Model Theory, basic test January 2023 **Prof. Rami Grossberg**

Answer only five questions. No notes or electronics are allowed. You have three hours for the test and can use up to another 15 minutes to upload the test. Please keep your cameras on while taking the test. When done send it to rami@cmu.edu

(1) Let M be a saturated model of cardinality greater than the number of formulas of its language. Show that for every $N \prec M$ and every $N' \succ N$ if $\|N\| < \|M\|$ and N' of cardinality at most $\|M\|$ there exists an elementary embedding $f : N' \to M$ such that $f \restriction N = \mathrm{id}_N$.

(2) Let T be the theory of the field of real numbers (in the language $\{+, \cdot, 0, 1\}$). Prove that T is not categorical in $\aleph_1$, state the theorems you are using.

(3) A theory T has the *independence property* provided there exists a formula $\varphi(\mathbf{x}; \mathbf{y})$ with $\ell(\mathbf{x}) = \ell(\mathbf{y})$ such that for every $n < \omega$ there exists $\{\mathbf{a}_i \mid i < n\} \subset {}^{\ell(\mathbf{x})}|\mathfrak{C}|$ such that

$$\mathfrak{C} \models \exists \mathbf{x} \bigwedge_{i<n} \varphi^{i \in S}(\mathbf{x}; \mathbf{a}_i) \text{ for every } S \subseteq n.$$

Prove that a theory with the independence property is unstable.

(4) Let T be a complete first order theory in a countable language and $\mathfrak{C} \models T$ a monster model. Suppose that $A \subseteq |\mathfrak{C}|$ is countable and $B \subseteq |\mathfrak{C}|$ (we don't require that A is a subset of B). If $p \in \mathrm{S}(B)$ is not isolated over A then there exists a countable $B_0 \subseteq B$ such that $p \restriction B_0$ is not isolated over A.

(5) If T is an $\aleph_0$-stable theory in a countable language then the isolated types are dense.

(6) Let T be a complete theory in a countable language and T is $\aleph_0$-stable. If $\lambda > \aleph_0$ is regular and $M \models T$, $A \subseteq |M|$ of cardinality less than λ then there exists $\{a_i \mid i < \lambda\} \subseteq |M|$ an indiscernible sequence over A.

(7) Let T be a complete theory in a countable language and T is $\aleph_0$-stable. Suppose that p is a type and $A \subseteq \operatorname{dom} p$. Show that if $R[p] = R[p \restriction A]$ then p does not split over A. State all the theorems you are using.

(8) Let T be a countable first-order theory. Suppose that $\mathrm{R}^1[x = x]$ is an ordinal. Prove that T is λ-stable for all $\lambda \geq \aleph_0$.