CATEGORICITY OF AN ABSTRACT ELEMENTARY CLASS IN TWO SUCCESSIVE CARDINALS SH576

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Abstract. We investigate categoricity of abstract elementary classes without any remnants of compactness (like non-definability of well ordering, existence of E.M. models or existence of large cardinals). We prove (a weak version of GCH around λ) that if $\mathfrak{K}$ is categorical in λ, λ^+, $LS(\mathfrak{K}) \le \lambda$ and $1 \le I(\lambda^{++}, \mathfrak{K}) < 2^{\lambda^{++}}$ <u>then</u> $\mathfrak{K}$ has a model in λ^{+++}.

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Anotated Content

§0 Introduction

[In addition to background and an explanation of what is done, $\langle 0.1\rangle$ - $\langle 0.23\rangle$, we review the relevant knowledge of abstract elementary classes to help make this paper self-contained, including representation by PC-classes defined by omission of quantifiers free types; types (based on $\leq_{\mathfrak{K}}$); and saturation equivalent to model homogeneity.]

§1 Weak Diamond

[We mainly present necessary material on the weak diamond, a combinatorial principle where the main variant holds for λ^+ if $2^\lambda < 2^{\lambda^+}$.
We define the variants needed and state cases provable from ZFC + suitable cardinal arithmetic assumptions. We then present application to the number of models, and define super limit models (and variants). We deal also with the definable weak diamond, and introduce a smaller ideal of "small" sets: $\mathrm{UDmId}^{\mathcal{F}}(\lambda)$.]

§2 First Attempts

[We define the class K^3_λ of triples (M,N,a) representing types in $\mathcal{S}(M)$, $M \in K_\lambda$; and start to investigate it dealing with the weak extension property, the extension property, being minimal, being reduced (except minimality, for the first order case, they hold trivially). Our aims are to have the extension or at least the weak extension property (for all triples in K^3_λ) and the density of the minimal triples. The first makes the model theory more like the first order case, the second is connected with categoricity. We start by proving the weak extension property under reasonable assumptions. We prove the density of minimal triples under the assumption $K_{\lambda^{+3}} = \emptyset$ and extra cardinal arithmetic assumption ($2^{\lambda^+} > \lambda^{++}$). In the end under the extra assumption $K_{\lambda^{+3}} = 0$ we prove that all triples have the extension property and we have disjoint amalgamation in K_λ. Now the assumption $K_{\lambda^{+3}} = 0$ does not harm if we just want to prove Theorem $\langle 0.\mathrm{B}\rangle$. This enables the reader willing to accept these assumptions to skip some proofs later.]

§3 The non-structure

[We try to present clearly the construction of many models in λ^{++} by our knowledge of models of size λ^+, using weak diamonds on λ^+ and on λ^{++}. This is done by forming a tree $\langle \bar M^\eta : \eta \in {}^{\lambda^{++}>}2\rangle$ with $\bar M^\eta$ an $\leq_{\mathfrak{K}}$-increasing continuous sequence of members of K_λ with limit, $\bigcup\limits_{i<\lambda^+} M^\eta_i$ increasing with η (and an additional restriction). Actually λ^+ can be replaced by a regular uncountable λ' (so $\|M^\eta_i\| = \lambda$ is replaced by $\|M^\eta_i\| < \lambda'$)].

§4 Minimal types

[We prove that every member of K^3_λ has the extension property, by proving it for minimal triples. We use for it: if $M_\ell \in K_\lambda$, for every minimal

$p \in \mathcal{S}(M_0)$ the set $\mathcal{S}_{\geq p}(M_1)$ has cardinality $\leq \lambda^+$ and then the $M \in K_{\lambda^+}$ is saturated for minimal types hence the number of minimal types in $\mathcal{S}(M_1)$ (for $M_1 \in K_\lambda$) is $\leq \lambda^+$, a step toward stability in λ.]

§5 Inevitable types and stability in λ

[We continue to "climb the ladder", using the amount of structure we already have (and sometimes categoricity) to get more. We start assuming there are minimal types, show that some minimal types are inevitable, construct $p_i \in \mathcal{S}(N_i)$ minimal ($i \leq \lambda^+$) both strictly increasing continuous and p_0, p_δ inevitable, and then as in the proof of saturativity = model homogeneity we show N_δ is universal over N_0. We can then deduce stability in λ, so the model in λ^+ is saturated. We finish noting that we have disjoint amalgamation in K_λ.]

§6 Proving for $\mathfrak{K}$ categorical in λ^{+2}

[We give a shortcut to proving the main theorem by using stronger assumptions. If $I(\lambda^{+2}, K) = 1, I(\lambda^{+3}, K) = 0$, then in some triple $(M, N, a) \in K_{\lambda^+}$, a is essentially algebraic over M, i.e. this is a maximal triple. Now first assuming for some pair $M_0 \leq_{\mathfrak{K}} M_2$ in K_λ we have unique amalgamation for every possible $M_1, M_0 \leq_{\mathfrak{K}} M_1 \in K_1$ (and using stability), we get models in λ^+ thus contradicting the existence of maximal triples. We then use the methods of §3 to prove there are enough cases of unique amalgamation.]

§7 Extensions and Conjugacy

[We investigate types, we prove that (in $\mathcal{S}(N), N \in K_\lambda$) reduced implies inevitable and non-algebraic extensions preserve the conjugacy classes (so solving the realize/ materialize problem). So if $\langle N_i : i < \alpha \rangle$ is $<_{\mathfrak{K}}$-increasing in K_λ, λ divides α then N_α is $(\lambda, \mathrm{cf}(\alpha))$-saturated over N_0.]

§8 Uniqueness of Amalgamation in $\mathfrak{K}_\lambda$

[We have by §6 only $M_0^* <_{\mathfrak{K}} M_2^*$ in $\mathfrak{K}_\lambda$ such that $M_0^* \leq_{\mathfrak{K}} M_1 \Rightarrow M_0^*, M_1, M_2^*$ has unique (disjoint) amalgamation. Now if we have a $\leq_{\mathfrak{K}}$-increasing continuous sequence $\langle N_i : i \leq \alpha \rangle$ such that $(N_i, N_{i+1}) \cong (M_0^*, M_2^*)$, we can amalgamate N_0, M_1, N_α whenever $N_0 \leq_{\mathfrak{K}} M_1$ step by step. So some uniqueness is preserved and N_α can be any $(\lambda, \mathrm{cf}(\alpha))$-saturated model over N_0. When we specify also saturativity of M_i and of the resulting model we get the nonforking relation $\mathrm{NF}_{\lambda,\bar{\delta}}$ (on models), closing down we define the general nonforking relation NF (on models). So we succeed to define a relation which will behave as a nice nonforking relation - if there is one. But we have to work proving that the relations satisfy the expected properties, first for the "saturated" version and then in the general case (chasing lots of arrows, of course).]

§9 Nice extensions in K_{λ^+}

[As we have "nonforking" amalgamations in K_λ, we can use them to build $\leq_{\mathfrak{K}}$-extensions $M_1 \in K_{\lambda^+}$ for any pregiven $M_0 \in K_{\lambda^+}$. This defines naturally a two-place relation $\leq^*_{\lambda^+}$ on K_{λ^+}; "being a nice $\leq_{\mathfrak{K}}$-submodel". We investigate it and variants. In particular, we prove the existence of disjoint amalgamation.]

§10 Non-structure for $<^*_{\lambda^+}$

[Instead proving all disjoint amalgamations in K_λ are nonforking ones, we prove that on K_{λ^+} that $<^*_{\lambda^+}$ is the same as $\leq^*_\lambda$, which is just as good for our purpose. Toward this we assume a failure and get many pairwise non-isomorphic models in $K_{\lambda^{+2}}$, contradicting an assumption of $\langle 0.\mathrm{B}\rangle(2)$. But once we have that $\leq_{\mathfrak{K}}$ agrees on K_{λ^+} with $\leq^*_{\lambda^+}$ we have disjoint amalgamation, which suffices for building a model in $K_{\lambda^{+3}}$.]

§0 Introduction

Makowski [Mw85] is a readable and good exposition concerning categoricity (in abstract elementary classes), etc., around $\aleph_1$.
Our real issue is

Problem A. Can we have some (not necessarily much) classification theory on reasonable non-first order class $\mathfrak{K}$ of models, with no traces of compactness and only soft set theoretic assumptions?

Below $I(\mu, \mathfrak{K})$ is the number of model of $\mathfrak{K}$ of cardinality μ up to isomorphisms and if $\lambda = \aleph_\alpha$ let $\lambda^{+\beta} = \aleph_{\alpha+\beta}$.

Let me try to clarify the meaning of the problem A.
What is the meaning of "<u>soft set theoretic assumptions</u>?" We are allowing requirements on cardinal arithmetic (like GCH and weaker relatives). Preferably, assumptions like diamonds and squares and mild large cardinals will not be used (this does not include cases provable in ZFC or in ZFC plus allowable assumptions). Certainly, we will not like to use morasses, $V = L$, special universes constructed by forcing to fit our proof or non-mild large cardinals.

Of course, such ordering reflects taste, but I think the one above is widely held; note in particular, that in fact we try to continue [Sh 88] where results about the number of non-isomorphic models in $\aleph_1$ and $\aleph_2$ of say a sentence $\psi \in L_{\omega_1,\omega}$ are obtained. Now in [Sh 88] the theorem parallel to the present one is proved assuming $2^{\aleph_0} < 2^{\aleph_1}$, so it is quite natural to use such assumptions here.

What is the meaning of "some classification theory?" The dream is to have a classification theory as "full" as the one obtained in [Sh:c]. But we will be glad to have theorems speaking just on having unique models in some cardinals and at least one in others. E.g. by [Sh 88] if $\psi \in L_{\omega_1,\omega}$ satisfies $1 \leq I(\aleph_1, \psi) < 2^{\aleph_1}$ (and $2^{\aleph_0} < 2^{\aleph_1}$) then $I(\aleph_2, \psi) > 0$.

What are "reasonable non first order classes?" This means we allow "locally finite", "atomic", "omitting a type" or more generally the class of models of a sentence in $L_{\kappa,\omega}$, (i.e. conjunction $< \kappa$ but quantification only over a finite string) but not one restricting ourselves to e.g. well ordering. In fact, we use "abstract elementary classes" from [Sh 88] (but presented below).

What is the meaning of "(without) traces of compactness?" Allowing non first order (i.e. elementary) classes means we cannot use the (very powerful) compactness theorem. But there are many ways to get weak forms of it: one way is using large cardinals (compact cardinals in Makkai Shelah [MaSh 285], or just measurable cardinals as in Kolman Shelah [KlSh 362], or in [Sh 472]). Another way is to have "non-definability of well ordering" which follows from the existence of Ehrenfeucht Mostowski models, and also from $\psi \in L_{\omega_1,\omega}$ (used extensively in [Sh 88]). Our aim is to use <u>none of this</u> and we would like to see if any theory is left.

What is not said explicitly is that we hope the proofs will help to start classification theory in this case, so we hope the flavour will be of introducing and investigating notions of model theoretic character, which look relevant to this hope. Proofs of, say, descriptive theory character, will not satisfy this hope.

It seems to us that this is accomplished here. It is proved.

Theorem ⟨0.B⟩. $(2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}})$. *Let $\mathfrak{K}$ be an abstract elementary class. If $\mathfrak{K}$ categorical in $\lambda, \lambda^+, \lambda^{++}$ <u>then</u> $I(\lambda^{+3}, \mathfrak{K}) > 0$.*

Of course, the categoricity in three successive cardinals is a strong assumption. Now note that in [Sh 88], the categoricity in $\aleph_0$ is gained "freely", so the gap is smaller than seems at first glance. Still it is better to have

Theorem ⟨0.C⟩. $(2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}})$. *Let $\mathfrak{K}$ be an abstract elementary class. If $\mathfrak{K}$ is categorical in λ, λ^+ and $1 \le I(\lambda^{++}, \mathfrak{K}) < 2^{\lambda^{++}}$ <u>then</u> $I(\lambda^{+3}, \mathfrak{K}) > 0$. Here we do not fully prove ⟨0.C⟩ but we prove a slightly weaker version (see explanation and discussion below), particularly on $\mu_{\rm wd}(\lambda^{++})$).*

Theorem ⟨0.D⟩. *If $\mathfrak{K}$ is categorical in $\lambda, \lambda^+, \lambda > \aleph_0$ and $1 \le I(\lambda^{++}, \mathfrak{K}) < \mu_{\rm wd}(\lambda^{++})$, <u>then</u> $I(\lambda^{+3}, \mathfrak{K}) > 0$.*

Note however

- (α) A silly point; at exactly one point in the proof of ⟨0.D⟩ we use $\lambda > \aleph_0$ (in the proof of ⟨4.2B⟩), this is silly as our intent is to prove for general λ what we know for $\lambda = \aleph_0$ by [Sh 88]; however, there we assume $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_1,\aleph_0}$, a reasonable assumption but which is not assumed here. So we shall complete this in [Sh 600, §2], so we do not mention the assumption $\lambda > \aleph_0$ in theorems relying on ⟨4.2B⟩.
- (β) More serious, at some point we assume toward contradiction $K_{\lambda^{+3}} = \emptyset$. This is fine for proving theorem ⟨0.B⟩, but is not desirable if we want to develop a classification theory. Again this will be dealt with in [Sh 600, §2],
- (γ) Concerning $\mu_{\rm wd}(\lambda^{+2})$, i.e. in ⟨6.6⟩ we get $I(\lambda^{++}, K) \ge \mu_{\rm wd}(\lambda^{++})$ instead $I(\lambda^{++}, K) = 2^{\lambda^{++}}$ (eliminated in [Sh 600], too), note $\mu_{\rm wd}(\chi)$ is defined in ⟨1.1⟩(6), it is almost 2^χ; i.e. if $\chi \ge \beth_\omega$ equality holds and always $2^\chi < \mu_{\rm wd}(\chi^+) \le 2^{\chi^+} \le \mu_{\rm wd}(\chi^+)^{\aleph_0}$. The point is that already in [Sh 87b] sometimes instead of getting 2^χ we get only $\mu_{\rm wd}(\chi)$ (again see [Sh 600]). Note that if you are satisfied with $I(\chi, K) \ge \mu_{\rm wd}(\chi)$ in those cases, you can ignore some proofs.

We present below, as background, the following open questions which appeared in [Sh 88], $\mathfrak{K}$ an abstract elementary class, of course, e.g. the class of models of $\psi \in L_{\lambda^+,\omega}$ with $M \le_{\mathfrak{K}} N$ being $M \prec_{\mathcal{L}} N, \mathcal{L}$ a fragment of $L_{\kappa^+,\omega}$ to which ψ belongs of course and <u>where</u> in [Sh 88] we consider the statements: (proved there)

- $(*)_3$ categoricity (of $\psi \in L_{\omega_1,\omega}(Q)$) in $\aleph_1$ implies the existence of a model of ψ of cardinality $\aleph_2$
- $(*)_4$ if $n > 0, 2^{\aleph_0} < 2^{\aleph_1} < \cdots < 2^{\aleph_n}, \psi \in L_{\omega_1,\omega}$ and $1 \le I(\aleph_\ell, \psi) < \mu_{wd}(\aleph_\ell)$ for $\ell \le n, \ell \ge 1$ (see (γ) above) <u>then</u> ψ has a model of cardinality $\aleph_{n+1}$.

Now the problems were:

<u>Problem</u> 1) Proving $(*)_3, (*)_4$ in our context (meant: for abstract elementary class $\mathfrak{K}$ which is $PC_{\aleph_0}$).

Problem 2) Parallel results in ZFC; e.g. prove $(*)_3$ when $n = 1, 2^{\aleph_0} = 2^{\aleph_1}$.

Problem 3) Construct examples; e.g. $\mathfrak{K}$ (or $\psi \in L_{\omega_1,\omega}$), categorical in $\aleph_0, \aleph_1, \dots, \aleph_n$ but not in $\aleph_{n+1}$.

Problem 4) If $\mathfrak{K}$ is PC_λ (abstract elementary class), categorical in λ, λ^+, does it necessarily have a model in λ^{++}?

Concerning Problem 3, by Hart Shelah [HaSh 323],$\langle 2.10\rangle(2)$ + $\langle 3.8\rangle$ there is $\psi_n \in \mathcal{L}_{\omega_1,\omega}$ categorical in $\aleph_0, \aleph_1, \dots, \aleph_{k-1}$, but not categorical in λ if $2^\lambda > 2^{\aleph_{k-1}}$. The direct motivation for this work, is that Grossberg asked me (Oct.94) some questions in this neighborhood, in particular

Problem 5) Assume $K =$ Mod(T), (i.e. K is the class of models of T), $T \subseteq L_{\omega_1,\omega}, |T| = \lambda$, $I(\lambda, K) = 1$ and $1 \le I(\lambda^+, K) < 2^{\lambda^+}$. Does it follow that $I(\lambda^{++}, K) > 0$?
We think of this as a test problem and much prefer a model theoretic to a set theoretic solution. This is closely related to Problem 4 above and to [Sh 88, Theorem 3.7] (where we assume categoricity in λ^+, do not require $2^\lambda < 2^{\lambda^+}$ but $\lambda = \aleph_0$ or some similar cases) and [Sh 88, Theorem 5.17(4)] (and see [Sh 88, 5.1,4.5] for the assumption) (there we require $2^\lambda < 2^{\lambda^+}, 1 \le I(\lambda^+, K) < 2^{\lambda^+}$ and $\lambda = \aleph_0$).
As said above, we are dealing with a closely related problem. Problem A was stated a posteriori but is, I think, the real problem.

In a first try we use more set theory, i.e. we use the definitional weak diamond on both λ^+ and λ^{++} (see Definition $\langle 1.7\rangle$) and things like "a nice equivalence relation on $\mathcal{P}(\lambda)$ has few or many classes" (see §1).
Here we use a model theoretic approach.

We feel that this paper is a reasonable positive answer of problem A, the solution having classification theory flavor. We shall continue it in [Sh 600] toward getting a parallel of [Sh 87a], [Sh 87b]. Grossberg Shelah, in the mid-eighties, started to write a paper [GrSh 266] to prove that: if $\psi \in L_{\lambda^+,\omega}$ has models of arbitrarily large cardinality, categorical in μ^{+n} for each $n, \mu \ge \lambda$ and $2^{\mu^{+n}} < 2^{\mu^{+n+1}}$ for $n < \omega$, then ψ is categorical in every $\mu' \ge \mu$; this is a weak form of the upward part of Los conjecture. See Makkai Shelah [MaSh 285] on $T \subseteq \mathcal{L}_{\kappa,\omega}$ where κ is a compact cardinal where we get downward and upward theorems for successor cardinals which are enough bigger than $\kappa + |T|$. On the downward part, see [KlSh 362],[Sh 472] which deals with downward theorem for successor cardinals which are larger enough than $\kappa + |T|$ when the theory T is in the logic $\mathcal{L}_{\kappa,\omega}$ and κ is measurable. See also [Sh 394] which deals with abstract elementary classes with amalgamation getting similar results (with no large cardinals).
Part of §1 and mainly §3 have combinatorial character. Most of the paper is from the context of a course in Fall '94 (essentially without §3, §9, §10). The paper is written having in mind more developing the model theory than just proving Theorem $\langle 0.B\rangle$.

* * *

We try to make the paper self-contained.
On abstract elementary classes see [Sh 88] and [Sh 300, II,§3]. To make the paper

self-contained, we quote here some relevant definitions and results.
This work is continued in [Sh 600].

Abstract elementary classes

⟨*0.1*⟩ *Conventions.* Here $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$, where K is a class of τ-models for a fixed vocabulary $\tau = \tau_K = \tau_{\mathfrak{K}}$ and $\leq_{\mathfrak{K}}$ is a two-place relation on the models in K. We do not always strictly distinguish between K and $(K, \leq_{\mathfrak{K}})$. We shall assume that $K, \leq_{\mathfrak{K}}$ are fixed, and $M \leq_{\mathfrak{K}} N \Rightarrow M, N \in K$; and we assume that the following axioms hold. When we use $<$ in the $\prec$ sense (elementary submodel for first order logic), we write $<_{\mathcal{L}_{\omega,\omega}}$.

⟨**0.2**⟩ **Definition.** $Ax0$: The holding of $M \in K, N \leq_{\mathfrak{K}} M$ depends on N, M only up isomorphism i.e. $[M \in K, M \cong N \Rightarrow N \in K]$, and [if $N \leq_{\mathfrak{K}} M$ and f is an isomorphism from M onto the τ-model M' mapping N onto N' then $N' \leq_{\mathfrak{K}} M'$].

AxI: If $M \leq_{\mathfrak{K}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

$AxII$: $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$ implies $M_0 \leq_{\mathfrak{K}} M_2$ and $M \leq_{\mathfrak{K}} M$ for $M \in K$.

$AxIII$: If λ is a regular cardinal, $M_i\,(i < \lambda)$ is $\leq_{\mathfrak{K}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{K}} M_j$) and continuous (i.e. for limit ordinal $\delta < \lambda$ we have $M_\delta = \bigcup\limits_{i<\delta} M_i$) then $M_0 \leq_{\mathfrak{K}} \bigcup\limits_{i<\lambda} M_i$.

$AxIV$: If λ is a regular cardinal, $M_i\,(i < \lambda)$ is $\leq_{\mathfrak{K}}$-increasing continuous, $M_i \leq_{\mathfrak{K}} N$ then $\bigcup\limits_{i<\lambda} M_i \leq_{\mathfrak{K}} N$.

AxV: If $M_0 \subseteq M_1$ and $M_\ell \leq_{\mathfrak{K}} N$ for $\ell = 0, 1$, then $M_0 \leq_{\mathfrak{K}} M_1$.

$AxVI$: $LS(\mathfrak{K})$ exists[1], where $LS(\mathfrak{K})$ is the minimal cardinal λ such that: if $A \subseteq N$ and $|A| \leq \lambda$ then for some $M \leq_{\mathfrak{K}} N$ we have $A \subseteq |M| \leq \lambda$ and we demand for simplicity $|\tau| \leq \lambda$.

Notation: 1) $K_\lambda = \{M \in K : \|M\| = \lambda\}$ and $K_{<\lambda} = \bigcup\limits_{\mu<\lambda} K_\mu$.
2) For a logic $\mathcal{L}$ and vocabulary τ, $\mathcal{L}(\tau)$ is the set of $\mathcal{L}$-formulas in this vocabulary.
3) $\mathcal{L}_{\omega,\omega}$ is first order logic.
4) A theory in $\mathcal{L}(\tau)$ is a set of sentences from $\mathcal{L}(\tau)$.

⟨**0.3**⟩ **Definition.** The embedding $f : N \to M$ is a $\mathfrak{K}$-embedding or a $\leq_{\mathfrak{K}}$-embedding if its range is the universe of a model $N' \leq_{\mathfrak{K}} M$, (so $f : N \to N'$ is an isomorphism onto).

Very central in [Sh 88] but peripherial here is:

[1] We normally assume $M \in \mathfrak{K} \Rightarrow \|M\| \geq LS(\mathfrak{K})$, here there is no loss in it. It is also natural to assume $|\tau(\mathfrak{K})| \leq LS(\mathfrak{K})$ which means just increasing $LS(\mathfrak{K})$, but no real need.

⟨0.4⟩ Definition. Let T_1 be a theory in $\mathcal{L}_{\omega,\omega}(\tau_1), \tau \subseteq \tau_1$ vocabularies, Γ a set of types in $\mathcal{L}_{\omega,\omega}(\tau_1)$; (i.e. for some m, a set of formulas $\varphi(x_0,\dots,x_{m-1}) \in \mathcal{L}_{\omega,\omega}(\tau_1)$).
1) $EC(T_1,\Gamma) = \{M : M$ a τ_1-model of T_1 which omits every $p \in \Gamma\}$.
(So τ_1 is reconstructible from T_1,Γ) and

$$PC_\tau(T_1,\Gamma) = PC(T_1,\Gamma,\tau) = \{M : M \text{ is a } \tau\text{-reduct of some } M_1 \in EC(T_1,\Gamma)\}.$$

2) We say that $\mathfrak{K}$ is PC^μ_λ if for some $T_1,T_2,\Gamma_1,\Gamma_2$ and τ_1 and τ_2 we have: (T_ℓ a first order theory in the vocabulary τ_ℓ,Γ_ℓ a set of types and) $K = PC(T_1,\Gamma_1,\tau_{\mathfrak{K}})$ and $\{(M,N) : M \leq_{\mathfrak{K}} N, M,N \in K\} = PC(T_2,\Gamma_2,\tau')$ where
$\tau' = \tau_1 \cup \{P\}$ (P a new one place predicate), $|T_\ell| \leq \lambda, |\Gamma_\ell| \leq \mu$ for $\ell = 1,2$. If $\mu = \lambda$, we may omit it.

⟨0.5⟩ Example. If $\tau_1 = \tau, T_1, \Gamma$ as above, and $(K,\leq_{\mathfrak{K}})$ are defined by
$K =: EC(T_1,\Gamma)$, $\leq_{\mathfrak{K}} =: \prec_{\mathcal{L}_{\omega,\omega}}$, then the pair satisfies the Axioms from ⟨1.2⟩ (so $LS(\mathfrak{K}) \leq |T_1| + \aleph_0$).

⟨0.6⟩ Lemma. *Let I be a directed set (i.e. partially ordered by $\leq$, such that any two elements have a common upper bound).*
1) If M_t is defined for $t \in I$, and $t \leq s \in I$ implies $M_t \leq_{\mathfrak{K}} M_s$ <u>then</u> for every $t \in I$ we have $M_t \leq_{\mathfrak{K}} \bigcup\limits_{s\in I} M_s$.
2) If in addition $t \in I$ implies $M_t \leq_{\mathfrak{K}} N$ <u>then</u> $\bigcup\limits_{s\in I} M_s \leq_{\mathfrak{K}} N$.

Proof. By induction on $|I|$ (simultaneously for 1) and 2)) (or see [Sh 88, 1.6])).

⟨0.7⟩ Lemma. *1) Let $\tau_1 = \tau \cup \{F^n_i : i < LS(\mathfrak{K})$ and $n < \omega\}, F^n_i$ an n-place function symbol (assuming, of course, $F^n_i \notin \tau$).*
Every model M (in K) can be expanded to a τ_1-model M_1 such that:

- (*A*) *$M_{\bar a} \leq_{\mathfrak{K}} M$ when for some $n < \omega, \bar a \in {}^n|M|$ and where $M_{\bar a}$ is the submodel of M with universe $\{F^n_i(\bar a) : i < LS(\mathfrak{K})\}$,*
- (*B*) *if $\bar a \in {}^n|M|$ then $\|M_{\bar a}\| \leq LS(\mathfrak{K})$,*
- (*C*) *if $\bar b$ is a subsequence of $\bar a$ (i.e. for some one-to-one function $h : (\ell g\bar b) \to (\ell g\bar a)$ we have $b_\ell = a_{h(\ell)}$), <u>then</u> $M_{\bar b} \leq_{\mathfrak{K}} M_{\bar a}$,*
- (*D*) *for every $N_1 \subseteq M_1$ we have $N_1 \restriction \tau \leq_{\mathfrak{K}} M$.*

2) If $N \leq_{\mathfrak{K}} M$ is given in the beginning, then we can choose $M_{\bar a}$ for $\bar a \in \bigcup\limits_{n<\omega} {}^n|M|$ such that clause (A) - (D) holds and

- (*E*) *$N = N_1 \restriction \tau$ for some $N_1 \subseteq M_1$.*

Proof. We define by induction on n, the values of $F^n_i(\bar a)$ for every $i < LS(\mathfrak{K})$, $\bar a \in {}^n|M|$ such that $\bar a \subseteq N \Rightarrow M_{\bar a} \subseteq N$ when we are proving (2). By $AxVI$ there is an $M_{\bar a} \le_{\mathfrak{K}} M$, $\|M_{\bar a}\| \le LS(\mathfrak{K})$, $|M_{\bar a}|$ includes $\bigcup\{M_{\bar b} : \bar b$ a subsequence of $\bar a$ of length $< n\} \cup \bar a$ and $M_{\bar a}$ does not depend on the order of $\bar a$. Let $|M_{\bar a}| = \{c_i : i < i_0 \le LS(\mathfrak{K})\}$, and define $F^n_i(\bar a) = c_i$ for $i < i_0$ and c_0 for $i_0 \le i < LS(\mathfrak{K})$ (so we can demand "F^n_i is symmetric").

Clearly our conditions are satisfied if $\bar b$ is a subsequence of $\bar a$ and $M_{\bar b} \le_{\mathfrak{K}} M_{\bar a}$ by AxV. $\square_{\langle 0.7\rangle}$

⟨0.8⟩ Lemma. *1) There is a set Γ of types in $\mathcal{L}_{\omega,\omega}(\tau_1)$ (where τ_1 from Lemma ⟨0.7⟩) such that $K = PC(\emptyset, \Gamma, \tau)$ so it is a $PC^{2^{LS(\mathfrak{K})}}_{LS(\mathfrak{K})}$-class.*
2) Moreover, if $N_1 \subseteq M_1 \in EC(\emptyset, \Gamma), N, M$ the τ-reducts of N_1, M_1 respectively <u>then</u> $N \le_{\mathfrak{K}} M$. Also if $N \le_{\mathfrak{K}} M$ then there is an τ_1-expansion M_1 of M and a submodel N_1 of M_1 such that $M_1 \in EC(\emptyset, \Gamma)$ and $N_1 \restriction \tau = N$.
3) So also $\{(M, N) : N \le_{\mathfrak{K}} M\}$ is a $PC^{(2^{LS(\mathfrak{K})})}_{LS(\mathfrak{K})}$-class, hence $\mathfrak{K}$ is.

Proof. 1) Let Γ_n be the set of complete quantifier free n-types in $\mathcal{L}_{\omega,\omega}(\tau_1), p(x_0, \dots, x_{n-1})$, such that: if M_1 is an L_1-model, $\bar a$ realizes p in M_1 and M is the L-reduct of M_1, then $M_{\bar b} \le_{\mathfrak{K}} M_{\bar a}$ for any subsequence of $\bar b$ of $\bar a$; where $M_{\bar c}$ (for $\bar c \in {}^m|M_1|$) is the submodel of M whose universe is $\{F^m_i(\bar c) : i < LS(\mathfrak{K})\}$ (and there are such submodels).

Let Γ be the set of p which for some $n < \omega$ are complete quantifier free n-types (in $\mathcal{L}_{\omega,\omega}(\tau_1)$) which do not belong to Γ_n. So if M^1 is in $PC(\emptyset, \Gamma, \tau_1)$ then by ⟨0.6⟩ we have $M_1 \restriction \tau \in K$ hence $PC(\emptyset, \Gamma, L) \subseteq K$ and by ⟨0.7⟩(1) we have $K \subseteq PC(\emptyset, \Gamma, L)$.
2) The first phrase is proven as in (1). For the second phrase, use ⟨0.7⟩(2).
3) By ⟨0.7⟩(2). $\square_{\langle 0.8\rangle}$

⟨0.9⟩ Conclusion. There is $\tau_1, \tau \subseteq \tau_1$ satisfying $|\tau_1| \le LS(\mathfrak{K})$ such that: for any $M \in K$ and any τ_1-expansion M_1 of M which is in $EC(\emptyset, \Gamma)$,

$$N_1 \subseteq M_1 \Rightarrow N_1 \restriction \tau \le_{\mathfrak{K}} M$$

$$N_1 \subseteq N_2 \subseteq M_1 \Rightarrow N_1 \restriction \tau \le_{\mathfrak{K}} N_2 \restriction \tau.$$

⟨0.10⟩ Conclusion. If for every $\alpha < (2^{LS(\mathfrak{K})})^+$, K has a model of cardinality $\ge \beth_\alpha$ <u>then</u> K has a model in every cardinality $\ge LS(\mathfrak{K})$.

Proof. Use ⟨0.8⟩ and the value of the Hanf number for: first order theory and omitting any set of types, for languages of cardinality $LS(K)$ (see [Sh:c, VII,§5]). $\square_{\langle 0.10\rangle}$

⟨0.11⟩ Definition. For λ regular $> LS(\mathfrak{K})$ for $N \in \mathfrak{K}_\lambda$ we say $\bar{N} = \langle N_\alpha : \alpha < \lambda \rangle$ is a representation of N if $\bar{N}$ is $\leq_{\mathfrak{K}}$-increasing continuous, $\|N_\alpha\| < \lambda$ and $N = \bigcup\limits_{\alpha<\lambda} N_\alpha$ and if $\lambda = \mu^+$ then if not said otherwise, $\|N_\alpha\| = \mu$.

How will we define types, in particular, the set $\mathcal{S}(M)$ of complete types over M, when no formulas are present? If we have a "monster" $\mathfrak{C}$ by using automorphism; any "monster" is far down the road. So we "chase arrows" in K_λ (be careful, better not use too large models), this gives E^{at}_μ, but this relation in general is not transitive (as long as we do not have amalgamation in K_λ). So E_μ will be the transitive closure of E^{at}_μ.

⟨0.12⟩ Definition. 1) For $\mu \geq LS(\mathfrak{K})$ and $M \in K_\mu$ we define $\mathcal{S}(M)$ as $\{\text{tp}(a, M, N) : M \leq_{\mathfrak{K}} N \in K_\mu \text{ and } a \in N\}$ where $\text{tp}(a, M, N) = (M, N, a)/E_M$ where E_M is the transitive closure of E^{at}_M, and the two-place relation E^{at}_M is defined by:

$$\begin{aligned}(M, N_1, a_1)E^{at}_M(M, N_2, a_2) \text{ } \underline{\text{iff}} \text{ there is } N \in K_\mu \text{ and } \leq_{\mathfrak{K}}\text{-embeddings}\\ f_\ell : N_\ell \to N \text{ for } \ell = 1, 2 \text{ such that:}\\ f_1 \restriction M = \text{id}_M = f_2 \restriction M \text{ and } f_1(a_1) = f_2(a_2).\end{aligned}$$

$$\text{(of course } M \leq_{\mathfrak{K}} N_1, M \leq_{\mathfrak{K}} N_2 \text{ and } a_1 \in N_1, a_2 \in N_2)$$

2) We say "a realizes p in N" if $a \in N, p \in \mathcal{S}(M)$ and for some $N' \in K_\mu$ we have $M \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} N$ and $a \in N'$ and $p = \text{tp}(a, M, N')$; so $M, N' \in K_\mu$ but possibly $N \notin K_\mu$.
3) We say "a_2 strongly realizes $(M, N^1, a^1)/E_M$ in N" if for some N^2, a^2 we have $M \leq_{\mathfrak{K}} N^2 \leq_{\mathfrak{K}} N$ and $a_2 \in N^2$ and $(M, N^1, a^1)\, E^{at}_M\, (M, N^2, a^2)$.
4) We say $M_0 \in \mathfrak{K}_\lambda$ is an amalgamation base if: for every $M_1, M_2 \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings $f_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) there is $M_3 \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings $g_\ell : M_\ell \to M_3$ (for $\ell = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$.
5) We say $\mathfrak{K}$ is stable in λ if $LS(\mathfrak{K}) \leq \lambda$ and $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda$.

⟨0.13⟩ Observation. 1) E^{at}_M is symmetric and reflexive, so E_M is an equivalence relation.
2) If M is an amalgamation base then

(α) $E_M = E^{at}_M$ and

(β) "a strongly realizes $(M, N, b)/E_M$ in N" iff "a realizes $(M, N, b)/E_M$.

⟨0.16⟩ Definition. 1) $\mathfrak{K}$ has the λ-amalgamation property or has amalgamation in λ, if every $M_0 \in \mathfrak{K}_\lambda$ is an amalgamation base (see ⟨0.12⟩(4) above).
2) N is universal over M if: $M \le_{\mathfrak{K}} N$ and if $M \le_{\mathfrak{K}} N' \in K_{\le \|N\|}$ then N' can be $\le_{\mathfrak{K}}$-embedded into N over M.
3) $\mathfrak{K}$ has universal extensions in λ <u>if</u> for every $M \in K_\lambda$ there is N such that:

- (a) $M \le_{\mathfrak{K}} N \in K_\lambda$
- (b) N is universal over M.

4) N_1, N_2 has joint embeddings if for some $N \in K$ there are $\le_{\mathfrak{K}}$-embeddings h_ℓ of N_ℓ into N for $\ell = 1, 2$. Let JEP$_\mu$ (JEP) means this holds for $N_1, N_2 \in K_\mu$ $(\in K)$.
5) $\mathfrak{K}_\lambda$ has unique amalgamation (or $\mathfrak{K}$ has unique amalgamation in λ) when: if $M^\ell_i \in K_\lambda$ for $\ell < 2, i < 4$, and for $i = 1, 2$ we have $h^\ell_{i,0}$ is a $\le_{\mathfrak{K}}$-embedding of M^ℓ_0 into $M^\ell_i, h^\ell_{3,i}$ is a $\le_{\mathfrak{K}}$-embedding of M^ℓ_i into $M^\ell_3, h^\ell_{3,1} \circ h^\ell_{1,0} = h^\ell_{3,2} \circ h^\ell_{2,0}$ and $\text{Rang}(h^\ell_{3,1}) \cap \text{Rang}(h^\ell_{3,2}) = \text{Rang}(h^\ell_{3,1} \circ h^\ell_{1,0})$ and for $i < 3, f_i$ is an isomorphism from M^0_i onto $M^1_i, f_0 \subseteq f_1, f_0 \subseteq f_2$ then for some $N \in K_\lambda$ there are $\mathfrak{K}$-embedding $h_\ell : M^\ell_3 \to N$ such that $f_0 \circ h^0_{3,i}(M^0_i) = f_1 \circ h^1_{3,i}(M^1_i)$, for $i = 1, 2$.

⟨0.17⟩ Claim. *1) If $\mathfrak{K}$ is categorical in λ and $LS(\mathfrak{K}) \le \lambda$ then: there is a model in K_{λ^+} <u>iff</u> for some ($\equiv$ every) model $M \in K_\lambda$ there is N such that $M <_{\mathfrak{K}} N \in K_\lambda$ <u>iff</u> for some ($\equiv$ every) $M \in K_\lambda$ there is N such that $M <_{\mathfrak{K}} N \in K_{\lambda^+}$. Assume $LS(\mathfrak{K}) \le \lambda$; then $K_{\lambda^+} \ne \emptyset$ iff $(\forall M \in K_\lambda)(\exists N \in K_\lambda)(M <_{\mathfrak{K}} N)$.*
2) If $\mathfrak{K}$ has amalgamation in λ and $LS(\mathfrak{K}) \le \lambda$ and $M_0 \le_{\mathfrak{K}} M_1$ are in K_λ and $M_0 \le_{\mathfrak{K}} N_0 \in K_{\lambda^+}$ <u>then</u> we can find h and N_1 such that $N_0 \le_{\mathfrak{K}} N_1 \in K_{\lambda^+}$ and h is a $\le_{\mathfrak{K}}$-embedding of M_1 into N_1 extending id_{M_0}. We can allow $N_0 \in K_\mu, \mu > \lambda$ if $\mathfrak{K}$ has amalgamation in every $\lambda' \in [\lambda, \mu)$.
3) Assume K has amalgamation in λ. If $M_0 \le M_1$ are from K_λ and $p_0 \in \mathcal{S}(M_0)$ then we can find an extension $p_1 \in \mathcal{S}(M_1)$ of p_0 (see Definition ⟨0.14⟩.

Proof. 1) As then we can choose by induction on $i < \lambda^+$ models $M_i \in K_\lambda, \le_{\mathfrak{K}}$-increasing continuous, $M_i \ne M_{i+1}$, for $i = 0$ use $K_{\lambda^+} \ne \emptyset$, for i limit take union, for $i = j + 1$ use the previous sentence; so $M_{\lambda^+} = \cup\{M_i : i < \lambda^+\} \in K_{\lambda^+}$ as required.
2), 3) Left to the reader. $\square_{\langle 0.17 \rangle}$

⟨0.17A⟩ Remark. We can here add the context of ⟨6.4⟩, ⟨6.5⟩, ⟨6.6⟩.

⟨0.18⟩ Definition. 1) For $\lambda > LS(\mathfrak{K})$ we say "$N \in \mathfrak{K}$ is λ-saturated" <u>if</u> for every $M \le_{\mathfrak{K}} N$ of cardinality $< \lambda$, every $p \in \mathcal{S}(M)$ is realized in N (i.e. for some $a \in M$ we have $\text{tp}(a, N, M) = p$). Moreover, if $M \le_{\mathfrak{K}} N' \in K_{<\lambda}$ and $a' \in N'$ then some $a \in M$ strongly realizes (M, N', a').
2) We say "$N \in \mathfrak{K}$ is λ-saturated above μ (or is λ-saturated $\ge \mu$)" <u>if</u> above we restrict ourselves to M of cardinality $\ge \mu$.
(On (λ, κ)-saturated see Definition ⟨0.21⟩, ⟨0.22⟩, also they are unique).

⟨0.18A⟩ Fact. 1) If $\mu + LS(\mathfrak{K}) < \lambda$ and λ is regular and $\mathfrak{K}$ has the amalgamation property in every $\mu' \in [\mu, \lambda)$ and $M \in K_{[\mu,\lambda)} \Rightarrow |\mathcal{S}(M)| \le \lambda$ and $\lambda =$ cf(λ) <u>then</u> there is $M \in K_\lambda$ saturated above μ.

⟨0.14⟩ Definition. 1) Let $p_\ell \in \mathcal{S}(M_\ell)$ for $\ell = 0, 1$. We say $p_0 \le p_1$ <u>if</u> $M_0 \le_{\mathfrak{K}} M_1$ and for some N and a we have $M_1 \le_{\mathfrak{K}} N \in K_{\|M_1\|+LS(\mathfrak{K})}, a \in N$ and $\mathrm{tp}(a, M_\ell, N) = p_\ell$ for $\ell = 0, 1$. We also write $p_0 = p_1 \restriction M_0$ (really p_0 is unique knowing M_0, M_1, p_1 hence $p_1 \restriction M_0$ is well defined).
2) The type $p \in \mathcal{S}(M)$ is local when: if for any directed partial order I and $M_t \le_{\mathfrak{K}} M$ for $t \in I, I \models t \le s \Rightarrow M_t \le_{\mathfrak{K}} M_s$ and $M = \bigcup\limits_{t \in I} M_t$ and $p' \in \mathcal{S}(M)$ and $(\forall t \in I)(p \restriction M_t = p' \restriction M_t)$ then $p = p'$. We say M is local if every $p \in \mathcal{S}(M)$ is. We say $\mathcal{K}$ is local if every M is. We can add "above μ_0" as in Definition ⟨0.18⟩(2).

⟨0.15⟩ Definition. 1) $I(\lambda, K) = I(\lambda, \mathfrak{K})$ is the number of $M \in K_\lambda$ up to isomorphism. 2) $\mathfrak{K}$ (or K) is categorical in λ if $I(\lambda, K) = 1$.
3) $IE(\lambda, \mathfrak{K}) = \sup\{|K'| : K' \subseteq K_\lambda$ and if $M \ne N$ are from K'_λ then M cannot be $\le_{\mathfrak{K}}$-embeddable into $N\}$. Saying $IE(\lambda, K)$ is $\ge \mu$ we mean that for some $K' \subseteq K_\lambda$ as above, $|K'| \ge \mu$, similarly for $= \mu$, (if there is a problem with attainment of the supremum we shall say explicitly).

⟨0.19⟩ The Model-homogeneity = Saturativity Lemma. *Let $\mu > \mu_0 \ge LS(\mathfrak{K})$ and $M \in K$.*
1) If $LS(\mathfrak{K}) \le \mu_0 \le \mu'_0 \le \mu' \le \mu$ and M is μ-saturated above μ_0, then it is μ'-saturated above μ'_0. If $LS(\mathfrak{K}) \le \mu_0 < \mu$ then: M is μ-saturated above μ_0 <u>iff</u> for every $\lambda \in [\mu_0, \mu), M$ is λ^+-saturated above λ.
2) M is μ-saturated above μ_0 <u>iff</u> M is $(\mathcal{D}_{\mathfrak{K}}, \mu)$-homogeneous above μ_0 which means: for every $N_1 \le_{\mathfrak{K}} N_2 \in K$ such that $\mu_0 \le \|N_1\| \le \|N_2\| < \mu$ and $N_1 \le_{\mathfrak{K}} M$ there is $\le_{\mathfrak{K}}$-embedding f of N_2 into M over N_1.
3) If $M_1, M_2 \in K_\mu$ are μ-saturated above $\mu^* < \mu$ and for some $N_1 \le_{\mathfrak{K}} M_1, N_2 \le_{\mathfrak{K}} M_2$ both of cardinality $\in [\mu^*, \mu)$ we have $N_1 \cong N_2$ <u>then</u> $M_1 \cong M_2$, in fact, any isomorphism f from N_1 onto N_2 can be extended to an isomorphism from M_1 onto M_2.
4) If in (3) we demand only "M_2 is μ-saturated" and $M_1 \in K_{\le \mu}$ <u>then</u> f can be extended to a $\le_{\mathfrak{K}}$-embedding from M_1 into M_2.
5) In part (3) instead $N_1 \cong N_2$ it suffices to assume N_1, N_2 can be $\le_{\mathfrak{K}}$-embedded into some $N \in K$ which holds if $\mathfrak{K}$ has the JEP (so JEP_{μ_0} suffices).

Proof. 1) Check definitions.
2) The "if" direction is easy as $\mu > LS(\mathfrak{K})$. Let us prove the other direction, so without loss of generality $\|N_1\| = \mu_0$.

By ⟨0.19⟩(1) without loss of generality μ is regular, moreover $\mu = \mu_0^+$.

Let $|N_2| = \{a_i : i < \kappa\}$, and we know $\kappa < \mu$. We define by induction on $i \le \kappa, N_1^i, N_2^i, f_i$ such that:

(a) $N_1^i \le_{\mathfrak{K}} N_2^i$ and $\|N_2^i\| < \mu$

(b) N_1^i is $\leq_{\mathfrak{K}}$-increasing continuous in i
(c) N_2^i is $\leq_{\mathfrak{K}}$-increasing continuous in i
(d) f_i is a $\leq_{\mathfrak{K}}$-embedding of N_1^i into M
(e) f_i is increasing continuous in i
(f) $a_i \in N_1^{i+1}$
(g) $N_1^0 = N_1, N_2^0 = N_2, f_0 = id_{N_1}$

For $i = 0$, clause (g) gives the definition. For i limit let:

$$N_1^i = \bigcup_{j<i} N_1^j \text{ and}$$

$$N_2^i = \bigcup_{j<i} N_2^j \text{ and}$$

$$f_i = \bigcup_{j<i} f_j.$$

Now (a)-(f) continues to hold by continuity.

For i successor we use our assumption; more elaborately, let $M_1^{i-1} \leq_{\mathfrak{K}} M$ be $f_{i-1}(N_1^{i-1})$ and M_2^{i-1}, g_{i-1} be such that g_{i-1} is an isomorphism from N_2^{i-1} onto M_2^{i-1} extending f_{i-1}, so $M_1^{i-1} \leq_{\mathfrak{K}} M_2^{i-1}$, now apply the saturation assumption with $M, M_1^{i-1}, M_2^{i-1}, g_{i_1}(a_{i-1})$ here standing for M, N_1, N_2 there (note: $a_{i-1} \in N_2 = N_2^0 \subseteq N_2^{i-1}$ and $\mu > \|N_2^{i-1}\| = \|M_2^{i-1}\|$ and $\|M_1^{i-1}\| = \|N_1^{i-1}\| \geq \|N_1^0\| = \|N_1\| \geq \mu_0$ so the requirements and also the cardinality in Definition $\langle 0.18 \rangle$ holds). So, remembering the second sentence in $\langle 0.18 \rangle(1)$ (about strongly realizes) there are $M_3^{i,*}, M_2^{i,*}, f_i^*$ such that:

(i) $M_1^{i-1} \leq_{\mathfrak{K}} M_1^{i,*} \leq_{\mathfrak{K}} M_2^{i,*}, \|M_2^{i,*}\| < \mu$
(ii) $M_1^{i-1} \leq_{\mathfrak{K}} M_2^{i-1} \leq_{\mathfrak{K}} M_2^{i,*}$
(iii) $g_{i-1}(a_{i-1}) \in M_1^{i,*}$
(iv) f_i^* a $\leq_{\mathfrak{K}}$ -embedding of $M_1^{i,*}$ into M
(v) $f_i^* \restriction M_1^{i-1} =$ id.

Let N_2^i, h_i be such that $N_2^{i-1} <_{\mathfrak{K}} N_2^i, h_i$ an isomorphism from N_2^i onto $M_2^{i,*}$ extending g_{i-1}. Let $N_1^i = h_i^{-1}(M_1^{i,*}), f_i = f_i^* \circ (h_i \restriction N_1^i)$.

We have carried the induction. Now f_κ is a $\leq_{\mathfrak{K}}$- embedding of N_1^κ into M over N_1, but $|N_2| = \{a_i : i < \kappa\} \subseteq N_1^\kappa$, so $f_\kappa \restriction N_2 : N_2 \to M$ is as required.
3), 4) By the hence and forth argument (or see [Sh 300, II,§3] = [Sh:e, II,§3]).
5) Easy, too. $\square_{\langle 0.19 \rangle}$

$\langle 0.20 \rangle$ *Remark.* 1) Note that by $\langle 0.19 \rangle(2)$ if M is μ-saturated above μ_0 and $\mathfrak{K}$ has the JEP$_{\mu_0}$ then $\mathfrak{K}$ has λ-amalgamation for each $\lambda \in [\mu_0, \mu)$.
2) We could have put here the existence and uniqueness of (λ, κ)- saturated models when $\mathfrak{K}$ is stable in λ (from [Sh 300]) (see $\langle 1.5 \rangle(3)$).

$\langle$*0.21X*$\rangle$ *Example.* $(V = L)$.

Let $\text{cf}(\lambda) \geq \aleph_1, n < \omega$ <u>then</u> for some $\psi \in L_{\lambda^+,\omega}$ we have: ψ has no model of cardinality $\lambda^{+(n+1)}$, and is categorical in λ^{+n} (i.e. has one and only one model up to isomorphism).
Let $M^* = (L_{\lambda^{+n}}, \in, i)_{i \leq \lambda}$ and let ψ be

$$\bigwedge\Big\{\varphi : \varphi \text{ is a first order sentence which } M^* \text{ satisfies}\Big\}$$
$$\wedge\, (\forall x)\left(x \in \lambda \equiv \bigvee_{i<\lambda} x = i\right).$$

$\langle$**0.21**$\rangle$ **Definition.** 1) We say that $N_1 \in K_\lambda$ is (λ,κ)-saturated over N_0 or $(N_1, c)_{c \in N_\delta}$ is (λ,κ)-saturated if:
there is a sequence $\langle M_i : i < \kappa\rangle$ which is $\leq_{\mathfrak{K}}$-increasing continuous $M_0 = N, M_\kappa = N_1, M_{i+1} \in K_\lambda$ universal over M_i (see $\langle 0.16\rangle(2)$) (see below). We say $(N, c)_{c \in N}$ saturated of cofinality κ.
2) If we omit κ, we mean $\kappa = \text{cf}(\lambda)$ and (λ,α)-saturated means $(\lambda, \text{cf}(\alpha))$-saturated and N_1 is $(\lambda, 1)$-saturated over N_0 means $N_0 \leq_{\mathfrak{K}} N_1$ are in K_λ.
3) N is universal over M if $M \leq_{\mathfrak{K}} N$ and if $M \leq_{\mathfrak{K}} N'$ and $\|N'\| \leq \|N\|$ then N can be $\leq_{\mathfrak{K}}$-embedded over N.

$\langle$**0.22**$\rangle$ **Claim.** *1) If N_1 is (λ,κ)-saturated over N, then we know N_1 is unique over N for λ,κ given.*
2) If $K_\lambda \neq \emptyset, \lambda \geq \kappa =$ cf(κ) and over every $M \in K_\lambda$ there is $N, M \leq_{\mathfrak{K}} N \in K_\lambda$ universal over M, <u>then</u> for every $N \in K_\lambda$ there is $N_1 \in K_\lambda$ which is (λ,κ)-saturated over N.
3) If $\mathfrak{K}_\lambda$ has amalgamation and $\mathfrak{K}$ is stable in λ (i.e. $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda$) <u>then</u> every $M \in K_\lambda$ has a universal extension (so part (2)'s conclusion holds).

Proof. See [Sh 300, Ch.II] or check.

We do not need at present but remember

$\langle$**0.23**$\rangle$ **Claim.** *There is $\tau' \supseteq \tau \cup \{P_0, P_1, P_2, c\}$ of cardinality $\leq LS(\mathfrak{K})$ with c individual constant, with P_ℓ unary predicates and we can find a set Γ' of quantifier free types such that:*

- (a) *if $M' \in PC_{\tau'}(\emptyset, \Gamma)$ and $M_\ell = (M' \restriction \tau) \restriction P_\ell^{M'}$ for $\ell = 0, 1, 2$, then $M_\ell \in K, M_0 \leq_{\mathfrak{K}} M_1, M_0 \leq_{\mathfrak{K}} M_2, c^{M'} \in M_2$ such that $N' \subseteq M' \Rightarrow (N' \restriction \tau) \restriction P_\ell^{N'} \leq_{\mathfrak{K}} M_\ell$ and there is no $b \in M_2$ satisfying:*
  - $\otimes$ *for every $\bar a \in {}^{\omega>}(P_0^{M'})$, letting $N_{\bar a}$ be the τ'-submodel of M' generated by $\bar a$ and $M_{\bar a}^\ell = M_\ell \restriction (M_\ell \cap N_{\bar a})$, we have $M_{\bar a}^\ell \leq_{\mathfrak{K}} M_\ell$, and b strongly realizes $tp(c^{M'}, M_{\bar a}^0, M_{\bar a}^2)$ in $M_{\bar a}^2$*

(b) *if* M^*_ℓ, c *is as in* $(*), M_0 = M_1 \cap M_2$ *then for some* M' *we have: clause (a).*

Proof. Should be clear or see [Sh 88].

⟨0.24⟩ Remark. 1) Claim ⟨0.23⟩ enables us to translate two cardinals with omitting types theorem in λ_2 implies existence of one in λ_1, provided that types are local i.e. $p \in \mathcal{S}(M)$ is determined by $\langle p \restriction N : N \leq_{\mathfrak{K}} M, \|N\| \leq \lambda \rangle$.
2) This enables us to prove implications between cases of λ-categoricity, if we have nice enough theory of types as in [Sh:c, VIII,§4]; if we have in λ_2 a saturated models; categoricity in λ_1 implies categoricity in λ_2. Also (if we know a little more) categoricity in λ_2 is equivalent to the existence of a non-saturated model in λ_2.

⟨1.28⟩ Claim. *1) Assume* $M_n \leq_{\mathfrak{K}} M_{n+1}, M_n \in K_\lambda, \mathfrak{K}$ *has amalgamation in* λ. *If* $p_n \in \mathcal{S}(\mathcal{M}_n), p_n \leq p_{n+1}$ *(i.e.* $p_n = p_{n+1} \restriction M_n$*, see* ⟨**1.x**⟩*), then there is*
—> *scite{1.x} undefined*
$p \in \mathcal{S}(\bigcup\limits_{n<\omega} M_n)$ *such that* $n < \omega \Rightarrow p_n \leq p$.
2) If $\langle M_i : i \leq \delta \rangle$ *is* $\leq_{\mathfrak{K}}$*-increasing continuous,* $p_i \in \mathcal{S}(M_i), (j < i \Rightarrow p_j \leq p_i)$, $p_i = tp(a_i, M_i, N_i)$ *and* $h_{i,j}$ *is a* $\leq_{\mathfrak{K}}$*-embedding of* N_j *into* N_i *(for* $j < i < \delta$*) such that* $h_{i,j} \restriction M_j = id_{M_j}, h_{i,j}(a_j) = a_i$*, then there is* $p_\delta \in \mathcal{S}(M_\delta), i \leq \delta \Rightarrow p_i \leq p_\delta$.
3) If $\mathfrak{K}$ *has amalgamation in* λ*, is stable in* λ *(i.e.* $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda$*), then*

(a) *every* $M \in K_\lambda$ *has a universal extension*

(b) *the uniqueness of amalgamation implies true uniqueness of amalgamation.*

§1 Weak Diamond and $\bar{\chi} = \langle \chi_\alpha : \alpha < \lambda \rangle, \chi_\alpha \geq 2$

$\langle$**1.1**$\rangle$ **Definition.** For $\lambda = \mathrm{cf}(\lambda) > \aleph_0$, and $S \subseteq \lambda$ let (note: WDmTId stand for weak diamond target ideal)[2].

1)

$$\mathrm{WDmTId}(\lambda, S, \bar{\chi}) = \Big\{ A : A \subseteq \prod_{\alpha \in S} \chi_\alpha, \text{ and for some function } F \text{ with domain } \bigcup_{\alpha<\lambda} {}^{\alpha}(2^{<\lambda}) \text{ mapping } {}^{\alpha}(2^{<\lambda}) \text{ into } \chi_\alpha, \text{ for every } \eta \in A, \text{ for some } f \in {}^{\lambda}(2^{<\lambda}) \text{ the set } \{\delta \in S : \eta(\delta) = F(f \restriction \delta)\} \text{ is not stationary} \Big\}.$$

note: we can replace $2^{<\lambda}$ by any set of this cardinality, and so can replace $f \in {}^{\lambda}(2^{<\lambda})$ by $f_1, \ldots, f_n \in {}^{\lambda}(2^{<\lambda})$ with F being n-place.

2) $$\mathrm{cov}_{\mathrm{wdmt}}(\lambda, S, \bar{\chi}) = \mathrm{Min}\Big\{ |\mathcal{P}| : \mathcal{P} \subseteq \mathrm{WDmTId}(\lambda, S) \text{ and } {}^{S}2 \subseteq \bigcup_{A \in \mathcal{P}} A \Big\}$$

3)

$$\mathrm{WDmTId}_{<\mu}(\lambda, S, \bar{\chi}) = \Big\{ A : \text{for some } i^* < \mu \text{ and } A_i \in \mathrm{WDmTId}(\lambda, S) \text{ for } i < i^* \text{ we have } A \subseteq \bigcup_{i<i^*} A_i \Big\}$$

4) $$\mathrm{WDmId}_{<\mu}(\lambda, \bar{\chi}) = \Big\{ S \subseteq \lambda : \mathrm{cov}_{\mathrm{wdmt}}(\lambda, S) < \mu \Big\}.$$

5) Instead of "$< \mu^+$" we write μ, if we omit μ we mean $(2^{<\lambda})$. If $\bar{\chi}$ is constantly 2 we may omit it, if $\chi_\alpha = 2^{|\alpha|}$ we may write pow instead of $\bar{\chi}$.

6) $$\text{Let } \mu_{\mathrm{wd}}(\lambda, \bar{\chi}) = \mathrm{cov}_{\mathrm{wdmt}}(\lambda, \lambda, \bar{\chi}).$$

7) We say that the weak diamond holds on λ if $\lambda \notin \mathrm{WDmId}(\lambda)$.

By Devlin Shelah [DvSh 65], Shelah [Sh:b, XIV,1.18(2),1.8] (better presented [Sh:f, AP,§1], note: 1.2(4) below rely on [Sh 460]) we have:

[2]in [Sh:b, AP,§1], [Sh:f, AP,§1] we express $\mathrm{cov}_{\mathrm{Wdmt}}(\lambda, S) > \mu^*$ by allowing $f(0) \in \mu^* < \mu$

⟨1.2⟩ Theorem.
1) If $\lambda = \aleph_1, 2^{\aleph_0} < 2^{\aleph_1}, \mu \le 2^{\aleph_0}$ *or even* $2^\theta = 2^{<\lambda} < 2^\lambda, \mu = (2^\theta)^+$ *or just* $2^\theta = 2^{<\lambda} < 2^\lambda$ *and* $\mu = \mu^{<\lambda}$, *then* $\lambda \notin WDmId_{<\mu}(\lambda)$. *If in addition* $(*)_{<\mu,\lambda}$ *below then* $\lambda \notin WDmId_{<\mu}(\lambda, pow)$, *where:*

$$(*)_{<\mu,\lambda} = \bigwedge_{\chi<\mu} (*)_{\chi,\lambda} \text{ where}$$

$(*)_{\mu,\lambda}$ *there are no* $A_i \in [\mu]^{\lambda^+}$ *for* $i < 2^\lambda$ *such that* $i \neq j \Rightarrow |A_i \cap A_j| < \aleph_0$.

2) If $\mu \le \lambda^+$, $cf([\mu_1]^{\le\lambda}, \subseteq) < \mu$ *for* $\mu_1 < \mu$ *or* $\mu = \aleph_0$ *then* $WDmId_{<\mu}(\lambda, \bar{\chi})$ *is a normal ideal on* λ. *If this ideal is not trivial, then* $(\lambda = cf(\lambda) > \aleph_0, 2^{<\lambda} < 2^\lambda$.

3) A sufficient condition for $(*)_{<\mu,\lambda}$ *is:*

(a) $\mu \le 2^\lambda$ & $(\forall \alpha < \mu)(|\alpha|^{\aleph_0} < 2^\lambda)$.

4) Another sufficient condition for $(*)_{\mu,\lambda}$ *is:*

(b) $\mu \le 2^\lambda$ & $\lambda \ge \beth_\omega$.

⟨*1.2A*⟩ *Remark.* 1) So if $\mathrm{cf}(2^\lambda) < \mu$ (which holds if 2^λ is singular $\mu = 2^\lambda$) then $(*)_{<\mu,\lambda}$ implies that then there is $A \subseteq {}^\lambda 2, |A| < 2^\lambda, A \notin \mathrm{WDmTId}(\lambda)$.
2) Some related definitions appear in ⟨1.7⟩; we use them below (mainly $\mathrm{DfWD}_{<\mu}(\lambda)$), but as in first reading it is recommended to ignore them, the definition is given later.
3) We did not look again at the case $(\forall \sigma < \lambda)(2^\sigma < 2^{<\lambda} < 2^\lambda)$.

As in [Sh 88, 3.5], ([Sh 87a, 2.7], [Sh 87b, 6.3]):

⟨1.3⟩ Claim. *Assume* $\lambda \notin WdmId_{<\mu}(\lambda)$ *or at least* $DfWD_{<\mu}(\lambda)$ *(where* $\lambda = cf(\lambda) > \aleph_0$, *of course) and* $\mathfrak{K}$ *an abstract elementary class.*
1) Assume $\mathfrak{K}$ *is categorical in* $\chi, \lambda = \chi^+$, *and* $\mathfrak{K}$ *have a model in* λ *(if* $LS(\mathfrak{K}) \le \chi$ *this is equivalent to: the model* $M \in K_\chi$ *is not* $\le_{\mathfrak{K}}$*-maximal). Assume further* $\mathfrak{K}$ *fails the* λ*-amalgamation property in* χ. *Then for any* $M_i \in \mathfrak{K}_\lambda, i < i^* < \mu$, *there is* $N \in \mathfrak{K}_\lambda$ *not* $\le_{\mathfrak{K}}$*-embeddable into any* M_i *(and the assumptions of part (2) below holds).*
2) Assume $M_\eta \in K_{<\lambda}$ *for* $\eta \in {}^{\lambda>}2$ *and* $M_\eta = \bigcup_{\alpha<\ell g(\eta)} M_{\eta\restriction(\alpha+1)}$ *and* $\nu \triangleleft \eta \Rightarrow M_\nu \le_{\mathfrak{K}} M_\eta$ *and* $M_{\eta\hat{}\langle 0\rangle}, M_{\eta\hat{}\langle 1\rangle}$ *cannot be amalgamated over* M_η *(hence* $M_\eta \neq M_{\eta\hat{}\langle \ell\rangle}$*), and let* $M_\eta =: \bigcup_{\alpha<\lambda} M_{\eta\restriction\alpha}$ *for* $\eta \in {}^\lambda 2$, *clearly* M_η *belongs to* K_λ *and, for the* $DfWD_{<\mu}(\lambda)$ *version*

(∗) $\langle M_\eta : \eta \in {}^{\lambda>}2\rangle$ *is definable (even just by* $\mathcal{L}_{\lambda,\lambda}$*) in* $\mathfrak{B} = (\mathcal{H}(\chi), \in, <^*_\chi, \mathfrak{K}_{<\chi}, \lambda, \mu)$.

Then for any $N_i \in \mathfrak{K}_\lambda, i < i^* < \mu$ *there is* $\eta \in {}^\lambda 2$ *such that:* M_η *is not* $\le_{\mathfrak{K}}$*-embeddable into any* N_i.

3) In part (2), if $LS(\mathfrak{K}) \leq \lambda$ we can allow $N_i \in K_{\kappa_i}$ if $\sum\limits_{i<i^*} cf([\kappa_i]^\lambda, \subseteq) < \mu$. And if we use $DfWD_{<\mu}(\lambda)$.
4) Assume only $\lambda \notin Unif_{<\mu}(\lambda, \bar{\chi})$. Part (2) holds if M_η is defined for $\eta \in \bigcup\limits_{\alpha<\lambda} \prod\limits_{i<\alpha} \chi_i$, and $\varepsilon < \zeta < \chi_i, \eta \in \prod\limits_{j<i} \chi_j \Rightarrow M_{\eta\hat{}\langle\varepsilon\rangle}, M_{\eta\hat{}\langle\zeta\rangle}$ cannot be amalgamated over M_η. Part (1) holds if we assume that for $M \in K_\chi, i < \lambda$ there are χ_i $\leq_{\mathfrak{K}}$-extensions of M in K_λ, pairwise cannot be amalgamated over M.

Proof. 1) It is straight to choose $M_\eta \in K_\chi$ for $\eta \in {}^\alpha 2$ by induction on α, as required in part (2). Then use part (2) to get the desired conclusion.
2) Without loss of generality the universe of N_i is λ and the universe of M_η is an ordinal γ_η such that $\eta \in {}^{\lambda>}2 \Rightarrow \gamma_\eta < \lambda$ and $\eta \in {}^\lambda 2 \Rightarrow \gamma_\eta = \lambda$. The reader can ignore the "DfWD$_{<\mu}(\lambda)$" version (ignoring the h_η's,g) if he like. Let for $\alpha < \lambda$ and $\eta \in {}^\alpha 2$ the function h_η be $h_\eta(i) = M_{\eta\restriction(i+1)}$ for $i < \ell g(\eta)$. Note $\langle M_\eta : \eta \in {}^{\lambda>}2\rangle$ is the $<^*_\chi$-first such object. For each $i < i^*$ we define $A_i \subseteq {}^\lambda 2$ by $A_i = \{\eta \in {}^\lambda 2 : M_\eta$ can be $\leq_{\mathfrak{K}}$-embedded into $N_i\}$.

For $\eta \in A_i$ choose $f_\eta : M_\eta \to N_i$, a $\leq_{\mathfrak{K}}$- embedding hence $f_\eta \in {}^\lambda 2$. We also define a function F_i with domain ${}^{\lambda>}2 \times {}^{\lambda>}\lambda$ to $\{0,1\}$ (really just $\bigcup\limits_{\alpha<\lambda}({}^\alpha 2 \times {}^\alpha\lambda)$ matters) by:

$F_i(\eta, f)$ is : 0 $\quad$ <u>if</u> f is a $\leq_{\mathfrak{K}}$-embedding of M_η into N_i with range $\subseteq \ell g(\eta)$ which can be extended to a $\leq_{\mathfrak{K}}$-embedding of $M_{\eta\hat{}\langle 0\rangle}$ into N_i

$F_i(\eta, f)$ is 1 $\quad$ otherwise.

Now for any $\eta \in A_i$, the set
$E = \{\delta < \lambda : \gamma_{\eta\restriction\delta} = \delta$ and $f_\eta \restriction \delta$ is a function from δ to $\delta\}$ is a club of λ. For every $\delta \in E$ clearly $F(\eta \restriction \delta, f_\eta \restriction \delta) = \eta(\delta)$ (as $M_{\eta\hat{}\langle 0\rangle}, M_{\eta\hat{}\langle 1\rangle}$ cannot be amalgamated over M_η).

Hence (you may ignore the "Def"; for it see Definition $\langle 1.7\rangle(2)$ using $\langle 1.8\rangle(1)$, $\langle 1.8\rangle(3)$) we have $A_i \in$ WDmTId$^{\rm Def}_{<\mu}(\lambda)$. As $i^* < \mu$ clearly $\bigcup\limits_{i<i^*} A_i \in$ WDmId$^{\rm Def}_{<\mu}(\lambda)$ hence by the assumption of the claim equality does not hold so there is $\eta \in {}^\lambda 2 \setminus \bigcup\limits_{i<i^*} A_i$. Now M_η is as required.
3) Similarly. Without loss of generality the universe of N_i is κ_i, let $\mathcal{P}_i \subseteq [\kappa_i]^\lambda$ be a set of minimal cardinality such that $(\forall B)[B \subseteq \kappa_i \;\&\; |B| \leq \lambda \to (\exists B' \in \mathcal{P}_i)(B \subseteq B')]$. As $LS(\mathfrak{K}) \leq \lambda$ we can find for each $A \in \mathcal{P}_i$, a model $N^i_A \leq_{\mathfrak{K}} N_i$ of cardinality λ^+ whose universe includes A. Now apply part (2) so $\{N^i_A : i < i^*$ and $A \in \mathcal{P}_i\}$.
4) Same proof. $\square_{\langle 1.3\rangle}$

⟨1.3A⟩ Claim. *1) Assume*

$(*)_1$ $\lambda = cf(\lambda) > \aleph_0$
   (a) *M_η is a τ-model of cardinality $< \lambda$ for $\eta \in {}^{\lambda>}2$ and*
   (b) *for each $\eta \in {}^\lambda 2, \langle M_{\eta \restriction \alpha} : \alpha < \lambda \rangle$ is $\subseteq$-increasing continuous with union, called $M_\eta \in K$, of cardinality λ*
   (c) *if $\eta \in {}^{\lambda>}2, \eta \triangleleft \rho_\ell \in {}^\lambda 2$ for $\ell = 1,2$ then M_{ρ_1}, M_{ρ_2} are not isomorphic over M_η*

$(*)_2$ $\lambda \notin WDmId_{<\mu}(\lambda)$

Then $I(\lambda, K) \geq \mu$, in fact we can find $X \subseteq {}^\lambda 2$ of cardinality $\geq \mu$ such that $\{M_\rho : \rho \in X\}$ are pairwise non-isomorphic.
2) Assume

$(*)^d_1$ *like $(*)_1$ in part (1) but in addition*
   (d) *$\langle M_\eta : \eta \in {}^{\lambda>}2 \rangle$ is definable in $\mathfrak{B} = \mathfrak{B}_\chi$*

$(*)^d_2$ *$DfWD_{<\mu}(\lambda)$.*

Then the conclusion of part (1) holds.
3) The parallel of ⟨1.3⟩(4) holds.

Proof. 1) Let $\{N_i : i < i^*\} \subseteq \{M_\rho : \rho \in {}^\lambda 2\}$ be a maximal subset of $\{M_\rho : \rho \in {}^\lambda 2\}$ of pairwise non-isomorphic models, and use the proof of ⟨1.3⟩(2).
2) Left to the reader. $\square_{\langle 1.3A \rangle}$

⟨1.4⟩ Claim. *1) Assume*

$(*)_1$ *$M_\eta \in K_{<\lambda}$ for $\eta \in {}^{\lambda>}2$ and $\langle M_{\eta \restriction \alpha} : \alpha \leq \ell g(\eta) \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous and $M_{\eta \char`^ \langle 1 \rangle}$ cannot be $\leq_{\mathfrak{K}}$-embedded into $M_\nu$ over $M_\eta$ when $\eta \char`^ \langle 0 \rangle \trianglelefteq \nu \in {}^{\lambda>}2$ (there are such M_η's if the assumptions of ⟨1.3⟩(1) hold, by its proof*

$(*)_2$ *$\lambda = cf(\lambda) > \aleph_0, \lambda \notin WDmId_{<\mu}(\lambda)$ and λ is a successor cardinal or at least there is no λ-saturated normal ideal on λ or at least $WDmId(\lambda)$ is not λ-saturated (which holds if for some $\theta < \lambda, \{\delta < \lambda : cf(\delta) = \theta\} \notin WDmId(\lambda)$).*

Then there is $A \subseteq {}^\lambda 2, |A| = 2^\lambda$ such that: if $\eta_1 \neq \eta_2$ are from A then (taking $M_\eta = \bigcup\limits_{\alpha<\lambda} M_{\eta \restriction \alpha}$ if $\eta \in {}^\lambda 2$)

(α) *$M_{\eta_1} \ncong M_{\eta_2}$ and*
(β) *if $\chi < \lambda \Rightarrow (2^\chi)^+ < 2^\lambda$ then: M_{η_1} cannot be $\leq_{\mathfrak{K}}$-embedded into M_{η_2}.*

2) Under the assumptions of $\langle 1.3\rangle(1)$ we can find $\langle M_\eta : \eta \in {}^{\lambda>}2\rangle$ as in the assumption of $\langle 1.3\rangle(2)$.
3) Under the assumption of $\langle 1.3\rangle(2)$ the assumption of $\langle 1.4\rangle(1)$ holds.
4) Under the assumption of $\langle 1.4\rangle(1)$ we have $I(\lambda, \mathfrak{K}) = 2^\lambda$ and if $(2^\chi)^+ < 2^\lambda$ then $IE(\lambda, \mathfrak{K}) = 2^\lambda$.
5) In $\langle 1.3\rangle(1)$ we can replace the categoricity of $\mathfrak{K}$ in χ by "$\mathfrak{K}$ has a super limit model in χ" which is not an amalgamation base (see Definition $\langle 1.5\rangle$ below). In this case the assumption of $\langle 1.3\rangle(2)$, and of $\langle 1.4\rangle(1)$ holds.
6) We can weaken (in $\langle 1.4\rangle(5)$ the existence of superlimit to) "for some $\kappa = cf(\kappa) \le \chi$ there is a $(\chi, \{\kappa\})$-super limit model which is not an amalgamation base"; provided that we add $\{\delta < \lambda : cf(\delta) = \kappa\} \notin WDmId_{<\mu}(\lambda)$ (but for $\langle 2.6\rangle(1)$ we need now "$WDmId_{<\mu}(\lambda) + \{\delta < \lambda : cf(\delta) = \kappa\}$ is not λ-saturated". If there is $S \subseteq \{\delta < \lambda : cf(\delta) = \kappa\}$, which belongs to $I[\lambda]$ but not to $WDmId_{<\mu}(\lambda)$ we can weaken the model theoretic requirement to: there is a $(\chi, \{\kappa\})$-medium limit (see [Sh 88, Definition §3]) but not used here.
7) The parallel of $\langle 1.3\rangle(4)$ holds.

$\langle$*1.6A*$\rangle$ *Conclusion.* 1) Assume

- $(*)_1$ for $\eta \in {}^{\lambda>}2$, $M_\eta \in K_{<\lambda}$ and $\langle M_{\eta\restriction\alpha} : \alpha \le \ell g(\eta)\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous and $M_{\eta\frown\langle 1\rangle}$ cannot be $\le_{\mathfrak{K}}$-embedded into M_ν over M_η when $\eta\frown\langle 0\rangle \trianglelefteq \nu \in {}^{\lambda>}2$ and $\langle M_\eta : \eta \in {}^{\lambda>}2\rangle$ is definable in $\mathfrak{B}$
- $(*)_2^d$ $\mathrm{WDmId}^{\mathrm{Def}}(\lambda)$ or $\mathrm{DfWD}^+(\lambda)$ is not λ-saturated (which holds if there is no normal λ-saturated ideals on λ (which holds for non Mahlo λ) and holds if for some θ, $\{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is not in the ideal).

<u>Then</u> the conclusion of $\langle 1.4\rangle$ holds.

Proof of $\langle$*1.4*$\rangle$. 1) The proof of [Sh 88, 3.5] works (see implications earlier). More elaborately, we divide the proof according to the answer to

<u>Question</u>: Is there $\eta^* \in {}^{\lambda>}2$ such that for every ν satisfying $\eta^* \trianglelefteq \nu \in {}^{\lambda>}2$ there are $\rho_0, \rho_1 \in {}^{\lambda>}2$ such that: $\nu \triangleleft \rho_0, \nu \trianglelefteq \rho_1$, and M_{ρ_0}, M_{ρ_1} cannot be amalgamated over M_{η^*}?

We can find a function $h : {}^{\lambda>}2 \to {}^{\lambda>}2$, such that:

- (a) the function h is one-to-one, preserving $\triangleleft$ and $(h(\nu))\frown\langle\ell\rangle \trianglelefteq h(\nu\frown\langle\ell\rangle)$
- $(b)_{\mathrm{yes}}$ <u>when</u> the answer to the question is yes, it is exemplified by η^*,$h(\langle\rangle) = \eta^*$ and $M_{h(\nu\frown\langle 0\rangle)}, M_{h(\nu\frown\langle 1\rangle)}$ cannot be amalgamated over M_{η^*} (for every $\nu \in {}^{\lambda>}2$)
- $(b)_{\mathrm{no}}$ when the answer to the question above is no, $h(\langle\rangle) = \langle\rangle$ and if $\nu\frown\langle 0\rangle \trianglelefteq \rho_0, \nu\frown\langle 1\rangle \triangleleft \rho_1$ then $M_{h(\rho_0)}, M_{h(\rho_1)}$ can be amalgamated over $M_{h(\nu)}$.

Without loss of generality h is the identity, by renaming (and we can preserve $(**)$ of $\langle 1.3\mathrm{A}\rangle(2)$ in the relevant case). Also clearly $M_{\eta\frown\langle\ell\rangle} \ne M_\eta$ (by the non-amalgamation assumption).

Case 1 The answer is yes.
For any $\eta \in {}^{\lambda}2$ and $\leq_{\mathfrak{K}}$-embedding g of $M_{\langle\rangle}$ into $M_\eta =: \bigcup_{\alpha<\lambda} M_{\eta\restriction\alpha}$, let

$$A_{\eta,g} =: \{\nu \in {}^{\lambda}2 : \text{there is a } \leq_{\mathfrak{K}} \text{-embedding of } M_\nu \text{ into } M_\eta \text{ extending } g\}$$

$$A_\eta =: \{\nu \in {}^{\lambda}2 : \text{there is a } \leq_{\mathfrak{K}} \text{-embedding of } M_\nu \text{ into } M_\eta\}.$$

So:

$(*)_1$ $|A_{\eta,g}| \leq 1$
(as if $\nu_1, \nu_2 \in A_{\eta,g}$ are distinct then for some ordinal $\alpha < \lambda$ and $\nu \in {}^{\alpha}2$ we have $\nu =: \nu_0 \restriction \alpha = \nu_1 \restriction \alpha, \nu_0(\alpha) \neq \nu_1(\alpha)$ and use the choice of $h(\nu\hat{\ }\langle\ell\rangle))$

$(*)_2$ $A_\eta = \cup\{A_{\nu,g} : g \text{ is a } \leq_{\mathfrak{K}} \text{-embedding of } M_{\langle\rangle} \text{ into } M_\eta\}$

$(*)_3$ $|A_\eta| \leq 2^{<\lambda}$

$(*)_4$ $\eta \in A_{\eta,\mathrm{id}_{M_{<>}}}$ hence $\eta \in A_\eta$.

Hence we can choose by induction on $\zeta < 2^\lambda$, $\eta_\zeta \in {}^{\lambda}2 \setminus \bigcup_{\xi<\zeta} A_{\eta_\xi}$ (exists by cardinality considerations as $2^{<\lambda} < \lambda$), so $\xi < \zeta \Rightarrow M_{\eta_\xi} \ncong M_{\eta_\zeta}$ so we have proved clause (α).
If $(2^{<\lambda})^+ < 2^\lambda$, then use Hajnal free subset theorem to choose $\{\eta_\zeta : \zeta < 2^\lambda\} \subseteq {}^{\lambda}2$ (of cardinality 2^λ) such that $\zeta \neq \xi \Rightarrow \eta_\zeta \notin A_{\eta_\xi}$ so we have proved clause (β), too.

Note we did not use the non λ-saturation of $\mathrm{WDmId}(\lambda)$ in this case.

Case 2: The answer is no.
Again, without loss of generality M_η has universe the ordinal γ_η.
Let $\langle S_i : i < \lambda\rangle$ be a partition of λ to sets, none of which is in $\mathrm{WDmId}(\lambda)$ (we leave the case of $\mathrm{DfWD}^+(\lambda)$ to the reader). For each i we define a function F_i as follows:

if $\delta \in S_i$ and $\eta, \nu \in {}^{\delta}2$ and $\gamma_\eta = \gamma_\nu = \delta$ and $f : \delta \to \delta$ then

$$F_i(\eta,\nu,f) = 0 \text{ if } f \text{ can be extended to an embedding of } M_{\nu\hat{\ }\langle 0\rangle} \text{ into some } M_\rho \text{ with } \eta\hat{\ }\langle 0\rangle \trianglelefteq \rho.$$

$$F_i(\eta,\nu,f) = 1 \text{ otherwise.}$$

So as $S_i \notin \mathrm{WDmId}(\lambda)$, for some $\eta_i^* \in {}^{\lambda}2$ we have:

$(*)$ for every $\eta \in {}^{\lambda}2, \nu \in {}^{\lambda}2$ and $f \in {}^{\lambda}\lambda$ the following set of ordinals $i < \lambda$ is stationary:

$$\{\delta \in S_i : F_i(\eta \restriction \delta, \nu \restriction \delta, f \restriction \delta) = \eta_i^*(\delta)\}.$$

Now for any $X \subseteq \lambda$ let $\eta_X, \rho_X \in {}^\lambda 2$ be defined by:

if $\alpha \in S_i$ then $i \in X \Rightarrow \eta_X(\alpha) = 1 - \eta^*_i(\alpha), i \notin X \Rightarrow \eta_X(\alpha) = 0$ and
$\rho_X = \eta_{\{2i:i \in X\} \cup \{2i+1:i \notin X\}}$.

Now

$(*)$ if $X, Y \subseteq \lambda$, and $X \neq Y$ then M_{ρ_X} cannot be $\leq_{\mathfrak{K}}$-embedded into M_{ρ_Y}.

Clearly $(*)$ suffices for finishing the proof; but why does it hold? Assume f is a $\leq_{\mathfrak{K}}$-embedding of M_{ρ_X} into M_{ρ_Y}; as $X \neq Y$ there is i such that $i \in X \Leftrightarrow i \notin Y$ so there is $j \in \{2i, 2i+1\}$ such that $\rho_X \restriction S_j = \langle 1 - \eta^*_j(\alpha) : \alpha \in S_j \rangle$ and $\rho_Y \restriction S_j$ is identically zero. Clearly $E = \{\delta : f \text{ maps } \delta \text{ into } \delta\}$ is a club of λ and hence $S_j \cap E \neq \emptyset$.

So if $\delta \in S_j \cap E$ then $f \restriction M_{(\rho_Y \restriction (\delta+1))} = f \restriction M_{(\rho_Y \restriction \delta)\char`\^\langle 0 \rangle}$ extend $f \restriction M_{\rho_Y \restriction \delta}$ and is a $\leq_{\mathfrak{K}}$-embedding of it into some $M_{(\rho_X \restriction \alpha)}, \alpha < \lambda$ large enough.

Now by the choice of h we get

$$\delta \in S_j \cap E \Rightarrow F(\eta_{\rho_X} \restriction \delta, \eta_{\rho_Y} \restriction \delta, f \restriction \delta) = \eta_{\rho_X}(\delta) - 1 - \eta^*_j(\delta).$$

But this contradicts the choice of η^*_j.

2),3),4),5). Check. Similarly.

7) Just note that in Case I we do not need the weak diamond just $2^{<\lambda} < 2^\lambda$. $\square_{\langle 1.4 \rangle}$

We use here mainly "superlimit" from the Definition below.

$\langle$1.5$\rangle$ Definition. 1) $M \in \mathfrak{K}_\lambda$ is superlimit if

- (a) for every $N \in \mathfrak{K}_\lambda$ such that $M \leq_{\mathfrak{K}} N$ there is $M' \in K_\lambda$ such that $N \leq_{\mathfrak{K}} M'$ and $N \neq M'$ and $M \cong M'$
- (b) if $\delta < \lambda^+$ is limit, $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing and (for $i < \delta$) $M_i \cong M$ then $\bigcup\limits_{i<\delta} M_i \cong M$.

2) For $\Theta \subseteq \{\mu : \aleph_0 \leq \mu \leq \lambda, \mu \text{ regular}\}$ we say $M \in \mathfrak{K}_\lambda$ in (λ, Θ)-superlimit if:

- (a) clause (α) from part (1) holds and
- (b) if $M_i \cong M$ is $(\leq_{\mathfrak{K}})$-increasing for $i < \mu \in \Theta$ <u>then</u> $\bigcup\limits_{i<\mu} M_i \cong M$.

3) For $S \subset \lambda^+$ we say $M \in \mathfrak{K}_\lambda$ is (λ, S)-strongly limit if:

- (a) clause (α) from part (1) holds
- (b) there is a function F from $\bigcup\limits_{\alpha<\kappa} {}^\alpha(K_\lambda)$ to K_λ such that:
  - (α) for any sequence $\langle M_i : i < \alpha \rangle$ if $\alpha < \kappa, M_0 = M, M_i$ is $\leq_{\mathfrak{K}}$-increasing, $M_i \in \mathfrak{K}_\lambda$, <u>then</u> $j < \alpha \Rightarrow M_j \leq_{\mathfrak{K}} F(\langle M_i : i < \alpha \rangle)$
  - (β) if $\langle M_i : i < \lambda^+ \rangle$ is $\leq_{\mathfrak{K}}$-increasing, $M_0 = M, M_i \in \mathfrak{K}_\lambda$, and for $i < \kappa, M_{i+1} \leq_{\mathfrak{K}} F(\langle M_j : j \leq i+1 \rangle) \leq_{\mathfrak{K}} M_{i+2}$ <u>then</u> $\{\delta \in S | \bigcup\limits_{i<\delta} M_i \ncong M\}$ is not stationary.

4) M is (λ,κ)-limit if there is a function F such that:

(a) if $\langle M_i : i<\kappa\rangle$ is a $<_{\mathfrak{K}}$-increasing continuous sequence in K_λ, $F(\bar M \restriction (i+1)) \le_{\mathfrak{K}} M_{i+1}$ then $\bigcup\limits_{i<\kappa} M_i \cong M$

(b) there is at least one sequence.

5) M is (λ,κ)-superlimit is defined similarly but F omitted and $M_{i+1} \cong M$.

$\langle 1.6\rangle$ Claim. *Assume $2^\lambda < 2^{\lambda^+}$.*
0) If $\mathfrak{K}$ (an abstract elementary class) is categorical in λ, $LS(\mathfrak{K}) \le \lambda, I(\lambda^+,\mathfrak{K}) < 2^{\lambda^+}$, <u>then</u> $\mathfrak{K}_\lambda$ has amalgamation.
1) If $\mathfrak{K}$ (an abstract elementary class) is categorical in λ and $1 \le IE(\lambda^+, K) < 2^{\lambda^+}$ but $K_{\lambda^{++}} = \emptyset$ <u>then</u> $\mathfrak{K}$ has amalgamation in λ.
2) Assume $\mathfrak{K}$ has amalgamation in $\lambda, LS(\mathfrak{K}) \le \lambda, K_{\lambda^+} \neq \emptyset$ and $K_{\lambda^{++}} = \emptyset$ <u>then</u> there is $M \in K_{\lambda^+}$ saturated above λ.
3) If M is μ-saturated above $\lambda, LS(\mathfrak{K}) \le \lambda_0 < \lambda$ and $\mathfrak{K}$ has amalgamation in every $\lambda_0' \in [\lambda_0,\lambda)$ <u>then</u> M is μ-saturated above λ_0.

$\langle 1.6B\rangle$ Remark. Of course, if $I(\lambda^+,\mathfrak{K}) < 2^{\lambda^+}$, then the assumption $\mathfrak{K}_{\lambda^{++}} = \emptyset$ is not used in part (1) of $\langle 1.6\rangle$; this is $\langle 1.4\rangle(1)$ + (2). Also if $(2^\lambda)^+ < 2^{\lambda^+}$ then the assumption $K_{\lambda^{++}} = \emptyset$ is not needed in part (1) of $\langle 1.6\rangle$; by $\langle 1.4\rangle(1)$ + (2) (note (b) of $\langle 1.4\rangle(1)$).

Proof. 0) By $\langle 1.4\rangle(1)$.
1) If not, we can choose for $\eta \in {}^{\lambda^+>}2$ a model $M_\eta \in \mathfrak{K}_\lambda$ such that $[\nu \triangleleft \eta \Rightarrow M_\nu \le_{\mathfrak{K}} M_\eta]$, and $M_{\eta\hat{}\langle 0\rangle}, M_{\eta\hat{}\langle 1\rangle}$ cannot be amalgamated over M_η. If $(2^\lambda)^+ < 2^{\lambda^+}$ we are done by $\langle 1.4\rangle(1)$, so assume $(2^\lambda)^+ = 2^{\lambda^+}$. For each $\eta \in {}^{\lambda^+}2$ let $M_\eta =: \bigcup\limits_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$, and let $N_\eta \in \mathfrak{K}_{\lambda^+}$ be such that $M_\eta \le_{\mathfrak{K}} N_\eta, N_\eta$ is $\le_{\mathfrak{K}}$-maximal (exists as $\mathfrak{K}_{\lambda^{++}} = \emptyset$). Now we choose by induction $\zeta < (2^\lambda)^+, \eta_\zeta \in {}^{\lambda^+}2$ such that M_{η_ζ} is not $\le_{\mathfrak{K}}$-embeddable into N_{η_ξ} for $\xi < \zeta$ (exists by $\langle 1.3\rangle(2)$). So necessarily for $\xi < \zeta, N_\zeta$ is not $\le_{\mathfrak{K}}$-embeddable into N_ξ (as $M_{\eta_\zeta} \le N_{\eta_\zeta}$). Also, for $\xi < \zeta, N_{\eta_\xi}$ is not $\le_{\mathfrak{K}}$-embeddable into N_{η_ζ} (otherwise by the maximality of N_ξ this implies $N_\xi \cong N_\zeta$ which contradict the previous sentence). So $\{N_\zeta : \zeta < 2^{\lambda^+}\}$ exemplifies $IE(\lambda^+,\mathfrak{K}) = 2^{\lambda^+}$, contradicting an assumption.
2), 3) Should be clear. $\square_{\langle 1.6\rangle}$

* * *

<u>$\langle 1.6C\rangle$ Discussion</u> Instead W.D. we discuss Definable W.D. which is weaker but suffice.
On The Definable Weak Diamond: see Mekler Shelah [MkSh 313], where many

Cohen subsets are added to λ and a combinatorial principle abound amalgamation of $\langle M_s : s \subseteq n, s \neq n\rangle$ are gotten.

We are interested here in the case $n = 1$; in §3 - also $n = 2$ - really more definability can be required.

This was particularly interesting when we look at results under some set theory, so having both $2^\lambda = 2^{\lambda^+}$ and definable weak diamond on λ^+ was helpful (this plays a major role in the preliminary form of this work).

⟨1.7⟩ Definition. 1) In Definition ⟨1.1⟩ we add the superscript $\mathcal{F}$ if we restrict ourselves to $F \in \mathcal{F}$.
2) For a model $\mathfrak{B}$ whose universe include λ and have a definable pairing function on λ and a logic $\mathcal{L}$ ($\mathcal{L}$ closed under first order operations and substitution and, for our context, allow "$M \in \mathfrak{K}$", "$M \leq_{\mathfrak{K}} N$" in the formulas if not said otherwise) let

$$\mathcal{F}^{\mathrm{Def}}_{\mathfrak{B},\mathcal{L}} = \Big\{ F : \text{for some } g \in {}^\lambda\lambda \text{ and } \bar{h} = \langle h_\eta : \eta \in {}^{\lambda>}2\rangle \text{ where}$$
$$h_\eta : \ell g(\eta) \to \lambda \text{ and } h_{\eta\restriction\beta} \subseteq h_\eta \text{ and some sequence } \bar{\psi} = \langle \psi_\alpha : \alpha < \lambda\rangle,$$
$$\psi_\alpha \in \mathcal{L} \text{ for } \alpha < \lambda \text{ the following holds: for every } \alpha < \lambda \text{ and}$$
$$f \in {}^\alpha(2^{<\lambda}) \text{ we have}$$
$$F(f) = 1 \ \underline{\text{iff}}\ (\mathfrak{B}, \alpha, g, h_f) \models \psi_\alpha \ \underline{\text{iff}}\ F(f) \neq 0 \Big\}$$

3) A version of weak diamond from ⟨1.1⟩, with $\mathcal{F}$ as in ⟨1.7⟩(2) (see ⟨1.7⟩(1)) is called the $(\mathfrak{B}, \mathcal{L})$-definitional version. If $\mathcal{L}$ is $\mathcal{L}_{\lambda,\lambda}$ we may omit it. If $\mathfrak{B}$ has the form $(\mathcal{H}(\chi), \in, <^*_\chi, \lambda)$ we write $\mathcal{F}^{\mathrm{Def}[(\chi)]}_{\mathcal{L}}$ or $\mathcal{F}^{\mathrm{Def}(\chi)}$ instead $\mathcal{F}^{\mathrm{Def}}_{\mathfrak{B},\mathcal{L}}$ or $\mathcal{F}^{\mathrm{Def}}_{\mathfrak{B}}$ respectively. If we omit χ, we mean $\chi = (2^\lambda)^+$ and we may put $\mathrm{Def}(\chi)$ or Def instead of $\mathcal{F}^{\mathrm{Def}(\chi)}$ or $\mathcal{F}^{\mathrm{Def}}$ in the superscript. Having the definitional version or the definable weak diamond for λ means $\lambda \notin \mathrm{WDmId}^{\mathrm{Def}}(\lambda)$.
4) Let $\mathrm{DfWD}_{<\mu}(\lambda)$ mean that with $\mathfrak{B} = (\mathcal{H}(\chi), \in, <^*_\chi, \lambda, \mu)$ we have $\lambda \notin \mathrm{WDmId}^{\mathrm{Def}}_{<\lambda}(\lambda)$. Instead "$< \mu^+$" we may write "$\mu$" and instead of "$< 2^{<\lambda^+}$" we may write nothing.
5) Let $\mathrm{DfWD}^+_{<\mu}(\lambda)$ mean $\mathrm{DfWD}_{<\mu}(\lambda)$ and the principle $\otimes_\lambda$ below; and we adopt the same conventions as in (4) concerning μ

$\bigotimes_\lambda$ if for $\eta \in {}^{\lambda>}2$, M_η is a $\tau_{\mathfrak{K}}$-model of cardinality $< \lambda$, $\langle M_{\eta\restriction\alpha} : \alpha \leq \ell g(\eta)\rangle$ is $\leq$-increasing continuous, for $\eta \in {}^\lambda 2$ we let $M_\eta = \bigcup\limits_{\alpha<\lambda} M_{\eta\restriction\alpha}$ and $\eta \neq \nu \in {}^\lambda 2 \Rightarrow M_\eta, M_\nu$ not isomorphic over $M_{\langle\rangle}$ then $\{M_\eta/\cong : \eta \in {}^\lambda 2\}$ has cardinality 2^λ
(note that $2^\theta = 2^{<\lambda} < 2^\lambda$ implies that).

⟨1.8⟩ Claim. *1) Assume $\mathcal{L}$ first order or at least definable enriching first order.*

In the definition of $\mathcal{F}^{\mathrm{Def}}_{\mathfrak{B},\mathcal{L}}$, we can replace "for every $\alpha < \lambda$" by "for a club of $\alpha < \lambda$" (we use the assumption on $\mathcal{L}$; anyhow not serious: reread the definition

⟨1.1⟩(1)). In the definition of $\mathcal{F}^{\mathrm{Def}}$ *we can let* $g \in {}^{\lambda}({}^{\lambda>}2)$ *and* $h_\eta : \ell g(\eta) \to {}^{\lambda>}2$. *In any case* $WDmId^{\mathcal{F}}_{<\mu}(\lambda)$ *increases with* $\mathcal{F}$ *and is* $\subseteq WDmId_{<\mu}(\lambda)$, *similarly for* $WDmTId^{\mathcal{F}}_{<\mu}(\lambda)$.
2) If $\mathcal{F} = \mathcal{F}^{\mathrm{Def}}_{\mathfrak{B}}$ *and* $cf(\mu) > \lambda$ *then* $WDmId_{<\mu}(\lambda)$ *is a normal ideal (but possibly is equal to* $\mathcal{P}(\lambda)$*).*
3) Assume $V \models$ "$\lambda = \chi^+, \chi^{<\chi} = \chi, \mu > \lambda$" *and* P *is the forcing notion of adding* μ *Cohen subsets to* χ *(i.e.*
$\{g : g$ *a partial function from* μ *to* $\{0\}$ *with domain of cardinality* $< \chi\}$*).*
Then $WDmId^{\mathrm{Def}}_{<\mu}(\lambda)$ *is the ideal of non-stationary subsets of* λ*; i.e. with* $\mathfrak{B} = (\mathcal{H}(\chi), \in, <^*_\mu)^{V^P}$ *for any* χ. *Also* $\otimes_\lambda$ *of Definition ⟨1.7⟩(5) holds.*

Proof. 1); 2) By manipulating the h's (using the pairing function on λ).
3) See [MkSh 313] or think (the point being that we can break the forcing, first adding $\bar{\psi}$ and g (or the $< \mu$ ones) and then (read ⟨1.1⟩(1)) choose $\eta \in {}^{\lambda}2$ as $\underset{\sim}{g} \restriction [\gamma, \gamma + \lambda)$ not "used before", now for any candidate $f \in {}^{\lambda}({}^{\lambda>}2)$ for a club of $\delta < \lambda, \eta(\delta) = g(\gamma + \delta)$ is not used in the definition of $f \restriction \delta, h_f \restriction \delta$ so stationarily often $\eta(\delta)$ - "guess" rightly.) $\square_{\langle 1.8 \rangle}$

⟨2.12A⟩ Claim. *1) If* $2^\theta = 2^{<\lambda} < 2^\lambda$ *then* $DfWD^+(\lambda)$.
2) $DfWD_{<\mu}(\lambda)$ *holds when* $\lambda \notin WDmId_{<\mu}(\lambda)$ *(see ⟨1.2⟩ for sufficient conditions).*

<u>⟨1.8B⟩ Discussion</u>: We hope to get successful "guessing" not just on a stationary but on a positive set for the same ideal for which have guess; i.e. there is I normal ideal on λ such that for $A \in I^+$ there is $\eta \in {}^{\varsigma}2$ guessing I-positively; this is connected to questions on λ^+-saturation. See more [Sh 638].

We phrased the following notion originally in hope of later eliminating $\mu_{\mathrm{wd}}(\lambda)$ (i.e. using 2^λ instead $\mu_{\mathrm{wd}}(\lambda)$).

⟨1.9⟩ Definition. 1)

$$\mathrm{UDmId}^{\mathcal{F}}_{<\mu}(\lambda) = \Big\{ S \subseteq \lambda : \text{for some } i^* < \mu \text{ and } F_i \in \mathcal{F} \text{ (for } i < i^*)$$
$$\text{for every } \eta \in {}^{S}2 \text{ there is } f \in {}^{\lambda}(2^{<\lambda}) \text{ and } i < i^*$$
$$\text{and club } E \text{ of } \lambda \text{ such that:}$$
$$\text{for every } \delta \in E \text{ we have:}$$
$$\delta \in S \Rightarrow \eta(\delta) = F_i(f \restriction \delta)$$
$$\delta \in \lambda \backslash S \Rightarrow 0 = F_i(f \restriction \delta) \Big\}.$$

2) We omit μ if $\mu = 1$.
3) $BA^{\mathcal{F}}(\lambda)$ is defined as the family of $S \subseteq \lambda$ such that for some $F \in \mathcal{F}$ and $\eta = O_S$ the condition above holds.

$\langle 1.10\rangle$ Claim. *Assume $\mathcal{F} = \mathcal{F}^{\rm Def}_{\mathfrak{B}}$.*
1) In the definition $\langle 1.9\rangle(1)$ we can replace $f \in {}^\lambda(2^{<\lambda})$ by $f \in {}^\lambda 2$ or $f \in {}^\lambda({}^{\lambda>}2)$.
2) $UDmId^{\mathcal{F}}(\lambda)$ is a normal ideal on λ (but possibly is $\mathcal{P}(\lambda)$).
3) $BA^{\mathcal{F}}(\lambda)$ is a Boolean algebra of subsets of λ including all non-stationary subsets of λ and even $UDmId^{\mathcal{F}}(\lambda)$ and is closed under union of $< \lambda$ and even diagonal union.
4) If $S \in BA^{\mathcal{F}}(\lambda)$ and $F \in \mathcal{F}$ then for some $\eta \in {}^S 2$ we have:

- $(*)$ *for every $f \in {}^\lambda({}^{\lambda>}2)$ we have*

$$\{\delta \in S : \eta(\delta) = F(f \restriction \delta)\} \neq \emptyset \ mod \ UDmId^{\mathcal{F}}(\lambda).$$

5) $UDmId^{\mathcal{F}}_{<\mu}(\lambda) \subseteq WDmId^{\mathcal{F}}_{<\mu}(\lambda) \subseteq WDmId_{<\mu}(\lambda)$ and they increase with $\mathcal{F}$ and $\lambda \in UDmId^{\mathcal{F}}_{<\mu}(\lambda) \Leftrightarrow \lambda \in WDmId^{\mathcal{F}}_{<\mu}(\lambda)$.

Proof. Straight.

$\langle 1.12\rangle$ Discussion Remember

- $(*)_1$ If $V =$ "$\chi = \chi^{<\chi}$ & $2^\chi = \chi^+$", P is the forcing notion of adding $\mu > \chi^+$ Cohen subsets to χ then in V^χ, any equivalence relation on $\mathcal{P}(\lambda)$ definable with parameter $X \subseteq \chi$ and ordinals which has at least μ equivalence classes and (see [Sh:f, XVI,§2])
- $(*)_2$ ZFC is consistent with $CH+$ for some stationary, costationary $S \subseteq \omega_1$ we have
  - (a) $\mathrm{WDmId}(\aleph_1) = \{A \subseteq \omega_1 : A \backslash S$ is not stationary$\}$
  - (b) $\mathcal{D}_{\omega_1} + S$ is $\aleph_2$-saturated.
- $(*)_3$ $ZFC + GCH$ is consistent with $\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\} \in \mathrm{WDmId}(\aleph_2)$
- $(*)_4$ $ZFC + 2^{\aleph_1} < 2^{\aleph_2}$ is consistent with $\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\} \in \mathrm{WDmId}(\aleph_2)$.

§2 First attempts

Starting with amalgamation (reasonable by $\langle 1.4\rangle$) we try to analyze types $p \in \mathcal{S}(M), M \in K_\lambda$, so because types here (as in [Sh 300]) are not set of formulas, we deal with triples $(M, N, a), M \leq_{\mathfrak{K}} N$ and $a \in N \backslash M$. We look for nice such triples and try to prove mainly the density of the set of minimal triples (i.e. ones representing minimal types).

To help the reader, here we allow uses of stronger assumptions than desired (e.g. $2^{\lambda^+} > \lambda^{++}$ and/or $K_{\lambda^{+3}} = \emptyset$), which later will be eliminated. Note: the first extra assumption is still a "soft set theoretic assumption" and the second is harmless as long as we think of just proving our main theorem $\langle 0.\mathrm{B}\rangle$ (and not on continuations).

So the aim of this section is to show we can start on our way in analyzing such classes.

$\langle 2.1\rangle$ *Hypothesis.* $\mathfrak{K}$ is an abstract elementary class.

$\langle \mathbf{2.2}\rangle$ **Claim.** *Assume*

$(*)^2_\lambda$ *K categorical in λ and $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$ and $LS(\mathfrak{K}) \leq \lambda$ and: $2^\lambda < 2^{\lambda^+}$ or at least the definable weak diamond for λ^+ holds.*

Then
1) K_λ has amalgamation.
2) If $I(\lambda^{++}, K) = 0$ then $\mathfrak{K}$ has a universal homogeneous above λ model in λ^+, so this model is saturated above λ (see $\langle 0.18\rangle(2)$).
3) If $I(\lambda^{++}, K) = 0$ then $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda^+$.

Proof. 1) We know that there is no λ^+-saturated ideal on λ^+; i.e. clause (d) of $\langle 2.6\rangle(1)$ holds (with λ^+ here standing for λ there). If the conclusion fails and $2^\lambda < 2^{\lambda^+}$, then the assumptions of $\langle 1.3\rangle(1)$ hold $(*)_1$ of $\langle 1.4\rangle(1)$ (see there) and easily also $(*)_2$ of $\langle 1.4\rangle(1)$, hence by $\langle 1.4\rangle(4)$ we have $I(\lambda^+, K) = 2^{\lambda^+}$ contradiction. If $2^\lambda = 2^{\lambda^+}$ we use the variants from $\langle 1.7\rangle$.
2) As $I(\lambda^{++}, K) = 0 < I(\lambda^+, K)$, clearly there is $M \in K_{\lambda^+}$ which is maximal, let $M = \bigcup\limits_{i<\lambda^+} M_i$ with $M_i \in K_\lambda$ increasing continuously; if M is not universal homogeneous above λ then there are $N_0 \leq_{\mathfrak{K}} M$ satisfying $\|N_0\| = \lambda$ and $N_1, N_0 \leq_{\mathfrak{K}} N_1 \in K_\lambda$ such that N_1 cannot be embedded into M over N_0. Without loss of generality $N_0 = M_0$ and use $\langle 0.17\rangle(2)$ to get contradiction.
3) Follows. $\square_{\langle 2.2\rangle}$

$\langle \mathbf{2.3}\rangle$ **Claim.** *Assume*

$(*)^3_\lambda$ *K is categorical in λ, λ^+ and $1 \leq I(\lambda^{++}, K)$ and $LS(\mathfrak{K}) \leq \lambda$.*

<u>Then</u>:
If $M_0 \leq_{\mathfrak{K}} M_1$ *are in* K_λ *and* $a \in M_1 \backslash M_0$, *then we can find* M_0', M_1' *from* $\mathfrak{K}_\lambda$ *such that:* $M_0 \neq M_0'$ *and* $\otimes$ *below holds (denoted by* $(M_0, M_1, a) \leq (M_0', M_1', a)$ *in* $\langle 2.3A\rangle(1)$ *below) where:*

$\bigotimes$(α) $M_0 \leq_{\mathfrak{K}} M_0' \leq_{\mathfrak{K}} M_1' \in K_\lambda$
(β) $M_1 \leq_{\mathfrak{K}} M_1'$
(γ) $a \in M_1' \backslash M_0'$.

Proof. Assume (M_0^*, M_1^*, a^*) is a counterexample. We can choose $\langle N_i^0, a_i^0 : i < \lambda^+\rangle$ such that:

(a) $N_i^0 \in K_\lambda$ is $\leq_{\mathfrak{K}}$ -increasing continuous in i
(b) h_i is an isomorphism from M_1^* onto N_{i+1}^0 such that $h_i(M_0^*) = N_i^0, h_i(a^*) = a_i^0$.

Now as $a^* \in M_1^* \backslash M_0^*$ clearly $i < j < \lambda^+ \Rightarrow a_i^0 \in N_{i+1}^0 \leq_{\mathfrak{K}} N_j^0$ & $a_j^0 \notin N_{i+1}^0$ hence $\bigcup\limits_{i<\lambda^+} N_i^0 \in K_{\lambda^+}$.

By $\langle 0.17\rangle(1)$ applied to λ^+ there are $M_0^{\otimes} \leq_{\mathfrak{K}} M_1^{\otimes}$ in $K_{\lambda^+}, M_0^{\otimes} \neq M_1^{\otimes}$, and there is $b \in M_1^{\otimes} \backslash M_0^{\otimes}$. As K is categorical in λ^+, without loss of generality $M_0^{\otimes} = \bigcup\limits_{i<\lambda^+} N_i^0$.
Let χ be large enough and $\mathfrak{B} \prec (\mathcal{H}(\chi) \in, <_\chi^*)$ be such that $\lambda \subseteq \mathfrak{B}, \|\mathfrak{B}\| = \lambda$ and $\langle N_i^0, a_i^0 : i < \lambda^+\rangle, M_0^{\otimes}, M_1^{\otimes}, b$ and the definition of $\mathfrak{K}$ belong to $\mathfrak{B}$.
Let $\delta = \mathfrak{B} \cap \lambda^+$, so $\delta \in (\lambda, \lambda^+)$ is a limit ordinal and

$$N_\delta^0 \leq_{\mathfrak{K}} N_{\delta+1}^0 \leq_{\mathfrak{K}} M_1^{\otimes},$$

$$N_\delta^0 \leq_{\mathfrak{K}} (M_1^{\otimes} \cap \mathfrak{B}) \leq_{\mathfrak{K}} M_1^{\otimes}$$

$$\mathfrak{B} \cap M_0^{\otimes} = N_\delta^0$$

$$N_{\delta+1}^0 \cap (M_1^{\otimes} \cap \mathfrak{B}) = N_\delta^0,$$

so for some N we have:

$$N \in \mathfrak{K}_\lambda, N \leq_{\mathfrak{K}} M_1^{\otimes}, \text{ and } (N_{\delta+1}^0 \cup (M_1^{\otimes} \cap \mathfrak{B})) \subseteq N$$

so (see Definition $\langle 2.3A\rangle(1)$ below)

$$(N_\delta^0, N_{\delta+1}^0, a_\delta^0) \leq_{\mathfrak{K}} (M_1^{\otimes} \cap \mathfrak{B}, N, a_\delta^0),$$

and b witnesses $N_\delta^0 \neq M_1^{\otimes} \cap \mathfrak{B}$.
Contradiction to choice of h_δ (and (M_0^*, M_1^*, a^*)). $\square_{\langle 2.3\rangle}$

⟨2.3A⟩ Definition.
1)(a) $K^3_\lambda = \{(M_0, M_1, a) : M_0 \le_{\mathfrak{K}} M_1$ are both in K_λ and $a \in M_1 \backslash M_0\}$.

$$(b) \qquad (M_0, M_1, a) \le (M'_0, M'_1, a') \text{ \underline{if} } a = a', M_0 \le_{\mathfrak{K}} M'_0, M_1 \le_{\mathfrak{K}} M'_1$$

$$(c) \qquad \begin{aligned}(M_0, M_1, a) \le_h (M'_0, M'_1, a') \text{ \underline{if} } h(a) = a', \text{ and for } \ell = 0, 1 \text{ we have:}\\ h \restriction M_\ell \text{ is a } \le_{\mathfrak{K}} \text{-embedding of } M_\ell \text{ into } M'_\ell.\end{aligned}$$

$$(d) \qquad \begin{aligned}(M_0, M_1, a) < (M'_0, M'_1, a') \text{ \underline{if} } (M_0, M_1, a)\\ \le (M'_0, M'_1, a) \text{ and } M_0 \ne M'_0.\end{aligned}$$

$$(e) \qquad \text{similarly } <_h$$

2) $(M_0, M_1, a) \in K^3_\lambda$ has the weak extension property if the conclusion of ⟨2.3⟩ holds.
3) $(M_0, M_1, a) \in K^3_\lambda$ has the extension property <u>if</u>: for every $N_0 \in \mathfrak{K}_\lambda$ and $\le_{\mathfrak{K}}$-embedding f of M_0 into N_0 there are N_1, b and g such that: $(M_0, M_1, a) \le_g (N_0, N_1, b) \in K^3_\lambda$ and $g \supseteq f$ (so $g(a) = b$ and g is a $\le_{\mathfrak{K}}$-embedding of M_1 into N_1).
4) $(M_0, M_1, a) \in K^3_\lambda$ is minimal when:

$$\text{if } (M_0, M_1, a) \le_{h_\ell} (M'_0, M^\ell_1, a_\ell) \in K^3_\lambda \text{ for } \ell = 1, 2$$

$$\underline{\text{then}} \ \text{tp}(a_1, M'_0, M^1_1) = \text{tp}(a_2, M'_0, M^2_1)$$

5) $(M_0, M_1, a) \in K^3_\lambda$ is reduced <u>if</u>

$$(M_0, M_1, a) \le (M'_0, M'_1, a) \in \mathfrak{K}^3_\lambda \Rightarrow M'_0 \cap M_1 = M_0.$$

6) We say $p \in \mathcal{S}(M_0)$ is minimal, where $M_0 \in K_\lambda$ if for some a, M_1 we have: $p = \text{tp}(a, M_0, M_1)$ and $(M_0, M_1, a) \in K^3_\lambda$ is minimal.
7) We say $p \in \mathcal{S}(M_0)$ is reduced where $M_0 \in K_\lambda$ if for some a, M_1 we have $p = \text{tp}(a, M_0, M_1)$ and $(M_0, M_1, a) \in K^3_\lambda$ is reduced.

⟨2.4⟩ Fact. 1) For every $(M_0, M_1, a) \in K^3_\lambda$ there is a reduced (M'_0, M'_1, a) such that: $(M_0, M_1, a) \le (M'_0, M'_1, a) \in K^3_\lambda$.
2) Assume $\langle (M_{0,\alpha}, M_{1,\alpha}, a) : \alpha < \delta \rangle$ is an increasing sequence of members of K^3_λ. If $\delta < \lambda^+$ <u>then</u> $(M_{0,\alpha}, M_{1,\alpha}, a) \le (\bigcup_{\beta<\delta} M_{0,\beta}, \bigcup_{\beta<\delta} M_{1,\beta}, a) \in K^3_\lambda$. If $\delta = \lambda^+$ the result

may be in $K^3_{\lambda^+}$: if $\{\alpha < \delta : M_{0,\alpha} \neq M_{0,\alpha+1}\}$ is cofinal, this holds.
2A) In part (2) if $\delta < \lambda^+$ and each $(M_{0,\alpha}, M_{1,\alpha}, a)$ is reduced <u>then</u> so is $(\bigcup_{\beta<\delta} M_{0,\beta}, \bigcup_{\beta<\delta} M_{1,\beta}, a)$.
3) If $(M_0, M_1, a) \leq (M'_0, M'_1, a)$ are in K^3_λ and the first triple is minimal <u>then</u> so is the second.
4) If $(M_0, M_1, a) \leq (M'_0, M'_1, a)$ are in K^3_λ <u>then</u> $\mathrm{tp}(a, M_0, M_1) \leq \mathrm{tp}(a, M'_0, M'_1)$; see Definition ⟨0.14⟩(1).
5) If K_λ has amalgamation, then: $(M_0, M_1, a) \in K^3_\lambda$ is minimal if and only if:

(∗) if $(M_0, M_1, a) \leq_{h_\ell} (M'_0, M'_1, a_\ell) \in K^3_\lambda$ for $\ell = 1, 2$ and $h_1 \restriction M_0 = h_2 \restriction M_0$ <u>then</u> $\mathrm{tp}(a_1, M'_0, M'_1) = \mathrm{tp}(a_2, M'_0, M'_1)$.

6) If there is no maximal member[3] of K^3_λ then there are $N^0 <_{\mathfrak{K}} N^1$ in $\mathfrak{K}_{\lambda^+}$.
7) If every triple in K^3_λ has the weak extension property, and there are $N_0 <_{\mathfrak{K}} N_1$ in $\mathfrak{K}_\lambda$ <u>then</u> there are $N^0 <_{\mathfrak{K}} N^1$ in $\mathfrak{K}_{\lambda^+}$.
8) If $\mathrm{LS}(\mathfrak{K}) \leq \lambda$ and every triple in K^3_λ has the extension property <u>then</u> no $M \in K_{\lambda^+}$ is $<_{\mathfrak{K}}$-maximal hence $K_{\lambda^{++}} \neq \emptyset$.
9) If $LS(\mathfrak{K}) \leq \lambda$ & $K_{\lambda^+} \neq \emptyset$ or there is no maximal member of K_λ <u>then</u> $K^3_\lambda \neq \emptyset$.

Proof. Easy. (Note that part (7) is ⟨2.3⟩).

⟨2.5⟩ Claim. *Assume* $(*)^2_{\lambda^+} + (*)^3_\lambda + 2^{\lambda^+} > \lambda^{++} + K_{\lambda^{+3}} = \emptyset$.
Then in $\mathfrak{K}^3_\lambda$ *the minimal triples are dense (i.e. above every triple in* K^3_λ *there is a minimal one).*

Remark. We do not intend to adopt "$2^{\lambda^+} > \lambda^{++}$", $K_{\lambda^{+3}} = \emptyset$. This will be eliminated in §3.

Proof. If not, we can choose by induction on $\alpha < \lambda^+$, for $\eta \in {}^\alpha 2$ a triple $(M^0_\eta, M^1_\eta, a_\eta)$ and $h_{\eta,\nu}$ for $\nu \trianglelefteq \eta$ such that:

(*i*) $(M^0_\eta, M^1_\eta, a_\eta) \in K^3_\lambda$
(*ii*) $\nu \triangleleft \eta \Rightarrow (M^0_\nu, M^1_\nu, a_\nu) \leq_{h_{\eta,\nu}} (M^0_\eta, M^1_\eta, a_\eta)$
(*iii*) $\nu_0 \triangleleft \nu_1 \triangleleft \nu_2 \Rightarrow h_{\nu_2,\nu_0} = h_{\nu_2,\nu_1} \circ h_{\nu_1,\nu_0}$
(*iv*) $(M^0_{\eta\char94\langle\ell\rangle}, M^1_{\eta\char94\langle\ell\rangle}, h_{\eta\char94\langle\ell\rangle,\eta} \restriction M^0_\eta)$ for $\ell = 0, 1$ are equal
(*v*) $\mathrm{tp}(a_{\eta\char94\langle 0\rangle}, M^0_{\eta\char94\langle\ell\rangle}, M^1_{\eta\char94\langle 0\rangle}) \neq \mathrm{tp}(a_{\eta\char94\langle 1\rangle}, M^0_{\eta\char94\langle\ell\rangle}, M^1_{\eta\char94\langle 1\rangle})$ (value of ℓ is immaterial)
(*vi*) if $\eta \in {}^\delta 2$ and $\delta < \lambda^+$ a limit ordinal, <u>then</u> $M^\ell_\eta = \bigcup_{\alpha<\delta} h_{\eta,\eta\restriction\alpha}(M^\ell_{\eta\restriction\alpha})$ for $\ell = 0, 1$,
(*vii*) $(M^0_{<>}, M^1_{<>}, a_{<>}) \in K^3_\lambda$ is a triple above which there is no minimal one.

[3]will be applied for λ^+

This is straight: in limit α use limit of diagrams (h's); in successor α, use non-minimality and $\langle 2.4\rangle(5)$; for $\alpha = 0$ choose a triple in K^3_λ above which supposedly there is no minimal triple.

Let $M^* \in \mathfrak{K}_{\lambda^{++}}$ be saturated above λ (exists by $\langle 2.2\rangle(2)$ so it is necessarily homogeneous universal above λ^+ hence above λ; note: λ there stands for λ^+ here).

We choose by induction on $\alpha < \lambda^+$ for $\eta \in {}^\alpha 2$, a $\le_{\mathfrak{K}}$-embedding g_η of M^0_η into M^* such that:

$$\nu \triangleleft \eta \Rightarrow g_\nu = g_\eta \circ h_{\eta,\nu}.$$

$$g_{\eta\char`^\langle 0\rangle} = g_{\eta\char`^\langle 1\rangle}.$$

Clearly possible. Let $N^0_\eta = M^* \restriction \text{Rang}(g_\eta)$. For $\eta \in {}^{\lambda^+}2$ let $g_\eta = \bigcup\limits_{\alpha<\lambda^+} g_{\eta\restriction\alpha}$ and $N^0_\eta = M^* \restriction \text{Rang}(g_\eta)$. Chasing arrows we can find for $\eta \in {}^{\lambda^+}2$ a limit to $(\langle M^0_{\eta\restriction\alpha}, M^1_{\eta\restriction\alpha}, a_{\eta\restriction\alpha}\rangle, h_{\eta\restriction\beta,\eta\restriction\alpha} : \alpha < \beta < \lambda^+)$ say $(M^0_\eta, M^1_\eta, a_\eta) \in K^3_{\lambda^+}$ and $h_{\eta,\nu}$ for $\nu \triangleleft \eta$ as usual. Let f_η be the function from M^0_η into M^* such that for $\alpha < \lambda^+$ we have $f_\eta \circ h_{\eta,\eta\restriction\alpha} = g_\eta$. So f_η is a $\le_{\mathfrak{K}}$-embedding of M^0_η into M^*. So we can extend f_η to f^+_η, a $\le_{\mathfrak{K}}$-embedding of M^1_η into M^*.

Let $a^*_\eta = f^+_\eta(a_\eta)$ for $\eta \in {}^{\lambda^+}2$.

As $2^{\lambda^+} > \lambda^{++}$ for some $\eta_0 \neq \eta_1$ we have $a^*_{\eta_0} = a^*_{\eta_1}$. So for some $\alpha < \lambda^+$, $\eta_0 \restriction \alpha = \eta_1 \restriction \alpha$ but $\eta_0(\alpha) \neq \eta_1(\alpha)$, without loss of generality $\eta_\ell(\alpha) = \ell$ and by clause (v) above we get a contradiction. $\square_{\langle 2.5\rangle}$

$\langle$**2.6**$\rangle$ **Claim.** *Assume $(*)^2_\lambda$ or just:*

$(*)^{2^-}_\lambda$ *$\mathfrak{K}$ has amalgamation for λ and $LS(\mathfrak{K}) \le \lambda$.*

1) Assume that above $(M_0, M_1, a) \in K^3_\lambda$ there is no minimal member of K^3_λ <u>then</u> (M_0, M_1, a) itself has the extension property.
2) If $(M_0, M_1, a) \in K^3_\lambda, M_0 \le_{\mathfrak{K}} N \in K$ and the number of $c \in N$ such that $tp(c, M_0, N) = tp(a, M_0, M_1)$ is $> \lambda$ <u>then</u> (M_0, M_1, a) has the extension property.
3) Assume above $(M_0, M_1, a) \in K^3_\lambda$ there is no minimal member of K^3_λ <u>then</u>

- (a) *for some $M^+_0, M_0 \le_{\mathfrak{K}} M^+_0 \in K_\lambda$ and $tp(a, M_0, M_1)$ has $> \lambda$ extensions in $\mathcal{S}(M^+_0)$ (in fact $\ge$ $min\{2^\mu : 2^\mu > \lambda\}$)*
- (b) *for some N we have: $M_0 \le_{\mathfrak{K}} N$ and N is as required in part (2).*

4) If $M_0 \le_{\mathfrak{K}} N_0 \in K_{\lambda^+}$ and $(M_0, M_1, a) \in K_\lambda$ <u>then</u> there is $N \in K_{\lambda^+}$ such that: $N_0 \le_{\mathfrak{K}} N$ and for every $c \in N$ satisfying $tp(c, M_0, N) = tp(a, M_0, M_1)$, there is a $\le_{\mathfrak{K}}$-embedding of M_1 into N extending id_{M_0} such that $h(a) = c$.

$\langle 2.6A\rangle$ Remark. 1) See $\langle 3.16\rangle$.
2) Note that $(*)^2_\lambda$ is from $\langle 2.2\rangle$ and $(*)^2_\lambda \Rightarrow (*)^{2^-}_\lambda$ by $\langle 2.2\rangle$.

Proof. 1) Follows by part (2) and (3) (part (b)).
2) Without loss of generality N has cardinality λ^+ and also is as in part (4) proved below.
By $\langle 0.17\rangle(2)$ for any M_0' such that $M_0 \leq_{\mathfrak{K}} M_0' \in K_\lambda$ there is $N_1, N \leq_{\mathfrak{K}} N_1 \in K_{\lambda^+}$ and a $\leq_{\mathfrak{K}}$-embedding h of M_0' into N_1 extending id_{M_0}. Now some $c \in N \backslash h(M_0')$ realizes $\mathrm{tp}(a, M_0, M_1)$ and there is an embedding h_c of M_1 into N extending id_{M_0}. Lastly let $N_1' \leq_{\mathfrak{K}} N_1$ be of cardinality λ and include $\mathrm{Rang}(h_c) \cup$ $\mathrm{Rang}(h)$ (send a to c via k).
3) We first prove clause (a) and then clause (b).

Proof of (a). Let $M^0_\eta, M^1_\eta, a_\eta, h_{\eta,\nu}$ ($\eta \in {}^{\lambda^+ >}2$ and $\nu \trianglelefteq \eta$) be as in the proof of $\langle 2.5\rangle$ (i.e. satisfy $(i) - (vi)$ there) and $M^0_{<>} = M^0, M^1_{<>} = M^1$. Let $\mu =$ $\mathrm{Min}\{\mu : 2^\mu > \lambda\}$, so ${}^{\mu >}2$ has cardinality $\leq \lambda$ and $\mu \leq \lambda$. Let ${}^{\mu >}2 = \{\eta_\zeta : \zeta < \zeta^*\}$ be such that $\eta_\xi \triangleleft \eta_\zeta \Rightarrow \xi < \zeta$ and so $\zeta^* < \lambda^+$ and without loss of generality is a limit ordinal. Now we can choose by induction on $\zeta \leq \zeta^*$ a model $M^*_\zeta \in K_\lambda$ and, if $\zeta < \zeta^*$ also a function g_{η_ζ} such that:

- (α) M^*_ζ is $\leq_{\mathfrak{K}}$-increasing continuous in ζ
- (β) $M^*_0 = M_{\langle\rangle}$
- (γ) g_{η_ζ} is a $\leq_{\mathfrak{K}}$-embedding of $M^0_{\eta_\zeta}$ into M^*_ζ
- (δ) if $\eta_\xi \triangleleft \eta_\zeta$ then $g_{\eta_\zeta} \circ h_{\eta_\zeta, \eta_\xi} = g_{\eta_\xi}$
- (ε) if $\xi < \zeta_0, \xi < \zeta_1, \eta_{\zeta_0} = \eta_\xi{}^\frown\langle 0\rangle, \eta_{\zeta_1} = \eta_\xi{}^\frown\langle 1\rangle$ then
  $(M^*_{\zeta_0} = M^*_{\zeta_1}$ and) $g_{\eta_{\zeta_0}} = g_{\eta_{\zeta_1}}$.

So for $\eta \in {}^\mu 2$ we can find g_η, a $\leq_{\mathfrak{K}}$-embedding of M^0_η into $M^*_{\zeta^*}$ such that
$g_\eta \circ h_{\eta, \eta\restriction\alpha} = g_{\eta\restriction\alpha}$ for every $\alpha < \mu$, we also can let
$p^0_\eta = g_\eta[\mathrm{tp}(a_\eta, M^0_\eta, M^1_\eta)] \in \mathcal{S}(M^*_{\zeta^*} \restriction \mathrm{Rang}(g_\eta))$, and find $p_\eta, p^0_\eta \leq p_\eta \in \mathcal{S}(M^*_{\zeta^*})$ (possible as $\mathfrak{K}_\lambda$ has amalgamation by $\langle 2.2\rangle(1)$ if $(*)^2_\lambda$ holds and by $(*)^{2^-}_\lambda$ otherwise).

For $\eta \in {}^\mu 2$ and $\alpha \leq \mu$ let $N^0_{\eta\restriction\alpha} = g_\eta(M^*_{\zeta^*} \restriction \mathrm{Rang}\ (h_{\eta,\eta\restriction\alpha}))$, clearly for
$\eta \in {}^{\mu \geq}2, N^0_\eta$ is well defined, $\eta \triangleleft \nu \in {}^{\mu\geq}2 \Rightarrow N^0_\eta \leq_{\mathfrak{K}} N^0_\nu$ and $N^0_{\eta^\frown\langle 0\rangle} = N^0_{\eta^\frown\langle 1\rangle}$. Also letting for $\eta \in {}^\mu 2$, and $\alpha \leq \mu$, the type $p^0_{\eta\restriction\alpha}$ be $p^0_\eta \restriction N^0_{\eta\restriction\alpha}$ we have: $p^0_\eta \in \mathcal{S}(N^0_\eta)$ is well defined, $\eta \triangleleft \nu \in {}^{\mu\geq}2 \Rightarrow p^0_\eta \leq p^0_\nu$ and $p^0_{\eta^\frown\langle 0\rangle} \neq p^0_{\eta^\frown\langle 1\rangle}$. Clearly for $\eta_0 \neq \eta_1$ from ${}^\mu 2$ we have $p_{\eta_0} \neq p_{\eta_1}$. So $|\mathcal{S}(M^*_{\zeta^*})| \geq 2^\mu > \lambda$.

Proof of (b). We choose by induction on $i < \lambda^+, N_i \in K_{\lambda^+}$ which is $\leq_{\mathfrak{K}}$-increasing continuous, $N_0 = M^+_0$ (M^+_0 is from clause (a)) and for each i some $c_i \in N_{i+1}$ realizes over N_0 a (complete) extension of $p =$ $\mathrm{tp}(a, M_0, M_1)$ not realized in N_i, note that there is such type by clause (a) above and there is such N_{i+1} as $\mathfrak{K}$ has amalgamation in λ. Clearly $c_i \notin N_i$ and so $\bigcup\limits_{i<\lambda^+} N_i$ is as required.
4) By using $\langle 0.17\rangle(2)$ repeatedly λ^+ times. $\square_{\langle 2.6\rangle}$

⟨2.7⟩ Claim. *Assume* $(*)^{2^-}_{\lambda^+}$ *(from ⟨2.6⟩).*

If $(M_0, M_1, a) \le (M'_0, M'_1, a)$ *are from* K^3_λ, *and the second has the extension property, <u>then</u> the first has the extension property.*

Proof. Use amalgamation over M_0: if $M_0 \le_{\mathfrak{K}} N_0 \in K_\lambda$ we can find N'_0 such that $M'_0 \le_{\mathfrak{K}} N'_0 \in K_\lambda$ and there is a $\le_{\mathfrak{K}}$-embedding of N_0 into N'_0 over M_0. Now use "(M'_0, M'_1, a) has the extension property" for N'_0. $\square_{\langle 2.7\rangle}$

⟨2.8⟩ Claim. *Assume* $(*)^{2^-}_\lambda + (*)^2_{\lambda^+} + (*)^3_\lambda + 2^{\lambda^+} < 2^{\lambda^{++}} + K_{\lambda^{+3}} = \emptyset$.
If $(M_0, M_1, a) \in K^3_\lambda$ *is minimal <u>then</u> it has the extension property.*

Remark. Instead of $2^{\lambda^+} < 2^{\lambda^{++}}$, we can just demand the definable weak diamond.

Proof. Assume not. As $\mathfrak{K}$ has amalgamation in λ^+ by ⟨1.6⟩(1) there is $M^* \in K_{\lambda^{++}}$ which is saturated above λ^+ hence above λ (by ⟨1.6⟩(3)). By ⟨2.4⟩(1) + ⟨2.7⟩, without loss of generality

$\bigotimes_0$ (M_0, M_1, a) is reduced.

Let $h : M_0 \to M^* \in K_{\lambda^+}$ be a $\le_{\mathfrak{K}}$-embedding and let $p = \text{tp}(a, M_0, M_1)$. If $h(p)$ is realized in M^* by $\ge \lambda^+$ elements by ⟨2.6⟩(2) we are done. So assume

$\bigotimes_1$ $h(p)$ is realized by $\le \lambda$ members of M^* more generally true for $g(\text{tp}(a, M'_0, M'_1))$ if g is a $\le_{\mathfrak{K}}$-embedding of M'_0 into M^* and $(M'_0, M'_1, a) \ge (M_0, M_1, a)$.

Now we introduce

⟨2.9⟩ Definition. For any models $M, M_0 \in K_\lambda$, a type $p \in \mathcal{S}(M_0)$ and $f_0 : M_0 \xrightarrow[\text{iso}]{\text{onto}} M$ we let $\mathcal{S}_p(M) = \mathcal{S}^p_M = \{f_0(f(p)) : f \in \text{AUT}(M_0)\}$. Note: $\mathcal{S}^p_M$ does not depend on f_0. If K is categorical in $\lambda, \mathcal{S}_p(M)$ is well defined for every $M \in K_\lambda$. We write also $\mathcal{S}_{\text{tp}(a,M_0,M_1)}(M)$ or $\mathcal{S}_{(M_0,M_1,a)}(M)$ when $(M_0, M_1, a) \in K^3_\lambda$.

Continuation of the proof of ⟨2.8⟩. : Next we prove

$\bigotimes_2$ for some reduced $(M'_0, M'_1, a) \ge (M_0, M_1, a)$ from K^3_λ we have $|\mathcal{S}_{(M'_0,M'_1,a)}(M'_0)| > \lambda^+$.

Proof of $\bigotimes_2$. If not, we build two non-isomorphic members of K_{λ^+}.

<u>First</u>: Choose by induction on $i < \lambda^+, (N_{0,i}, N_{1,i}, a) \in K^3_\lambda$ reduced (⟨2.4⟩(1)), increasing continuously (see ⟨2.4⟩(2)), $N_{0,i} \ne N_{0,i+1}, (N_{0,0}, N_{1,0}, a) = (M_0, M_1, a)$; this is possible as $(N_{0,i}, N_{1,i}, a) \in K^3_\lambda$ has the weak extension property (by ⟨2.3⟩ see ⟨2.3A⟩(1)). Let $N^1 = \bigcup\limits_{i<\lambda^+} N_{0,i}$.

Second: Choose by induction on $i < \lambda^+$, $N^0_i \leq_{\mathfrak{K}} M^*$, $\|N^0_i\| = \lambda$, N^0_i strictly increasing continuous such that:

(∗) for every $\alpha < \lambda^+, \beta < \lambda^+$, and $q \in \mathcal{S}_{(M'_0,M'_1,a)}, N_{1,\alpha}, a(N_\beta)$ for some $\gamma \in (\beta, \lambda^+)$ for no $N', N^0_\gamma \leq_{\mathfrak{K}} N' \in K_\lambda$ and $c \in N' \backslash N^0_\gamma$ does c realize q.

This is straightforward by $\otimes_1 + \otimes_2$ and bookkeeping. Let $N^0 = \bigcup\limits_{i<\lambda^+} N^0_i$. By categoricity of K_{λ^+} there is an isomorphism g from N^1 onto N^0, so $E = \{\delta < \lambda^+ : g \text{ maps } N^0_\delta \text{ onto } N_{0,\delta}\}$ is a club of λ^+. Now let $\delta^* \in E$, so apply (∗) for $\beta = \delta^*, q = g(\text{tp}(a, N_{0,\delta^*}, N_{1,\delta^*}))$ to get γ. Choose $\delta \in E \backslash \gamma$. Now $N_{1,\delta}$ gives a contradiction. $\square_{\otimes_2}$

Without loss of generality

$\bigotimes_3$ $|\mathcal{S}_{(M_0,M_1,a)}(M_0)| > \lambda^+$ and $p = \text{tp}(a, M_0, M_1)$.

Next

$\bigotimes_4$ assume $M \in K_\lambda, M \leq_{\mathfrak{K}} M^*, \Gamma \subseteq \bigcup\{\mathcal{S}_p(M') : M' \leq_{\mathfrak{K}} M, \|M'\| = \lambda\}$, $|\Gamma| \leq \lambda^+$, then

$$\Gamma^* =: \Big\{ q \in \mathcal{S}_p(M) : \text{there is } M', M \leq_{\mathfrak{K}} M', \|M'\| = \lambda, \quad M' \text{ realizes } q \text{ but no } r \in \Gamma \Big\}$$

has cardinality λ^{++}, in fact $|\mathcal{S}_p(M) \backslash \Gamma^*| \leq \lambda^+$.

Proof of $\otimes_4$. Without loss of generality $|M^*| = \lambda^{++}$ (i.e. the universe of M^* is λ^{++}), now for every $q \in \mathcal{S}_p(M)$ there is a triple $(M_0, M_{1,q}, a_q)$ isomorphic to (M_0, M_1, a) (hence reduced) such that $\text{tp}(a_q, M_0, M_{1,q}) = q$. As M^* is saturated above λ, by ⟨0.19⟩ without loss of generality $M_{1,q} \leq_{\mathfrak{K}} M^*$.

Without loss of generality $\delta < \lambda^{++}$ & (λ^+ divides δ) $\Rightarrow M_\delta =: M^* \restriction \delta \leq_{\mathfrak{K}} M^*$. Now

$(*)_0$ $q_1 \neq q_2 \Rightarrow a_{q_1} \neq q_2$ and

$(*)_1$ $a_q \notin \delta, \delta < \lambda^{++}$ & λ^+ divides $\delta \Rightarrow M_{1,q} \cap M_\delta = M$.

[Why? As $(M, M_{1,q}, a_q)$ is reduced].
Now if $r \in \Gamma$, say $r \in \mathcal{S}_p(M'')$ then by $\otimes_1$ we know $A_r = \{c \in M^* : c \text{ realizes } r\}$ has cardinality $\leq \lambda$ hence $A = \bigcup\{A_r : r \in \Gamma\}$ has cardinality $\leq \lambda^+$, so we can find $\delta < \lambda^{++}$ divisible by λ^+ such that $A \subseteq \delta$. So as (by $\otimes_3$) we have $|\mathcal{S}_p(M)| > \lambda^+$ hence we can find $q[\delta] \in \mathcal{S}_p(M)$ such that $a_{q(\delta)} \notin \delta$ hence $(M, M_{1,q[\delta]}, a_{q[\delta]})$, exemplifies the conclusion of $\otimes_4$. $\square_{\otimes_4}$

Final contradiction: By $\otimes_4$ we can construct 2^{λ^+} non-isomorphic members of K_{λ^+} by ⟨1.4⟩(1) as follows. We choose by induction on $\alpha < \lambda^+$, for every $\eta \in {}^\alpha 2$, $M_\eta, p^0_\eta, p^1_\eta$ such that:

(a) $M_{<>} = M_0$
(b) $M_\eta \in K_\lambda$
(c) $\langle M_{\eta\restriction\beta} : \beta \leq \ell g(\eta)\rangle$ is $<_{\mathfrak{K}}$-increasing continuous
(d) $p_\eta \in \mathcal{S}_p(M_{\eta\restriction\beta})$
(e) for $\beta \leq \alpha$, we have $p^0_\eta, p^1_\eta \in \mathcal{S}(M_\eta)$ such that M_η realizes $p^\ell_{\eta\restriction\beta}$ iff $\ell = \eta(\beta)$.

If $\alpha = 0$ or α limit, no problem to define M_η (for $\eta \in {}^\alpha 2$). If M_η is defined, we can choose by induction on $i < \lambda^+, (N_{\eta,i}, a_{\eta,i})$ such that $(M_\eta, N_{\eta,i}, a_{\eta,i}) \in K^3_\lambda$, $\mathrm{tp}(a_{\eta,i}, M_\eta, N_{\eta,i}) \in \mathcal{S}_p(M_\eta)$ and $N_{\eta,i}$ omit any
$q \in \{p^\ell_{\eta\restriction\beta} : \beta < \ell g(\eta), \ell \neq \eta(\beta)\} \cup \{\mathrm{tp}(b, M_\eta, N_{\eta,j}) : j < i$ and
$b \in N_{\eta,j}$ and $\mathrm{tp}(b, M_\eta, N_{\eta,j}) \in \mathcal{S}_p(M_\eta)\}$. By $\otimes_4$ we can choose $(N_{\eta,j}, a_{\eta,j})$.
So $W_{\eta,i} = \{j :$ for some $b \in N_{\eta,i}$ we have $\mathrm{tp}(b, M_\eta, N_{\eta,i}) = \mathrm{tp}(a_{\eta,j}, M_\eta, N_{\eta,j})\}$, so $|W_{\eta,i}| \leq \lambda$.
Hence we can find $i < j < \lambda^{++}$ such that $i \notin w_j \;\;\&\;\; j \notin w_i$. Let $M_{\eta\hat{}\langle 0\rangle} = N_{\eta,i}, M_{\eta\hat{}\langle 1\rangle} = N_{\eta,j}, p^0_\eta = \mathrm{tp}(a_{\eta,i}, M_\eta, N_{\eta,i}), p^1_\eta = \mathrm{tp}(a_{\eta,j}, M_\eta, N_{\eta,j})$.
Let for $\eta \in {}^{\lambda^+}2, M_\eta = \bigcup\limits_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$, and apply $\langle 1.4\rangle(1)$. $\square_{\langle 2.8\rangle}$

$\langle$*2.10*$\rangle$ *Conclusion.* $[(*)^{2^-}_\lambda + (*)^2_{\lambda^+} + (*)^3_\lambda + 2^{\lambda^+} < 2^{\lambda^{++}} + K_{\lambda^{+3}} = \emptyset]$.
Every $(M_0, M_1, a) \in K^3_\lambda$ has the extension property.

Proof. By $\langle 2.7\rangle$ and $\langle 2.6\rangle$ + $\langle 2.8\rangle$. $\square_{\langle 2.10\rangle}$

$\langle$**2.11**$\rangle$ **Claim.** $[(*)^3_\lambda]$.
If $M_0 \leq_{\mathfrak{K}} M_1$ are in K_λ __then__ we can find $\alpha < \lambda^+$ and $\langle N_i : i \leq \alpha\rangle$ which is $\leq_{\mathfrak{K}}$-increasing continuously, $N_i \in K_\lambda, (N_i, N_{i+1}, a_i) \in K^3_\lambda$ is reduced, $M_0 = N_0$ and $M_1 \leq_{\mathfrak{K}} N_\alpha$.

Proof. If not, we can contradict categoricity in K_{λ^+} (similar to the proof of $\otimes_2$ during the proof of $\langle 2.8\rangle$).

Without loss of generality $M_0 \neq M_1$. We choose by induction on $i < \lambda^+$, $N^0_i \in K_\lambda, \leq_{\mathfrak{K}}$-increasing continuous such that $(N^0_i, N^0_{i+1}) \cong (M_0, M_1)$ (possible by $\langle 2.4\rangle(9)$ and the categoricity of K in λ). Let $N^0 = \bigcup\limits_{i<\lambda^+} N^0_i$.

We choose by induction on $i < \lambda^+, N^1_i \in K_\lambda, \leq_{\mathfrak{K}}$-increasing continuous and a_i such that $(N^1_i, N^1_{i+1}, a_i) \in K^3_\lambda$ is reduced and let $N^1 = \bigcup\limits_{i<\lambda^+} N^1_i$ (possible by $\langle 2.4\rangle(1)$ and the categoricity of K in λ). So by the categoricity in λ^+ without loss of generality $N^1 = N^0$, hence for some $\delta_1 < \delta_2 < \lambda^+$ we have

$$N^0_{\delta_1} = N^1_{\delta_1}, N^0_{\delta_2} = N^1_{\delta_2}.$$

By changing names $(N^0_{\delta_1}, N^0_{\delta_1+1}) = (M_0, M_1)$ and so $\langle N_{\delta_2+i} : i \leq \delta_2 - \delta_1\rangle$ is as required. $\square_{\langle 2.11\rangle}$

$\langle 2.12 \rangle$ *Conclusion.* $[(*)^{2^-}_\lambda + (*)^2_{\lambda^+} + (*)^3_\lambda + 2^{\lambda^+} < 2^{\lambda^{++}} + K_{\lambda^{+3}} = \emptyset]$.
K_λ has disjoint amalgamation (M_2, M_1 are in disjoint amalgamation over M_0 in M_1 if $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_4, M_1 \cap M_2 = M_0$).

Proof. By $\langle 2.11 \rangle$ + $\langle 2.10 \rangle$.

§3 The non-structure

The first major aim of this section is to prove the density of minimal types using as set theoretic assumption only $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ from cardinal arithmetic. The second aim is to prepare for a proof of a weak form of uniqueness of amalgamation in $\mathfrak{K}_\lambda$. Our aim is also to explain various methods. The proofs are similar to the ones in [Sh 87b, §6].

The immediate role of this section is to get many models in λ^{++} from the assumption "the minimal triples in K^3_λ are not dense," now in $\langle 3.16\rangle$ we get this under some additional assumptions, in $\langle 3.17\rangle$ we get it using only the additional assumption $I(\lambda, K^{+3}) = 0$, which suffices for our main theorem (it does not for the theorem of [Sh 600] hence is eliminated there).

But the section is done in a more general fashion, so let us first explain two general results.

In $\langle 3.12\rangle$, we give a proof assuming the ideal of small subsets of λ^+ (that is WDmId(λ^+)) is not λ^{++}-saturated as exemplified by $\langle S_\alpha : \alpha < \lambda^{++}\rangle$. We build for $\eta \in {}^{(\lambda^{+2})>}2$ the model $M_\eta \in K_{\lambda^+}$ such that $M_\eta = \bigcup\limits_{\alpha<\lambda^+} M_{\eta,\alpha}, |M_\eta| = \lambda \times (1 + \ell g\eta)$ and $\nu \triangleleft \eta \Rightarrow M_\nu \le_{\mathfrak{K}} M_\eta$. Building $M_{\eta\hat{}\langle\ell\rangle}$ by choosing $\langle M_{\eta\hat{}\langle\ell\rangle,\alpha} : \alpha < \lambda^+\rangle$ a representation of $M_{\eta\hat{}\langle\ell\rangle}$, usually we act in a way predetermined simply except when $\alpha \in S_{\ell g(\eta)}, \ell = 1$ and then we consult a weak diamond sequence. This is like $\langle 1.4\rangle(1)$ but there we use our understanding of models in K_λ to build many models in K_{λ^+} and here in $K_{\lambda^{++}}$, thus getting $2^{\lambda^{+2}}$ models in λ^{+2} and we even get $2^{(\lambda^{+2})}$ models from $K_{\lambda^{+2}}$ no one $\le_{\mathfrak{K}}$-embeddable into another.

A second proof $\langle 3.17\text{B}\rangle$ is like $\langle 1.3\rangle(1)$ in the sense that we get only close to $2^{\lambda^{++}}$ models. It is similar to [Sh 87b, 6.4], the parallel to [Sh 87b, 6.3] holds here. So we have to make parallel of [Sh 87b, definition 6.5,6.7]. But there we use fullness on the side (the meaning there is: $M \in K_\lambda$ is full over $N \in K_\lambda$ if $N \le_{\mathfrak{K}} M$, and $(M, c)_{c\in N}$ is saturated but we do not have this yet).

We still have not explained the framework of this section; now in $\langle 3.0\rangle$ - $\langle 3.2\rangle$ we present framework $\mathbf{C}$, speaking on sequences of models of length $\le \lambda$ each of cardinaltiy $< \lambda$ (later we apply it to λ^+) in particular, define local and nice $\mathbf{C}$ and note implications.

Then in $\langle 3.3\rangle$ - $\langle 3.4\rangle$ we present examples we have in mind. Our intention is to use sequences $\langle M_\alpha : \alpha < \lambda\rangle$ as approximations to models of cardinality λ^+, for this we see in $\langle 3.5\rangle$ - $\langle 3.6\rangle$ define being a successor, modulo a "$< \lambda$-amalgamation choice function", this is $\bar M^1 \le^{\text{at}}_{F_1} \bar M^2$, and continuing it we get the quasi order $\le_F$ (see $\langle 3.7\rangle$). In $\langle 3.8\rangle$ we define the coding properties, (of F for $\mathbf{C}$). The intention being that they suffice to build many non-isomorphic models in λ^+. In $\langle 3.10\rangle$ we give the "atomic step" for it.

In $\langle 3.11\rangle$ we prove the existence of 2^{λ^+} non-isomorphic models, using the λ-coding property, which however we do not know in some applications we have in mind, so we turn to the weak λ-coding property in $\langle 3.12\rangle$ (described above) define another version (weak (local) λ-coding property) and corresponding properties of F (all in Definition $\langle 3.13\rangle$, $\langle 3.17\text{A}\rangle$) connect them, $\langle 3.14\rangle$, and prove there are many

models (in ⟨3.17B⟩) described above).
Lastly, ⟨3.16⟩, ⟨3.17⟩ deal with our concrete case: the minimal triples in $\mathfrak{K}^3_\lambda$ are not dense.

Note generally that we mainly axiomatize the construction in λ^+, not how we get $\bar M', \bar M \le^{\rm at}_{F,a} \bar M' \in \mathbf{S}eq_\lambda$.

Later, in ⟨**6.5B**⟩, we shall need again to use the machinery from this section, in
—> scite{6.5B} ambiguous
trying to prove that there are enough cases of disjoint amalgamation in $\mathfrak{K}_\lambda$.
Also, we may want to turn the framework presented here into a more general one; see more in [Sh 600].

⟨*3.0*⟩ *Context.* 1) $\mathfrak{K}$ is an abstract elementary class.
2) But $=_M$ or $=_{\mathfrak{K}}$ is just an equivalence relation, i.e. for $M \in K, =^M$ is an equivalence relation on $|M|$, moreover a congruence relation (i.e. if $R \in \tau(M)$ an n-ary relation, then

$$\bigwedge_{\ell<n} a_i =^M b_i \Rightarrow \langle a_0,\dots,a_{n-1}\rangle \in R^M \equiv \langle b_0,\dots,b_{n-1}\rangle \in R^M).$$

We let $\|M\| = |(M)/=^M|$ and

$$K_{\lambda,\mu} = \{M : |M| \text{ has } \mu \text{ elements and } |M|/=^M \text{ has } \lambda \text{ elements}\}.$$

3) Now the meaning of $\le_{\mathfrak{K}}$ should be clear but $M <_{\mathfrak{K}} N$ means ($M \in K, N \in K$ and) $M \subseteq N$ and $M/=^M \le_{\mathfrak{K}} N/=^N$ and $\exists a \in N[a/=^N \notin (M/=^N)]$.
4) $K^3_\lambda = \{(M,N,a) : M \le_{\mathfrak{K}} N$ are from $K_{\lambda,\lambda}$ and $a \in N, (a/=^N) \notin (M/=^N)\}$.

⟨*3.0A*⟩ *Explanation.* The need of ⟨3.0⟩(2) is just to deal with amalgamations which are not necessarily disjoint. If we use disjoint amalgamation, we can omit ⟨3.0⟩(2) below on Definition ⟨3.5⟩, a disappears so F is four place and use K_λ instead of $K_{\lambda,\lambda}$. This is continued in [Sh 600, 2.17].

⟨**3.OB**⟩ **Definition.** Let λ be regular uncountable.
1) A λ-construction framework $\mathbf{C} = \mathbf{C}(\mathfrak{K}^+, \mathbf{S}eq, \le^*)$ means (we shall use it below with λ^+ playing the role of λ):

(*a*) $\tau^+ = \tau^+(\mathfrak{K}^+)$ is a vocabulary extending $\tau, \mathfrak{K}^+$ an abstract class satisfying axioms I,II,III from ⟨0.2⟩ and $M \le_{\mathfrak{K}^+} N \Rightarrow M \restriction \tau \le_{\mathfrak{K}} N \restriction \tau, \mathfrak{K}^+ = \mathfrak{K}^+_{<\lambda}$ (note: $= \in \tau$ is considered a two-place relation, and we can have a true equality say $\cong$)

(*b*) $\mathbf{S}eq = \bigcup\limits_{\alpha\le\lambda} \mathbf{S}eq_\alpha$ where for $\alpha \le \lambda, \mathbf{S}eq_\alpha$ is a subset of
$\{\bar M : \bar M = \langle M_i : i < \alpha\rangle, M_i \in \mathfrak{K}^+$ is $\le_{\mathfrak{K}^+}$-increasing continuous$\}$ and
$\alpha = \lambda$ implies that $M/=^M$ has cardinality λ where $M = \bigcup\limits_{i<\lambda} M_i$

(*c*) $\le^*$ is a relation written $x \le^*_t y$ which implies $x, y \in \mathbf{S}eq$ and t a set of pairwise disjoint closed intervals of $\ell g(x)$

such that:

(d) $\mathbf{S}eq$ is closed under isomorphisms and initial segments

(e) if $\bar{M}^1 \leq^*_t \bar{M}^2$ and $[\alpha, \beta] \in t$ then

$$\alpha \leq \gamma \leq \beta \Rightarrow M^1_\gamma \leq_{\mathfrak{K}^+} M^2_\gamma \text{ hence } M^1_\gamma \restriction \tau \leq_{\mathfrak{K}} M^2_\gamma \restriction \tau$$

(f) if $\bar{M}^1 \leq^*_t \bar{M}^2$ and $s \subseteq t$ and $\bar{M}^2 \trianglelefteq \bar{M}^3 \in \mathbf{S}eq$ then $\bar{M}^1 \leq^*_s \bar{M}^3$

(g) if t is a set of closed pairwise disjoint intervals of $\ell g(\bar{M})$ and $\bar{M} \in \mathbf{S}eq$ then $\bar{M} \leq^*_t \bar{M}$.

<u>$\langle 3.1\rangle$ Convention/Definition</u>: 1) From now on $\mathbf{C}$ will be a λ-construction framework.
2) If $\bar{M}^x \in \mathbf{S}eq_\lambda$ then we let $\bar{M}^x = \langle M^x_i : i < \lambda\rangle$ and $M^x =: \bigcup\limits_{i<\lambda} M^x_i$.
3) $K^{\rm qr}_\lambda = \{(\bar{M}, \mathbf{f}) : \bar{M} \in \mathbf{S}eq_\lambda$ and $\mathbf{f} : \lambda \rightarrow \lambda\}$.
4) If $(\bar{M}^\ell, \mathbf{f}^\ell) \in K^{\rm qr}_\lambda$ for $\ell = 1,2$ then $(\bar{M}^1, \mathbf{f}^1) \leq (\bar{M}^2, \mathbf{f}^2)$ means:

(a) for some club E of λ, we have
$\delta \in E \Rightarrow \mathbf{f}^1(\delta) \leq \mathbf{f}^2(\delta)$ and
$\bar{M}^1 \leq^*_t \bar{M}^2$ where
$t = t_{E,\mathbf{f}} = \{[\delta, \delta + \mathbf{f}^1(\delta)] : \delta \in E\}$.

5) $\mathbf{S}eq^s = \{\bar{M} \in \mathbf{S}eq : \bigcup\limits_i |M_i|$ is a set of ordinals $< \lambda^+\}$; similarly for $\mathbf{S}eq_\alpha$.
6) $K^{\rm qs}_\lambda = \{(\bar{M}, \mathbf{f}) \in K^{\rm qr}_\lambda : \bar{M} \in \mathbf{S}eq^s_\lambda\}$.
7) $\mathbf{C}$ is local (or revised local) if:

(a) $\bar{M} = \langle M_i : i < \alpha\rangle \in \mathbf{S}eq_\alpha$ iff:
- (α) $\bar{M}$ is $\leq_{\mathfrak{K}^+}$-increasing continuous in $\mathfrak{K}^+_{<\lambda}$
- (β) $i + 1 < \alpha \Rightarrow \langle M_i, M_{i+1}\rangle \in \mathbf{S}eq_2$
- (γ) if $\alpha = \lambda$ then $|M/=^M| = \lambda$ (where $M = \bigcup\limits_{i<\lambda} M_i$)

(b) for $\bar{M}^1, \bar{M}^2 \in \mathbf{S}eq$ and t a set of pairwise disjoint closed intervals $\subseteq \ell g(\bar{M}^1)$ we have:

$$\bar{M}^1 \leq^*_t \bar{M}^2 \text{ \underline{iff} } [\gamma_1, \gamma_2] \in t \text{ implies}$$

- (α) $\gamma \in [\gamma_1, \gamma_2] \Rightarrow M^1_\gamma \leq_{\mathfrak{K}^+} \bar{M}^2_\gamma$
- (β) in the local case $\gamma \in [\gamma_1, \gamma_2) \Rightarrow \langle M^1_\gamma, M^1_{\gamma+1}\rangle \leq^*_{\{[0,1]\}} \langle M^2_\gamma, M^2_{\gamma+1}\rangle$
in the revised local case: if $\ell g(\bar{M}^1), \ell g(\bar{M}^2) < \lambda$ then
$\gamma \in [\gamma_1, \gamma_2] \Rightarrow \langle M^1_\gamma, M^1\rangle \leq^*_{\{[0,1]\}} \langle M^2_\gamma, M^2\rangle$, generally for some club E of $\lambda, \gamma \in [\gamma_1, \gamma_2]$ & $\gamma < \delta \in E \Rightarrow \langle M^1_\gamma, \bigcup\limits_{\beta<\delta} M^1_\beta\rangle \leq^*_{\{[0,1]\}} \langle M^2_\gamma, \bigcup\limits_{\beta<\delta} M^2_\beta\rangle$
(and if $\ell g(\bar{M}^\ell) = \alpha_\ell < \delta, \bigcup\limits_{\beta<\delta} M^\ell_\beta$ means $\bigcup\limits_{\beta<\alpha^\ell} M^\ell_\beta$)

(c) if $\langle M^\zeta_0, M^\zeta_1\rangle \in \mathbf{S}eq_2$ for $\zeta < \zeta^* < \lambda, \langle M^\zeta_0 : \zeta \le \zeta^*\rangle$ is $\le_{\mathfrak{K}^+}$-increasing continuous, $\langle M^\zeta_1 : \zeta \le \zeta^*\rangle$ is $\le_{\mathfrak{K}^+}$-increasing continuous, and $\zeta < \zeta^* \Rightarrow \langle M^\zeta_0, M^\zeta_1\rangle \le^*_{\{[0,1]\}} \langle M^{\zeta+1}_0, M^{\zeta+1}_1\rangle$ <u>then</u> $\langle M^0_0, M^0_1\rangle \le^*_{\{[0,1]\}} \langle M^{\zeta^*}_0, M^{\zeta^*}_1\rangle \in \mathbf{S}eq$.
So intervals $[\alpha,\alpha)$ do not count for the local version. In the revised local version it is natural to add monotonicity for $\le_{\{[0,1]\}}$.

8) For $\alpha \le \lambda$ we say $\mathbf{C}$ is closed for α if:

(α) $\bar M = \langle M_i : i < \alpha\rangle \in \mathbf{S}eq$ <u>iff</u> $\beta < \alpha \Rightarrow \bar M \restriction (\beta+1) \in \mathbf{S}eq$
(β) if $\bar M^\ell = \langle M^\ell_i : i < \alpha_\ell\rangle \in \mathbf{S}eq$ for $\ell = 1,2$ and $\alpha = \alpha_1 < \alpha_2$, <u>then</u>
$$\bar M^1 \le^*_t \bar M^2 \Leftrightarrow \bar M^1 \le^*_t \bar M^2 \restriction \alpha.$$

9) $\mathbf{C}$ is disjoint if $\bar M^1 \le^*_t \bar M^2, [\gamma_1,\gamma_2] \in t, \gamma \in [\gamma_1,\gamma_2)$ implies $M^1_\gamma = M^1_{\gamma+1} \cap M^2_\gamma$.
$\mathbf{C}$ is truly disjoint if $\bar M^1 \le^*_t \bar M^2, [\gamma_1,\gamma_2] \in t, \gamma \in [\gamma_1,\gamma_2]$ implies $M^1_\gamma = M^1 \cap M^2_\gamma$.

10) In $K^{\rm qr}_\lambda$, we say $(\bar M, \mathbf{f})$ is a m.u.b. (minimal upper bound) of $\langle(\bar M^\xi, \mathbf{f}^\xi) : \xi < \delta\rangle$ if $\xi < \delta \Rightarrow (\bar M^\xi, \mathbf{f}^\xi) \le (\bar M, \mathbf{f})$ and for any other $(\bar M, \mathbf{f}')$ satisfying it for some club E of λ we have $\alpha \in E \ \& \ j \le \mathbf{f}(\alpha) \Rightarrow \mathbf{f}(\alpha) \le \mathbf{f}'(\alpha) \ \& \ M_{\alpha+j} \le_{\mathfrak{K}^+} M'_{\alpha+j}$. Saying increasing continuous sequence in $K^{\rm qr}_\lambda$ we mean this.

11) We say $\mathbf{C}$ is explicitly local if it is local and

(d) if $\zeta^* < \lambda$ is a limit ordinal, $\langle M^\zeta_0, M^\zeta_1\rangle \in \mathbf{S}eq_2$ for $\zeta \le \zeta^*$ and for $\ell = 0,1$ the sequence $\langle M^\zeta_\ell : \zeta < \zeta^*\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, and $M^\zeta_\ell \le_{\mathfrak{K}^+} M^{\zeta^*}_\ell$, and $\zeta < \xi \le \zeta^* \Rightarrow \langle M^\zeta_0, M^\zeta_1\rangle \le^*_{\{[0,1]\}} \langle M^\xi_0, M^\xi_1\rangle$ <u>then</u> $\langle \bigcup\limits_{\zeta<\zeta^*} M^\zeta_0, \bigcup\limits_{\zeta<\zeta^*} M^\zeta_1\rangle$ is $\le_{\{[0,1]\}} (M^{\zeta^*}_0, M^{\zeta^*}_1)$.

12) $\mathbf{C}$ is closed if it is closed for every ordinal $\le \lambda$.

13) $\mathbf{C}$ is semi (or almost) closed as witnessed by G if:

(a) $\mathbf{C}$ is closed for every limit ordinal $\delta < \lambda$
(b) G is a function from $\mathbf{S}eq_{<\lambda}$ to $\mathbf{S}eq_{<\lambda}$ such that $\bar M \triangleleft G(\bar M)$,
(c) $\bar M = \langle M_\alpha : \alpha < \lambda\rangle$ belongs to $\mathbf{S}eq_\lambda$ if $\bar M$ obeys G which means: $\beta < \lambda \Rightarrow \bar M \restriction \beta \in \mathbf{S}eq_\beta$ and $\{\alpha < \lambda : G(\bar M \restriction \alpha) \triangleleft \bar M\}$ is unbounded in λ
(d) in the almost close version, we add: $G(\langle M_i : i < \alpha\rangle)$ depends on $\bigcup\limits_{i<\alpha} M_i$ only.

14) $\mathbf{C}$ is λ-nice if

(a) $\le$ is a quasi order on K^{qr}_λ
(b) any increasing continuous sequence in K^{qr}_λ of length $< \lambda^+$ has a m.u.b. (see part (10)) (not necessarily unique)
(c) $\mathbf{C}$ is closed (see part (12)).

15) $\mathbf{C}$ is almost λ-nice (as witnessed by G) is defined similarly replacing closed by almost closed (witnessed by G).

$\langle 3.2 \rangle$ Claim. *Let $\mathbf{C}$ be a local (or revised local) λ-construction framework.*
1) If $(\bar{M}^\ell, \mathbf{f}^\ell) \in K^{\rm qr}_\lambda$ for $\ell = 1, 2$ and E is a club of λ^+ then
$(\bar{M}^1, \mathbf{f}^1) \leq (\bar{M}^2, \mathbf{f}^2)$ when

$$(*) \quad \text{if } \delta \in E \text{ then } \bigwedge_{i \leq \mathbf{f}^1(\delta)} M^1_{\delta+i} = M^2_{\delta+i} \text{ and } \mathbf{f}^1(\delta) = \mathbf{f}^2(\delta)$$
(or just $\mathbf{f}^1(\delta) \leq \mathbf{f}^2(\delta)$), in the "revised local" version add $M^1 = M^2$.

2) $\leqq$ is a quasi-order on K^{qr}_λ (i.e, transitive and $(\bar{M}, \mathbf{f}) \leqq (\bar{M}, \mathbf{f})$).
3) Any increasing continuous sequence of pairs from K^{qr}_λ of length $< \lambda^+$ has a minimal upper bound.
4) If in addition $\mathbf{C}$ is explicitly local (see Definition $\langle 3.1 \rangle$(11)) then any increasing sequence in K^{qr}_λ of length $< \lambda^+$ has a lub.
5) $\mathbf{C}$ is λ-closed and is λ-nice.

Proof. 1) Trivial. (Using clause (b) of Definition $\langle 3.1 \rangle$(7) so here "local" is not used).
2) Easy (using clause (c) of Definition $\langle 3.1 \rangle$(7)).
3) Without loss of generality the elements of the sequence are $(\bar{M}^\xi, \mathbf{f}^\xi) \in K^{\rm qr}_\lambda$ for $\xi < \mu$, where μ is a regular cardinal $\leq \lambda$. For $\xi < \zeta < \mu$, let $E_{\xi,\zeta}$ be a closed unbounded subset of λ exemplifying Definition $\langle 3.1 \rangle$(4) (see clause (b)) hold for $(\bar{M}^\xi, \mathbf{f}^\xi) \leqq (\bar{M}^\zeta, \mathbf{f}^\zeta)$. First assume $\mu < \lambda$, let $E \subseteq \bigcap\limits_{\xi<\zeta<\mu} E_{\xi,\zeta} \subset \lambda$ be a closed unbounded subset of λ, such that: $\alpha \in E \Rightarrow \alpha + (\sup\limits_{\xi<\mu} \mathbf{f}^\xi(\alpha)) + 1 < \text{Min}(E \backslash (\alpha+1))$.
Let $E = \{\alpha_i : i < \lambda\}$ with α_i increasing continuously with i. Notice that for every $i, \xi < \zeta < \mu \Rightarrow \mathbf{f}^\xi(\alpha_i) \leqq \mathbf{f}^\zeta(\alpha_i)$. Let $E^* = \{i : \alpha_i = i\}$.

First assume $i \in E^*$. Now we define $M_{\alpha_i} = \bigcup\limits_{\xi<\mu} M^\xi_{\alpha_i}$, and if $\bigvee\limits_{\xi<\mu} 0 < j \leq \mathbf{f}^\xi(\alpha_i)$ then for some $\xi_{i,j} < \mu$ we have $\xi_{i,j} < \xi < \mu \Rightarrow j < \mathbf{f}^\xi(\alpha_i)$, and we let $M_{\alpha_i+j} = \bigcup\limits_{\xi_{i,j}<\xi<\mu} M^\xi_{\alpha_i+j}$. If $j = \sup\limits_{\xi<\mu} \mathbf{f}^\xi(\alpha_i)$ and for no $\xi < \mu$ do we have $j < \mathbf{f}^\xi(\alpha_i)$, then we let $M_{\alpha_i+j} = \bigcup\limits_{\varepsilon<j} M_{\alpha_i+\varepsilon}$. Second, if $i < \lambda$ but M_i has not been defined yet, then we let $M_i = \bigcup\limits_{\xi<\mu} M^\xi_{\alpha_i}$. Let $\bar{M} = \langle \bar{M}_\beta : \beta < \lambda^+ \rangle$, and $\mathbf{f}$ be defined by $\mathbf{f}(\alpha_i) = \sup\limits_{\xi<\mu} \mathbf{f}^\xi(\alpha_i)$ if $i \in E^*$ and $\mathbf{f}(\alpha_i) = 0$ otherwise. Now $i \in E^*$ & $j < \mathbf{f}_\xi(i) \Rightarrow \langle M^\xi_{i+j}, M^\xi_{i+j+1} \rangle \leq_{\{[0,1]\}} \langle M_{i+j}, M_{i+j+1} \rangle$ by clause (c) of Definition $\langle 3.1 \rangle$(7). Clearly $(\bar{M}, \mathbf{f})$ is as required. The proof for $\mu = \lambda$ is similar, using diagonal intersection.

What about being a $\leq$-mub (from Definition $\langle 3.1 \rangle$(10)). So assume that $(\bar{M}', \mathbf{f}') \in K^{qr}_\lambda$ and $\xi < \mu \Rightarrow (\bar{M}^\xi, \mathbf{f}^\xi) \leq (\bar{M}', \mathbf{f}')$. So for each $\xi < \mu$ some club E'_ξ of λ exemplifies Definition $\langle 3.1 \rangle$(4), and let $E' =: \bigcap\limits_{\xi<\mu} E'_\xi \cap E^*$ clearly it is a club of λ.

Now for $\delta \in E'$ we have $(\forall \xi < \mu)(\mathbf{f}^\xi(\delta) \le \mathbf{f}'(\delta))$ hence $\sup\limits_{\xi<\mu} \mathbf{f}^\xi(\delta) \le \mathbf{f}'(\delta)$ but the former is just $\mathbf{f}(\delta)$ as $i \in E^*$, so $\delta \in E' \Rightarrow \mathbf{f}(\delta) \le \mathbf{f}'(\delta)$. Similarly $\delta \in E'$ & $j \le \mathbf{f}(i) \Rightarrow M_{\delta+j} \le_{\mathfrak{K}} M'_{\delta+j}$. So clearly $(\bar M, \mathbf{f}) \le (\bar M', \mathbf{f}')$ finishing the proof of $\bar M$ being a minimal u.b. (see Definition $\langle 3.1\rangle(10)$).
4) As in the proof of part (3), let $\langle (\bar M^\xi, \mathbf{f}^\xi) : \xi < \mu\rangle$ be as there and let $(\bar M, \mathbf{f})$ be as there. For proving it is a lub, let $(M^\xi, \mathbf{f}^\xi) \le (\bar M', \mathbf{f}')$ for $\xi < \mu$, and define E' as there for $\delta \in E', j < \mathbf{f}(\delta)$ we have $\langle (M^\xi_{\delta+j}, M^\xi_{\delta+j+1}) : \xi \in (\xi_{\delta,j+1}, \mu)\rangle$ is $\le^*_{\{[0,1]\}}$-increasing continuous and $\langle M^\xi_{\delta+j}, M^\xi_{\delta+j+1}\rangle \le^*_{\{[0,1]\}} (M'_{\delta+j}, M'_{\delta+j+1})$ for $\xi \in (\xi_{\delta,j+1}, \mu)$, so as $\mathbf{C}$ is explicitly local by clause (d) in Definition $\langle 3.1\rangle(11)$ we have

$$\langle M_{\delta+j}, M_{\delta+j+1}\rangle = \left\langle \bigcup_{\xi\in(\xi_{\delta,j+1},\mu)} M^\xi_{\delta+j}, \bigcup_{\xi\in(\xi_{\delta,j+1},\mu)} M^\xi_{\delta+j+1} \right\rangle \le_{\{[0,1]\}} (M'_{\delta+j}, M'_{\delta+j+1})$$

is as required.
5) Left to the reader. $\square_{\langle 3.2\rangle}$

* * *

It may clarify if we introduce some natural cases of $\mathbf{C}$, we shall use the forthcoming $\mathbf{C}^0$ in our construction of many models in $\mathfrak{K}_{\lambda^{+2}}$.

$\langle$**3.3**$\rangle$ **Definition.** For $\ell \in \{0,1,2\}$ let $\mathbf{C} = \mathbf{C}^\ell_{\mathfrak{K},\lambda}$ where $\lambda = \text{cf}(\lambda) > LS(\mathfrak{K})$ consists of

(a) $\tau^+ = \tau$,
$\mathfrak{K}^+ = \{M \in \mathfrak{K}_{<\lambda}:$ if λ is a successor cardinal then $\|M\|^+ = \lambda\}$
(with $=^M$ being equality)

(b) $\mathbf{S}eq_\alpha = \{\bar M : \bar M = \langle M_i : i < \alpha\rangle$ is a $\le_{\mathfrak{K}}$-increasing continuous sequence of members of $\mathfrak{K}_{<\lambda}$ and if $\alpha = \lambda$ then $\bigcup\limits_{i<\alpha} M_i$ has cardinality $\lambda\}$

(c) $\bar M <^*_t \bar N$ when:
  (α) $\bar M = \langle M_i : i < \alpha^*\rangle, \bar N = \langle N_i : i < \beta^*\rangle$ are from $\mathbf{S}eq$
  (β) if $[\gamma_1, \gamma_2] \in t$ then:
    (i) $\gamma \in [\gamma_1, \gamma_2] \Rightarrow M_\gamma \le_{\mathfrak{K}} N_\gamma$
    (ii) if $\ell = 1$, then in addition $\gamma \in [\gamma_1, \gamma_2) \Rightarrow M_\gamma = M_{\gamma+1} \cap N_\gamma$
    (iii) if $\ell = 2$, then in addition
    $\gamma \in [\gamma_1, \gamma_2) \Rightarrow M_\gamma = M \cap N_\gamma$ where $M = \bigcup\limits_{i<\alpha^*} M_i$.

Note that $\mathbf{C}^1_{\mathfrak{K},\lambda}$ is interesting when we have disjoint amalgamation in the appropriate cases.

$\langle$*3.3A*$\rangle$ *Fact.* 1) If $\ell = 0,1$, then $\mathbf{C}^\ell_{\mathfrak{K},\lambda}$ is an explicitly local λ-construction framework, (hence closed and λ-nice by $\langle 3.2\rangle(5)$) and $\mathfrak{K}^+$ satisfies axioms I-VI.
2) If $\ell = 0,2$, then $\mathbf{C}^\ell_{\mathfrak{K},\lambda}$ is an explicitly revised local λ-construction framework (hence closed and λ-nice by $\langle 3.2\rangle(5)$) and $\mathfrak{K}^+$ satisfies axioms I-VI.

⟨3.4⟩ Definition. $\mathbf{C} = \mathbf{C}^3_{\mathfrak{K},\lambda}$ where $\lambda = \mathrm{cf}(\lambda) > LS(\mathfrak{K})$ consists of

(a) $\tau^+ = \tau \cup \{P, <\}, \mathfrak{K}^+$ is the set of $(M, P^M, <^M)$ where $M \in \mathfrak{K}, P^M \subseteq M, <^M$ a linear ordering of P^M (but $=^M$ may be as in ⟨3.0⟩(2)) and $M_1 \leq_{\mathfrak{K}} M_2$ iff $(M_1 \restriction \tau) \leq_{\mathfrak{K}} (M_2 \restriction \tau)$ and $M_1 \subseteq M_2$

(b) $\mathbf{S}eq_\alpha = \{\bar{M} : \bar{M} = \langle M_i : i \leq \alpha \rangle$ is an increasing continuous sequence of members of $\mathfrak{K}^+$ and $\langle M_i \restriction \tau : i \leq \alpha \rangle$ is $\leq_{\mathfrak{K}}$-increasing, and for
$i < j < \alpha : P^{M_i}$ is a proper initial segment of $(P^{M_j}, <^{M_j})$ and there is a first element in the difference$\}$
we denote the $<^{M_{i+1}}$-first element of $P^{M_{i+1}} \backslash P^{M_i}$, if there is one, by $a_i[\bar{M}]$

(d) $\bar{M} <^*_t \bar{N}$ iff
$\bar{M} = \langle M_i : i < \alpha^* \rangle, \bar{N} = \langle N_i : i < \alpha^{**} \rangle$ are from $\mathbf{S}eq, t$ is a set of pairwise disjoint closed intervals of α^* and for any $[\alpha, \beta] \in t$ we have ($\beta < \alpha^{**}$ and):

$\gamma \in [\alpha, \beta] \Rightarrow M_\gamma \leq_{\mathfrak{K}} N_\gamma$ and $\gamma \in [\alpha, \beta] \Rightarrow \alpha_\gamma[\bar{M}] \notin N_\gamma$ moreover
$a_\gamma[\bar{M}] = a_\gamma[\bar{N}]$.

⟨3.4A⟩ Fact. $\mathbf{C}^3_{\mathfrak{K},\lambda}$ is an explicitly local (and revised local) λ-construction framework (hence closed and λ-nice by ⟨3.2⟩(5)) and $\mathfrak{K}^+$ satisfies axioms I-VI.

The use of "F a λ-amalgamation choice function" is to help use the weak diamond, so as to take out some freedom.

⟨3.5⟩ Definition. 1) We say that F is a λ-amalgamation choice function for $\mathbf{C}$ if F is a five place function satisfying

(a) if $M_\ell \in K^+_{<\lambda}$ for $\ell < 3, M_0 \leq_{\mathfrak{K}^+} M_1, M_0 \leq_{\mathfrak{K}^+} M_2, M_1 \cap M_2 = M_0$ (before dividing by $=^{M_\ell}$), $a \in M_2$ and $(\forall b \in M_0)[\neg a =^{M_1} b], A$ is a set such that $A \cup M_1 \cup M_2$ is a set of ordinals then
$F(M_0, M_1, M_2, A, a)$, if defined, is a member N of $\mathfrak{K}^+$ with universe
$A \cup |M_1| \cup |M_2|$, which $\leq_{\mathfrak{K}^+}$-extends M_1 and M_2 and $(a/=^N) \notin (M_1/=^N)$

(b) [uniqueness]
if (M_0, M_1, M_2, A, a) and $(M'_0, M'_1, M'_2, A', a')$ are as above and in the domain of F, f an order preserving mapping from $A \cup M_1 \cup M_2$ onto $A' \cup M'_1 \cup M'_2$ and $f \restriction M_\ell$ is an isomorphism from M_ℓ onto M'_ℓ for $\ell = 0, 1, 2$ and $f(a) = a'$ then f is an isomorphism from $F(M_0, M_1, M_2, A, a)$ onto $F(M'_0, M'_1, M'_2, A', a')$

(c) if $F(x_0, x_1, x_2, x_3, x_4)$ is well defined then x_0, x_1, x_2, x_3, x_4 are as in part (a).

2) If F is defined whenever the conditions in part (a) holds and $A \backslash M_1 \backslash M_2$ has large enough cardinality then we say F is full (if $\lambda = \mu^+$, it suffices to demand $A \backslash M_1 \backslash M_2$ has cardinality μ).

⟨3.6⟩ Definition. Assume $\mathbf{C}$ is a λ-construction framework and F is a λ-amalgamation choice function (for $\mathbf{C}$). Let $(\bar M^\ell, \mathbf{f}^\ell) \in K^{\rm qr}_\lambda$ for $\ell = 1,2$.
1) $(\bar M^1, \mathbf{f}^1) \le^{\rm at}_{F,a} (\bar M^2, \mathbf{f}^2)$ (if we omit a, this means for some a; "at" stands for atomic extension) means that:

- (a) $(\bar M^1, \mathbf{f}^1) \le (\bar M^2, \mathbf{f}^2)$
- (b) for some club E of λ^+ for every $\delta \in E$ for $e_\delta =: [\delta, \delta + \mathbf{f}^1(\delta)]$ we have

  - $(*)$ if $\delta \in E$ and $\beta < \gamma$ are successive members of e_δ then: $M^1_\gamma = F(M^0_\beta, M^0_\gamma, M^1_\beta, |M^1_\gamma|, a)$

- (c) $a \in M^2$ and $(a/ =^{M^2}_{\mathfrak{K}}) \notin (M^1/ =^{M^2}_{\mathfrak{K}})$ (essentially implicit in $(b)(*)$).

2) $(\bar M', \mathbf{f}') \le_F (\bar M'', \mathbf{f}'')$ means that: there is a sequence $\langle (\bar M^\zeta, \mathbf{f}^\zeta) : \zeta \le \xi\rangle$ such that:

- (a) $\xi < \lambda^+$
- (b) $(\bar M^\zeta, \mathbf{f}^\zeta) \in K^{\rm qr}_\lambda$ is $\le$-increasing continuous in ζ (remember Definition ⟨3.1⟩(10))
- (c) for each $\zeta < \xi$ we have $(\bar M^\zeta, \mathbf{f}^\zeta) <^{\rm at}_F (\bar M^{\zeta+1}, \mathbf{f}^{\zeta+1})$
- (d) $(\bar M', \mathbf{f}') = (\bar M, \mathbf{f}^0)$ and $(\bar M'', \mathbf{f}'') = (\bar M^\xi, \mathbf{f}^\xi)$.

3) $<^{\rm at,*}_{F,a}$ is defined similarly to part (1) but we demand only $e_\delta \subseteq [\delta, \delta + \mathbf{f}^2(\delta)]$ is closed and $\{\delta, \delta + \mathbf{f}^2(\delta)\} \subseteq e_\delta$; the requirement from Definition ⟨3.2⟩(4) i.e. for every $\delta \in E$ and $i \le \mathbf{f}^1(\delta)$ we have:

$$i \le \mathbf{f}^1(\delta) \Rightarrow M^1_{\delta+i} \le_{\mathfrak{K}} M^2_{\delta+i}$$

$$i < \mathbf{f}^1(\delta) \Rightarrow M^1_{\delta+i} = M^1_{\delta+i+1} \cap M^2_{\delta+i}$$

$$\mathbf{f}^1(\delta) \le \mathbf{f}^2(\delta)$$

$$\text{if } \beta \in [\delta, \delta + \mathbf{f}^2(\delta)] \text{ then } M^0_\beta = M^0_{\max(e_\delta \cap (\beta+1))}$$

and then using it define $\le^*_F$.
4) We replace F by $\mathbf{F}$, a family of such functions, if in each case (i.e. each $\zeta < \xi$ in ⟨3.6⟩(2)(e) we use one of them (not necessarily the same for distinct α). $\mathbf{F}^*$ is the family of all such F.
5) $(\bar M^1, \mathbf{f}^1) <_{F,a} (\bar M^2, \mathbf{f}^2)$ means that for some $(\bar M, \mathbf{f})$ we have $(\bar M^1, \mathbf{f}^1) \le^{at}_{F,a} (\bar M, \bar f) \le_F (\bar M^2, \mathbf{f}^2)$.
6) We define mub as in ⟨3.1⟩(10).

⟨3.6A⟩ Remark. What we prove below on $<^{\rm at}_{F,a}, \le_F$ also holds for $<^{\rm at,*}_{F,a}, \le^*_F$.

⟨3.7⟩ Claim. *[**C** is nice]. On $K^{\mathrm{qr}}_\lambda, \leq_F$ is a quasi-order, and every increasingly continuous sequence of length $< \lambda^+$ has a mub.*

Proof. Check.

⟨3.8⟩ Definition. 1) We say F has the λ-coding property (for **C**) if: $\mathbf{S}eq_\lambda \neq \emptyset$ and, F is a λ-amalgamation choice function for $\mathfrak{K}^+$, and for every $\bar M^1 \in \mathbf{S}eq_\lambda$, function $\mathbf{f}^1 : \lambda \to \lambda$ and $S \subseteq \lambda$ we can find $\bar M^{2,\eta} \in \mathbf{S}eq_\lambda$, a function
$\mathbf{f}^2 : \lambda \to \lambda$ such that $\mathbf{f}^1 \leq_{\mathcal{D}_\lambda} \mathbf{f}^2, \mathbf{f}^2 \restriction (\lambda \backslash S) = \mathbf{f}^1 \restriction (\lambda \backslash S)$ and club E of λ^+ and element a such that:

- $(*)_1$ $(\bar M^1, \mathbf{f}^1) \leq^{\mathrm{at}}_{F,a} (\bar M^{2,\eta}\mathbf{f}^1)$
  $\eta \restriction \alpha = \nu \restriction \alpha \Rightarrow M^{2,\eta} \restriction \alpha = M^{2,\nu} \restriction \alpha$
- $(*)_2$ for some club E of λ the following is impossible: for some $\eta_3, \eta_4 \in {}^\lambda 2$, for $\ell = 3,4$ we have $(\bar M^1, \mathbf{f}^2) \leq_F (\bar M^\ell, \mathbf{f}^\ell)$ witnessed by E_ℓ and f_ℓ is a $\leq_{\mathfrak{K}}$-embedding of $M^{2,\eta_\ell} \restriction \tau$ into $M^\ell \restriction \tau$ over M^1_λ and for some $\delta \in E \cap E_3 \cap E_4 \cap S$ we have $\bar M^3 \restriction (\delta + \mathbf{f}^2(\delta)) = \bar M^4 \restriction (\delta + \mathbf{f}^2(\delta)), \mathbf{f}^3 \restriction \delta = \mathbf{f}^4 \restriction \delta, f_3 \restriction M^{2,\eta_3}_\delta = f_4 \restriction M^{2,\eta_4}_\delta$ and $\ell \in \{3,4\} \Rightarrow \mathrm{Rang}(f_\ell \restriction M^2_\delta) \subseteq M^\ell_\delta, \eta_3 \restriction \delta = \eta_4 \restriction \delta, \eta_3(\delta) \neq \eta_4(\delta)$.

2) We say that F has the weak λ-coding property if above we restrict ourselves to the cases $\mathbf{f}^1 \restriction S = 0_S$. We can even restrict ourselves to the cases $\mathbf{f}^1 \in \mathcal{F} \subseteq {}^\lambda\lambda$ provided that $0_\lambda \in \mathcal{F}$ and we demand $\mathbf{f}^2 \in \mathcal{F}$.
3) If we replace F by $\mathbf{F}$, a family of such functions it means we use Definition ⟨3.3⟩(3), we say $\mathfrak{K}$ has a property if $\mathbf{F}^*$ has this property.

⟨3.9⟩ Remark. 1) We can weaken the demand in ⟨3.8⟩(1) to extensions which actually arise, but this seems more cumbersome, we have no application in mind and the adaptation is straight. We may demand in $(*)_2$ of ⟨3.8⟩(1) more: the E_ℓ clubs depend on $(\bar M^\ell, \mathbf{f}^\ell)$ separately.
2) The next claim translates the criterion in ⟨3.8⟩(1) to one which really speaks on $\mathfrak{K}_\lambda$ and which is natural for checking in the examples we have in mind, to another, natural for use in a non-structure theorem.
3) For understanding Definition ⟨3.8⟩(1) you may look at the place it is verified.
4) Note that ⟨3.8⟩(1) implies: for every $(\bar M^1, \mathbf{f}^1)$ there is $(\bar M^2, \mathbf{f}^2)$ such that $(\bar M^1, \mathbf{f}^1) \leq_F (\bar M^2, \mathbf{f}^2)$ and $M^1_\lambda \neq M^2_\lambda$, hence by the definition of $<_F$, for some $(\bar M^3, \mathbf{f}^3)$ we have $(\bar M^1, \mathbf{f}^1) <^{at}_{F,a} (\bar M^3, \mathbf{f}^3)$.
5) In Definition ⟨3.8⟩(1) we can replace $<^{at}_{F,a}$ by $<_F$ or $<_{F,a}$ with no harm as $<_{F,a}$ satisfies the requirement on $<^{at}_{F,a}$ and starting from it we again get $<_F$.
6) In $(*)_2$ of Definition ⟨3.8⟩(1) we may demand that, for some function H depending on $(\bar M^1, f^1)$ and add the further restriction $\ell \in \{3,4\}$ & $\alpha < \delta \Rightarrow \delta \in H(\bar M^\ell \restriction \alpha, f_\ell \restriction M^\ell_\alpha)$. I.e. this weakening of the demand does not change the desired conclusions.
7) For local **C**, ⟨3.8⟩(1),(2) can be translated to a "local" demand.

⟨3.10⟩ Lemma. *Assume* $(\exists \mu)(\lambda = \mu^+ \ \ \& \ \ 2^\mu < 2^{\mu^+})$ *or at least the definitional weak diamond on* λ *holds. Assume* $\mathbf{C}$ *is* λ*-nice,* $J = WDmId^{\mathrm{Def}}(N)$*.*
If F *has the* λ*-coding property, it has the explicit* $9\lambda, J)$*- coding property which means:* $(\bar{M}^1, \bar{\mathbf{f}}^1) \in K^{\mathrm{qs}}_\lambda$ *and* $S \subseteq \lambda^+$ *satisfies* $S \notin J$ *<u>then</u> we can find* $(\bar{M}^2, \mathbf{f}^2) \in K^{\mathrm{qs}}_\lambda$ *such that:*

- (a) $(M^1, \mathbf{f}^1) <^{\mathrm{at}}_F (\bar{M}^2, \mathbf{f}^1)$ *and* $\mathbf{f}^1 \restriction (\lambda \backslash S) = \mathbf{f}^2 \restriction (\lambda \backslash S)$
- (b) *if* $(M^1, \mathbf{f}^2) \leq_F (\bar{M}^3, \mathbf{f}^3) \in K^{\mathrm{qs}}_\lambda$ *<u>then</u>* $M^2 \restriction \tau$ *cannot be* $\leq_{\mathfrak{K}}$*-embedded into* $M^3 \restriction \tau$ *over* M^1*.*

Proof. The proof is straight once you digest the meaning of weak diamond. $\square_{\langle 3.10\rangle}$

Now we can give a reasonable non-structure theorem.

⟨3.11⟩ Theorem. *Assume* $\mathbf{C}$ *is* λ*-nice,* $(\exists \mu)(\lambda = \mu^+ \ \ \& \ \ 2^\mu < 2^{\mu^+}) \ \ \& \ \ 2^\lambda < 2^{\lambda^+}$ *or at least* $DWD(\lambda)$ *and* $DWD^+(\lambda^+)$ *and* $J = WDmId^{\mathrm{Def}}(\lambda)$*.*

If F *has the* (λ, J)*-coding property, then* $I(\lambda^+, \mathfrak{K}) \geq 2^{\lambda^+}$ *and equality holds if* $|\tau(\mathfrak{K})| \leq \lambda^+$*; also if* $2^{\lambda^+} > (2^\lambda)^+$ *then* $IE(\lambda^+, \mathfrak{K}) \geq 2^{\lambda^+}$*.*

Proof. We choose by induction on $\alpha < \lambda^+$, for every $\eta \in {}^\alpha 2$, a pair $(\bar{M}^\eta, \mathbf{f}^\eta)$ such that:

- (a) $(\bar{M}^\eta, \mathbf{f}^\eta) \in K^{\mathrm{qs}}_\lambda$
- (b) if $\nu \triangleleft \eta$ then $(\bar{M}^\nu, \mathbf{f}^\nu) \leq_F (\bar{M}^\eta, \bar{\mathbf{f}}^\eta)$
- (c) $(\bar{M}^\eta, \mathbf{f}^\eta) <^{\mathrm{at}}_{F, a_{\eta\hat{}\langle 0\rangle}} (\bar{M}^{\eta\hat{}\langle 0\rangle}, \mathbf{f}^{\eta\hat{}\langle 0\rangle})$
- (d) $(\bar{M}^\eta, \bar{\mathbf{f}}^\eta) <^{\mathrm{at}}_{F, a_{\eta\hat{}\langle 1\rangle}} (\bar{M}^{\eta\hat{}\langle 1\rangle}, \mathbf{f}^{\eta\hat{}\langle 1\rangle})$
- (e) if $\ell g(\eta)$ is a limit ordinal and $(\bar{M}^\eta, \bar{f}^{\eta\hat{}\langle 0\rangle}) \leq_F (\bar{M}', \mathbf{f}')$ <u>then</u> $M^{\eta\hat{}\langle 1\rangle} \restriction \tau$ cannot be $\leq_{\mathfrak{K}}$-embedded into $M' \restriction \tau$ over M^η
- (f) if α is limit ordinal, then $(\bar{M}^\eta, \mathbf{f}^\eta)$ is a $\leq_F$-mub of $\langle (\bar{M}^{\eta \restriction i}, \mathbf{f}^{\eta\restriction i}) : i < \alpha \rangle$.

For $\alpha = 0$ note that as $\mathbf{S}eq_\lambda \neq \emptyset$ also $\mathbf{S}eq^s_\lambda \neq \emptyset$ hence $K^{\mathrm{qs}}_\lambda \neq \emptyset$. For α limit use ⟨3.3⟩, ⟨3.5⟩. For $\alpha = \beta + 1$, β a limit ordinal and $\nu \in {}^\beta 2$, define $(\bar{M}^{\nu\hat{}\langle \ell\rangle}, \mathbf{f}^{\nu\hat{}\langle \ell\rangle})$ for $\ell = 0, 1$ by Definition ⟨3.8⟩. If $\alpha = \beta + 1$, β non-limit use ⟨3.9⟩(4). Let $M^\eta = \bigcup_{\alpha < \lambda^+} M^{\eta\restriction\alpha} \restriction \tau$ for $\eta \in {}^{(\lambda^+)}2$. Now note $\{a_{\eta\restriction(i+1)}/=^{M^\eta} : i < \lambda^+\} \subseteq M^\eta/=^{M^\eta}$ are pairwise distinct so $M^\eta \in K_{\lambda^+}$. Now we can apply ⟨1.4⟩ (with λ^+ here standing for λ there). $\square_{\langle 3.11\rangle}$

* * *

Unfortunately, in some interesting cases we get only weak coding.

⟨3.12⟩ Theorem. *Assume* $\mathbf{C}$ *is* λ*-nice,* $(\exists\mu)(\lambda = \mu^+ \ \ \& \ \ 2^\mu < 2^{\mu^+})$ *and* $2^\lambda < 2^{\lambda^+}$ *and* $WDmId(\lambda)$ *is not* λ^+*-saturated (or at least* $DfWD(\lambda)$, $DfWD^+(\lambda^+)$ *and* $WDmId^{\rm Def}(\lambda)$ *is not* λ^+*-saturated).*
If F *has the weak* λ*-coding property (see Definition ⟨3.8⟩(2), or at least the parallel of the conclusion of ⟨3.10⟩),* *then* $I(\lambda^+, K) \geq 2^{\lambda^+}$.

Proof. We can find $\langle S^*_\alpha : \alpha < \lambda^+\rangle$ such that:

$$S^*_\alpha \subseteq \lambda^+$$

$$[\alpha < \beta \Rightarrow |S^*_\alpha \backslash S^*_\beta| < \lambda^+]$$

and

$$S'_\alpha =: S^*_{\alpha+1} \backslash S^*_\alpha \notin \ \mathrm{WDmId}^{\rm def}(\lambda^+).$$

We again choose by induction on $\alpha < \lambda^+$ for every $\eta \in {}^\alpha 2$ a pair $(\bar M^\eta, \mathbf{f}^\eta)$ such that:

- (a) $(\bar M^\eta, \mathbf{f}^\eta) \in K^{\rm qs}_\lambda$
- (b) if $\nu \lhd \eta$, then $(\bar M^\nu, \mathbf{f}^\nu) \leq_F (\bar M^\eta, \mathbf{f}^\eta)$
- (c) $f^\eta \restriction (\lambda \backslash S_{\ell g(\eta)}) = 0_{\lambda \backslash S_{\ell g(\eta)}}$
- (d) $(\bar M^\eta, \mathbf{f}^\eta) <^{\rm at}_{F, a_{\eta\hat{}\langle 0\rangle}} (\bar M^{\eta\hat{}\langle 0\rangle}, \mathbf{f}^{\eta\hat{}\langle 0\rangle})$
- (e) $(\bar M^\eta, \mathbf{f}^\eta) \leq^{\rm at}_{F, a_{\eta\hat{}\langle 1\rangle}} (\bar M^{\eta\hat{}\langle 1\rangle}, \mathbf{f}^\eta)$
- (f) if $(\bar M^\eta, \mathbf{f}^{\eta\hat{}\langle 0\rangle}) \leq_F (\bar M', \mathbf{f}')$ then $M^{\eta\hat{}\langle 1\rangle} \restriction \tau$ cannot be $\leq_{\mathfrak{K}}$-embedded into $M' \restriction \tau$ over M^η
- (g) if α is a limit ordinal, then $(\bar M^\eta, \mathbf{f}^\eta)$ is a $\leq_F$-mub of $\langle (\bar M^{\eta \restriction i}, \mathbf{f}^{\eta\restriction i}) : i < \alpha\rangle$.

Again there are no problems (the difference is in clause (c)).
Then we apply ⟨1.4⟩(1) (or ⟨1.6A⟩(1)). $\square_{\langle 3.12\rangle}$

⟨3.13⟩ Definition. 1) We say F has the weak local λ-coding property for $\mathbf{C}$ if: F is a λ-amalgamation choice function for $\mathbf{C}$ and

$(*)_1$ assume $\langle M_0, M_1\rangle \in \mathbf{S}eq_2$, $M_0 \leq_{\mathfrak{K}^+} N_0 \in \mathfrak{K}^+_{<\lambda}$, $M_1 \cup N_0 \subseteq \lambda^+$, and $M_1 \cap N_0 = M_0$ and $a \in N_0$, $a \notin (M_0/=^{N_0})$ (i.e. $(\forall b \in M_0)(\neg a =^{N_0} b)$).
Then we can find $N^1, N^2 \in \mathfrak{K}^+_{<\lambda}$ such that:
- (a) $N^1 = F(M_0, M_1, N_0, |N^1|, a)$
- (b) $\langle N_0, N^1\rangle \in \mathbf{S}eq_2$ and $\langle N_0, N^2\rangle \in \mathbf{S}eq_2$
- (c) $M_1 \leq_{\mathfrak{K}^+} N^1$ and $M_1 \leq_{\mathfrak{K}^+} N^2$

(d) $N^1 \restriction \tau, N^2 \restriction \tau$ are contradicting[4] amalgamations of $M_1 \restriction \tau, N \restriction \tau$ over $M_0 \restriction \tau$; i.e. for no N', h do we have: $(N^1 \restriction \tau) \leq_{\mathfrak{K}} N' \in \mathfrak{K}_{<\lambda}$ and h is a $\leq_{\mathfrak{K}}$-embedding of N^2 into N' over $M_1 \cup N_0$ or at least

(d)$^-$ (N^1, N^2) is a τ-contradicting pair of amalgamations of M_1, N_0 over M_0 which just says: if $N^1 \leq_{\mathfrak{K}^+} N \in \mathfrak{K}^1_{<\lambda}$ then there is no $\leq_{\mathfrak{K}}$-embedding h of $N^2 \restriction \tau$ into $N \restriction \tau$ over $M_1 \cup N_0$ (i.e. is the identity on M_1 and on N_0)

(note: this is not necessarily symmetric; and we use just the τ-reducts of N^2, M_0, M_1, N_0 so we can replace them (or some of them) by
$N^2 \restriction \tau, M_0 \restriction \tau, M_1 \restriction \tau, N_0 \restriction \tau$ respectively).
2) We say that F has the local λ-coding property if:
F is a λ-amalgamation choice function for $\mathbf{C}$ and

$(*)_2$ if $\bar{M} = \langle M_i : i < \lambda \rangle \in \mathbf{S}eq_\lambda, \bar{N} = \langle N_i : i \leq \delta + i \rangle \in \mathbf{S}eq_{\delta+i}, a \in N_0$, and $\bar{M} \restriction (\delta + i + 1) \leq^*_{\{[\delta,\delta+i]\}} \bar{N}$, and $N_{\delta+j+1} = F(M_{\delta+i}, M_{\delta+i+1}, N_{\delta+j}, |N_{\delta+j+1}|, a)$ for $j < i$ <u>then</u> for some $i_1, i_2 \in (i, \lambda)$ and $\bar{N}^\ell = \langle N^\ell_\alpha : \alpha < \delta + i_\ell \rangle \in \mathbf{S}eq_{\delta+i_\ell}$ we have:

(a) $\bar{N}^\ell \restriction (\delta + i + 1) = \bar{N}$

(b) for $j \in [\delta + i, \delta + i_1)$ we have $N^1_{\delta+j+1} = F(M_{\delta+j}, M_{\delta+j+i}, N^1_{\delta+j}, |N^1_{\delta+j+1}|, a)$

(c) $M_{\delta+i_2} \leq_{\mathfrak{K}^+} N^2_{\delta+i_2}$

(d) $N^1_{\delta+i_1} \restriction \tau, N^2_{\delta+i_2} \restriction \tau$ are contradictory amalgamations of $M_{\delta+i_1} \restriction \tau$ and $N_\delta \restriction \tau$ over $M_\delta \restriction \tau$ or at least

(d)$^-$ $(N^1_{\delta+i_1}, N^2_{\delta+i_2})$ are τ-contradictory amalgamations of $M_{\delta+i_1}$ and N_0 over M_δ.

(So if $i = 0$, this gives us a possibility to amalgamate, helpful for $i \in \lambda \backslash \bigcup\limits_{\delta \in E} [\delta, \delta + \mathbf{f}(\delta))$).

3) We say that F has the weaker local λ-coding property for $\mathbf{C}$ if: F is a λ-amalgamation choice function for $\mathbf{C}$ and

$(*)_3$ as in part (2) but $i = 1$.

4) In 1), 2), 3) above we say $\mathbf{C}$ has weakly local or the local or the weaker local coding property respectively, if we omit the mention of F (clause (a) in $(*)_1$, clause (b) in $(*)_2$.)

[4]Of course, we may consider only ones in "legal" extensions. We can also note that for the intended use, the disjointness is automatic (so $\langle 3.0 \rangle(2)$ not needed)

⟨3.14⟩ Claim. *1) If F has the weak local λ-coding property for* $\mathbf{C}$ *or F has the local λ-coding property for* $\mathbf{C}$ *then F has the weaker local λ-coding property for* $\mathbf{C}$.
2) Assume

- (a) $\mathbf{C}$ *has a weak local λ-coding property*
- (b) *F has the λ-coding (or weaker λ-coding) (or the weak λ-coding) property for* $\mathbf{C}$.

Then for some F' we have:

- (α) *F', too, is a λ-amalgamation choice function*
- (β) *if $F(N_0, N_1, N_2, A, a)$ is well defined, then for some $A', A \subseteq A' \nsubseteq |A'| < \lambda$, ($A'$ a set of ordinals) and $F'(N_0, N_1, N_2, A', a)$ is well defined and $F(N_0, N_1, N_2, A, a) \leq_{\mathfrak{K}^+} F'(N_0, N_1, N_2, A', a)$*
- (γ) *F' has the local or weak local or weaker local (respectively as in (b)) λ-coding property for* $\mathbf{C}$.

3) If $\mathbf{C}$ *is a local (λ-construction framework), F a λ-amalgamation choice function, with the weak (or just weaker) local λ-coding property, then F has the weak λ-coding property (hence under the set theoretic assumptions of ⟨3.12⟩, $I(\lambda^+, \mathfrak{K}) \geq 2^{\lambda^+}$).*
4) If $\mathbf{C}$ *is a local (λ-construction framework), F a λ-amalgamation choice function, with the local λ-coding property, then F has the λ-coding property (hence under the set theoretic assumptions of ⟨3.11⟩, we have $I(\lambda^+, \mathfrak{K}) \geq 2^{\lambda^+}$ and $(2^\lambda)^+ < 2^{\lambda^+} \Rightarrow IE(\lambda^+, \mathfrak{K}) \geq 2^{\lambda^+}$).*

Remark. The parallel of part (2) holds for local and weaker local property if F acts on sequences.

Proof. 1) Check.
2) Here we use clause $(d)^-$ rather than (d).
3) Similar.
4) Check. $\square_{\langle 3.14 \rangle}$

⟨3.17A⟩ Definition. We say that F is a λ-amalgamation choice function for sequences, for $\mathbf{C}$ if:

- (a) if $x = F(x_1, x_2, x_3, x_4, x_5, x_6)$ is defined then for some $\alpha_1, \alpha_2, \alpha_3, \alpha < \lambda$ we have $x_\ell = \bar{M}^\ell \in \mathbf{S}eq^s_{\alpha_\ell}$ for $\ell < 3, \bar{M}^1 \triangleleft \bar{M}^2, t = x_4$ is a set of pairwise disjoint intervals $\subseteq \alpha_i, \bar{M}^1 <^*_t \bar{M}^3, A = x_5$ a set of $< \lambda$ ordinals $< \lambda^+$, $x = \bar{M} \in \mathbf{S}eq^s_\alpha, M_\alpha$ has universe $A, \bar{M}^1 \leq^*_{t \cup \{[\alpha_1, \alpha_2]\}} \bar{M}$
- (b) [uniqueness] as in Definition ⟨3.5⟩.

⟨3.17B⟩ Claim. *Assume:*

- (a) $(\exists \mu < \lambda)(2^\mu = 2^{<\mu} < 2^\lambda), 2^\lambda < 2^{\lambda^+}$ *or at least we have the definable weak diamond for* λ *and* λ^+*; and* $2^{<\lambda} \geq \lambda^+$
- (b) $\mathbf{C}$ *is a nice (λ-construction framework)*
- (c) F *is a* λ*-amalgamation choice function for* $\mathbf{C}$
- (d) F *has the weaker* λ*-coding property for* λ*.*

<u>*Then*</u> $I(\lambda, \mathfrak{K}) \geq \mu_{\rm wd}(\lambda^+)$.

⟨3.13A⟩ Remark. 1) If we are just interested in $I(\lambda^+, K)$ rather than also in $IE(\lambda^+, K)$, then we can change the definition of τ-contradictory to:

> N^1, N^2 are τ-contradictory amalgamations of M_1, N over M_0 if $(M_0 \leq_{\mathfrak{K}^+} M_1 \leq_{\mathfrak{K}^+} N^\ell, M_0 \leq_{\mathfrak{K}^+} N_0 \leq_{\mathfrak{K}^+} N^\ell, M_0 = M_1 \cap N_0$ and there are no $N^1_*, N^2_* \in \mathfrak{K}^+_{<\lambda}$ such that: $N^\ell \leq_{\mathfrak{K}^+} N^\ell_*$ and $N^1_* \restriction \tau, N^2_* \restriction \tau$ are isomorphic over $M_1 \cup N_0$.

2) Note that it is unreasonable to assume that we will always use the local versions: e.g. if we have a superlimit model in $\mathfrak{K}_{\lambda^+}$ and we want to have $\bar{M} \in \mathbf{S}eq_\lambda \Rightarrow M_\lambda$ superlimit, we have to add something (see [Sh 600]). Also possibly we will have in the construction (i.e. in ⟨3.11⟩ or ⟨3.12⟩) that $\bar{M}_\eta$ have two "contradictory" extensions $\bar{M}_{\eta\hat{}\langle 0\rangle}, \bar{M}_{\eta\hat{}\langle 1\rangle}$, (see clauses (e) and (f) in their proof) only when $\mathrm{cf}(\ell g(\eta)) = \theta$ where $S = \{\delta : \delta < \lambda^+ \text{ and } \mathrm{cf}(\delta) = \theta\} \notin$ WDmId(λ), even $\ell g(\eta) \in S$, $S \in$ WDmId$(\lambda)^+$. We shall phrase when needed.

Proof. Straight using ⟨1.3⟩.

Let $\langle M_i : i < i^* \rangle$ list the models in $\mathfrak{K}_{\lambda^+}$ up to isomorphisms and $i^* < \mu_{\rm wd}(\lambda^+)$. It is enough to choose by induction on $\alpha < \lambda^+$ a $\bar{M}^\eta_0$ for $\eta \in {}^\alpha 2$ such that:

- (a) $\bar{M}^\eta \in \mathbf{S}eq_\lambda$
- (b) M^η_λ has universe $\gamma_\eta < \lambda^+$
- (c) $\nu \triangleleft \eta \Rightarrow \bar{M}^\nu \leq_F \bar{M}^\eta$
- (d) if $\delta = \ell g(\eta)$ is a limit ordinal then $\bar{M}^\eta$ is a mub of $\langle \bar{M}^{\eta \restriction \alpha} : \alpha < \delta \rangle$
- (e) if $\eta\hat{}\langle 0\rangle \trianglelefteq \nu \in {}^{\lambda^+}2$ then $M^{\eta\hat{}\langle 1\rangle}_\lambda \restriction \tau$ cannot be $\leq_{\mathfrak{K}}$-embedded into $M^\nu_\lambda \restriction \tau$ over $M^\eta_\lambda \restriction \tau$.

This is possible by ⟨3.14⟩(3).
Having the $M_\eta, \eta \in {}^{\lambda >}2$ we get the conclusion by ⟨1.3A⟩. $\square_{\langle 3.17B\rangle}$

Remark. We will continue this in [Sh 600].

⟨3.16⟩ Theorem. *Assume* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *or at least the definitional weak diamond holds for* λ^+ *and* λ^{++}.

The minimal triples are dense in K^3_λ *if one of the following holds*

$(*)_1$ $\mathfrak{K}$ *(an abstract elementary class with* $LS(\mathfrak{K}) \le \lambda$*) is categorical in* λ, λ^+ *and* $I(\lambda^{++}, K) = 1$ *and* $K_{\lambda^{+3}} = \emptyset$

$(*)_2$ *there is* $M \in K_{\lambda^+}$ *saturated over* λ *and* $1 \le I(\lambda^{++}, K) < 2^{\lambda^{++}}$.

⟨3.17AB⟩ Remark. 1) The assumptions $K_{\lambda^{+3}} = \emptyset, I(\lambda^{++}, K) = 1$ will be eliminated in [Sh 600] (however already here in ⟨3.17⟩, we eliminate the second) or use $2^{\lambda^+} > \lambda^{++}$; i.e. ⟨2.5⟩, anyhow it does not harm the proof of Theorem ⟨0.B⟩(1).

Proof. Assume that the conclusion fails, so let $(M^*, N^*, a^*) \in K^3_\lambda$ be such that no triple above it is minimal. By ⟨2.6⟩(1) we know

$\bigotimes_1$ if $(M^*, N^*, a^*) \le (M, N, a^*) \in K^3_\lambda$, then (M, N, a^*) has the extension property.

Let $\mathbf{C} = \mathbf{C}^0_{\mathfrak{K},\lambda^+}$ (the λ^+-construction framework see Definition ⟨3.3⟩). Consider the statements

(α) there is a $M \in K_{\lambda^+}$ saturated over λ and in K^3_λ above $(M^*, N^*, a) \in K^3_\lambda$ there is no minimal triple or

(β) if $M \in K_{\lambda^+}, (M^0, N^0, a^0) \in K^3_\lambda$ and $M^0 \le M$ then we cannot find h, $\bar M, \langle M_i : i < \lambda^+\rangle, \bar N = \langle N_i : i < \lambda^+\rangle$ and a such that:

(a) $(M^0, N^0, a^0) \le_h (M_0, N_0, a)$ and $h \restriction M^0 = \mathrm{id}_{M^0}$

(b) $(M_i, N_i, a) \in K^3_\lambda$ is increasing

(c) $\langle M_i : i < \lambda^+\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous with union M

(d) $\langle N_i : i < \lambda^+\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous

(e) $S =: \big\{\delta < \lambda^+ :$ for some $j \in (\delta, \lambda^+)$ for every $i \in [j, \lambda^+)$ we have: if for $\ell = 1, 2, (M_j, N_j, a) \le_{h_\ell} (M^\ell, N^\ell, a)$, $h_\ell \restriction M_j = \mathrm{id}_{M_j}, M^1 = M_i = M^2$ and $h \restriction N_\delta = \mathrm{id}_{N_\delta}$ then for some g_1, g_2 and $(M', N', a') \in K^3_\lambda$ we have $(M^\ell, N^\ell, a) \le_{g_\ell} (M', N', a)$ for $\ell = 1, 2, g_\ell \restriction M_j = \mathrm{id}_{M_i}$ and $g_\ell \restriction N_\delta = \mathrm{id}_{N_\delta}\big\}$

is a stationary subset of λ^+.

Now some λ-amalgamation choice function F for $\mathbf{C}$ satisfies the statement (a) and (b) where:

$$(a) \qquad \mathrm{Dom}(F) = \left\{ (M_0, M_1, M_2, A, b) : \begin{array}{ll} (i) & \text{for some } h \text{ we have} \\ & (M^*, N^*, a) \leq_h (M_0, M_2, b) \\ (ii) & A \cup |M_1| \cup |M_2| \subseteq \lambda^{+2} \\ (iii) & A \backslash |M_1| \backslash |M_2| \text{ has cardinality } \lambda \\ (iv) & M_0 \leq_{\mathfrak{K}} M_\ell \text{ are in } K_\lambda \text{ for } \ell = 1, 2 \end{array} \right\}$$

(b) F has the λ^+-coding property if (α) holds (not using $I(\lambda^{++}, K) = 1, K_{\lambda^{+3}} = \emptyset$).

[Why? A point is $\bigotimes_1$ i.e. that $(M^*, N^*, a) \leq_h (M, N, b) \in K^3_\lambda \Rightarrow (M, N, b)$ has the extension property (see $\langle 2.6 \rangle(1)$). A second point is that if $(M^*, N^*, a) \leq (M_i, N_i, a)$ are in $K^3_\lambda, M_i \leq M^+ \in K_{\lambda^+}$ <u>then</u> for some $M' \in K_\lambda$ we have $M_i \leq_{\mathfrak{K}} M' \leq_{\mathfrak{K}} M^+$, and $\mathrm{tp}(a, M_i, N_i)$ has more than one extension in $\mathcal{S}(M')$. Let us prove in detail. So let $\bar{M}' \in \mathbf{S}eq_{\lambda^+}$ and $\mathbf{f}^1$ be a function from λ to λ to λ. Clearly $M^1 = \bigcup\limits_{\alpha < \lambda^+} M^1_\alpha \in K_{\lambda^+}$ hence (by assumption $(*)_1$) M^1 is saturated over λ. So there is an embedding $h : M^* \to M^1$ and without loss of generality$|M^1| \leq \lambda^{++}, M^1 \subseteq A^2 \subseteq \lambda^{++}, |A^2 \backslash M^1| = \lambda^+$, let $A^2 = \bigcup\limits_{\alpha < \lambda} A^2_\alpha$ such that $A^2_\alpha \cap |M^1| = |M^1_\alpha|, A^2_\alpha$ increasing continuously $A^2_0 \backslash M^1_0| = \lambda, |A^2_{\alpha+1} \backslash A^2_\alpha \backslash M^1_{\alpha+1}| = \lambda$. Let $E^* \subseteq \lambda$ a club $\alpha < \delta \in E \Rightarrow \mathbf{f}^1(\alpha) < \delta$.

We choose by induction on $i < \lambda^+, (M^*_\eta, N^*_\eta)$ for $\eta \in {}^\lambda 2$ such that:

- (a) $(M^*_\eta, N^*_\eta, a^*) \in K^3_\lambda$ is $\leq$-increasingly continuous
- (b) $(M^*_{<0>}, N^*_{<\alpha>}) \cong (M^*, N^*)$
- (c) $\alpha_\eta < \lambda^+$ is increasingly continuous so $\nu \triangleleft \eta \Rightarrow \alpha_\nu < \alpha_\eta$
- (d) $M^*_\alpha = M^1_{\alpha_\eta}, |N^*_\eta| = A^2_{\alpha_\eta}$
- (e) if $\eta \in {}^\delta 2, \alpha_\eta = \delta \in E$
  - (α) $i < \mathbf{f}^1(\delta), \eta \triangleleft \nu \in {}^{\delta+i+1}2 \Rightarrow N^*_\nu = F(M_{\delta+i}, M_{\delta+i+1}, N^*_{\nu \restriction (\delta+i)}, A^2_{\delta+i+1}, b^*)$,

    $\alpha_\nu = \ell g \nu$
  - (β) for all $\nu \in {}^{\delta+\mathbf{f}^1(\delta)+1}2, \alpha_\nu$ is $\beta_{\nu \restriction \delta}$ and if $\nu_1 \neq \rho \in {}^{\delta+\mathbf{f}^1(\delta)+1}2$, $\nu \restriction \delta = \eta = \rho \restriction \delta$, then $\mathrm{tp}(b^*, M_{\beta_\eta}, N^*_\nu) \neq \mathrm{tp}(b^*, M_{\beta_\eta}, N^*_\rho)$.

Now for $\eta \in {}^{\lambda^+}2$ we let $\bar{M}^{2,\eta} = \langle M_{\alpha_{\eta \restriction i}} : i < \lambda^+ \rangle, \mathbf{f}^2 = \mathbf{f}^1 + 1$. We can check. (Note: this is essentially the proof hinted at in $\langle 3.9 \rangle(7)$.)]
So if clause (α), i.e. assumption $(*)_2$ of $\langle 3.16 \rangle$ holds, then we get the desired conclusion. So we can assume $(*)_1$ and $\neg(*)_2$, i.e. statement (α) above fails. Now without loss of generality

(c) $\quad F$ has the λ^+-coding property for $\mathbf{C}$ if the statement (β) fails.

[Why? Let $M, (M^0, N^0, a)$ be a counterexample to condition (β). Now if $\langle M_i : i < \lambda^+\rangle \in \mathbf{S}eq_{\lambda^+}(\mathbf{C}^1_{\mathfrak{K}})$, and $\mathbf{f} : \lambda^+ \to \lambda^+$ be given, <u>then</u> (as K is categorical in λ^+) without loss of generality $M = \bigcup\limits_{i<\lambda^+} M_i$. Now we try to choose $N_\eta \in K_\lambda$ for $\eta \in {}^i 2$ by induction on i, naturally (really with embedding) $(M_0, N_{\langle\rangle}, a) \cong (M^0, N^0, a^0)$ and failure contradicts the choice of $M, (M^0, N^0, a)$.]

So assuming towards contradiction that our conclusion fails by $\langle 3.11\rangle$ necessarily condition (α) of our proof fails and condition (β) of our proof holds.

By $\langle 2.4\rangle(8)$ with λ^+ here standing for λ there (or see proof of $\langle 6.4\rangle$) it suffices (in order to get the conclusion of $\langle 3.16\rangle$) to prove:

$\bigotimes_2$ if $(M, N, a) \in K^3_{\lambda^+}$ then for some M^+, N^+
we have $N \leq_{\mathfrak{K}} N^+ \in K_\lambda, M \leq_{\mathfrak{K}} M^+ \leq_{\mathfrak{K}} N^+$ and $a \notin M^+$.

By the failure of (α) necessarily for some $M' \in K_\lambda$ we have $|\mathcal{S}(M')| > \lambda^+$, so by the categoricity of K in λ^+ we can find $(M^1_0, N^1_0, b) \in K^3_\lambda, M^1_0 \leq_{\mathfrak{K}} M \in K_{\lambda^+}$ and $\mathrm{tp}(b, M^1_0, N^1_0)$ not realized in M.

By the holding of (β) we can find $\langle M^1_i : i < \lambda^+\rangle, \langle N^1_i : i < \lambda^+\rangle$ such that:

$(*)_2$ $M, \langle M^1_i : i < \lambda^+\rangle, \langle N^1_i : i < \lambda^+\rangle, (M^1_0, N^1_0, b)$ satisfies the requirement in (β) above for $M, \langle M_i : i < \lambda^+\rangle, \langle N_i : i < \lambda^+\rangle, (M^0, N^0, a^0)$ and let S be as there.

So $M = \bigcup\limits_{i<\lambda^+} M^1_i$.

Without loss of generality (as $\mathfrak{K}$ has amalgamation in λ^+) for some M^2, $N^2 \in K_{\lambda^+}$ we have $M \leq_{\mathfrak{K}} M^2 \leq_{\mathfrak{K}} N^2 \in K_{\lambda^+}, N \leq_{\mathfrak{K}} N^2$, and $M^2 = \bigcup\limits_{i<\lambda^+} N^1_i$.

We can also choose $(M^3_i, N^3_i, a^*) \in K^3_\lambda$ increasing continuous, $(M^3_0, N^3_0, a^*) = (M^*, N^*, a^*)$, each (M^3_i, N^3_i, a^*) is reduced (see Definition $\langle 2.3\mathrm{A}\rangle(5)$) and $\mathrm{tp}(a^*, M^3_i, N^3_i)$ has more than one extension in $\mathcal{S}(M^3_{i+1})$ (again using the definition of minimality and $\langle 2.4\rangle(1)$).

Again using categoricity in λ^+ and amalgamation in λ^+ without loss of generality $M = \bigcup\limits_{i<\lambda^+} M^3_i, M \leq_{\mathfrak{K}} M^3 \leq_{\mathfrak{K}} N^3 \in K_{\lambda^+}, N^2 \leq_{\mathfrak{K}} N^3$ and $M^3 = \bigcup\limits_{i<\lambda^+} N^3_i$. As $M \neq M^2, M \neq M^3$ clearly if $a \notin M^3$ or $a \notin M^2$ (i.e. $a \notin M^3 \cap M^2$), then we have proved $\bigotimes_2$ so we are done. So assume toward contradiction that $a \in M^3 \cap M^2$.

Let $\langle N^*_i : i < \lambda^+\rangle \in \mathbf{S}eq_{\lambda^+}[\mathbf{C}^1_{\mathfrak{K},\lambda^+}], N^3 = \bigcup\limits_{i<\lambda^+} N^*_i$. So for some $i < \lambda^+$ we have $a \in N^*_i$, and as $E = \{i < \lambda^+ : N^*_i \cap M = M^1_i$ and $M^3_i = M^1_i$ and $N^*_i \cap M^3 = N^3_i$ and $N^*_i \cap M^2 = N^1_i\}$ is a club of λ^+, without loss of generality $i \in E \cap S$ hence $a \in N^3_i \cap N^1_i$ and $a \notin M^1_i$ as $a \notin M$. Now as (M^3_i, N^3_i, a^*) is reduced, and $N^3_i \cap N^1_i \backslash M^1_i \neq \emptyset$ (as a exemplifies) clearly $a^* \in N^1_i$.

Now $\mathrm{tp}(a^*, M^3_i, N^3_i)$ has more than one extension in $\mathcal{S}(M^3_{i+1})$ hence in $\mathcal{S}(M^3_j)$ for $j =: \mathrm{Min}(E\backslash(i+1))$. But $M^3_i = M^1_i, M^3_j = M^1_j$ (as $i, j \in E$) hence this means that

$\mathrm{tp}(a^*, M_i^1, N_i^3) = \mathrm{tp}(a^*, M_i^1, N_i^*) = \mathrm{tp}(a^*, M_i^1, N_i^1)$ has more than one extension in $\mathcal{S}(M_j^1)$; hence M_j^1, N_j^1 can be amalgamated in two contradictory ways over M_i^1, in both cases b is not mapped into M_j^1 (as $\mathrm{tp}(b, M_0^1, N_0^1)$ is not realized in M), so we get a contradiction to $i \in E \cap S$, i.e. to our previous assumption "β fails for (M^0, N^0, a)" and it means also clause (e) of (β) holds. $\square_{\langle 3.16 \rangle}$

Remark. In clause (β) we could demand "$\mathrm{tp}(a_0, M^0, N^0)$ has no minimal extension", this is okay as it implies that for every $M^0 \in K_\lambda, \{p \in \mathcal{S}(M^0) : p$ has no minimal extension (i.e. in any $\mathcal{S}(N), M^0 \leq_{\mathfrak{K}} N)\}$ has cardinality $> \lambda$ (see §2).

* * *

⟨3.17⟩ Claim. *The minimal triples are dense in K_λ^3 if we assume $I(\lambda^{+3}, K) = 0$ and $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$ (so we do not assume "$I(\lambda^{+2}, K) = 1$"), but we assume K is categorical in λ, λ^+, $LS(K) \leq \lambda$ and $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$.*

Proof. If $2^{\lambda^+} > \lambda^{++}$ we get the conclusion by ⟨2.5⟩. If $2^{\lambda^+} = \lambda^{++}$ necessarily $2^\lambda = \lambda^+$ (as $\lambda < 2^\lambda < 2^{\lambda^+}$), so as $\mathfrak{K}$ has amalgamation in λ, there is a model $M \in K_{\lambda^+}$ saturated over λ, but then during the proof of ⟨3.16⟩ (the possibility $(*)_2$) gives the result. $\square_{\langle 3.17 \rangle}$

§4 Minimal types

We return to the analysis of minimal types in $\langle 2.8\rangle$.
We use from §2 only $\langle 2.1\rangle$, $\langle 2.4\rangle$, $\langle 2.6\rangle$ so there are repetitions.

$\langle 4.0\rangle$ Hypothesis.

(*a*) $\mathfrak{K}$ is an abstract elementary class with $LS(\mathfrak{K}) \leq \lambda$ (for simplicity $K_{<\lambda} = \emptyset$).

(*b*) $\mathfrak{K}$ is categorical in λ, λ^+ with $K_{\lambda^{+2}} \neq \emptyset$ (note: $(a) + (b) = (*)^3_\lambda$ of $\langle 2.3\rangle$).

(*c*) $\mathfrak{K}$ has amalgamation in λ ($\langle 2.2\rangle(1)$),
so by $(a)+(c)$, we have $(*)^2_\lambda$ from $\langle 2.6\rangle$ hence $\mathfrak{K}$ satisfies the model theoretic properties which were deduced in $\langle 2.3\rangle$-$\langle 2.4\rangle$,$\langle 2.6\rangle$ in particular:

- (i) every $(M, N, a) \in K^3_\lambda$ has the weak extension property $\langle 2.3\rangle$
- (ii) criterions for the extension property $\langle 2.6\rangle$
- (iii) basic Definitions and properties $\langle 2.3\text{A}\rangle$, $\langle 2.4\rangle$.

$\langle$**4.1**$\rangle$ **Claim.** *If $(M_0, M_1, a) \in K^3_\lambda$ is minimal, <u>then</u> it has the extension property.*

Proof. Let $p^* = \text{tp}(a, M_0, M_1)$, and assume it is a counter-example. We let (continuing Definition $\langle 2.9\rangle$):

$\langle$**4.1A**$\rangle$ **Definition.** 1) If $p \in \mathcal{S}(N), N \in K_\lambda$ and $N' \in K_\lambda$ remember

$$\mathcal{S}_p(N') = \{f(p) : f \text{ is an isomorphism from } N \text{ onto } N'\}$$

and let

$$\mathcal{S}_{\geq p}(N') = \Big\{ q \in \mathcal{S}(N') : q \text{ not algebraic (i.e. not realized by any } c \in N') \text{ and, for some } N'' \in K_\lambda, \ N'' \leq_{\mathfrak{K}} N', \text{ we have } q \restriction N'' \in \mathcal{S}_p(N'') \Big\}.$$

2) We say the type $p \in \mathcal{S}(N)$ is λ-algebraic if $\|N\| \leq \lambda$ and for every M such that $N \leq_{\mathfrak{K}} M$ we have: $\lambda \geq |\{c \in M : \text{tp}(c, N, M) = p\}|$.

Continuation of the Proof of $\langle 4.1\rangle$.

$\bigotimes_1$ for some M^* we have $M_0 \leq_{\mathfrak{K}} M^* \in K_\lambda$ but for no M^+ and b do we have $M^* \leq_{\mathfrak{K}} M^+ \in K_\lambda, b \in M^+ \backslash M^*$ and b realizes p^*.
[Why? If not for every $N, M_0 \leq_{\mathfrak{K}} N \in K_\lambda$, we can find N_1,

$N \leq_{\mathfrak{K}} N_1 \in K_\lambda$ and $b \in N_1 \backslash N$ which realizes p^*. Hence (as $\mathfrak{K}_\lambda$ has amalgamation in λ) we can find N_2 such that $N_1 \leq_{\mathfrak{K}} N_2 \in K_\lambda$, and g a $\leq_{\mathfrak{K}}$-embedding of M_1 into N_2 extending id_{M_0} such that $g(a) = b$.
This proves the extension property].

$\bigotimes_2$ if $p \in \mathcal{S}_{\geq p^*}(N)$ and $N \in K_\lambda$ and $N \leq_{\mathfrak{K}} N^* \in K$, <u>then</u> the set of elements of $b \in N^*$ realizing p has cardinality $\leq \lambda$.
[Why? by $\langle 4.0 \rangle$(c)(ii); so indirectly $\langle 2.6 \rangle$(2)].

$\bigotimes_3$ if $N \in K_\lambda$, then $|\mathcal{S}_{\geq p^*}(N)| > \lambda^+$.

Proof of $\bigotimes_3$. If N forms a counterexample, as K is categorical in λ and using $\otimes_2$ we can find $\langle N_i : i < \lambda^+ \rangle$, $\leq_{\mathfrak{K}}$-increasing continuous sequence of members of K_λ such that:

(*) for every $\alpha < \lambda^+$ and $q \in \mathcal{S}_{\geq p^*}(N_\alpha)$, for some $\beta = \beta_q < \lambda^+$ we have: for no N', b do we have $N_\beta \leq_{\mathfrak{K}} N' \in K_\lambda, b \in N' \backslash N_\beta$ and b realizes q.

So $N_{\lambda^+} = \bigcup\limits_{i<\lambda^+} N_i$ has the property

(**) if $\bar{N}' = \langle N'_\alpha : \alpha < \lambda^+ \rangle$ is a representation of N_{λ^+} then for a club of $\delta < \lambda^+$ for every $q \in \mathcal{S}_{\geq p^*}(N'_\delta)$ for a club of $\beta \in (\delta, \lambda^+)$, for no N', b do we have: $N_\beta \leq_{\mathfrak{K}} N' \in K_\lambda, b \in N' \backslash N_\beta$ and b realizes q.

On the other hand, we can choose by induction on $\alpha < \lambda^+$ a triple $(N_{0,\alpha}, N_{1,\alpha}, a) \in K^3_\lambda$ increasing continuous in α such that $(N_{00}, N_{10}, \alpha) = (M_0, M_1, a)$ and $N_{0,\alpha} \neq N_{0,\alpha+1}$ (existence by the weak extension property; i.e. $\langle 2.3 \rangle = \langle 4.0 \rangle$(c)(i)).
Now $N_{0,\lambda^+} = \bigcup\limits_{\alpha<\lambda^+} N_{0,\alpha} \in K_{\lambda^+}$ does not satisfy the statement (**):
$\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$ is a representation of N_{0,λ^+}, and for every $\alpha, \mathrm{tp}(a, N_{0,\alpha}, N_{1,\alpha})$ extend $\mathrm{tp}(a, N_{0,0}, N_{1,0}) = \mathrm{tp}(a, M_0, M_1) = p^*$ hence $\mathrm{tp}(a, N_{0,\alpha}, N_{1,\alpha}) \in \mathcal{S}_{\geq p^*}(N_{0,\alpha})$ satisfies: for every $\beta \in (\alpha, \lambda^+)$, there is $N', N_{0,\beta} \leq_{\mathfrak{K}} N'$ and some $b \in N' \backslash N_{0,\beta}$ realizes $\mathrm{tp}(a, N_{0,\alpha}, N_{1,\alpha})$; simply choose $(N', b) = (N_{1,\beta}, a)$. So $N_{0,\lambda^+}, N_{\lambda^+}$ cannot be isomorphic (as one satisfies (**) the other not). But both are in K_{λ^+}, contradicting the categoricity of $\mathfrak{K}$ in λ^+. $\square_{\otimes_3}$

To finish the proof of $\langle 4.1 \rangle$ it is enough to prove

$\langle 4.2 \rangle$ Claim. *If $p^* \in \mathcal{S}(M_0)$ is minimal, $M_0 \in K_\lambda$, <u>then</u> $N \in K_\lambda \Rightarrow |\mathcal{S}_{\geq p^*}(N)| \leq \lambda^+$.*

Proof. By $\langle 4.2\mathrm{B} \rangle$, $\langle 4.2\mathrm{C} \rangle$ below. Note that $\mathcal{S}_{\geq p^*}(N)$ has the same cardinality for every $N \in K_\lambda$.

⟨4.2A⟩ Claim. *1) In $N_1 \leq_{\mathfrak{K}} N_2$ are in K_λ and $p_1 \in \mathcal{S}(N_1)$ is minimal and is omitted by N_2 <u>then</u> p_1 has a unique extension in $\mathcal{S}(N_2)$ call it p_2 and $p_1 \in \mathcal{S}_{\geq p^*}(N_1) \Rightarrow [p_2 \in \mathcal{S}_{\geq p^*}(N_2)$ and p_2 is minimal].*
2) If $N_1 \leq_{\mathfrak{K}} N_2$ are in $K_\lambda, p_1 \in \mathcal{S}(N_1)$ minimal, <u>then</u> p_1 has a unique non-algebraic extension in $\mathcal{S}(N_2)$ called p_2, it is minimal and $p_1 \in \mathcal{S}_{\geq p^*}(N_1) \Rightarrow p_2 \in \mathcal{S}_{\geq p^*}(N_2)$.
3) (Continuity) if $\langle N_i : i \leq \alpha\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous sequence of members of $K_\lambda, p_i \in \mathcal{S}(N_i), p_0$ minimal, $p_i \in \mathcal{S}(N_i)$ extends p_0 and is non-algebraic then $\langle p_i : i \leq \alpha\rangle$ is increasing continuously.

Proof of ⟨4.2A⟩. Easy. E.g.,
3) If $i < j \leq \alpha$ then $p_j \restriction N_i$ is well defined, it belongs to $\mathcal{S}(N_i)$, also it is non-algebraic and extending p_0 hence by the uniqueness (=⟨4.2A⟩(1)) we have $p_i = p_j \restriction N_i$. If $\delta \leq \alpha, p_\delta \in \mathcal{S}(N_\delta)$ extend p_i for $i < \delta$; if $p^1_\delta \in \mathcal{S}(N_\delta)$ extend each $p_i (i < \delta)$ then it extends p_0 and is non-algebraic hence by uniqueness $p'_\delta = p_\delta$.

⟨4.2B⟩ Claim. *If $N \in K_\lambda, \mathcal{S} \subseteq \mathcal{S}(N)$ and $|\mathcal{S}| > \lambda^+$, <u>then</u> we can find N^*, N_i in K_λ, (for $i < \lambda^{++}$) such that:*

- (α) *$N \leq_{\mathfrak{K}} N^* <_{\mathfrak{K}} N_i$*
- (β) *for no $i_0 < i_1 < \lambda^{++}$ and $c_\ell \in N_{i_\ell} \backslash N^*$ (for $\ell = 0, 1$) do we have $tp(c_0, N^*, N_{i_0}) = tp(c_1, N^*, N_{i_1})$*
- (γ) *there are $a_i \in N_i$ (for $i < \lambda^{++}$) such that $tp(a_i, N, N_i) \in \mathcal{S}$ (and they are pairwise distinct).*

Remark. We use here less than Hypothesis ⟨4.0⟩:

- ($*$) $\mathfrak{K}$ is abstract elementary class with amalgamation in λ, categorical in λ, $K_{\lambda^+} \neq \emptyset$.

The same applies to ⟨4.2C⟩.

Proof. Without loss of generality $|N| = \lambda$; now choose by induction on $\alpha < \lambda^{++}, \bar{N}^\alpha, N_\alpha, a_\alpha$ such that:

- (A) $N_\alpha \in K_{\lambda^+}$ has a set of elements $\lambda \times (1 + \alpha)$ and N_α is $\leq_{\mathfrak{K}}$-increasing continuous in α
- (B) $\bar{N}^\alpha = \langle N^\alpha_i : i < \lambda^+\rangle$ is a representation of N_α (i.e. is $\leq_{\mathfrak{K}}$-increasing continuous, $\|N^\alpha_i\| \leq \lambda$ and $N_\alpha = \bigcup\limits_{i<\lambda^+} N^\alpha_i$)
- (C) for $\alpha < \lambda^{++}$ successor, if $i < j < \lambda^+, p \in \mathcal{S}(N^\alpha_i)$ is realized in N^α_j and is λ-algebraic (see Definition ⟨4.1A⟩(2)) <u>then</u> for no N', b do we have $N^\alpha_j \leq_{\mathfrak{K}} N' \in K_\lambda$ and $b \in N' \backslash N^\alpha_j$ realizes p (actually not needed)
- (D) $N = N^0_0 \leq_{\mathfrak{K}} N_0$ and $a_\alpha \in N_{\alpha+1} \backslash N_\alpha$ realizes some $p_\alpha \in \mathcal{S}$ not realized in N_α.

(E) If $\aleph_0 <$ cf$(\alpha) \le \lambda$ then let for $j < \lambda$:
$M^\alpha_j = \cap\{\bigcup\limits_{\beta \in C} N^\beta_j : C$ a club of $\alpha\}$ whenever possible, i.e. the result is in K_λ and $\le_{\mathfrak{K}} N^\alpha$

(F) for each $\alpha < \lambda^{++}$, for a club E^0_α of ordinals $i < \lambda^+$ we have $(N^\alpha_i, N^{\alpha+1}_i, a_\alpha)$ is reduced;
hence (as tp$(a_\alpha, N^\alpha_i, N^{\alpha+1}_i)$ extends tp$(a_\alpha, N, N^{\alpha+1}_i)$ which is not realized in N_α):

(G) for every $(i \in E^0_\alpha$ and$)$ $b \in N^{\alpha+1}_i \backslash N^\alpha_i$ the type tp$(b, N^\alpha_i, N^{\alpha+1}_i)$ is not realized in N_α
(a key point).

There is no problem to carry the construction ($\mathfrak{K}$ has amalgamation in λ).
Let $w^\alpha_i =: \{\beta : N^{\alpha+1}_i \cap N_{\beta+1} \nsubseteq N_\beta\}$, so necessarily $|w^\alpha_i| \le \lambda, w^\alpha_i$ is increasing continuous in $i < \lambda^+$ and $\alpha = \bigcup\limits_{i<\lambda^+} w^\alpha_i$ and for $\beta < \alpha$ let
$\mathbf{i}(\beta, \alpha) =$ Min$\{i : \beta \in w^\alpha_i\}$.
Now for every $\alpha \in S^* =: \{\delta < \lambda^{++} : \text{cf}(\delta) = \lambda^+\}$, the set

$$\begin{aligned} E_\alpha =: \Big\{ i < \lambda^+ : & i \text{ limit }, N \le_{\mathfrak{K}} N^{\alpha+1}_i, a_\alpha \in N^{\alpha+1}_i, \\ & \text{and for every } \beta < \alpha \text{ if} \\ & \beta \in w^\alpha_i \text{ then} \\ & N^\beta_i = N^\alpha_i \cap N_\beta \text{ and for } j < i \\ & \text{the closure of } w^\alpha_j \text{ (in } \alpha) \text{ is included in } w^\alpha_i \\ & \text{and } \beta_1 < \beta_2 \ \& \ \beta_1 \in w^\alpha_j \ \& \ \beta_2 \in w^\alpha_j \Rightarrow \mathbf{i}(\beta_1, \beta_2) < i \Big\} \end{aligned}$$

is a club of λ^+.

As we can assume $\lambda > \aleph_0$ (ignoring $\lambda = \aleph_0$ as was treated earlier in [Sh 88] though for a PC$_{\aleph_0}$ class, or see [Sh 600],§2), we can choose $j_\alpha \in E_\alpha$ such that cf$(j_\alpha) = \aleph_1$ and let $\delta_\alpha = \sup(w^\alpha_{j_\alpha})$, now $w^\alpha_{j_\alpha}$ is closed under ω-limits (as $\langle w^\alpha_j : j \le \alpha\rangle$ is increasing continuous, $j < \alpha \Rightarrow$ closure$(w^\alpha_j) \subseteq w^\alpha_{j+1}$) and $\aleph_1 =$ cf(otp $w^\alpha_{j_\alpha}$) so there is $\langle \beta_\varepsilon : \varepsilon < \omega_1\rangle$ increasing continuous with limit $\delta_\alpha, \beta_\varepsilon \in w^\alpha_{j_\alpha}$ so $\varepsilon < \zeta < \omega_1 \Rightarrow N^{\beta_\varepsilon}_j = N^{\beta_\zeta}_j \cap N_{\beta_\varepsilon}$ and easily $N^\alpha_j \cap N_{\beta_\varepsilon} = N^{\beta_\varepsilon}_j$ hence

$$\bigoplus \quad M^{\delta_\alpha}_{j_\alpha} = \cap \Big\{ \bigcup_{j \in C} N^\beta_j : C \text{ a club of } \delta_\alpha \Big\}$$

so $N^\alpha_{j_\alpha} = M^{\delta_\alpha}_{j_\alpha}$ (see [Sh 351, §4]).

By Fodor lemma for some j^*, α^* and stationary $S \subseteq S^*, \alpha \in S^* \Rightarrow j_\alpha = j^*$ & $\delta_\alpha = \delta^*$. So for all $\alpha \in S, N^\alpha_{j_\alpha}$ are the same say N^*. So $N^* \in K_\lambda$, for $\alpha \in S$,
$q_\alpha =$ tp$(a_\alpha, N^*, N^{\alpha+1}_{j_\alpha})$ extend $p_\alpha (\in \mathcal{S})$. Also if $r \in \mathcal{S}(N^*)$ is realized in $N^{\alpha+1}_{j_\alpha}$ say

by b (for some $\alpha \in S$) then no member of $\bigcup\{N^{\beta+1}_{j^*} \backslash N^{\beta}_{j^*} : \beta \in S \cap \alpha\}$ realizes it (holds by clause (G), see clause (F)). So the sets $\Gamma_\alpha = \{\mathrm{tp}(b, N^{\delta^*}_{j^*}, N^{\alpha+1}_{j^*}) : b \in N^{\alpha+1}_{j^*} \backslash N^*\}$ are pairwise disjoint and each has a member extending $p_\alpha \in \mathcal{S}$ (as exemplified by a_α and p_α is not extended by any $p \in \bigcup\limits_{\beta<\alpha} \Gamma_\beta$ (as p_α is not realized in N_α)). $\square_{\langle 4.2B\rangle}$

$\langle$**4.2C**$\rangle$ **Claim.** *Assume p^* is a counterexample to $\langle 4.2\rangle$.*
1) If $N \in K_\lambda, \Gamma \subseteq \mathcal{S}_{\geq p^*}(N), |\Gamma| \leq \lambda^+$ <u>then</u>

$$\{p \in \mathcal{S}_{\geq p^*}(N) : \textit{for some } N', N \leq_{\mathfrak{K}} N' \in K_\lambda \textit{ and some } b \in N' \textit{ realizes } p \textit{ but no } b \in N' \textit{ realizes any } q \in \Gamma\}$$

has cardinality $\geq \lambda^{++}$.
2) We can find $N \in K_{\lambda^+}$, and $N_i, N <_{\mathfrak{K}} N_i \in K_{\lambda^+}$ for $i < 2^{\lambda^+}$ such that the set $\Gamma_i = \{tp(a, N, N_i) : a \in N_i\}$ are pairwise distinct, in fact, no one embeddable into another (so we get $I(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$ and if $(2^\lambda)^+ < 2^{\lambda^+}$ then $IE(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$ thus contradicting the categoricity in λ^+ from the assumptions).

Proof. 1) Apply $\langle$4.2B$\rangle$ with N and $\mathcal{S}_{\geq p^*}(N)$ here standing for N and $\mathcal{S}$ there, so we get $N^*, N_i (i < \lambda^{++})$ and M^* such that $N^* \leq_{\mathfrak{K}} N_i \in \mathfrak{K}_\lambda$, and $\Gamma_i = \{\mathrm{tp}(a, N^*, N_i) : a \in N_i \backslash N^*\}$ are pairwise disjoint and there are $p_i \in \Gamma_i, p_i \restriction N \in \mathcal{S}_{\geq p^*}(N)$ pairwise distinct; now p_i is not algebraic hence $p_i \in \mathcal{S}_{\geq p^*}(N^*)$. As K is categorical in λ, without loss of generality $N^* = N$, so all but $\leq \lambda^+$ of the models N_i can serve as the required N'.
2) Now by part (1) of $\langle$4.2C$\rangle$ we can choose by induction on $i < \lambda^+$, $\langle (N_\eta, \Gamma_\eta) : \eta \in {}^i 2\rangle$ such that:

- (a) $N_\eta \in K_\lambda$ and $\Gamma_\eta \subseteq \bigcup\limits_{j<i} \mathcal{S}_{\geq p^*}(N_{\eta \restriction j})$ and $|\Gamma_\eta| \leq \lambda$
- (b) if $\nu \triangleleft \eta$ then $N_\nu \leq_{\mathfrak{K}} N_\eta$ and $\Gamma_\eta \subseteq \Gamma_\nu$
- (c) some $p \in \Gamma_{\eta \frown \langle 0\rangle}$ is from $\mathcal{S}(N_\eta)$ and is realized in $N_{\eta \frown \langle 1\rangle}$ (similarly for $\Gamma_{\eta \frown \langle 1\rangle}, N_{\eta \frown \langle 0\rangle}$)
- (d) if i is a limit ordinal, then $N_\eta = \bigcup\limits_{j<i} N_{\eta \restriction j}$ and $\Gamma_\eta = \bigcup\limits_{j<i} \Gamma_{\eta \restriction j}$.

The successor case is done by $\langle$4.2C$\rangle$(1) (you may object that the type in Γ_η are not from $\mathcal{S}_{\geq p^*}(N_\eta)$ but from $\bigcup\limits_{j<i} \mathcal{S}_{\geq p^*}(N_{\eta \restriction j})$? However they are minimal - see $\langle$4.2A$\rangle$(1)).
For $\eta \in {}^{\lambda^+}2$ let $N_\eta = \bigcup\limits_{i<\lambda^+} N_{\eta \restriction i}$. Now by 1.4(1), $\{N_\eta / \cong : \eta \in {}^{\lambda^+}2\}$ has cardinality 2^{λ^+}, contradiction. $\square_{\langle 4.2C\rangle}, \square_{\langle 4.2\rangle}, \square_{\langle 4.1\rangle}$

⟨4.3⟩ Claim. *1) If $M_0 \in K_\lambda, M_1 \in K_{\lambda^+}$, and $M_0 \leq_{\mathfrak{K}} M_1$ then every minimal $p \in \mathcal{S}(M_0)$ is realized in M.*
2) Every $M_1 \in K_{\lambda^+}$ is saturated at least for minimal types (i.e. if $M_0 \leq_{\mathfrak{K}} M_1$, $M_0 \in K_\lambda$ and $M_1 \in K_{\lambda^+}$ then every minimal $p \in \mathcal{S}(M_0)$ is realized in M_1).
3) If $M \in K_\lambda$ then $\{p \in \mathcal{S}(M) : p \text{ minimal}\}$ has cardinality $\leq \lambda^+$.

Proof. 1) Let $\bar{N} = \langle N_\alpha : \alpha < \lambda^+ \rangle$ be a representation of $N_{\lambda^+} \in K_{\lambda^+}$. Let $N \in K_\lambda, p \in \mathcal{S}(N)$ be minimal we ask

$(*)_p$ is there a club of $\alpha < \lambda^+$ such that every $q \in \mathcal{S}_{\geq p}(N_\alpha)$ is realized in λ^+?

By ⟨4.2⟩ there is $N'_{\lambda^+} \in K_{\lambda^+}$ for which the answer is yes, hence, as $\mathfrak{K}$ is categorical in λ^+, this holds for N_{λ^+}. So this holds for every minimal p. Now if $N' \leq_{\mathfrak{K}} N_{\lambda^+}, N' \in K_\lambda$ and $p \in \mathcal{S}(N')$ is minimal then for some $\alpha, N' \leq_{\mathfrak{K}} N_\alpha$ and for every $\beta \in [\alpha, \lambda^+)$, p has a unique non-algebraic extension $p_\beta \in \mathcal{S}(N_\beta)$ (which necessarily is minimal). Now $p_\beta \in \mathcal{S}_{\geq p}(N_\beta)$ hence for a club of $\beta < \lambda^+, p_\beta$ is realized in N_{λ^+}, so we have finished the proof of part (1).
2) By part (1).
3) Follows by part (1). $\square_{\langle 4.3 \rangle}$

From ⟨4.1⟩, ⟨2.6⟩(1) we can conclude

⟨4.4⟩ Conclusion. Every $(M_0, M_1, a) \in K^3_\lambda$ has the extension property.

§5 Inevitable types and stability in λ

$\langle$*5.0*$\rangle$ *Hypothesis.* Assume the model theoretic assumptions from $\langle 4.0\rangle$ and

(*d*) there is a minimal member of K^3_λ (follows from the conclusion of $\langle 3.17\rangle$).

$\langle$**5.1**$\rangle$ **Definition.** We call $p \in \mathcal{S}(N)$ inevitable if:
$N \leq_{\mathfrak{K}} M \ \ \& \ \ N \neq M \Rightarrow$ some $c \in M$ realizes p.
We call $(M, N, a) \in K^3_\lambda$ inevitable if $\mathrm{tp}(a, M, N)$ is inevitable.

So by $\langle 4.1\rangle$ - $\langle 4.3\rangle$ we shall deduce

$\langle$**5.2**$\rangle$ **Claim.** *1) If there is a minimal triple in K^3_λ, <u>then</u> there is an inevitable $p = tp(a, N, N_1)$ with $(N, N_1, a) \in K^3_\lambda$ minimal.*
2) Moreover, if $p_0 \in \mathcal{S}(N_0)$ is minimal, $N_0 \in K_\lambda$ <u>then</u> we can find $N_1, N_0 \leq_{\mathfrak{K}} N_1 \in K_\lambda$ such that the unique non-algebraic extension p_1 of p_0 in $\mathcal{S}(N_1)$ is inevitable.

Proof of $\langle$*5.2*$\rangle$. 1) Follows by part (2).
2) Let $(M_0, M_1, a) \in K^3_\lambda$ be minimal and $p_0 = \mathrm{tp}(a, M_0, M_1)$. We try to choose by induction on i a model N_i such that: $N_0 = M_0, N_i \in K_\lambda$ is $\leq_{\mathfrak{K}}$-increasing continuously and N_i omits $p_0, N_i \neq N_{i+1}$. If we succeed, $\bigcup\limits_{i<\lambda^+} N_i$ is a member of K_{λ^+} which is non-saturated for minimal types, contradicting $\langle 4.3\rangle(2)$. As for $i = 0$, i limit we can define, necessarily for some i we have N_i but not N_{i+1}. Now p_0 has a unique extension in $\mathcal{S}(N_i)$ which we call p_i and p_0 has no algebraic extension in $\mathcal{S}(N_i)$.
[why? as N_i omits p_0]. So p_i is the unique extension of p_0 in $\mathcal{S}(N_i)$ [by $\langle 4.2\mathrm{A}\rangle(1)$], and so

$(*)$ if $N_i \leq_{\mathfrak{K}} N' \in K_\lambda$ and $N' \neq N$, <u>then</u> p_i is realized in N'.

By L.S. we can omit "$N' \in K_\lambda$", so (N_i, p_i) are as required. $\square_{\langle 5.2\rangle}$

$\langle$*5.2A*$\rangle$ *Fact.* Inevitable types have few ($\leq \lambda$) conjugates (i.e. for $p \in \mathcal{S}(M_0)$ inevitable $M_0 \in K_\lambda, M_1 \in K_\lambda$ we have $|\mathcal{S}_p(M_1)| \leq \lambda$), moreover $|\{p \in \mathcal{S}(N) : p \text{ inevitable}\}| \leq \lambda$ for $N \in K_\lambda$.

Proof. Easy.

The following construction plays a central role in what remains of this paper.

⟨5.3⟩ Claim. *For any limit $\alpha < \lambda^+$, we can find $\langle N_i : i \leq \alpha\rangle$ and $\langle p_i : i \leq \alpha\rangle$ such that:*

(i) $N_i \in K_\lambda$,
(ii) N_i *is* $\leq_{\mathfrak{K}}$*-increasing continuous*
(iii) $p_i \in \mathcal{S}(N_i)$ *is minimal,*
(iv) p_i *increases continuously (see ⟨4.2A⟩(3))*
(v) p_0 *is inevitable*
(vi) p_α *is inevitable*
(vii) $N_i \neq N_{i+1}$ *moreover some* $c \in N_{i+1} \backslash N_i$ *realizes* p_0 *(hence* p_i*).*

Remark. Why not just build a non-saturated model in order to prove ⟨5.3⟩? Works, too.

Proof. Choose $N^0 <_{\mathfrak{K}} N^1$ in K_{λ^+} (so $N^0 \neq N^1$), such a pair exists as $K_{\lambda^{+2}} \neq \emptyset$. Let $N^\ell = \bigcup\limits_{i<\lambda^+} N^\ell_i$ with $N^\ell_i \in K_\lambda$ being $\leq_{\mathfrak{K}}$-increasing continuously in i. Now $E_0 = \{\delta < \lambda^+ : N^1_\delta \neq N^0_\delta \text{ and } N^1_\delta \cap N^0 = N^0_\delta\}$ is a club of λ^+. Without loss of generality $E_0 = \lambda^+$.

For each $c \in N^1 \backslash N^0$, the set

$$X_c =: \{i < \lambda^+ : c \in N^1_i \text{ and } (N^0_i, N^1_i, c) \text{ is minimal}\}$$

is empty or an end segment of λ^+ hence

$$E_1 = \left\{ \delta < \lambda^+ : \begin{array}{l} \text{(i) } \delta \text{ limit} \\ \text{(ii) if } i < \delta \text{ and } p \in \mathcal{S}(N^0_i) \text{ is minimal inevitable} \\ \quad \text{and realized in } N^0 \backslash N^0_\delta \text{ then it is} \\ \quad \text{realized in } N^0_\delta \backslash N^0_i \text{ (actually automatic)} \\ \text{(iii) if } c \in N^1_\delta \backslash N^0 \text{ (hence } \exists i < \delta, c \in N^1_i) \text{ and } X_c \\ \quad \text{is non-empty } \underline{\text{then}}\ \delta \in X_c \text{ and } \min(X_c) < \delta \end{array} \right\}$$

is a club of λ^+ (see ⟨5.2A⟩).

Now for $\delta \in E_1$, we have $N^0_\delta <_{\mathfrak{K}} N^1_\delta$, so by ⟨5.2⟩(1) there is $c_\delta \in N^1_\delta \backslash N^0_\delta$ such that:

$$(N^0_\delta, N^1_\delta, c_\delta) \text{ is minimal}$$
$$\text{tp}(c_\delta, N^0_\delta, N^1_\delta) \text{ is inevitable}$$

As δ is limit, for some $i < \delta, c \in N^1_i$, also $\delta \in X_c$ hence there is j such that: $i < j < \delta \quad \& \quad j \in X_c$ hence (N^0_j, N^1_j, c) is minimal; choose such j_δ, c_δ. Let $\kappa = \mathrm{cf}(\kappa) = \mathrm{cf}(\alpha) \leq \lambda$, so for some j^*, c^* we have

$$S = \{\delta \in E_1 : \mathrm{cf}(\delta) = \kappa, j_\delta = j^*, c_\delta = c^*\}$$

is stationary in λ^+.

Choose e closed $\subseteq E_1$ of order type $\alpha + 1$ with first element and last element in S; for $\zeta \in [j^*, \lambda^+)$ let $p_\zeta = \mathrm{tp}(c^*, N^0_\zeta, N^1_\zeta)$. (In fact, we could have: all non-accumulation member of e are in S; no real help.)

Now $\langle N^0_\zeta, p_\zeta : \zeta \in e\rangle$ is as required (up to re-indexing)(clause (viii) holds by clause (ii) in the definition of E_1). $\square_{\langle 5.3\rangle}$

⟨5.4⟩ Claim. *Assume $\langle N_i, p_i : i \leq \alpha\rangle$ is as in ⟨5.3⟩, $\alpha < \lambda$ divisible by λ. Then any $p \in \mathcal{S}(N_0)$ is realized in N_α, moreover N_α is universal in K_λ over N_0.*

Proof. (Similar to the proof of ⟨0.15⟩; which is [Sh 300, II,§3]).

Let $N_0 \leq_{\mathfrak{K}} M_0 \in K_\lambda, a \in M_0 \backslash N_0$ we shall show that $\mathrm{tp}(a, N_0, M_0)$ is realized in N_α.

Let $\alpha = \bigcup\limits_{i<\lambda} S_i, \langle S_i : i < \lambda\rangle$ pairwise disjoint, each S_i unbounded in α, λ divides $\mathrm{otp}(S_i)$ and $\mathrm{Min}(S_i) \geq i$. We choose by induction on $i \leq \alpha$ the following:

$$N^1_i, M^1_i, h_i, \langle a_\zeta : \zeta \in S_i\rangle \quad \text{(the last one only if } i < \alpha)$$

such that:

- (a) $N^1_i \leq_{\mathfrak{K}} M^1_i$ are in K_λ
- (b) N^1_i is $\leq_{\mathfrak{K}}$-increasing continuous in i
- (c) M^1_i is $\leq_{\mathfrak{K}}$-increasing continuous in i
- (d) $(N^1_0, M^1_0) = (N_0, M_0)$
- (e) $\langle a_\zeta : \zeta \in S_i\rangle$ is a list of $\{c \in M^1_i : c \text{ realizes } p_0\}$
- (f) h_i is an isomorphism from N_i onto N^1_i
- (g) $j < i \Rightarrow h_j \subseteq h_i$ and $h_0 = \mathrm{id}_{N_0}$
- (h) $a_i \in N^1_{i+1}$ (note: $M^1_i \cap N^1_{i+1} \neq N^1_i$ in general).

<u>For $i = 0$</u>: See clauses $(d), (g)$

$$N^1_0 = N_0,\ M^1_0 = M_0,\ h_0 = \mathrm{id}_{N_0}.$$

<u>For $i = \text{limit}$</u>: Let $N^1_i = \bigcup\limits_{j<i} N^1_j$ and $M^1_i = \bigcup\limits_{j<i} M^1_j$ and $h_i = \bigcup\limits_{j<i} h_j$ and lastly choose $\langle a_\zeta : \zeta \in S_i\rangle$ by clause (e).

For $i = j+1$: Note a_j is already defined, it belongs to M^1_j and it realizes p_0.

Case 1: $a_j \in N^1_j$ (so clause (h) is no problem).
Use amalgamation on N_j, N_i, M^1_j and the mapping id_{N_j}, h_i,i.e.

$$\begin{tikzcd} N_i \arrow[r] & M^1_i \\ N_j \arrow[u, "id_{N_j}"] \arrow[r, "h_i"] & N^1_j \arrow[u, "id_{N^1_0}"'] \end{tikzcd}$$

Case 2: $a_j \notin N^1_j$.

Then $\mathrm{tp}(a_j, N^1_j, M^1_j)$ is not algebraic, extending the minimal type $p_0 \in \mathcal{S}(N_0)$. Also by clause $(viii)$ of $\langle 5.3\rangle$ there is $c \in N_i \backslash N_j$ which realizes p_0. As $p_0 \in \mathcal{S}(N)$ is minimal

$$h_j\left(\mathrm{tp}(c, N_j, N_i)\right) = \ \mathrm{tp}(a_j, N^1_j, M^1_j)$$

so acting as in Case 1 we can also guarantee $h_i(c) = a_j$, so $a_j \in \ \mathrm{Rang}(h_i) = N^1_i$ as required.

* * *

In the end we have $N^1_\alpha \leq_{\mathfrak{K}} M^1_\alpha$. If $N^1_\alpha = M^1_\alpha$, then $h^{-1}_\alpha \restriction M_0 = h^{-1}_\alpha \restriction M^1_0 = h^{-1}_\alpha \restriction N^1_0$ show that M_0 can be embedded into N_α over N_0 as required. So assume $N^1_\alpha <_{\mathfrak{K}} M^1_\alpha$. Now $p_\alpha \in \mathcal{S}(N_\alpha)$ is inevitable hence $h_\alpha(p_\alpha) \in \mathcal{S}(N^1_\alpha)$ is inevitable. Hence some $d \in M^1_\alpha \backslash N^1_\alpha$ realizes $h_\alpha(p_\alpha)$ hence d realizes $h_\alpha(p_\alpha) \restriction N^1_0 = p_0$; also α is a limit ordinal so for some $i < \alpha, d \in M^1_i$ hence for some $\zeta \in S_i$ we have $a_\zeta = d$, hence

$$d = a_\zeta \in N^1_{\zeta+1} \subseteq N^1_\alpha,$$

contradicting the choice of d.
So we are done. $\square_{\langle 5.4\rangle}$

$\langle$*5.5*$\rangle$ *Conclusion.* If $N \in K_\lambda$ then:

- (a) $|\mathcal{S}(N)| = \lambda$
- (b) there is $N_1, N <_{\mathfrak{K}} N_1 \in K_\lambda$ such that N_1 is universal over N in K_λ
- (c) for any regular $\kappa \leq \lambda$ we can demand that $(N_1, c)_{c \in N}$ is (λ, κ)-saturated (see $\langle 0.2\rangle(1)$)

$\langle$*5.5R*$\rangle$ *Remark.* In fact amalgamation in λ and stability in λ (i.e. (a) of $\langle 5.5\rangle$) implies (b) and (c) of $\langle 5.5\rangle$.

$\langle$*5.5A*$\rangle$ *Conclusion.* The $N \in K_{\lambda^+}$ is saturated above λ (i.e. over models in K_λ!).

$\langle 5.6\rangle$ Claim. *Assume $\kappa = cf(\kappa) \le \lambda$.*
There are $N_0, N_1, a, N_0^+, N_1^+$ such that

(*i*) $(N_0, N_1, a) \in K_\lambda^3$ *and*

(*ii*) $(N_0, N_1, a) \le (N_0^+, N_1^+, a) \in K_\lambda^3$ *and*

(*iii*) $(N_0^+, c)_{c \in N_0}$ *is (λ, κ)-saturated,*

(*iv*) $tp(a, N_0, N_1)$ *is minimal inevitable and*

(*v*) $tp(a, N_0^+, N_1^+)$ *is minimal inevitable.*

Proof. As in the proof of $\langle 5.3\rangle$ as

$$E_2 = \Big\{\delta : \text{for every } i < \delta, (N_\delta^0, c)_{c \in N_i} \text{ is saturated of cofinality } \mathrm{cf}(\delta)\Big\}$$

is a club of λ^+. $\square_{\langle 5.6\rangle}$

$\langle 5.7\rangle$ Claim. *1) In K_λ we have disjoint amalgamations.*
2) If $M \le_{\mathfrak{K}} N$ are in K_λ and $p \in \mathcal{S}(M)$ non-algebraic then for some N', c we have: $N \le_{\mathfrak{K}} N' \in K_\lambda$ and $c \in N' \backslash N$ realizes p.

Proof. 1) First note

$\bigotimes$ if $M \le_{\mathfrak{K}} N$ in K_λ we can find $\alpha < \lambda^+$,and sequence $\langle M_i : i \le \alpha\rangle$ which is $\le_{\mathfrak{K}}$-increasing continuous, and $\langle a_i : i < \alpha\rangle$ such that (M_i, M_{i+1}, a_i) is minimal and reduced and $N \le_{\mathfrak{K}} M_\alpha, M = M_0$.
[Why? There is a minimal reduced pair, hence we can find $\langle M_i : i < \lambda^+\rangle$ $\le_{\mathfrak{K}}$-increasing continuous, (M_i, M_{i+1}, a_i) minimal reduced and $M = M_0$. So by $\langle 5.5\mathrm{A}\rangle$ we know $\bigcup\limits_{i<\lambda^+} M_i \in K_{\lambda^+}$ is saturated, hence we can embed N into $\bigcup\limits_{i<\lambda^+} M_i$ over N so this embedding is into some $M_\alpha, \alpha < \lambda^+$.]

So given $M \le_{\mathfrak{K}} M^1, M^2$, without loss of generality $M^\ell = M^\ell_{\alpha_\ell}, \langle (M_i^\ell, a_i^\ell) : i \le \alpha_\ell\rangle$ as above, and start to amalgamate using the extension property and "reduced". $\square_{\langle 5.7\rangle}$

$\langle 5.8\rangle$ Remark. We could prove $\langle 5.7\rangle$ earlier using "reduced triples". I.e. note that for some $\langle M_i^1 : i < \lambda^+\rangle \in \mathbf{S}eq_{\lambda^+}[\mathbf{C}^1_{\mathfrak{K},\lambda^+}]$, for each i for some a the triple $(M_i^1, M_{i+1}^1, a) \in K_\lambda^3$ is reduced. Hence if $M \le_{\mathfrak{K}} N$ from K_λ, for some $\bar{M} = \langle M_i : i \le \alpha\rangle$, $\le_{\mathfrak{K}}$-increasing continuous, $\langle M_i, M_{i+1}, b_i\rangle \in K_\lambda^3$ is reduced, $M_0 = M, N \le M_\alpha \in$

K_λ (otherwise find $\langle M^2_i : i < \lambda^+\rangle \in \mathbf{S}eq_{\lambda^+}[\mathbf{C}^1_{\mathfrak{K},\lambda^+}]$ with $(M_i, M_{i+1}) \cong (M, N)$, hence $M^1 = \bigcup\limits_{i<\lambda^+} M^1_i, M^2 = \bigcup\limits_{i<\lambda^+} M^2_i$ are non-isomorphic members of K_{λ^+}, contradiction). Now prove by induction on $\beta \le \alpha$ that if $M \le_{\mathfrak{K}} N_0 \in K_\lambda$ then N_0, M_β has disjoint amalgamation over $M_0 = M$ (i.e. we need to decompose only one side).

<u>$\langle 5.9\rangle$ Question</u>: If $M \in K_\lambda, p \in \mathcal{S}(M)$ is minimal, is it reduced? Or at least, if $M_0 \le_{\mathfrak{K}} M_1$ are in $K_\lambda, p_1 \in \mathcal{S}(M_\ell)$, no algebraic $p_0 = p_1 \restriction M_0, p_0$ is minimal and reduced is also p_1 reduced?
Probably true, but not needed for our aims (here and in [Sh 600], $\langle 7.4\mathrm{B}\rangle(2)$) is an approximation. Can prove it if $\lambda < \lambda^{\aleph_0}$ or there are E.M. models.

§6 Proving for $\mathfrak{K}$ categorical in λ^{+2}

⟨*6.0*⟩ *Hypothesis.* Assume the model theoretic assumptions from ⟨4.0⟩ + ⟨5.0⟩ and so the further model theoretic properties deduced in §4 + §5.
We use heavily ⟨4.3⟩.

⟨**6.1**⟩ **Definition.** 1) We say $(M_0, M_1, M_2) \in K_\lambda^{\rm uniq}$ (has unique amalgamation in K_λ) when

(*a*) $M_0, M_1, M_2 \in K_\lambda$

(*b*) $M_0 \le_{\mathfrak{K}} M_1$ and $M_0 \le_{\mathfrak{K}} M_2$

(*c*) <u>if</u> for $i = 1, 2$ we have $g^i_\ell : M_\ell \to N_i \in K_\lambda$ such that:

 (i) g^i_ℓ a $\le_{\mathfrak{K}}$-embedding,

 (ii) $g^i_0 \subseteq g^i_1$ and $g^i_0 \subseteq g^i_2$ for $i = 1, 2$

 (iii) $\mathrm{Rang}(g^i_1) \cap \mathrm{Rang}(g^i_2) = \mathrm{Rang}(g^i_0)$ (disjoint amalgamation) for $i = 1, 2$

 <u>then</u> we can find $N \in K_\lambda$ and $\le_{\mathfrak{K}}$-embeddings

$$f^i : N_i \to N \text{ for } i = 1, 2$$

 <u>such that</u>

$$\bigwedge_{\ell < 3} f^1 \circ g^1_\ell = f^2 \circ g^2_\ell.$$

2) Let $K_\lambda^{2,\rm uq}$ is the class of pairs (M_0, M_2) such that $M_0 \le_{\mathfrak{K}} M_2$ are both in K_λ and $[M_0 \le_{\mathfrak{K}} M_1 \in K_\lambda \Rightarrow (M_0, M_1, M_2) \in K_\lambda^{\rm uniq}]$ and let $K_\lambda^{3,\rm uq}$ be the class of pairs $(M_0, M_2) \in K_\lambda^{2,\rm uq}$ satisfying $M_0 \ne M_2$.

⟨**6.2**⟩ **Claim.** *1) If $(M_0, M_1, M_2) \in K_\lambda^{\rm uniq}$ <u>then</u>*

(*a*) $(M_0, M_2, M_1) \in K_\lambda^{\rm uniq}$

(*b*) *if $M_0 \le_{\mathfrak{K}} M'_2 \le_{\mathfrak{K}} M_2$ <u>then</u> $(M_0, M_1, M'_2) \in K_\lambda^{\rm uniq}$.*

2) Assume $M_0 \le_{\mathfrak{K}} M_2$ are from K_λ and $M_1 \in K_\lambda$ is universal over M_0. Then $(M_0, M_2) \in K_\lambda^{2,\rm uq} \Leftrightarrow (M_0, M_1, M_2) \in K_\lambda^{\rm uniq}$.

Proof. 1)a) Trivial.
b) Chase arrows (using disjoint amalgamation; i.e. ⟨5.7⟩).
2) Follows by ⟨6.2⟩(1)(a)+(b) and the definition. $\square_{\langle 6.2\rangle}$

⟨6.3⟩ Lemma. *Suppose*

$\bigotimes$ *there is* $(M_0, M_1, M_2) \in K_\lambda^{\text{uniq}}$ *such that* $M_0 \neq M_2$ *and* M_1 *is universal over* M_0.

Then: there are $N^0 <_{\mathfrak{K}} N^1$ *in* K_{λ^+} *such that:*

- (a) $N^0 \neq N^1$
- (b) *for every* $c \in N^1 \backslash N^0$ *there is* $M = M_c$ *satisfying* $N^0 \leq_{\mathfrak{K}} M \leq_{\mathfrak{K}} N^1$ *and* $N^0 \neq M$ *and* $c \in N^1 \backslash M$.

Proof. Choose $\langle N_i^0 : i < \lambda^+ \rangle$, a sequence of members of K_λ which is $\leq_{\mathfrak{K}}$-increasing continuous, such that:

$$(N_i^0, N_{i+1}^0) \cong (M_0, M_2).$$

So $N_i^0 \neq N_{i+1}^0$ hence $N_0 = \bigcup_{i<\lambda^+} N_i^0 \in K_{\lambda^+}$ and without loss of generality $|N_0| = \lambda^+$.
We now choose by induction $i < \lambda^+, N_i^1$ and $M_{i,c}$ for $c \in N_i^1 \backslash N_i^0$ such that:

- (a) $N_i^0 \leq_{\mathfrak{K}} N_i^1 \in K_\lambda$ and $N_i^0 \neq N_i^1$
- (b) N_i^1 is $\leq_{\mathfrak{K}}$-increasing continuous in i
- (c) $j < i \Rightarrow N_j^1 \cap N_i^0 = N_j^0$; moreover $N_i^1 \cap |N_0| = N_i^0$
- (d) $N_i^0 \leq_{\mathfrak{K}} M_{i,c} \leq_{\mathfrak{K}} N_i^1$
- (e) $c \notin M_{i,c}$
- (f) $N_i^0 \neq M_{i,c}$
- (g) if $j < i$ and $c \in N_j^1 \backslash N_j^0$ <u>then</u> $M_{i,c} \cap N_j^1 = M_{j,c}$.

<u>For $i = 0$</u>: Choose N_i^1 such that
$N_i^0 \leq_{\mathfrak{K}} N_i^1, (N_i^1, c)_{c \in N_i^0}$ saturated (any cofinality will do) then by disjoint amalgamation easy to define the $M_{0,c}$ (remembering clause (c)).

<u>For i limit</u>: Straightforward.

<u>For $i = j + 1$</u>: First we disjointly amalgamate getting $N_i' \in K_\lambda$ such that $N_i^0 \leq N_i', N_j^1 \leq_{\mathfrak{K}} N_i'$ and $|N_i'| \cap |N_0| = |N_i^0|$ (as set of elements).
Let N_i^1 be such that:

$$N_i' \leq_{\mathfrak{K}} N_i^1 \in K_\lambda$$
$$(N_i^1, c)_{c \in N_i'} \text{ is saturated (any cofinality will do)}$$
$$|N_i^1| \cap |N_0| = |N_i^0|.$$

Lastly we shall find the $M_{i,c}$'s, the point is that $(N^0_j, N^0_i, N^1_j) \in K^{\text{uniq}}_\lambda$ (by $\langle 6.2\rangle(2)$).
By $\bigotimes$ and Claim $\langle 6.2\rangle$ we could have done the amalgamation in two steps and use uniqueness. Then by uniqueness of saturated extensions embed the result inside N^1_i and similarly deal with new c's.

Now let $N_1 =: \bigcup\limits_{1<\lambda^+} N^1_i$ and for $c \in N_1 \backslash N_0$ let $M_c = \bigcup\{M_{i,c} : c \in N'_i\}$ they are as required. $\square_{\langle 6.3\rangle}$

Remark. The proof of $\langle 6.4\rangle$ below is like [Sh 88, proof of 3.8 stage(c)]. The aim is to contradict that under $I(\lambda^+, \mathfrak{K}) = 0$ there are maximal triples.

$\langle 6.4\rangle$ Conclusion. Assume $\mathfrak{K}$ has amalgamation in λ^+. If $\bigotimes$ of $\langle 6.3\rangle$, then there is no maximal triple (M, N, a) in $K^3_{\lambda^+}$.

Proof. We can get by $\langle 6.3\rangle$ a contradiction.
[Why? Assume $(N_0, N_2, a) \in K^3_\lambda$ maximal, (N^0, N^1) as in the conclusion (i.e. (a) + (b)) of $\langle 6.3\rangle$; by categoricity in λ^+ without loss of generality $N_0 = N^0$ and let $N_1 = N^1$. Now $\mathfrak{K}$ has amalgamation for λ^+ so there are $N \in K_{\lambda^+}$ and f such that $f : N_2 \to N$ is a $\leq_{\mathfrak{K}}$-embedding of N_2 into N over N_0 and $N_1 \leq_{\mathfrak{K}} N$. If $f(a) \notin N_1$, then $(N_0, N_2, a) <_f (N_1, N, f(a))$ contradict maximality. If $f(a) \in N_1$, then $M_{f(a)}$ is well defined (see $\langle 6.3\rangle$) and $(N_0, N_2, a) <_f (M_{f(a)}, N, f(a))$ contradicts maximality.] $\square_{\langle 6.4\rangle}$

$\langle 6.4A\rangle$ Remark. 1) Another proof is to replace the assumption "$\mathfrak{K}$ has amalgamation in λ^+" by $I(\lambda^+, K) < 2^{\lambda^{+2}}$. We start with N_0, N_1, N_2, a as above and building, for every $S \subseteq \lambda^{+2}$, a sequence $\langle M^S_\alpha : \alpha < \lambda^{+2}\rangle$ of members of K_{λ^+}, which is $\leq_{\mathfrak{K}}$-increasing continuous, and $\alpha \in S \Rightarrow (M^S_\alpha, M^S_{\alpha+1}, a^S_\alpha) \cong (N_0, N_2, a)$, and $\alpha \in \lambda^{+2}\backslash S \Rightarrow (M^S_\alpha, M^S_{\alpha+1}) \cong (N_0, N_1)$ which are as in (a) + (b) of $\langle 6.3\rangle$. Let $M^S = \bigcup\limits_{\alpha<\lambda^{+2}} M^S_\alpha \in K_{\lambda^{+2}}$ and from $M^S/\cong$ we can reconstruct $S/\mathcal{D}_{\lambda^{+2}}$. So here we use $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$ but not the definitional weak diamond for λ^{++}.
2) Note that if $2^{\lambda^+} < 2^{\lambda^+}$ then the assumption of $\langle 6.4\text{A}\rangle(1)$ implies the assumption of $\langle 6.4\rangle$.

$\langle 6.5\rangle$ Claim. *Assume*

$(*)$ $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$
(or at least the definitional weak diamond for λ^+, λ^{++}).

If $\bigotimes$ of $\langle 6.3\rangle$ fails, we get $I(\lambda^{+2}, K) \geq \mu_{\text{wd}}(\lambda)$.

Proof. By $\langle 3.17\text{B}\rangle$ and $\langle 6.5\text{A}\rangle$ below.

The following serves to prove $\langle 6.5\rangle$

⟨6.5A⟩ Claim. *Assume $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda$.*

If $K^{3,\mathrm{uq}}_\lambda = \emptyset$ (see Definition ⟨6.1⟩(2)), <u>then</u> there is F, an amalgamation choice function F for $\mathbf{C} = \mathbf{C}^0_{\mathfrak{K},\lambda^+}$ with the weak λ^+-coding property.

Proof. The point is that if $\bar M = \langle M_\alpha : \alpha < \lambda^+\rangle \in \mathbf{S}eq_{\lambda^+}[\mathbf{C}]$ and $\alpha < \lambda^+, M_\alpha <_{\mathfrak{K}} N \in K_\lambda$, <u>then</u> for some $\beta \in (\alpha, \lambda^+)$ we have:

> M_β is universal over M_α, so as $K^{3,\mathrm{uq}}_\lambda = \emptyset$ necessarily
> $(M_\alpha, M_\beta, N) \notin K^{\mathrm{uniq}}_\lambda$ and rest should be clear.

Of course, we use the extension property. $\square_{\langle 6.5A\rangle}$

—> MARTIN WARNS: Label 6.5B on next line is also used somewhere else (Perhaps should have used scite instead of stag?

⟨6.5B⟩ Remark. We can work in the context of §3, we need the existence of a saturated (equivalent by super limit) $M \in K_{\lambda^+}$. We now say how to replace $\mu_{\mathrm{wd}}(\lambda^{+2})$ by $2^{\lambda^{+2}}$.

—> MARTIN WARNS: Label 6.5B on next line is also used somewhere else (Perhaps should have used scite instead of stag?

⟨6.5B⟩ Claim. *1) Assume each $M \in K_{\lambda^+}$ is saturated above λ.*

If $(M, N, a) \in K^3_\lambda$ it and every $(M', N', a) \in K^3_\lambda$ above it has the extension property but for every $(M'', N'', a) \geq (N, N, a)$ (all in $K^3_{\lambda^+}$ for some $M^* \geq_{\mathfrak{K}} M''$ from K_{λ^+}, in amalgamation $(M^*, N^*, a) \geq (M'', N'', a)$ the type of $M^* \cup N$ inside N^* is not determined <u>then</u> some F (actually $\mathbf{F}^*$) has the λ^+-coding.
2) If above we just require that the type of $M^* \cup N''$ inside N^* is not determined, <u>then</u> some F (actually $\mathbf{F}^*$) has weak λ^+-coding.
3) We can restrict ourselves to disjoint embedding; i.e. use $(K^3_{\lambda^+}, \leq_{\mathrm{dj}})$.

⟨6.5C⟩ Discussion. 1) We get $IE(\lambda^{+2}, \mathfrak{K}) = 2^{\lambda^{+2}}$ when $(2^{\lambda^+})^+ < 2^{\lambda^{+2}}$. See more in [Sh 600].

⟨6.6⟩ Theorem. *Assume $I(\lambda^{+2}, K) < \mu_{\mathrm{wd}}(\lambda^{+2})$ and $(*)$ of ⟨6.5⟩ (or at least the conclusion of ⟨6.5⟩). Then $I(\lambda^{+2}, K) = 1 \Rightarrow I(\lambda^{+3}, K) > 0$.*

Remark. As in [Sh 88, §3].

Proof. By ⟨0.17⟩(1) it is enough to show that for some $M \in K_{\lambda^{++}}$ there is M', $M \leq_{\mathfrak{K}} M' \in K_{\lambda^{++}}, M \neq M'$.
[Why? As then we can choose by induction on $i < \lambda^{+3}$ models $M_i \in K_{\lambda^{+2}}$, $\leq_{\mathfrak{K}}$-increasing continuous, $M_i \neq M_{i+1}$, for $i = 0$ use $K_{\lambda^{+2}} \neq \emptyset$, for i limit take union, for $i = j+1$ use the previous sentence; so $M_{\lambda^{+3}} = \cup\{M_i : i < \lambda^{+3}\} \in K_{\lambda^{+3}}$ as required.]

By $\langle 6.5\rangle$, the statement $\bigotimes$ of $\langle 6.3\rangle$ holds so we can find (N^0, N^1) as there so by $\langle 6.4\rangle$ there is in $K^3_{\lambda^+}$ no maximal member. This implies (easy, see $\langle 2.4\rangle(6)$) that there are $M^* \leq_{\mathfrak{K}} N^*$ from $K_{\lambda^{+2}}$ such that $M^* \neq N^*$ which as said above (by categoricity in λ^+), suffices. $\square_{\langle 6.6\rangle}$

§7 Extensions and conjugacy

⟨*7.0*⟩ *Hypothesis.* Assume the model theoretic assumptions from ⟨4.0⟩ + ⟨5.1⟩ and the further model theoretic properties deduced since then (but not in ⟨6.5⟩,⟨6.6⟩) or just

- (a) $\mathfrak{K}$ abstract elementary class
- (b) $\mathfrak{K}$ has amalgamation in λ
- (c) $\mathfrak{K}$ is categorical in λ (can be weakened)
- (d) $\mathfrak{K}$ is stable in λ (see ⟨5.5⟩, clause (a))
- (e) there is an inevitable $p \in \mathcal{S}(N)$ for $N \in K_\lambda$ (holds by ⟨5.2⟩)
- (f) the basic properties in type theory.

We now continue toward eliminating the use of $I(\lambda^{++}, K) = 1$ (in ⟨6.6⟩), and give more information. We first deal with the nice types in $\mathcal{S}(N), N \in K_\lambda$ in particular the realize/materialize problem which is here: if $N_1 \leq_{\mathfrak{K}} N_2$ are in $K_\lambda, p_\ell \in \mathcal{S}(N_\ell)$ is minimal, $p_1 \leq p_2$ are they conjugate? (i.e. $p_2 \in \mathcal{S}_{p_1}(N_2)$).

⟨**7.1**⟩ **Claim.** *If $N \in K_\lambda$ and $p \in \mathcal{S}(N)$ is minimal and reduced or just p is reduced (see Definition ⟨2.3A⟩(7)), then p is inevitable.*

Proof. Suppose N, p form a counterexample. We can then find N_1 and a such that $N \leq_{\mathfrak{K}} N_1 \in K_\lambda, a \in N_1 \backslash N$ and $p = \mathrm{tp}(a, N, N_1)$ and (N, N_1, a) is reduced. As p is not inevitable, there is N_2 such that: $N \leq_{\mathfrak{K}} N_2 \in K_\lambda, N \neq N_2$ but no element of N_2 realizes p. By amalgamation in K_λ, without loss of generality there is $N_3 \in K_\lambda$ such that $\ell \in \{1, 2\} \Rightarrow N_\ell \leq_{\mathfrak{K}} N_3$. By ⟨5.2⟩ (i.e.⟨7.0⟩(e)) there is $q \in \mathcal{S}(N)$ which is inevitable so there are $c_\ell \in N_\ell$ with $q = \mathrm{tp}(c_\ell, N, N_\ell)$ for $\ell \in \{1, 2\}$. By the equality of types (and amalgamation in K_λ) there is $N^+ \in K$, a $\leq_{\mathfrak{K}}$-extension of N_1 and a $\leq_{\mathfrak{K}}$-embedding of N_2 into N^+ over N such that $f(c_2) = c_1$; so without loss of generality $N^+ = N_3$ and f is the identity, hence $c_1 = c_2$. Now $a \notin N_2$ as $p = \mathrm{tp}(a, N, N_1)$ is not realized in N_2. So $(N, N_1, a) \leq (N_2, N_3, a)$ and $N_2 \cap N_1 \backslash N \neq \emptyset$ contradicting "(N, N_1, a) is reduced". $\square_{\langle 7.1 \rangle}$

⟨**7.2**⟩ **Claim.** *1) If $\kappa = cf(\kappa) \leq \lambda$ and $\bar{N} = \langle N_i : i \leq \omega\kappa \rangle$ is an $\leq_{\mathfrak{K}}$-increasingly continuous sequence, $N_i \in K_\lambda, N_{i+1}$ universal over N_i, and $p \in \mathcal{S}(N_{\omega\kappa})$ is minimal reduced (or minimal inevitable) then for some $i < \omega\kappa$ we have $p \restriction N_i \in \mathcal{S}(N_i)$ is minimal (so p is the unique, non-algebraic extension of $p \restriction N_i$ in $\mathcal{S}(N_{\omega\kappa})$ (and of course, there is one)).*
2) If $\lambda \geq \kappa = cf(\kappa), \bar{N} = \langle N_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous in K_λ and $p \in \mathcal{S}(N_\kappa)$ is minimal and reduced and the set $Y =: \{i < \kappa : N_{i+1}$ is (λ, κ)-saturated over $N_i\}$ is unbounded in κ then for every large enough $i \in Y$ there is an isomorphism f from N_{i+1} onto N_κ which is the identity on N_i and[5]

- $(*)$ *f maps $p \restriction N_{i+1} \in \mathcal{S}(N_{i+1})$ to $p \in \mathcal{S}(N_\kappa)$.*

[5]in fact, this implies $(*)$

Hence as p is minimal reduced, so is $p \restriction N_{i+1}$.

Proof. 1) We can choose $(N^0_i, N^1_i, a) \in K^3_\lambda$ for $i < \lambda^+$ reduced, $\leq$-increasing continuous such that $N^0_i \neq N^0_{i+1}$. Let $N_\ell = \bigcup\limits_{i<\lambda^+} N^\ell_i$. As in the proof of $\langle 5.3 \rangle$ for $c \in N_1 \backslash N_0$

$$I^*_c = \{j < \lambda^+ : c \in N^1_j \text{ and } \mathrm{tp}(c, N^0_j, N^1_j) \text{ is minimal}\}$$

is empty or is an end segment of λ^+ and

$$E = \Big\{\delta < \lambda^+ : \underline{\text{if}}\ c \in N^1_\delta \text{ and } I^*_c \neq \emptyset\ \underline{\text{then}}\ I^*_c \cap \delta \text{ is an unbounded subset of } \delta; \text{and if } \alpha < \delta \text{ then for some } \beta \in (\alpha, \delta), N^0_\beta \text{ is universal over } N^0_\alpha \text{ and if } Pr \text{ is one of the properties reduced and/or inevitable and/or minimal and there is } i \geq \delta \text{ such that } (N^0_i, N^1_i, c) \text{ has } Pr, \text{ then there are arbitrarily large such } i < \delta\Big\}$$

is a club of λ^+; for the universality demand in the definition of E use categoricity in λ^+. Let $\delta \in \mathrm{acc}(\mathrm{acc}(E)), \mathrm{cf}(\delta) = \kappa$, let $\langle \alpha_\zeta : \zeta < \omega\kappa \rangle$ be an increasing continuous sequence of ordinals from E with limit δ, now set $\alpha_{\omega\kappa} = \delta$ and $N'_\zeta =: N^0_{\alpha_\zeta}$.

So there is an isomorphism f from $N_{\omega\kappa}$ onto $N^0_{\alpha_{\omega\kappa}}$ such that for every $\zeta < \omega\kappa$ we have $N^0_{\alpha_{2\zeta}} \leq_{\mathfrak{K}} f(N_{\alpha_{2\zeta}}) \leq_{\mathfrak{K}} N^0_{\alpha_{2\zeta+1}}$ (so if ζ is a limit ordinal, then $N^0_{\alpha_\zeta} = N^0_{\alpha_{2\zeta}} = f(N_\zeta)$), so without loss of generality f is the identity. As $p \in \mathcal{S}(N_{\omega\kappa})$ is inevitable (by assumption or by $\langle 7.1 \rangle$) and $N_{\omega\kappa} = N^0_{\alpha_{\omega\kappa}} <_{\mathfrak{K}} N^1_{\alpha_{\omega\kappa}}$, for some $c \in N^1_{\alpha_{\omega\kappa}} \backslash N^0_{\alpha_{\omega\kappa}}$ we have $p = \mathrm{tp}(c, N^0_{\alpha_{\omega\kappa}}, N^1_{\alpha_{\omega\kappa}})$, so for some $\beta < \alpha_{\omega\kappa}$ we have $c \in N^1_\beta$. As p is minimal (by assumption) clearly $\delta \in I_c$, but $\delta \in E$ so $\mathrm{Min}(I_c) < \delta$, but I_c is an end segment of λ^+ hence without loss of generality for some $\zeta < \omega\kappa$ we have $\beta = \alpha_\zeta \in I_c$. So for $\xi \in (\zeta, \omega\kappa)$, both $p \in \mathcal{S}(N_{\omega\kappa})$ and $p \restriction N_\xi \in \mathcal{S}(N_\xi)$ are non-algebraic extensions of the minimal $p \restriction N^0_{\alpha_\zeta} \in \mathcal{S}(N^0_{\alpha_\zeta})$ and $N^0_{\alpha_\zeta} \leq_{\mathfrak{K}} N_\xi \leq_{\mathfrak{K}} N_{\omega\kappa}$, all in K_λ, so we have proved part (1).

2) Without loss of generality every $\zeta < \kappa$ is in Y. We can find $\langle N'_\zeta : \zeta \leq \omega\kappa \rangle$ as in part (1), moreover satisfying "$N'_{\zeta+1}$ is (λ, κ)-saturated over N'_ζ" and such that: for every $\zeta \leq \kappa$ we have $N_\zeta = N'_{\omega\zeta}$. So again choose $\zeta < \kappa$ as there, we set $\beta = \alpha_{\omega\zeta} \in I^*_c$. If $\xi \in Y$ & $\omega\xi > \zeta$ clearly by the uniqueness of (λ, κ)-saturated models there is an isomorphic f from $N_{\xi+1} = N'_{\omega(\xi+1)}$ onto $N_\kappa = N'_{\omega\kappa}$ over $N_\xi = N'_{\omega\kappa}$, and $f(p \restriction N_{\xi+1}) = p$ is proved as above. $\square_{\langle 7.2 \rangle}$

⟨7.3⟩ Claim. *1) If $M_0 \leq_{\mathfrak{K}} M_1$ are in K_λ and the types $p_\ell \in \mathcal{S}(M_\ell)$ are minimal reduced, for $\ell = 0,1$ and $p_0 = p_1 \restriction M_0$ <u>then</u> p_0, p_1 are conjugate; (i.e. there is an isomorphic f from M_0 onto M_1 such that $f(p_0) = p_1$).*
2) If in addition $M \leq_{\mathfrak{K}} M_0$ and M_0, M_1 are (λ,κ)-saturated over M, <u>then</u> p_0, p_1 are conjugate over N.

Remark. Note that p minimal (or reduced) implies that p is not algebraic.

Proof. 1) Let $\langle (N^0_i, N^1_i, a) : i < \lambda^+\rangle$ and E be as in the proof of ⟨7.2⟩ and $\kappa = \operatorname{cf}(\kappa) \leq \lambda$. For each $\delta \in S_\kappa =: \{\alpha < \lambda^+ : \alpha \in E \text{ and } \operatorname{cf}(\alpha) = \kappa\}$, and minimal reduced $p \in \mathcal{S}(N^0_\delta)$, we know that for some $i_p < \delta, p \restriction N^0_{i_p}$ is minimal reduced [why? by ⟨7.2⟩(1),(2)] and some $q_p \in \mathcal{S}(N^0_{i_p})$ is conjugate to p. For $\kappa = \operatorname{cf}(\kappa) \leq \lambda, q \in \mathcal{S}(N^0_i), i < \lambda^+, r \in \mathcal{S}(N^0_i)$ minimal let

$$\begin{aligned} A^{\kappa,i}_{q,r} = \big\{\delta < \lambda^+ :\ & \text{there is a type } p \text{ such that } r \subseteq p \in \mathcal{S}(N^0_\delta), p \text{ non-algebraic} \\ & \text{(this determines } p), p \text{ minimal reduced, } i_p = i, q_p = q \\ & \text{(and clearly } p \restriction N^0_i = r)\big\}. \end{aligned}$$

Next let

$$\begin{aligned} E_1 = \Big\{\delta < \lambda^+ :\ & \text{for every } \kappa = \operatorname{cf}(\kappa) \leq \lambda, \\ & r, q \in \mathcal{S}(N^0_i) \text{ and } i < \delta, \ \underline{\text{if}}\ A^{\kappa,i}_{q,r} \text{ is well defined and} \\ & \text{unbounded in } \lambda^+ \ \underline{\text{then}} \text{ it is unbounded in } \delta\Big\}. \end{aligned}$$

So if $\delta_1 \in E_1, \kappa = \operatorname{cf}(\delta_1), p_1 \in \mathcal{S}(N^0_\delta)$ is minimal reduced, <u>then</u> we can find $\delta_0 < \delta_1, \operatorname{cf}(\delta_0) = \kappa$, and $p_0 \in \mathcal{S}(N^0_{\delta_0})$ minimal reduced with $q_{p_1} = q_{p_0}, i_{p_1} = i_{p_0}, p_0 \restriction N^0_{i_{p_0}} = p_1 \restriction N^0_{i_{p_1}}$ call it r, it is necessarily minimal.

As p_1, p_0 extend $r, N^0_{i_{p_0}} = N^0_{i_{p_1}} \leq_{\mathfrak{K}} N^0_{\delta_0} \leq_{\mathfrak{K}} N^0_{\delta_1}$, necessarily $p_1 = p_0 \restriction N^0_{\delta_0}$, and also they are both conjugate to $q_{p_0} = q_{p_1}$ hence they are conjugate.
Next we prove

(∗) if $M_0 <_{\mathfrak{K}} M_1$ are in K_λ, M_1 is (λ,κ)-limit over M_0, $p'_0 \in \mathcal{S}(M_0)$ is minimal reduced and $p'_0 \leq p'_1 \in \mathcal{S}(M_1), p'_1$ non-algebraic, <u>then</u> p'_0, p'_1 are conjugate.

Above we have a good amount of free choice in choosing $p_1 \in \mathcal{S}(N^0_{\delta_1})$ (it should be minimal and reduced) so we could have chosen p_1 to be conjugate to p'_0 (i.e. is in $\mathcal{S}_{p'_0}(N^0_{\delta_1})$; now also the corresponding p_0 is conjugate to p_1 hence p_0 is conjugate to p'_0, hence we can find an isomorphism f_0 from M_0 onto $N^0_{\delta_0}, f_0(p'_0) = p_0$, and extend it to an isomorphism f_1 from M_1 onto $N^0_{\delta_1}$, so necessarily $f(p'_1) = p_1$ (as p_1 is the unique non-algebraic extension). As p_0, p_1 are conjugate through f_1 also p'_0, p'_1 are conjugate. So (∗) holds.

Now assume just $M_0 \leq_{\mathfrak{K}} M_1$ are in $K_\lambda, p_0 \in \mathcal{S}(M_0)$ minimal reduced, $p_1 \in \mathcal{S}(M_1)$ the unique non-algebraic extension of p_0 and it is reduced (and necessarily minimal). There is $M_2, M_1 \leq_{\mathfrak{K}} M_2 \in K_\lambda, M_2$ is (λ,κ)-limit over M_1 hence also over M_0 and let p_2 be the unique non-algebraic extension of p_1 in $\mathcal{S}(M_2)$ hence p_2 is also the unique non-algebraic extension of p_0 in $\mathcal{S}(M_2)$.
Using $(*)$ on (M_0, M_2, p_0, p_2) and on (M_1, M_2, p_1, p_2) and get that p_0, p_2 are conjugate and that p_1, p_2 are conjugate hence p_1, p_2 are conjugate, the required result.
2) Similar proof. $\square_{\langle 7.3\rangle}$

$\langle$7.4A$\rangle$ Claim. *1) Assume $M_1 \leq_{\mathfrak{K}} M_2$ are in K_λ and M_2 is (λ,κ)-saturated over M_1. If $p_1 \in \mathcal{S}(M_1)$ is minimal and reduced, <u>then</u> p_2, the unique non-algebraic extension of p_1 in $\mathcal{S}(M_2)$, is reduced (and, of course, minimal).*
2) There is no need to assume "p_1 reduced.

Proof. 1) We can find $\langle N_i : i \leq \kappa\rangle$, an $\leq_{\mathfrak{K}}$-increasingly continuous sequence in K_λ such that N_{i+1} is (λ,κ)-saturated over N_i and $N_\kappa = M_1$. So by $\langle$**7.x**$\rangle$, for some
—> scite{7.x} undefined
$\zeta < \kappa$ we hve: $p_2 \restriction N_\zeta$ is minimal and for some isomorphism f from $N_{\zeta+1}$ onto N_κ we have $f(p_1 \restriction N_\zeta) = p_1$ and $f \restriction N_\zeta = \mathrm{id}_{N_\zeta}$. Also $M_1, N\zeta+1$ are isomorphic over N_ζ (as both are (λ,κ)-saturated over it) hence there is an isomorphism g from $N_{\zeta+1}$ onto M_2 over N_ζ. Now $p_1 = f(p_1 \restriction N_{\zeta+1})$ and $f_2 =: g(p_1 \restriction N_{\zeta+1})$ are non-algebraic extensions of $p_1 \restriction N_\zeta$ which is minimal, hence $p_1 = p_2 \restriction M_1$ and p_2 is as mentioned in $\langle$7.4A$\rangle$. Now $g \circ f^{-1}$ show that p_1, p_2 are conjugate so as p_1 is reduced also p_2 is reduced.
2) Easy as we can find $N, M_1 \leq_K N, q \in \mathcal{S}(N)$ minimal reduced; without loss of generality $N \leq_{\mathfrak{K}} M_2$ and M_2 is (λ,κ)-saturated over N, and apply part (1). $\square_{\langle 7.4A\rangle}$

$\langle$7.4B$\rangle$ Claim. *Assume*

(a) *$N_{i,j} \in K_\lambda$ for $i \leq \delta_1, j \leq \delta_2$*
(b) *$\langle N_{i,j} : j \leq \delta_2\rangle$ is $\leq_{\mathfrak{K}}$-increasingly continuous for each $i \leq \delta_1$*
(c) *$\langle N_{i,j} : i < \delta_1\rangle$ is $\leq_{\mathfrak{K}}$-increasingly continuous for each $j \leq \delta_2$*
(d) *$\langle N_{i,j} : i \leq \delta_1, j \in \delta_2\rangle$ is smooth, i.e. $N_{i_1,j_1} \cap N_{i_2,j_2} = N_{\min\{i_1,i_2\}} \cap N_{\min\{j_1,j_2\}}$*
(e) *$N_{i+1,j+1}$ is universal over $N_{i,j+1} \cup N_{i+1,j}$ (i.e. $N_{i+1,j+1}$ is universal over some $N'_{i+1,j+1}$ where $N_{i,j+1} \cup N_{i+1,j} \subseteq N'_{i+1,j+1}$,*
(f) *δ_1 is divisible by*
$cf(\delta_2) \times \lambda \times \omega$
(and even easier if $\delta_1 = 1$!).

<u>Then</u> *N_{δ_1,δ_2} is $(\lambda, cf(\delta_1))$-saturated over N_{i,δ_1} for $i < \delta_1$.*

Proof. Without loss of generality $\delta_2 = \mathrm{cf}(\delta_2)$. (Why? let $\langle \alpha_\varepsilon : \varepsilon \leq \mathrm{cf}(\delta_1)\rangle$ be increasingly continuous with limit δ_1 such that [ε limit $\leftrightarrow \alpha_\varepsilon$ limit], and use $N'_{i,\varepsilon} = N_{i,\alpha_\varepsilon}$).

For $i < \delta_1, j < \delta_2$ let $M_{i,j}, M'_{i,j}$ be such that $M_{i+1,j} \cup M_{i,j+1} \subseteq M_{i,j} \leq_{\mathfrak{K}} M'_{i,j} \leq M_{i+1,j+1}$ and $M'_{i,j}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $M_{i,j}$.

Now let $p \in \mathcal{S}(N)$ be minimal and reduced; and for $i \leq \delta_1, j \leq \delta_2$ let $p_{i,j} \in \mathcal{S}(N_{i,j})$ be the unique non-algebraic extension of p in $\mathcal{S}(N_{i,j})$ so it is minimal. Now for $i < \delta_1$, note that $\langle M'_{i+\varepsilon,\varepsilon} : \varepsilon < \delta_2 \rangle$ is $\leq_{\mathfrak{K}}$-increasing (not continuous!) and $M'_{i+\varepsilon+1,\varepsilon+1}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $M'_{i+\varepsilon,\varepsilon}$ and $\bigcup\limits_{\varepsilon<\delta} M'_{i+\varepsilon,\varepsilon} = \bigcup\limits_{\varepsilon<\delta_1} N_{i+1,\varepsilon+1} = N_{i+\delta_2,\delta_2}$, and $N_{0,0} \leq_{\mathfrak{k}} M'_{1,0}$, hence by $\langle 7.4\mathrm{A} \rangle$ we know that $p_{i+\delta_2,\delta_2}$ is reduced (and minimal). In fact, similarly $\alpha < \delta_1 \ \ \& \ \ \mathrm{cf}(\alpha) = \mathrm{cf}(\delta_2) \Rightarrow p_{\alpha,\delta_2}$ is reduced. As $N_{i+1,j+1} \neq N_{i+1,j} \cup N_{i,j+1}$ and clause (d) (smoothness) necessarily $N_{i+\delta_2,\delta_2} < N_{i+\delta_2+1,\delta_2}$, hence some $c \in N_{i+\delta_2+i,\delta_2} \backslash N_{i+\delta_2,\delta_2}$ realizes $p_{i+\delta_2,\delta_2}$. So if $\alpha \leq \delta_1$ is divisible by $\delta_2 \times \lambda$ and has cofinality $\mathrm{cf}(\delta_2)$ and $\beta < \alpha$, then by $\langle 5.4 \rangle$ N_{α,δ_2} is universal over N_{β,δ_2}. As δ_1 is divisible by $\mathrm{cf}(\delta_2) \times \lambda \times \omega$ we are done. $\square_{\langle 7.4B \rangle}$

$\langle$**7.5**$\rangle$ **Lemma.** *1) For every $N \in K_{\lambda^+}$ we can find a representation $\bar{N} = \langle N_i : i < \lambda^+ \rangle$, with N_{i+1} being $(\lambda, cf(\lambda))$-saturated over N_i.*
2) If for $\ell = 1,2$ we have $N^\ell = \langle N^\ell_i : i < \lambda^+ \rangle$ as in part (1) then there is an isomorphism f from N^1 onto N^2 mapping N^1_i onto N^2_i for each $i < \lambda^+$. Moreover for any $i < \lambda^+$ and isomorphism g from N^1_i onto N^2_i we can find an isomorphism f from N^1 onto N^2 extending g and mapping N^1_j onto N^2_j for each $j \in [i, \lambda^+)$.
3) If $N^0 \leq_{\mathfrak{K}} N^1$ are in K_{λ^+} then we can find representation $\bar{N}^\ell$ of N^ℓ as in (1) with $N^0_i = N^0 \cap N^1_i$, (so $N^0_i \leq_{\mathfrak{K}} N^1_i$).
4) For any strictly increasing function $\mathbf{f} : \lambda^+ \to \lambda^+$, we can find $N_{i,\varepsilon}$ for $i < \lambda^+, \varepsilon \leq \lambda \times (1 + \mathbf{f}(i))$ such that:

- (a) $N_{i,\varepsilon} \in K_\lambda$
- (b) $\langle N_{i,\varepsilon} : \varepsilon \leq \lambda \times (1 + \mathbf{f}(i)) \rangle$ *is strictly $\leq_{\mathfrak{K}}$-increasing continuous*
- (c) *for each $\varepsilon, \langle N_{i,\varepsilon} : i \in [i_\varepsilon, \lambda^+) \rangle$ is a representation as in (1) where $i_\varepsilon =$ $Min\{i : \varepsilon \leq \lambda \times (1 + \mathbf{f}(i))\}$*
- (d) *if $\varepsilon < \lambda \times (1 + f(i))$ and $i < j < \lambda^+$ then $N_{j,\varepsilon} \cap N_{i,\lambda \times (1+\mathbf{f}(i))} = N_{i,\varepsilon}$*
- (e) *$N_{i+1,\varepsilon+1}$ is $(\lambda, \aleph_0)$-saturated over $N_{i+1,\varepsilon} \cup N_{i,\varepsilon+1}$.*

Proof. Straight.
4) First use $\mathbf{f}' : \lambda^+ \to \lambda^+$ which is $\mathbf{f}'(i) = \lambda^\omega \times \mathbf{f}(i)$. Then define the $N_{i,\varepsilon}$ ($\varepsilon < \lambda \times (1 + \mathbf{f}'(i); i < \lambda^+)$. "Forget" about "$N_{i+1,\varepsilon}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $N_{i,\varepsilon}$", remember we have disjoint amalgamation by $\langle 5.7 \rangle$. Now by $\langle 7.4\mathrm{B} \rangle$, even for ε limit divisible by λ^3 we get $N_{i+\lambda,\varepsilon}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $N_{i,\varepsilon}$, so renaming all is O.K.). $\square_{\langle 7.5 \rangle}$

We can deduce from $\langle 7.4 \rangle$, but to keep the door open to other uses we shall not use

$\langle$**7.6**$\rangle$ **Claim.** *If $\kappa_\ell =$ $cf(\kappa_\ell) \leq \lambda$, and N_ℓ is (λ, κ_ℓ)-saturated over N for $\ell = 1,2$ then N_1, N_2 are isomorphic over N.*

Proof. We can define by induction on $i \leq \lambda \times \kappa_1$, and then by induction on $j \leq \lambda \times \kappa_2$, $M_{i,j}$ such that:

(a) $M_{i,j} \in K_\lambda$

(b) $M_{0,0} = N$

(c) $i_1 \leq i$ & $j_1 \leq j \Rightarrow M_{i_1,j_1} \leq_{\mathfrak{K}} M_{i,j}$

(d) $M_{i_1,j_1} \cap M_{i_2,j_2} = M_{\min\{i_1,i_2\},\min\{j_1,j_2\}}$

(e) $M_{i,j}$ is $\leq_{\mathfrak{K}}$-increasing continuous in i

(f) $M_{i,j}$ is $<_{\mathfrak{K}}$-increasing continuous in j

(g) $M_{0,j} \neq M_{0,j+1}$

(h) $M_{i+1,0} \neq M_{i,0}$

(i) $M_{i+1,j+1}$ is universal over $M_{i+1,j} \cup M_{i,j+1}$.

There is no problem by $\langle 5.7 \rangle(1)$ (using the existence of disjoint amalgamation). Now $M_{\lambda\times\kappa_1,\lambda\times\kappa_2}$ is the union of the strictly $<_{\mathfrak{K}}$-increasing sequence $\langle M_{0,0}\rangle\hat{}\langle M_{\lambda\times i,\lambda\times\kappa_2} : i < \kappa_1\rangle$ hence by $\langle 7.4\text{B}\rangle$ is (λ, κ_1)-saturated over $M_{0,0} = N$ hence $M_{\lambda\times\kappa_1,\lambda\times\kappa_2} \cong_N N_1$. Similarly $M_{\lambda\times\kappa_1,\lambda\times\kappa_2}$ is the union of the strictly $\leq_{\mathfrak{K}}$-increasing sequence $\langle M_{0,0}\rangle\hat{}\langle M_{\lambda\times\kappa_1,\lambda\times j} : j < \kappa_2\rangle$ hence is (λ, κ_2)-saturated over $M_{0,0} = N$, hence $M_{\lambda\times\kappa_1,\lambda\times\kappa_2} \cong_N N_2$. Together N_1, N_2 are isomorphic over N.

$\square_{\langle 7.6\rangle}$

§8 Uniqueness of amalgamation in $\mathfrak{K}_\lambda$

We deal in this section only with K_λ.
We want to, at least, approximate unique amalgamation using as starting point $\bigotimes$ of $\langle 6.3\rangle$ (see also $\langle 6.5\rangle$), i.e. $K^{3,\mathrm{uq}}_\lambda \neq \emptyset$.

$\langle 8.0\rangle$ Hypothesis. 1) Assume hypothesis $\langle 7.0\rangle$, so

- (a) $\mathfrak{K}$ abstract elementary class
- (b) $\mathfrak{K}$ has amalgamation in λ
- (c) $\mathfrak{K}$ is categorical in λ (can be weakened)
- (d) $\mathfrak{K}$ is stable in λ (see $\langle 5.5\rangle$, clause (a))
- (e) there is an inevitable $p \in \mathcal{S}(N)$ for $N \in K_\lambda$ (holds by $\langle 5.2\rangle$)
- (f) the basic properties in type theory.

2) (M^*, N^*) is some pair in
$K^{3,\mathrm{uq}}_\lambda = \{(M_0, M_2) : M_0 \leq_{\mathfrak{K}} M_2$ are in K_λ and for every $M_1, M_0 \leq_{\mathfrak{K}} M_1 \in K_\lambda \Rightarrow (M_0, M_1, M_2) \in K^{\mathrm{uniq}}_\lambda$; equivalently for some $M_1, (M_0, M_1, M_2)$ are as in $\bigotimes$ of $\langle 6.3\rangle\}$ (eventually the choice does not matter; if each time instead of $\cong (M^*, N^*)$ we write $\in K^{2,\mathrm{uq}}_\lambda$, see $\langle 8.9\rangle$; but if we start with this definition then the uniqueness theorems will be more cumbersome).

$\langle 8.1\rangle$ Definition. Assume $\bar\delta = \langle \delta_1, \delta_2, \delta_3\rangle, \delta_2$ a limit ordinal $< \lambda^+$ but δ_1, δ_3 are $< \lambda^+$ and may be 1. We say that $NF_{\lambda,\bar\delta}(N_0, N_1, N_2, N_3)$ (we say N_1, N_2 are saturated and smoothly amalgamated in N_3 over N_0 for $\bar\delta$) when:

- (a) $N_\ell \in K_\lambda$ for $\ell \in \{0, 1, 2, 3\}$
- (b) $N_0 \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} N_3$ for $\ell = 1, 2$
- (c) $N_1 \cap N_2 = N_0$ (i.e. in disjoint amalgamation)
- (d) N_1 is $(\lambda,\mathrm{cf}(\delta_1))$-saturated over N_0
- (e) N_2 is $(\lambda,\mathrm{cf}(\delta_2))$-saturated over N_0, if $\delta_1 = 1$ this means just $N_0 \leq_{\mathfrak{K}} N_2$
- (f) there are $N_{1,i}, N_{2,i}$ for $i \leq \lambda \times \delta_1$ (called the witness) such that:
  - (α) $N_{1,0} = N_0, N_{1,\lambda\times\delta_1} = N_1$
  - (β) $N_{2,0} = N_2$
  - (γ) $\langle N_{\ell,i} : i \leq \lambda \times \delta_1\rangle$ is $<_{\mathfrak{K}}$-increasing continuous for $\ell = 1, 2$
  - (δ) $(N_{1,i}, N_{1,i+1}) \cong (M^*, N^*)$
  - (ε) $N_{2,i} \cap N_1 = N_{1,i}$
  - (ζ) N_3 is $(\lambda,\mathrm{cf}(\delta_3))$-saturated over $N_{2,\lambda\times\delta_1}$; if $\delta_3 = 1$ this means just $N_{2,\lambda\times\delta_1} \leq_{\mathfrak{K}} N_3$

<u>Discussion</u>: Why this definition of NF? We need a nonforking notion with the usual properties. We first describe a version depending on $\langle \delta_0, \delta_1, \delta_2 \rangle$ and get $NF = NF_{\lambda,\bar{\delta}}$, $\bar{\delta}$ works like a scaffold – eventually $\bar{\delta}$ disappears.

$\langle$**8.2**$\rangle$ **Definition.** 1) We say $N_1 \bigcup\limits_{N_0}^{N_3} N_2$ (or N_1, N_2 are smoothly amalgamated over N_0 inside N_3 or $NF_\lambda(N_0, N_1, N_2, N_3)$) <u>if</u> we can find $M_\ell \in K_\lambda$ (for $\ell < 4$) such that:

- (a) $NF_{\lambda,\langle \lambda,\lambda,\lambda \rangle}(M_0, M_1, M_2, M_3)$
- (b) $N_\ell \leq_{\mathfrak{K}} M_\ell$ for $\ell < 4$
- (c) $N_0 = M_0$
- (d) M_1, M_2 are $(\lambda, \mathrm{cf}(\lambda))$-saturated over N_0 (follows by (a) see clauses (d), (e) of $\langle 8.1 \rangle$).

$\langle$**8.3**$\rangle$ **Claim.** *1) If $\bar{\delta} = \langle \delta_1, \delta_2, \delta_3 \rangle, \delta_\ell$ a limit ordinal $< \lambda^+$ and $N_\ell \in K_\lambda$ for $\ell < 3$, and N_1 is $(\lambda,\ cf(\delta_1))$- saturated over N_0 and N_2 is $(\lambda, cf(\delta_2))$-saturated over N_0 and $N_0 \leq_{\mathfrak{K}} N_1, N_0 \leq_{\mathfrak{K}} N_2$ and for simplicity $N_1 \cap N_2 = N_0$. <u>Then</u> we can find N_3 such that $NF_{\lambda,\bar{\delta}}(N_0, N_1, N_2, N_3)$.*
2) Moreover, we can choose any $\langle N_{1,i} : i \leq \lambda \times \delta_1 \rangle$ as in $\langle 8.1 \rangle (f)(\alpha), (\gamma), (\delta)$ as part of the witness.

Proof. Straight (remembering $\langle 7.4 \rangle$ (and uniqueness of the $(\lambda, \mathrm{cf}(\delta_1))$-saturated model over N_0)). $\square_{\langle 8.3 \rangle}$

$\langle$**8.4**$\rangle$ **Claim.** *If Definition $\langle 8.1 \rangle$, if δ_3 is a limit ordinal, <u>then</u> without loss of generality (even without changing $\langle N_{1,i} : i \leq \lambda \times \delta_1 \rangle$)*

- (g) *$N_{2,i+1}$ is $(\lambda, cf(\delta_2))$-saturated over $N^1_{i+1} \cup N^2_i$ (which means it is $(\lambda, cf(\delta_2))$-saturated over some N, where $N^1_{i+1} \cup N^2_i \subseteq N \leq_{\mathfrak{K}} N_{2,i+1}$).*

Proof. So assume $NF_{\lambda,\bar{\delta}}(N_0, N_1, N_2, N_3)$ holds as witnessed by $\langle N_{\ell,i} : i \leq \lambda \times \delta_\ell \rangle$ for $\ell = 1, 2$. Now we choose by induction on $i \leq \lambda \times \delta_1$ a model $M_{2,i} \in K_\lambda$ such that:

- (i) $N_{2,i} \leq M_{2,i}$
- (ii) $M_{2,0} = N_2$
- (iii) $M_{2,i}$ is $\leq_{\mathfrak{K}}$-increasing continuous
- (iv) $M_{2,i} \cap N_{2,\lambda \times \delta_1} = N_{2,i}$ moreover $M_{2,i} \cap N_3 = N_{2,i}$
- (v) $M_{2,i+1}$ is $(\lambda, \mathrm{cf}(\delta_2))$-saturated over $M_{2,i} \cup N_{2,i+1}$.

There is no problem to carry the definition. Let M_3 be such that $M_{2,\lambda\times\delta_1} \leq_{\mathfrak{K}} M_3 \in K_\lambda$ and M_3 is $(\lambda,\text{cf}(\delta_3))$-saturated over $M_{2,\lambda\times\delta_1}$. So both M_3 and N_3 are $(\lambda,\text{cf}(\delta_3))$-saturated over $N_{2,\lambda\times\delta_1}$, hence they are isomorphic over $N_{2,\lambda\times\delta_1}$ so let f be an isomorphism from M_3 onto N_3 which is the identity over $N_{2,\lambda\times\delta_1}$.
Clearly $\langle N_{1,i} : i \leq \lambda\times\delta_1\rangle, \langle f(M_{2,i}) : i \leq \lambda\times\delta_1\rangle$ are also witnesses for $NF_{\lambda,\bar\delta}(N_0,N_1,N_2,N_3)$ satisfying the extra demand (g). $\square_{\langle 8.4\rangle}$

⟨8.5⟩ Claim. *(Weak Uniqueness).*
Assume that for $x \in \{a,b\}$, we have $NF_{\lambda,\bar\delta^x}(N^x_0,N^x_1,N^x_2,N^x_3)$ holds as witnessed by $\langle N^x_{1,i} : i \leq \lambda\times\delta^x_1\rangle, \langle N^x_{2,i} : i \leq \lambda\times\delta^x_1\rangle$ and $\delta_1 =: \delta^a_1 = \delta^b_1$, $cf(\delta^a_2) = cf(\delta^b_2)$ and $cf(\delta^a_3) = cf(\delta^b_3) \geq \aleph_0$.

Suppose further that f_ℓ is an isomorphism from N^a_ℓ onto N^b_ℓ for $\ell = 0,1,2$, moreover: $f_0 \subseteq f_1, f_0 \subseteq f_2$ and $f_1(N^a_{1,i}) = N^b_{1,i}$.
<u>Then</u> we can find an isomorphism f from N^a_3 onto N^b_3 extending $f_1 \cup f_2$.

Proof. Without loss of generality $N^x_{2,i+1}$ is $(\lambda,\text{cf}(\delta_2))$-saturated over $N^x_{1,i+1} \cup N^x_{2,i}$ (by ⟨8.4⟩, note "without changing the $N_{1,i}$'s). Now we choose by induction on $i \leq \lambda\times\delta_1$ an isomorphism g_i from $N^a_{2,i}$ onto $N^b_{2,i}$ such that: g_i is increasing in i and g_i extends $(f_1 \restriction N^a_{1,i}) \cup f_2$.
For $i = 0$ choose $g_0 = f_2$ and for i limit let g_i be $\bigcup_{j<i} g_j$ and for $i = j+1$ use $(N_{1,i},N_{1,i+1}) \cong (M^*,N^*)$ (see ⟨8.1⟩) and the extra saturation clause (g). Now we can extend $g_{\lambda\times\delta_1}$ to an isomorphism from N^a_3 onto N^b_3 as N^x_3 is $(\lambda,\text{cf}(\delta_3))$-saturated from $N^x_{2,\lambda\times\delta_1}$ (for $x \in \{a,b\}$); note that knowing ⟨8.5⟩ possibly the choice of $\langle N_{1,i} : i \leq \lambda\times\delta_1\rangle$ matters. $\square_{\langle 8.5\rangle}$

Now we prove an "inverted" uniqueness

⟨8.6⟩ Claim. *Suppose that*

(a) *for $x \in \{a,b\}$ we have $NF_{\lambda,\bar\delta^x}(N^x_0,N^x_1,N^x_2,N^x_3)$*
(b) *$\bar\delta^x = \langle\delta^x_1,\delta^x_2,\delta^x_3\rangle, \delta^a_1 = \delta^b_2$, $\delta^a_2 = \delta^b_1, cf(\delta^a_3) = cf(\delta^b_3)$ all limit ordinals*
(c) *f_0 is an isomorphism from N^a_0 onto N^b_0*
(d) *f_1 is an isomorphism from N^a_1 onto N^b_2*
(e) *f_2 is an isomorphism from N^a_2 onto N^b_1*
(f) *$f_0 \subseteq f_1$ and $f_0 \subseteq f_2$.*

<u>Then</u> there is an isomorphism from N^a_3 onto N^b_3 extending $f_1 \cup f_2$.

Before proving

⟨8.6A⟩ Subclaim. *1) For any limit ordinals $\delta_1^a, \delta_2^a, \delta_3^a \le \lambda$ we can find $M_{i,j}$ (for $i \le \lambda \times \delta_1^a$ and $j \le \lambda \times \delta_2^a$) and M_3 such that:*

(A) $M_{i,j} \in K_\lambda$

(B) $i_1 \le i_2 \ \& \ j_1 \le j_2 \Rightarrow M_{i_1,j_1} \le_{\mathfrak{K}} M_{i_2,j_2}$

(C) *if $i \le \lambda \times \delta_1$ is a limit ordinal and $j \le \lambda \times \delta_2$ then* $M_{i,j} = \bigcup\limits_{\zeta<i} M_{\zeta,j}$

(D) *if $i \le \lambda \times \delta_1$ and $j \le \lambda \times \delta_2$ is a limit ordinal then* $M_{i,j} = \bigcup\limits_{\xi<j} M_{i,\xi}$

(E) $(M_{0,j}, M_{0,j+1}) \cong (M^*, N^*)$

(F) $(M_{i,0}, M_{i+1,0}) \cong (M^*, N^*)$

(G) *for $i_1, i_2 \le \lambda \times \delta_1$ and $j_1, j_2 \le \lambda \times \delta_2$ we have*
$M_{i_1,j_1} \cap M_{i_2,j_2} = M_{\min\{i_1,i_2\},\min\{j_1,j_2\}}$,

(H) $M_{\lambda\times\delta_1,\lambda\times\delta_2} \le_{\mathfrak{K}} M_3 \in K_\lambda$ *moreover*
M_3 *is* $(\lambda, cf(\delta_3))$*-saturated over* $M_{\lambda\times\delta_1,\lambda\times\delta_2}$.

2) Moreover, it is O.K. if $\langle M_{0,j} : j \le \lambda \times \delta_2^a\rangle, \langle M_{i,0} : i \le \lambda \times \delta_1^a\rangle$ are pregiven as long as both $\le_{\mathfrak{K}}$-increasing continuous in $\mathfrak{K}_\lambda$ satisfying (E) + (F) and $M_{0,\lambda\times\delta_2^a} \cap M_{\lambda\times\delta_2^a,0} = M_{0,0}$.

Proof. 1) This is done by induction on i and for fixed i by induction on j. For $i = 0, j = 0$ let $M_{0,0} \in K_\lambda$ be arbitrary, for $i = 0, j$ limit use clause (D), for $i = 0, j = \xi + 1$ use clause (E). For i limit use clause (C) (and check). For $i = \zeta + 1$, if $j = 0$ use clause (F), if j limit use clause (D) and if $j = \xi + 1$ use the existence of disjoint amalgamation (i.e. ⟨5.7⟩).

Lastly, choose $M_3 \in K_\lambda$ which is $(\lambda, \mathrm{cf}(\delta_3^a))$-saturated over $M_{\lambda\times\delta_1^a,\lambda\times\delta_2^a}$.
2) Similarly. $\square_{\langle 8.6A\rangle}$

Proof of ⟨8.6⟩. Let $M_{i,j}, M_3$ be as in ⟨8.6A⟩. For $x \in \{a, b\}$ as $NF_{\lambda,\bar{\delta}^x}(N_0^x, N_1^x, N_2^x, N_3^x)$, we know that there are witnesses $\langle N_{1,i}^x : i \le \lambda \times \delta_1^x\rangle, \langle N_{2,i}^x : i \le \lambda \times \delta_1^x\rangle$ for this, so $\langle N_{1,i}^x : i \le \lambda \times \delta_1^x\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous and $(N_{1,i}^x, N_{1,i+1}^x) \cong (M^*, N^*)$.
So $\langle N_{1,i}^a : i \le \lambda \times \delta_1^a\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous sequences with each successive pair isomorphic to (M^*, N^*) hence by ⟨8.6A⟩(2) without loss of generality there is an isomorphism g_1 from $N_{1,\lambda\times\delta_1^a}^a$ onto $M_{\lambda\times\delta_1^a}$, mapping $N_{1,i}^a$ onto $M_{i,0}$; remember $N_{1,\lambda\times\delta_1^a}^a = N_1^a$. Let $g_0 = g_1 \restriction N_0^a = g_1 \restriction N_{1,0}^a$ so $g_0 \circ f_0^{-1}$ is an isomorphism from N_0^b onto $M_{0,0}$.

As $\delta_1^b = \delta_2^a$, using ⟨8.6A⟩(2) fully without loss of generality there is an isomorphism g_2 from $N_{1,\lambda\times\delta_2^a}^b$ onto $M_{0,\lambda\times\delta_2^a}$ mapping $N_{1,j}^b$ onto $M_{0,j}$ (for $j \le \lambda \times \delta_2^a$) and g_2 extends $g_0 \circ f_0^{-1}$.
Now we want to use the weak uniqueness ⟨8.5⟩ and for this note:

(α) $NF_{\lambda,\bar{\delta}^a}(N_0^a, N_1^a, N_2^a, N_3^a)$ as witnessed by $\langle N_{1,i}^a : i \le \lambda \times \delta_1^a\rangle$,
$\langle N_{2,i}^a : i \le \lambda \times \delta_1\rangle$
[why? an assumption]

(β) $NF_{\lambda,\bar{\delta}^a}(M_{0,0}, M_{\lambda\times\delta_1^a,0}, M_{0,\lambda\times\delta_2^a}, M_3)$ as witnessed by the sequences $\langle M_{i,0} : i \le \lambda \times \delta_1\rangle, \langle M_{i,\lambda\times\delta_2^a} : i \le \lambda \times \delta_2^a\rangle$
[why? check]

(γ) g_0 is an isomorphism from N_0^a onto $M_{0,0}$
[why? see its choice]

(δ) g_1 is an isomorphism from N_1^a onto $M_{\lambda\times\delta_1^a,0}$ mapping $N_{1,i}^a$ onto $M_{i,0}$ for $i \le \lambda \times \delta_1^a$ and extending g_0
[why? see the choice of g_1 and of g_0]

(ε) $g_2 \circ f_2$ is an isomorphism from N_2^a onto $M_{0,\lambda\times\delta_2^a}$ extending g_0
[why? f_2 is an isomorphism from N_2^a onto N_1^b and g_2 is an isomorphism from N_1^b onto $M_{0,\lambda\times\delta_1^a}$ extending $g_0 \circ f_0^{-1}$ and $f_0 \subseteq f_2$].

So there is by $\langle 8.5\rangle$ an isomorphism g_3^a from N_3^a onto M_3 extending g_1 and $g_2 \circ f_2$.

We next want to apply $\langle 8.5\rangle$ to the N_1^b's; so note:

(α)′ $NF_{\lambda,\bar{\delta}^b}(N_0^b, N_1^b, N_2^b, N_3^b)$ as witnessed by $\langle N_{1,i}^b : i \le \lambda \times \delta_2^a\rangle$, $\langle N_{2,i}^b : i \le \lambda \times \delta_2^a\rangle$

(β)′ $NF_{\lambda,\bar{\delta}^b}(M_{0,0}, M_{0,\lambda\times\delta_2^a}, M_{\lambda\times\delta_1^a,0}, M_3)$ as witnessed by the sequences $\langle M_{0,j} : j \le \lambda \times \delta_2^a\rangle, \langle M_{\lambda\times\delta_1^a,j} : j \le \lambda \times \delta_1^a\rangle$

(γ)′ $g_0 \circ (f_0)^{-1}$ is an isomorphism from N_0^b onto $M_{0,0}$
[why? Check.]

(δ)′ g_2 is an isomorphism from N_1^b onto $M_{0,\lambda\times\delta_2^a}$ mapping $N_{1,j}^b$ onto $M_{0,j}$ for $j \le \lambda \times \delta_2^a$ and extending $g_0 \circ (f_1)^{-1}$
[why? see the choice of g_2: it maps $N_{1,j}^b$ onto $M_{0,j}$]

(ε)′ $g_1 \circ (f_1)^{-1}$ is an isomorphism from N_2^b onto $M_{\lambda\times\delta_0^a}$ extending g_0
[why? remember f_1 is an isomorphism from N_1^a onto N_2^b extending f_0 and the choice of g_1: it maps N_1^a onto $M_{\lambda\times\delta_{1,0}^a}$].

So there is an isomorphism g_3^b form N_3^b onto M_3 extending $g_2, f_1 \circ (f_1)^{-1}$.
Lastly $(g_3^b)^{-1} \circ g_3^a$ is an isomorphism from N_3^a onto N_3^b (chase arrows). Also

$$\begin{aligned}((g_b^b)^{-1} \circ g_3^a) \restriction N_1^a &= (g_3^b)^{-1}(g_3^a \restriction N_1^a)\\ &= (g_3^b)^{-1} g_1 = ((g_3^b)^{-1} \restriction M_{\lambda\times\delta_{1,0}^a}) \circ g_1\\ &= (g_3^b \restriction N_2^b)^{-1} \circ g_1 = ((g_1 \circ (f_1)^{-1})^{-1}) \circ g_1\\ &= (f_1 \circ (g_1)^{-1}) \circ g_1 = f_1.\end{aligned}$$

Similarly $((g_3^b)^{-1} \circ g_3^a) \restriction N_2^a = f_2$.
So we have finished. $\square_{\langle 8.6\rangle}$

⟨8.7⟩ Claim. *[Uniqueness]. Assume for $x \in \{a,b\}$ we have $NF_{\lambda,\bar\delta^x}(N^x_0,N^x_1,N^x_2,N^x_3)$ and $cf(\delta^a_1) = cf(\delta^b_1), cf(\delta^a_2) = cf(\delta^b_2), cf(\delta^a_3) = cf(\delta^b_3)$, all δ^x_ℓ limit ordinals.*

If f_ℓ is an isomorphism from N^a_ℓ onto N^b_ℓ for $\ell < 3$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$ then there is an isomorphism f from N^a_3 onto N^b_3 extending f_1,f_2.

Proof. Let $\bar\delta^c = \langle \delta^c_1,\delta^c_2,\delta^c_3\rangle = \langle \delta^a_2,\delta^a_1,\delta^a_3\rangle$; by ⟨8.3⟩ there are N^c_ℓ (for $\ell \le 3$) such that $NF_{\lambda,\bar\delta^c}(N^c_0,N^c_1,N^c_2,N^c_3)$. There is for $x \in \{a,b\}$ an isomorphism g^x_0 from N^a_0 onto N^c_0 (as K_λ is categorical in λ) and without loss of generality $g^b_0 = g^a_0 \circ f_0$. Similarly for $x \in \{a,b\}$ there is an isomorphism g^x_1 from N^x_1 onto N^c_2 extending g^x_0 (as N^x_1 is $(\lambda,\mathrm{cf}(\delta^x_1))$-saturated over N^x_0 and also N^c_2 is $(\lambda,\mathrm{cf}(\delta^c_2))$-saturated over N^c_0 and $\mathrm{cf}(\delta^c_2) = \mathrm{cf}(\delta^a_1) = \mathrm{cf}(\delta^x_1)$) and without loss of genrality $g^b_1 = g^a_1 \circ f_1$). Similarly for $x \in \{a,b\}$ there is an isomorphism g^x_2 from N^x_2 onto N^c_1 extending g^x_0 (as N^x_2 is $(\lambda,\mathrm{cf}(\delta^x_2))$-saturated over N^x_0 and also N^c_1 is $(\lambda,\mathrm{cf}(\delta^c_1))$-saturated over N^c_0 and $\mathrm{cf}(\delta^c_1) = \mathrm{cf}(\delta^a_2) = \mathrm{cf}(\delta^x_2)$) and without loss of generality $g^b_2 = g^b_2 \circ f_2$.
So by ⟨8.6⟩ for $x \in \{a,b\}$ there is an isomorphism g^x_3 from N^x_3 onto N^c_3 extending g^x_1 and g^x_2. Now $(g^b_3)^{-1} \circ g^a_3$ is an isomorphism from N^a_3 onto N^b_3 extending f_1,f_2 as required. $\square_{\langle 8.7\rangle}$

⟨8.8⟩ Conclusion. [Symmetry].
If $NF_{\lambda,\langle\delta_1,\delta_2,\delta_3\rangle}(N_0,N_1,N_2,N_3)$ then $NF_{\lambda,\langle\delta_2,\delta_1,\delta_3\rangle}(N_0,N_2,N_1,N_3)$.

Proof. By ⟨8.6⟩ (and ⟨8.7⟩).

⟨8.9⟩ Claim. *1) In Definition ⟨8.1⟩ we can replace $(N_{1,i},N_{1,i+1}) \cong (M^*,N^*)$ by $(N_{1,i},N_{1,i+1}) \in K^{3,\mathrm{uq}}$.*

Proof. Like the proof of ⟨8.6⟩ (get $(M_{i,0},M_{i+1,0}) \cong (M^*,N^*)$, $(M_{0,j},M_{0,j+1}) \in K^{3,\mathrm{uq}}_\lambda$), but as we shall not use it, we do not elaborate. $\square_{\langle 8.9\rangle}$

Now we turn to smooth amalgamation (not necessarily saturated, see Definition ⟨8.2⟩).

⟨8.10⟩ Claim. *1) If $NF_{\lambda,\bar\delta}(N_0,N_1,N_2,N_3)$ each δ_ℓ limit then $NF_\lambda(N_0,N_1,N_2,N_3)$ (see Definition ⟨8.2⟩).*
2) In Definition ⟨8.2⟩ we can add:

- *$(d)^+$ M_ℓ is $(\lambda,cf(\lambda))$-saturated over N_0 and moreover over N_ℓ,*
- *(e) M_3 is $(\lambda,cf(\lambda))$-saturated over $M_1 \cup M_2$ (actually this is given by $(f)(\zeta)$ of Definition ⟨8.1⟩).*

Proof. 1) By ⟨8.6A⟩ we can find $M_{i,j}$ for $i \le \lambda \times (\delta_1+\lambda), j \le \lambda \times (\delta_2+\lambda)$ for $\bar\delta' =: \langle \delta_1+\lambda,\delta_2+\lambda,\delta_3\rangle$ and choose $M'_3 \in K_\lambda$ which is $(\lambda,\mathrm{cf}(\delta_3))$-saturated over

$M_{\lambda\times\delta_1,\lambda\times\delta_2}$. So $NF_{\lambda,\bar\delta}(M_{0,0}, M_{\lambda\times\delta_1,0}, M_{0,\lambda\times\delta_2}, M_3)$, hence by $\langle 8.7\rangle$ without loss of generality $M_{0,0} = N_0$, $M_{\lambda\times\delta_1,0} = N_1$, $M_{0,\lambda\times\delta_2} = N_2$, and $N_3 = M'_3$. Lastly, let M_3 be $(\lambda, \mathrm{cf}(\lambda))$-saturated over M'_3. Now clearly also
$NF_{\lambda,\langle\delta_1+\lambda,\delta_2+\lambda,\delta_3+\lambda\rangle}(M_{0,0}, M_{\lambda\times(\delta_1+\lambda),0}, M_{0,\lambda\times(\delta_2+\lambda)}, M_3)$ and
$N_0 = M_{0,0}, N_1 = M_{\lambda\times\delta_2,0} \leq_{\mathfrak{K}} M_{\lambda\times(\delta_2+\lambda),0}, N_2 = M_{0,\lambda\times\delta_2} \leq_{\mathfrak{K}} M_{0,\lambda\times(\delta_2+\lambda)}$
and $M_{\lambda\times(\delta_1+\lambda),0}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $M_{\lambda\times\delta_1,0}$ and $M_{0,\lambda\times(\delta_2+\lambda)}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $M_{0,\lambda\times\delta_2}$ and $N_3 = M'_3 \leq_{\mathfrak{K}} M_3$. So we get all the requirements for $NF_\lambda(N_0, N_1, N_2, N_3)$ (as witnessed by $\langle M_{0,0}, M_{\lambda\times(\delta_1+\lambda),0}, M_{0,\lambda\times(\delta_2+\lambda)}, M_3\rangle$).

2) Similar proof. $\square_{\langle 8.10\rangle}$

$\langle$**8.11**$\rangle$ **Claim.** *[Uniqueness of smooth amalgamation]: If* $NF_\lambda(N_0^x, N_1^x, N_2^x, N_3^x)$ *for* $x \in \{a, b\}$, f_ℓ *an isomorphism from* N_ℓ^a *onto* N_ℓ^b *for* $\ell < 3$ *and* $f_0 \subseteq f_1, f_0 \subseteq f_2$ *<u>then</u>* $f_1 \cup f_2$ *can be extended to a* $\leq_{\mathfrak{K}}$*-embedding of* N_3^a *into some* $\leq_{\mathfrak{K}}$*-extension of* N_3^b *(so if* N_3^x *is* (λ, κ)*-saturated over* $N_1^x \cup N_2^x$ *for* $x = a, b$, *we can extend* $f_1 \cup f_2$ *to an isomorphism from* N_3^a *onto* N_3^b.

Proof. For $x \in \{a, b\}$ let the sequence $\langle M_\ell^x : \ell < 4\rangle$ be a witness to
$NF_\lambda(N_0^x, N_1^x, N_2^x, N_3^x)$ as in $\langle 8.2\rangle$, $\langle 8.10\rangle(2)$ so in particular
$NF_{\lambda,\langle\lambda,\lambda,\lambda\rangle}(M_0^x, M_1^x, M_2^x, M_3^x)$. By chasing arrows and uniqueness, i.e. $\langle 8.6\rangle$ without loss of generality $M_\ell^a = M_\ell^b$ for $\ell < 4$ and $f_0 = \mathrm{id}_{N_0^a}$. As M_1^a is $(\lambda, \mathrm{cf}(\lambda))$-saturated over N_1^a and also over N_1^b and f_1 is an isomorphism from N_1^a onto N_1^b, clearly there is an automorphism g_1 of M_1^a such that $f_1 \subseteq g_1$, hence also $\mathrm{id}_{N_0^a} = f_0 \subseteq f_1 \subseteq g_1$. Similarly there is an automorphism g_2 of M_1^a extending f_1 hence f_0. So $g_\ell \in \mathrm{AUT}(M_\ell^a)$ for $\ell = 1, 2$ and $g_1 \restriction M_0^a = f_0 = g_2 \restriction M_0^a$. By the uniqueness of $NF_{\lambda,\langle\lambda,\lambda,\lambda\rangle}$ (i.e. Claim $\langle 8.7\rangle$) there is an automorphism g_4 of M_4^a extending $g_1 \cup g_2$. This proves the desired conclusion. $\square_{\langle 8.11\rangle}$

$\langle$**8.12**$\rangle$ **Claim.** *Assume*

- (a) $\bar\delta = \langle\delta_1, \delta_2, \delta_3\rangle, \delta_\ell < \lambda^+$ *is a limit ordinal for* $\ell = 1, 2, 3$;
  $N_0 \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} N_3$ *for* $\ell = 1, 2$ *and*
- (b) N_ℓ *is* $(\lambda, cf(\delta_\ell))$*-saturated over* N_0
- (c) N_3 *is* $cf(\delta_3)$*-saturated over* $N_1 \cup N_2$.

<u>Then</u> $NF_\lambda(N_0, N_1, N_2, N_3)$ *iff* $NF_{\lambda,\bar\delta}(N_0, N_1, N_2, N_3)$.

Proof. The "if" direction holds by $\langle 8.11\rangle$. For the "only if" direction, by the proof of $\langle 8.10\rangle(1)$ (and Definition $\langle 8.1\rangle$, $\langle 8.2\rangle$) we can find $M_\ell (\ell \leq 3)$ such that $NF_{\lambda,\bar\delta}(M_0, M_1, M_2, M_3)$ and clauses (b), (c), (d) of Definition $\langle 8.2\rangle$ holds so by $\langle 8.10\rangle$ also $NF_\lambda(M_0, M_1, M_2, M_3)$. Easily there are for $\ell < 3$, an isomorphism f_ℓ from M_ℓ onto N_ℓ such that $f_0 = f_\ell \restriction M_\ell$. By the uniqueness for smooth amalgamation (i.e. $\langle 8.11\rangle$) we can find an isomorphism f_3 from M_3 onto N_3 extending $f_1 \cup f_2$. So as $NF_{\lambda,\bar\delta}(M_0, M_1, M_2, M_3)$ holds also $NF_{\lambda,\bar\delta}, (f_0(M_0), f_3(M_1), f_3(M_2), f_3(M_3))$; i.e. $NF_{\lambda,\bar\delta}(N_0, N_1, N_2, N_3)$ as required. $\square_{\langle 8.12\rangle}$

⟨8.13⟩ Claim. *[Monotonicity]: If* $NF_\lambda(N_0, N_1, N_2, N_3)$ *and* $N_0 \le_{\mathfrak{K}} N'_1 \le_{\mathfrak{K}} N_1$ *and* $N_0 \le_{\mathfrak{K}} N'_2 \le_{\mathfrak{K}} N_2$ *and* $N'_1 \cup N'_2 \subseteq N'_3 \le_{\mathfrak{K}} N_3$ *then* $NF_\lambda(N_0, N'_1, N'_2, N'_3)$.

Proof. Read Definition ⟨8.2⟩.

⟨8.14⟩ Claim. *[Symmetry]:* $NF_\lambda(N_0, N_1, N_2, N_3)$ *holds if and only if* $NF_\lambda(N_0, N_2, N_1, N_3)$ *holds.*

Proof. By Claim ⟨8.8⟩ (and Definition ⟨8.2⟩).

⟨8.15⟩ Claim. *Assume* $\alpha < \lambda^+$ *is an ordinal and for* $x \in \{a, b, c\}$ *the sequence* $\langle N^x_i : i \le \alpha \rangle$ *is a* $\le_{\mathfrak{K}}$*-increasing sequence of members of* K_λ, *for* $x = a, b$ *the sequence is* $\le_{\mathfrak{K}}$*-increasing continuous for* $i \le \alpha$, $N^b_i \cap N^a_\alpha = N^a_i$, $N^c_i \cap N^a_\alpha = N^a_i$, $N^a_i \le_{\mathfrak{K}} N^b_i \le_{\mathfrak{K}} N^c_i$ *and* N^c_i *is* (λ, κ_i)*-saturated over* N^b_i *and* $NF_{\lambda,\bar\delta^i}(N^a_i, N^a_{i+1}, N^c_i, N^b_{i+1})$ *(so* $i < \alpha \Rightarrow N^c_i \le_{\mathfrak{K}} N^b_{i+1}$*) where* $\bar\delta^i = \langle \delta^i_1, \delta^i_2, \delta^i_3 \rangle$ *sequence of limit ordinals,* $i < \alpha \Rightarrow \delta^{i+1}_2 = \delta^3_i$, *and for* $i < 0$ *limit,* $cf(\delta^3_i) = \sum\limits_{j<i} \delta^3_j, \delta_1 = \sum\limits_{\beta<\alpha} \delta^1_\beta$ *and* $\delta_3 = \kappa_\alpha, \bar\delta = \langle \delta_1, \delta^0_2, \delta_3 \rangle$. *Then* $NF_{\lambda,\bar\delta}(N^a_0, N^a_\alpha, N^b_0, N^c_\alpha)$.

Proof. Use uniqueness of ⟨8.7⟩, lastly use ⟨8.7⟩ to show N^b_α is $(\lambda, \mathrm{cf}(\alpha))$-saturated over $N^a_\alpha \cup N^b_0$. $\square_{\langle 8.15 \rangle}$

⟨8.16⟩ Claim. *Assume that* $\alpha < \lambda^+$ *and for* $x \in \{a, b\}$ *we have* $\langle N^x_i : i \le \alpha \rangle$ *is* $\le_{\mathfrak{K}}$*-increasing continuously sequence of members of* K_λ.
1) If $NF_\lambda(N^a_i, N^a_{i+1}, N^b_i, N^b_{i+1})$ *for each* $i < \alpha$ *then* $NF_\lambda(N^a_0, N^a_\alpha, N^b_0, N^b_\alpha)$.
2) If $\alpha_1 < \lambda^+, \alpha_2 < \lambda^+$ *and* $M_{i,j}$ $(i \le \alpha_1, j \le \alpha_2)$ *are as in ⟨8.6A⟩, and for each* $i < \alpha_1, j < \alpha_2$ *we have:*

$$M_{i,j+1} \quad \underset{M_{i,j}}{\overset{M_{i+1,j+1}}{\biguplus}} \quad M_{i+1,j}$$

$$\underline{\text{then}}\ M_{i,0} \quad \underset{M_{0,0}}{\overset{M_{\alpha_1,\alpha_2}}{\biguplus}} \quad M_{0,j} \ \text{for}\ i \le \alpha_1, j \le \alpha_2.$$

Proof. 1) We first prove special cases and use them to prove more general cases.

Case A: N^a_{i+1} is (λ, δ^1_i)-saturated over N^a_i and N^b_{i+1} is (λ, δ^2_i)-saturated over $N^a_{i+1} \cup N^b_i$ for $i < \alpha$.

We can choose by induction on $i \le \alpha$, $N^c_i \in K_\lambda$ such that

(a) $N^b_0 \leq_{\mathfrak{K}} N^c_0 \leq_{\mathfrak{K}} N^b_{i+1}, N^c_0$ is (λ, δ^2_0)-saturated over N^b_0, and $NF_{\lambda, \langle \delta^0_1, \delta^0_2, \delta^0_2 \rangle}(N^a_0, N^a_1, N^c_0, N^b_1)$

(b) $N^c_\alpha \in K_\lambda$ is $(\lambda, \delta^\alpha_3)$-saturated over N^b_α.

(Possible by uniqueness; i.e. $\langle 8.11 \rangle$ and monotonicity, i.e. $\langle 8.13 \rangle$). Now we can use $\langle 8.15 \rangle$.

Case B: For each $i < \alpha$ we have: N^a_{i+1} is (λ, κ_i)-saturated over N^a_i.
Let $\bar{\delta}^i = (\kappa_i, \lambda, \lambda)$.
We can find a $\leq_{\mathfrak{K}}$-increasing sequence $\langle M^x_i : i \leq \alpha \rangle$ for $x \in \{a, b, c\}$, continuous for $x = a, b$ such that $i < \alpha \Rightarrow M^b_i \leq_{\mathfrak{K}} M^c_i \leq_{\mathfrak{K}} M^b_{i+1}$ and $M^b_\alpha \leq_{\mathfrak{K}} M^c_\alpha$ and $NF_{\lambda, \bar{\delta}^i}(M^a_i, M^a_{i+1}, M^c_i, M^b_{i+1})$ by choosing M^a_i, M^b_i, M^c_i by induction on i. By Case A we know that $NF_\lambda(M^a_0, M^a_\alpha, M^b_0, M^c_\alpha)$ holds.

We can now choose an isomorphism f^a_0 from N^a_0 onto M^a_0 (exists as K is categorical in λ) and then a $\leq_{\mathfrak{K}}$-embedding of N^b_0 into M^b_0 extending f^a_0. Next we choose by induction on $i \leq \alpha, f^a_i$ an isomorphism from N^a_i onto M^a_i such that: $j < i \Rightarrow f^a_j \subseteq f^a_i$, possible by "uniqueness of the (λ, κ_i)-saturated model over M^a_i". ($\langle 0.22 \rangle$)

Now we choose by induction on $i \leq \alpha$, a $\leq_{\mathfrak{K}}$-embedding f^b_i of N^b_i into M^b_i extending f^a_i and f^b_j for $j < i$. For $i = 0$ we have done it, for i limit use $\bigcup_{j<i} f^b_j$, lastly for i a successor ordinal let $i = j + 1$, now we have

$(*)_2$ $NF_\lambda(M^a_i, M^a_{i+1}, f^b_i(N^b_i), M^b_{i+1})$
[why? because $NF_{\lambda, \bar{\delta}^i}(M^a_i, M^a_{i+1}, M^c_i, M^b_{i+1})$ by the choice of the M^x_ζ's hence by $\langle 8.12 \rangle$ we have $NF_\lambda(M^a_i, M^a_{i+1}, M^c_i, M^b_{i+1})$ and as $M^a_i \leq_{\mathfrak{K}} f^b_i(N^b_i) \leq M^b_i, M^c_i$ by $\langle 8.13 \rangle$ we get $(*)_2$].

By $(*)_2$ and the uniqueness of smooth amalgamation $\langle 8.11 \rangle$ there is f^b_i as required.

So without loss of generality f^b_α is the identity, so we have $N^a_0 = M^a_0, N^a_\alpha = M^a_\alpha, N^b_0 \leq_{\mathfrak{K}} M^b_0, N^b_\alpha \leq_{\mathfrak{K}} M^b_\alpha$; also as said above $NF_\lambda(M^a_0, M^a_\alpha, M^b_0, M^b_\alpha)$ holds so by monotonicity i.e. $\langle 8.13 \rangle$ we get $NF_\lambda(N^a_0, N^a_\alpha, N^b_0, N^b_\alpha)$ as required.

Case C: General case.
We can find M^ℓ_i for $\ell < 3, i \leq \alpha$ such that:

(a) $M^\ell_i \in K_\lambda$

(b) for each $\ell < 3, M^\ell_i$ is $\leq_{\mathfrak{K}}$-increasing in i

(c) $M^0_i = N^a_i$

(d) $M^{\ell+1}_{i+1}$ is (λ, λ)-saturated over $M^\ell_{i+1} \cup M^{\ell+1}_i$ for $\ell < 2, i < \alpha$

(e) $NF_\lambda(M^\ell_i, M^\ell_{i+1}, M^{\ell+1}_i, M^{\ell+1}_i)$ for $\ell < 2, i < \alpha$

(f) $M^{\ell+1}_0$ is (λ, λ)-saturated over M^ℓ_0

(g) for $\ell < 2$ and $i < \alpha$ limit we have

$$M_i^{\ell+1} \text{ is } (\lambda, \lambda)\text{-saturated over } \bigcup_{j<i} M_j^{\ell+1} \cup M_i^\ell$$

(h) for $i < \alpha$ limit we have

$$NF_\lambda \left(\bigcup_{j<i} M_j', M_i^1, \bigcup_{j<i} M_j^2, M_i^2 \right).$$

[how? as in the proof of $\langle 8.6A \rangle$].

Now note:

$(*)_4$ $M_i^{\ell+1}$ is $(\lambda, \mathrm{cf}(\lambda \times (1+i)))$-saturated over M_i^ℓ
[why? If $i = 0$ by clause (f), if i a successor ordinal by clause (d) and if i is a limit ordinal then by clause (g)]

$(*)_5$ for $i < \alpha, NF_\lambda(M_i^0, M_{i+1}^0, M_i^2, M_{i+1}^2)$
[why? we use Case B for $\alpha = 2$ with $M_i^0, M_{i+1}^0, M_i^1, M_{i+1}^1$, M_i^2, M_{i+1}^2 here standing for $N_0^a, N_0^b, N_1^a, N_1^b, N_2^a, N_2^b$ there].

Now we continue as in Case B (using $f_i^a = \mathrm{id}_{N_i^a}$ and defining by induction on i a $\leq_{\mathfrak{K}}$-embedding f_i^b of N_i^b into M_i^c).

2) By part (1) for each i the sequences $\langle M_{\beta,i} : \beta \leq \alpha_1 \rangle, \langle M_{\beta,i+1} : \beta \leq \alpha_1 \rangle$ we get $M_{\alpha_1} \underset{M_{0,i}}{\overset{M_{\alpha_1,i+1}}{\bigcup\!\!\!\!\bigcup}} M_{0,i+1}$ hence by symmetry (i.e. $\langle 8.11 \rangle$) we have $M_{0,i+1} \underset{M_{0,i}}{\overset{M_{\alpha_1,i+1}}{\bigcup\!\!\!\!\bigcup}} M_{\alpha_1,i}$. Applying part (1) to the sequences $\langle M_{0,j} : j \leq \alpha_2 \rangle, \langle M_{\alpha_1,j} : j \leq \alpha_2 \rangle$ we get $M_{0,\alpha_2} \underset{M_{0,0}}{\overset{M_{\alpha_1,\alpha_2}}{\bigcup\!\!\!\!\bigcup}} M_{\alpha_1,0}$ hence by symmetry (i.e. $\langle 8.11 \rangle$) we have $M_{\alpha_1,0} \underset{M_{0,0}}{\overset{M_{\alpha_1,\alpha_2}}{\bigcup\!\!\!\!\bigcup}} M_{0,\alpha_2}$; by monotonicity, i.e $\langle 8.13 \rangle$ (or restriction of the matrix) we get the desired conclusion. $\square_{\langle 8.16 \rangle}$

$\langle 8.17 \rangle$ Conclusion. Assume $\langle N_i^\ell : i \leq \alpha \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous for $\ell = 0, 1$ where $N_i^\ell \in K_\lambda$ and N_{i+1}^1 is (λ, κ_ℓ)-saturated over $N_{i+1}^0 \cup N_i^1$ and $\mathrm{NF}_\lambda(N_i^0, N_i^1, N_{i+1}^0, N_{i+1}^1)$.

Then N_α^1 is $(\lambda, \mathrm{cf}(\sum_{i<\alpha} \kappa_i))$-saturated over $N_\alpha^0 \cup N_0^1$ (if α is a limit ordinal, "N_{i+1}^1 is universal over $N_{i+1}^0 \cup N_i^1$" suffice).

Proof. The case α not limit is trivial so assume α is a limit ordinal. We choose by induction on $i \leq_x \alpha$, a sequence $\langle M_{i,\varepsilon}' : \varepsilon \leq \varepsilon(i) \rangle$ such that:

(a) $\langle M_{i,\varepsilon} : \varepsilon \leq \varepsilon(i) \rangle$ is (strictly) $<_{\mathfrak{K}}$-increasing continuous

(b) $N^0_i \leq_{\mathfrak{K}} M_{i,\varepsilon} \leq_{\mathfrak{K}} N^1_i$

(c) $N^0_i = M_{i,0}$

(d) $\varepsilon(i)$ is (strictly) increasing continuous in i,

(e) $j < i \ \& \ \varepsilon \leq \varepsilon(j) \Rightarrow M_{i,\varepsilon} \cap N^1_j = M_{j,\varepsilon}$

(f) $\varepsilon(0) = 1, M_{i,1} = N^1_0$

(g) for $i > 0, \lambda$ divide $\varepsilon(i)$

(h) $N^i_\ell \leq_{\mathfrak{K}} M_{i+1,\varepsilon(i)+1}$.

If we succeed, then $\varepsilon(\alpha)$ is divisible by λ and $\langle M_{i,\varepsilon} : \varepsilon \leq \varepsilon(\alpha)\rangle$ is (strictly) $<_{\mathfrak{K}}$-increasing continuous, $M_{\alpha,0} = N^0_\alpha$, and $M_{\alpha,\varepsilon(\alpha)} \leq_{\mathfrak{K}} N^1_\alpha$ but it includes N^1_i for $i < \alpha$ hence (as α is a limit ordinal) it includes $\bigcup\limits_{i<\alpha} N^1_i = N^1_\alpha$; and by $\langle 7.4\rangle$ we conclude that $N^1_\alpha = M_{\alpha,\varepsilon(\alpha)}$ is $(\lambda, \mathrm{cf}(\alpha))$-saturated over $M_{\alpha,1}$ hence over $N^0_\alpha \cup N^1_0$ (both $\prec M_{\alpha,i}$).

For $i = 0$, and i limit there is not much to do. For i successor we use $\langle 8.18\rangle$ below.

$\langle 8.18\rangle$ Conclusion. 1) If $NF_\lambda(N_0, N_1, N_2, N_3)$ and $\langle M_{0,\varepsilon} : \varepsilon \leq \varepsilon(*)\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous, $N_0 \leq_{\mathfrak{K}} M_{0,\varepsilon} \leq_{\mathfrak{K}} M_2$ <u>then</u> we can find $\langle M_{1,\varepsilon} : \varepsilon \leq \varepsilon(*)\rangle$ and N'_3 such that:

(a) $N_3 \leq_{\mathfrak{K}} N'_3 \in K_\lambda$

(b) $\langle M_{1,\varepsilon} : \varepsilon \leq \varepsilon(*)\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous

(c) $M_{1,\varepsilon} \cap N_2 = M_{0,\varepsilon}$

(d) $N_1 \leq_{\mathfrak{K}} M_{1,\varepsilon} \leq_{\mathfrak{K}} N'_3$

(e) if $M_{0,0} = N_0$ then $M_{1,0} = N$.

2) if N_3 is universal over $N_1 \cup N_2$, then without loss of generality $N_3 = N_2$.

Proof. 1) Straight by uniqueness.
2) Follows by (1). $\square_{\langle 8.17\rangle}$, $\square_{\langle 8.18\rangle}$

§9 Nice extensions in K_{λ^+}

$\langle 9.0\rangle$ *Hypothesis.* Assume the hypothesis $\langle 8.0\rangle$.

So by §8 we have reasonable control on smooth amalgamation in K_λ. We use this to define "nice" extensions in K_{λ^+}. This is treated again in §10.

$\langle$**9.1**$\rangle$ **Definition.** 1) Let $M_0 \le^*_{\lambda^+} M_1$ mean:

(a) $M_\ell \in K_{\lambda^+}$, for $\ell = 0, 1$

(b) we can find $\bar M^\ell = \langle M^\ell_i : i < \lambda^+\rangle$, a representation of M^ℓ, so $M^\ell_i \in K_\lambda$ (and M^ℓ_i is $\le_{\mathfrak{K}}$-increasing continuously and $M_\ell = \bigcup\limits_{i<\lambda^+} M^\ell_i$) such that:

$NF_\lambda(M^0_i, M^0_{i+1}, M^1_i, M^1_{i+1})$ for $i < \lambda^+$.

2) Let $M_0 <^+_{\lambda^+,\kappa} M_1$ mean $M_0 \le^*_{\lambda^+} M_1$ by some witnesses M^ℓ_i (for $i < \lambda^+, \ell < 2$) such that $NF_{\lambda,\langle\kappa,1,\kappa\rangle}(M^0_i, M^0_{i+1}, M^1_i, M^1_{i+1})$. If $\kappa = \lambda$, we omit it.

$\langle$**9.2**$\rangle$ **Claim.** *1) If $M_0 \le^*_{\lambda^+} M_1$ and $\bar M^\ell = \langle M^\ell_i : i < \lambda^+\rangle$ is a representation of M_ℓ (as in $\langle 8.16\rangle$) then for some club E of λ^+, for every $\alpha < \beta$ from E we have $NF_\lambda(M^0_\alpha, M^0_\beta, M^1_\alpha, M^1_\beta)$.*
2) Similarly for $<^+_{\lambda^+,\kappa}$; if $M_0 <^*_{\lambda^+,\kappa_\ell} M_1, \bar M^\ell = \langle \bar M^\ell_i : i < \lambda^+\rangle$ a representation of M_ℓ for $\ell = 1, 2$ then for some club E of λ^+ for every $\alpha < \beta$ from E we have $NF_{\lambda,\langle \mathrm{cf}(\alpha),1,\mathrm{cf}(\alpha)\rangle}(M^0_\alpha, M^0_\beta, M^1_\alpha, M^0_\beta)$.
3) The κ in Definition $\langle 9.1\rangle(2)$ does not matter. In fact, if $\langle M^\ell_i : i < \lambda^+\rangle$ are as in $\langle 9.1\rangle(1)$ then for some club E of λ^+ we have: $\alpha \in E \Rightarrow M^1_\alpha \cap M_0 = M^0_\alpha$ and $\alpha < \beta\ \&\ \alpha \in E\ \&\ \beta \in E \Rightarrow [M^1_\beta$ is cf(β)-saturated over $M^0_\beta \cup M^1_\alpha]$.

Proof. 1) Straight by $\langle 8.16\rangle$.
2) Easy.
3) By $\langle 8.17\rangle$. (Could have used $\langle 7.6\rangle$). $\square_{\langle 9.2\rangle}$

$\langle$**9.3**$\rangle$ **Claim.** *1) For every $\kappa =$ cf(κ) $\le \lambda$ and $M_0 \in K_{\lambda^+}$ for some $M_1 \in K_{\lambda^+}$ we have $M_0 <^+_{\lambda^+,\kappa} M_1$.*
2) $<^*_{\lambda^+}$ are $<^+_{\lambda^+,\kappa}$ are transitive.
3) If $M_0 \le_{\mathfrak{K}} M_1 \le_{\mathfrak{K}} M_2$ and $M_0 \le^*_{\lambda^+} M_2$ then $M_0 \le^*_{\lambda^+} M_1$.
4) [transitivity] If $M_0 \le^*_{\lambda^+} M_1 <^+_{\lambda^+,\kappa} M_2$ then $M_0 <^*_{\lambda^+,\kappa} M_2$.

Proof. 1) Let $\langle M^0_i : i < \lambda^+\rangle$ be a representation of M_0 such that M^0_{i+1} is (λ,κ)-saturated over M^0_i. We choose by induction on $i, M^1_i \in K_\lambda$ such that $\langle M^1_i : i < \lambda^+\rangle$ is $<_{\mathfrak{K}}$-increasing continuously, $M^0_i \le_{\mathfrak{K}} M^1_i, M^1_i \cap M_0 = M^0_i$ and $NF_{\lambda,\langle\kappa,1,\kappa\rangle}(M^0_i, M^0_{i+1}, M^1_i, M^1_{i+1})$. We can do it by $\langle 7.5\rangle(4)$.

2) Concerning $<^+_{\lambda^+,\kappa}$ use $\langle 9.2\rangle$ and $\langle 8.16\rangle$ (i.e. transitivity for smooth amalgamations). Now the proof for $<^*_{\lambda^+}$ is similar.
3) By monotonicity for smooth amalgamations; i.e. $\langle 8.13\rangle$.
4) Check. $\square_{\langle 9.3\rangle}$

$\langle \mathbf{9.4}\rangle$ **Claim.** *1) If $M_0 \le^*_{\lambda^+} M_\ell$ for $\ell = 1,2, \kappa = \text{cf}(\kappa) \le \lambda$ and $a \in M_2 \backslash M_0$ <u>then</u> for some M_3 and f we have: $M_1 <^+_{\lambda^+,\kappa} M_3$ and f is an $\le_{\mathfrak{K}}$-embedding of M_2 into M_3 over M_0 with $f(a) \notin M_1$, moreover, $f(M_2) \le^*_{\lambda^+} M_3$.*
2) [uniqueness] Assume $M_0 <^+_{\lambda^+,\kappa} M_\ell$ for $\ell = 1,2$ <u>then</u> there is an isomorphism f from M_1 onto M_2 over M_0.

Proof. We first prove part (2).
2) By $\langle 9.2\rangle(1) + (2)$ there are representations $\bar M^\ell = \langle M^\ell_i : i < \lambda^+\rangle$ of M_ℓ for $\ell < 3$ such that: $M^\ell_i \cap M_0 = M^\ell_0$ and $NF_{\lambda,\langle\kappa,\text{cf}(\kappa\times i),\kappa\rangle}(M^0_i, M^0_{i+1}, M^\ell_i, M^\ell_{i+1})$.

Now we choose by induction on $i < \lambda$ an isomorphism f_i from M^1_i onto M^2_i, increasing with i and being the identity over M^0_i. For $i = 0$ use "M^ℓ_0 is (λ,κ)-saturated over M^0_0 for $\ell = 1,2$" which holds by $\langle 8.1\rangle$. For i limit take unions, for i successor ordinal use uniqueness Claim $\langle 8.7\rangle$.

Proof of part (1). Let $\kappa = \aleph_0$, by $\langle 9.3\rangle(1)$ there are for $\ell = 1,2$ models $N^*_\ell \in K_{\lambda^+}$ such that $M_\ell <^+_{\lambda^+,\kappa} N^*_\ell$. Now let $\bar M^\ell = \langle M^\ell_i : i < \lambda^+\rangle$ be a representation of M_ℓ for $\ell = 0,1,2$ and let $\bar N^\ell = \langle N^\ell_i : i < \lambda^+\rangle$ be a representation of N^*_ℓ for $\ell = 1,2$. By $\langle 9.3\rangle(4)$ and $\langle 9.2\rangle(3)$ without loss of generality $NF_\lambda(M^0_i, M^0_{i+1}, M^\ell_i, M^\ell_{i+1})$ for $\ell = 1,2$ and $NF_{\lambda,\langle\kappa,1,\kappa\rangle}(M^\ell_i, M^\ell_{i+1}, N^\ell_i, N^\ell_{i+1})$ respectively. Now clearly N^ℓ_0 is (λ,κ)-saturated over M^ℓ_0 hence over M^0_0 (for $\ell = 1,2$) so there is an isomorphism f_0 from N^2_0 onto N^1_0 extending $\text{id}_{M^0_0}$ and $f(a) \notin M^1_0$.

We continue as in the proof of part (2). In the end $f = \bigcup\limits_{i<\lambda^+} f_i$ is an isomorphism of N_2 onto N_1 over M_0 and as $f^1_0(a)$ is well defined and in $N^1_0 \backslash M^1_0$ clearly $f(a) = f_0(a) \notin M_1$, as required. $\square_{\langle 9.4\rangle}$

$\langle \mathbf{9.5}\rangle$ **Claim.** *If δ is a limit ordinal $< \lambda^{+2}$ and $\langle M_i : i < \delta\rangle$ is a $\le^*_{\lambda^+}$-increasing continuous <u>then</u> $M_i \le^*_{\lambda^+} \bigcup\limits_{j<\delta} M_j$ for each $i < \delta$.*

Proof. We prove it by induction on δ. Now if C is a club of δ with $i \in C$, then we can replace $\langle M_j : j < \delta\rangle$ by $\langle M_j : j \in C\rangle$ so without loss of generality $\delta = \text{cf}(\delta)$, so $\delta \le \lambda^+$; clearly it is enough to prove $M_0 \le^*_{\lambda^+} \bigcup\limits_{j<\delta} M_j$. Let $\langle M^i_\zeta : \zeta < \lambda^+\rangle$ be a representation of M_i.

Case A: $\delta < \lambda^+$.
Without loss of generality (see $\langle 9.2\rangle(1)$) for every $i < j < \delta$ and $\zeta < \lambda^+$ we have: $M^j_\zeta \cap M_i = M^i_\zeta$ and $NF_\lambda(M^i_\zeta, M^i_{\zeta+1}, M^j_\zeta, M^j_{\zeta+1})$. Let $M^\delta_\zeta = \bigcup\limits_{i<\delta} M^i_\zeta$, so $\langle M^\delta_\zeta : \zeta < \lambda^+\rangle$ is $\le_{\mathfrak{K}}$-increasingly continuous sequence of members of K_λ with limit M_δ, and for $i < \delta, M^\delta_\zeta \cap M_i = M^i_\zeta$. By symmetry (see $\langle 8.14\rangle$) we have $NF_\lambda(M^i_\zeta, M^{i+1}_\zeta, M^i_{\zeta+1}, M^{i+1}_{\zeta+1})$ so as $\langle M^i_\zeta : i \le \delta\rangle$,$\langle M^i_{\zeta+1} : i \le \delta\rangle$ are $\le_{\mathfrak{K}}$-increasingly continuous by $\langle 8.16\rangle$ we know $NF_\lambda(M^0_\zeta, M^\delta_\zeta, M^0_{\zeta+1}, M^\delta_{\zeta+1})$ hence by symmetry ($\langle 8.14\rangle$) we have $NF_\lambda(M^0_\zeta, M^0_{\zeta+1}, M^\delta_\zeta, M^\delta_{\zeta+1})$.
So $\langle M^0_\zeta : \zeta < \lambda^+\rangle, \langle M^\delta_\zeta : \zeta < \lambda^+\rangle$ are witnesses to $M_0 \le^*_{\lambda^+} M_\delta$.

Case B: $\delta = \lambda^+$.

By $\langle 9.2\rangle(1)$ (using normality of the club filter, restricting to a club of λ^+ and renaming), without loss of generality for $i < j \le 1 + \zeta < 1 + \xi < \lambda^+$ we have $M^j_\zeta \cap M_i = M^i_\zeta$, and $NF_\lambda(M^i_\zeta, M^i_\xi, M^j_\zeta, M^j_\xi)$. Let us define $M^{\lambda^+}_i = \bigcup\limits_{j<1+i} M^j_i$. So $\langle M^{\lambda^+}_i : i < \lambda^+\rangle$ is a representation of $M^\lambda_{\lambda^+} = M_\delta$ and continue as before. $\square_{\langle 9.5\rangle}$

$\langle 9.6\rangle$ Claim. *Assume $M_0 <^+_{\lambda^+,\kappa} M_2$ and $a \in M_2 \backslash M_0$, and for some $N \le_{\mathfrak{K}} M_0$ we have: $N \in K_\lambda$ and $tp(a, N, M_2)$ is minimal. Then we can find M_1, $\bar{M}^0 = \langle M_{0,i} : i < \lambda^+\rangle, \bar{M}^1 = \langle M_{1,i} : i < \lambda^+\rangle$ such that:*

- (a) *$\bar{M}^0$ is a $\le_{\mathfrak{K}}$-representation of M_0*
- (b) *$\bar{M}^1$ is a representation of $M_1 (\in K_{\lambda^+}), a \in M_{1,i}$, for all i*
- (c) $M_0 \le_{\mathfrak{K}} M_1 \le_{\mathfrak{K}} M_2$
- (d) *for $i < \lambda^+$ we have $NF_{\lambda,\langle\lambda,1,1\rangle}(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$ (hence $M_{\ell,i} = M_{\ell+1,i} \cap M_\ell$)*
- (e) *$(M_{0,i}, M_{1,i}, a) \in K^3_\lambda$ is minimal and reduced.*

Proof. Let $\langle M_{0,i} : i < \lambda^+\rangle, \langle M_{2,i} : i < \lambda^+\rangle$ be representations of M_0, M_2 respectively, as required in $\langle 9.1\rangle(2)$ and without loss of generality $N \le_{\mathfrak{K}} M_{0,0}$ and $a \in M_{2,0}$. We now choose by induction on $\zeta < \lambda^+$, an ordinal $i(\zeta)$ and models $M_{1,i(\zeta)}, M_{3,i(\zeta)}$ such that:

- (A) $i(\zeta) < \lambda^+$ is increasing continuous in ζ and $a \in M_{2,i(0)} \backslash M_{0,i(0)}, N \le_{\mathfrak{K}} M_{0,i(0)}$
- (B) $M_{0,i(\zeta)} \le_{\mathfrak{K}} M_{1,i(\zeta)} \le_{\mathfrak{K}} M_{3,i(\zeta)}$ and $M_{2,i(\zeta)} \le_{\mathfrak{K}} M_{3,i(\zeta)}$
- (C) $a \in M_{1,i(0)}$ and $(M_{0,i(\zeta)}, M_{1,i(\zeta)}, a)$ is minimal and reduced
- (D) for $\xi < \zeta$ and $(\ell, m) \in \{(0,1), (0,2), (1,3), (2,3)\}$ we have $NF_\lambda(M_{\ell,i(\xi)}, M_{\ell,i(\zeta)}, M_{m,i(\xi)}, M_{m,i(\zeta)})$
- (E) $M_{1,i(\zeta)}, M_{3,i(\zeta)}$ is $\le_{\mathfrak{K}}$-increasing continuous in ζ.

For $\zeta = 0$ note that for $i(0) < \lambda^+, a \in M_{2,i(0)}$ and $M_{2,i(0)}$ is universal over $M_{0,i(0)}$.

For ζ limit let $i(\zeta) = \bigcup_{\xi<\zeta} i(\xi)$ and $M_{1,i(\zeta)} = \bigcup_{\xi<\zeta} M_{1,i(\zeta)}$.

For $\zeta = \xi + 1$, there is $i(\zeta) \in (i(\xi), \lambda^+)$ and a model N_ζ such that $M_{1,i(\xi)} \leq_{\mathfrak{K}} N_\zeta \in K_\lambda$ and $\leq_{\mathfrak{K}}$-embedding f of $M_{0,i(\zeta)}$ into $N_\zeta, f \restriction M_{0,i(\zeta)}$ the identity and $(f(M_{0,i(\xi)}), N_\zeta, a)$ is minimal and reduced. By uniqueness (i.e. Claim $\langle 8.1\rangle$) we can find such N satisfying $(\exists M)(N \leq_{\mathfrak{K}} M \in K_\lambda \ \ \& \ \ M_{1,i(\zeta)} \leq_{\mathfrak{K}} M)$. So we can carry the induction.

Lastly, by uniqueness of $<^+_{\lambda,\kappa}$ we can make $M_3 = \bigcup_{\zeta<\lambda^+} M_{3,i(\zeta)}$ to be $\leq_{\mathfrak{K}} M_2$ as required. $\square_{\langle 9.6\rangle}$

$\langle$**9.7**$\rangle$ **Definition.** If (M_0, M_1, a) are as in $\langle 9.6\rangle$ (a)-(e) we say (M_0, M_1, a) is λ^+-locally reduced nice and minimal (λ^+-l.r.n.m.). We omit "nice" if we omit clause (d).

$\langle$**9.8**$\rangle$ **Claim.** *If* (M_0, M_1, a) *is* λ^+*-l.r.n.m.* <u>*then*</u> $(M_0, M_1, a) \in K^3_{\lambda^+}$ *is reduced.*

Proof. Check.

Using also $\langle 7.4\rangle$:

$\langle$**9.9**$\rangle$ **Claim.** $M_0 <^+_{\lambda^+,\kappa} M_1$ <u>*if and only if*</u> *we can find* $\langle M^*_j, a_j : j < \lambda^+ \times \kappa\rangle$ *such that:*

- (a) M^*_j *is* $\leq_{\mathfrak{K}}$*-increasing continuous (in* K_{λ^+}*)*
- (b) (M^*_j, M^*_{j+1}, a_j) *is* λ^+*-l.r.n.m.*
- (c) $M^*_0 = M_0, M^*_j \cup M^*_j = M_1$
- (d) *for some* $N \leq M_0, N \in K_\lambda$ *and minimal reduced* $p \in S(N)$*, for every* j*.*

Proof. We can find $\langle M^*_j : j \leq \lambda^+ \times \kappa\rangle$ satisfying clauses $(a), (b)$ and (d). By $\langle 5.4\rangle$ easily $M^*_0 <^+_\lambda M^*_{\lambda^+}$. Now by the uniqueness ($= \langle 9.4\rangle(2)$) + categoricity of K in λ^+, we are done. $\square_{\langle 9.9\rangle}$

$\langle$**9.10**$\rangle$ **Claim.** *In* $(K_{\lambda^+}, <^*_{\lambda^+})$ *we have disjoint amalgamation.*

Proof. First redo $\langle 9.4\rangle$ assuming (M_0, M_ℓ, a_ℓ) for $\ell = 1, 2$ is λ^+-l.r.n.m., and getting $a_1 \notin f(M_2), f(a_2) \notin M_1$ (and we can start with this). By $\langle 9.8\rangle$ we get $M_1 \cap f(M_2) = M_3$, so we have disjoint amalgamation. By $\langle 9.9\rangle$ and chasing arrows we get it in general. $\square_{\langle 9.10\rangle}$

Remark. This is like the proof of disjoint amalgamations in $\langle 5.7\rangle$.

§10 Non-structure for $\leq^*_{\lambda^+}$

$\langle 10.0 \rangle$ *Hypothesis.* Assume the hypothesis $\langle 8.0 \rangle$ and the further model theoretic properties deduced since then (including $\langle 6.5 \rangle$ but not $\langle 6.6 \rangle$.)

It would have been nice to prove all disjoint amalgamations in K_λ are NF_λ, but this is, at this point, not clear. But as we look upward (i.e. want to prove statement on $K_{>\lambda^+}$) and $<^*_{\lambda^+}$ is very nice, it will be essentially just as well if for $M, N \in K_{\lambda^+}$ we have $M \leq_{\mathfrak{K}} N \Rightarrow M \leq^*_{\lambda^+} N$. Our intention is to assume $M^* \leq_{\mathfrak{K}} N^*$ is a counterexample of this statement and we would like to say that in a sense this implies the existence of many types over M^* so that we can construct many models in λ^{+2}. Note: building models in $K_{\lambda^+}, K_{\lambda^{++}}$ by approximations in K_λ is nice if we use the smooth amalgamation but we do not have it for non-smooth ones. So we shall use $M^* \in K_{\lambda^+}$ being saturated so it has many automorphisms.

$\langle$**10.1**$\rangle$ **Claim.** *1) Assume $M_1 \leq_{\mathfrak{K}} M_2$ are in K_{λ^+}. Then we can find $M_0 \in K_{\lambda^+}$ such that $M_0 <^+_{\lambda,\kappa} M_1$ and $M_0 \leq^*_{\lambda^+} M_2$.*
2) Also we can find $\langle M_{0,i} : i < \lambda^+ \rangle$, an $\leq_{\mathfrak{K}}$-increasingly continuous sequence of members of K_{λ^+} such that $M_{0,i} <^*_{\lambda^+} M_{0,i+1}$ and $\bigcup\limits_{i<\lambda^+} M_{0,i} = M_1$ and $i < \lambda^+ \Rightarrow M_{0,i} \leq^*_{\lambda^+} M_2$.

Proof. Let $(M^*, N^*) \in K^{3,\mathrm{uq}}_\lambda$ be from $\langle 8.0 \rangle(2)$. Let $\langle M_{\ell,i} : i < \lambda^+ \rangle$ be a representation of M_ℓ for $\ell = 1, 2$ and without loss of generality $M_{\ell,i+1}$ is (λ,λ)-saturated over $M_{\ell,i}$ for $\ell = 1, 2$ and $M_{2,i} \cap M_1 = M_{1,i}$.
1) Now choose by induction on i, $M_{0,i}$ such that:

- (a) $M_{0,i} \leq_{\mathfrak{K}} M_{1,i}$
- (b) $M_{0,i}$ is $\leq_{\mathfrak{K}}$-increasing continuous
- (c) $M_{0,i+1} \cap M_{1,i} = M_{0,i}$
- (d) $M_{1,i+1}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $M_{1,i} \cup M_{0,i+1}$
- (e) $(M_{0,i}, M_{0,i+1}) \cong (M^*, N^*)$.

There is no problem to carry the definition. Now let $M_0 = \bigcup\limits_{i<\lambda^+} M_{0,i}$ so $M_0 <^+_{\lambda^+} M_1$ and $M_0 \leq^*_{\lambda^+} M_2$ are checked by their definitions.
2) We choose by induction on $i < \lambda^+$, $\langle M^*_{\varepsilon,i} : \varepsilon \leq 1+i \rangle$ such that:

- (a) $M^*_{1+i,i} = M_{1,\lambda\times(1+\varepsilon)\times i}$
- (b) For each ε the sequence $\langle M^*_{\varepsilon,j} : \varepsilon \leq j \leq i \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous
- (c) For each i, the sequence $\langle M^*_{\varepsilon,i} : \varepsilon \leq 1+i \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous
- (d) $M^*_{\varepsilon,i} \cap M^*_{\zeta,j} = M^*_{\min\{\varepsilon,\zeta\},\min\{i,j\}}$
- (e) $M^*_{\varepsilon+1,i+1}$ is $(\lambda, \mathrm{cf}(\lambda \times (1+\varepsilon)))$-saturated over $M^*_{\varepsilon,i+1} \cup M^*_{\varepsilon+1,i}$
- (f) $NF_{\lambda,\langle\lambda,1,\lambda\rangle}(M^*_{\varepsilon,i}, M^*_{\varepsilon+1,i}, M^*_{\varepsilon,i+1}, M^*_{\varepsilon+1,i+1})$
- (g) for $\varepsilon < 1+i$ we have $NF_\lambda(M^*_{\varepsilon,i}, M^*_{\varepsilon,i+1}, M_{2,\lambda\times\lambda\times i}, M_{2,\lambda\times\lambda\times(i+1)})$.

For $i = 0, i$ limit no problem, for $i = j + 1$ first choose $N_{i,\varepsilon,\zeta} \leq_{\mathfrak{K}} M_{1,\lambda\times\lambda\times i+\lambda\times\zeta}$ for $\zeta \leq \lambda \times (1 + i), \leq_{\mathfrak{K}}$-increasing continuous, $N_{i,\varepsilon,\zeta} \in K_\lambda, N_{i,\varepsilon,0} = M_{1,j} (= M^*_{1+i,i})$, $(N_{i,\varepsilon,\zeta}, N_{i,\varepsilon,\zeta+1}) \cong (M^*, N^*)$ and $N_{i,\varepsilon,\zeta+1} \cap M_{2,\lambda\times\lambda\times i+\lambda\times\zeta} = N_{i,\varepsilon,\zeta}$.

Now by $\langle 5.4\rangle$, without loss of generality $N_{i,\varepsilon,\lambda}$ is (λ, λ)-saturated over $M_{1,j}$, and we choose it as $M^*_{1+j,i}$, and we choose $M^*_{1,i+1}$ as $M_{1+i,i}$ note that clauses (a) and (f) holds. Now we can find N_{ε,j_1} for $j_1 < 1 + j$ as in $\langle 8.6\text{A}\rangle$ and use uniqueness of the $(\lambda, \lambda \times (1 + i))$-saturated model over $M_{1,j}$. $\square_{\langle 10.2\rangle}$

$\langle$*10.2*$\rangle$ *Conclusion.* Assume $M \leq_{\mathfrak{K}} N$ are from K_{λ^+}. If $\langle M_i : i < \lambda^+\rangle$ is $\leq^*_{\lambda^+}$-increasing continuous and for each i for some N_i we have $M_i <^+_{\lambda^+} N_i \leq^*_{\lambda^+} M_{i+1}$ then for some (M', N') we have:

$$(M', N') \cong (M, N)$$

$$M' = \bigcup_{i<\lambda} M_i$$

$$i < \lambda^+ \Rightarrow M_i \leq^*_{\lambda^+} N'.$$

Proof. By $\langle 10.1\rangle(2)$ and as we know $M^a \leq^*_{\lambda^+} M^b \leq^+_{\lambda^+} M^c \Rightarrow M^a <^+_{\lambda^+} M^c$ and the uniqueness of M'' over M' when $M' <^+_{\lambda^+} M''$. $\square_{\langle 10.2\rangle}$

$\langle$**10.3**$\rangle$ **Lemma.** *If* $\leq_{\mathfrak{K}}\restriction K_{\lambda^+}$ *is not* $\leq^*_{\lambda^+}$ *then* $I(\lambda^{+2}, K) = 2^{\lambda^{+2}}$.

Proof. Let $S \subseteq \{\delta < \lambda^{+2} : \text{cf}(\delta) = \lambda^+\}$ be stationary. We shall below construct a model $M_S \in K_{\lambda^{+2}}$ such that from the isomorphism type of M^S_1 we can reconstruct $S/\mathcal{D}_{\lambda^{+2}}$; this clearly suffices. Choose $M \leq_{\mathfrak{K}} N$ from K_{λ^+} such that $\neg(M \leq^*_{\lambda^+} N)$ so by $\langle 9.3\rangle(3)$ without loss of generality $|N\backslash M| = \lambda^+$.

We choose by induction on $\alpha < \lambda^{+2}$ a model M^S_α such that:

- (a) $M^S_\alpha \in K_{\lambda^+}$ has universe $\lambda \times (1 + \alpha)$
- (b) for $\beta < \alpha$ we have $M^S_\beta \leq_{\mathfrak{K}} M^S_\alpha$
- (c) if $\alpha = \beta + 1$, $\beta \notin S$ then $M^S_\beta <^+_{\lambda^+} M^S_\alpha$
- (d) if $\alpha = \beta + 1, \beta \in S$ then $(M^S_\beta, M^S_\alpha) \cong (M, N)$
- (e) if $\beta < \alpha, \beta \notin S$ then $M^S_\beta \leq^*_{\lambda^+} M^S_\alpha$.

We use freely the transitivity $(\langle 9.3\rangle(4))$ and continuity $(\langle 9.5\rangle)$ of $\leq^*_\lambda$ and $[M^a \leq_{\mathfrak{K}} M^b \leq_{\mathfrak{K}} M^c$ in $K_\lambda, \neg(M^a \leq^*_{\lambda^+} M^b \Rightarrow \neg(M^a \leq^*_{\lambda^+} M^c)]$ $\langle 9.3\rangle(3)$.
For $\alpha = 0$ no problem.

For α limit no problem.

For $\alpha = \beta + 1, \beta \notin S$ no problem.

For $\alpha = \beta + 1, \beta \in S$ so $\mathrm{cf}(\beta) = \lambda^+$, let $\langle \gamma_i : i < \lambda^+ \rangle$ be increasing continuous with limit β and $\mathrm{cf}(\gamma_i) \le \lambda^+$, hence $\gamma_i \notin S$. By $\langle 9.2\rangle(2)$ without loss of generality $M_{\gamma_i} <^*_{\lambda^+} M_{\gamma_{i+1}}$. Now use $\langle 10.2\rangle$ (and the uniqueness ($\langle 9.4\rangle(2)$)). $\square_{\langle 10.3\rangle}$

⟨10.4⟩ Conclusion. Assume $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$ (in addition to hypothesis $\langle 10.0\rangle$).
1) $\le^*_{\lambda^+} = \le_{\mathfrak{K}} \restriction K_{\lambda^+}$.
2) $(K_{\lambda^+}, \le_{\mathfrak{K}})$, has disjoint amalgamation so no $M \in K_{\lambda^{+2}}$ is $\le_{\mathfrak{K}}$-maximal.
3) $K_{\lambda^{+3}} \ne \emptyset$.

Proof. 1) By $\langle 10.3\rangle$.
2) By $\langle 9.10\rangle$ (and part (1)).
3) By $\langle 10.4\rangle(2)$ and $\langle 2.4\rangle(8)$ with λ there replaced by λ^+ here. $\square_{\langle 10.4\rangle}$

* * *

So we have finally proved the main theorem. Though not directly contributing to our main theme, we remark some more consequences of $\le_{\mathfrak{K}} \restriction K_{\lambda^+} \ne \le^*_{\lambda^+}$.

⟨10.5⟩ Claim. $(*)_0 \Leftrightarrow (*)_1$ *where*

$(*)_0$ *for some $M \le_{\mathfrak{K}} N$ from K_{λ^+}, we do not have $M \le^*_{\lambda^+} N$*

$(*)_1$ *for some $M \le_{\mathfrak{K}} N$ from K_{λ^+} we have:*
if $a \in N \backslash M$ <u>then</u> $tp(a, M, N)$ is not realized in any M such that $M^* \le^*_{\lambda^+} M \in K_{\lambda^+}$

⟨10.6⟩ Definition. Assume $M_0 <_{\mathfrak{K}} M_1$ are in K_{λ^+},and $\bar{M}^\ell = \langle M_{\ell,i} : i < \lambda^+ \rangle$ is a $\le_{\mathfrak{K}}$-representation of M_ℓ for $\ell = 0, 1$. Let

(a) $$S_0(\bar{M}^0, \bar{M}^1) = \Big\{ \delta < \lambda^+ : M_{1,\delta} \cap M_0 = M_{0,\delta} \text{ and not } NF_\lambda(M_{0,\delta}, M_{0,\delta+1}, M_{1,\delta}, M_1) \Big\}$$

(b) $$S_1(M_0, M_1) = S_0(\bar{M}^0, \bar{M}^1)/D_{\lambda^+} \text{ (well defined).}$$

(c) J is the normal ideal on λ^{++} generated by sets of the form $S_0(M^0, M^1)$ where M^0, M^1 are as above.

⟨10.7⟩<u>Comment</u>: An earlier try for $\langle 10.3\rangle$ was:
1) For every $S \in J$ for some $\bar{M}^0, M^1$ as in $\langle 10.6\rangle$ we have

$S = S_0(\bar{M}^0, \bar{M}^1)$.
2) If $S_1 = S_1(M^0, M^1)$ is stationary then for some $\bar{M} = \langle M_i : i < \lambda^+ \rangle$ a representation of $M = \bigcup_{i<\lambda^+} M_i \in K_{\lambda^+}$ for every $S \subseteq S_1$ for some $\bar{M}'$, we have
$\bar{M}^1 = \langle M^1_i : i < \lambda^+ \rangle, M_1 <_{\mathfrak{K}} M^1 = \bigcup_{i<\lambda^+} M^1_i, M^1_i \cap M = M_i$ and
$S_0(\bar{M}, \bar{M}^1) = S \bmod \mathcal{D}_{\lambda^+}$.
3) If $\leq^*_{\lambda^+} \neq \leq_{\mathfrak{K}} \restriction K_{\lambda^+}$ and $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$ <u>then</u> for some stationary $S \subseteq \lambda^+$ we have:

- (a) $\mathcal{D}_{\lambda^+} \restriction S$ is λ^{++}-saturated
- (b) $\bar{M}^0, \bar{M}^1$ as in $\langle 10.3 \rangle$ implies $S_1(\bar{M}^0, \bar{M}^1) \subseteq S \bmod D_{\lambda^+}$.

4) If $\leq^*_{\lambda^+} \neq \leq_{\mathfrak{K}} \restriction K_{\lambda^+}$ and $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$ then $I(\lambda^{+2}, K) = 2^{\lambda^{+2}}$.

Proof. 1) First we prove only "$S \subseteq S_0(\bar{M}^0, \bar{M}^1)$". Easy as $\mathfrak{K}_{\lambda^+}$ has amalgamation and

$\bigotimes$ if $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$ are in $K_{\lambda^+}, \bar{M}^\ell$
representing the $S_0(\bar{M}^0, \bar{M}^1) \subseteq S_0(\bar{M}^0, \bar{M}^2)$.

Now for equality use part (2).
2) Similar to the proof of $\langle 10.1 \rangle$.
3) Suppose $S^* = S_1(\bar{M}^0, \bar{M}^2)$ is stationary, let $\bar{S} = \langle S'_\alpha : \alpha < \lambda^{++} \rangle$ be such that $S'_\alpha \in J$. We can build a model $M^{\bar{S}} \in K_{\lambda^{+2}}$ and a representation $\langle M^{\bar{S}}_\alpha : \alpha < \lambda^{++} \rangle$ such that

$$S_1(M_\alpha, M_{\alpha+1}) = S'_\alpha / D_{\lambda^{++}}.$$

4) By §3 (using the proof of part (2).

Private Appendix

$\langle$**2.4X**$\rangle$ **Claim.** $(*)_2 + (*)_3$
1) If $(M_0, M_1, a) \in K^3_\lambda, M_0 \leq M^+_0 \in K_{\lambda^+}$ *then we can find:* M^+_1 *such that:* $M^+_0 \leq_K M^+_1 \in K_{\lambda^+}$ *and* h *is a* $\leq_K$*-embedding of* M_1 *into* $M^+_q, h \restriction M_0 = \ id_{M_0}$*,*
$h(a) \in M^+_1 \backslash M^+_0$*.*
2) For every $(M_0, M_1, a) \in K^3_\lambda$ *(i.e.* $M_0 \leq_K M_1$ *in* $K_3, a \in M$*) and* $M_0 \leq M^\otimes \in K_\lambda$ *then we can find* $(M'_0, M'_1, a) \geq (M_0, M_1, a)$ *in* K^3_λ *such that* $M^\otimes$ *can be* $\leq_K$*-embedded into* M'_0 *over* M_0*.*

Proof. 1) Just use repeatedly the conclusion of $\langle 2.3\rangle(1)$. (λ^+ times).
2) Let $M_0 \leq_K M^* \in K_{\lambda^{++}}, M^*$ saturated (see $\langle 2.2\rangle(2)$).

Proof. 1) We choose inductively $N_i \in K_\lambda$ for $i < \lambda^+, \leq_K$-increasing continuous, $N_0 = M_0$ and a $\leq_K$-embedding h_i of M_i into N_{i+1} such that
$h_i \restriction M_0 = \ \mathrm{id}_{M_0}, h_i(a) \notin M_i$ (possible by $\langle 2.3\rangle(1)$).

§3

Here we shall prove the density of minimal types using only $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ from cardinal arithmetic. Our aim is also to explain various methods.
First, we give a proof assuming the ideal of small subsets of λ^+ (W Dm Id(λ^+)) is not λ^{++}-saturated as exemplified by $\langle S_\alpha : \alpha < \lambda^{++}\rangle$. We build $M_\eta \in K_{\lambda^+}, M_\eta = \bigcup\limits_{\alpha<\lambda^+} M_{\eta,\alpha}, |M_\eta| = \lambda \times |1 + \ell g \eta), \nu \triangleleft \eta \Rightarrow M_\nu \triangleleft M_\eta$. Building $M_{\eta\char`^\langle\ell\rangle}$ choosing $M_{\eta\char`^\langle\ell\rangle,\alpha}$, usually we act in a way predetermined simply (as in [Sh 87b], except when $\alpha \in S_{\ell g(\eta)}, \ell = 1$ then we consult a weak diamond sequence. This is like $\langle 1.4\rangle(1)$ getting $2^{\lambda^{+2}}$ models in λ^{+2}.

A second proof (2) is like $\langle 1.3\rangle(1)$ in the sense that we get only close to $2^{\lambda^{++}}$ models. It is similar to [Sh 87b, 6.4], the parallel to [Sh 87b, 6.3] holds here. So we have to make parallel of [Sh 87b, definition 6.5,6.7]. But we need fullness on the side which we shall get by the λ^{++}-saturation?

<u>$\langle 3.0\text{AX}\rangle$ Question</u>: 1) Assume $M_0 \leq_K M_1, M_0 \leq_K M_2$ all in K_λ is the disjoint amalgamation unique? (up to extension ... [Sh 300, II]).
2) Assume in $K_\lambda, M_0 \leq_K M_1, M_0 \leq_K M_2, M_0 \leq_K M_{1,i} \leq_K M_1$ for $i < \lambda$) and without loss of generality $M_1 \cap M_2 = M_0$ can we find $M_3, M_{3_i} (\in K_\lambda)$ such that:

$$\begin{gathered} M_1 \leq_K M_3, \\ M_1 \leq_K M_{3,i} \leq_K M_3 \\ M_{3,i} \cap M_1 = M_{1,i} \end{gathered}$$

$\langle 3.0\text{BX}\rangle$ Claim. *Positive answer to $\langle 3.1\rangle$(1) implies positive answer to $\langle 3.1\rangle$(2).*

Proof. Amalgamate in row

M_2	first	second
M_0	$M_{1,i}$	M_1

and now use uniqueness.

$\langle 3.0CX\rangle$ Discussion. On play is to use double W.D. construction as in [Sh 87b], end. So we have $N \in K_{\lambda^+}, N^0 = \bigcup\limits_{i<\lambda^+} N_i^0$ usual, we choose inductively $N_i^1, \leq_K$-increasing continuously, $N_i^0 \leq N_i^1, N_i^1 \cap N = N_i^0$.
For $\delta < \lambda^+$ if for some $\beta_\delta \in (\delta, \lambda^+)$, for the disjoint amalgamation of $N_\beta^0 \bigcup\limits_{N_\delta^0} N_\delta^1$ we have non-uniqueness, see ($\langle 3.1\rangle(1)$), we have two contradictory choices.

Possibility A: If we have it on a non-small (in the relevant sense (of weak diamond or a definable W.D.) we can get a contradiction to $I(\lambda^{++}, K) < 2^{\lambda^{++}}$, (as in [? Sh:87 ? Sh:87, §6]).

Possibility B: If on a club we have uniqueness, we should be through $\langle 3.1\rangle(2)$ (see $\langle 3.2\rangle$) find N^1 such that:

$$N_0 \leq_K N_1 \in K_{\lambda^+}$$

$$\forall a \in N_1 \backslash N_0 \exists N (N_0 \cup \{a\} \subseteq N \leq_K N_1, (\text{so } N_0 \leq N)).$$

We use $\diamond^+$ (= from building Kurepa trees).
If we succeed, then as in [Sh 88] we get $I(\lambda^{++}, K) < 2^{\lambda^{++}}$ that the saturated model in λ^{++} is not maximal and contradicts.

Such consideration will be necessary for generalizing [Sh 87b] fully to this context (only starting with cardinals with categoricity) but have not pursued.

* * *

⟨3.1X⟩ Definition. Let

$$\begin{aligned} K^{\mathrm{qr}}_\lambda = \Big\{ (\bar M, f) : & \bar M = \langle M_i : i \leq \lambda^+ \rangle, M_i \leq_K \text{-increasing continuously} \\ & M_i \in K_\lambda, M_i \neq M_{i+1}, \text{ and } f : \lambda^+ \to \lambda^+, \\ & f(\alpha) \geq \alpha (\bar M^1, f) \leq (\bar M^2, f) \text{ iff} \\ & \{\delta < \lambda^+ : \text{for every } \alpha \in [\delta, f(\delta)] \text{ we have} \\ & M^2_\alpha \cap \bigcup_{\gamma < \lambda^+} M^1_\delta = M^1_\alpha \text{ and} \\ & M^1_\alpha \leq_K M^2_\alpha\} \text{ contains a club} \Big\}. \end{aligned}$$

If $f(\alpha) = \alpha + 1$ we may write $\bar M$.

⟨*3.2X*⟩ *Fact.* We can find a function F, n-place such that:

(*a*) if $M_\ell \in K_\lambda$ for $\ell < 3, M_0 \leq_K M_\ell, |M_1| \cap |M_2| = |M_0|, |M_1| \cup |M_2| \subseteq A, A \backslash (|M_1| \cup |M_2|)$ has cardinality λ then $F(M_0, M_1, M_2, A)$ is a model M with universe $A, M_\ell \leq_K M$

(*b*) if (M_0, M_1, M_2, A) and (M'_0, M'_1, M'_2, A) are as aobve, A, A' are sets of ordinals of the same order type, f the order preserving map from A onto A' and $f \restriction M_\ell$ is an isomorphism from M_ℓ onto M'_ℓ then f is an isomorphism form $F(M_0, M_1, M_2, A)$ onto $F(M'_0, M'_1, M'_2, A)$.

⟨3.3X⟩ Lemma. *Assume WDmId(λ^+) is not λ^{++}-saturated.*
If

(a) K_λ, K_{λ^+} *are co?*
(b) $(M^*, N^*, a^*) \in K^3_\lambda$ *exemplifies non-minimality of density of the family of minimal triples.*

Then $I(\lambda^{+2}, \mathfrak{K}) = 2^{\lambda^{++}}$.

Proof. Let $S^*_\zeta \subseteq \lambda^+, S^*_\zeta \neq \emptyset$ mod WDmId$(\lambda^+), \xi < \zeta \Rightarrow |S^*_\xi \cap S^*_\zeta| < \lambda^+$.

We can find $\bar{M}^\eta = \langle M^{\langle\rangle}_\alpha : \alpha < \lambda^+\rangle \in K^{\rm qr}_{\lambda^+}$ and $\langle a_\alpha : \alpha < \lambda^+\rangle$ such that $(M^{\langle\rangle}_\alpha, M^{\langle\rangle}_{\alpha+1}, a_\alpha) \cong (M^*, N^*, a^*)$ and the set of elements of $M^{\langle\rangle}_\alpha$ is $\lambda \times (1-\alpha)$.
We choose by induction on $\zeta < \lambda^{+2}, \langle \bar{M}^\eta : \eta \in {}^\alpha 2\rangle, \langle b^*_\eta : \eta \in {}^\zeta 2, \zeta$ non-limit$\rangle$ such that:

(a) $\bar{M}^\eta \in K^{\rm qr}_{\lambda^+}$, let $M^\eta = \bigcup\limits_{\alpha<\lambda^+} M^\eta_\alpha$
(b) $\bigcup\limits_{\alpha<\lambda^+} M^\eta_\alpha$ has set of elements $\lambda^+ \times (1 + \ell g(\eta))$, and $(M^{\eta\char`^\langle\ell\rangle}_\alpha)$ includes $\lambda^+ \times (1 + \ell g(\eta)) + \alpha$
(c) $\nu \trianglelefteq \eta \Rightarrow \bar{M}^\nu \le \bar{M}^\eta$
(d) $b_{\eta\char`^\langle\ell\rangle} \in M^{\eta\char`^\langle 0\rangle}_0 \backslash \bigcup\limits_{\alpha<\lambda^+} M^\eta_\alpha$
(e) $\{\alpha < \lambda^+ : M^\eta_\alpha \le_K M^{\eta\char`^\langle\ell\rangle}_\alpha$ and $(M^\eta_\alpha, M^{\eta\char`^\langle\ell\rangle}_\alpha, b_{\eta\char`^\langle\ell\rangle})$ is reduced$\}$ contains a club.
Moreover, for each α, for some $\beta_{\eta,\alpha,\ell}, (M^\eta_{\beta_{\eta,\alpha,ell}}, M^{\eta\char`^\langle\ell\rangle}_\alpha, b_{\eta\char`^\langle\ell\rangle})$ is reduced to $\beta_{\eta,\alpha,\ell}$ increasing continuously in $\alpha < \lambda^+$.
(f) For $\eta \in {}^\zeta 2, \ell < 2$

$$\Big\{\delta < \lambda^+ : |M^{\eta\char`^\langle\ell\rangle}_\delta| \cap [\lambda^+ \times (1 + \ell g(\eta)), \lambda^+ \times (1 + \ell g(\eta)) + \lambda) =$$
$$[\lambda^+ \times (1 + \ell g(\eta)), \lambda^+ \times (1 + \ell g(\eta)) + \delta)$$
$$\text{and if } \delta \notin S_{23+\ell} \text{ then } M^{\eta\char`^\langle\ell\rangle}_{\delta+1} = F(M^\eta,_\delta, M^\eta_{\delta+1}, M^{\eta\char`^\langle\ell\rangle}_\delta, A^{\eta,\ell}_\delta)$$
$$\text{where } A^{\eta,\ell}_\delta = [\lambda^+ \times (1 + \ell g(\eta)) + \delta,$$
$$\lambda^+ \times (1 + \ell g(\eta)) + \delta + \lambda) \cup |M^\eta_\delta| \cup |M^\eta_{\delta+1}| \cup |M^{\eta\char`^\langle\ell\rangle}_\delta|\Big\}$$
contains a club.

We still have a freedom left, defining $\eta\char`^\langle\ell\rangle$ in stages $\delta \in S_{\ell g(\eta)+\ell}$ arriving to $\bar{M}^\eta, \eta \in {}^\zeta 2$ for $\ell < 2$ letting
$\Gamma^\alpha_{\eta\char`^\langle\ell\rangle} = \{\rho : \text{for} \rho \in {}^\alpha 2$ and $\rho \restriction (\alpha \backslash S_{2,\zeta+1})$ is constantly zero$\}$.
We can define $M^{\eta\char`^\langle\ell\rangle,\rho}_\rho$ for $\rho \in \Gamma^\alpha_{\eta\char`^\langle\ell\rangle}$ by induction on α they are as required on

$M_\alpha^{\eta\char`\^\langle\ell\rangle}$, but if $\alpha = \beta+1, \beta \in S_{2,\zeta+1}, \rho \in \Gamma^\beta_{\eta\char`\^\langle\ell\rangle}$ and
$|M_\rho^{\eta\char`\^\langle\ell\rangle}| \cap \bigcup\limits_\gamma M_\gamma^\eta = M_\beta^{\eta\char`\^\langle\ell\rangle}, |M_\rho^{\eta\char`\^\langle\ell\rangle}| \cap [\lambda^+ \times (1+\ell g(\eta)), \lambda^+ \times (1+\ell g(\eta))+\lambda) =$
$[\lambda^+ \times (1+\ell g(\eta)),$
$\lambda^+ \times (1+\ell g(\eta))+\beta)$, the two possibilities
$M^{\eta\char`\^\langle\ell\rangle}_{\rho\char`\^\langle 0\rangle}, M^{\eta\char`\^\langle\ell\rangle}_{\rho\char`\^\langle 1\rangle}$ are incompatible as amalgamations of $M_\alpha^\eta, M_\rho^{\eta\char`\^\langle\ell\rangle}$ over M_β^η.

Fact: Now we act as in the proof of $\langle 1.4\rangle(2)$.

$\langle$3.4X$\rangle$ Definition.

$$K^{\rm pg}_{\lambda^+} = \left\{(\bar M, \bar a, f) : (\bar M, f) \in K^{\rm qr}_{\lambda^+} \text{ and } a_i \in M_{i+1}\backslash M_i\right\}$$

$$\begin{aligned} K^{\rm rg}_{\lambda^+,D} = \Big\{(\bar M, \bar a, f) \in K^{\rm pg}_{\lambda^+} : &\text{and for the } D\text{-majority of } \delta < \lambda^+, \\ &\text{for every } \alpha \in [\delta, f(\delta)), \text{ the triple} \\ &(M_\alpha, M_{\alpha+1}, a_\alpha) \in K^3_\lambda \text{ is reduced}\Big\} \end{aligned}$$

$$\begin{aligned} (\bar M^1, \bar a^1, f^1) \le (\bar M^2, \bar a^2, f) \ \underline{\text{iff}}\ &(\bar M^1, f^1) \le (M^2, f^2) \\ &\text{and for a club of } \delta < \lambda^+, \text{ for every} \\ &\alpha \in [\delta, f^1(\delta)), \text{ we have } a^1_\alpha = a^2_\alpha \end{aligned}$$

$$(\bar M^1, \bar a^1, f^1) \le_F (\bar M^2, \bar a^2, f) \ \underline{\text{iff}}\ (\bar M^1, f^1) \le_F (\bar M^2, f^2)$$

and (for some $\langle(\bar N^i, \bar f^i) : i \le \alpha\rangle, (N^0, f^0) = (\bar M^1, f^1), (\bar M^2, f^2) = (\bar N^2, f^1) =$ $(N$, and for $\beta < \alpha$ for club of $\delta < \lambda^+$, for $\zeta \in [\delta, f(\delta))$ we have used F) and $(\bar M^1, \bar a^1, f^1) \le (\bar M^2, \bar a^2, f^2)$.

$\langle$3.5X$\rangle$ Claim. *If $\xi \le \lambda^{+2}, \langle(\bar M^\zeta, \bar a^\zeta, f^\zeta) : \zeta < \xi\rangle$ is $\le_F$-increasing in $K^{\rm pg}$ (in $K^{\rm rg}_{\lambda^+}$ $\underline{\text{then}}$ it has a $\le_F$-lub.*

$\langle$3.6X$\rangle$ Claim. *Assume D is a λ^{++}-saturated filter on λ^+. If $(\bar M^*, \bar a^*, f^*) \in K^{\rm pg}_{\lambda^+}$, $\underline{\text{then}}$ for some $(\bar M^*, \bar a^\ell, f^\ell) \in K^{\rm rg}_{\lambda^+,D}$ we have $(\bar M^*, \bar a^*, f^*) \le_F (\bar M^\ell, \bar a^\ell, f^\ell)$.*

Proof. For each $\alpha < \lambda^+$ let $\{\beta^i_i, b^\alpha_i : i < \lambda\}$ list the pairs $(\beta, b), \beta \in [\alpha, f(\alpha)]$, $b \in M^1_{f(\alpha)}\backslash M^1_\beta$.
For any $(\bar M, \bar a, f) \ge_F (\bar M^*, \bar a^*, f^*)$, let

$$S_i[(\bar{M}, \bar{a}, f)] = \{\delta < \lambda^+ : b_i^\alpha \in M_\beta\}.$$

Clearly $(\bar{M}^*, \bar{a}^*, f^*) \leq_F (\bar{M}^1, \bar{a}^1, f^1) \leq (\bar{M}^2, \bar{a}^2, f^2)$,
$i < \lambda$ implies that
$S_i[(\bar{M}^1, \bar{a}^1, f^1)] \subseteq S_i[(\bar{M}^2, \bar{a}^2, f^2)] \bmod D_{\lambda^+}$. Hence for some
$(\bar{M}, \bar{a}, f) \geq_F (\bar{M}^*, \bar{a}^*, f)$, we have $S_i[(\bar{M}, \bar{a}, f)]/D$ is maximal, in fact, we can get this for every $i < \lambda$.

This $(\bar{M}, \bar{a}, f)$ is not necessarily in $K^{\mathrm{rg}}_{\lambda^+}$ as in $M_{f(0)} \backslash M_\delta$ there are elements not in $M^*_{f(\delta)} \backslash M_\delta$. But repeating the argument ω times in the limit we get a triple as desired.

* * *

$\langle$**3.7X**$\rangle$ **Claim.** *If there is a $(M, N, a) \in K^3_\lambda$ exemplifying non-density of minimal triples* <u>*then*</u>

(a) $2^\lambda = \lambda^+$
(b) $2^{\lambda^+} = \lambda^+$.

Proof. Let $\bar{M} \in K^{\mathrm{gr}}_{\lambda^+}, M = \bigcup\limits_{i<\lambda^+} M_i$, we choose by induction on $\alpha < \lambda, i_\eta, N_\eta, a_\eta$ such that:

$$(M_0, N_{\langle\rangle}, a_{\langle\rangle}) \cong (M, N), (M_{i_\eta}, N_\eta, a_\eta) \in K^3_\lambda,$$

$$\nu \triangleleft \eta \Rightarrow (M_\nu, N_\nu, a_\nu) \leq (M_{i_\eta}, N_\eta, a_\eta),$$

$$i_{\eta\hat{}\langle 0\rangle} = i_{\eta\hat{}\langle 1\rangle}$$

and

$$(M_{i_{\eta\hat{}\langle\ell\rangle}}, N_{\eta\hat{}\langle\ell\rangle}, a_{\eta\hat{}\langle\ell\rangle})$$

for $\ell = 1, 2$ exemplifies non-minimality of $(M_{i_\eta}, N_\eta, a_\eta)$.
So $M \in K_\lambda \Rightarrow |S(M)| = 2^\lambda$.

$\langle$**3.8X**$\rangle$ **Claim.** *(Minimal not dense in K^3_λ).*
$I(\lambda^{+2}, K) = 2^{\lambda^{++}}$.

Proof. We choose by induction on $\zeta < \lambda^{+2}$ for $\eta \in {}^\zeta 2, (\bar{M}^\eta, f^\eta)$ as in the proof of $\langle 3.3\rangle$, but in addition

$(a)'$ we demand $\bar{M}^\eta$ is reduced except when $\ell g(\eta) = 1 \bmod \omega$

(f) $\nu \lhd \eta \Rightarrow (\bar{M}^\nu, f^\nu) \leq_F (\bar{M}^\eta, f^\eta)$

(g) $(\bar{M}^\eta, f^\eta + 1) \leq_F (\bar{M}^{\eta\hat{\ }\langle 0\rangle}, f^{\eta\hat{\ }\langle 0\rangle})$

(h) $(\bar{M}^{\eta\hat{\ }\langle 1\rangle}, f^{\eta\hat{\ }\langle 1\rangle})$ is chosen such that $\mathrm{tp}(b_{\eta\hat{\ }\langle 0\rangle}, M^\eta, M^{\eta\hat{\ }\langle 2\rangle})$ is not realized in M^ν when $\eta\hat{\ }\langle 0\rangle \trianglelefteq \nu \in {}^{\lambda^{+2}\geq}2$ (use weak diamond on λ^+, think similar to [Sh 87b, 6.4] but did not check).

This gives us:

$(*)$ for every $\eta \in {}^{\lambda^{+2}}2$ some $p_\eta \in S(M^\eta)$ is realized in $M^{\eta\hat{\ }\langle 1\rangle}$ but not in M^ν when $\eta\hat{\ }\langle 0\rangle \trianglelefteq \nu \in {}^{\lambda^{+2}>}2$.

Now we can use in $\langle 1.4\rangle(2)$ - isomorphism we have no problem.
$\langle$3.9X$\rangle$ Fact: This argument applies to [Sh 87b, 6.4], too, considering the type of $M^{\eta\hat{\ }\langle 1\rangle}$ on M^η.

Discussion: We get $IE(\lambda^{+2}, \mathfrak{K} = 2^{\lambda^{+2}}$ when $(2^{\lambda^+})^+ < 2^{\lambda^{+2}}$. Also if $\mathrm{cf}([2^{\lambda^+}]^{\leq\lambda^+}, \subseteq) = 2^\lambda$ we can deal with this in another way. As in $\langle 6.4\rangle$, if $\lambda \geq \beth_\omega$ also we are done. Now x.x.x.x.

§7 Old section 7

After then categoricity in λ^{++} case (moved April 95)

⟨7.1X⟩ Definition.
1) $K^{3,\mathrm{uq}}_\lambda = \{(M_0, M_1, a) \in K^3_\lambda : (M_0, M_1, a)$ is an amalgamation base in K^3_λ (the reasonable definition)$\}$.
2) $K^{3,\mathrm{red}}_\lambda = \{(M_0, M_1, a) \in K^3_\lambda : (M_0, M_1, a)$ is reduced$\}$.
3) For $M_0 \in K_\lambda, p \in S(M_0)$ is minimal κ-inevitable and: <u>if</u> it is inevitable, for some $M_1, a, (M_0, M_1, a) \in K^3_\lambda$ is minimal, $\mathrm{tp}(a, M_0, M_1) = p$ and if $(M_0, M_1, a) < (M'_0, M'_1, a), (M'_0, c)_{c \in M_0}$ is (λ, κ)-limit saturated <u>then</u> $\mathrm{tp}(a, M'_0, M'_1)$ is (minimal and) conjugate to p; i.e. for some $h \in \mathrm{ISO}(M_0, M'_0)$ we have $h(\mathrm{tp}(a, M'_0, M'_1)) = p$ hence is inevitable.
4) $(M_0, M_1, a) <^*_\kappa (M'_0, M'_1, a)$ (or $<^{**}$) <u>if</u> $(M_0, M^*_1, a) \le (M'_0, M'_1, a), (M'_0, c)_{c \in M_0}$ is (λ, κ)- limit saturated over M_0 and for $<^{**}_\kappa$ there is M''_1, $M'_0 \cup M_1 \subseteq M''_1 < M'_1 < M'_1, M'_1$ is saturated of cofinality κ over M''_1.
5) $K^{3,\mathrm{rf}}_\lambda = \{(M_0, M_1, a) \in K^{3,\mathrm{red}}_\lambda$: for every $\kappa = \mathrm{cf}(\kappa) \le \lambda$, for some $b \in M_1 \backslash M_0$, $\mathrm{tp}(b, M_0, M_1)$ is minimal κ-inevitable$\}$.
6) $K^{3,f}_\lambda = \{(M_0, M_1, a) \in K^3_\lambda$: there is $(M'_0, M'_1, a) \le (M_0, M_1, a)$ which belongs to $K^{3,\mathrm{rf}}_\lambda\}$.

From earlier we have

⟨7.2X⟩ Claim. *1) For every $(M_0, M_1, a) \in K^3_\lambda$ there is $(M'_0, M'_1, a) \in K^{3,\mathrm{rf}}_\lambda$, $(M'_0, M'_1, a) \ge (M_0, M_1, a)$.*
2) $K^{3,\mathrm{rf}}_\lambda \subseteq K^{3,\mathrm{red}}_\lambda \cap K^{3,f}_\lambda$.
3) $(M_0, M_1, a) \in K^{3,f}_\lambda, (M_0, M_1, a) \le (M'_0, M'_1, a) \Rightarrow (M'_0, M'_1, a) \in K^{3,f}_\lambda$.
4) If $(M_0, M_1, a) \le (M'_0, M'_1, a) <^*_\kappa (M''_0, M''_1, a)$ then $(M_0, M_1, a) <^*_\kappa (M''_0, M''_1, a)$.
5) For $(M_0, M_1, a) \in K^3_\lambda$ and $\kappa = \mathrm{cf}\,\kappa \le \lambda$ there is $(M'_0, M'_1, a) \in K^3_\lambda$ such that $(M_0, M_1, a) <^*_\kappa (M'_0, M'_1, a)$.
6) If $(M_0, M_1, a) \in K^{3,\mathrm{uq}}_\lambda, (M_0, M_1, a) <^*_\kappa (M^\ell_0, M^\ell_1, a)$ for $\ell = 1, 2$ then (M^ℓ_0, M^ℓ_1, a) for $\ell = 1, 2$ are isomorphic over (M_0, M_1, a).

⟨7.3X⟩ Definition. $(M_0, M_2) \in K^{2,\mathrm{uq}}_\lambda$ if $(\forall M_1)(M_0 \le_K M_1 \in K_\lambda \Rightarrow (M_0, M_1, M_2) \in K^{\mathrm{unique}}_\lambda)$.

Remark. Below instead $K^{2,\mathrm{uq}}_\lambda \ne \emptyset$ we can assume $[\kappa = \mathrm{cf}\kappa \le \lambda \Rightarrow S^{\lambda^+}_\kappa$ has weak diamond$]$.

⟨7.4X⟩ Definition. Assume $(M^*_0, M^*_2) \in K^{2,\mathrm{uq}}_\lambda, \kappa = \mathrm{cf}\,\kappa \le \lambda$.
We call (M'_0, M'_1, a) a κ-limit triple over (M_0, M_1, a) if we can find $\langle (M_{0,i}, M_{1,i}, a) : i \le \kappa \rangle$, such that:

(a) it is an increasing continuous sequence in K^3_λ

(*b*) $(M_{0,i}, M_{1,i}, a) <^*_\lambda (M_{0,i+1}, M_{1,i+1}, a)$
(*c*) if $i < \kappa$ is limit then we can find
$\langle (M_{0,i,\zeta}, M_{1,i,\zeta}, a) : \zeta \leq \zeta(*)\rangle$ is increasing continuous in K^3_λ
$(M_{0,i,0}, M_{1,i,0}, a) = (M_{0,i}, M_{1,i}, a)$
$(M_{0,i,\zeta(*)}, M_{1,i,\zeta(*)}, a) = (M_{0,i+1}, M_{1,i+1}, a)$
$(M_{0,i,\zeta}, M_{0,i,\zeta+1}) \cong (M^*_0, M^*_2)$
(*d*) $(M_{0,\kappa}, M_{1,\kappa}, a) = (M'_0, M'_1, a), (M_{0,0}, M_{1,0}, a) = (M_0, M_1, a)$.

$K^{3,\mathrm{uq}}_\lambda$ is dense in $K^{3,\mathrm{uq}}_\lambda$ and for some $\kappa =$ cf $\kappa = \lambda$ (just W.D. $(S^{\lambda^+}_\kappa)$.

⟨7.5X⟩ Claim. *For every (M_0, M_1, a) for some $(M'_0, M'_1, a) \in K^{3,\mathrm{uq}}_\lambda, (M_0, M_1, a) <^*_\kappa (M'_0, M'_1, a)$.*

⟨7.6X⟩ Claim. *1) For every minimal (M_0, M_1, a) there is (M'_0, M'_1, a) κ-limit over it.*
2) If $(M_0, M_1, a) \in K$???

⟨*7.7X*⟩ *Discussion.* We can decompose $N^*_0 \in K_{\lambda^+}$ to $\bar N^0 = \langle N^0_i : i < \lambda^+\rangle$ for $(M^*_0, M^*_2) \in K^{2,\mathrm{uq}}_\lambda : (N^0_i, N^0_{i+1}) \cong (M^*_0, M^*_1)$. Consider $\bar N^0 < \bar N^1, (N^1_i \cap N_0 = N^0_i, N^0 \leq N^1)$. Here all is nice, we can get a universal $\bar N^1$ but not all $S(N_0)$ can be gotten this way.
On the other hand, we can consider extensions $\langle N^\varepsilon_i : i < \lambda^+\rangle$ for $\varepsilon < \lambda, (N^\varepsilon_i \leq_K N^\varepsilon_{i_1} \leq N^{\varepsilon_1}_{i_1}$ if $\varepsilon \leq \varepsilon_i, i \leq i_1, a_\varepsilon \in N^{\varepsilon+1}_i \backslash N^\varepsilon_i, (N^\varepsilon_i, N^{\varepsilon+1}_i, a_\varepsilon) \in K^{3,f}_\lambda$, so it is easy to guarantee $\bigcup\limits_{\varepsilon<\lambda} N^\varepsilon_i$ saturated over N^0_i, so for which $i < \lambda^+$ do we have freedom = non-unique amalgamation?

We know weak diamond holds on $\{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa\}$ for sme κ. Assuming it for all κ is not so desirable, but passable.

Now ⟨7.1⟩ - ⟨7.6⟩ above went after this: we want to work as in §3 (= [`Sh:87`, §6]). ? Sh:87 ?
We can build (N^0_i, N^1_i) inductively, and it is enough that for a club of $\delta \in S^{\lambda^+}_\kappa$ we have a real choice (over N^0_{i+1}, N^1_i) so we need that for all such δ we have the same situation will be nice if instead W.D. We code a stationary subset of $S^{\lambda^+}_\kappa$, but we still have problems on uniform behaviour.

Without uniform behaviour strong set theory (as in [Sh 133]) help.

The next step should be analyzing $(N_0, N_1, a) \in K^3_\lambda$ such that for some $(M_0, M_1, a) \leq (N_0, N_1, a), (M_0, M_1, a) \in K^3_\lambda$ is minimal and show essentially it has unique extension in $S(N_0)$.

7.5 Claim - end of old §8. *6)?*
In part (5), if $i_0 < \lambda^+, \mathbf{f}_0 \le \mathbf{f}(i), \mathbf{f}_0$ *as in (4) and* g *is a* $\le_K$*-embedding* N^1_i *into* $N_{i,\lambda \times (1+\mathbf{f}_0(i))}$ *<u>then</u> we can extend* g *to a* $\le_K$*-embedding of* N^1 *into* $\cup\{N_{j,\varepsilon} : j < \lambda^+ \varepsilon \le \lambda \times (1 + \mathbf{f}_0(i))$ *such that*

- (f') $\mathbf{f}(N^2_i) \le N_{i,\lambda \times (1+\mathbf{f}_0(i))}$ *for* $i \in [i_0, \lambda^+)$
- $(g)'$ *for* $i < j < \lambda, f(N^2_i) \cap N_{i,\lambda \times (1+\mathbf{f}_0(i))}$.

7)? Every $(M, N, a) \in K^3_{\lambda^+}$ *has the extension property.*

Proof. Straight. Saharon!

7.6 Conclusion?. $K_{\lambda^{+3}} = \emptyset$.

§9 Old section 9: after current §7, moved April 95

We return to the analysis of minimal types in 2.8 (natural to assume $2^{\lambda^+} > \lambda^{++}$ we get minimal types).

So we want to get $I(\lambda^3, K) \neq \emptyset$ by a shortcut

9.1 Claim. $(*)_8$ *(= 3.10).*

Assume $(M_0, M_1, a) \in K^3_\lambda$ *is reduced and minimal.*
Then

$$|S_{(M_0,M_1,a)}(M_0) \leq \lambda^+|.$$

Proof. Assume not.

We need to prove the parallel of $\bigotimes_4$ in 2.8(3.9). So

(∗) we are given $M \in K_\lambda$,
$\Gamma \subseteq \cup\{S_{(M_0,M_1,a)}(M') : M' \leq M, M' \in K_\lambda\}, |\Gamma| \leq \lambda$,
then there are $M', M \leq_\lambda M' \in K_\lambda, M'$ omit all types in Γ but realizes some $q \in S_{(M_0,M_1,a)}(M)$.

Proof. Without loss of generality $M \leq M^* \in M^* \in K_{\lambda^{++}}$ saturated.
Each $r \in \Gamma$ has a unique extension r^+ of the form $\text{tp}(b_r, M, M^*)$ (b_r not unique).
For each $q \in S_{(M_0,M,a)}(M)$ there are M_q, a_q such that (M, M_q, r_q) is $\cong (M_0, M_q, a)$ hence reduced and minimal and $a_q = \text{ tp}(a_q, M, M_q)$ and we have λ^{++} candidates.
Now if each M_q realizes some $r \in \Gamma$ using M^* saturated without loss of generality

$$q \in S_{(M_0,M_1,a)}(M) \Rightarrow (M_q \backslash M) \cap \{b_r : r \in \Gamma \neq \emptyset\}.$$

Choose $N_0, M \cup \{b_r : r \in \Gamma\} \subseteq N_0 \leq_K M^*, N_0 \in K_\lambda$.
Choose $q \in S_{(M_0,M_1,a)}(M)$ not realized in N_0, so $a_q \notin N_0$.
Choose $N_1, M_q \cup N \subseteq N_1 \prec_K M^*, N_1 \in K_\lambda$.

So

$$(M, M_q, a_q) \leq (N_0, N_1, a_q) \in K^3_\lambda$$

$$M_q \cap N_1 \neq N_0$$

contradicting (M, M_q, a_q) is reduced. Contradiction - we finished proving the parallel of $\bigotimes_4$ hence 9.1.

9.2 Conclusion. 1) If $M \in K_{\lambda^+}, M_0 \leq_K M, M_0 \in K_\lambda, (M_0, M_1, a) \in K_{\lambda^+}$ minimal then $\text{tp}(a, M_0, M_1)$ is realized in M.
2) [So if 2^{λ^+} is minimal, types are dense (hold if $2^{\lambda^+} > \lambda^{++}$)] then every type in $S(M_0)$) is realized in M so M is saturated.

We now redo 4.1 (so 2.8) without using the Def W.D.
See §5.

9.3 Claim. $(*)_8$ *In 4.1 (and 3.9 and 2.8) we can avoid using Def. W.D.; i.e. we prove:*

$$(M_0, M_1, a) \textit{ is reduced minimal and } M \in K_\lambda \Rightarrow |S_{(M_0,M_1,a)}(M)| \leq \lambda^+.$$

Proof. We can find $N^0 = \bigcup\limits_{i<\lambda^+} N^0_i, N^0_i \in K_\lambda$ is $\leq_K$-increasing continuous in i and $p_i = \text{tp}(a, N^0_i, N^0_{i+1})$ minimal increasing with i.

Then we construct N^1_i $(i < \lambda^+)$ strictly $\leq_K$-increasing continuous such that:

- $(*)$ if $j < i, b \in N^1_i$, and for some $M, b \cup \{N^1_j\} \subseteq M \leq N^1_i, (M^1_j, M, b)$ is reduced and minimal, $r^j_b \in S(N^1_i)$ unique extension of $\text{tp}(b, N_j, N_i)$ which is not algebraic. Then N^1_{i+1} omits r^j_b (M not important for this). Let $N^1 = \bigcup\limits_i N^1_i$.

We can do it by $\bigotimes_4$ or parallel as there is a minimal reduced pair.
Assume $g : N^0 \to N^1$ isomorphic onto (by K categorical in $\lambda = 1$,
$E = \{S : g \text{ maps } N^0_i \text{ onto } N^1_i\}$. Let $\delta_1 < \delta_2$ be in E then $g(a_{\delta_2})$ realizes the type $r^{\delta_1}_{g(a_{\delta_1})}$ contradiction.
Can continue in §5 - concept 4.8, still read. $\square_{4.3}$

Question 9.4: If $N_0 \in K_{\lambda^+}, N_0 = \bigcup\limits_{i<\lambda^+} N^0_i, N^0_i \in K_\lambda$ increasing continuous, $N_0 \leq M^* \in K_{\lambda^{++}}, a \in M^* \backslash N_0, p = \text{tp}(a, N_0, M^*)$ is minimal, does this determine $\text{tp}(a, N_0, M^*)$?

We can find $N_1 \in K_{\lambda^+}, N_1 \leq_K M^*, N_0 \cup \{a\} \subseteq N_i$ and without loss of generality $\langle N^1_i : i < \lambda^+\rangle, (N^0_i, N^1_i, a) \leq_{\text{suitably}} (N_0, N_1, a)$, problem N^1_i is not unique.

Semi-Proof: Repeat the paper Rami wrote up for me quickly (so $V = L$) or like. Goes nowhere.

We can try to eliminate the use of $2^{\lambda^{++}} > \lambda^{++}$. We could use reduced triples or not by 2.10.

Claim $(*)_8$. *1) Assume there is no minimal (M_0, M_1, a) above some triple.*
For every $M_0 <_K M_q$ in K_λ we can find $M^+_0, M_0 \leq_K M^+_0 \in K_\lambda$ and $M^1_\eta, M^+_0 \leq_K M^1_\eta \in K_\lambda$ such that $\langle\{tp(a, M^+_0, M^1_\eta) : a \in M^1_\eta \backslash M\} : \eta \in {}^\lambda 2\rangle$ are pairwise disjoint.
2) Assume for every $M_0 <_K M_1 <_K M_2$ from $K_\lambda, a \in M_2 \backslash M_1$ we can find $(M_0, M_1, M_2) \leq_{h_\ell} (M'_0, M'_1, M'_2)$ such that $h_\ell(a) = a_\ell, M_0 = M'_0$, $tp(a_1, M_0, M_2) \neq tp(a_2, M_0, M_2)$.
Then we can ?

9.5 Claim. $(*)_8$ *In 4.3 we really get* $|S_{(M_0,M_1,a)}(M)| \leq \lambda$.

Proof. Same (by the model in λ^{++}?)

STILL INTERESTING

9.6 Definition. $(M_0, M_1, a) \in K^3_\lambda$ is weakly minimal if: if
$(M_0, M_1, a) \leq_{h_\ell} (M^\ell_0, M^\ell_1, a_\ell), (M^1_1, M^1_0, h_0 \restriction M_0) = (M^2_1, M^2_0, h_2 \restriction M_2)$ and
$M^\ell_0 \cap M_1 = M_0$ then $\text{tp}(a_1, M^\ell_0, M^\ell_1) = \text{tp}(a_2, M^\ell_0, M^\ell_1)$.

STILL INTERESTING

9.7 Claim. $(*)_8$.
1) There are $M_0 <_K M_1$ in K_λ such that for every $a \in M_1 \backslash M_0, (M_0, M_1, a)$ is weakly minimal and for some $a \in (M_0, M_1, a)$ is reduced and minimal.
2)

9.8 Claim. *If $(M_0, M_1, a) \in K^3_\lambda, p =$ $tp(a, M_0, M_1), \delta < \lambda^+$,*
$\alpha < \beta < \delta < M_0 \leq M_{0,\alpha} \leq M_{0,n,\beta} \leq M_{0,\delta} = \bigcup\limits_{\alpha<\delta} M_0, p \subseteq p_\alpha \in S(M_{0,\alpha})$ a non-algebraic extension then there is a unique $p, \delta, \bigwedge\limits_{\alpha<\delta} p_\alpha \subseteq p_\delta \in S(M_{0,\omega}), p_\delta$ not algebraic.

Proof. As there is one and only one non-algebraic extension $p' \in S(M_\alpha)$ of p for $\alpha \leq \omega$.
Note: $\varphi \in S(N_\delta)$ is algebraic $\Rightarrow q$ realized in $N_\delta \Rightarrow \bigvee\limits_{\alpha<\delta}$ (q realized in N_α) $\Rightarrow \bigvee\limits_{\alpha<\delta}$ ($q \restriction N_\alpha$ algebraic).

9.9 Claim. *1) There is $(M_0, M_1, a) \in K^3_\lambda$ minimal such that:*

$(*)_{(M_0,M_1,a)}$ *for every $N_0 <_K N_1$ in K_λ for every $q \in S_{(M_0,M_1,a)}(N_0)$ is realized in N_1.*

2) In fact every minimal p_0 has such extension p; i.e. $tp(a, M_0, M_1) = p$.

Proof. If not, we build by induction on $\alpha < \lambda^+, N_\alpha, p_2$ such that:

- (a) $N_\alpha \in K_\lambda$ $<_K$-increasing continuous
- (b) $N_\alpha \neq N_{\alpha+1}$
- (c) $N_0 = M_0$
- (d) $p_\alpha \in S(N_\alpha)$ the unique non-algebraic extension of $\text{tp}(a, M_\ell, M_1)$ (hence $\beta < \alpha \Rightarrow p_\beta \subseteq p_\alpha$
- (e) p_α is not realized in $N_{\alpha+1}$
- (f) moreover, p_α is the unique extension of p_0 of p_0 in $S(N_\alpha)$.

For $\alpha = 0$ start with M_0, for limit use 4.8, for successor $\alpha = \beta+1$, if in some proper extension $N_{\alpha+1}$ some conjugate of p_α is omitted in another p_α is omitted and easy. If not: N_β, p_β are as required in $(*)$.

Situation parallel to [Sh 88] with <u>realize</u> and <u>materialize Question</u>: are the unique extension and a conjugate; i.e. if $M_0 < M'_0$ in $K_\lambda, p_0 \in S(M_0)$ minimal,
$p_0 \subseteq p_1 \in S(M_1), p_1$ non-algebraic is there $q \in$ ISO(M_0, M'_0) such that $g(p_0) = p_1$?

<u>STILL INTERESTING</u>

9.10 Definition. Let $p^* \in S(N^*), N^* \in K_\lambda$ be as in 4.9.
We define $\mathrm{rk}_{p^*}(M_0, M_1, a)$ for
$(M_0, M_1, a) \in K^3_\lambda[N^*] = \{(M_0, M_1, a) \in K^3_\lambda : N^* \leq_K M_0\}$ by defining when $\mathrm{rk}_{p^*}(M_0, M_1, a) \geq \alpha$ for α an ordinal

<u>$\alpha = 0$</u>: Always.

<u>$\alpha = \beta + 1$</u>: If there is (M'_0, M'_1, a) such that:

- (i) $(M_0, M_1, a) \leq (M'_0, M'_1, a) \in K^3_\lambda$
- (ii) if $b \in M_1 \backslash M_0, \mathrm{tp}(b^*, M_1) \in p^*$ then $b \in M'_0$
- (iii) $\mathrm{rk}_{p^*}(M'_0, M'_1, a) \geq \beta$
- (iv) $\alpha = \delta$ limit if $\bigwedge\limits_{\beta<\delta} \mathrm{rk}_{p^*}(M_0, M_1, a) \geq \beta$.

<u>Question</u>: $N = \bigcup\limits_{i<\lambda^+} N_i, N_i \in K_\lambda, N \in K_{\lambda^+}$

$$E_j = \{(\alpha, \beta) : N_\alpha \cong_{N_j} N_\beta\}?$$

9.11 Definition. $N_0 <_{p^*} N_1$ if $p^* \in S(N_0)$ nice realized and
$\forall c \in p^*(N_0) \exists (N_0, N', c) \in K^3_\lambda$ reduced $N' \leq_K N_1$.
See [Sh 300],II,§3.

12.1 Lemma - OLD from "non-structure for $\leq^*_\lambda$". *Assume for some $M^* \leq N^*$ from K_λ, we do not have $M^* \leq^*_{\lambda^+} N^*$.*

If $M_0 \in K_{\lambda^+}, \langle M^0_i : i < \lambda^+\rangle$ is a representation of M_0 then we can find $\langle (M^1_\eta, \alpha_\eta) : \eta \in {}^{\lambda^+ >}2\rangle$ such that:

$\bigotimes$
- (a) $\alpha_\eta < \lambda^+$
- (b) $[\nu \triangleleft \eta \in {}^{\lambda^+ >}2 \Rightarrow \alpha_\nu < \alpha_\eta]$
- (c) *if $\eta \in {}^\alpha 2, \alpha < \lambda^+$ a limit order <u>then</u>*

$$\alpha_\eta = \bigcup_{\gamma<\alpha} \alpha_{\eta \restriction \gamma}$$

- (d) $M^0_{\alpha_\eta} \leq_{\mathfrak{K}} M^1_\eta$
- (e) $\langle M^1_{\eta \restriction \alpha} : \alpha \leq \ell g(\eta)\rangle$ *is $\leq_{\mathfrak{K}}$-increasing continuous*
- (f) $M^1_\eta \cap M_0 = M^0_{\alpha_\eta}$

§13 - THIS IS SUPPOSE TO BECOME §11

13.1 Definition.
$\mathrm{Rep}_{\lambda^+} = \{\bar{M} : \bar{M} = \langle M_i : i < \lambda^+ \rangle$ is $<_K$-increasing continuous, $M_i \in K_\lambda\}$.

$\bar{M}^1 <^r_{\lambda^+} \bar{M}^2$ <u>if</u> for every i large enough $M^1_i \leq_K M^2_i$ and
$j \in (i, \lambda^+) : M^2_i \cap M^1_j = M^1_i$ and $\mathrm{NF}_\lambda(M^1_i, M^1_j, M^2_i, M^2_j\}$.

We say $\bar{M}$ rep $\bigcup\limits_{i<\lambda^+} M_i$.

13.2 Claim. *1) (Existence).*
2) (Transitive).
3) (Upper Bound).
4) (Amalgamation) (see 11.4!).

13.3 Definition.

$$\mathrm{Rep}_{\lambda^{+2}} = \Big\{ \mathbf{t} = (\bar{M}, \langle \bar{M}^\alpha : \alpha < \lambda^{+2} \rangle) : \begin{array}{l} \bar{M} = \langle M_\alpha : \alpha < \lambda^{+2} \rangle \\ \text{is } <_K \text{-increasing continuous}, M_\alpha \in K_{\lambda^+}, \\ \bar{M}^\alpha \text{ represent } M_\alpha, \\ \alpha < \beta \Rightarrow \bar{M}^\alpha <^r_{\lambda^+} \bar{M}^\beta \end{array} \Big\}$$

$\mathbf{s} \leq \mathbf{t}$ <u>if</u>: for every i large enough $< \lambda^{+2} : M^{\mathbf{s}}_i \leq_K M^s_j$ and
$j \in (i, \lambda) \Rightarrow M^{\mathbf{s}}_i = M^{\mathbf{t}}_i \cap M^{\mathbf{s}}_i$ and $\bar{M}^{\mathbf{s};\alpha} \leq^*_{\lambda^+} \bar{M}^{\mathbf{t};\alpha}$.

This will enable us to build (by an increasing sequence of members of $\mathrm{Rep}_{\lambda^{+2}}$) a model in $K_{\lambda^{+2}}$.

5 Feb '95 - Discussion

Definition. For $M \in K_{\lambda^+}, S^{\rm nice}(M) = \{{\rm tp}(a, M, N) : M <^*_{\lambda^+} N$ and $a \in N\}$.

Question: Try in §12 to treat types in $S(M), M \in K_{\lambda^+}$ (not in $S^{\rm nice}(M)$).

Question: Can we find 3-dimensional grid $M_{\alpha_1,\alpha_2,\alpha_3} (\alpha_\ell \le \delta_3)$ with nice amalgamation, then define nice amalgamation in K_{λ^+}?

Answer: Start with 2-dimensional grid $M_{i,j}\, (i \le \delta_1, j \le \delta_1)$ in K_{λ^+} with neighbors being λ^+-locally reduced nice minimal. Now reflect down; we get $\bigcup$ because of minimal reduced.

9/4/95
From Sunday afternoon proof reading with this yesterday:
in Feb $\le^*_{\lambda^+}, <^+_\lambda$ were done properly, now add the proof of $\le_{\mathfrak{K}} \restriction K_{\lambda^+} = \le^*_{\lambda^+}$ - see also §3 finish it (Feb - only non-saturation).

Also §3 bad will redo.
Next: generalizing [Sh 87b]. First the (strong) real model theoretic condition used in first proof (increase f) then the one with non-saturated; deduce as hinted the existence of maximal invariant, then try to eliminate.

For this we need: $\otimes$ of 6.9(0.2) is frequent enough. Or use atomic extension in K_{λ^+}:use nice down of M_0.

Question: In §12 get the existence of $N^0_i\, (i \le \delta + 1), N^1_i\, (i \le \delta + 1)$ increasing continuous in $K_{\lambda^+}, N^0_{\delta+1} \bigcup\limits_{N^0_i}^{N^1_\delta} N^1_i$ for $i < \delta$.

$N^0_{\delta+1} \bigcup\limits_{N^0_i}^{N^1_{\delta+1}} N^1_i$ for $i < \delta$, does not imply $N^0_{\delta+1} \bigcup\limits_{N^0_\delta}^{N^1_{\delta+1}} N^1_\delta$ all in K_λ.
From this ??

Try

11 Definition. ${\rm NF}_{\lambda^+}(M_0, M_1, M_2, M_3)$ if: $M_0 \le^*_{\lambda^+} M_\ell \le^*_{\lambda^+} M_3$, and $\langle M_{0,i}, M_{1,i} : i < \lambda^+\rangle$ a nice decomposition of (M_0, M_1) (so $\langle M_{\ell,i} : i < \lambda^+\rangle$ a representation of M_ℓ for $\ell = 0, 1, i < j \Rightarrow {\rm NF}_\lambda(M_{0,i}, M_{0,j}, M_{1,i}, M_{1,j})$) then we can extend it to a nice decomposition of (M_2, M_3).

* * *

Another avenue (Recommended) Let

$$\mathfrak{K} = \Big\{ \mathbf{t} = (M, \bar M, \langle (\bar M^a_i, M^b_i) : \ell < \lambda^{+2}\rangle : M \in K_{\lambda^{+2}},$$
$$\bar M \text{ a rep of } M, M_i, (\bar M^a_i, M^b_i) \text{ is a nice}$$
$$\text{rep of } (M_i, M_{i+1}) \Big\}$$

$\mathbf{s} \leq \mathbf{t}$ if: $M^{\mathbf{s}} \leq_K M^{\mathbf{t}}, (\bar{M}^{\mathbf{s}}, \bar{M}^{\mathbf{t}})$ is a rep of $(M^{\mathbf{s}}, M^{\mathbf{t}})$ for every $i < \lambda^{+2}$ large enough $(\bar{M}^a_i, \bar{M}^b_i)^{\mathbf{s}} \leq (\bar{M}^a_i, \bar{M}^b_i, M^s_i <^*_{\lambda^+} M^t_i)^{\mathbf{t}}$.

(Natural Definitions) and all nice.

<u>Question</u>: If $\mathbf{s}$ gives ? we want $\mathbf{t} > s$, choose $M^{\mathbf{t}}_i$ arriving to i limit - why?

p.26
1995/10/17 - was part of Def of λ-coding

$(*)_2$ for some club E of λ^+, for every $\delta \in S \cap E$ we have $M^1_\delta \leq_{\mathfrak{K}^+} M^2_\delta, M^1_{\delta+\mathbf{f}^1(\delta)} \leq_{\mathfrak{K}^+} M^2_{\delta+\mathbf{f}^2(\delta)}$ and:

$\bigotimes$ there are no $\bar N^\ell = \langle N_\zeta : \zeta \leq \mathbf{f}^2(\delta)\rangle$, for $\ell = 1, 2, N, h$ such that :

(a) $M^2_{\delta+\mathbf{f}^2(\delta)\restriction\tau} \leq_{\mathfrak{K}} N \in \mathfrak{K}_{<\lambda}$
(b) $N_\zeta \in \mathfrak{K}_\lambda$ has universe $\subseteq \lambda^+$
(c) $\bar N$ is $\leq_{\mathfrak{K}}$-increasing continuous (by $\leq_{\mathfrak{K}}$)
(d) $M^1_{\delta+\zeta} \leq_{\mathfrak{K}^+} N_\zeta$ for $\zeta \leq \mathbf{f}^2(\delta)$
(e) $N_{\zeta+1} = F(M^1_{\delta+\zeta}, M^1_{\delta+\zeta+1}, N_\zeta, |N_{\zeta+1}|, a)$ (so is well defined) for $\zeta < \mathbf{f}^2(\delta)$
(f) h is a $\leq_{\mathfrak{K}}$-embedding of $N_{\mathbf{f}^2(\delta)} \restriction \tau$ into $N \restriction \tau$ over $M^1_{\delta+\mathbf{f}^2(\delta)} \cup M^2_\delta$.

From try to improve $\mu_{wd}(\lambda)$ to 2^λ

p.28-30

3.13 Definition. Assume $(\bar M^0, \mathbf{f}) \leq (\bar M^1, \mathbf{f}) \leq (\bar M^2, \mathbf{f})$ are from $K^{\rm qs}_\lambda$ (for simplicity). We define

$$S((\bar M^0, \mathbf{f}), (\bar M^1, \mathbf{f}), (\bar M^2, \mathbf{f})) =: \Big\{\delta < \lambda^+ \text{:for every } i \leq \mathbf{f}(\delta), M^0_{\delta+i} \leq_{\mathfrak{K}} M^1_{\delta+i} \leq_{\mathfrak{K}} M^2_{\delta+i}, \text{ and } M^0_\delta = M^1_\delta \cap M^0, M^1_\delta = M^2_\delta \cap M^1 \text{ and disjoint amalgamation of } M^0_{\delta+\mathbf{f}(\delta)}, M^2_\delta \text{ over } M^0_\delta \text{ is not unique even having } M^1_{\delta+\mathbf{f}(\delta)} \text{ or any } M^1_\alpha, \alpha \geq \delta + \mathbf{f}(\delta)\Big\}.$$

3.12 Claim. *All $(\bar M, \mathbf{f})$ below are from $K^{\rm qs}_\lambda$ (or $K^{\rm qr}_\lambda$ over λ^+ and does not matter as F is not present).*
1) If $(\bar M^\ell, \mathbf{f}) \leq (\bar M^{\ell+1}, \mathbf{f})$ for $\ell < 5$ <u>then</u>

$$S\left((\bar M^0, \mathbf{f}), (\bar M^2, \mathbf{f}), (\bar M^3, \mathbf{f})\right) \subseteq S\left((\bar M^0, \mathbf{f}), (\bar M^1, \mathbf{f}), (\bar M^4, \mathbf{f})\right) \ mod\ D_{\lambda^+}.$$

2) If D is a λ^{+2}-saturated normal filter over λ^+ and $(\bar M^\ell, \mathbf{f}) \leq (\bar M^{\ell+1}, \mathbf{f})$ over $\ell = 0, 1, 2$ <u>then</u> for some $(\bar M^3, \mathbf{f})$ we have:

(a) $(\bar{M}^2, \mathbf{f}) \leq (\bar{M}^3, \mathbf{f})$
(b) *if* $(\bar{M}^3, \mathbf{f}) \leq (\bar{M}^4, \mathbf{f}) (\in K^{\mathrm{qr}}_\lambda)$ *then*

$$S\left((\bar{M}^0, \mathbf{f}), (\bar{M}^1, \mathbf{f}), (\bar{M}^3, \mathbf{f})\right) = S\left((\bar{M}^0, \mathbf{f}), (\bar{M}^1, \mathbf{f}), (\bar{M}^4, \mathbf{f})\right) \ \textit{mod } D.$$

3) [K has amalgamation in λ^+ *if we take first variant of S]*

$$S\left((\bar{M}^0, \mathbf{f}), (\bar{M}^1, \mathbf{f}), (\bar{M}^2, \mathbf{f})\right) / D_{\lambda^+}$$

depend just on $\bar{M}^0, \mathbf{f}, M^1, M^2$ *(and we can weaken the demand in part (1) to* $M^\ell \leq_{\mathfrak{K}} M^{\ell+1}$*).*

4) [$\mathfrak{K}$ *has amalgamation in* λ^+, D *a* λ^{++}*-saturated normal filter on* λ^+*].*
If $(\bar{M}^0, \mathbf{f}) \leq (\bar{M}^1, \mathbf{f})$*, <u>then</u> for some* $S^*/D = S^*_D\left((\bar{M}^0, \mathbf{f}), (\bar{M}^1, \mathbf{f})\right)$*:*

(a) *if* $(\bar{M}^1, \mathbf{f}) \leq (\bar{M}^2, \mathbf{f})$ *then*
$S\left((\bar{M}^0, \mathbf{f}), (\bar{M}^1, \mathbf{f}), (\bar{M}^2, \mathbf{f})\right) \subseteq S^*$ *mod D*
(b) *if* $(\bar{M}^1, \mathbf{f}) \leq (\bar{M}^2, \mathbf{f})$ *then for some* $(\bar{M}^3, \mathbf{f})$ *we have*
$(\bar{M}^2, \mathbf{f}) \leq (\bar{M}^3, \mathbf{f})$ *and for* $(\bar{M}^3, \mathbf{f})$ *we get equality.*

5) [Assumption as in (4)].
If $(\bar{M}^\ell, \mathbf{f}) \in K^{\mathrm{qr}}_\lambda$ *and* $(\bar{M}^\ell, \mathbf{f}) \leq (\bar{M}^{\ell+1}, \mathbf{f})$ *for* $\ell = 0, 1$ *<u>then</u>*

$$S\left((\bar{M}^0, \mathbf{f}), (\bar{M}^1, \mathbf{f})\right) \supseteq S\left((\bar{M}^0, \mathbf{f}), (\bar{M}^2, \mathbf{f})\right) \ \textit{mod } D.$$

6) [D a λ^{++}*-saturated normal filter on* λ^+*].*
For every $(\bar{M}^0, \mathbf{f}) \in K^{\mathrm{qr}}_\lambda$ *for some* $(\bar{M}^1, \mathbf{f}) \geq (\bar{M}^0, \mathbf{f})$ *for every* $(\bar{M}^2, f) \geq (\bar{M}^1, f)$ *we have*

$$S_D\left((\bar{M}^0, \mathbf{f}), (\bar{M}^1, f)\right) = S_D\left((\bar{M}^0, \mathbf{f}), \bar{M}^2, \mathbf{f})\right) \ \textit{mod } D$$

as earlier this is called $S\left((\bar{M}^0, \mathbf{f})\right) / D$*.*

Proof. Straight.

3.13 Discussion. Now the proof splits into cases. In the novel one we can repeat the proof of 3.10 with the complimentary assumption "$\mathrm{WDmId}_{\mathrm{def}}(\lambda^+)$ is λ^{+2}-saturated", but we need success on a positive set not just on a stationary set. So we need to be more careful on which ideal we are dealing with (this will also improve the result in [Sh 87b]).

As it happens, the small set which is missing is "small" enough for the weak diamond to "overcome".

3.14 Claim. *Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ or at least the definitional weak diamond for λ^+ and λ^{++}.*

Assume further:

(a) $(\bar{M}^*, \mathbf{f}) \in K^{\rm qs}_\lambda$
(b) *The ideal $\mathcal{I}$ associated with $(\bar{M}^*, \mathbf{f})$ is λ^{++}-saturated (not like 3.6?!) and D be the dual filter*
(c) $S((\bar{M}^*, \mathbf{f})) \neq \lambda^+/D$
(d) *$\mathfrak{K}$ has the weak λ^+-coding.*

<u>*Then*</u> $I(\lambda^{+2}, K) = 2^{\lambda^{+2}}$.

Proof. We can find $\langle S^*_\alpha : \alpha < \lambda^{++}\rangle$ such that:

$$S_\alpha \subseteq \lambda^+$$

$$[\alpha < \beta \Rightarrow |S^*_\alpha \backslash S^*_\beta| < \lambda^+]$$

and

$$S'_\alpha =: S_{\alpha+1} \backslash S_{\alpha+1} \notin \text{ WDmId}_{\rm def}(\lambda^+).$$

We again choose by induction on $\alpha < \lambda^{+2}$ for every $\eta \in {}^\alpha 2$ a pair $(\bar{M}^\eta, \mathbf{f}^\eta)$ such that:

(a) $(\bar{M}^\eta, \mathbf{f}^\eta) \in K^{\rm qs}_\lambda$
(b) if $\nu \triangleleft \eta$ then $(\bar{M}^\nu, \mathbf{f}^\nu) \leq_F (\bar{M}^\eta, \mathbf{f}^\eta)$
(c) $f^\eta \restriction (\lambda \backslash S_{\ell g(\eta)}) = 0_{\lambda \backslash S_{\ell g(\eta)}}$
(d) $(\bar{M}^\eta, \mathbf{f}^\eta) <^{\rm at}_{F, a_{\eta\char`\^\langle 0\rangle}} (\bar{M}^{\eta\char`\^\langle 0\rangle}, \mathbf{f}^{\eta\char`\^\langle 0\rangle})$
(e) $(\bar{M}^\eta, \mathbf{f}^\eta) \leq^{\rm at}_{F, a_{\eta\char`\^\langle 1\rangle}} (\bar{M}^{\eta\char`\^\langle 1\rangle}, \mathbf{f}^{\eta\char`\^\langle 1\rangle})$
(f) if $(\bar{M}^\eta, \mathbf{f}^{\eta\char`\^\langle 0\rangle}) \leq (\bar{M}', \mathbf{f}')$ <u>then</u> $M^{\eta\char`\^\langle 0\rangle}$ cannot be $\leq_{\mathfrak{K}}$-embedded into M' over M^η.

Again there are no problems (the difference is in clause (c)).

3.15 Discussion. For the remaining case we produce M_η (for $\eta \in {}^{\lambda^{++}}2$) as usual such that

$$\eta\char`\^\langle \ell\rangle \triangleleft \nu_\ell \in {}^{(\lambda^{+2})}2 \Rightarrow M_{\nu_0} \ncong_{M_{\langle\rangle}} M_{\nu_1} [Saharon].$$

From proof of 10.1(2)
p.31
The following serves to prove 6.5.

6.5A Claim. *Assume $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda$.*

If $K^{3,\mathrm{uq}}_\lambda = \emptyset$ (see Definition 6.1(2)) <u>then</u> there is F an amalgamation choice function F for $\mathbf{C} = \mathbf{C}_{\mathfrak{K},\lambda^+}$ with the weak λ^+-coding property.

Proof. The point is that for $\bar M \in \mathbf{S}eq_{\lambda^+}[\mathbf{C}]; \alpha < \lambda^+, M_\alpha <_{\mathfrak{K}} N$ then for some $\beta \in (\alpha, \lambda^+)$ we have:

> M_β is universal over M_α, so as $K^{3,\mathrm{uq}}_\lambda$ necessarily
> $(M_\alpha, M_\beta, N) \notin K^{\mathrm{uniq}}_\lambda$ and rest should be clear.

6.5B Claim. *1) Assume each $M \in K_{\lambda^+}$ is saturated above λ.*

If $(M, N, a) \in K^3_\lambda$ it and every $(M', N'a) \in K^3_\lambda$ above it has the extension property but for every $(M'', N'', a) \geq (M', N', a) \geq (N, N, a)$ (all in $K^3_{\lambda^+}$) for some $M^* \geq_{\mathfrak{K}} M''$ from K_{λ^+}, in amalgamation $(M^*, N^*, a) \geq (M'', N'', a)$ the type of $M^* \cup N$ inside N^* is not determined <u>then</u> some F (actually $\mathbf{F}^*$) has λ^+-coding.
2) If above we just require that the type of $M^* \cup N''$ inside N^* is not determined, <u>then</u> some F (actually $\mathbf{F}^*$) has weak λ^+-coding.
3) We can restrict ourselves to disjoint embedding; i.e. use $(K^3_{\lambda^+}, \leq_{dj})$.

6.5C Discussion. 1) We get $IE(\lambda^{+2}, \mathfrak{K}) = 2^{\lambda^{+2}}$ when $(2^{\lambda^+})^+ < 2^{\lambda^{+2}}$. Also if $\mathrm{cf}([2^{\lambda^+}]^{\leq \lambda^+}, \subseteq) = 2^\lambda$ we can deal with this in another way. As in 6.4, if $\lambda \geq \beth_\omega$ help. See more in [Sh 600].
2) Now we can use UDmId to get $I(\lambda^{++}, K)$ large also when the ideal is saturated. We need that advancing horizontally is unique.

1995/10/21 - 1) each section/th on what it depend
2) §8 change consistently to allow δ_2, δ_3 be 1
p.62
Choose $N_{j,\zeta} \leq_{\mathfrak{K}} M_{1,i+\zeta}$ for $\zeta \leq \lambda \times (1+i)$ such that $N_{j,\zeta}$ is $\leq_{\mathfrak{K}}$-increasing continuous, $\xi < \zeta \Rightarrow N_{j,\zeta} \cap M_{1,i+\xi} = N_{j,\xi}$ and $(N_{j,\zeta}, N_{j,\zeta+1}) \cong (M^*, N^*)$. So $N_{j,\lambda}$ is $(\lambda, \mathrm{cf}(\lambda \times (1+i)))$-saturated. Now choose $\langle N_{\varepsilon,i} : \varepsilon < 1+i \rangle$ such that
$\bigcup\limits_{\varepsilon < 1+i} N_{\varepsilon,i} = \bigcup\limits_{\zeta < \lambda \times (1+i)} N_{j,\zeta}$, as required in clause (f) and lastly let
$M_{i,i+1} = M_{1,\lambda \times \lambda \times (i+1)}$.

Moved from p.23

⟨1.8AX⟩ Claim. *1) We can in ⟨1.3⟩, ⟨1.3A⟩ and ⟨1.4⟩ (omitting (β)) (allowing replacing* $\bar{\mu} = \langle M_\eta : \eta \in {}^{\lambda >}2\rangle$ *by a* $\bar{\mu}'$ *with similar properties) replace the assumption on the weak diamond (ideal) by one on the definitional variant; thus omitting* $2^{<\lambda} < 2^\lambda$ *from the assumption.*

Proof. Same proof. (Use $\chi > 2^{(2^\lambda)}$, so the $<^*_\chi$-first $\langle M_\eta : \eta \in {}^{\lambda >}2\rangle$ as required is definable) except in the proof of ⟨1.4⟩(1)(α) case I. The conclusion $|A_{\eta,g}| \leq 2^\lambda$ is no longer of value. However, as we are speaking of isomorphisms; i.e. onto embedding the proof of case II work in case I, too.

moved from p.46
<u>Remark to end of proof of ⟨3.13A⟩ - 1996/3/17</u>
(UDmId for full models; use of [Sh 430, 3.4]).

1995/12/3

<u>moved from</u> 3.14: toward minimal pairs are dense

<u>moved from p.36</u>

(β) $(M^*, N^*, a) \in K^3_\lambda$ and above it there is no minimal pair, moreover? there is $M^\otimes \in K_{\lambda^+}$ such that $M^* \leq_{\mathfrak{K}} M^\otimes$ and: <u>if</u> $(M^*, N^*, a) \leq_h (M, N, a) \in K_\lambda$ and $M^* \leq_{\mathfrak{K}} M <_{\mathfrak{K}} M^\otimes, h \restriction M^* = \mathrm{id}_{M^*}$ <u>then</u> for some $M' \in K_\lambda, M \leq_{\mathfrak{K}} M' \leq M^\otimes$, and
- (i) $\mathrm{tp}(a, M, N)$ has more than one extension in $\mathcal{S}(M')$ or at least
- (ii) there are h_ℓ, N_ℓ for $\ell = 1, 2$ such that $(M, N, a) \leq_{h_\ell} (M', N_\ell, a)$ and $h_\ell \restriction M = \mathrm{id}_M$, and those amalgamations are contradictory; i.e. for no N', f do we have $N_2 \leq_{\mathfrak{K}} N' \in K_\lambda, f$ is a $\leq_{\mathfrak{K}}$-embedding of N_1 into N_2 such that $f \restriction M' = \mathrm{id}_{M'}, h_2 = f \circ h_0$.

<u>moved from later-same proof - moved from p.37</u>

- $(*)_0$ $M^1_i \leq_{\mathfrak{K}} M, M^1_i \leq_{\mathfrak{K}} N^1_i$ (so automatically $a \notin M^1_i$)
- $(*)_1$ for no i are there j and N such that $(M^1_i, N^1_i, a) \leq_h (M^1_j, N, a), h \restriction M^1_i = \mathrm{id}_{M^1_i}, M^1_i \neq h(N^1_i) \cap M^1_j$
- $(*)_2$ for each $i < j, (M^1_i, M^2_i, a), M^1_j$ are not as in the statement (β) above
- $(*)_3$ $\bigcup_{i<\lambda^+} M^1_i = M$.

[we can use the amalgamation property (which holds by ⟨2.4⟩(1)) and the failure of (β)].

On References moved w. 96/6/27
1) [Sh 300] in ⟨0.19⟩, ⟨0.22⟩, ⟨0.23⟩, ⟨5.4⟩.
2) [Sh 88] in p.19 ⟨1.6⟩ to ⟨1.4⟩, p.64 ⟨**6.13**⟩ to ⟨6.6⟩, ⟨1.4⟩(6), ⟨6.6⟩,p.65.
—> scite{6.13} undefined
3) Inside ⟨**6.8**⟩ to ⟨6.5⟩.
—> scite{6.8} undefined
4) [Sh:c]: (⟨0.24⟩) (weak).

Moved from §7 in March 1997:

⟨**7.4**⟩ **Claim.** *? If $\alpha < \lambda^+$ is divisible by λ, and $\langle N_i : i \le \alpha\rangle$ is strictly increasing continuous then N_α is $(\lambda, cf(\alpha))$-saturated over N_0.*

Proof. Let $p_0 \in \mathcal{S}(N_0)$ be minimal reduced (exists e.g. by ⟨7.0⟩(d)) hence inevitable (by ⟨7.1⟩), and $p_i \in \mathcal{S}(N_i)$ be the unique non-algebraic extension of p_0 in $\mathcal{S}(N_i)$. So p_i is increasing continuous (⟨4.2A⟩(3)) by ⟨7.3⟩? each p_i is conjugate to p_0 hence is inevitable by ⟨7.1⟩ hence if $i < \alpha$ then p_i is realized by some $c_i \in N_{i+1}\backslash N_i$, and in particular p_α is inevitable. So $\langle N_i : i \le \alpha\rangle, \langle p_i : i \le \alpha\rangle$ are as in ⟨5.3⟩. So by ⟨5.4⟩ we have N_α is universal over N_0.

The saturativity is proved as in the proof of ⟨5.4⟩, taking care that M^1_{i+1} is $(\lambda, \mathrm{cf}(\lambda))$-saturated over M_i, $M^1_\alpha = N^1_\alpha \cong_{N_0} N_\alpha$ is $(\lambda, \mathrm{cf}(\alpha))$-saturated as required. (Note: if α is divisible by $\lambda\omega$, just quote ⟨5.4⟩). $\square_{\langle 7.4\rangle}$

Add to: Justifying the comment in the end of §5 (97/3/30):

Assume

$\bigotimes$ $p \in S(M), p$ is minimal but reduced, $M \in K_\lambda$.

Claim. *If I is a linear order $|I| = \lambda$, with $> \lambda$ Dedekind cuts, then we cannot find $\langle M_T : t \in I\rangle$ such that:*

(i) $M_t \in K_\lambda$

(ii) $I \models s < t \Rightarrow M_t$ *is a universal extension of* M_s.

Proof. For every Dedkind cut of I with $J \ne \emptyset, J$ with no last element, let $\langle t^x_i : i < \kappa_\lambda\rangle$ be $<_I$-increasing cofinal in J. Let $M_x = \bigcup\limits_{i<\kappa} M_{t^x_i} = \bigcup\limits_{t\in J} M_t$. Let $p_x \in \mathcal{S}(M_x)$ be minimal not reduced (exists by $\boxtimes$).
We claim:

(∗) $t \in J \Rightarrow p \restriction M_t$ not minimal
Why? if so, then by ⟨7.4⟩(2), p_x is reduced continuously (first there is $i < \kappa_x, t <_I t^x_i$).

Let $M^* = \bigcup_{t \in J} M_t$, let $q_x \in \mathcal{S}(M)$ be a non-algebraic extension of p_x. So $q_x \restriction M_t$ is minimal iff $t \notin J$ (if $t \in J$, use $(*)$, if $t \notin J$, then $q_x \restriction M_t$ is a non-algebraic extension of p_x.

So $|\mathcal{S}(M)| \geq |\{q_x : x$ a Dedekind cut of I, low part non-empty with no last element$\}| >$ $\mathrm{Ded}(I) - \lambda > \lambda$.

We can build $\langle M_t : t \in I \rangle$for: $I = ({}^{\omega >}\lambda, <_{\ell x})$ and for any I of cardinality λ if there are E.M. models (e.g. $\{\|M\| : M \in K\}$ unbounded.

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