EXCELLENT ABSTRACT ELEMENTARY CLASSES ARE TAME

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Abstract. We prove the statement from the title. As an application we conclude (using a theorem of Shelah):

Corollary 0.1. *Suppose* **V=L**. *Let T be a countable $L_{\omega_1,\omega}$ theory in a countable language. If* $\mathrm{I}(\aleph_{n+1}, \mathrm{T}) < 2^{\aleph_{n+1}}$ *for every $n < \omega$ then* $\mathcal{K} := \mathrm{Mod}(T)$ *is $\aleph_0$-tame (i.e. for any p and q distinct Galois types there exist a countable $M \in \mathcal{K}$ such that $p \restriction M \neq q \restriction M$).*

Introduction

In 1977 Shelah influenced by earlier work of Jónsson ([Jo1] and [Jo2]) in [Sh 88] introduced a semantic generalization of Keisler's [Ke] treatment of $L_{\omega_1,\omega}(\mathbf{Q})$. It is the notion of *Abstract Elementary Class*:

Definition 0.2. Let $\mathcal{K}$ be a class of structures all in the same similarity type $\mathrm{L}(\mathcal{K})$, and let $\prec_{\mathcal{K}}$ be a partial order on $\mathcal{K}$. The ordered pair $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an *abstract elementary class, AEC for short* iff

- A0 (Closure under isomorphism)
  - (a) For every $M \in \mathcal{K}$ and every $\mathrm{L}(\mathcal{K})$-structure N if $M \cong N$ then $N \in \mathcal{K}$.
  - (b) Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.
- A1 For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.
- A2 Let M, N, M^* be $\mathrm{L}(\mathcal{K})$-structures. If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$.
- A3 (Downward Löwenheim-Skolem) There exists a cardinal $\mathrm{LS}(\mathcal{K}) \geq \aleph_0 + |\,\mathrm{L}(\mathcal{K})|$ such that for every $M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \supseteq A$ and $\|N\| \leq |A| + \mathrm{LS}(\mathcal{K})$.
- A4 (Tarski-Vaught Chain)

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(a) For every regular cardinal μ and every $N \in \mathcal{K}$ if $\{M_i \prec_{\mathcal{K}} N \ : \ i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing (i.e. $i < j \Longrightarrow M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $\bigcup_{i<\mu} M_i \prec_{\mathcal{K}} N$.
(b) For every regular μ, if $\{M_i \ : \ i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i<\mu} M_i$.

For M and $N \in \mathcal{K}$ a monomorphism $f : M \to N$ is called an $\mathcal{K}$-*embedding* iff $f[M] \prec_{\mathcal{K}} N$. Thus, $M \prec_{\mathcal{K}} N$ is equivalent to "id_M is a $\mathcal{K}$-embedding from M into N".

Many of the fundamental facts on AECs were introduced in [Sh 88], [Sh 394] and [Sh 576]. For a survey of some of the basics see [Gr1] or Chapter 13 of [Gr2].

In the late seventies Shelah proposed the following as a test propblem:

Conjecture 0.3 (Shelah's conjecture). *Let $\psi \in L_{\omega_1,\omega}$ be a sentence in a countable language. If ψ is λ-categorical in some $\lambda \geq \beth_{\omega_1}$ then ψ is μ-categorical for every $\mu \geq \beth_{\omega_1}$.*

In 1990 Shelah proposed a generalization for AECs:

Conjecture 0.4 (see [Sh c]). *Let $\mathcal{K}$ be an AEC. If $\mathcal{K}$ is categorical in some $\lambda \geq \mathrm{Hanf}(\mathcal{K})$ then $\mathcal{K}$ is μ-categorical for every $\mu \geq \mathrm{Hanf}(\mathcal{K})$.*

Notation 0.5. Let μ be a cardinal number and $\mathcal{K}$ a class of models. By $\mathcal{K}_\mu$ we denote the subclass $\{M \in \mathcal{K} \ : \ \|M\| = \mu\}$.

Two classical concepts that introduced in the fifties and studied extensively by Fraisee, Robinson and Jonsson paly also an important role in AECs:

Definition 0.6. Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an AEC nd suppose $\mu \geq \mathrm{LS}(\mathcal{K})$. We say that $\mathcal{K}$ *has the μ-amalgamation property* iff for all $M_\ell \in \mathcal{K}_\mu$ (for $\ell = 0, 1, 2$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell = 1, 2$) there exists $N^* \in \mathcal{K}_\mu$ and $f_\ell : M_\ell \to N^*$ (for $\ell = 1, 2$) such that $f_1 \restriction M_0 = f_2 \restriction M_0$, i.e. the following diagram commutes:

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] & N^* \\
M_0 \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & M_2 \arrow[u, "f_2"']
\end{tikzcd}$$

M_0 as above is called *amalgamation base*.

$\mathcal{K}$ *has the μ-joint mapping property* iff for any $M_\ell \in \mathcal{K}_\mu$ for $\ell = 1, 2$ there are $N^* \in \mathcal{K}_\mu$ and $\mathcal{K}$-embeddings $f_\ell : M_\ell \to N^*$.

We say that $\mathcal{K}$ has the *amalgamation property* iff it has the μ-amalgamation property for all $\mu \geq \mathrm{LS}(\mathcal{K})$.

Using Axiom A0 from the definition of AEC it follows that both a stronger-looking and a weaker-looking amalgamation properties are equivalent to what we call above the amalgamation property:

Lemma 0.7. *Let $\mathcal{K}$ be an AEC. The following are equivalent*

1. *$\mathcal{K}$ has the μ-amalgamation property,*
2. *for all $M_\ell \in \mathcal{K}_\mu$ (for $\ell = 0, 1, 2$) such that $M_0 \prec_{\mathcal{K}} M_\ell$ (for $\ell = 1, 2$) there exists $N^* \in \mathcal{K}_\mu$ such that $N^* \succ_{\mathcal{K}} N_2$ and there is $f : M_1 \to N$ satisfying $f \restriction M_0 = \mathrm{id}_{M_0}$, i.e. the following diagram commutes:*

$$\begin{tikzcd} M_1 \arrow[r, "f"] & N^* \\ M_0 \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & M_2 \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

3. *for all $M_\ell \in \mathcal{K}_\mu$ (for $\ell = 0, 1, 2$) such that $g_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) are $\mathcal{K}$-embeddings there are $N^* \in \mathcal{K}_\mu$ and there is $f_\ell : M_\ell \to N^*$ satisfying $f_1 \circ g_1 \restriction M_0 = f_2 \circ g_2 \restriction M_0$ i.e. the next diagram commutes:*

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] & N^* \\ M_0 \arrow[u, "g_1"] \arrow[r, "g_2"'] & M_2 \arrow[u, "f_2"'] \end{tikzcd}$$

There are classical theorems of Robinson stating that if T is a complete first-order theory than $\mathrm{Mod}(T)$ has both the amalgamation and the joint mapping properties.

Galois types. In the theory of AECs the notion of complete first-order type is replaced by that of a *Galois type*:

Definition 0.8. Let $\beta > 0$ be an ordinal. For triples $(\bar{a}_l, M, N_l)$ where $\bar{a}_l \in {}^{\beta}\mathrm{N}_l$ and $M_l \prec_{\mathcal{K}} N_l \in \mathcal{K}$ for $l = 0, 1$, we define a binary relation E as follows: $(\bar{a}_0, M, N_0)E(\bar{a}_1, M, N_1)$ iff and there exists $N \in \mathcal{K}$ and $\mathcal{K}$-mappings f_0, f_1 such that $f_l : N_l \to N$ and $f_l \restriction M = \mathrm{id}_M$ for $l = 0, 1$ and $f_0(\bar{a}_0) = f_1(\bar{a}_1)$:

$$\begin{tikzcd} N_1 \arrow[r, "f_1"'] & N \\ M \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & N_2 \arrow[u, "f_2"'] \end{tikzcd}$$

Remark 0.9. E is an equivalence relation on the class of triples of the form $(\bar{a}, M, N)$ where $M \prec_{\mathcal{K}} N$, $\bar{a} \in N$ and both M and N are amalgamation bases. When N is not an amalgamation base, E may fail to be transitive, but the transitive closure of E could be used instead.

Definition 0.10. Let β be a positive ordinal.

1. For $M, N \in \mathcal{K}$ and $\bar{a} \in {}^{\beta}\mathrm{N}$. The *Galois type of $\bar{a}$ in N over M*, written $\text{ga-tp}(\bar{a}/M, N)$, is defined to be $(\bar{a}, M, N)/E$.

(2) We abbreviate ga-tp$(\bar{a}/M, N)$ by ga-tp$(\bar{a}/M)$.
(3) For $M \in \mathcal{K}$,

$$\text{ga-S}^{\beta}(M) := \{\text{ga-tp}(\bar{a}/M, N) \mid M \prec N \in \mathcal{K}_{\|M\|}, \bar{a} \in {}^{\beta}\mathrm{N}\}.$$

We write ga-S(M) for ga-S$^1(M)$.
(4) Let $p :=$ ga-tp$(\bar{a}/M', N)$ for $M \prec_{\mathcal{K}} M'$ we denote by $p \restriction M$ the type ga-tp$(\bar{a}/M, N)$. The *domain of* p is denoted by $\operatorname{dom} p$ and it is by definition M'.
(5) Let $p =$ ga-tp$(\bar{a}/M, N)$, suppose that $M \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} N$ and let $\bar{b} \in {}^{\beta}\mathrm{N}'$ we say that $\bar{b}$ *realizes* p iff ga-tp$(\bar{b}/M, N') = p \restriction M$.
(6) For types p and q, we write $p \leq q$ if $\operatorname{dom}(p) \subseteq \operatorname{dom}(q)$ and there exists $\bar{a}$ realizing p in some N extending $\operatorname{dom}(p)$ such that $(\bar{a}, \operatorname{dom}(p), N) = q \restriction \operatorname{dom}(p)$.

In [GrV1] Grossberg and VanDieren introduced the notion of *tameness* as a candidate for a further "reasonable" assumption an an AEC that permits development of stability-like theory. In [GrV2] they recently proved the last step Shelah's categorcity conjecture for tame AECs with the amalgamation property.

Definition 0.11. Let $\mathcal{K}$ be an AEC with the amalgamation property and let $\chi \geq \mathrm{LS}(\mathcal{K})$. The class $\mathcal{K}$ is called χ*-tame* iff

$$p \neq q \implies \exists N \prec_{\mathcal{K}} M \text{ of cardinality } \leq \chi \text{ such that } p \restriction N \neq q \restriction N$$

for any $M \in \mathcal{K}_{>\chi}$ and every $p, q \in$ ga-S(M)

$\mathcal{K}$ is *tame* iff it is χ-tame for some $\chi < \mathrm{Hanf}(\mathcal{K})$

Suppose $\mu > \chi$. The class is (χ, μ)*-tame* iff

$$p \neq q \implies \exists N \prec_{\mathcal{K}} M \text{ of cardinality } \leq \chi \text{ such that } p \restriction N \neq q \restriction N$$

for any $M \in \mathcal{K}_{\mu}$ and every $p, q \in$ ga-S(M)

In [Sh 394] Shelah proved that for an AEC with the amalgamation property. If $\mathcal{K}$ is λ-categorical for some $\lambda > \mathrm{Hanf}(\mathcal{K})$ then it is $(< \mathrm{Hanf}(\mathcal{K}), \mu)$-tame for all $\mathrm{Hanf}(\mathcal{K}) < \mu < \lambda$.

Definition 0.12. Let I be a subset of $\mathcal{P}(n)$ for some $n < \omega$ that is downward closed (i.e. $t \in I$ and $s \subseteq t$ implies $s \in I$).

For an $\mathbf{S} = \langle M_s \mid s \in I \rangle$ is an I*-system* iff for all $s, t \in I$

(1) $s \leq t \implies M_s \prec_{\mathcal{K}} M_t$ and
(2) $M_{s \cap t} = M_s \cap M_t$

$\mathbf{S}$ is a (λ, I)*-system* iff in addition all the models are of cardinality λ.

Denote by

$$A_t^{\mathbf{S}} := \bigcup_{s<t} M_s$$

Some sets that are amalgamation bases play in important role since they permit existence of Galois-types over them. Here is the formal

Definition 0.13. Let $\mathcal{K}$ be an AEC and suppose $\mu \geq \mathrm{LS}(\mathcal{K})$. Suppose $\mathbf{S} = \langle M_s \in \mathcal{K}_\mu \mid s \in I \rangle$ is an I-system for $I \subseteq \mathcal{P}^-(n)$ and $t \in \mathcal{P}(n)$. For $A := A_t^{\mathbf{S}}$ we say that *the set A is an amalgamation base* iff for all $M_\ell \in \mathcal{K}_\mu$ (for $\ell = 0, 1, 2$) such that $A \subseteq |M_\ell|$ (for $\ell = 1, 2$) there exists $N^* \in \mathcal{K}_\mu$ such that $N^* \succ_{\mathcal{K}} N_2$ and there is a $\mathcal{K}$-embedding $f : M_1 \to N$ satisfying $f \restriction M_0 = \mathrm{id}_A$, i.e. the following diagram commutes:

$$\begin{tikzcd} M_1 \arrow[r, "f"] & N^* \\ A \arrow[u, "\mathrm{id}_A"] \arrow[r, "\mathrm{id}_A"'] & M_2 \arrow[u, "\mathrm{id}_{M_2}"'] \end{tikzcd}$$

By $\mathrm{id}_A : A \to M_\ell$ we mean that $M_s \prec_{\mathcal{K}} M_\ell$ holds for $\ell = 1, 2$ and every $s < t$.

Notation 0.14. Denote by $\mathrm{Ab}(\mathcal{K})$ the class $\{A \mid A \text{ is an amalgamation base}\}$.

Thus $\mathcal{K}$ has the λ-amalgamation property iff $\mathcal{K}_\lambda \subseteq \mathrm{Ab}(\mathcal{K})$.

Clearly under the assumption that $\mathcal{K}$ has the amalgamation property the notion of Galois-type can be extended to include also ga-tp$(\bar{a}/A, M)$ for $A \in \mathrm{Ab}(\mathcal{K})$.

Examples 0.15.
(1) Let T be a complete first-order theory and $\mathfrak{C}$ its monster model. By Robinson's consistency lemma any $A_t^{\mathbf{S}}$ for an I-system is an element of $\mathrm{Ab}(\mathrm{Mod}(T)))$.
(2) On can prove that if $\mathcal{K}$ is the class of atomic models of a first-order T satisfying all the assumptions of [Sh 87a] then

$$A \in \mathrm{Ab}(\mathcal{K}) \quad \text{iff} \quad A \text{ is good.}$$

1. The basic framework and concepts

Definition 1.1. A pair $\langle \mathcal{K}, \mathop{\smile}\!\!\!\!\!|\,\, \rangle$ is a *weak forking notion* iff $\mathcal{K}$ is an AEC, where $\mathop{\smile}\!\!\!\!\!|\,\,$ is a three-place relation called *non-forking* $\bar{a} \underset{A}{\mathop{\smile}\!\!\!\!\!|\,\,} B$ for $\bar{a} \in {}^{\beta}\mathrm{M}$ for some $M \in \mathcal{K}$ and $A \subseteq B$ both elements of $\mathrm{Ab}(\mathcal{K})$ such that $\mathop{\smile}\!\!\!\!\!|\,\,$ is *invariant under automorphisms* which means for any $\bar{a}, A, B$ as above for all $N \in \mathcal{K}$ containing $A \cup B \cup \bar{a}$ we have that

$$\bar{a} \underset{A}{\mathop{\smile}\!\!\!\!\!|\,\,} B \iff f(\bar{a}) \underset{f(A)}{\mathop{\smile}\!\!\!\!\!|\,\,} f(B) \quad \text{for all } f \in \mathrm{Aut}(N).$$

the following conditions hold:

(0) *Definability:* There exists a cardinal number κ such that the relation $\bar{a} \underset{A}{\mathop{\smile}\!\!\!\!\!|\,\,} B$ is (set-theoretically) *definable over* κ i.e. there is a f.o. formula $\varphi(\mathbf{x})$ in the similarity type $\mathrm{LS}(\mathcal{K}) \cup \{\in, P, Q\}$ such that

$$\langle H(\chi), \in, \kappa, A, B, \psi(\mathbf{y})\rangle_{\psi(y) \in \mathrm{Fml}(\mathrm{L}(\mathcal{K}))} \models \varphi[\mathbf{a}] \iff \mathbf{a} \underset{A}{\mathop{\smile}\!\!\!\!\!|\,\,} B \quad \text{for all finite } \mathbf{a} \in \bar{a}.$$

(1) *Disjointness:*
$$\bar{a} \underset{A}{\perp} B \implies \bar{a} \cap B \subseteq A.$$
(2) *Existence:* Let $A \in \mathrm{Ab}(\mathcal{K})$ if $\bar{a}$ is such that there exists a model N containing B but disjoint to $\bar{a}$ then $\bar{a} \underset{A}{\perp} A$.
(3) *Extension property:* If $\bar{a} \underset{A}{\perp} B$ then for all $C \in \mathrm{Ab}(\mathcal{K})$ such that $C \supseteq B$ there exists $\bar{a}'$ in some $M \in \mathcal{K}$ such that
$$\bar{a}' \underset{A}{\perp} C \quad \text{and} \quad \text{ga-tp}(\bar{a}/A) = \text{ga-tp}(\bar{a}'/A).$$
(4) *Symmetry:* if $\bar{a} \underset{A}{\perp} A\bar{b}$, then $\bar{b} \underset{A}{\perp} A\bar{a}$.

Examples 1.2.

(1) Let $\mathcal{K} := \mathrm{Mod}(T)$ when T is a first-order complete theory, $\prec_{\mathcal{K}}$ is the usual elementary submodel relation and $\perp$ is the non-forking relation.

Clearly $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is a weak forking notion iff T is simple. κ in this case is $\kappa(T)$.

(2) Let $\mathcal{K} := \mathrm{Mod}(T)$ when T is a first-order complete theory, $\prec_{\mathcal{K}}$ is the usual elementary submodel relation and $\perp$ is the non-dividing relation. It is not difficult to show that $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is a weak forking notion with $\kappa = \aleph_0$ iff T is supersimple.

(3) Let T be a countable first-order theory, suppose that T is $\aleph_0$-*atomically stable*, i.e. for $\mathrm{R}[p] < \infty$ for every atomic type, let
$$\mathcal{K} := \{M \models T \mid \text{ga-tp}(\mathbf{a}/\emptyset, M) \text{ is an isolated type for every } \mathbf{a} \in |M|\}.$$
Where $p \in S(A)$ is called *atomic* iff $A \cup \{\mathbf{a}\}$ is atomic subset of $\mathfrak{C}$ and $\mathbf{a} \models p$. An atomic type is *stationary* iff there is a finite $B \subseteq A$ and a countable model N containing the set B and an atomic realization $\mathbf{a}$ of p we have that
$$\mathrm{R}[p] = \mathrm{R}[\text{ga-tp}(\mathbf{a}/B)] = \mathrm{R}[\text{ga-tp}(\mathbf{a}/|N|)].$$
An atomic set $A \subseteq \mathfrak{C}$ is *good* iff for every consistent $\varphi(\mathbf{x}; \mathbf{a})$ (with $\mathbf{a} \in A$) there is an isolated type $p \in S(A)$ containing $\varphi(x; \mathbf{a})$.

Definition 1.3. For $M \in \mathcal{K}^a$ and $\mathbf{a} \in M$ define by induction of α when $\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a})] \geq \alpha$

$\alpha = 0$; $M \models \exists \mathbf{x} \varphi(\mathbf{x}; \mathbf{a})$

For $\alpha = \beta + 1$;

There are $\mathbf{b} \supseteq \mathbf{a}$ and $\psi(\mathbf{x}; \mathbf{b})$ such that
$$\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b})] \geq \beta$$
$$\mathrm{R}[\varphi(\mathbf{x}; \mathbf{a}) \wedge \neg\psi(\mathbf{x}; \mathbf{b})] \geq \beta \quad \text{and for every } \mathbf{c} \supseteq \mathbf{a}$$

there is $\chi(\mathbf{x};\mathbf{c})$ complete s.t.

$$\mathrm{R}[\varphi(\mathbf{x};\mathbf{a})\wedge\chi(\mathbf{x};\mathbf{c})]\geq\beta$$

Notation 1.4.

$$D_A:=\{\text{ga-tp}(\mathbf{a}/A)\mid A\cup\{\mathbf{a}\}\text{ is atomic}\}.$$

Fact 1.5 ([Sh 87a]). *If $|D_A|<2^{\aleph_0}$ then A is good.*

(4) Let $\mathcal{K}$ be the class of elementary submodels of a sequentially homogeneous model. Let $M_1 \underset{M_0}{\perp} M_2$ stand for ga-tp$(\mathbf{a}/M_2)$ does not strongly-split over M_0 for every $\mathbf{a}\in|M_1|$.

Compare with XII.2 of [Sh c].

Definition 1.6 (Stable systems). Let $\langle\mathcal{K},\perp\rangle$ be weak forking notion. Suppose $I\subseteq\mathcal{P}^-(n)$, suppose $\mathbf{S}=\{M_s\mid s\in I\}$ is a (λ,n)-system. The system $\mathbf{S}$ is called (λ,n)-*stable* iff for every enumeration $\bar{s}:=\langle s(i)\mid i<m\rangle$ of I (always without repetitions such that $s(i_1)<_I s(i_2)\implies i_1<i_2$)

(1) $A^{\mathbf{S}}_{s(i)}$ is good for all i,
(2) for every $\langle\mathbf{b}_i\in|M_s(i)|\ :\ i\leq j\leq m\rangle$ there are $\langle\mathbf{b}'_i\in|M_{s(i)\cap s(j)}|\ :\ i\leq m\rangle$ such that
   (a)
   $$\text{ga-tp}(\mathbf{b}_0,\mathbf{b}_1,\dots/|M_\emptyset|)=\text{ga-tp}(\mathbf{b}'_0,\mathbf{b}'_1,\dots/|M_\emptyset|)$$
   and
   (b) $s(i)\leq s(j)\implies\mathbf{b}'_i=\mathbf{b}_i$.
(3)
$$A^{\mathbf{S}}_{s(j)}\underset{|M_{s(j)}|}{\perp}\bigcup_{i<j}|M_{s(i)}|.$$

Axiom 1.7 (Generalized Symmetry). *Let $\langle\mathcal{K},\perp\rangle$ be weak forking notion. We say that $\langle\mathcal{K},\perp\rangle$ has the* (λ,n)-symmetry property *iff for every $I\subseteq\mathcal{P}^-(n)$ and every $\mathbf{S}=\{M_s\mid s\in I\}$ (λ,n)-system $\mathbf{S}$. The system is (λ,n)-stable iff there exists an enumeration $\bar{s}$ of I satisfying requirements* (1), (2) *and* (3) *of the previous definition.*

CHECK if follows from symmetry.

Definition 1.8 (n-dimensional amalgamation). Let $\langle\mathcal{K},\perp\rangle$ be weak forking notion, it has the (λ,n)-*existence property* iff for every stable system $\mathbf{S}=\langle M_s\mid s\in\mathcal{P}^-(n)\rangle$ of models of cardinality λ, there exists a model over the set $A^{\mathbf{S}}_n$.

Definition 1.9 (systems are amalgamation bases). Let $\langle\mathcal{K},\perp\rangle$ be weak forking notion, it has the (λ,n)-*non-uniqueness property* iff for every stable system $\mathbf{S}=\langle M_s\mid s\in\mathcal{P}^-(n)\rangle$ we have that $A^{\mathbf{S}}_n\in\mathrm{Ab}(\mathcal{K})$.

Definition 1.10 (goodness)**.** Let $\langle \mathcal{K}, \perp \rangle$ be weak forking notion, it has the (λ, n)-*goodness property* iff $\langle \mathcal{K}, \perp \rangle$ has the (λ, n)-symmetry property and for every stable system $\mathbf{S} = \langle M_s \mid s \in \mathcal{P}^-(n) \rangle$ of models of cardinality λ, has the (λ, n)-existence property and the (λ, n)-non-uniqueness property.

Theorem 1.11 (characterizing goodness for f.o.)**.** *Let T be a complete countable f.o theory. Suppose T is superstable without dop If $\mathbf{S} = \langle M_s \mid s \in \mathcal{P}^-(n) \rangle$ is a stable system of models of cardinality $\aleph_0$ then TFAE*

(1) *the set $A_n^{\mathbf{S}}$ is an amalgamation base*
(2) *Thee is a prime and minimal model over $A_n^{\mathbf{S}}$.*

Definition 1.12 (excellence)**.** Let $\langle \mathcal{K}, \perp \rangle$ be weak forking notion and let $\lambda \geq \mathrm{LS}(\mathcal{K})$. $\langle \mathcal{K}, \perp \rangle$ is λ-*excellent* iff $\langle \mathcal{K}, \perp \rangle$ has the (λ, n)-goodness property for every $n < \omega$. When $\lambda = \mathrm{LS}(\mathcal{K})$ we say that $\mathcal{K}$ excellent instead of λ-excellent.

Theorem 1.13 (Shelah 1982)**.** *Let T be a complete countable f.o theory. Suppose T is superstable without dop. TFAE*

(1) $\langle \mathrm{Mod}(T), \prec \rangle$ *is excellent.*
(2) $\mathrm{Mod}(T)$ *has the* $(\aleph_0, 2)$*-goodness property.*
(3) T *does not have the otop.*

For proof see [Sh c]....

Remark 1.14. Even for complete first-order theories in general the (λ, n)-amalgamation property may fail. Failure of $(\aleph_0, 3)$-amalgamation is witnessed by the example of a triangle-free random graph. Start with a triple of models M_i, $i < 3$, and fix some elements $a_i \in M_i$. Take a triple of models M_{01}, M_{02}, and M_{12} that form an $(\aleph_0, \mathcal{P}^-(3))$-system over M_i, and such that $M_{ij} \models R(a_i, a_j)$ for $i < j < 3$. The system cannot be amalgamated since the amalgam would witness a triangle.

COMMENT: this example was suggested by Shelah. It is an example of a non-simple theory. It can be generalized to a failure of $(\aleph_0, n+1)$-amalgamation by using n-dimensional tetrahedron-free graphs. Those examples will be simple first order theories.

There is an example of a triple of totally categorical theories T_{ij}, $i < j < 3$, that are pairwise coherent, but cannot be "amalgamated" into a consistent first order theory, i.e., 3-dimensional Robinson's consistency test fails:

For $i < 3$, let $T_\emptyset$ be the theory of an infinite set. For $i < 3$, let $L_i := \{P_i, f_i\}$, and T_i says that the model is divided by P_i into two parts of equal size, as witnessed by f_i. For $i < j < 3$, T_{ij} contains the union of T_i and T_j, and says that $P_i(x) \iff \neg P_j(x)$. Then clearly the union $\bigcup_{i<j<3} T_{ij}$ is inconsistent.

Fact 1.15 (Hart and Shelah 1986)**.** *For every $n < \omega$ there is an $\aleph_0$-atomically stable class $\mathcal{K}_n$ of atomic models of a countable f.o. theory such that $\mathcal{K}$ is has the $(\aleph_0, k)$-goodness property for all $k < n$ but is not excellent.*

Theorem 1.16. *If $\langle \mathcal{K}, \perp \rangle$ is excellent then it has the (λ, n)-goodness property for every $n < \omega$ and every $\lambda \geq \mathrm{LS}(\mathcal{K})$.*

Proof. Will be added. $\dashv$

$\mathcal{K}$ is

2. Tameness of AEC with n-amalgamation

In this section, $\mathcal{K}$ is an AEC with 2-amalgamation and arbitrarily large models.

Theorem 2.1. *If $\langle \mathcal{K}, \perp \rangle$ is excellent then $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-tame.*

Proof. Let κ be the least uncountable cardinal witnessing that $\mathcal{K}$ is not $\mathrm{LS}(\mathcal{K})$-tame.

Thus there are M, N_0, N_1 be of size κ, $\mathbf{a}_i \in N_i$, $i = 0, 1$, realize the same Galois type over every $\mathcal{K}$-submodel of M of size $\mathrm{LS}(\mathcal{K})$ such that the Galois types of $\mathbf{a}_0$, $\mathbf{a}_1$ over M are different. By renaming some of the elements we may assume that $N_0 \cap N_1 = M$.

By the existence and invariance properties we may assume that

$$N_0 \underset{M}{\perp} N_1.$$

Let χ be a regular cardinal large enough so that $N_1, N_0, M \in H(\chi)$ and also the definition of $\mathcal{K}$ is there as well as $\langle H(\chi), \in \rangle$ reflects all the relevant information e.g.

$$\langle H(\chi), \in \rangle \models N_0 \underset{M}{\perp} N_1.$$

Now pick $\{\mathcal{B}_i \prec \langle H(\chi), M, N_0, N_1 \ldots \in \rangle \mid i < \kappa\}$ such that $\|\mathcal{B}_i\| = |i| + \mathrm{LS}(\mathcal{K})$ and $\langle \mathcal{B}_j \mid j \leq i \rangle \in \mathcal{B}_{i+1}$ for all $i < \kappa$

By minimality of κ, the Galois types of $\mathbf{a}_0$, $\mathbf{a}_1$ are the same over every $\mathcal{K}$-submodel of M of size less than κ. To get a contradiction, we construct a model N_{01} and embeddings $f_0 : N_0 \to N_{01}$ and $f_1 : N_1 \to N_{01}$ that fix M and map $\mathbf{a}_0$, $\mathbf{a}_1$ to the same sequence.

Let $\{(M^i, N_0^i, N_1^i) \mid i < \kappa\}$ be the interpretation of the corresponding models in $\mathcal{B}_i$ so we have

(1) $\mathbf{a}_\ell \in N_\ell^0$ for $\ell = 0, 1$;
(2) M^i, N_0^i, and N_1^i are $\prec_{\mathcal{K}}$-increasing continuous chains of $\mathcal{K}$-models with union M, N_0, and N_1 respectively.
(3) $\|M^i\| = \|N_0^i\| = \|N_1^i\| = |i| + \mathrm{LS}(\mathcal{K})$, for all $i < \kappa$.
(4) $N_0^i \underset{M^i}{\perp} N_1^i$ for all $i < \kappa$, $\ell = 0, 1$.

By induction on $i \leq \kappa$, define a model N_{01}^i and embeddings $f_\ell^i : N_\ell^i \to N_{01}^i$, $\ell = 0, 1$. In addition, we need to keep track of embeddings $f_{01}^{ij} : N_{01}^i \to N_{01}^j$ such that $\{N_{01}^i, f_{01}^{ij} \mid i < j < \alpha\}$ form a direct system of $\mathcal{K}$-submodels.

Base $i = 0$: since the Galois types of $\mathbf{a}_0$, $\mathbf{a}_1$ over M^0 coincide, there is a model N^0_{01} and embeddings $f^0_\ell : N^0_\ell \to N^0_{01}$ that map $\mathbf{a}_0$, $\mathbf{a}_1$ together.

Successor step. We have $f^i_\ell : N^i_\ell \to N^i_{01}$ for $\ell = 0, 1$. We also have the identity embeddings $M^i \to M^{i+1}$, $N^i_\ell \to N^{i+1}_\ell$, $\ell = 0, 1$. the picture is:

$$
\begin{tikzcd}
 & & N_0^{i+1} \arrow[r, dotted, "f_0^{i+1}"] & N_{01}^{i+1} \\
M^{i+1} \arrow[urr, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & N_1^{i+1} \arrow[urr, dotted, "f_1^{i+1}"'] & & \\
 & & N_0^i \arrow[uu, "\mathrm{id}"] \arrow[r, "f_0^i"] & N_{01}^i \arrow[uu, dotted, "\mathrm{id}"'] \\
M^i \arrow[uu, "\mathrm{id}"] \arrow[urr, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & N_1^i \arrow[uu, "\mathrm{id}"] \arrow[urr, "f_1^i"'] & &
\end{tikzcd}
$$

Let $\lambda := |i| + \mathrm{LS}(\mathcal{K})$. By 3-amalgamation, we get N^{i+1}_{01} and embeddings $f^{i+1}_\ell : N^{i+1}_\ell \to N^{i+1}_{01}$ for $\ell = 0, 1$. For the direct system part, 3-amalgamation gives $N^{i+1}_{01} \succ_{\mathcal{K}} N^i_{01}$ and $\mathcal{K}$-embeddings $f^{i+1}_\ell : N^{i+1}_\ell \to N^{i+1}_{01}$.

Limit step. We have that $\{N^i_{01}, f^{ij}_{01} \mid i < j < \alpha\}$ form an $\prec_{\mathcal{K}}$-chain. Let N^α_{01} be the union and f^i_ℓ be the union of the corresponding chain of $\prec_{\mathcal{K}}$-embeddings. By Axiom A4 this is what we need.

Finally, the model N^κ_{01}, and the maps f^κ_ℓ, $\ell = 0, 1$ are as needed. The image of $\mathbf{a}_0$ under f^κ_0 is

$$f^\kappa_0(\mathbf{a}_0) = f^\kappa_1(\mathbf{a}_1),$$

i.e., is the same as the image of $\mathbf{a}_1$ under f^κ_1. $\dashv$

A similar proof gives several related theorems, e.g.:

Theorem 2.2. *Let $\mathcal{K}$ be an AEC, and $\mu_0 > \mathrm{LS}(\mathcal{K})$ if $\mathcal{K}$ has the $(\lambda, 3)$-AP for all $\mathrm{LS}(\mathcal{K}) \leq \lambda < \mu_0$ then given $M \in \mathcal{K}_{\mu_0}$ for any $p \neq q \in \text{ga-S}(M)$ there is $N \prec_{\mathcal{K}} M$ of cardinality $\mathrm{LS}(\mathcal{K})$ such that $p \restriction N \neq q \restriction N$.*

Theorem 2.3. *Suppose that $\mathcal{K}$ has $(\aleph_0, n)$-amalgamation property for all $n < \omega$. Then $\mathcal{K}$ has (λ, n)-amalgamation for all λ.*

Proof. The statement follows from the two claims:

Claim 2.4. *Suppose that $\mathcal{K}$ has $(\lambda, n + 1)$-amalgamation. Then $\mathcal{K}$ has (λ^+, n)-amalgamation.*

Claim 2.5. *Suppose that λ is a limit cardinal and $\mathcal{K}$ has $(<\lambda, n+1)$-amalgamation. Then $\mathcal{K}$ has (λ, n)-amalgamation.*

Indeed, $(\aleph_\alpha, n)$-amalgamation property for all $n<\omega$ for $\mathcal{K}$ implies $(\aleph_{\alpha+1}, n)$-amalgamation property for all $n<\omega$ for $\mathcal{K}$ by Claim 2.4. Claim 2.5 gives $(\aleph_\alpha, n)$-amalgamation property for limit α, for all $n<\omega$.

Proof of Claim 2.4. Let $\{M_s \mid s \in \mathcal{P}^-(n)\} \subset \mathcal{K}_\lambda$ be an incomplete n-diagram of models in $\mathcal{K}$. Our goal is to find M_n and the embeddings $\{f_s \mid s \subset_{n-1} n\}$, $f_s : M_s \to M_n$ that make the diagram commute.

Take $\{M_s^i \mid i<\lambda^+, s \in \mathcal{P}^-(n)\}$ a resolution of the incomplete n-diagram. We may assume that $|M_s^i| = \lambda$ for all s, i.

By induction on $i \leq \lambda^+$, define a model M_n^i and embeddings $f_s^i : M_s^i \to M_n^i$, for each $s \subset_{n-1} n$. As before, we will keep track of embeddings $f_n^{ij} : M_n^i \to M_n^j$ such that $\{M_n^i, f_n^{ij} \mid i<j<\alpha\}$ form a direct system of $\mathcal{K}_\lambda$-submodels.

For the base case, we just take a completion of the n-diagram $\{M_s^0 \mid s \in \mathcal{P}^-(n)\}$. It exists since we are assuming $(\lambda, n+1)$-amalgamation.

Successor step. We have $f_s^i : M_s^i \to M_n^i$ for $s \subset_{n-1} n$. We also have the identity embeddings $M_s^i \to M_s^{i+1}$, $s \subsetneq n$. By $(\lambda, n+1)$-amalgamation, we get M_n^{i+1} and embeddings $f_s^{i+1} : M_s^{i+1} \to M_n^{i+1}$ for $s \subset_{n-1} n$. For the direct system part, $(\lambda, n+1)$-amalgamation also gives $f_n^i : M_n^i \to M_n^{i+1}$. So we let $f_n^{j,i+1} := f_n^i \circ f_n^{ji}$ for $j<i$, and $f_n^{i,i+1} := f_n^i$.

Limit step. We have that $\{M_n^i, f_n^{ij} \mid i<j<\alpha\}$ form a direct system. Let M_n^α be the direct limit of the system. As before, we define the maps from M_s^α to M_n^α by

$$f_s^\alpha := \bigcup_{i<\alpha} f_n^{i,\alpha} \circ f_s^i$$

for $s \subset_{n-1} n$.

Finally, the model $M_n^{\lambda^+}$, and the maps $f_s^{\lambda^+}$, $s \subset_{n-1} n$, are as needed. ⊣

Proof of Claim 2.5. Is almost exactly the same, the only difference is that the cardinality of models in the resolution will be $|i| + \aleph_0$. ⊣

⊣

From the two theorems above we easily get

Corollary 2.6. *If $\mathcal{K}_{\aleph_0}$ has n-amalgamation property for all $n<\omega$, then $\mathcal{K}$ is $\aleph_0$-tame.*

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