(\lambda, 2)$-uniqueness we may demand that $M^* \succ M_2^1$, i.e., $f^1$ is the identity embedding.

(2) In the first-order case, weak 2-uniqueness says that $\mathrm{tp}(M_1 \cup M_2/M_0)$ is uniquely determined by $\mathrm{tp}(M_1/M_0) \cup \mathrm{tp}(M_2/M_0)$ as long as $M_1 \underset{M_0}{\downarrow} M_2$. So it really is the analog of stationarity.

(3) Using the isomorphism axioms in the definition of AEC, weak 2-uniqueness can be viewed as an amalgamation property for 2 isomorphisms.

(4) The property is called *weak* uniqueness since the property $M_1 \underset{M_0}{\overset{N}{\downarrow}} M_2$ (all models of cardinality $\lambda$) implies existence of $N' \prec N$ such that $|N'| \supseteq$

$|M_1| \cup |M_2|$ and $N'$ is prime and minimal over the set $|M_1| \cup |M_2|$. This stronger property occurs in two different situations: Superstable countable elementary classes without the OTOP (see Chapter XII in [Sh c]) and also in excellent classes of atomic models of a first-order theory (in the sense of [Sh 87b]).

(5) Compare Definition 2.4 with Definition 6.2 and Claim 6.7 from [Sh 576].

**Lemma 2.6.** *Suppose $(\lambda, 2)$-uniqueness holds. Let $\mathbf{S}^\ell$, $\ell = 1, 2$, be stable and piecewise isomorphic $\lambda$-systems, $\mathbf{S}^\ell = \{M_0^\ell, M_1^\ell, M_2^\ell, M_3^\ell\}$. Then there are $\mathcal{K}_\lambda$-models $N^\ell \succ M_3^\ell$, $\ell = 0, 1$, and $\bar{f} : N^1 \cong N^2$ that extends the isomorphisms $f_i : M_i^1 \to M_i^2$, $i = 0, 1, 2$.*

*Proof.* Let $N^2 \succ M_3^2$ and $f : M_3^1 \to N^2$ be as guaranteed by the weak $(\lambda, 2)$-uniqueness, in the sense of Remark 2.5(1). Using Fact 0.8, we get the needed $N^1 \succ M_3^1$ and the isomorphism $\bar{f}$ extending $f$, and therefore all the mappings $f_i$, $i = 0, 1, 2$. ⊣

**Lemma 2.7.** *Let $\lambda \geq \mathrm{LS}(\mathcal{K}) + \kappa(\mathcal{K})$, and suppose $\lambda$-extension property holds. Let $\|M\| = \lambda^+$, $M_0 \prec M, N_0$, where $M_0, N_0 \in \mathcal{K}_\lambda$. Then there is $N \succ M$ and $f : N_0 \to N$ over $M_0$ such that $M \underset{M_0}{\overset{N}{\downarrow}} f(N_0)$.*

In short, $\lambda$-extension implies extension when one of the models has size $\lambda^+$.

*Proof.* Let $\{M_i \mid i < \lambda^+\}$ be an increasing continuous chain of models with $\bigcup_{i<\lambda^+} M_i = M$, $\|M_i\| = \lambda$, and $M_0$ given in the statement of the lemma.

By induction on $i < \lambda^+$, we build models $N_i$, $\|N_i\| = \lambda$ and $\mathcal{K}$-embeddings $f_{ij} : N_i \to N_j$ such that:

(1) $N_i \succ M_i$ for all $i < \lambda^+$;
(2) $\{N_i, f_{ij}\}$ form a directed system;
(3) $f_{ij} \restriction M_i = \mathrm{id}_{M_i}$;
(4) $M_i \underset{M_0}{\overset{N_i}{\downarrow}} f_{0i}(N_0)$.

This is clearly sufficient: letting $N$ be the direct limit of $\{N_i, f_{ij} \mid i < \lambda^+\}$, we have $M \prec N$ by (1) and (3) and letting $f := f_{0\lambda^+}$ we have $M \underset{M_0}{\overset{N}{\downarrow}} f(N_0)$ and $f \restriction M_0 = \mathrm{id}_{M_0}$ by (3) and (4).

Now the construction: $N_0$ is given; having constructed $N_i$ and $f_{jk}$ for $j \leq k \leq i$ satisfying (1)–(4), build $N_{i+1}$ and $f_{j,i+1}$.

By $\lambda$-extension applied to $M_{i+1} \succ M_i$ and $N_i \succ M_i$, there is a model $N_{i+1} \succ M_{i+1}$ and embedding $f_{i,i+1} : N_i \to N_{i+1}$ such that $f_{i,i+1} \restriction M_i = \mathrm{id}_{M_i}$ and $M_{i+1} \underset{M_i}{\overset{N_{i+1}}{\downarrow}} f_{i,i+1}(N_i)$. For $j < i$ we define $f_{j,i+1} := f_{i,i+1} \circ f_{ji}$, and $f_{i+1,i+1}$ is the identity. Thus, we have met (1)–(3).

We prove that we have (4). Since $M_{i+1} \overset{N_{i+1}}{\underset{M_i}{\downarrow}} f_{i,i+1}(N_i)$ and $f_{0,i+1}(N_0) \prec f_{i,i+1}(N_i)$ by monotonicity we have $M_{i+1} \overset{N_{i+1}}{\underset{M_i}{\downarrow}} f_{0,i+1}(N_0)$. By induction hypothesis $M_i \overset{N_i}{\underset{M_0}{\downarrow}} f_{0i}(N_0)$, so by invariance (applying $f_{i,i+1}$) and monotonicity $M_i \overset{N_{i+1}}{\underset{M_0}{\downarrow}} f_{0,i+1}(N_0)$. Symmetry and transitivity now give the desired $M_{i+1} \overset{N_{i+1}}{\underset{M_0}{\downarrow}} f_{0,i+1}(N_0)$.

Suppose now that $i$ is a limit ordinal. Let $N_i$ be the direct limit of the system $\{N_j, f_{jk} \mid j \leq k < i\}$. By induction hypothesis $M_j \overset{N_j}{\underset{M_0}{\downarrow}} f_{0j}(N_0)$ for all $j < i$. Applying $f_{ji}$ and noting $f_{ji}(f_{0j}(N_0)) = f_{0i}(N_0)$, by invariance and monotonicity we get $M_j \overset{N_i}{\underset{M_0}{\downarrow}} f_{0,i}(N_0)$ for all $j < i$. By continuity of independence we finally have $M_i \overset{N_i}{\underset{M_0}{\downarrow}} f_{0,i}(N_0)$. ⊣

**Remark 2.8.** The proof is actually a five-line argument if we phrase its key element this way:

Given $N_i$ such that $M_i \overset{N_i}{\underset{M_0}{\downarrow}} N_0$, by $\lambda$-extension property, *we may assume* that there is $N_{i+1} \succ N_i, M_{i+1}$ such that $M_{i+1} \overset{N_{i+1}}{\underset{M_i}{\downarrow}} N_i$. By monotonicity $M_{i+1} \overset{N_{i+1}}{\underset{M_i}{\downarrow}} N_0$, and since also $M_i \overset{N_{i+1}}{\underset{M_0}{\downarrow}} N_0$, symmetry and transitivity give $M_{i+1} \overset{N_{i+1}}{\underset{M_0}{\downarrow}} N_0$.

So below we agree to use an appropriate "we may assume" in the place of a directed system argument. This makes the proofs much more transparent and does not limit the generality.

**Corollary 2.9.** *Let $\chi \geq \mathrm{LS}(\mathcal{K}) + \kappa(\mathcal{K})$, and suppose $\mu$-extension property holds for all $\chi \leq \mu < \lambda$. Let $\|M\| = \lambda$, $M_0 \prec M, N_0$, where $M_0, N_0 \in \mathcal{K}_\chi$. Then there is $N \succ M$ and $f : N_0 \to N$ over $M_0$ such that $M \overset{N}{\underset{M_0}{\downarrow}} f(N_0)$.*

*Proof.* The same argument as in Lemma 2.7; the only difference is that the sequence $\{N_i \mid i < \lambda\}$ is such that $\|N_i\| = \chi + |i|$. ⊣

**Theorem 2.10.** *Suppose that $\mathcal{K}$ is an AEC with a weak forking notion. Suppose for some $\chi \geq \mathrm{LS}(\mathcal{K}) + \kappa(\mathcal{K})$ for all $\mu \in [\chi, \lambda)$ weak $(\mu, 2)$-uniqueness and $\mu$-extension hold. Then $\mathcal{K}$ is $(\chi, \lambda)$-tame.*

**Corollary 2.11.** *Suppose that $\mathcal{K}$ is an AEC with a weak forking notion. Suppose for some $\lambda \geq \mathrm{LS}(\mathcal{K}) + \kappa(\mathcal{K})$ weak $(\lambda, 2)$-uniqueness and $\lambda$-extension hold. Then $\mathcal{K}$ is $(\lambda, \lambda^+)$-tame.*

*Proof of the theorem.* Let $M \in \mathcal{K}$ be of size $\lambda$, and let $a_2$, $a_1$ have the same Galois type over every $\mathcal{K}_\chi$-submodel of $M$. We are constructing models $\bar{N}^\ell$ extending $N^\ell$ and a $\mathcal{K}$-isomorphism $\bar{f} : \bar{N}^2 \to \bar{N}^1$ such that $\bar{f}(a_2) = a_1$ and $\bar{f} \restriction M = \mathrm{id}_M$.

Since this is the first time we are using our agreement from Remark 2.8, let us note that, strictly speaking, the models $\bar{N}^\ell$, $\ell = 1, 2$, arise as certain direct limits, $N^\ell$ embed into $\bar{N}^\ell$ via $f^\ell$, and the condition is $\bar{f}(f^2(a_2)) = f^1(a_1)$.

Let $\{M_i \mid i < \lambda\}$ and $\{N^\ell_i \mid i < \lambda\}$, $\ell = 1, 2$ be increasing continuous chains such that

(1) $M_i \prec M$ for all $i < \lambda$ and $\bigcup_{i<\lambda} M_i = M$;
(2) $\|M_i\| = \chi + |i|$ for all $i < \lambda$;
(3) $N^\ell_i \prec N^\ell$ for all $i < \lambda$ and $\bigcup_{i<\lambda} N^\ell_i = N^\ell$;
(4) $M_i \prec N^\ell_i$ for all $i < \lambda$ and $\|N^\ell_i\| = \chi + |i|$;
(5) $a_\ell \in N^\ell_0$ and $M \underset{M_0}{\overset{N^\ell}{\downarrow}} N^\ell_0$.

This is easy since by local character we can find $N^\ell_0 \prec N^\ell$ containing $a_\ell$ and $M_0 \prec M$, $M_0 \prec N^\ell_0$, such that $\|M_0\| = \|N^\ell_0\| = \chi$ and $M \underset{M_0}{\overset{N^\ell}{\downarrow}} N^\ell_0$. The rest is immediate.

By induction on $i < \lambda$ we build increasing continuous chains $\{\bar{N}^\ell_i \mid i < \lambda\} \subseteq \mathcal{K}_{<\lambda}$ and $\{\hat{N}^\ell_i \mid i < \lambda\} \subseteq \mathcal{K}_\lambda$ as well as isomorphisms $\bar{f}_i : \bar{N}^2_i \cong \bar{N}^1_i$ such that

(1) $N^\ell_i \prec \bar{N}^\ell_i \prec \hat{N}^\ell_i$ for all $i < \lambda$, $\ell = 1, 2$;
(2) $\|N^\ell_i\| = \|\bar{N}^\ell_i\| = \chi + |i|$ and $\|\hat{N}^\ell_i\| = \lambda$ for all $i < \lambda$;
(3) $\bar{f}_0(a_2) = a_1$ and $\bar{f}_i \subset \bar{f}_j$ for $i < j < \lambda$;
(4) $\bar{f}_i \restriction M_i = \mathrm{id}_{M_i}$ for all $i < \lambda$;
(5) $M \underset{M_i}{\overset{\hat{N}^\ell_i}{\downarrow}} \bar{N}^\ell_i$.

Begin with $i = 0$. Since ga-tp$(a_2/M_0)$ = ga-tp$(a_1/M_0)$, there is a model $\bar{N}^1_0 \succ N^1_0$ and an embedding $g_0 : N^2_0 \to \bar{N}^1_0$. Let $\bar{N}^2_0 \in \mathcal{K}$ be such that $\bar{N}^2_0 \succ N^2_0$ and $\bar{N}^2_0$ is isomorphic to $\bar{N}^1_0$ via some $\bar{f}_0$ such that $\bar{f}_0 \restriction N^2_0 = g_0^{-1}$ (possible by Fact 0.8).

By extension, we may assume that there are $\hat{N}^\ell_0 \succ M, \bar{N}^\ell_0$ such that $M \underset{M_0}{\overset{\hat{N}^\ell_0}{\downarrow}} \bar{N}^\ell_0$.

For $\alpha$ a limit ordinal, let $\bar{N}^\ell_\alpha := \bigcup_{i<\alpha} \bar{N}^\ell_i$, $\hat{N}^\ell_\alpha := \bigcup_{i<\alpha} \hat{N}^\ell_i$, and $\bar{f}_\alpha := \bigcup_{i<\alpha} \bar{f}_i$. It is routine to check that (1)–(4) hold, and we need to establish

(5). By the induction hypothesis and monotonicity, for all $i < \alpha$ we have $M \underset{M_\alpha}{\overset{\hat{N}^\ell_\alpha}{\downarrow}} \bar{N}^\ell_i$. So by continuity we get $M \underset{M_\alpha}{\overset{\hat{N}^\ell_\alpha}{\downarrow}} \bar{N}^\ell_\alpha$.

For the successor case, let $\mu := \chi + |i|$. Since $N^\ell_i \prec N^\ell_{i+1}, \hat{N}^\ell_i$ by $(<\lambda)$-extension and Lemma 2.7, we can find $\check{N}^\ell_{i+1}$ of cardinality $\lambda$ such that $\hat{N}^\ell_i \underset{N^\ell_i}{\overset{\check{N}^\ell_{i+1}}{\downarrow}} N^\ell_{i+1}$. (Of course $N^\ell_{i+1}$ embeds into $\check{N}^\ell_{i+1}$, and as in Remark 2.8 we assume the embedding is identity.)

Let $\tilde{N}^\ell_{i+1} \prec \check{N}^\ell_{i+1}$ be of cardinality $\mu$ such that $|\tilde{N}^\ell_{i+1}| \supseteq |\bar{N}^\ell_i| \cup |N^\ell_{i+1}|$. By monotonicity, we still have $M_{i+1} \underset{M_i}{\overset{\tilde{N}^\ell_{i+1}}{\downarrow}} \bar{N}^\ell_i$. By weak $(\mu, 2)$-uniqueness (the systems $\{M_{i+1}, M_i, \bar{N}^2_i\}$ and $\{M_{i+1}, M_i, \bar{N}^1_i\}$ are piecewise isomorphic inside $\tilde{N}^2_{i+1}$ and $\tilde{N}^1_{i+1}$), there is a model $\bar{N}^1_{i+1} \succ \tilde{N}^1_{i+1}$ and an embedding $g_{i+1} : \tilde{N}^2_{i+1} \to \bar{N}^1_{i+1}$ that extends the identity map on $M_{i+1}$ and the isomorphism $\bar{f}_i$. Using Fact 0.8 again, we get $\bar{N}^2_{i+1} \in \mathcal{K}_\mu$ such that $\bar{N}^2_{i+1} \succ \tilde{N}^2_{i+1}$ and $\bar{N}^2_{i+1}$ is isomorphic to $\bar{N}^1_{i+1}$ via some $\bar{f}_{i+1}$ such that $\bar{f}_{i+1} \restriction \tilde{N}^2_{i+1} = g^{-1}_{i+1}$. By $(<\lambda)$-extension and Lemma 2.7, we may assume that there are $\hat{N}^\ell_{i+1} \succ \hat{N}^\ell_i, \bar{N}^1_{i+1}$ such that $M \underset{M_{i+1}}{\overset{\hat{N}^\ell_{i+1}}{\downarrow}} \bar{N}^\ell_{i+1}$ for $\ell = 1, 2$.

Having finished the construction, it remains to note that $a_\ell \in \bar{N}^\ell := \bar{N}^\ell_\lambda$, $N^\ell \prec \bar{N}^\ell$, $\ell = 1, 2$, and the isomorphism $\bar{f}_\lambda : \bar{N}^2 \cong \bar{N}^1$ fixes $M$ and sends $a_2$ to $a_1$. Thus ga-tp$(a_2/M)$ = ga-tp$(a_1/M)$. ⊣

The following is a variation on Definition 0.23 from [Sh 576]:

**Definition 2.12.** Let $\mu > \text{LS}(\mathcal{K})$. The class $\mathcal{K}$ is called *$\mu$-local* iff for every $M \in \mathcal{K}_\mu$ and every resolution $\{M_i \prec_{\mathcal{K}} M \mid i < \mu\} \subseteq \mathcal{K}_{<\mu}$ we have that

$$(\forall i < \mu)[p \restriction M_i = q \restriction M_i \implies p = q] \text{ for all } p, q \in \text{ga-S}(M).$$

It is easy to see that if an AEC is $\lambda^+$-local then it is $(\lambda, \lambda^+)$-tame. Notice that the proof of Theorem 2.10 gives us the slightly stronger result:

**Corollary 2.13.** *Suppose that $\mathcal{K}$ is an AEC with a weak forking notion. Suppose for some $\lambda \geq \text{LS}(\mathcal{K}) + \kappa(\mathcal{K})$ weak $(\lambda, 2)$-uniqueness and $\lambda$-extension hold. Then $\mathcal{K}$ is $\lambda^+$-local.*

## 3. Stepping up

For this section, $\mathcal{K}$ is $\chi$-excellent; $\chi \geq \text{LS}(\mathcal{K}) + \kappa(\mathcal{K})$. Our goal is to show that a $\chi$-excellent AEC $\mathcal{K}$ is $(\chi, \infty)$-tame. For this, it is enough to establish that

In $\mathcal{K}$ $(\lambda, 2)$-existence and weak uniqueness hold for $\lambda \geq \chi$.

This will follow from two theorems:

**Theorem 3.1.** *Suppose that $\lambda > \chi$ and $\mathcal{K}$ has $(\mu, \leq n+1)$-existence and weak $(\mu, n)$-uniqueness for all $\chi \leq \mu < \lambda$. Then $\mathcal{K}$ has $(\lambda, \leq n)$-existence.*

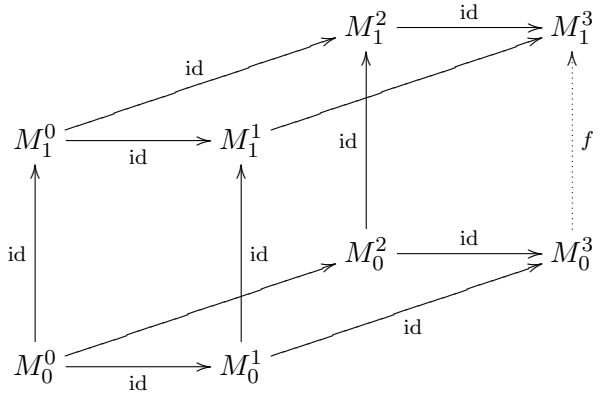
**Theorem 3.2.** *Suppose that $\lambda > \chi$ and $\mathcal{K}$ has weak $(< \lambda, \leq n+1)$-uniqueness. Then $\mathcal{K}$ has weak $(\lambda, \leq n)$-uniqueness.*

Let us start with some definitions and preliminary results.

**Definition 3.3.** Let $\mathbf{S}_\ell = \{M^s_\ell \mid s \in \mathcal{P}^-(n)\}$, $\ell = 1, 2$ be $\mathcal{P}^-(n)$-systems such that $M^s_1 \prec M^s_2$ for all $s \in \mathcal{P}^-(n)$. We then write $\mathbf{S}_1 \prec \mathbf{S}_2$.

If in addition for all $s \in \mathcal{P}^-(n)$, $|s| = n-1$, the $\mathbf{S}_1 \cup \mathbf{S}_2$-submodels of $M^s_2$ form a stable $(\lambda, n)$-system inside $M^s_2$, then we say that $\mathbf{S}_1 \prec \mathbf{S}_2$ are *independent* and write $\mathbf{S}_1 \mathop{\smile\!\!\!\!\!\!|} \mathbf{S}_2$.

Let us illustrate what the definition of $\mathbf{S}_1 \mathop{\smile\!\!\!\!\!\!|} \mathbf{S}_2$ says in the simplest case when the dimension is 2, so $\mathbf{S}_\ell = \{M^0_\ell, M^1_\ell, M^2_\ell, M^3_\ell\}$, $\ell = 1, 2$.




If $\mathbf{S}_1 \mathop{\smile\!\!\!\!\!\!|} \mathbf{S}_2$ then in $M^1_1$ we have $M^0_1 \mathop{\smile\!\!\!\!\!\!|}_{M^0_0}^{M^1_1} M^1_0$, and similar for $M^2_1$. However, this is not a 3-dimensional stable system in $M^3_1$ yet. Existence of embedding $f$ that makes the system stable is obtained in Lemma 3.4 below. This is really a generalized extension property.

**Lemma 3.4.** *Let $\lambda \geq \mathrm{LS}(\mathcal{K}) + \kappa(\mathcal{K})$, and suppose $(\lambda, \leq n+1)$-existence and weak $(\lambda, n)$-uniqueness hold, $n \geq 2$. Let $\mathbf{S}_1 \prec \mathbf{S}_2$ be independent stable $(\lambda, n)$-systems inside the models $M^n_1$, $M^n_2$ respectively. Then there is $\bar{M}^n_2 \succ M^n_2$ and an embedding $f : M^n_1 \to \bar{M}^n_2$ such that $f \restriction \bigcup_{M \in \mathbf{S}_1} M = \mathrm{id}$ and the system $\mathbf{S}_1 \cup \{f(M^n_1)\} \cup \mathbf{S}_2$ is a stable $(\lambda, n+1)$-system inside $\bar{M}^n_2$.*

*Proof.* By $(\lambda, n+1)$-existence, there is $\bar{M}^n_2$ and embeddings $f_s : M^s_2 \to \bar{M}^n_2$, $s \in \mathcal{P}^-(n)$, $|s| = n-1$, and $f : M^n_1 \to \bar{M}^n_2$ such that the images form a stable $(\lambda, n+1)$-system in $\bar{M}^n_2$.

Now in particular the image of $\mathbf{S}_2$ is a stable $(\lambda, n)$-system in $\bar{M}^n_2$. So by weak $(\lambda, n)$-uniqueness, the models $M^n_2$ and $\bar{M}^n_2$ can be amalgamated over $\mathbf{S}_2$. Thus by Fact 0.8 we may assume that actually $\bar{M}^n_2 \succ M^n_2$. Finally, $\bar{M}^n_2$ and $f$ are as needed. $\dashv$

**Lemma 3.5.** *Let $\chi \geq \mathrm{LS}(\mathcal{K}) + \kappa(\mathcal{K})$, $\lambda > \chi$. Let $\mathbf{S} = \{M^s \mid s \in \mathcal{P}^-(n)\}$ be a stable $(\lambda, \mathcal{P}^-(n))$-system inside some $M^n$. There is a sequence $\mathbf{S}_i = \{M_i^s \mid s \in \mathcal{P}^-(n)\}$, for $i < \lambda$ such that*

(1) *$\mathbf{S}_i$ is a $(\chi + |i|, \mathcal{P}^-(n))$-system;*
(2) *$\mathbf{S}_i \prec \mathbf{S}_{i+1}$ and $\mathbf{S}_i \downarrow \mathbf{S}_{i+1}$ for $i < \lambda$;*

*Proof.* Let $\chi_0$ be large enough regular so that $H(\chi_0)$ contains all the information about the system $\mathbf{S}$. Let $\mathcal{B}_i \prec \langle H(\chi_0), \in \ldots \rangle$ be an internal chain of models, with $\|\mathcal{B}_i\| = \chi + |i|$, and such that $(M^\emptyset)^{\mathcal{B}_i}$ has size $\chi + |i|$.

By definability of independence, $\mathbf{S}_i := \mathbf{S}^{\mathcal{B}_i}$ is a stable $\mathcal{P}^-(n)$-system. It remains to show (2). Let $s \in \mathcal{P}^-(n)$, $|s| = n - 1$, let $j := i + 1$, and let $\mu := \chi + |i|$. We are showing that $\{M_i^t \mid t \subseteq s\} \cup \{M_j^t \mid t \subsetneq s\}$ is a stable $(\mu, n+1)$-system in $M_j^s$.

By generalized symmetry, it is enough to show that $M_i^s \overset{M_j^s}{\underset{A_i^s}{\downarrow}} A_j^s$. But this follows from definability of independence: $A_i^s = A_j^s \cap M_i^s$ and $M_i^s$ is closed under the $\kappa$-many functions that define independence. ⊣

**Remark 3.6.** This is the only place where we had to use the generalized symmetry axiom.

**Lemma 3.7.** *Let $\lambda \geq \mathrm{LS}(\mathcal{K}) + \kappa(\mathcal{K})$, and suppose $(< \lambda, \leq n+1)$-existence and weak $(< \lambda, n)$-uniqueness hold, $n \geq 2$. Let $\mathbf{S}_1 \prec \mathbf{S}_2$ be independent stable $(\mu, n)$- and $(\lambda, n)$-systems inside some models $M_1^n$, $M_2^n$ respectively. Then there is $\bar{M}_2^n \succ M_2^n$ and an embedding $f : M_1^n \to \bar{M}_2^n$ such that $f \restriction \bigcup_{M \in \mathbf{S}_1} M = \mathrm{id}$ and the system $\mathbf{S}_1 \cup \{f(M_1^n)\} \cup \mathbf{S}_2$ is a stable $(\lambda, n+1)$-system inside $\bar{M}_2^n$.*

*Proof.* Iterate Lemma 3.4 $\lambda$-many times. ⊣

*Proof of Theorem 3.1.* Let $\mathbf{S} = \{M^s \mid s \in \mathcal{P}^-(n)\} \subset \mathcal{K}_\lambda$ be an (incomplete) system of models. Our goal is to find a model $M^n$ and the coherent embeddings $f^s : M^s \to M^n$.

Take $\mathbf{S}_i := \{M_i^s \mid i < \mu, s \in \mathcal{P}^-(n)\}$ a resolution of the system $\mathbf{S}$ such that for all $s \in \mathcal{P}^-(n)$ $\|M_i^s\| = \chi + |i|$ and $\mathbf{S}_i \prec \mathbf{S}_j$ for $i < j$.

For the base case, we just take a completion $M_0^n$ of the stable system $\{M_0^s \mid s \in \mathcal{P}^-(n)\}$. Namely, we get a system of mappings $f_0^s : M_0^s \to M_0^n$. It exists since we are assuming $(\chi, n)$-existence.

Successor step. We have the model $M_i^n$, in which $f_i^s(M_i^s)$, $s \in \mathcal{P}^-(n)$, form a stable $n$-system. And from the resolution we have $M_{i+1}^s$ for $s \in \mathcal{P}^-(n)$, $|s| = n - 1$, where $\{M_j^t \mid (\emptyset, i) \leq (t, j) < (s, i+1)\}$, form a stable $n$-system in size $\mu = \chi + |i|$.

By $(\mu, n+1)$-existence, we get $M_{i+1}^n$ and embeddings $f_{i+1}^s : M_{i+1}^s \to M_{i+1}^n$ for $s \subset_{n-1} n$. Now $(\mu, n+1)$-amalgamation also gives that $f_{i+1}^s \supset f_i^s$ for $s \in \mathcal{P}^-(n)$.

For the limit step we simply take the union. Finally, the model $M_n^\lambda$ is as needed. ⊣

*Proof of Theorem 3.2.* Let $\mathbf{S}^\ell$, $\ell = 1, 2$, be stable $(\lambda, n)$-systems that are piecewise isomorphic. We are constructing models $\bar{N}^\ell$ extending $M^\ell_n$ and a $\mathcal{K}$-isomorphism $\bar{f} : \bar{N}^1 \to \bar{N}^2$ that extends all $f_s : M^1_s \to M^2_s$ for $s \in \mathcal{P}^-(n)$.

By definability of forking we can find $M^\ell_{0,s} \prec M^\ell_s$ such that $\|M^\ell_{0,s}\| = \chi$ and $A^\ell_n \overset{N^\ell}{\underset{A^\ell_{0,n}}{\downarrow}} M^\ell_{0,n}$. Let $\{A^\ell_{i,n} \mid i < \lambda\}$ be an increasing continuous chain whose union is $A^\ell_n$ and $\|A^\ell_{i,n}\| = \chi + |i|$.

By induction on $i < \lambda$ build models $\bar{N}^\ell_i$, $\hat{N}^\ell_i$, and isomorphisms $\bar{f}_i : \bar{N}^1_i \to \bar{N}^2_i$ such that

(1) $\bar{N}^\ell_i \succ N^\ell_i$;
(2) $\bar{f}_i \restriction A^1_{i,n} = \bigcup_{s \in \mathcal{P}^-(n)} f_s(M^1_{s,n})$;
(3) $\bar{f}_{i+1}$ extends $\bar{f}_i$;
(4) $A^\ell_n \overset{\hat{N}^\ell_i}{\underset{A^\ell_{i,n}}{\downarrow}} \bar{N}^\ell_i$ $(\|\hat{N}^\ell_i\| = \lambda)$.

Begin with $i = 0$. By weak $(\chi, n)$-uniqueness there is a model $\bar{N}^2_0 \succ N^2_0$ and an embedding $f_0 : N^1_0 \to \bar{N}^2_0$. By extension, we may assume that there is $\hat{N}^2_0 \succ A^2_n, \bar{N}^2_0$ such that $A^2_n \overset{\hat{N}^2_0}{\underset{A^2_{0,n}}{\downarrow}} \bar{N}^2_0$. Let $\bar{N}^1_0 \in \mathcal{K}$ be such that $\bar{N}^1_0 \succ N^1_0$ and $\bar{N}^1_0$ is isomorphic to $\bar{N}^2_0$ via some $\bar{f}_0$ such that $\bar{f}_0 \restriction N^1_0 = f_0$. Using extension again, we get $\hat{N}^1_0$ such that $A^1_n \overset{\hat{N}^1_0}{\underset{A^1_{0,n}}{\downarrow}} \bar{N}^1_0$.

For $\alpha$ a limit ordinal, let $\bar{N}^\ell_\alpha := \bigcup_{i<\alpha} \bar{N}^\ell_i$; $\hat{N}^\ell_\alpha := \bigcup_{i<\alpha} \hat{N}^\ell_i$; and $\bar{f}_\alpha := \bigcup_{i<\alpha} \bar{f}_i$. It is routine to check that (1)–(3) hold, and we need to establish (4). By the induction hypothesis and monotonicity, for all $i < \alpha$ we have $A^\ell_n \overset{\hat{N}^\ell_\alpha}{\underset{A^\ell_{\alpha,n}}{\downarrow}} \bar{N}^\ell_i$. So by continuity we get $A^\ell_n \overset{\hat{N}^\ell_\alpha}{\underset{A^\ell_{\alpha,n}}{\downarrow}} \bar{N}^\ell_\alpha$.

For the successor case, let $\mu := \delta + |i|$. Let $N^\ell_{i+1} \succ N^\ell_i$ be a $\mathcal{K}$-submodel of $\hat{N}^\ell_i$ containing $A^\ell_{i+1,n}$; $\|N^\ell_{i+1}\| = \mu$. By monotonicity, $A^\ell_{i+1,n} \overset{N^\ell_{i+1}}{\underset{A^\ell_{i,n}}{\downarrow}} \bar{N}^\ell_i$, so the system $\mathbf{S}^\ell_{i+1} := A^\ell_{i+1,n} \cup A^\ell_{i,n} \cup \{\bar{N}^\ell_i\}$ is a $(\mu, n+1)$-stable system inside $N^\ell_{i+1}$. By weak $(\mu, n+1)$-uniqueness ($\mathbf{S}^\ell_{i+1}$, $\ell = 1, 2$ are piecewise isomorphic), there is a model $\bar{N}^2_{i+1} \succ N^2_{i+1}$ and an embedding $f_{i+1} : N^1_{i+1} \to \bar{N}^2_{i+1}$ that extends the "piecewise isomorphisms" $f_{i+1,s} : M^1_{i+1,s} \cong M^2_{i+1,s}$ as well as the isomorphism $\bar{f}_i$. By $(< \lambda, n)$-existence and Lemma 3.7, we may assume that there is $\hat{N}^2_{i+1} \succ \hat{N}^2_i, \bar{N}^2_{i+1}$ such that $A^2_n \overset{\hat{N}^2_{i+1}}{\underset{A^2_{i+1,n}}{\downarrow}} \bar{N}^2_{i+1}$.

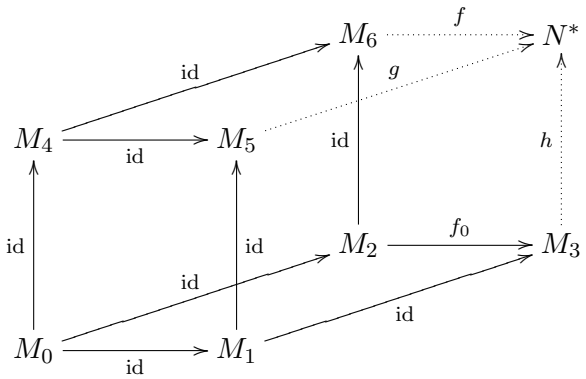
Using Fact 0.8, we get $\bar{N}^1_{i+1} \in \mathcal{K}_\mu$ such that $\bar{N}^1_{i+1} \succ N^1_{i+1}$ and $\bar{N}^1_{i+1}$ is isomorphic to $\bar{N}^2_{i+1}$ via some $\bar{f}_{i+1}$ such that $\bar{f}_{i+1} \restriction N^1_{i+1} = f_{i+1}$. Using $(< \lambda, n)$-existence again, we get $\hat{N}^1_{i+1}$ such that $A^1_n \underset{A^1_{i+1,n}}{\overset{\hat{N}^1_{i+1}}{\downarrow}} \bar{N}^1_{i+1}$. $\dashv$

## 4. Three dimensional amalgamation

A previous draft of this paper dealt with $n$-dimensional amalgamation properties. In this section, we state *a* definition of 3-dimensional amalgamation, outline the proof of $(\lambda, \lambda^+)$-tameness from $(\lambda, 3)$-amalgamation, and finally show that $(\lambda, 3)$-amalgamation implies the weak $(\lambda, 2)$-uniqueness property.

The outline of the proof was presented by Rami Grossberg in Bogotá model theory conference in the fall in 2003 and a preliminary version of this paper was posted on the web since December 2003. Recently, our idea of using 3-dimensional amalgamation was employed to show directly that a certain natural class of structures is tame. Using variants of 3-dimensional amalgamation Villaveces-Zambrano in [ViZa] and Baldwin in [Ba] managed to obtain tameness of certain abstract elementary classes arising naturally from Hrushovski's fusion of strongly minimal theories. As the work of Villaveces-Zambrano and Baldwin is still in progress the interested reader should consult them for their most recent results.

**Definition 4.1.** We say $\langle \mathcal{K}, \downarrow \rangle$ has $(\lambda, 3)$-amalgamation if for any system of seven $\mathcal{K}_\lambda$-models $\{M_i \mid i < 7\}$ such that $M_1 \underset{M_0}{\overset{M_5}{\downarrow}} M_4$ and $M_2 \underset{M_0}{\overset{M_6}{\downarrow}} M_4$ and a $\mathcal{K}$-embedding $f_0$, there is a model $N^*$ and embeddings $f : M_6 \to N^*$ $g : M_5 \to N^*$ and $h : M_3 \to N^*$ such that the following diagram commutes.

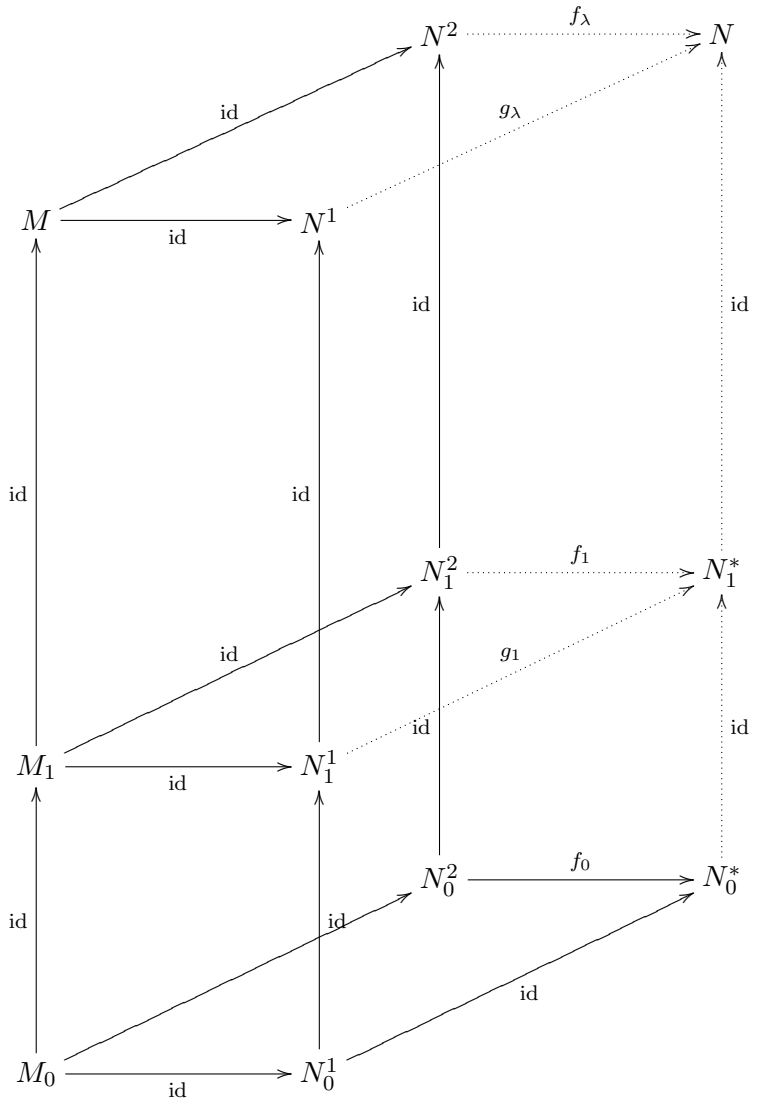



4.1. **Tameness from 3-dimensional amalgamation.** Let $p, q \in \text{ga-S}(M)$ such that $M \in \mathcal{K}_\lambda$ and $q \restriction N = p \restriction N$ for all $N \in \mathcal{K}_{<\lambda}$ enough to show that this condition implies $p = q$.

Suppose $(a_1, M, N^1) \in p$ and $(a_2, M, N^2) \in q$. By the LS-axiom pick $\{N^\ell_\alpha \prec_{\mathcal{K}} N^\ell \mid \alpha < \lambda\} \subseteq \mathcal{K}_{<\lambda}$ increasing and continuous resolutions of $N_\ell$

and $\{M^\alpha \prec_{\mathcal{K}} M \mid \alpha < \lambda\} \subseteq \mathcal{K}_{<\lambda}$ increasing resolution of $M$ such that $M_\alpha \prec N^\ell_\alpha$, require that $a_\ell \in N^0_\ell$.

By the assumption there exist $N^*_0 \succ_{\mathcal{K}} N^2_0$ of cardinality less than $\lambda$, an amalgam of $N^1_0$ and $N^2_0$ over $M_0$ mapping $a_1$ to $a_2$.




Clearly it is enough to find $N^*_1 \in \mathcal{K}_{<\lambda}$ a $\prec_{\mathcal{K}}$-extension of $N^*_0$ and $\mathcal{K}$-embeddings $g_1 : N^1_1 \to N^*_1$ and $f_1 : N^2_1 \to N^*_1$ such that the above diagram commutes. Continuing this by induction on $\alpha < \lambda$ gives $N^*_\alpha$ and $f_\alpha, g_\alpha$ such that the above diagram commutes.

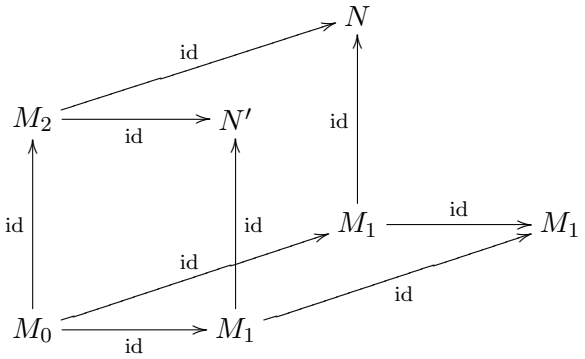
Let $N := \bigcup_{\alpha<\lambda} N^*_\alpha$ and $f_\lambda := \bigcup_{\alpha<\lambda} f_\alpha$. Since $f_\lambda(a_1) = a_2$ we have that $p = q$.

4.2. **3-amalgamation implies weak 2-uniqueness.** Rather than formalize the above argument, we show that $(\lambda, 3)$-amalgamation is a strong enough assumption, so that a particular case of $(\lambda, 3)$-amalgamation implies weak $(\lambda, 2)$-uniqueness.

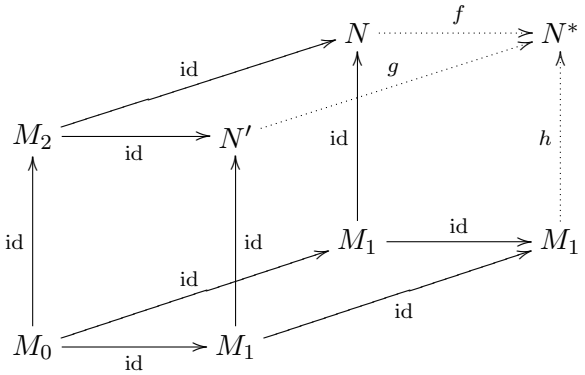
**Proposition 4.2.** *Suppose $\mathcal{K}$ has $(\lambda, 3)$-amalgamation, where all the given embeddings are identity (in other words, we allow only systems where $f_0 = \mathrm{id}$). Then weak $(\lambda, 2)$-uniqueness holds.*

*Proof.* Take $M_0 \prec_{N,N'} M_1, M_2 \in \mathcal{K}_\lambda$; let $N, N' \in \mathcal{K}_\lambda$ both contain $M_1 \cup M_2$ and suppose that $M_1 \underset{M_0}{\perp} M_2$. By Proposition 1.16, it is enough to show that $N$, $N'$ can be amalgamated over $M_1 \cup M_2$.

We have the following diagram:




By $(\lambda, 3)$-amalgamation, there is a model $N^*$ and embeddings $f : N \to N^*$ $g : N' \to N^*$ and $h : M_1 \to N^*$ such that the following diagram commutes.




Now $f(M_2) = g(M_2)$, and $f(M_1) = h(M_1) = g(M_1)$, so $f(M_i) = g(M_i)$ for $i = 0, 1, 2$. Thus $N^*$ is an amalgam of $N$ and $N'$ over $M_1 \cup M_2$. ⊣

## References

[Ba] John Baldwin. The Hrushovski construction and quasiminimal excellent classes. In preparation.

[Gr1] Rami Grossberg. Classification theory for non-elementary classes. **Logic and Algebra**, ed. Yi Zhang, *Contemporary Mathematics*, **302**, (2002) AMS, pp. 165–204.

[Gr2] Rami Grossberg. **A Course in Model Theory.** Book in Preparation. 740 pages

[Gr3] Rami Grossberg. **Cloassification theory for Abstract Elementary Classes.** Book in Preparation. 167 pages

[GrLe] Rami Grossberg and Olivier Lessmann. Shelah's Stability Spectrum and Homogeneity Spectrum, *Archive for mathematical Logic*, **41**, (2002) 1, 1-31.

[GrSh] Rami Grossberg and Saharon Shelah, On universal locally finite groups *Israel J. of Math.* , **4**4, 1983, 289–302.

[GrV1] Rami Grossberg and Monica VanDieren. Galois-stability for Tame Abstract Elementary Classes. (23 pages) submitted. URL: `www.math.cmu.edu/~rami`.

[GrV2] Rami Grossberg and Monica VanDieren. Upward Categoricity Transfer Theorem for Abstract Elementary Classes. (18 pages) preprint. URL: `www.math.cmu.edu/~rami`.

[Jo1] Bjarni Jónsson. Universal relational systems, *Math. Scand.* **4** (1956) 193–208.

[Jo2] Bjarni Jónsson. Homogeneous universal systems. *Math. Scand.*, **8**, 1960, 137–142.

[Ke] H. Jerome Keisler, $L_{\omega_1,\omega}(\mathbf{Q})$ *Ann of Math Logic*, **1**, 1969.

[MaSh] Michael Makkai and Saharon Shelah. Categoricity of theories in $L_{\kappa\omega}$, with $\kappa$ a compact cardinal. *Annals of Pure and Applied Logic*, **47**:41–97, 1990.

[Sh c] Saharon Shelah. **Classification Theory and the Number of Nonisomorphic Models**, Rev. Ed., North-Holland, 705 pages, 1990, Amsterdam.

[Sh 87a] Saharon Shelah. Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A. *Israel J. Math.*, **46**:212–240, 1983.

[Sh 87b] Saharon Shelah. Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B. *Israel J. Math.*, **46**:241–273, 1983.

[Sh 88] Saharon Shelah. Classification of nonelementary classes, II. Abstract Elementary Classes. In **Classification Theory (Chicago IL 1985)**, volume 1292 of *Lecture Notes in Mathematics*, pages 419–497. Springer, Berlin, 1987. Proceedings of the USA-Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T.

[Sh 394] Saharon Shelah. Categoricity of abstract classes with amalgamation. *Annals of Pure and Applied Logic*, **98**(1-3), pages 241–294, 1999.

[Sh 576] Saharon Shelah. Categoricity of an abstract elementary class in two successive cardinals. *Israel J. of Math*, **126**, (2001), 29–128.

[Sh 600] Saharon Shelah. Categoricity in abstract elementary classes: going up inductive step. Preprint. 100 pages.

[Sh 705] Saharon Shelah. Toward classification theory of good $\lambda$ frames and abstract elementary classes. 221 pages (August 2005 version).

[ViZa] Andrés Villaveces and Pedro Zambrano. Hrushovski constructions and tame abstract elementary classes. in preparation.

*E-mail address*, Rami Grossberg: `rami@cmu.edu`
*URL*: `http://www.math.cmu.edu/∼rami`

Department of Mathematics, Carnegie Mellon University, Pittsburgh PA 15213

*E-mail address*: `alexeik@umich.edu`
*URL*: `http://www.math.lsa.umich.edu/∼alexeik`

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109