K-QUASICONVEXITY REDUCES TO QUASICONVEXITY

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Abstract. The relation between quasiconvexity and k-quasiconvexity, $k \geq 2$, is investigated. It is shown that every smooth strictly k-quasiconvex integrand with p-growth at infinity, $p > 1$, is the restriction to k-th order symmetric tensors of a quasiconvex function with the same growth. When the smoothness condition is dropped, it is possible to prove an approximation result. As a consequence, lower semicontinuity results for k-th order variational problems are deduced as corollaries of well-known first order theorems. This generalizes a previous work by Dal Maso, Fonseca, Leoni and Morini, in which the case $k = 2$ was treated.

Keywords: quasiconvexity, higher order variational problems.

1. Introduction

We consider higher order variational problems, in which the energy functional has the expression

$$u \longmapsto \int_\Omega f(x, u, \nabla u, \dots, \nabla^k u)\, dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded, $N, k \geq 2$ are integer, and f is a scalar function satisfying suitable growth conditions. Although our treatment can be extended to the vectorial case, to keep the formulation as simple as possible we will treat the case of scalar functions $u : \Omega \to \mathbb{R}$. Functionals of this type appear in the study of elastic materials of grade k (see [23]), in the theory of second order structured deformations (see [21]), in the Blake-Zisserman model for image segmentation in computer vision (see [5]), in gradient theories of phase transitions within elasticity regimes (see [7], [15], [20]), and in the description of equilibria of micromagnetic materials (see [9], [6], [20], [22]). In order to study lower semicontinuity of functionals of this type, Meyers introduced in [18] the notion of k-quasiconvexity (see also [3] and [13]), extending the definition of quasiconvexity given by Morrey in [19].

Let $E_k \subset \overbrace{\mathbb{R}^N \times \ldots \times \mathbb{R}^N}^{k \text{ times}} = \mathbb{R}^{N^k}$ be the set of k-th order tensors of $\mathbb{R}^N$ that are symmetric with respect to all permutations of indices. In particular, E_2 coincides with the set of the symmetric $N \times N$ matrices. A function $f \in L^1_{loc}(E_k)$ is said to be *k-quasiconvex* if

$$\int_Q \big[f(A + \nabla^k \phi) - f(A)\big]\, dx \geq 0$$

for every $A \in E_k$ and every $\phi \in C^k_c(Q)$, where $Q = (0,1)^N$ is the open unit cube in $\mathbb{R}^N$, and $C^k_c(Q)$ is the set of functions of class C^k with compact support in Q. We recall that a function $F \in L^1_{loc}(\mathbb{R}^{N^k})$ is said to be 1-quasiconvex (or simply quasiconvex) if

$$\int_Q [F(A + \nabla\varphi) - F(A)]\, dx \geq 0$$

for every $A \in \mathbb{R}^{N^k}$ and every $\varphi \in C^1_c(Q; \mathbb{R}^{N^{k-1}})$. In [18], the author proved that k-quasiconvexity is a necessary and sufficient condition for sequential lower semicontinuity of (1.1) with respect to weak convergence in the Sobolev space $W^{k,p}(\Omega)$, under appropriate p-growth and continuity conditions on the integrand f. This result has been later extended to the case where f is a Carathéodory integrand by Fusco (see [13]) and by Guidorzi and Poggiolini (see [14]), for $p = 1$ and $p > 1$ respectively.

The aim of this paper is to investigate the relation between k-quasiconvexity and quasiconvexity. When $k = 2$ this problem has been studied by Dal Maso, Fonseca, Leoni and Morini. In [8], they prove that every strictly 2-quasiconvex function (see condition (a) below) of class C^1, whose gradient is locally Lipschitz continuous, is the restriction to symmetric matrices of a 1-quasiconvex function. We extend here this result to the case $k > 2$.

Theorem 1.1. *Let $k \in \mathbb{N}$, $k \geq 2$. Let $f \in C^1(E_k)$, and let $1 < p < \infty$, $\mu \geq 0$, $L > 0$, $\nu > 0$. Assume that*

(a) *(strict k-quasiconvexity)*

$$\int_Q \left[f(A + \nabla^k \phi) - f(A)\right] dx \geq \nu \int_Q \left(\mu^2 + |A|^2 + |\nabla^k \phi|^2\right)^{\frac{p-2}{2}} |\nabla^k \phi|^2 dx$$

for every $A \in E_k$ and every $\phi \in C_c^k(Q)$;

(b) *(Lipschitz condition for gradients)*

$$|\nabla f(A + B) - \nabla f(A)| \leq L \left(\mu^2 + |A|^2 + |B|^2\right)^{\frac{p-2}{2}} |B| \tag{1.2}$$

for every $A, B \in E_k$.

Then there exists a $1-$quasiconvex function $F : \mathbb{R}^{N^k} \to \mathbb{R}$ such that

$$F(A) = f(A) \qquad \forall\, A \in E_k, \tag{1.3}$$

$$|F(A)| \leq c_f(1 + |A|^p) \qquad \forall\, A \in \mathbb{R}^{N^k}, \tag{1.4}$$

for a suitable constant c_f depending on f.

Notice that the above conditions (a) and (b) together imply $L \geq \nu$ (see Proposition 2.8). When $p \geq 2$, we also give an explicit expression for the function F (see formula (3.9)). The proof of Theorem 1.1 (see Section 3) is obtained by iterating $k - 1$ times a refined version of [8, Theorem 1] (see Lemma 3.1 for the case $1 < p < 2$ and Lemma 3.2 for the case $p \geq 2$).

It is not clear whether Theorem 1.1 still holds true by weakening condition (1.2). However, if we substitute (1.2) with the milder (see Proposition 2.9) condition (1.5), we obtain an approximation result for the function f. More precisely, we show that a strictly k-quasiconvex function with p-growth at infinity can be obtained as pointwise limit of a sequence of 1-quasiconvex functions with the same growth (see [8, Theorem 2] for the case $k = 2$).

Theorem 1.2. *Let $k \in \mathbb{N}$, $k \geq 2$. Let $1 < p < \infty$, $\mu \geq 0$, $\nu > 0$, $M > 0$, and let $f : E_k \to \mathbb{R}$ be a measurable function such that*

(a) *(strict k-quasiconvexity)*

$$\int_Q \left[f(A + \nabla^k \phi) - f(A)\right] dx \geq \nu \int_Q \left(\mu^2 + |A|^2 + |\nabla^k \phi|^2\right)^{\frac{p-2}{2}} |\nabla^k \phi|^2 dx$$

for every $A \in E_k$ and every $\phi \in C_c^k(Q)$;

(b) *(p-growth condition)*

$$|f(A)| \leq M(1 + |A|^p) \tag{1.5}$$

for every $A \in E_k$.

Then there exists an increasing sequence $\{F_i\}_{i\in\mathbb{N}}$ of $1-$quasiconvex functions $F_i : \mathbb{R}^{N^k} \to \mathbb{R}$, such that

$$\lim_{i\to+\infty} F_i(A) = f(A) \quad \forall\, A \in E_k, \tag{1.6}$$

$$|F_i(A)| \leq M_i(1 + |A|^p) \quad \forall\, A \in \mathbb{R}^{N^k}, \quad \forall\, i \in \mathbb{N}, \tag{1.7}$$

where $\{M_i\}_{i\in\mathbb{N}}$ is a sequence of positive constants depending only on i and on the constants p, μ, ν, M, but not on the specific function f.

To show this, we use the property that every k-quasiconvex function with p-growth is locally Lipschitz. We give here a proof of this fact (see Proposition 2.7), that was already known in the cases $k = 1$ (see [17]) and $k = 2$ (see [14]). Thanks to Theorem 1.2, the study of lower semicontinuity of (1.1) reduces to a first order problem. Thus, when f is a k-quasiconvex normal integrand (see assumption (a) below) we can prove the following result (see [8, Theorem 3] for the case $k = 2$). Here we use the notation

$$E_1 := \mathbb{R}^N, \qquad E_{[k-1]} := \mathbb{R} \times E_1 \times ... \times E_{k-1},$$

and

$$SBH^{(k)}(\Omega) := \{u \in W^{k-1,1}(\Omega) : \nabla^{k-1}u \in SBV(\Omega; E_{k-1})\}.$$

Theorem 1.3. *Let $k \in \mathbb{N}, k \geq 2$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let*

$$f : \Omega \times E_{[k-1]} \times E_k \to [0, +\infty)$$

be a measurable function such that:

(a) *$f(x, \cdot, \cdot)$ is lower semicontinuous on $E_{[k-1]} \times E_k$ for $\mathcal{L}^N$-a.e. $x \in \Omega$;*
(b) *$f(x, \mathbf{v}, \cdot)$ is k-quasiconvex on E_k for $\mathcal{L}^N$-a.e. $x \in \Omega$ and every $\mathbf{v} \in E_{[k-1]}$;*
(c) *there exist a locally bounded function $a : \Omega \times E_{[k-1]} \to [0, +\infty)$ and a constant $p > 1$ such that*

$$0 \leq f(x, \mathbf{v}, A) \leq a(x, \mathbf{v})(1 + |A|^p)$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$ and every $(\mathbf{v}, A) \in E_{[k-1]} \times E_k$.

Then

$$\int_\Omega f(x, u, \nabla u, \dots, \nabla^k u)\, dx \leq \liminf_{j\to+\infty} \int_\Omega f(x, u_j, \nabla u_j, \dots, \nabla^k u_j)\, dx$$

for every $u \in SBH^{(k)}(\Omega)$ and any sequence $\{u_j\} \subset SBH^{(k)}(\Omega)$ converging to u in $W^{k-1,1}(\Omega)$ and such that

$$\sup_j \Big(\|\nabla^k u_j\|_{L^p(\Omega)} + \int_{S(\nabla^{k-1}u_j)} \theta(|[\nabla^{k-1}u_j]|)\, d\mathcal{H}^{N-1} \Big) < +\infty,$$

where $\theta : [0, +\infty) \to [0, +\infty)$ is a concave, nondecreasing function such that

$$\lim_{t\to 0^+} \frac{\theta(t)}{t} = +\infty,$$

$\nabla^k u$ is the density of the absolutely continuous part of $D\left(\nabla^{k-1}u\right)$ with respect to the N-dimensional Lebesgue measure, and $[\nabla^{k-1}u_j]$ denotes the jump of $\nabla^{k-1}u_j$ on the jump set $S(\nabla^{k-1}u_j)$.

This extends to the k-th order setting a lower semicontinuity property of 1-quasiconvex functions in $SBV(\Omega; \mathbb{R}^d)$ due to Ambrosio (see [2]) and later generalized by Kristensen (see [16]), and a lower semicontinuity theorem for 2-quasiconvex integrands in $SBH(\Omega; \mathbb{R}^d)$ proven by Dal Maso, Fonseca, Leoni and Morini (see [8]). As a corollary, we recover [14, Theorem 7.1].

Corollary 1.4. *Let Ω, f, k and p be as in Theorem 1.3. Then*

$$\int_\Omega f(x, u, \nabla u, \dots, \nabla^k u)\, dx \leq \liminf_{j\to\infty} \int_\Omega f(x, u_j, \nabla u_j, \dots, \nabla^k u_j)\, dx$$

for every $u \in W^{k,p}(\Omega)$ and any sequence $\{u_j\} \subset W^{k,p}(\Omega)$ weakly converging to u in $W^{k,p}(\Omega)$.

We remark that in [14] Guidorzi and Poggiolini require the function f to be locally Lipschitz continuous with respect to the last variable. As already mentioned, we do not need this hypothesis, since we prove here that this is a direct consequence of k-quasiconvexity and p-growth.

Finally, we mention that it remains still an open problem to prove the analogue of Theorem 1.3 for the case $p = 1$, even when $k = 2$, unless very special functions f are considered (see [11]). This will probably require new and original ideas. Indeed, we think that for $p = 1$ the fundamental Korn-type inequalities (see Lemma 2.13 and Lemma 2.14) used in the proofs of Theorem 1.1 and Theorem 1.2 fail, although we do not have any explicit counterexample.

The plan of the paper is as follows. In Section 2 we give the setting of the problem. Section 3 contains the proof of Theorem 1.1, while Theorem 1.2 and Theorem 1.3 are proved in Section

4. Finally, some auxiliary results that are extensively used in the paper can be found in the Appendix.

2. Setting

Throughout the paper N and k are fixed integer numbers, with $N, k \geq 2$. For this reason, we will often omit to indicate the explicit dependence on N and k. Also, $\Omega \subset \mathbb{R}^N$ is an open bounded set, and $Q = (0,1)^N$ denotes the open unit cube of $\mathbb{R}^N$. Since N and k are fixed,

Definition 2.1. Let $A \in \mathbb{R}^{N^k}$. We say that A is a *k-th order tensor in* $\mathbb{R}^N$.

The components of a tensor $A \in \mathbb{R}^{N^k}$ will be denoted with the symbols

$$A_{i_1 \dots i_k} \qquad\qquad i_1, \dots, i_k = 1, \dots, N.$$

Moreover, the scalar product of two tensors $A, B \in \mathbb{R}^{N^k}$ is given by

$$A \cdot B := \sum_{i_1, \dots, i_k = 1}^{N} A_{i_1 \dots i_k} B_{i_1 \dots i_k}.$$

Accordingly, the norm of a k-th order tensor $A \in \mathbb{R}^{N^k}$ is

$$|A| := \left[\sum_{i_1, \dots, i_k = 1}^{N} |A_{i_1 \dots i_k}|^2 \right]^{\frac{1}{2}}.$$

Let now $s \in \{1, \dots, k-1\}$ be fixed. For any $\zeta \in C^s(Q; \mathbb{R}^{N^{k-s}})$, we can regard the s-th order gradient $\nabla^s \zeta$ of ζ as a k-th order tensor in $\mathbb{R}^N$, by setting

$$(\nabla^s \zeta)_{i_1 \dots i_k} := \frac{\partial^s \zeta_{i_1 \dots i_{k-s}}}{\partial x_{i_{k-s+1}} \dots \partial x_{i_k}} \qquad\qquad i_1, \dots, i_k = 1 \dots, N.$$

Notice also that $\nabla^s \zeta$ is symmetric with respect to every permutation of the last s indices. To take account of this property, we introduce some additional notation.

Definition 2.2. Let $A \in \mathbb{R}^{N^k}$ be a k-th order tensor in $\mathbb{R}^N$, and let $j, r \in \{1, \dots, k\}$. The *(j,r)-transpose of* A is the element $A^{T_r^j}$ of $\mathbb{R}^{N^k}$ such that (assuming, for instance, $j \leq r$)

$$(A^{T_r^j})_{i_1 i_2 \dots i_k} = A_{i_1 i_2 \dots i_{j-1} i_r i_{j+1} \dots i_{r-1} i_j i_{r+1} \dots i_k} \qquad\qquad i_1, \dots, i_k = 1, \dots, N.$$

We then set

$$E_k^{N^{k-s}} := \{A \in \mathbb{R}^{N^k} : A = A^{T_j^r} \text{ for every } r, j = k-s+1, \dots, k\}.$$

In particular, we will make the identification $E_k^{N^{k-1}} = \mathbb{R}^{N^k}$. In this way, for every $\zeta \in C^s(Q; \mathbb{R}^{N^{k-s}})$ we have

$$\nabla^s \zeta \in E_k^{N^{k-s}}.$$

To include the case $s = k$, we define

$$E_k^1 := \{A \in \mathbb{R}^{N^k} : A = A^{T_j^r} \text{ for every } r, j = 1, \dots, k\}.$$

Very often we will simply write E_k instead of E_k^1. Hence, we have that

$$\nabla^k \phi \in E_k$$

for every $\phi \in C^k(Q)$, using the notation

$$(\nabla^k \phi)_{i_1 \dots i_k} := \frac{\partial^k \phi}{\partial x_{i_1} \dots \partial x_{i_k}} \qquad\qquad i_1, \dots, i_k = 1 \dots, N.$$

We are now going to define the symmetric part of an element of $E_k^{N^{k-s}}$.

Definition 2.3. The *symmetrization operator* $\mathcal{S}_{s+1} : E_k^{N^{k-s}} \to E_k^{N^{k-s-1}}$ is defined by

$$\mathcal{S}_{s+1}A := \frac{1}{s+1}\sum_{r=k-s}^{k} A^{T_r^{k-s}} = \frac{A + A^{T_{k-s+1}^{k-s}} + \cdots + A^{T_k^{k-s}}}{s+1} \qquad \text{for every } A \in E_k^{N^{k-s}}.$$

We will say that $\mathcal{S}_{s+1}A$ is the *symmetric part* of A.

The subscript $s+1$ denotes the fact that the tensor $\mathcal{S}_{s+1}A$ is symmetric in the last $s+1$ entries.

Definition 2.4. Accordingly, we define the *antisymmetric part* of a tensor $A \in E_k^{N^{k-s}}$ as the tensor $\mathcal{A}_{s+1}A \in E_k^{N^{k-s}}$ given by

$$\mathcal{A}_{s+1}A := A - \mathcal{S}_{s+1}A = \frac{sA - (A^{T_{k-s+1}^{k-s}} + \cdots + A^{T_k^{k-s}})}{s+1}.$$

We will use the notation

$$\mathcal{A}_{s+1}E_k^{N^{k-s}} := \{\mathcal{A}_{s+1}A : A \in E_k^{N^{k-s}}\} \subset E_k^{N^{k-s}}.$$

Next proposition generalizes the well-known fact that symmetric and antisymmetric matrices define orthogonal spaces. For the convenience of the reader, the proof is in the Appendix.

Proposition 2.5. *There holds*

$$A \cdot B = 0 \qquad \text{for every } A \in E_k^{N^{k-s-1}} \text{ and for every } B \in \mathcal{A}_{s+1}E_k^{N^{k-s}}.$$

We give now the definition of (higher order) quasiconvexity.

Definition 2.6. Let $j \in \{1, \ldots, k\}$. A function $f \in L^1_{loc}(E_k^{N^{k-j}})$ is said to be *j-quasiconvex* if

$$\int_Q \left[f(A + \nabla^j \phi) - f(A)\right] dx \geq 0$$

for every $A \in E_k^{N^{k-j}}$ and for every $\phi \in C_c^j(Q; \mathbb{R}^{N^{k-j}})$.

It is very well-known that every convex function is locally Lipschitz. This property still holds true for j-quasiconvex functions with p-growth. We give here a proof of this fact, that is in general explicitly stated only for the case $j = 2$ (see [14]).

Proposition 2.7. *Let $j \in \{2, \ldots, k\}$, and let $f \in L^1_{loc}(E_k^{N^{k-j}})$ be j-quasiconvex. Assume, in addition, that*

$$|f(A)| \leq M(1 + |A|^p) \qquad \textit{for every } A \in E_k^{N^{k-j}}, \tag{2.1}$$

for some $M > 0$ and $1 < p < \infty$. Then, there exists a constant $L = L(N, M, k, j, p) > 0$ such that

$$|f(A + B) - f(A)| \leq L\left(1 + |A|^{p-1} + |B|^{p-1}\right)|B| \qquad \text{for every } A, B \in E_k^{N^{k-j}}.$$

Proof. Let us set

$$X := \{b \otimes \overbrace{w \otimes \ldots \otimes w}^{j \text{ times}} : b \in \mathbb{R}^{N^{k-j}}, w \in \mathbb{S}^{N-1}\} \subset E_k^{N^{k-j}}, \qquad m = m(N, k, j) := \dim E_k^{N^{k-j}}.$$

Here, for every $b \in \mathbb{R}^{N^{k-j}}$ and $w \in \mathbb{S}^{N-1}$ the symbol $b \otimes w \otimes \ldots \otimes w$ denotes the element of $\mathbb{R}^{N^k}$ such that

$$(b \otimes w \otimes \ldots \otimes w)_{i_1 \ldots, i_k} = b_{i_1 \ldots i_{k-j}} w_{i_{k-j+1}} \ldots w_{i_k}, \qquad i_1, \ldots, i_k = 1 \ldots, N.$$

It can be proven that the orthogonal complement of X in $E_k^{N^{k-j}}$ is zero, so that

$$\operatorname{span} X = E_k^{N^{k-j}}.$$

Let now $\{\omega_1,\dots,\omega_m\}\subset X$ be a (not necessarily orthonormal) basis for $E_k^{N^{k-j}}$, with $|\omega_i|=1$ for $i=1,\dots,m$, and let $c_1,\dots,c_m\in\mathbb{R}$ be such that $B=\sum_{i=1}^m c_i\omega_i$. We have

$$|f(A+B)-f(A)|=\Big|f\Big(A+\sum_{i=1}^m c_i\omega_i\Big)-f(A)\Big|\le\Big|f\Big(A+\sum_{i=1}^m c_i\omega_i\Big)-f\Big(A+\sum_{i=1}^{m-1} c_i\omega_i\Big)\Big|$$
$$+\Big|f\Big(A+\sum_{i=1}^{m-1} c_i\omega_i\Big)-f\Big(A+\sum_{i=1}^{m-2} c_i\omega_i\Big)\Big|+\dots+|f(A+c_1\omega_1)-f(A)|.$$

It will be enough to prove that there exists $C=C(N,M,k,j,p)$ such that for every $l=1,\dots,m$

$$\Big|f\Big(c_l\omega_l+A+\sum_{i=0}^{l-1} c_i\omega_i\Big)-f\Big(A+\sum_{i=0}^{l-1} c_i\omega_i\Big)\Big|\le C\Big(1+|A|^{p-1}+|B|^{p-1}\Big)|B|, \tag{2.2}$$

where we set $c_0:=0$ and $\omega_0:=0$. Then, the conclusion will follow by defining $L:=mC$.

Thanks to [12, Proposition 3.4 and Example 3.10 (d)], for every $R\in E_k^{N^{k-j}}$ and every $\omega\in X$ the function

$$t\to f(t\,\omega+R)$$

is convex in $\mathbb{R}$. Hence, defining $G(t):=f\Big(t\,c_l\omega_l+A+\sum_{i=0}^{l-1}c_i\omega_i\Big)$ and using (2.1), for every $t\ge1$ we have

$$\Big|f\Big(c_l\omega_l+A+\sum_{i=0}^{l-1} c_i\omega_i\Big)-f\Big(A+\sum_{i=0}^{l-1} c_i\omega_i\Big)\Big|=G(1)-G(0)\le\frac{G(t)-G(0)}{t}$$
$$=\frac{f\Big(t\,c_l\omega_l+A+\sum_{i=0}^{l-1}c_i\omega_i\Big)-f\Big(A+\sum_{i=0}^{l-1}c_i\omega_i\Big)}{t}$$
$$\le\frac{M}{t}\Big(2+\Big|t\,c_l\omega_l+A+\sum_{i=0}^{l-1}c_i\omega_i\Big|^p+\Big|A+\sum_{i=0}^{l-1}c_i\omega_i\Big|^p\Big)$$
$$\le\frac{M}{t}\Big(2+2^{p-1}t^p|c_l|^p+(2^{p-1}+1)\Big|A+\sum_{i=0}^{l-1}c_i\omega_i\Big|^p\Big)$$
$$\le\frac{M}{t}\Big(2+2^{p-1}t^p\|B\|^p+2^{p-1}(2^{p-1}+1)|A|^p+2^{p-1}(2^{p-1}+1)m^{\frac p2}\|B\|^p\Big),$$

where we set

$$\|B\|:=\Big(\sum_{i=0}^m c_i^2\Big)^{\frac12}.$$

Let us now choose

$$t=\frac{\big(|A|^{p-1}+\|B\|^{p-1}\big)^{\frac{1}{p-1}}}{\|B\|}\ge1.$$

Noticing that

$$t^{p-1}\|B\|^p=(|A|^{p-1}+\|B\|^{p-1})\|B\|,\quad \frac{|A|^p}{t}\le|A|^{p-1}\|B\|,\quad \frac{\|B\|^p}{t}\le\|B\|^p,$$

and using the fact that $\|\cdot\|$ and $|\cdot|$ are equivalent norms, we obtain (2.2). □

Next proposition shows that conditions (a) and (b) of Theorem 1.1 necessarily imply $L\ge\nu$.

Proposition 2.8. *Let $f\in C^1(E_k)$ satisfy conditions (a) and (b) of Theorem 1.1 for some constants $\mu\ge0$, $L,\nu>0$ and $1<p<\infty$. Then $L\ge\nu$.*

Proof. Let $A\in E_k$, $\phi\in C_c^k(Q)$, and let $x\in Q$. By the Mean Value Theorem,

$$f(A+\nabla^k\phi(x))-f(A)=[\nabla f(A+t\,\nabla^k\phi(x))-\nabla f(A)]\cdot\nabla^k\phi(x)+\nabla f(A)\cdot\nabla^k\phi(x),$$

for some $t \in [0,1]$. Integrating last equality, since $\phi \in C_c^k(Q)$, we get

$$\int_Q [f(A + \nabla^k \phi(x)) - f(A)]\, dx = \int_Q [\nabla f(A + t\, \nabla^k \phi(x)) - \nabla f(A)] \cdot \nabla^k \phi(x)\, dx.$$

Hence, using property (b)

$$\begin{aligned}\int_Q \left[f(A + \nabla^k \phi(x)) - f(A)\right]\, dx &\le L \int_Q \left(\mu^2 + |A|^2 + t^2\, |\nabla^k \phi(x)|^2\right)^{\frac{p-2}{2}} t\, |\nabla^k \phi(x)|^2 dx \\ &\le L \int_Q \left(\mu^2 + |A|^2 + |\nabla^k \phi(x)|^2\right)^{\frac{p-2}{2}} |\nabla^k \phi(x)|^2 dx,\end{aligned}$$

since the function $t \mapsto \left(\mu^2 + |A|^2 + t^2\, |\nabla^k \phi|^2\right)^{\frac{p-2}{2}} t\, |\nabla^k \phi|^2$ is increasing. Comparing last relation and condition (a) we conclude that $L \ge \nu$. □

We prove now that condition (1.2) is stronger than (1.5).

Proposition 2.9. *Let $j \in \{2, \dots, k\}$, let $L > 0$, $\mu \ge 0$, $1 < p < \infty$, and let $f \in C^1(E_k^{N^{k-j}})$ be such that*

$$|\nabla f(A + B) - \nabla f(A)| \le L \left(\mu^2 + |A|^2 + |B|^2\right)^{\frac{p-2}{2}} |B| \tag{2.3}$$

for every $A, B \in E_k^{N^{k-j}}$. Then, there exists a positive constant c_f, depending on f, such that

$$|f(A)| \le c_f (1 + |A|^p) \qquad\qquad \forall\, A \in E_k^{N^{k-j}}.$$

Proof. Let $C \in E_k^{N^{k-j}} \setminus \{0\}$ be fixed. Then, by the Mean Value Theorem for every $A \in E_k^{N^{k-j}}$ we have

$$f(A) = f(C) + [\nabla f(C + t(A - C)) - \nabla f(C)] \cdot (A - C) + \nabla f(C) \cdot (A - C),$$

for some $t \in [0,1]$. Thanks to (2.3)

$$\begin{aligned}|f(A)| &\le |f(C)| + L \left(\mu^2 + |C|^2 + t^2 |A - C|^2\right)^{\frac{p-2}{2}} t |A - C|^2 + |\nabla f(C)||A - C| \\ &\le |f(C)| + L \left(\mu^2 + |C|^2 + |A - C|^2\right)^{\frac{p-2}{2}} |A - C|^2 + |\nabla f(C)||A - C|,\end{aligned} \tag{2.4}$$

since the function $t \mapsto \left(\mu^2 + |C|^2 + t^2 |A - C|^2\right)^{\frac{p-2}{2}} t |A - C|^2$ is increasing. Concerning the last term, using Young's inequality we have

$$|\nabla f(C)||A - C| \le \frac{|\nabla f(C)|^{p'}}{p'} + \frac{|A - C|^p}{p} \le \frac{|\nabla f(C)|^{p'}}{p'} + \frac{2^{p-1}}{p} \left(|A|^p + |C|^p \right), \tag{2.5}$$

where $p' = \frac{p}{p-1}$. Since the function $r \mapsto \left(\mu^2 + |C|^2 + r\right)^{\frac{p-2}{2}} r$ is increasing in $\mathbb{R}$, using inequality

$$|A - C|^2 \le 2|A|^2 + 2|C|^2,$$

we have

$$\begin{aligned}&\left(\mu^2 + |C|^2 + |A - C|^2\right)^{\frac{p-2}{2}} |A - C|^2 \le 2 \left(\mu^2 + 3|C|^2 + 2|A|^2\right)^{\frac{p-2}{2}} \left(|A|^2 + |C|^2\right) \\ &\le 2 \max\{1, |C|^2\} \left(\mu^2 + 3|C|^2 + 2|A|^2\right)^{\frac{p-2}{2}} \left(1 + |A|^2\right) \\ &\le 2\, K^{\frac{p-2}{2}} \max\{1, |C|^2\} \left(1 + |A|^2\right)^{\frac{p}{2}},\end{aligned}$$

where

$$K = \begin{cases} \min\{\mu^2 + 3|C|^2, 2\} & \text{if } 1 < p < 2, \\ \max\{\mu^2 + 3|C|^2, 2\} & \text{if } p \ge 2. \end{cases}$$

Thus, since

$$\left(1 + |A|^2\right)^{\frac{p}{2}} \le C_p \left(1 + |A|^p\right),$$

for some positive constant C_p depending only on p, we have

$$L \left(\mu^2 + |C|^2 + |A - C|^2\right)^{\frac{p-2}{2}} |A - C|^2 \le 2\, L\, C_p\, K^{\frac{p-2}{2}} \max\{1, |C|^2\} \left(1 + |A|^p\right). \tag{2.6}$$

Combining (2.4), (2.5) and (2.6) the conclusion follows. □

We now state some important results concerning periodic functions.

Definition 2.10. A function $w:\mathbb{R}^N\to\mathbb{R}^{N^{k-s}}$ is said to be *Q-periodic* if $w(x+e_i)=w(x)$ for a.e. $x\in\mathbb{R}^N$ and every $i=1,\dots,N$, where $\{e_1,\dots,e_N\}$ is the canonical basis of $\mathbb{R}^N$.

Let $d,r\in\mathbb{N}$. We will denote with $C^\infty_{per}(\mathbb{R}^N;\mathbb{R}^d)$ the space of Q-periodic functions of $C^\infty(\mathbb{R}^N;\mathbb{R}^d)$. Moreover, we will use the notation $C^r_c(Q;\mathbb{R}^d)$ for the space of functions of class C^r from Q to $\mathbb{R}^d$ with compact support in Q. Next lemma will be extensively used in the paper.

Lemma 2.11 (Helmholtz Decomposition). *For every $\varphi\in C^\infty_{per}(Q;\mathbb{R}^{N^{k-s}})$ there exist two functions $\phi\in C^\infty_{per}(Q;\mathbb{R}^{N^{k-s-1}})$ and $\psi\in C^\infty_{per}(Q;R^{N^{k-s}})$ such that*

$$\varphi_{i_1\dots i_{k-s}}=(\nabla\phi)_{i_1\dots i_{k-s}}+\psi_{i_1\dots i_{k-s}},\qquad \text{for } i_1,\dots,i_{k-s}=1,\dots,N,$$

with

$$\sum_{i_b=1}^N\frac{\partial\psi_{i_1\dots i_{b-1}i_bi_{b+1}\dots i_{k-s}}}{\partial x_{i_b}}=0\qquad\textit{for every } b\in\{1,\dots,k-s\}. \tag{2.7}$$

Proof. By applying the usual Helmholtz Decomposition Lemma (see [8, Lemma 1]) to each component $\varphi_{i_1\dots i_{k-s}}$ of the function φ, the lemma follows. □

Before stating next lemma, we need the following definition.

Definition 2.12. The *s-divergence* is the operator $s\text{-div}:C^s(Q;\mathbb{R}^{N^k})\to C(Q;\mathbb{R}^{N^{k-s}})$ defined by

$$(s\text{-div}\,\xi)_{i_1\dots i_{k-s}}:=\sum_{i_{k-s+1},\dots,i_k=1}^N\frac{\partial^s\xi_{i_1i_2\dots i_k}}{\partial x_{i_{k-s+1}}\dots\partial x_{i_k}}\qquad i_1,\dots,i_{k-s}=1,\dots,N,$$

for every $\xi\in C^s(Q;\mathbb{R}^{N^k})$. The definition is analogous when ξ is a Sobolev function.

We are now ready to state a fundamental Korn-type estimate.

Lemma 2.13. *For every $p>1$ there exists a constant $\gamma=\gamma(N,p,s)\ge 1$ such that*

$$\int_Q|\nabla^s\psi|^p dx\le\gamma\int_Q|\mathcal{A}_{s+1}\nabla^s\psi|^p dx$$

for every Q-periodic function $\psi:\mathbb{R}^N\to\mathbb{R}^{N^{k-s}}$ of class C^∞ satisfying condition (2.7).

Proof. Notice that, for every $r=k-s+1,\dots,k$, we have

$$\sum_{i_r=1}^N\frac{\partial}{\partial x_{i_r}}\left[(\nabla^s\psi)^{T_r^{k-s}}\right]_{i_1i_2\dots i_k}=\sum_{i_r=1}^N\frac{\partial}{\partial x_{i_r}}\left[\frac{\partial^s\psi_{i_1\dots i_{k-s-1}i_r}}{\partial x_{i_{k-s+1}}\dots\partial x_{i_{r-1}}\partial x_{i_{k-s}}\partial x_{i_{r+1}}\dots\partial x_{i_k}}\right]$$
$$=\frac{\partial^s}{\partial x_{i_{k-s+1}}\dots\partial x_{i_{r-1}}\partial x_{i_{k-s}}\partial x_{i_{r+1}}\dots\partial x_{i_k}}\left[\sum_{i_r=1}^N\frac{\partial\psi_{i_1\dots i_{k-s-1}i_r}}{\partial x_{i_r}}\right]=0.$$

Thus,

$$(s+1)\,\left[s\text{-div}\,(\mathcal{A}_{s+1}\nabla^s\psi)\right]_{i_1\dots i_{k-s}}$$
$$=\sum_{i_{k-s+1},\dots,i_k=1}^N\frac{\partial^s}{\partial x_{i_{k-s+1}}\dots\partial x_{i_k}}\left[s\nabla^s\psi-\left((\nabla^s\psi)^{T^{k-s}_{k-s+1}}+\dots+(\nabla^s\psi)^{T^{k-s}_k}\right)\right]_{i_1i_2\dots i_k}$$
$$=s\sum_{i_{k-s+1},\dots,i_k=1}^N\frac{\partial^s}{\partial x_{i_{k-s+1}}\dots\partial x_{i_k}}\,(\nabla^s\psi)_{i_1i_2\dots i_k}$$
$$=s\sum_{i_{k-s+1},\dots,i_k=1}^N\frac{\partial^s}{\partial x_{i_{k-s+1}}\dots\partial x_{i_k}}\left[\frac{\partial^s\psi_{i_1\dots i_{k-s}}}{\partial x_{i_{k-s+1}}\dots\partial x_{i_k}}\right]=s\,\Delta^s\psi_{i_1\dots i_{k-s}},$$

where with Δ^s we denoted the s-th power of the Laplace operator. Hence,

$$\Delta^s \psi_{i_1 \dots i_{k-s}} = \frac{s+1}{s} \left[s\text{-div} \left(\mathcal{A}_{s+1} \nabla^s \psi \right) \right]_{i_1 \dots i_{k-s}} \qquad i_1, \dots, i_{k-s} = 1, \dots, N.$$

The conclusion follows applying [1, Theorem 10.5 and following remark]. □

We will also need the following generalization of Lemma 2.13.

Lemma 2.14. *For every $p > 1$ there exists a constant $\tau = \tau(N, p, s) \geq 1$ such that*

$$\int_Q \left(\mu^2 + |\nabla^s \psi|^2 \right)^{\frac{p-2}{2}} |\nabla^s \psi|^2 dx \leq \tau \int_Q \left(\mu^2 + |\mathcal{A}_{s+1} \nabla^s \psi|^2 \right)^{\frac{p-2}{2}} |\mathcal{A}_{s+1} \nabla^s \psi|^2 dx$$

for every constant $\mu \geq 0$ and every Q-periodic function $\psi : \mathbb{R}^N \to \mathbb{R}^{N^{k-s}}$ of class C^∞ satisfying condition (2.7).

Proof. The proof simply follows by adapting the proof of [8, Lemma 11] and using Lemma 2.13. □

We conclude this section giving some definitions of higher order BV spaces. We set

$$\begin{aligned} BH^{(k)}(\Omega) : &= \{ u \in W^{k-1,1}(\Omega) : D^k u \text{ is a finite Radon measure } \} \\ &= \{ u \in W^{k-1,1}(\Omega) : \nabla^{k-1} u \in BV(\Omega; E_{k-1}) \}, \end{aligned}$$

where $D^k u$ stands for the k-th order distributional gradient of u, and

$$\begin{aligned} SBH^{(k)}(\Omega) : &= \{ u \in BH^{(k)}(\Omega) : \nabla^{k-1} u \in SBV(\Omega; E_{k-1}) \} \\ &= \{ u \in W^{k-1,1}(\Omega) : \nabla^{k-1} u \in SBV(\Omega; E_{k-1}) \} \subset BH^{(k)}(\Omega). \end{aligned}$$

3. Proof of Theorem 1.1

To prove Theorem 1.1 we will first show that, for every $j = 2, \dots, k$, every strictly j-quasiconvex function of class C^1 can be extended to a strictly $(j-1)$-quasiconvex function, provided we require the gradient to be Lipschitz continuous. In the case $1 < p < 2$, that we present below, we actually have to consider a "perturbed" strict j-quasiconvexity.

Lemma 3.1. *Let $j \in \{2, \dots, k\}$, $1 < p < 2$, $\mu \geq 0$, and let $M^{(j)}$, $\nu^{(j)}$, and ε be positive constants. Let $f^{(j)} \in C^1(E_k^{N^{k-j}})$ satisfy the following conditions:*

(a) *(strict j-quasiconvexity up to a perturbation)*

$$\begin{aligned} \int_Q \left[f^{(j)}(A + \nabla^j \phi) - f^{(j)}(A) \right] dx &\geq -\varepsilon h^{(j)}(A) \\ &+ \nu^{(j)} \int_Q \left(\mu^2 + |A|^2 + |\nabla^j \phi|^2 \right)^{\frac{p-2}{2}} |\nabla^j \phi|^2 dx \end{aligned}$$

for every $A \in E_k^{N^{k-j}}$ and every $\phi \in C_c^j(Q; \mathbb{R}^{N^{k-j}})$, where $h^{(j)} : E_k^{N^{k-j}} \to [0, +\infty)$;

(b) *(Lipschitz condition for gradients)*

$$|\nabla f^{(j)}(A + B) - \nabla f^{(j)}(A)| \leq M^{(j)} \left(\mu^2 + |A|^2 + |B|^2 \right)^{\frac{p-2}{2}} |B|$$

for every $A, B \in E_k^{N^{k-j}}$.

Then there exists a function $F^{(j)} \in C^1(E_k^{N^{k-j+1}})$, and a positive constant $L^{(j)} = L^{(j)}(p, \mu, M^{(j)}, \nu^{(j)}, j)$, such that

(a') *(strict $(j-1)$-quasiconvexity up to a perturbation)*

$$\begin{aligned} \int_Q \left[F^{(j)}(A + \nabla^{j-1} \varphi) - F^{(j)}(A) \right] dx \geq \frac{\nu^{(j)}}{4} \int_Q \left(\mu^2 + |A|^2 + |\nabla^{j-1} \varphi|^2 \right)^{\frac{p-2}{2}} |\nabla^{j-1} \varphi|^2 \, dx \\ - \varepsilon \left(\mu^2 + |\mathcal{A}_j A|^2 \right)^{\frac{p-2}{2}} |\mathcal{A}_j A|^2 - \varepsilon \, h^{(j)}(\mathcal{S}_j A) \end{aligned}$$

for every $A \in E_k^{N^{k-j+1}}$ and every $\varphi \in C_c^{j-1}(Q; \mathbb{R}^{N^{k-j+1}})$;

(b') *(Lipschitz condition for gradients)*

$$|\nabla F^{(j)}(A+B) - \nabla F^{(j)}(A)| \le L^{(j)} \left(\mu^2 + |A|^2 + |B|^2\right)^{\frac{p-2}{2}} |B|$$

for every $A, B \in E_k^{N^{k-j+1}}$*;*

(c) *(*$F^{(j)}$ *extends* $f^{(j)}$*)*

$$F^{(j)}(A) = f^{(j)}(A) \qquad\qquad \forall\, A \in E_k^{N^{k-j}}.$$

Proof. Let $\beta > 0$ be a constant to be chosen at the end of the proof and define $F^{(j)} : E_k^{N^{k-j+1}} \to \mathbb{R}$ as

$$F^{(j)}(A) := f^{(j)}(\mathcal{S}_j A) + \beta \left[\left(\mu^2 + |\mathcal{A}_j A|^2\right)^{\frac{p}{2}} - \mu^p \right] = f^{(j)}(\mathcal{S}_j A) + \beta \left[g(\mathcal{A}_j A) - \mu^p \right],$$

where g is given by relation (5.2) with $X = E_k^{N^{k-j+1}}$.

Relation (c) is clearly satisfied. Let us show that condition (a') holds true for a good choice of β. Let $\varphi \in C^\infty_{\rm per}(Q; \mathbb{R}^{N^{k-j+1}})$. By Lemma 2.11 we can write

$$\varphi = \nabla \phi + \psi,$$

where $\psi \in C^\infty_{\rm per}(Q; R^{N^{k-j+1}})$ satisfies condition (2.7) with $s = j-1$, and $\phi \in C^\infty_{\rm per}(Q; \mathbb{R}^{N^{k-j}})$. By differentiating $j-1$ times the previous relation we get

$$\nabla^{j-1}\varphi = \nabla^j \phi + \nabla^{j-1}\psi,$$

with $\nabla^{j-1}\varphi, \nabla^{j-1}\psi \in C^\infty_{\rm per}(Q; E_k^{N^{k-j+1}})$, and $\nabla^j \phi \in C^\infty_{\rm per}(Q; E_k^{N^{k-j}})$. We have

$$\begin{aligned}
&\int_Q \left[F^{(j)}(A + \nabla^{j-1}\varphi) - F^{(j)}(A) \right] dx \\
&\quad = \int_Q \left[f^{(j)}(\mathcal{S}_j A + \nabla^j \phi + \mathcal{S}_j \nabla^{j-1}\psi) - f^{(j)}(\mathcal{S}_j A + \nabla^j \phi) \right] dx \\
&\qquad + \int_Q \left[f^{(j)}(\mathcal{S}_j A + \nabla^j \phi) - f^{(j)}(\mathcal{S}_j A) \right] dx \\
&\qquad + \beta \int_Q [g(\mathcal{A}_j A + \mathcal{A}_j \nabla^{j-1}\psi) - g(\mathcal{A}_j A)] dx \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned}$$

Notice that $\nabla f^{(j)}(\mathcal{S}_j A) \in E_k^{N^{k-j}}$. Then, thanks to Proposition 2.5 and using the fact that ψ is Q-periodic

$$\int_Q \nabla f^{(j)}(\mathcal{S}_j A) \cdot \mathcal{S}_j \nabla^{j-1}\psi \, dx = \int_Q \nabla f^{(j)}(\mathcal{S}_j A) \cdot \nabla^{j-1}\psi \, dx = 0.$$

Hence

$$I_1 = \int_Q \left[f^{(j)}(\mathcal{S}_j A + \nabla^j \phi + \mathcal{S}_j \nabla^{j-1}\psi) - f^{(j)}(\mathcal{S}_j A + \nabla^j \phi) - \nabla f^{(j)}(\mathcal{S}_j A) \cdot \mathcal{S}_j \nabla^{j-1}\psi \right] dx.$$

Applying Lemma 5.6 with $\varepsilon = \nu^{(j)}/2$ there exists a positive constant $c_1 = c_1(\nu^{(j)}, p, M^{(j)}) > 0$ such that

$$\begin{aligned}
I_1 &\ge -\frac{\nu^{(j)}}{2} \int_Q \left(\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2\right)^{\frac{p-2}{2}} |\nabla^j \phi|^2 \, dx \\
&\qquad - c_1 \int_Q \left(\mu^2 + |\mathcal{S}_j \nabla^{j-1}\psi|^2\right)^{\frac{p-2}{2}} |\mathcal{S}_j \nabla^{j-1}\psi|^2 \, dx \\
&\ge -\frac{\nu^{(j)}}{2} \int_Q \left(\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2\right)^{\frac{p-2}{2}} |\nabla^j \phi|^2 \, dx \\
&\qquad - \tau\, c_1 \int_Q \left(\mu^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \, dx,
\end{aligned}$$

where $\tau = \tau(N, p, j-1)$ is given by Lemma 2.14. The perturbed strict j-quasiconvexity of $f^{(j)}$ gives

$$I_2 \geq \nu^{(j)} \int_Q \left(\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2\right)^{\frac{p-2}{2}} |\nabla^j \phi|^2 \, dx - \varepsilon h^{(j)}(\mathcal{S}_j A),$$

so that

$$\begin{aligned} I_1 + I_2 \geq \frac{\nu^{(j)}}{2} \int_Q \left(\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2\right)^{\frac{p-2}{2}} |\nabla^j \phi|^2 \, dx - \varepsilon h^{(j)}(\mathcal{S}_j A) \\ - \tau\, c_1 \int_Q \left(\mu^2 + |\mathcal{A}_j \nabla^{j-1} \psi|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j \nabla^{j-1} \psi|^2 \, dx. \end{aligned}$$

We apply now Lemma 5.3 to the first integral of the last expression with $\tilde{\mu}^2 = \mu^2 + |\mathcal{S}_j A|^2$, $x = \nabla^j \phi$, and $y = \nabla^{j-1} \psi$. Recalling that $\nabla^j \phi + \nabla^{j-1} \psi = \nabla^{j-1} \varphi$ we get

$$\begin{aligned} I_1 + I_2 \geq &\frac{\nu^{(j)}}{4} \int_Q \left(\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^{j-1} \varphi|^2\right)^{\frac{p-2}{2}} |\nabla^{j-1} \varphi|^2 \, dx - \varepsilon h^{(j)}(\mathcal{S}_j A) \\ &- \frac{\nu^{(j)}}{2} \int_Q \left(\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^{j-1} \psi|^2\right)^{\frac{p-2}{2}} |\nabla^{j-1} \psi|^2 \, dx \\ &- \tau\, c_1 \int_Q \left(\mu^2 + |\mathcal{A}_j \nabla^{j-1} \psi|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j \nabla^{j-1} \psi|^2 \, dx. \end{aligned} \tag{3.1}$$

Using the fact that $1 < p < 2$ and Lemma 2.14

$$\begin{aligned} - \frac{\nu^{(j)}}{2} &\int_Q \left(\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^{j-1} \psi|^2\right)^{\frac{p-2}{2}} |\nabla^{j-1} \psi|^2 \, dx \\ &\geq -\frac{\nu^{(j)}}{2} \int_Q \left(\mu^2 + |\nabla^{j-1} \psi|^2\right)^{\frac{p-2}{2}} |\nabla^{j-1} \psi|^2 \, dx \\ &\geq -\frac{\tau\, \nu^{(j)}}{2} \int_Q \left(\mu^2 + |\mathcal{A}_j \nabla^{j-1} \psi|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j \nabla^{j-1} \psi|^2 \, dx. \end{aligned} \tag{3.2}$$

Hence, collecting (3.1) and (3.2)

$$\begin{aligned} I_1 + I_2 \geq &\frac{\nu^{(j)}}{4} \int_Q \left(\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^{j-1} \varphi|^2\right)^{\frac{p-2}{2}} |\nabla^{j-1} \varphi|^2 \, dx - \varepsilon h^{(j)}(\mathcal{S}_j A) \\ &- \tau \left(c_1 + \frac{\nu^{(j)}}{2}\right) \int_Q \left(\mu^2 + |\mathcal{A}_j \nabla^{j-1} \psi|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j \nabla^{j-1} \psi|^2 \, dx \\ \geq &\frac{\nu^{(j)}}{4} \int_Q \left(\mu^2 + |A|^2 + |\nabla^{j-1} \varphi|^2\right)^{\frac{p-2}{2}} |\nabla^{j-1} \varphi|^2 \, dx - \varepsilon h^{(j)}(\mathcal{S}_j A) \\ &- \tau \left(c_1 + \frac{\nu^{(j)}}{2}\right) \int_Q \left(\mu^2 + |\mathcal{A}_j \nabla^{j-1} \psi|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j \nabla^{j-1} \psi|^2 \, dx, \end{aligned}$$

where we used once again the fact that $1 < p < 2$. Since $\nabla g(\mathcal{A}_j A) \in \mathcal{A}_j E_k^{N^{k-j+1}}$ and ψ is Q-periodic,

$$\int_Q \nabla g(\mathcal{A}_j A) \cdot \mathcal{A}_j \nabla^{j-1} \psi \, dx = \int_Q \nabla g(\mathcal{A}_j A) \cdot \nabla^{j-1} \psi \, dx = 0,$$

so that

$$I_3 = \beta \int_Q \left[g(\mathcal{A}_j A + \mathcal{A}_j \nabla^{j-1} \psi) - g(\mathcal{A}_j A) - \nabla g(\mathcal{A}_j A) \cdot \mathcal{A}_j \nabla^{j-1} \psi \right] dx.$$

Let $0 < \delta < 1$ to be chosen at the end of the proof. Thanks to Lemma 5.1

$$\begin{aligned} I_3 &\geq \beta\theta_p \int_Q \left(\mu^2 + |\mathcal{A}_j A|^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \, dx \\ &\geq \beta\theta_p \delta^{\frac{2-p}{2}} \int_Q \left(\mu^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \, dx \\ &\quad - \beta\theta_p \delta \left(\mu^2 + |\mathcal{A}_j A|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j A|^2, \end{aligned}$$

where in the second inequality we used Lemma 5.3 with $X = E_k^{N^{k-j+1}}$, $\tilde{\mu} = \mu$, $x = \mathcal{A}_j A$ and $y = \mathcal{A}_j \nabla^{j-1}\psi$. Choosing $\beta = \beta^{(j)} > 0$ and $\delta = \delta^{(j)} \in (0, 1)$ such that

$$\beta^{(j)}\theta_p (\delta^{(j)})^{\frac{2-p}{2}} \geq \tau \left(c_1 + \frac{\nu^{(j)}}{2}\right), \qquad \beta^{(j)}\theta_p \delta^{(j)} \leq \varepsilon,$$

we obtain

$$\begin{aligned} I_1 + I_2 + I_3 \geq \frac{\nu^{(j)}}{4} &\int_Q \left(\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2\right)^{\frac{p-2}{2}} |\nabla^{j-1}\varphi|^2 \, dx \\ &- \varepsilon h^{(j)}(\mathcal{S}_j A) - \varepsilon \left(\mu^2 + |\mathcal{A}_j A|^2\right)^{\frac{p-2}{2}} |\mathcal{A}_j A|^2, \end{aligned}$$

so that (a') holds. To check condition (b'), we observe that the function g satisfies the hypotheses of Lemma 5.5. Then, for every $A, B \in E_k^{N^{k-j+1}}$

$$|\nabla g(A + B) - \nabla g(A)| \leq C_p \left(\mu^2 + |A|^2 + |B|^2\right)^{\frac{p-2}{2}} |B|,$$

where C_p is a positive constant depending only on p. Using last relation, (b), and the fact that $\beta^{(j)}$ depends on $\nu^{(j)}, \tau$ and c_1, we conclude that (b') holds for some positive constant $L^{(j)} = L^{(j)}(p, \mu, M^{(j)}, \nu^{(j)}, j)$.

□

We pass now to the case $p \geq 2$.

Lemma 3.2. *Let $j \in \{2, \ldots, k\}$, $p \geq 2$, $\mu \geq 0$, $M^{(j)} > 0$, $\nu^{(j)} > 0$, and let θ_p and Θ_p be given by Lemma 5.1. Let $f^{(j)} \in C^1(E_k^{N^{k-j}})$ satisfy the following conditions:*

(a) *(strict j-quasiconvexity)*

$$\int_Q \left[f^{(j)}(A + \nabla^j \phi) - f^{(j)}(A)\right] dx \geq \nu^{(j)} \int_Q \left(\mu^2 + |A|^2 + |\nabla^j \phi|^2\right)^{\frac{p-2}{2}} |\nabla^j \phi|^2 dx$$

for every $A \in E_k^{N^{k-j}}$ and every $\phi \in C_c^j(Q; \mathbb{R}^{N^{k-j}})$;

(b) *(Lipschitz condition for gradients)*

$$|\nabla f^{(j)}(A + B) - \nabla f^{(j)}(A)| \leq M^{(j)} \left(\mu^2 + |A|^2 + |B|^2\right)^{\frac{p-2}{2}} |B|$$

for every $A, B \in E_k^{N^{k-j}}$.

Then there exists a function $F^{(j)} \in C^1(E_k^{N^{k-j+1}})$, and a positive constant $L^{(j)} = L^{(j)}(p, \mu, M^{(j)}, \nu^{(j)}, j)$, such that

(a') *(strict $(j-1)$-quasiconvexity)*

$$\int_Q \left[F^{(j)}(A + \nabla^{j-1}\varphi) - F^{(j)}(A)\right] dx \geq \nu^{(j)} \frac{\theta_p}{4\,\Theta_p} \int_Q \left(\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2\right)^{\frac{p-2}{2}} |\nabla^{j-1}\varphi|^2 \, dx$$

for every $A \in E_k^{N^{k-j+1}}$ and every $\varphi \in C_c^{j-1}(Q; \mathbb{R}^{N^{k-j+1}})$;

(b') *(Lipschitz condition for gradients)*

$$|\nabla F^{(j)}(A+B)-\nabla F^{(j)}(A)|\le L^{(j)}\left(\mu^2+|A|^2+|B|^2\right)^{\frac{p-2}{2}}|B|$$

for every $A,B\in E_k^{N^{k-j+1}}$;

(c) *($F^{(j)}$ extends $f^{(j)}$)*

$$F^{(j)}(A)=f^{(j)}(A)\qquad\qquad \forall\, A\in E_k^{N^{k-j}}.$$

Proof. Let $\lambda\in(0,\nu^{(j)}/\Theta_p]$ and $\beta>0$ be two constants to be determined at the end of the proof. We define $F^{(j)}:E_k^{N^{k-j+1}}\to\mathbb{R}$ as

$$F^{(j)}(A):=f^{(j)}(\mathcal{S}_jA)-\lambda\Big(\mu^2+|\mathcal{S}_jA|^2\Big)^{\frac{p}{2}}+\lambda\Big(\mu^2+|\mathcal{S}_jA|^2+\beta^2|\mathcal{A}_jA|^2\Big)^{\frac{p}{2}}.$$

Let g and g_β be defined by (5.2) and (5.3) respectively, with $X=E_k^{N^{k-j}}$ and $Y=\mathcal{A}_jE_k^{N^{k-j+1}}$. Setting for every $B\in E_k^{N^{k-j}}$

$$f_\lambda^{(j)}(B):=f^{(j)}(B)-\lambda g(B),$$

we have

$$F^{(j)}(A)=f_\lambda^{(j)}(\mathcal{S}_jA)+\lambda g_\beta(\mathcal{S}_jA\,,\,\mathcal{A}_jA).$$

Condition (c) is clear from the definition of $F^{(j)}$. In order to check (a'), let $\varphi\in C^\infty_{\rm per}(Q;\mathbb{R}^{N^{k-j+1}})$. By repeating the argument of the previous proof, we can write

$$\nabla^{j-1}\varphi=\nabla^j\phi+\nabla^{j-1}\psi,$$

with $\nabla^{j-1}\varphi,\nabla^{j-1}\psi\in C^\infty_{\rm per}(Q;E_k^{N^{k-j+1}})$, and $\nabla^j\phi\in C^\infty_{\rm per}(Q;E_k^{N^{k-j}})$, where $\psi\in C^\infty_{\rm per}(Q;R^{N^{k-j+1}})$ satisfies condition (2.7) with $s=j-1$. Hence,

$$\begin{aligned}
&\int_Q\Big[F^{(j)}(A+\nabla^{j-1}\varphi)-F^{(j)}(A)\Big]\,dx\\
&\quad=\int_Q\Big[f_\lambda^{(j)}(\mathcal{S}_jA+\mathcal{S}_j\nabla^{j-1}\varphi)-f_\lambda^{(j)}(\mathcal{S}_jA+\mathcal{S}_j\nabla^{j-1}\varphi-\mathcal{S}_j\nabla^{j-1}\psi)\Big]\,dx\\
&\qquad+\int_Q\Big[f_\lambda^{(j)}(\mathcal{S}_jA+\nabla^j\phi)-f_\lambda^{(j)}(\mathcal{S}_jA)\Big]\,dx\\
&\qquad+\lambda\int_Q\big[g_\beta(\mathcal{S}_jA+\mathcal{S}_j\nabla^{j-1}\varphi\,,\,\mathcal{A}_jA+\mathcal{A}_j\nabla^{j-1}\varphi)-g_\beta(\mathcal{S}_jA\,,\,\mathcal{A}_jA)\big]\,dx\\
&\quad=:I_1+I_2+I_3.
\end{aligned}$$

Concerning the second integral, since by periodicity

$$\int_Q\nabla g(\mathcal{S}_jA)\cdot\nabla^j\phi\,dx=0,$$

using condition (a) and Lemma 5.1 we have

$$\begin{aligned}
I_2&=\int_Q\big[f(\mathcal{S}_jA+\nabla^j\phi)-f(\mathcal{S}_jA)\big]\,dx-\lambda\int_Q\big[g(\mathcal{S}_jA+\nabla^j\phi)-g(\mathcal{S}_jA)\big]\,dx\\
&=\int_Q\big[f(\mathcal{S}_jA+\nabla^j\phi)-f(\mathcal{S}_jA)\big]\,dx-\lambda\int_Q\big[g(\mathcal{S}_jA+\nabla^j\phi)-g(\mathcal{S}_jA)+\nabla g(\mathcal{S}_jA)\cdot\nabla^j\phi\big]\,dx\\
&\ge\Big(\nu^{(j)}-\lambda\Theta_p\Big)\int_Q\big(\mu^2+|\mathcal{S}_jA|^2+|\nabla^j\phi|^2\big)^{\frac{p-2}{2}}|\nabla^j\phi|^2dx\ge 0.
\end{aligned}\tag{3.3}$$

Let us pass to the first integral. Noticing that $\nabla f_\lambda^{(j)}(\mathcal{S}_jA)\in E_k^{N^{k-j}}$, thanks to Proposition 2.5 and using the fact that ψ is Q-periodic,

$$\int_Q\nabla f_\lambda^{(j)}(\mathcal{S}_jA)\cdot\mathcal{S}_j\nabla^{j-1}\psi\,dx=\int_Q\nabla f_\lambda^{(j)}(\mathcal{S}_jA)\cdot\nabla^{j-1}\psi\,dx=0.$$

Hence,

$$I_1 = -\int_Q \Big[f_\lambda^{(j)}(\mathcal{S}_j A + \mathcal{S}_j \nabla^{j-1}\varphi - \mathcal{S}_j \nabla^{j-1}\psi) - f_\lambda^{(j)}(\mathcal{S}_j A + \mathcal{S}_j \nabla^{j-1}\varphi) \\ - \nabla f_\lambda^{(j)}(\mathcal{S}_j A) \cdot \mathcal{S}_j \nabla^{j-1}\psi \Big] \, dx.$$

As observed in the previous proof the function g satisfies condition (5.6), and so by Lemma 5.5 condition (b) still holds for the function f_λ for a suitable constant $\widetilde{M} = \widetilde{M}(p, M^{(j)}, \lambda)$ in place of $M^{(j)}$. Thus, applying Lemma 5.6 with $\varepsilon = \lambda\theta_p/2$, there exists a positive constant $\sigma = \sigma(p, M^{(j)}, \lambda)$ such that

$$I_1 \geq -\frac{\lambda\theta_p}{2} \int_Q \Big(\mu^2 + |\mathcal{S}_j A|^2 + |\mathcal{S}_j \nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}} |\mathcal{S}_j \nabla^{j-1}\varphi|^2 \, dx \\ - \sigma\Big(\mu^2 + |\mathcal{S}_j A|^2\Big)^{\frac{p-2}{2}} \int_Q |\mathcal{S}_j \nabla^{j-1}\psi|^2 \, dx - \sigma \int_Q |\mathcal{S}_j \nabla^{j-1}\psi|^p \, dx.$$

Thanks to Lemma 2.13 and using (3.3) we get

$$I_1 + I_2 \geq -\frac{\lambda\theta_p}{2} \int_Q \Big(\mu^2 + |\mathcal{S}_j A|^2 + |\mathcal{S}_j \nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}} |\mathcal{S}_j \nabla^{j-1}\varphi|^2 \, dx \tag{3.4} \\ - \sigma\,\gamma(N,2,j-1)\Big(\mu^2 + |\mathcal{S}_j A|^2\Big)^{\frac{p-2}{2}} \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^2 \, dx - \sigma\,\gamma(N,p,j-1) \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p \, dx.$$

Since φ is Q-periodic,

$$0 = \int_Q \nabla\, g_\beta(\mathcal{S}_j A\,,\, \mathcal{A}_j A) \cdot \nabla^{j-1}\varphi \, dx \\ = \int_Q \big[\nabla_x\, g_\beta(\mathcal{S}_j A\,,\, \mathcal{A}_j A) \cdot \mathcal{S}_j \nabla^{j-1}\varphi + \nabla_y\, g_\beta(\mathcal{S}_j A\,,\, \mathcal{A}_j A) \cdot \mathcal{A}_j \nabla^{j-1}\varphi\big] \, dx,$$

so that

$$I_3 = \lambda \int_Q \big[g_\beta(\mathcal{S}_j A + \mathcal{S}_j \nabla^{j-1}\varphi, \mathcal{A}_j A + \mathcal{A}_j \nabla^{j-1}\varphi) - g_\beta(\mathcal{S}_j A\,,\, \mathcal{A}_j A) \\ - \nabla_x\, g_\beta(\mathcal{S}_j A\,,\, \mathcal{A}_j A) \cdot \mathcal{S}_j \nabla^{j-1}\varphi - \nabla_y\, g_\beta(\mathcal{S}_j A\,,\, \mathcal{A}_j A) \cdot \mathcal{A}_j \nabla^{j-1}\varphi\big] \, dx. \tag{3.5}$$

We are now going to split I_3 into two terms. We will use the first term to compensate the sum $I_1 + I_2$, and the remaining one to get the strict $(j-1)$-quasiconvexity. Relation (5.5) of Lemma 5.2 gives

$$\frac{I_3}{2} \geq \frac{\lambda\theta_p}{2} \int_Q \Big(\mu^2 + |\mathcal{S}_j A|^2 + |\mathcal{S}_j \nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}} |\mathcal{S}_j \nabla^{j-1}\varphi|^2 \, dx \\ + \frac{\lambda\theta_p\beta^2}{4}\Big(\mu^2 + |\mathcal{S}_j A|^2\Big)^{\frac{p-2}{2}} \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^2 \, dx + \frac{\lambda\theta_p\beta^p}{4} \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p \, dx. \tag{3.6}$$

If we choose $\beta = \beta^{(j)} > 0$ so large that

$$\frac{\lambda\theta_p(\beta^{(j)})^2}{4} \geq \sigma\gamma(N,2,j-1) \qquad \text{and} \qquad \frac{\lambda\theta_p(\beta^{(j)})^p}{4} \geq \sigma\gamma(N,p,j-1),$$

using relations (3.4) and (3.6), we have

$$I_1 + I_2 + I_3 \geq \frac{I_3}{2}.$$

Let us estimate the last term. Without any loss of generality we can assume $\beta^{(j)} \geq 1$. Then, recalling (3.5) and using inequality (5.4) of Lemma 5.2

$$\int_Q \Big[F^{(j)}(A+\nabla^{j-1}\varphi) - F^{(j)}(A)\Big]\,dx = I_1 + I_2 + I_3$$

$$\geq \frac{\lambda\theta_p}{2}\int_Q \Big(\mu^2 + |\mathcal{S}_j A|^2 + |\mathcal{S}_j\nabla^{j-1}\varphi|^2 + (\beta^{(j)})^2|\mathcal{A}_j A|^2 + (\beta^{(j)})^2|\mathcal{A}_j\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}} \Big(|\mathcal{S}_j\nabla^{j-1}\varphi|^2 + (\beta^{(j)})^2|\mathcal{A}_j\nabla^{j-1}\varphi|^2\Big)\,dx$$

$$\geq \frac{\lambda\theta_p}{2}\int_Q \Big(\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx$$

$$= \nu^{(j)}\frac{\theta_p}{4\,\Theta_p}\int_Q \Big(\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx,$$

where we chose

$$\lambda = \frac{\nu^{(j)}}{2\,\Theta_p}.$$

One can show that $F^{(j)}$ satisfies condition (b') as it was done in the proof of Lemma 3.1. □

We can now pass to the proof of Theorem 1.1.

Proof of Theorem 1.1.

Step 1. Case $1 < p < 2$. To simplify the notation, for every $B \in \mathbb{R}^{N^k}$ we set

$$\mathcal{P}(B) := \Big(\mu^2 + |B|^2\Big)^{\frac{p-2}{2}}|B|^2, \qquad \mathcal{G}(B) := \Big[\Big(\mu^2 + |B|^2\Big)^{\frac{p}{2}} - \mu^p\Big].$$

Let $\varepsilon > 0$ be fixed. We start the proof by applying Lemma 3.1 with $j = k$, $\nu^{(k)} = \nu$, $h^{(k)} \equiv 0$ and

$$f^{(k)}(A) = f(A), \qquad \text{for every } A \in E_k.$$

Then, we apply again $k-2$ times Lemma 3.1 with $j = k-1, k-2, \ldots, 2$ respectively, with

$$\nu^{(j)} = \frac{\nu}{4^{k-j}},$$

and

$$f^{(j)}(A) = F^{(j+1)}(A), \qquad \text{for every } A \in E_k^{N^{k-j}},$$

while the functions $h^{(j)} : E_k^{N^{k-j}} \to [0,+\infty)$ will be chosen as

$$h^{(k-1)}(A) = \mathcal{P}(\mathcal{A}_k A), \qquad h^{(j)}(A) = \mathcal{P}(\mathcal{A}_{j+1}A) + \sum_{r=j+2}^{k}\mathcal{P}(\mathcal{A}_r\mathcal{S}_{r-1}\ldots\mathcal{S}_{j+1}A) \qquad j = k-2,\ldots,2.$$

In this way after the last step, corresponding to $j = 2$, we obtain a function $F^{(2)} : \mathbb{R}^{N^k} \to \mathbb{R}$ given by

$$F^{(2)}(A) := f(\mathcal{S}_k\mathcal{S}_{k-1}\ldots\mathcal{S}_2) + \beta^{(2)}\mathcal{G}(\mathcal{A}_2 A) + \sum_{r=3}^{k}\beta^{(r)}\mathcal{G}(\mathcal{A}_r\mathcal{S}_{r-1}\ldots\mathcal{S}_2 A). \tag{3.7}$$

Here, for every $j = 2,\ldots,k$, the constant $\beta^{(j)}$ is given by the proof of Lemma 3.1 with the correspondent index j. $F^{(2)}$ has the following properties:

(a') (strict 1-quasiconvexity up to a perturbation)

$$\int_Q \Big[F^{(2)}(A+\nabla\varphi) - F^{(2)}(A)\Big]\,dx \geq -\varepsilon\mathcal{P}(\mathcal{A}_2 A) - \varepsilon h^{(2)}(\mathcal{S}_2 A)$$
$$+ \frac{\nu}{4^{k-1}}\int_Q \Big(\mu^2 + |A|^2 + |\nabla\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla\varphi|^2\,dx$$

for every $A \in \mathbb{R}^{N^k}$ and every $\varphi \in C_c^1(Q;\mathbb{R}^{N^{k-1}})$;

(b') (Lipschitz condition for the gradient)

$$|\nabla F^{(2)}(A+B) - \nabla F^{(2)}(A)| \leq L\left(\mu^2 + |A|^2 + |B|^2\right)^{\frac{p-2}{2}} |B|,$$

for every $A, B \in \mathbb{R}^{N^k}$, with $L = L(p,\mu,M,\nu)$;

(c) ($F^{(2)}$ extends f)

$$F^{(2)}(A) = f(A) \qquad\qquad \forall\, A \in E_k.$$

Now, let us define

$$F(A) := \inf\left\{\int_Q F^{(2)}(A + \nabla\varphi(x))\,dx : \varphi \in C^\infty_{per}(Q;\mathbb{R}^{N^{k-1}})\right\}$$

for every $A \in \mathbb{R}^{N^k}$. Property (a') implies that for every $A \in \mathbb{R}^{N^k}$

$$F^{(2)}(A) - \varepsilon\mathcal{P}(\mathcal{A}_2 A) - \varepsilon h^{(2)}(\mathcal{S}_2 A) \leq F(A) \leq F^{(2)}(A). \tag{3.8}$$

Since for every $A \in E_k$

$$\mathcal{P}(\mathcal{A}_2 A) = h^{(2)}(\mathcal{S}_2 A) = 0,$$

from property (c) and relation (3.8) equality (1.3) follows. Let us check (1.4). Thanks to Proposition 2.9, from condition (b') we infer that there exists a positive constant c, depending on the function $F^{(2)}$ and in turn on f, such that

$$|F^{(2)}(A)| \leq c\,(1 + |A|^p) \qquad \forall\, A \in \mathbb{R}^{N^k}.$$

Recalling the definitions of the functions $\mathcal{P}$ and $h^{(2)}$, last relation and (3.8) give (1.4).

Step 2. Case $p \geq 2$. Repeating the strategy used for the case $1 < p < 2$, we first apply Lemma 3.2 with $j = k$, $\nu^{(k)} = \nu$ and

$$f^{(k)}(A) = f(A), \qquad \text{for every } A \in E_k.$$

Then, we apply again $k-2$ times Lemma 3.2 with $j = k-1, k-2, \ldots, 2$ respectively, with

$$\nu^{(j)} = \nu\left(\frac{\theta_p}{4\Theta_p}\right)^{k-j+1},$$

and

$$f^{(j)}(A) = F^{(j+1)}(A), \qquad \text{for every } A \in E_k^{N^{k-j}}.$$

Finally, when $j = 2$ we obtain a function $F^{(2)} : \mathbb{R}^{N^k} \to \mathbb{R}$ given by

$$F^{(2)}(A) = f(\mathcal{S}_k \ldots \mathcal{S}_2 A) + \mathcal{L}^{(2)}(\mathcal{S}_2 A\,,\, \mathcal{A}_2 A) + \sum_{r=3}^{k} \mathcal{L}^{(r)}(\mathcal{S}_r\mathcal{S}_{r-1}\ldots\mathcal{S}_2 A\,,\, \mathcal{A}_r\mathcal{S}_{r-1}\ldots\mathcal{S}_2 A), \tag{3.9}$$

where we set

$$\mathcal{L}^{(r)}(A,B) := -\frac{\nu^{(r)}}{2\Theta_p}\Big(\mu^2 + |A|^2\Big)^{\frac{p}{2}} + \frac{\nu^{(r)}}{2\Theta_p}\Big(\mu^2 + |A|^2 + (\beta^{(r)})^2|B|^2\Big)^{\frac{p}{2}}, \qquad r = 2,\ldots,k,$$

and for every $j = 2,\ldots,k$, the constant $\beta^{(j)}$ is given by the proof of Lemma 3.2 with the correspondent index j. The function $F^{(2)}$ just defined is such that

(a') (strict 1-quasiconvexity)

$$\int_Q \Big[F^{(2)}(A + \nabla\varphi) - F^{(2)}(A)\Big]\,dx \geq \frac{\nu}{4^k}\left(\frac{\theta_p}{\Theta_p}\right)^k \int_Q \Big(\mu^2 + |A|^2 + |\nabla\varphi|^2\Big)^{\frac{p-2}{2}} |\nabla\varphi|^2\,dx$$

for every $A \in \mathbb{R}^{N^k}$ and every $\varphi \in C_c^1(Q;\mathbb{R}^{N^{k-1}})$;

(b') (Lipschitz condition for the gradient)

$$|\nabla F^{(2)}(A+B) - \nabla F^{(2)}(A)| \le L\left(\mu^2 + |A|^2 + |B|^2\right)^{\frac{p-2}{2}} |B|,$$

for every $A, B \in \mathbb{R}^{N^k}$, with $L = L(p, \mu, M, \nu)$;

(c) ($F^{(2)}$ extends f)

$$F^{(2)}(A) = f(A) \qquad\qquad \forall\, A \in E_k.$$

We claim that the proof is concluded by setting $F := F^{(2)}$. Indeed, condition (c) gives (1.3), while (1.4) follows by applying Proposition 2.9 to $F^{(2)}$. □

4. Proof of Theorem 1.2

To prove the theorem, we first need two preliminary lemmas.

Lemma 4.1. *Let $j \in \{2, \dots, k\}$, $1 < p < 2$, $\mu \ge 0$, $\nu^{(j)} > 0$, and let $\{M_i^{(j)}\}_{i\in\mathbb{N}}$ be a sequence of positive constants. Let $\{f_i^{(j)}\}_{i\in\mathbb{N}}$ be a sequence of functions $f_i^{(j)} : E_k^{N^{k-j}} \to \mathbb{R}$ satisfying the following conditions:*

(a) *(strict j-quasiconvexity up to a perturbation)*

$$\int_Q \left[f_i^{(j)}(A + \nabla^j \phi) - f_i^{(j)}(A)\right] dx \ge -h_i^{(j)}(A) + \nu^{(j)} \int_Q \left(\mu^2 + |A|^2 + |\nabla^j \phi|^2\right)^{\frac{p-2}{2}} |\nabla^j \phi|^2 dx$$

for every $A \in E_k^{N^{k-j}}$, for every $\phi \in C_c^j(Q; \mathbb{R}^{N^{k-j}})$, and for every $i \in \mathbb{N}$, where $\{h_i^{(j)}\}_{i\in\mathbb{N}}$ is a sequence of functions $h_i^{(j)} : E_k^{N^{k-j}} \to [0, +\infty)$;

(b) *(p-growth condition)*

$$|f_i^{(j)}(A)| \le M_i^{(j)}(1 + |A|^p) \quad \forall\, A \in E_k^{N^{k-j}}, \quad \forall\, i \in \mathbb{N}.$$

Then there exists an increasing sequence $\{F_i^{(j)}\}_{i\in\mathbb{N}}$ of functions $F_i^{(j)} : E_k^{N^{k-j+1}} \to \mathbb{R}$, and two sequences $\{L_i^{(j)}\}_{i\in\mathbb{N}}$ and $\{\lambda_i^{(j)}\}_{i\in\mathbb{N}}$ of positive numbers, depending on $\nu^{(j)}, M_i^{(j)}, j, p, \mu$, such that

(a') *(strict $(j-1)$-quasiconvexity up to a perturbation)*

$$\int_Q \left[F_i^{(j)}(A + \nabla^{j-1}\varphi) - F_i^{(j)}(A)\right] dx \ge \frac{\nu^{(j)}}{4} \int_Q \left(\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2\right)^{\frac{p-2}{2}} |\nabla^{j-1}\varphi|^2\, dx - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_j A|^p - \lambda_i^{(j)} |\mathcal{A}_j A|^p - h_i^{(j)}(\mathcal{S}_j A)$$

for every $A \in E_k^{N^{k-j+1}}$, for every $\varphi \in C_c^{j-1}(Q; \mathbb{R}^{N^{k-j+1}})$, and for every $i \in \mathbb{N}$;

(b') *(p-growth condition)*

$$|F_i^{(j)}(A)| \le L_i^{(j)}(1 + |A|^p) \quad \forall\, A \in E_k^{N^{k-j+1}}, \quad \forall\, i \in \mathbb{N};$$

(c) *($F_i^{(j)}$ extends $f_i^{(j)}$)*

$$F_i^{(j)}(A) = f_i^{(j)}(A) \qquad\qquad \forall\, A \in E_k^{N^{k-j}}, \quad \forall\, i \in \mathbb{N}.$$

Proof. First we observe that, thanks to Proposition 2.7, there exists a positive constant $L = L(M_i^{(j)}, j, p)$ (we do not stress here the dependence on N and k), such that

$$|f_i^{(j)}(A+B) - f_i^{(j)}(A)| \le L\left(1 + |A|^{p-1} + |B|^{p-1}\right)|B| \tag{4.1}$$

for every $A, B \in E_k^{N^{k-j}}$. Let $\beta > 0$ be a constant to be chosen at the end of the proof. For every $A \in E_k^{N^{k-j+1}}$, we define

$$F_i^{(j)}(A) := f_i^{(j)}(\mathcal{S}_j A) + \beta |\mathcal{A}_j A|^p. \tag{4.2}$$

Condition (c) is clearly satisfied. In order to show (a'), let us consider a function $\varphi \in C^\infty_{\rm per}(Q;\mathbb{R}^{N^{k-j+1}})$. We can write

$$\nabla^{j-1}\varphi = \nabla^j\phi + \nabla^{j-1}\psi,$$

with $\nabla^{j-1}\varphi, \nabla^{j-1}\psi \in C^\infty_{\rm per}(Q;E_k^{N^{k-j+1}})$, and $\nabla^j\phi \in C^\infty_{\rm per}(Q;E_k^{N^{k-j}})$, where $\psi \in C^\infty_{\rm per}(Q;R^{N^{k-j+1}})$ satisfies condition (2.7) with $s = j-1$. Hence,

$$\begin{aligned}
&\int_Q \Big[F_i^{(j)}(A + \nabla^{j-1}\varphi) - F_i^{(j)}(A)\Big]\,dx \\
&\quad = \int_Q \Big[f_i^{(j)}(\mathcal{S}_jA + \nabla^j\phi + \mathcal{S}_j\nabla^{j-1}\psi) - f_i^{(j)}(\mathcal{S}_jA + \nabla^j\phi)\Big]\,dx \\
&\qquad + \int_Q \Big[f_i^{(j)}(\mathcal{S}_jA + \nabla^j\phi) - f_i^{(j)}(\mathcal{S}_jA)\Big]\,dx \\
&\qquad + \beta\int_Q \Big[\,\big|\mathcal{A}_jA + \mathcal{A}_j\nabla^{j-1}\psi\big|^p - |\mathcal{A}_jA|^p\,\Big]\,dx \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned}$$

By (4.1) and Young's inequality, for every $\delta > 0$ there exists a constant $C = C(M_i^{(j)}, j, p, \delta)$ such that

$$\begin{aligned}
I_1 &\geq -L\int_Q \big(1 + |\mathcal{S}_jA + \nabla^j\phi|^{p-1} + |\mathcal{S}_j\nabla^{j-1}\psi|^{p-1}\big)|\mathcal{S}_j\nabla^{j-1}\psi|\,dx \\
&\geq -\delta - \delta\,|\mathcal{S}_jA|^p - \delta\int_Q |\nabla^j\phi|^p\,dx - C\int_Q |\mathcal{S}_j\nabla^{j-1}\psi|^p\,dx.
\end{aligned}$$

Using Lemma 2.13

$$I_1 \geq -\delta - \delta\,|\mathcal{S}_jA|^p - \delta\int_Q |\nabla^j\phi|^p\,dx - C\,\gamma\int_Q |\mathcal{A}_j\nabla^{j-1}\psi|^p\,dx. \tag{4.3}$$

Thanks to Lemma 5.4, for every $0 < \varepsilon < 1$

$$\begin{aligned}
I_1 &\geq -\delta(1 + \varepsilon\mu^p) - \delta(1+\varepsilon)|\mathcal{S}_jA|^p - C\,\gamma\int_Q |\mathcal{A}_j\nabla^{j-1}\psi|^p\,dx \\
&\quad - 8\,\delta\,\varepsilon^{\frac{p-2}{p}}\int_Q \Big(\mu^2 + |\mathcal{S}_jA|^2 + |\nabla^j\phi|^2\Big)^{\frac{p-2}{2}}|\nabla^j\phi|^2\,dx.
\end{aligned}$$

Then, applying Lemma 5.3 with $\tilde\mu = 0$, $x = \mathcal{A}_jA$, and $y = \mathcal{A}_j\nabla^{j-1}\psi$,

$$\begin{aligned}
I_1 &\geq -\delta(1 + \varepsilon\mu^p) - \delta(1+\varepsilon)|\mathcal{S}_jA|^p - C\,\gamma\,\varepsilon^{\frac{p}{2}}|\mathcal{A}_jA|^p \\
&\quad - C\,\gamma\,\varepsilon^{\frac{p-2}{2}}\int_Q \Big(|\mathcal{A}_jA|^2 + |\mathcal{A}_j\nabla^{j-1}\psi|^2\Big)^{\frac{p-2}{2}}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,dx \\
&\quad - 8\,\delta\,\varepsilon^{\frac{p-2}{p}}\int_Q \Big(\mu^2 + |\mathcal{S}_jA|^2 + |\nabla^j\phi|^2\Big)^{\frac{p-2}{2}}|\nabla^j\phi|^2\,dx.
\end{aligned}$$

Thus, there exists a sequence of positive numbers $\{\lambda_i^{(j)}\}_{i\in\mathbb{N}}$, such that for every $i \in \mathbb{N}$

$$\begin{aligned}
I_1 &\geq -\frac{\nu^{(j)}}{2}\int_Q \Big(\mu^2 + |\mathcal{S}_jA|^2 + |\nabla^j\phi|^2\Big)^{\frac{p-2}{2}}|\nabla^j\phi|^2\,dx \\
&\quad - \lambda_i^{(j)}\int_Q \Big(|\mathcal{A}_jA|^2 + |\mathcal{A}_j\nabla^{j-1}\psi|^2\Big)^{\frac{p-2}{2}}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,dx \\
&\quad - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_jA|^p - \lambda_i^{(j)}|\mathcal{A}_jA|^p.
\end{aligned}$$

Here, for every fixed $i \in \mathbb{N}$, $\lambda_i^{(j)} = \lambda_i^{(j)}(\nu^{(j)}, M_i^{(j)}, j, p, \mu)$. By condition (a)

$$I_2 \geq \nu^{(j)}\int_Q \big(\mu^2 + |\mathcal{S}_jA|^2 + |\nabla^j\phi|^2\big)^{\frac{p-2}{2}}\,|\nabla^j\phi|^2 dx - h_i^{(j)}(\mathcal{S}_jA),$$

so that

$$
\begin{aligned}
I_1+I_2 \geq \frac{\nu^{(j)}}{2}\int_Q &\Big(\mu^2+|\mathcal{S}_jA|^2+|\nabla^j\phi|^2\Big)^{\frac{p-2}{2}}|\nabla^j\phi|^2\,dx\\
&-\lambda_i^{(j)}\int_Q\Big(|\mathcal{A}_jA|^2+|\mathcal{A}_j\nabla^{j-1}\psi|^2\Big)^{\frac{p-2}{2}}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,dx\\
&-\frac{1}{i}-\frac{1}{i}|\mathcal{S}_jA|^p-\lambda_i^{(j)}|\mathcal{A}_jA|^p-h_i^{(j)}(\mathcal{S}_jA),
\end{aligned}
\tag{4.4}
$$

We focus now on the first term of last expression. Applying Lemma 5.3 with $\tilde{\mu}=\mu^2+|A|^2$, $x=\nabla^j\phi$, and $y=\nabla^{j-1}\psi$, and recalling that $\nabla^j\phi+\nabla^{j-1}\psi=\nabla^{j-1}\varphi$ we get

$$
\begin{aligned}
&\frac{\nu^{(j)}}{2}\int_Q\Big(\mu^2+|\mathcal{S}_jA|^2+|\nabla^j\phi|^2\Big)^{\frac{p-2}{2}}|\nabla^j\phi|^2\,dx\\
&\quad\geq\frac{\nu^{(j)}}{2}\int_Q\Big(\mu^2+|A|^2+|\nabla^j\phi|^2\Big)^{\frac{p-2}{2}}|\nabla^j\phi|^2\,dx\\
&\quad\geq\frac{\nu^{(j)}}{4}\int_Q\Big(\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx\\
&\qquad-\frac{\nu^{(j)}}{2}\int_Q\Big(\mu^2+|A|^2+|\nabla^{j-1}\psi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\psi|^2\,dx,
\end{aligned}
$$

where in the first line we used the fact that $1<p<2$. By Lemma 2.14 last inequality becomes

$$
\begin{aligned}
&\frac{\nu^{(j)}}{2}\int_Q\Big(\mu^2+|\mathcal{S}_jA|^2+|\nabla^j\phi|^2\Big)^{\frac{p-2}{2}}|\nabla^j\phi|^2\,dx\\
&\quad\geq\frac{\nu^{(j)}}{4}\int_Q\Big(\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx\\
&\qquad-\frac{\nu^{(j)}}{2}\tau\int_Q\Big(\mu^2+|A|^2+|\mathcal{A}_j\nabla^{j-1}\psi|^2\Big)^{\frac{p-2}{2}}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,dx\\
&\quad\geq\frac{\nu^{(j)}}{4}\int_Q\Big(\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx\\
&\qquad-\frac{\nu^{(j)}}{2}\tau\int_Q\Big(|\mathcal{A}_jA|^2+|\mathcal{A}_j\nabla^{j-1}\psi|^2\Big)^{\frac{p-2}{2}}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,dx,
\end{aligned}
\tag{4.5}
$$

again exploiting that $1<p<2$. Collecting (4.4) and (4.5) we have

$$
\begin{aligned}
I_1+I_2\geq\frac{\nu^{(j)}}{4}\int_Q&\Big(\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx\\
&-\left(\lambda_i^{(j)}+\frac{\nu^{(j)}\tau}{2}\right)\int_Q\Big(|\mathcal{A}_jA|^2+|\mathcal{A}_j\nabla^{j-1}\psi|^2\Big)^{\frac{p-2}{2}}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,dx\\
&-\frac{1}{i}-\frac{1}{i}|\mathcal{S}_jA|^p-\lambda_i^{(j)}|\mathcal{A}_jA|^p-h_i^{(j)}(\mathcal{S}_jA).
\end{aligned}
\tag{4.6}
$$

Concerning I_3, using the periodicity of ψ and thanks to Lemma 5.1 with $\mu=0$

$$
\begin{aligned}
I_3&=\beta\int_Q\Big[\,\big|\mathcal{A}_jA+\mathcal{A}_j\nabla^{j-1}\psi\big|^p-|\mathcal{A}_jA|^p-p\,|\mathcal{A}_jA|^{p-2}\,\mathcal{A}_jA\cdot\mathcal{A}_j\nabla^{j-1}\psi\Big]\,dx\\
&\geq\beta\,\theta_p\int_Q\Big(|\mathcal{A}_jA|^2+|\mathcal{A}_j\nabla^{j-1}\psi|^2\Big)^{\frac{p-2}{2}}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,dx.
\end{aligned}
\tag{4.7}
$$

Choosing $\beta=\beta_i^{(j)}>0$ such that

$$
\beta_i^{(j)}\theta_p\geq\lambda_i^{(j)}+\frac{\nu^{(j)}\tau}{2},
$$

from (4.6) and (4.7) we obtain

$$I_1+I_2+I_3 \geq \frac{\nu^{(j)}}{4}\int_Q \Big(\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx$$
$$-\frac{1}{i}-\frac{1}{i}|\mathcal{S}_jA|^p-\lambda_i^{(j)}|\mathcal{A}_jA|^p-h_i^{(j)}(\mathcal{S}_jA),$$

so that (a') holds. From (4.2) condition (b') follows. □

The second lemma addresses the case $p\geq 2$.

Lemma 4.2. *Let $j\in\{2,\dots,k\}$, $p\geq 2$, $\mu\geq 0$, $\nu^{(j)}>0$, and let $\{M_i^{(j)}\}_{i\in\mathbb{N}}$ be a sequence of positive constants. Let moreover θ_p and Θ_p be given by Lemma 5.1, and let $\{f_i^{(j)}\}_{i\in\mathbb{N}}$ be a sequence of functions $f_i^{(j)}:E_k^{N^{k-j}}\to\mathbb{R}$ satisfying the following conditions:*

(a) *(strict j-quasiconvexity up to a perturbation)*

$$\int_Q\Big[f_i^{(j)}(A+\nabla^j\phi)-f_i^{(j)}(A)\Big]\,dx\geq -h_i^{(j)}(A)$$
$$+\nu^{(j)}\int_Q\left(\mu^2+|A|^2+|\nabla^j\phi|^2\right)^{\frac{p-2}{2}}|\nabla^j\phi|^2dx$$

for every $A\in E_k^{N^{k-j}}$, for every $\phi\in C_c^j(Q;\mathbb{R}^{N^{k-j}})$, and for every $i\in\mathbb{N}$, where $\{h_i^{(j)}\}_{i\in\mathbb{N}}$ is a sequence of functions $h_i^{(j)}:E_k^{N^{k-j}}\to[0,+\infty)$;

(b) *(p-growth condition)*

$$|f_i^{(j)}(A)|\leq M_i^{(j)}(1+|A|^p)\quad\forall A\in E_k^{N^{k-j}},\quad\forall i\in\mathbb{N}.$$

Then there exists an increasing sequence $\{F_i^{(j)}\}_{i\in\mathbb{N}}$ of functions $F_i^{(j)}:E_k^{N^{k-j+1}}\to\mathbb{R}$, and a sequence $\{L_i^{(j)}\}_{i\in\mathbb{N}}$ of positive numbers, depending on $\nu^{(j)},M_i^{(j)},j,p,\mu$, such that

(a') *(strict $(j-1)$-quasiconvexity up to a perturbation)*

$$\int_Q\Big[F_i^{(j)}(A+\nabla^{j-1}\varphi)-F_i^{(j)}(A)\Big]\,dx\geq -h_i^{(j)}(\mathcal{S}_jA)-\frac{1}{i}-\frac{1}{i}|\mathcal{S}_jA|^p$$
$$+\nu^{(j)}\frac{\theta_p}{4\Theta_p}\int_Q\Big(\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx$$

for every $A\in E_k^{N^{k-j+1}}$, for every $\varphi\in C_c^{j-1}(Q;\mathbb{R}^{N^{k-j+1}})$, and for every $i\in\mathbb{N}$;

(b') *(p-growth condition)*

$$|F_i^{(j)}(A)|\leq L_i^{(j)}(1+|A|^p)\quad\forall A\in E_k^{N^{k-j+1}},\quad\forall i\in\mathbb{N};$$

(c) *($F_i^{(j)}$ extends $f_i^{(j)}$)*

$$F_i^{(j)}(A)=f_i^{(j)}(A)\qquad\qquad\forall A\in E_k^{N^{k-j}},\quad\forall i\in\mathbb{N}.$$

Proof. Let $\alpha\in(0,\nu^{(j)}/\Theta_p]$ and $\beta>0$ to be determined at the end of the proof. We define

$$F_i^{(j)}(A):=f_i^{(j)}(\mathcal{S}_jA)-\alpha\Big(\mu^2+|\mathcal{S}_jA|^2\Big)^{\frac{p}{2}}+\alpha\Big(\mu^2+|\mathcal{S}_jA|^2+\beta^2|\mathcal{A}_jA|^2\Big)^{\frac{p}{2}}.\tag{4.8}$$

Condition (c) is clearly satisfied. Let now $\varphi\in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^{N^{k-j+1}})$. As usual, we can write

$$\nabla^{j-1}\varphi=\nabla^j\phi+\nabla^{j-1}\psi,$$

with $\nabla^{j-1}\varphi,\nabla^{j-1}\psi\in C^\infty_{\mathrm{per}}(Q;E_k^{N^{k-j+1}})$, and $\nabla^j\phi\in C^\infty_{\mathrm{per}}(Q;E_k^{N^{k-j}})$, where $\psi\in C^\infty_{\mathrm{per}}(Q;R^{N^{k-j+1}})$ satisfies condition (2.7) with $s=j-1$. Setting

$$(f_i^{(j)})_\alpha(B):=f_i^{(j)}(B)-\alpha\Big(\mu^2+|B|^2\Big)^{\frac{p}{2}}$$

for every $B \in E_k^{N^{k-j}}$, we have

$$\begin{aligned}
&\int_Q \Big[F_i^{(j)}(A+\nabla^{j-1}\varphi) - F_i^{(j)}(A)\Big]\,dx \\
&\quad = \int_Q \Big[(f_i^{(j)})_\alpha(\mathcal{S}_jA+\nabla^j\phi+\mathcal{S}_j\nabla^{j-1}\psi) - (f_i^{(j)})_\alpha(\mathcal{S}_jA+\nabla^j\phi)\Big]\,dx \\
&\qquad + \int_Q \Big[(f_i^{(j)})_\alpha(\mathcal{S}_jA+\nabla^j\phi) - (f_i^{(j)})_\alpha(\mathcal{S}_jA)\Big]\,dx \\
&\qquad + \alpha\int_Q \big[g_\beta(\mathcal{S}_jA+\mathcal{S}_j\nabla^{j-1}\varphi,\,\mathcal{A}_jA+\mathcal{A}_j\nabla^{j-1}\varphi) - g_\beta(\mathcal{S}_jA,\mathcal{A}_jA)\big]\,dx \\
&\quad =: I_1+I_2+I_3,
\end{aligned}$$

with g_β defined by (5.3), with $X = E_k^{N^{k-j}}$ and $Y = \mathcal{A}_jE_k^{N^{k-j+1}}$. By repeating the chain of inequalities (3.3), one can show $(f_i^{(j)})_\alpha$ is j-quasiconvex. In addition, applying Lemma 5.5 and Proposition 2.9 to the function

$$B \mapsto \alpha\Big(\mu^2+|B|^2\Big)^{\frac{p}{2}},$$

we have that $(f_i^{(j)})_\alpha$ satisfies condition (b), for some positive constant $\widetilde{M}_i^{(j)} = \widetilde{M}_i^{(j)}(\alpha,\mu,M_i^{(j)})$ in place of $M_i^{(j)}$. Thus, by applying Proposition 2.7, we can still conclude that relation (4.1) holds true for the function $(f_i^{(j)})_\alpha$, for a suitable constant $L = L(N, M_i^{(j)}, k, j, p, \alpha)$. By repeating the same argument of the previous proof, we get that for every $\delta > 0$ there exists a positive constant $c = c(M_i^{(j)}, j, p, \alpha, \mu, \delta)$ such that

$$\begin{aligned}
I_1 &\geq -\delta - \delta\,|\mathcal{S}_jA|^p - \delta\int_Q |\nabla^j\phi|^p\,dx - c\,\gamma\int_Q |\mathcal{A}_j\nabla^{j-1}\psi|^p\,dx \\
&\geq -\delta - \delta\,|\mathcal{S}_jA|^p - \delta\int_Q \Big(\mu^2+|\mathcal{S}_jA|^2+|\nabla^j\phi|^2\Big)^{\frac{p-2}{2}}|\nabla^j\phi|^2\,dx \\
&\quad - c\,\gamma\int_Q |\mathcal{A}_j\nabla^{j-1}\psi|^p\,dx.
\end{aligned}$$

Hence, we can find a sequence of positive numbers $\{\lambda_i^{(j)}\}_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$

$$\begin{aligned}
I_1 &\geq -\Big(\nu^{(j)} - \alpha\,\Theta_p\Big)\int_Q \Big(\mu^2+|\mathcal{S}_jA|^2+|\nabla^j\phi|^2\Big)^{\frac{p-2}{2}}|\nabla^j\phi|^2\,dx \\
&\quad - \lambda_i^{(j)}\int_Q |\mathcal{A}_j\nabla^{j-1}\psi|^p\,dx - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_jA|^p.
\end{aligned}$$

Here $\lambda_i^{(j)} = \lambda_i^{(j)}(M_i^{(j)}, j, p, \alpha, \mu)$ for every fixed $i \in \mathbb{N}$, . Adapting to the present situation inequality (3.3) we get

$$\begin{aligned}
I_2 &= \int_Q \Big[(f_i^{(j)})_\alpha(\mathcal{S}_jA+\nabla^j\phi) - (f_i^{(j)})_\alpha(\mathcal{S}_jA)\Big]\,dx \geq -h_i^{(j)}(\mathcal{S}_jA) \\
&\quad + \Big(\nu^{(j)} - \alpha\,\Theta_p\Big)\int_Q \big(\mu^2+|\mathcal{S}_jA|^2+|\nabla^j\phi|^2\big)^{\frac{p-2}{2}}|\nabla^j\phi|^2 dx.
\end{aligned}$$

Moreover, assuming without any loss of generality that $\beta \geq 1$,

$$\begin{aligned} I_3 \geq \frac{\alpha\,\theta_p}{2}&\int_Q \Big(\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx \\ &+\frac{\alpha\theta_p}{2}\int_Q \Big(\mu^2+|\mathcal{S}_jA|^2+|\mathcal{S}_j\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\mathcal{S}_j\nabla^{j-1}\varphi|^2\,dx \\ &+\frac{\alpha\theta_p\beta^2}{4}\Big(\mu^2+|\mathcal{S}_jA|^2\Big)^{\frac{p-2}{2}}\int_Q|\mathcal{A}_j\nabla^{j-1}\psi|^2\,dx+\frac{\alpha\theta_p\beta^p}{4}\int_Q|\mathcal{A}_j\nabla^{j-1}\psi|^p\,dx \\ &\geq \frac{\alpha\,\theta_p}{2}\int_Q \Big(\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx+\frac{\alpha\theta_p\beta^p}{4}\int_Q|\mathcal{A}_j\nabla^{j-1}\psi|^p\,dx. \end{aligned}$$

Let us now choose

$$\alpha=\alpha^{(j)}=\frac{\nu^{(j)}}{2\Theta_p},$$

and $\beta=\beta_i^{(j)}>0$ such that

$$\frac{\alpha^{(j)}\,\theta_p(\beta_i^{(j)})^p}{4}\geq\lambda_i^{(j)},$$

we obtain

$$\begin{aligned}\int_Q\Big[F_i^{(j)}(A+\nabla^{j-1}\varphi)-F_i^{(j)}(A)\Big]\,dx&=I_1+I_2+I_3\\ &\geq\nu^{(j)}\frac{\theta_p}{4\Theta_p}\int_Q\Big(\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla^{j-1}\varphi|^2\,dx\\ &\quad-h_i^{(j)}(\mathcal{S}_jA)-\frac{1}{i}-\frac{1}{i}|\mathcal{S}_jA|^p,\end{aligned}$$

so that (a') holds. Finally, condition (b') follows by (4.8).

□

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2.

Step 1. Case $1<p<2$.

We start by applying Lemma 4.1 with $j=k$, $\nu^{(k)}=\nu$ and

$$f_i^{(k)}(A)=f(A),\quad h_i^{(k)}(A)=0\qquad\text{for every } A\in E_k,\ i\in\mathbb{N}.$$

Then, we apply again $k-2$ times Theorem 4.1 with $j=k-1,k-2,\dots,2$ respectively, with

$$\nu^{(j)}=\frac{\nu}{4^{k-j}},$$

and, for every $A\in E_k^{N^{k-j}}$ and $i\in\mathbb{N}$,

$$f_i^{(j)}(A)=F_i^{(j+1)}(A).$$

Accordingly, the functions $h_i^{(j)}$ will be chosen as

$$h_i^{(k-1)}(A)=\frac{1}{i}+\frac{1}{i}|\mathcal{S}_kA|^p+\lambda_i^{(k)}|\mathcal{A}_kA|^p,$$

and, for $j=k-2,\dots,2$,

$$\begin{aligned}h_i^{(j)}(A)=\frac{k-j}{i}+\frac{1}{i}\sum_{r=j+1}^{k}|\mathcal{S}_r\mathcal{S}_{r-1}\dots\mathcal{S}_{j+1}A|^p+\lambda_i^{(j+1)}|\mathcal{A}_{j+1}A|^p\\ +\sum_{r=j+2}^{k}\lambda_i^{(r)}|\mathcal{A}_r\mathcal{S}_{r-1}\dots\mathcal{S}_{j+1}A|^p,\end{aligned}$$

where the sequences $\{\lambda_i^{(j)}\}_{i\in\mathbb{N}}$ are given by Lemma 4.1. In this way after the last step, corresponding to $j=2$, we obtain a sequence $\{F_i^{(2)}\}_{i\in\mathbb{N}}$ of functions $F_i^{(2)}:\mathbb{R}^{N^k}\to\mathbb{R}$ given by

$$F_i^{(2)}(A)=f(\mathcal{S}_k\dots\mathcal{S}_2A)+\beta_i^{(2)}|\mathcal{A}_2A|^p+\sum_{r=3}^{k}\beta_i^{(r)}|\mathcal{A}_r\mathcal{S}_{r-1}\dots\mathcal{S}_2A|^p.$$

Here for $r=2,\dots,k$, the sequence $\{\beta_i^{(r)}\}_{i\in\mathbb{N}}$ is that one given in the proof of Lemma 4.1. The functions $F_i^{(2)}$ just defined have the following properties:

(a') (strict 1-quasiconvexity up to a perturbation)

$$\begin{aligned}\int_Q\Big[F_i^{(2)}(A+\nabla\varphi)-F_i^{(2)}(A)\Big]\,dx&\ge -h_i^{(2)}(\mathcal{S}_2A)-\frac{1}{i}-\frac{1}{i}|\mathcal{S}_2A|^p\\&+\frac{\nu}{4^{k-1}}\int_Q\Big(\mu^2+|A|^2+|\nabla\varphi|^2\Big)^{\frac{p-2}{2}}|\nabla\varphi|^2\,dx-\lambda_i^{(2)}|\mathcal{A}_2A|^p\\&\ge -h_i^{(2)}(\mathcal{S}_2A)-\frac{1}{i}-\frac{1}{i}|\mathcal{S}_2A|^p-\lambda_i^{(2)}|\mathcal{A}_2A|^p\end{aligned}$$

for every $A\in\mathbb{R}^{N^k}$, for every $\varphi\in C_c^1(Q;\mathbb{R}^{N^{k-1}})$, and for every $i\in\mathbb{N}$;

(b') (growth condition)

$$|F_i^{(2)}(A)|\le L_i^{(2)}(1+|A|^p)\quad\forall A\in\mathbb{R}^{N^k},\forall i\in\mathbb{N},$$

with $L_i^{(2)}=L_i^{(2)}(\nu,M,p,\mu)$ for every fixed $i\in\mathbb{N}$;

(c) ($F_i^{(2)}$ extends f)

$$F_i^{(2)}(A)=f(A)\qquad\qquad \forall A\in E_k,\forall i\in\mathbb{N}.$$

Now, for every $A\in\mathbb{R}^{N^k}$ and $i\in\mathbb{N}$, we set

$$F_i(A):=\inf\left\{\int_Q F_i^{(2)}(A+\nabla\varphi(x))\,dx:\varphi\in C_{per}^\infty(Q;\mathbb{R}^{N^{k-1}})\right\}.$$

From property (a') it follows that for every $A\in\mathbb{R}^{N^k}$ and for every $i\in\mathbb{N}$

$$F_i^{(2)}(A)-h_i^{(2)}(\mathcal{S}_2A)-\frac{1}{i}-\frac{1}{i}|\mathcal{S}_2A|^p-\lambda_i^{(2)}|\mathcal{A}_2A|^p\le F_i(A)\le F_i^{(2)}(A).\tag{4.9}$$

Noticing that for every $A\in E_k$

$$\lim_{i\to+\infty}h_i^{(2)}(\mathcal{S}_2A)=\lim_{i\to+\infty}\left[\frac{k-2}{i}+\frac{1}{i}\sum_{r=3}^{k}|\mathcal{S}_r\dots\mathcal{S}_2A|^p\right]=0,$$

from property (c) and (4.9) we have (1.6). Finally, (1.7) follows from (b') and (4.9).

Step 2. Case $p\ge 2$.

We first apply Lemma 4.2 with $j=k$, $\nu^{(k)}=\nu$ and

$$f_i^{(k)}(A)=f(A),\quad h_i^{(k)}(A)=0\qquad\text{for every } A\in E_k,\ i\in\mathbb{N}.$$

At this point, we apply again $k-2$ times Lemma 4.2 with $j=k-1,k-2,\dots,2$ respectively, with

$$\nu^{(j)}=\nu\left(\frac{\theta_p}{4\Theta_p}\right)^{k-j+1},$$

and, for every $A \in E_k^{N^{k-j}}$ and $i \in \mathbb{N}$,

$$f_i^{(j)}(A) = F_i^{(j+1)}(A), \qquad h_i^{(j)}(A) = \frac{k-j}{i} + \frac{1}{i} \sum_{r=j+1}^{k} |\mathcal{S}_r \mathcal{S}_{r-1} \dots \mathcal{S}_{j+1} A|^p.$$

Finally, when $j = 2$, we obtain a sequence $\{F_i^{(2)}\}_{i\in\mathbb{N}}$ of functions $F_i^{(2)} : \mathbb{R}^{N^k} \to \mathbb{R}$ given by

$$\begin{aligned} F_i^{(2)}(A) = f(\mathcal{S}_k \dots \mathcal{S}_2 A) + \sum_{r=3}^{k} \mathcal{L}_i^{(r)}(\mathcal{S}_r \mathcal{S}_{r-1} \dots \mathcal{S}_2 A\,,\, \mathcal{A}_r \mathcal{S}_{r-1} \dots \mathcal{S}_2 A) \\ + \mathcal{L}_i^{(2)}(\mathcal{S}_2 A\,,\, \mathcal{A}_2 A), \end{aligned} \tag{4.10}$$

where we set

$$\mathcal{L}_i^{(r)}(A, B) := -\frac{\nu^{(r)}}{2\Theta_p}\Big(\mu^2 + |A|^2\Big)^{\frac{p}{2}} + \frac{\nu^{(r)}}{2\Theta_p}\Big(\mu^2 + |A|^2 + (\beta_i^{(r)})^2 |B|^2\Big)^{\frac{p}{2}}, \qquad r = 2, \dots, k.$$

The functions $F_i^{(2)}$ just defined have the following properties:

(a') (strict 1-quasiconvexity up to a perturbation)

$$\begin{aligned} \int_Q \Big[F_i^{(2)}(A + \nabla\varphi) - F_i^{(2)}(A)\Big]\, dx &\geq -h_i^{(2)}(\mathcal{S}_2 A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_2 A|^p \\ &\quad + \frac{\nu}{4^k}\left(\frac{\theta_p}{\Theta_p}\right)^k \int_Q \Big(\mu^2 + |A|^2 + |\nabla\varphi|^2\Big)^{\frac{p-2}{2}} |\nabla\varphi|^2\, dx \\ &\geq -h_i^{(2)}(\mathcal{S}_2 A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_2 A|^p \end{aligned}$$

for every $A \in \mathbb{R}^{N^k}$, for every $\varphi \in C_c^1(Q; \mathbb{R}^{N^{k-1}})$, and for every $i \in \mathbb{N}$;

(b') (growth condition)

$$|F_i^{(2)}(A)| \leq L_i^{(2)}(1 + |A|^p) \quad \forall\, A \in \mathbb{R}^{N^k}, \forall\, i \in \mathbb{N};$$

(c) ($F_i^{(2)}$ extends f)

$$F_i^{(2)}(A) = f(A) \qquad \forall\, A \in E_k, \forall\, i \in \mathbb{N}.$$

For every $i \in \mathbb{N}$, we define now F_i as the quasiconvexification of the function $F_i^{(2)}$:

$$F_i(A) := \inf\left\{\int_Q F_i^{(2)}(A + \nabla\varphi(x))\, dx : \varphi \in C_{per}^\infty(Q; \mathbb{R}^{N^{k-1}})\right\}$$

for every $A \in \mathbb{R}^{N^k}$. From property (a') and by the definition of F_i, we have

$$F_i^{(2)}(A) - h_i^{(2)}(\mathcal{S}_2 A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_2 A|^p \leq F_i(A) \leq F_i^{(2)}(A), \qquad \forall\, A \in \mathbb{R}^{N^k}. \tag{4.11}$$

Noticing that

$$\lim_{i\to+\infty} h_i^{(2)}(\mathcal{S}_2 A) = 0 \quad \text{for all } A \in \mathbb{R}^{N^k},$$

from property (c) and (4.11) we have (1.6). Finally, (1.7) follows from (b') and (4.11). □

4.1. **Proof of Theorem 1.3.** To conclude the section, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. It is enough to adapt the proof of [8, Theorem 3] and to use Theorem 1.3. □

5. Appendix

This section contains some auxiliary results used in the rest of the paper. First, we give the proof of Proposition 2.5.

Proof of Proposition 2.5. Since $A \in E_k^{N^{k-s-1}}$ and $B \in \mathcal{A}_{s+1}E_k^{N^{k-s}}$, we can write $A = \mathcal{S}_{s+1}A$ and $B = \mathcal{A}_{s+1}C$, for some $C \in E_k^{N^{k-s}}$. As a first step, let us prove that for every $r, l \in \{k-s+1, \dots, k\}$ with $r \neq l$ we have

$$A^{T_r^{k-s}} \cdot C^{T_l^{k-s}} = A \cdot C^{T_r^{k-s}} = A^{T_r^{k-s}} \cdot C. \tag{5.1}$$

To fix the ideas, let us assume $r < l$. By definition of transpose operators

$$A^{T_r^{k-s}}_{i_1 i_2 \dots i_k} C^{T_l^{k-s}}_{i_1 i_2 \dots i_k} = A_{i_1 i_2 \dots i_{k-s-1} i_r i_{k-s+1} \dots i_{r-1} i_{k-s} i_{r+1} \dots i_k} C_{i_1 i_2 \dots i_{k-s-1} i_l i_{k-s+1} \dots i_{l-1} i_{k-s} i_{l+1} \dots i_k},$$

for every $i_1, i_2, \dots, i_k = 1, \dots, N$. In the last expression, since $A \in E_k^{N^{k-s-1}}$ and $r, l > k-s$, we can exchange the indices in r-th and l-th position in the first factor, obtaining

$$A^{T_r^{k-s}}_{i_1 i_2 \dots i_k} C^{T_l^{k-s}}_{i_1 i_2 \dots i_k} = A_{i_1 i_2 \dots i_{k-s-1} i_r i_{k-s+1} \dots i_{r-1} i_l i_{r+1} \dots i_{l-1} i_{k-s} i_{l+1} \dots i_k} C_{i_1 i_2 \dots i_{k-s-1} i_l i_{k-s+1} \dots i_{l-1} i_{k-s} i_{l+1} \dots i_k}.$$

Summing last relation with respect to $i_1, \dots, i_k$ and renumerating the indices

$$\begin{aligned} A^{T_r^{k-s}} \cdot C^{T_l^{k-s}} &= \sum_{i_1 \dots, i_k}^{1,N} A^{T_r^{k-s}}_{i_1 i_2 \dots i_k} C^{T_l^{k-s}}_{i_1 i_2 \dots i_k} \\ &= \sum_{i_1 \dots, i_k}^{1,N} A_{i_1 i_2 \dots i_{k-s-1} i_r i_{k-s+1} \dots i_{r-1} i_l i_{r+1} \dots i_{l-1} i_{k-s} i_{l+1} \dots i_k} C_{i_1 i_2 \dots i_{k-s-1} i_l i_{k-s+1} \dots i_{l-1} i_{k-s} i_{l+1} \dots i_k} \\ &= \sum_{i_1 \dots, i_k}^{1,N} A_{i_1 \dots i_k} C_{i_1 i_2 \dots i_{k-s-1} i_r i_{k-s+1} \dots i_{r-1} i_{k-s} i_{r+1} \dots i_k} = \sum_{i_1 \dots, i_k}^{1,N} A_{i_1 i_2 \dots i_k} C^{T_r^{k-s}}_{i_1 i_2 \dots i_k} = A \cdot C^{T_r^{k-s}}. \end{aligned}$$

In the same way one can prove the second equality in (5.1).

Let us now prove the proposition. We have

$$\begin{aligned} (s+1)^2 A \cdot B &= (s+1)^2 \left(\mathcal{S}_{s+1}A \cdot \mathcal{A}_{s+1}C\right) \\ &= \left(A + A^{T_{k-s+1}^{k-s}} + \dots + A^{T_k^{k-s}}\right) \cdot \left[s\, C - (C^{T_{k-s+1}^{k-s}} + \dots + C^{T_k^{k-s}})\right] \\ &= sA \cdot C - \sum_{r=k-s+1}^{k} A^{T_r^{k-s}} \cdot C^{T_r^{k-s}} - \sum_{r=k-s+1}^{k} A \cdot C^{T_r^{k-s}} \\ &\quad + s \sum_{r=k-s+1}^{k} A^{T_r^{k-s}} \cdot C - \sum_{l=k-s+1}^{k} \sum_{\substack{r \neq l \\ r=k-s+1}}^{k} A^{T_r^{k-s}} \cdot C^{T_l^{k-s}} \end{aligned}$$

Since the sum of the first two terms is zero, using relation (5.1) we get

$$\begin{aligned} (s+1)^2 A \cdot B &= (s-1) \sum_{r=k-s+1}^{k} A \cdot C^{T_r^{k-s}} - \sum_{l=k-s+1}^{k} \sum_{\substack{r \neq l \\ r=k-s+1}}^{k} A \cdot C^{T_r^{k-s}} \\ &= (s-1) \sum_{r=k-s+1}^{k} A \cdot C^{T_r^{k-s}} - (s-1) \sum_{r=k-s+1}^{k} A \cdot C^{T_r^{k-s}} = 0. \end{aligned}$$

□

In the remaining part of the section we state some lemmas that are proved in [8].

Lemma 5.1. *Let X be a Hilbert space, and let $g : X \to \mathbb{R}$ be given by*

$$g(x) := \left(\mu^2 + |x|^2\right)^{\frac{p}{2}}. \tag{5.2}$$

For every $p > 1$, there exist two constants $\theta_p > 0$ and $\Theta_p > 0$ such that for every $\mu \geq 0$ the function g defined in (5.2) *satisfies the following inequalities*

$$\theta_p(\mu^2 + |x|^2 + |y|^2)^{\frac{p-2}{2}}|y|^2 \leq g(x+y) - g(x) - \nabla g(x) \cdot y$$
$$\leq \Theta_p(\mu^2 + |x|^2 + |y|^2)^{\frac{p-2}{2}}|y|^2$$

for every $x, y \in X$.

Lemma 5.2. *Let X, Y be Hilbert spaces and let $p > 1, \mu \geq 0, \beta \geq 0$. Let $g_\beta : X \times Y \to \mathbb{R}$ be given by*

$$g_\beta(x, y) := \Big(\mu^2 + |x|^2 + \beta^2|y|^2\Big)^{\frac{p}{2}}. \tag{5.3}$$

Then

$$g_\beta(x + \xi, y + \eta) - g_\beta(x, y) - \nabla_x g_\beta(x, y) \cdot \xi - \nabla_y g_\beta(x, y) \cdot \eta \tag{5.4}$$
$$\geq \theta_p\Big(\mu^2 + |x|^2 + |\xi|^2 + \beta^2|y|^2 + \beta^2|\eta|^2\Big)^{\frac{p-2}{2}}\Big(|\xi|^2 + \beta^2|\eta|^2\Big)$$

for every x, $\xi \in X$, y, $\eta \in Y$, where θ_p is the first constant in Lemma 5.1. Therefore, if $p \geq 2$, we have

$$g_\beta(x + \xi, y + \eta) - g_\beta(x, y) - \nabla_x g_\beta(x, y) \cdot \xi - \nabla_y g_\beta(x, y) \cdot \eta \tag{5.5}$$
$$\geq \theta_p\Big(\mu^2 + |x|^2 + |\xi|^2\Big)^{\frac{p-2}{2}}|\xi|^2 + \frac{\theta_p\beta^2}{2}\Big(\mu^2 + |x|^2\Big)^{\frac{p-2}{2}}|\eta|^2 + \frac{\theta_p\beta^p}{2}|\eta|^p$$

for every x, $\xi \in X$, y, $\eta \in Y$.

Lemma 5.3. *Let X be a Hilbert space and let $1 < p \leq 2$. Then for every $\tilde{\mu} \geq 0$ and every $0 < \delta < 1$ we have*

$$\Big(\tilde{\mu}^2 + |x+y|^2\Big)^{\frac{p-2}{2}}|x+y|^2 \leq 2\Big(\tilde{\mu}^2 + |x|^2\Big)^{\frac{p-2}{2}}|x|^2 + 2\Big(\tilde{\mu}^2 + |y|^2\Big)^{\frac{p-2}{2}}|y|^2,$$
$$\delta^{\frac{2-p}{2}}\Big(\tilde{\mu}^2 + |y|^2\Big)^{\frac{p-2}{2}}|y|^2 \leq \Big(\tilde{\mu}^2 + |x|^2 + |y|^2\Big)^{\frac{p-2}{2}}|y|^2 + \delta\Big(\tilde{\mu}^2 + |x|^2\Big)^{\frac{p-2}{2}}|x|^2$$

for every $x, y \in X$.

Lemma 5.4. *Let $1 < p \leq 2$. Then*

$$b^p \leq 8\varepsilon^{\frac{p-2}{p}}(\mu^2 + a^2 + b^2)^{\frac{p-2}{2}}b^2 + \varepsilon a^p + \varepsilon\mu^p$$

for every $a \geq 0$, $b \geq 0$, $\mu \geq 0$, and $0 < \varepsilon < 1$.

Lemma 5.5. *Let X be a Hilbert space, and let $f \in C^1(X) \cap C^2(X \setminus \{0\})$. Assume that there exist $p > 1$, $C > 0$, and $\mu \geq 0$ such that*

$$|\nabla^2 f(x)| \leq C\big(\mu^2 + |x|^2\big)^{\frac{p-2}{2}} \tag{5.6}$$

for every $x \in X \setminus \{0\}$. Then

$$|\nabla f(x+y) - \nabla f(x)| \leq K_p C\Big(\mu^2 + |x|^2 + |y|^2\Big)^{\frac{p-2}{2}}|y| \tag{5.7}$$

for every $x, y \in X$, where $K_p \geq 1$ is a constant depending only on p.

Lemma 5.6. *Let X be a Hilbert space and let $f \in C^1(X)$. Assume that there exist $p > 1$ and $\mu \geq 0$ such that*

$$|\nabla f(x+y) - \nabla f(x)| \leq \Big(\mu^2 + |x|^2 + |y|^2\Big)^{\frac{p-2}{2}}|y|$$

for every $x, y \in X$. If $1 < p \leq 2$, then for every $\varepsilon > 0$ there exists a constant $c_1 = c_1(\varepsilon, p) > 0$, depending only on ε and p, such that

$$|f(x+y+z) - f(x+y) - \nabla f(x) \cdot z|$$
$$\leq \varepsilon\Big(\mu^2 + |x|^2 + |y|^2\Big)^{\frac{p-2}{2}}|y|^2 + c_1\Big(\mu^2 + |z|^2\Big)^{\frac{p-2}{2}}|z|^2$$

for every $x, y, z \in X$.

If $p \geq 2$, *then for every* $\varepsilon > 0$ *there exists a constant* $c_2 = c_2(\varepsilon, p) > 0$, *depending only on* ε *and* p, *such that*

$$|f(x+y+z) - f(x+y) - \nabla f(x) \cdot z| \\ \leq \varepsilon\Big(\mu^2 + |x|^2 + |y|^2\Big)^{\frac{p-2}{2}} |y|^2 + c_2\Big(\mu^2 + |x|^2\Big)^{\frac{p-2}{2}} |z|^2 + c_2|z|^p$$

for every $x, y, z \in X$.

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