Minimizing Shortfall Risk Using Duality Approach - An Application to Partial Hedging in Incomplete Markets

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This work is dedicated to my mother Ximei Wu.

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Contents

Introduction

Option pricing and hedging in a complete market are well-studied with nice results using martingale theories. However, there remain many open questions in incomplete markets. In particular, when the underlying processes involve jumps, there could be infinitely many martingale measures which give an interval of no-arbitrage prices instead of a unique one. Consequently, there is often no martingale representation theorem to produce a perfect hedge. The question of picking a particular price and executing a hedging strategy according to some reasonable criteria becomes a non-trivial issue and an interesting question.

One conservative choice and a natural extension of the Black-Scholes theory, as studied in Karoui and Quenez (1995) and Kramkov (1996), is to eliminate any risk for selling the option at expiration by super-replicating the option payoff. However, this often turns out to be too expensive to be practical. For example, as shown in Eberlein and Jacod (1997) and Bellamy and Jeanblanc (2000), in a wide range of pure-jump models and in the case of jump diffusion models, the super-hedging prices for European options are equal to the trivial upper bound of the no-arbitrage interval. Take the most common example of a call option. The super-hedging strategy is to buy and hold, and therefore the price of the call is equal to the initial stock price.

Föllmer and Leukert (2000) proposed an interesting partial-hedging strategy for European type options to reduce the initial capital charged while bearing some residual risk, as financial institutions usually function. Their optimality criterion for measuring a hedging strategy is to minimize the shortfall risk at expiration. More formally, assume at time T the option payoff is a nonnegative random variable H. The final wealth X_T is produced by an admissible self-financing strategy trading between a money market account and a stock starting with initial capital $X_0 \equiv x$. Then the optimal strategy is the one which minimizes the expected shortfall under the physical measure at expiration:

$$\min_X E^{\mathbb{P}} \left[l \left((H - X_T)^+ \right) \right],$$

where the loss function $l(x)$ is increasing and convex. Notice that, unlike variance minimization, there is no penalty in case the hedging portfolio overshoots the option payoff. The existence of the optimal trading strategy when the stock price follows a semimartingale process is proved in Föllmer and Leukert (2000) using the Neyman-Pearson lemma. In a complete market, explicit solutions in terms of the Radon-Nikodym derivative between the risk-neutral measure and the physical measure were given. As an example, they showed how to compute the hedging strategy in the Black-Scholes model with insufficient initial capital. In the more interesting case of incomplete markets, they took the convex duality approach and provided an example of a geometric Brownian motion model where the volatility jumps from one constant to another according to some distribution at a deterministic time.

We are interested in doing some explicit computations in a simple incomplete market model. Our stock price follows a jump diffusion process:

$$dS_t = S_{t-}[\, \mu_t dt + \sigma_t dW_t - (1 - \alpha_t)(dN_t - \lambda_t dt)\,],$$

where W_t is a standard Brownian motion, and N_t is a Poisson process with intensity process λ_t. Assume $\mu_t > 0, \sigma_t > 0, 0 < \alpha_t < 1, \lambda_t > 0$ to be predictable processes. Note that the stock price is an Itô process driven by the Brownian motion until a jump occurs in the Poisson process. When that happens, the price jumps to a fraction of itself. We study the particular case where the loss function is linear: $l(x) = x$. Adopting the convex duality approach as in Föllmer and Leukert (2000), we define a random function

$$U(x, \omega) = H(\omega) - (H(\omega) - x)^+ = H(\omega) \wedge x.$$

The shortfall minimization problem can then be transformed into the utility maximization problem:

$$u(x) = \max_X E^{\mathbb{P}} [\, U(\, X_T\,)\,].$$

First we extend the duality results in Kramkov and Schachermayer (1999) to utility functions which are state dependent and not necessarily strictly concave, as our model requires, and in the generality of a semimartingale setting.[1] Then we specialize the results to the problem of minimizing shortfall. The by-product of the duality result is an alternative way of proving the existence of optimal solutions. In addition, it gives the structure of the optimal primal solution in terms of the dual solution.[2] We compute the optimal strategy, check the duality relationship and derive the HJB equation from the dynamic programming principle in three complete market cases.

We next focus on the more interesting jump process case where we explicitly characterize the primal and dual sets in terms of the characteristics. We provide upper bounds for the value function $u(x)$ using duality results. Each upper bound produced in this way corresponds to a dual element. In particular, in the case of constant parameters, we provide two solutions in closed form that correspond to the Radon-Nikodym derivatives

$$Z_T = e^{-\theta W_T - \frac{1}{2}\theta^2 T}, \quad \text{where} \quad \theta = \frac{\mu}{\sigma},$$

and

$$Z_T = \left(\frac{\lambda^*}{\lambda}\right)^{N_T} e^{-(\lambda^* - \lambda)T}, \quad \text{where} \quad \lambda^* = \frac{\mu + (1-\alpha)\lambda}{1-\alpha},$$

with which we are familiar from the complete cases. For lower bounds, we pick a particular strategy which we call the 'bold strategy' and compute the corresponding value function in closed form. Numerical results are shown for the cases of bonds and call options.

This research provides for the first time a method of checking the quality of a hedging strategy according to the principle of minimizing shortfall in an incomplete market model. Although in order to get closed-form solutions, we provide numerical examples in the case of constant parameters, the theory for choosing upper and lower bounds works for general semimartingale models. Better upper

[1] There have been a few articles on extending duality theory so it could be applied to the shortfall minimization problem, although we are not aware of any results that cover the linear case in the semimartingale setting we study here. Cvitanic (1998) worked out the duality results of the linear case $l(x) = x$ under a multidimensional Brownian motion model with portfolio constraints. Föllmer and Leukert (2000) and Leukert (1999) proved duality results for state dependent utility function resulting from strictly convex loss function $l(x)$. Bouchard, Touzi and Zeghal (2002) extended duality results to a finite deterministic utility function defined on the real line that is not necessarily smooth, but in their shortfall minimizing case the loss function $l(x)$ cannot be linear near infinity.

[2] It is well-known in the complete market case that the primal optimal terminal portfolio value is a constant times the marginal utility of the Radon-Nikodym derivative.

bounds can be chosen by picking the dual element using a more sophisticated technique, and better lower bounds can be chosen by varying investment strategies. The results can be easily extended to more general convex loss function $l(x)$.

Chapter 1

Duality Theory for Shortfall Risk Minimization

1.1 Setup

Suppose the discounted asset price process is a real-valued semimartingale $S = (S_t)_{0\leq t\leq T}$ on the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ that satisfies the usual conditions. Assume $\mathcal{F}_0$ is trivial. Let $\mathcal{M}$ denotes the set of equivalent local martingale measures.

Assumption 1.1. $\mathcal{M} \neq \emptyset$.

Remark 1.2. *This assumption is closely related to a version of the no-arbitrage condition for continuous-time semimartingale models; see Delbaen and Schachermayer (1994).*

Assumption 1.3. *The utility function $U(x,\omega) : ([0,\infty) \times \Omega) \to [0,\infty)$ satisfies: $U(\cdot,\omega)$ is a continuous, increasing and concave function for any fixed ω and $U(0,\omega) = 0$. The right-hand derivative also satisfies* [1] *$U^r(0,\omega) > 0$ and $U^r(\infty,\omega) = \lim_{x\to\infty} U^r(x,\omega) = 0$ for all $\omega \in \Omega$.*

Since we are mainly interested in extending the duality theory to solve the particular problem of shortfall risk minimization, the following assumption is reasonable to make instead of posing restrictions on asymptotic elasticity of the utility function as in some other references.[2] It simplifies the task of existence proofs because we can invoke the dominated convergence theorem.

[1] Its existence is guaranteed by the concavity of U.

[2] They are mentioned in the footnote on page vii.

Assumption 1.4. *The utility function is dominated by an integrable random variable measurable with respect to $\mathcal{F}_T$, i.e., there exists an $\mathcal{F}_T$-measurable random variable H such that $E^{\mathbb{P}}[H] < \infty$ and $U(x, \omega) \leq H(\omega)$ for all $\omega \in \Omega$ and $x \geq 0$.*

As usual, we use stochastic integrals to represent the wealth process from investment strategies and impose non-bankruptcy condition for admissibility. There will be no endowments and consumptions. The process ξ_s represents the number of shares invested in the underlying asset, and is assumed to be integrable with respect to S. The set of admissible self-financing portfolios starting at initial capital x is defined as:

$$\mathcal{X}(x) = \left\{ X \;\middle|\; X_t = x + \int_0^t \xi_s dS_s \geq 0 \;\; \mathbb{P} - a.s., \text{ for } \quad 0 \leq t \leq T \right\}. \tag{1.1}$$

The primal problem is to maximize the expected utility at final time T:

$$u(x) = \sup_{X \in \mathcal{X}(x)} E^{\mathbb{P}} \left[U(X_T) \right], \quad \text{for} \quad x \geq 0. \tag{Primal}$$

The stochastic conjugate function of $U(x, \omega)$ is

$$V(y, \omega) = \sup_{x \geq 0} \left[U(x, \omega) - xy \right], \quad \text{for} \quad y \geq 0. \tag{1.2}$$

Remark 1.5. *$V(\cdot, \omega)$ is a continuous, decreasing and convex function for any fixed ω. Note that $V(0) = U(\infty) = \lim_{x \to \infty} U(x)$, and $V(y) \geq U(0) = 0$.*

As in Kramkov and Schachermayer (1999), define the dual space to be

$$\mathcal{Y}(y) = \{ Y \geq 0 \mid Y_0 = y \text{ and } XY \text{ is a } \mathbb{P}\text{-supermartingale for any } X \in \mathcal{X}(1) \}. \tag{1.3}$$

Then the dual problem is to minimize the expected value of the conjugate function:

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} E^{\mathbb{P}} \left[V(Y_T) \right] \tag{Dual}$$

There are a few natural questions that arise at this point:
1. Are the primal and dual value functions $u(x)$ and $v(y)$ conjugates?
2. Do optimal solutions to the primal and dual problems exist? (Since we do not have strict concavity in the utility function, it will be too optimistic to expect uniqueness.) How are they related to each other? What are sufficient and necessary conditions for optimality?

1.2 Bipolar theorem

To give positives answers to both above questions, we need a bipolar theorem proved in Kramkov and Schachermayer (1999). Let us first fix some definitions. $L^0(\Omega, \mathcal{F}, \mathbb{P})$ denotes the set of random variables and $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ includes bounded ones only. Let

$$\mathcal{C}(x) = \left\{ g \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid 0 \leq g \leq X_T \text{ for some } X \in \mathcal{X}(x) \right\}$$

denote the set of contingent claims super-replicable by some admissible self-financing strategies with initial capital x. Let

$$\mathcal{D}(y) = \left\{ h \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y) \right\}.$$

Then the primal and dual value functions can be written as:

$$u(x) = \sup_{g \in \mathcal{C}(x)} E^{\mathbb{P}} \left[U(g) \right] \qquad \text{and} \qquad v(y) = \inf_{h \in \mathcal{D}(y)} E^{\mathbb{P}} \left[V(h) \right].$$

Notice the scaling property $\mathcal{C}(x) = x\,\mathcal{C}(1)$ and $\mathcal{D}(y) = y\mathcal{D}(1)$. We will define $\mathcal{C} := \mathcal{C}(1)$ and $\mathcal{D} := \mathcal{D}(1)$. The novel choice of sets $\mathcal{X}$ and $\mathcal{Y}$ by Kramkov and Schachermayer (1999) gives the perfect bipolar relation between the solid sets $\mathcal{C}$ and $\mathcal{D}$. Recall a set $C \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ is solid if $0 \leq f \leq g$ and $g \in C$ implies $f \in C$.

Proposition 1.6 (Proposition 3.1 in Kramkov & Schachermayer (1999)). *Suppose Assumption 1.1 hold. The sets $\mathcal{C}$ and $\mathcal{D}$ have the following properties:*
(i) $\mathcal{C}$ and $\mathcal{D}$ are subsets of $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ which are convex, solid and closed in the topology of convergence in measure.
(ii)

$$\begin{aligned} &g \in \mathcal{C} \quad \textit{iff} \quad E[gh] \leq 1, \quad \textit{for all} \quad h \in \mathcal{D} \quad \textit{and} \\ &h \in \mathcal{D} \quad \textit{iff} \quad E[gh] \leq 1, \quad \textit{for all} \quad g \in \mathcal{C}. \end{aligned}$$

(iii) The constant function 1 is in $\mathcal{C}$.

Remark 1.7. *Notice Assumption 1.1 used in the bipolar theorem assumes the existence of some equivalent local martingale measures, and therefore is only dependent on the exclusion of certain arbitrage opportunities in the model. It has nothing to do with utility functions.*

The following lemma is quite powerful, yet surprisingly not very hard to prove. For interested readers, a proof is given in the appendix of Delbaen and Schachermayer (1994).

Lemma 1.8. *Let* $(f_n)_{n\geq 1}$ *be a sequence of non-negative random variables. Then there is a sequence* $g_n \in conv(f_n, f_{n+1}, \dots)$, $n \geq 1$[3], *which converges almost surely to a random variable* g *with values in* $[0, \infty]$.

1.3 Duality theorems

Now we are ready to prove the first major theorem.

Theorem 1.9. *Suppose Assumptions 1.1, 1.3 and 1.4 hold. Then*
(i) For $x > 0$ *and* $y > 0$*, the optimal solution* $\hat{g}(x)$ *to (Primal) exists, and the optimal solution* $\hat{h}(y)$ *to (Dual) exists. The set of optimal solutions is convex.*
(ii) $u(x) < \infty, v(y) < \infty$ *for all* $x > 0$ *and* $y > 0$*. The value functions* u *and* v *are conjugates:*

$$v(y) = \sup_{x>0} [u(x) - xy] \quad \text{for any} \quad y > 0, \quad \text{and} \tag{3.4}$$

$$u(x) = \inf_{y>0} [v(y) + xy] \quad \text{for any} \quad x > 0. \tag{3.5}$$

u is concave,continuous and increasing; v is convex, continuous and decreasing. The right-hand derivatives exist and satisfy

$$u^r(\infty) = \lim_{x\to\infty} u^r(x) = 0, \quad v^r(\infty) = \lim_{y\to\infty} v^r(y) = 0$$

Remark 1.10. *Since the proof is fairly long, we will divide it into a few steps and throw some tricky lemmas into the Appendix.*

Proposition 1.11. *Suppose Assumptions 1.1, 1.3 and 1.4 hold. Then for* $x > 0$ *and* $y > 0$*, both the optimal solution* $\hat{g}(x)$ *to (Primal) and the optimal solution* $\hat{h}(y)$ *to (Dual) exist. Furthermore,* $u(x) \leq E^{\mathbb{P}}[H] < \infty$ *is concave and increasing on* $[0, \infty)$*, and therefore is a continuous function on* $(0, \infty)$*. The right-hand derivative* u^r *exists and satisfies*

$$u^r(\infty) = \lim_{x\to\infty} u^r(x) = 0.$$

Proof. Let $(f_n)_{n\geq 1}$ be a sequence in $\mathcal{C}(x)$ such that the expectation of the utility function increases to the value function $u(x)$:

$$E^{\mathbb{P}}\left[U(f_n)\right] \nearrow u(x) \quad \text{as} \quad n \to \infty. \tag{3.6}$$

[3] This is the notation for finite convex combination.

By Lemma 1.8, there exists a sequence

$$g_n \in conv(f_n, f_{n+1}, \dots), \quad \text{and} \quad g_n \to \hat{g} \quad \text{a.s.}$$

By Theorem 1.6, we know $\mathcal{C}(x)$ is closed and convex, and therefore $g_n, \hat{g} \in \mathcal{C}(x)$. Let $g_n = \sum_m a_m f_m$. By the concavity of U,

$$E^{\mathbb{P}}\left[U(g_n)\right] \geq \sum_m a_m E^{\mathbb{P}}\left[U(f_m)\right] \geq E^{\mathbb{P}}\left[U(f_n)\right].$$

By Assumption 1.4 and the Dominated Convergence Theorem,

$$E^{\mathbb{P}}\left[U(g_n)\right] \to E^{\mathbb{P}}\left[U(\hat{g})\right].$$

Comparing to equation (3.6), we get

$$E^{\mathbb{P}}\left[U(\hat{g})\right] \geq u(x).$$

By definition, $u(x) = \sup_{g \in \mathcal{C}(x)} E^{\mathbb{P}}\left[U(g)\right]$. We conclude that $\hat{g}$ is optimal. The existence proof in the case of (Dual) is similar using Fatou's lemma and the fact $V \geq 0$ by Remark 1.5.
By the definition in (Primal) for $u(x)$ and Assumption 1.4, we know $u(x) \leq E^{\mathbb{P}}[H] < \infty$. Notice that for any optimal $\hat{g}_1 \in \mathcal{C}(x_1)$ and $\hat{g}_2 \in \mathcal{C}(x_2)$, we have $\frac{\hat{g}_1+\hat{g}_2}{2} \in \mathcal{C}(\frac{x_1+x_2}{2})$. By the concavity of U,

$$\begin{aligned} u\left(\tfrac{x_1+x_2}{2}\right) &\geq E^{\mathbb{P}}\left[U\left(\tfrac{\hat{g}_1+\hat{g}_2}{2}\right)\right] \\ &\geq \tfrac{1}{2}\left(E^{\mathbb{P}}\left[U(\hat{g}_1)\right] + E^{\mathbb{P}}\left[U(\hat{g}_2)\right]\right) \\ &= \tfrac{u(x_1)+u(x_2)}{2} \end{aligned}$$

Therefore $u(x)$ is concave on $[0, \infty)$. It is a trivial consequence that $u(x)$ is continuous on $(0, \infty)$. Suppose $0 < x_1 < x_2$. We know that $x_1 \mathcal{C}(1) = \mathcal{C}(x_1)$, $x_2 \mathcal{C}(1) = \mathcal{C}(x_2)$ and $\mathcal{C}(1)$ is solid. Therefore, $\mathcal{C}(x_1) \subseteq \mathcal{C}(x_2)$. By definition, $u(x_1) \leq u(x_2)$. It remains to show that $u^r(\infty) = 0$. Suppose not, i.e., there exists some $\epsilon > 0$ such that $u^r(x) > \epsilon$ for some large constant X and any $x \geq X$. Then the following would be true

$$u(x) = u(X) + \int_X^x u^r(z)\,dz > u(X) + \epsilon(x - X).$$

This is a contradiction to Assumption 1.4, $u(x) \leq E^{\mathbb{P}}[H] < \infty$. We conclude $u^r(\infty) = 0$. ◇

Proposition 1.12. *Suppose Assumptions 1.1, 1.3 and 1.4 hold. Then*

$$v(y) = \sup_{x>0} [u(x) - xy] \quad \textit{for any} \quad y > 0, \quad \textit{and}$$
$$u(x) = \inf_{y>0} [v(y) + xy] \quad \textit{for any} \quad x > 0.$$

Also, $v(y) < \infty$ is convex, continuous and decreasing for all $y > 0$.

Proof. We follow the proof of Lemma 3.6 in Kramkov and Schachermayer (1999) very closely, filling in some details, and extending the results to state-dependent utility function $U(\cdot, \omega)$. For $n > 0$, define

$$\mathcal{B}_n = \{ g \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) : 0 \le g \le n \}.$$

Lemma A.1 gives the following equality for fixed n: [4]

$$\sup_{g\in\mathcal{B}_n} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}} [\, U(g) - gh \,] = \inf_{h\in\mathcal{D}(y)} \sup_{g\in\mathcal{B}_n} E^{\mathbb{P}} [\, U(g) - gh \,] . \tag{3.7}$$

Remark A.2, Lemma A.3 and A.5 give that

$$\lim_{n\to\infty} \sup_{g\in\mathcal{B}_n} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}} [\, U(g) - gh \,] = \sup_{x>0} \sup_{g\in\mathcal{C}(x)} E^{\mathbb{P}} [\, U(g) - xy \,] . \tag{3.8}$$

Then by the definition of $u(x)$, equation (3.8) and equation (3.7), we have

$$\begin{aligned} \sup_{x>0} [u(x) - xy] &= \sup_{x>0} \sup_{g\in\mathcal{C}(x)} E^{\mathbb{P}} [\, U(g) - xy \,] \\ &= \lim_{n\to\infty} \sup_{g\in\mathcal{B}_n} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}} [\, U(g) - gh \,] \\ &= \lim_{n\to\infty} \inf_{h\in\mathcal{D}(y)} \sup_{g\in\mathcal{B}_n} E^{\mathbb{P}} [\, U(g) - gh \,] . \end{aligned}$$

Now define

$$V^n(y,\omega) = \sup_{0<x\le n} [\, U(x,\omega) - xy \,] \quad \text{and} \quad v^n(y) = \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}} [\, V^n(h) \,] .$$

Then we have

$$\inf_{h\in\mathcal{D}(y)} \sup_{g\in\mathcal{B}_n} E^{\mathbb{P}} [\, U(g) - gh \,] = \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}} [\, V^n(h) \,] = v^n(y).$$

To finish the first part of the proof, it is sufficient to show that

$$v(y) = \lim_{n\to\infty} v^n(y).$$

[4] The sets $\mathcal{B}_n$ are $\sigma(L^\infty, L^1)$-compact. By (i)and (iii) of Proposition 1.6, $\mathcal{D}(y)$ is a closed convex subset of $L^1(\Omega, \mathcal{F}, \mathbb{P})$. The proof in Kramkov and Schachermayer (1999) uses the Minmax Theorem (Theorem 45.8 in Strasser (1985)) to get the desired equality.

Evidently, v^n is an increasing sequence and $v^n \le v$, for $n \ge 1$, so $\lim_{n\to\infty} v^n(y) \le v(y)$. Let f_n be a sequence in $\mathcal{D}(y)$ such that

$$\lim_{n\to\infty} E^{\mathbb{P}}\left[V^n(f_n)\right] \nearrow \lim_{n\to\infty} v^n(y). \tag{3.9}$$

Lemma 1.8 implies the existence of $h_n \in conv(f_n, f_{n+1}, \dots)$ such that $h_n \to \hat{h}$ a.s. By Proposition 1.6, we know $\mathcal{D}(y)$ is closed and convex, and therefore $\hat{h} \in \mathcal{D}(y)$. Let $h_n = \sum_m f_m$. Since $V^n(y) \nearrow V(y)$ are convex, we have

$$\begin{aligned} E^{\mathbb{P}}\left[V^n(h_n)\right] &\le \sum_m a_m E^{\mathbb{P}}\left[V^n(f_m)\right] \\ &\le \sum_m a_m E^{\mathbb{P}}\left[V^m(f_m)\right] \le \sup_{m\ge n} E^{\mathbb{P}}\left[V^m(f_m)\right]. \end{aligned} \tag{3.10}$$

Applying (3.9) and (3.10), and Fatou's lemma, we have

$$\begin{aligned} \lim_{n\to\infty} E^{\mathbb{P}}\left[V^n(f_n)\right] &= \sup_{m\ge n} E^{\mathbb{P}}\left[V^m(f_m)\right] \ge \liminf_{n\to\infty} E^{\mathbb{P}}\left[V^n(h_n)\right] \\ &\ge E^{\mathbb{P}}\left[\liminf_{n\to\infty} V^n(h_n)\right] = E^{\mathbb{P}}\left[V(\hat{h})\right] \ge v(y). \end{aligned}$$

So we have the desired inequality $\lim_{n\to\infty} v^n(y) \ge v(y)$ and therefore

$$v(y) = \sup_{x>0}\left[u(x) - xy\right] \quad \text{for any} \quad y > 0.$$

By Proposition 1.11, we know that $u(x)$ is an increasing and concave function such that $u^r(\infty) = 0$. In the case $u^r(0+) > 0$, Theorem B.6 gives

$$u(x) = \inf_{y>0}\left[v(y) + xy\right] \quad \text{for any} \quad x > 0,$$

as well as $v(y) < \infty$ is convex, continuous and decreasing for all $y > 0$. Suppose $u^r(0+) = 0$. Since $u(x)$ is concave and increasing, $u(x) \equiv u(0+)$ is a constant function on $(0, \infty)$. We have just proved that $v(y) = \sup_{x>0}[u(x) - xy]$, therefore $v(y) \equiv u(0+)$ is also a constant function for all $y > 0$. Then $\inf_{y>0}[v(y) + xy] = v(0+) = u(0+) = u(x)$, and all the above results hold. $\diamond$

Proof of theorem 1.9. We still need to prove $v^r(\infty) = 0$. Since v is convex, the right-hand derivative $v^r(y)$ exists and is increasing. We've proved that v is decreasing; therefore $v^r(y) \le 0$. Suppose there exists some $\epsilon > 0$ such that $v^r(y) < -\epsilon$ for some large constant Y and any $y \ge Y$, then the following should be true

$$v(y) = v(Y) + \int_Y^y v^r(z)\,dz < v(Y) - \epsilon(y - Y).$$

This is a contradiction to the fact $v(y) \ge 0$. We conclude $v^r(\infty) = 0$. $\diamond$

The next theorem gives a sufficient and necessary condition for optimality. Note that we define $\partial U(0) = [U^r(0), \infty)$ and $\partial V(0) = [V^r(0), \infty)$.

Theorem 1.13. *Suppose Assumptions 1.1, 1.3 and 1.4 hold. Suppose $x > 0$ and $y > 0$ such that $y \in \partial u(x)$. Let $\hat{g} \in \mathcal{C}(x)$ and $\hat{h} \in \mathcal{D}(y)$. Then the following two statements are equivalent.*
(i) $\hat{g}$ is an optimal solution to (Primal) and $\hat{h}$ is an optimal solution to (Dual).
(ii) $E^{\mathbb{P}}[\,\hat{g}\,\hat{h}\,] = xy$ and $\hat{h} \in \partial U(\hat{g}) - \mathbb{P}$ a.s., or equivalently, $\hat{g} \in -\partial V(\hat{h}) - \mathbb{P}$ a.s.

PROOF. "*(i)* $\Rightarrow$ *(ii)*": From Theorem B.6, when $y \in \partial u(x)$,

$$v(y) = \sup_{z>0} [u(z) - zy] = u(x) - xy.$$

(i) implies,

$$E^{\mathbb{P}}[\,V(\hat{h})\,] = E^{\mathbb{P}}[\,U(\hat{g})\,] - xy.$$

Since $\hat{g} \in \mathcal{C}(x)$ and $\hat{h} \in \mathcal{D}(y)$, we have $E^{\mathbb{P}}[\,\hat{g}\,\hat{h}\,] \le xy$. Therefore,

$$E^{\mathbb{P}}[\,V(\hat{h})\,] \le E^{\mathbb{P}}[\,U(\hat{g})\,] - E^{\mathbb{P}}[\,\hat{g}\hat{h}\,]. \tag{3.11}$$

By definition $V(y) = \sup_{x>0} [U(x) - xy]$. Therefore

$$V(y) \ge U(x) - xy \quad \text{for any} \quad x > 0,\ y > 0.$$

Consequently,

$$V(\hat{h}) \ge U(\hat{g}) - \hat{g}\hat{h} \quad \mathbb{P} \text{ a.s.} \tag{3.12}$$

Equations (3.11) and (3.12) gives the equality:

$$V(\hat{h}) = U(\hat{g}) - \hat{g}\hat{h} \quad \mathbb{P} \text{ a.s.}$$

By Assumption 1.3 and Theorem B.6, the above equation is equivalent to $\hat{h} \in \partial U(\hat{g}) - \mathbb{P}$ a.s., or equivalently, $\hat{g} \in -\partial V(\hat{h}) - \mathbb{P}$ a.s. Taking expectation of the above equation, we get $E^{\mathbb{P}}[\,\hat{g}\,\hat{h}\,] = xy$.
"*(ii)* $\Rightarrow$ *(i)*": Since $\hat{h} \in \partial U(\hat{g}) - \mathbb{P}$ a.s. , we have

$$U(\hat{g}) = V(\hat{h}) + \hat{g}\hat{h} \quad \mathbb{P} \text{ a.s.}$$

Taking expectation, and using the definition of $v(y)$, *(ii)*, Theorem 1.9 and Theorem B.6 , we get

$$E^{\mathbb{P}}[\,U(\hat{g})\,] = E^{\mathbb{P}}[\,V(\hat{h})\,] + E^{\mathbb{P}}[\,\hat{g}\,\hat{h}\,] \ge v(y) + xy = u(x).$$

Therefore, $\hat{g}$ is an optimal solution to (Primal). Similarly we can prove that $\hat{h}$ is an optimal solution to (Dual). ◇

1.4 Minimizing shortfall risk in a semimartingale model

We will follow the set up in Section 1.1. Consider a contingent claim given by a nonnegative, $\mathcal{F}_T$-measurable random payoff H. We would like to hedge this option optimally by solving the problem of minimizing shortfall risk [5]

$$\min_{X \in \mathcal{X}(x)} E^{\mathbb{P}}\left[\,(\,H - X_T\,)^+\,\right]. \tag{4.13}$$

where $\mathcal{X}(x)$ is the set of admissible self-financing portfolios defined in (1.1). Recall Assumption 1.1 that the set of equivalent local martingale measures $\mathcal{M}$ is not empty. We make the following assumptions to ensure that the value functions will turn out to be finite.

Assumption 1.14. *The super-hedging price $\bar{x}$ for this claim is finite, i.e.,*

$$\bar{x} \stackrel{\Delta}{=} \sup_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[H] < \infty.$$

Assumption 1.15. $E^{\mathbb{P}}[H] < \infty$.

Kramkov (1996) and Föllmer and Kabanov (1998) showed that there exist both a wealth and a consumption process to super-hedge the claim with initial capital $\bar{x}$:

$$W_t = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}\left[\,H \,|\, \mathcal{F}_t\,\right] = X_t - C_t, \quad W_0 = \bar{x}, \quad W_T = H,$$

where $X \in \mathcal{X}(\bar{x})$ and C_t is a nonnegative, increasing optional process with $C_0 = 0$.

To fit into the utility maximization framework, we define the state-dependent utility function

$$U(x, \omega) = H(\omega) - (H(\omega) - x)^+ = H(\omega) \wedge x.$$

By definition in (1.2), the stochastic conjugate function is

$$V(y, \omega) = (1 - y)^+ H(\omega).$$

Our primal problem in the case of minimizing shortfall risk can be written as

$$\text{(Primal-Shortfall)} \qquad u(x) = \sup_{X \in \mathcal{X}(x)} E^{\mathbb{P}}\left[\,H \wedge X_T\,\right] = \sup_{g \in \mathcal{C}(x)} E^{\mathbb{P}}\left[\,H \wedge g\,\right].$$

[5] Föllmer and Leukert (2000) and Leukert (1999) studied the duality problem of minimizing shortfall risk for a strictly convex loss function $l(x)$: $\min_{X \in \mathcal{X}(x)} E^{\mathbb{P}}\left[\,l((H - X_T)^+)\,\right]$.

The dual problem is

(Dual-Shortfall)
$$v(y) = \inf_{Y \in \mathcal{Y}(y)} E^{\mathbb{P}}\left[(1-Y)^+ H\right] = \inf_{h \in \mathcal{D}(y)} E^{\mathbb{P}}\left[(1-h)^+ H\right].$$

Remark 1.16. *By Assumption 1.14 and the discussion of super-hedging, we can easily see that*

$$\bar{x} = \min\{\, x : u(x) \equiv E^{\mathbb{P}}[H] \text{ is a constant function on } [x, \infty) \,\}.$$

Notice that $U(x,\omega) \leq H(\omega)$. With Assumption 1.15, U satisfies both Assumption 1.3 and 1.4. Therefore, all the duality results (Theorem 1.9 and 1.13) developed in section 1.3 apply in this case. We will restate them in a stronger sense with the following assumption. Recall that we define $\partial u(0) = [u^r(0), \infty)$ and $\partial v(0) = [v^r(0), \infty)$.

Assumption 1.17. *The discounted asset price process S is locally bounded.*

Theorem 1.18. *Suppose Assumptions 1.1, 1.14, 1.15 and 1.17 hold. Then*
(i) For $x \geq 0$ and $y \geq 0$, the optimal solution $\hat{g}(x)$ to (Primal-Shortfall) exists, and the optimal solution $\hat{h}(y)$ to (Dual-Shortfall) exists.
(ii) $u(x) < \infty, v(y) < \infty$ for all $x \geq 0$ and $y \geq 0$. The value functions u and v are conjugates:

$$v(y) = \max_{x \geq 0} [u(x) - xy] \quad \text{for any} \quad y \geq 0, \quad \text{and}$$
$$u(x) = \min_{y \geq 0} [v(y) + xy] \quad \text{for any} \quad x \geq 0.$$

$u(x)$ is concave,continuous and increasing for $x \in [0, \infty)$; $v(y)$ is convex, continuous and decreasing for $y \in [0, \infty)$. In particular,

$$u(x) \equiv E^{\mathbb{P}}[H] = v(0) \text{ is a constant function on } [\bar{x}, \infty);$$
$$v(y) \equiv 0 = u(0) \text{ is a constant function on } [u^r(0), \infty);$$

The right-hand derivatives satisfy

$$v^r(0+) = \lim_{y \searrow 0} v^r(y) = -\bar{x} \quad \text{and} \quad v^r(\infty) = \lim_{y \to \infty} v^r(y) = 0$$

(iii) $v(y) = u(x) - xy$ if and only if $y \in \partial u(x)$, or equivalently, $x \in -\partial v(y)$.

Proof. (i) and (ii): Recall the results from Theorem 1.9 and Theorem B.6. We need to take care of the cases at 0.

When $x = 0$, $X \equiv 0$ for all $X \in \mathcal{X}(0)$. The optimal solution to (Primal-Shortfall) is $\hat{X}_T = 0$, and $u(x) = 0$. Corollary 1.2 in Delbaen and Schachermayer (1994) says, under Assumption 1.17, Assumption 1.1 is equivalent to the condition that S satisfies No Free Lunch with Vanishing Risk (NFLVR). Corollary 3.7 in the same paper implies that, for any $x > 0$ and for all $X \in \mathcal{X}(x)$, $x \to 0$ implies $X_T \to 0$ in probability. In particular, the optimal solution $\hat{X} \in \mathcal{X}(x)$ such that $u(x) = E^{\mathbb{P}}[H \wedge \hat{X}_T]$ satisfies $\hat{X}_T \to 0$ in probability. By Assumption 1.15 and the Dominated Convergence Theorem, $u(x) \to 0 = u(0)$ as $x \to 0$, and therefore, $u(x)$ is right continuous at $x = 0$.
When $y = 0$, $Y \equiv 0$ for all $Y \in \mathcal{Y}(0)$ because they are nonnegative supermartingales. Therefore, the optimal solution to (Dual-Shortfall) is $\hat{Y}_T = 0$, and $v(0) = E^{\mathbb{P}}[H]$. For any fixed $y > 0$ and for all $Y \in \mathcal{Y}(y)$, since Y are nonnegative supermartingales, $y \to 0$ implies $Y_T \to 0$ in probability. In particular, the optimal solution $\hat{Y} \in \mathcal{Y}(y)$ such that $v(y) = E^{\mathbb{P}}[(1 - \hat{Y}_T)^+ H]$ satisfies $\hat{Y}_T \to 0$ in probability. By Assumption 1.15 and the Dominated Convergence Theorem, $v(y) \to E^{\mathbb{P}}[H] = v(0)$ as $y \to 0$, and therefore, $v(y)$ is right continuous at $y = 0$.
Remark 1.16 tells us that $u(x) \equiv E^{\mathbb{P}}[H]$ for $x \geq \bar{x}$, and since $u(x)$ is continuous and increasing on $[0, \infty)$, the sup in (3.4) is obtained and therefore can be written as max. Since $v(y)$ is continuous and decreasing on $[0, \infty)$, and $v^r(\infty) = 0$, the inf in (3.5) is obtained and can be written as min.
Lemma B.4 and Theorem B.6 shows that $v^r(0+) = -\bar{x}$.
(iii) This is true by Remark B.7. ◇

Theorem 1.19. *Suppose Assumptions 1.1, 1.14, 1.15 and 1.17 hold. Suppose $y \in \partial u(x)$ where $x \geq 0$ and $y \geq 0$.*
(i) If $\hat{g} \in \mathcal{C}(x)$ is an optimal solution to (Primal-Shortfall), then there is an optimal solution to (Dual-Shortfall) $\hat{h} \in \mathcal{D}(y)$ such that

$$\hat{h} = 1_{\{0 \leq \hat{g} < H\}} + \delta_1 1_{\{\hat{g} = H\}}, \quad \textit{and} \quad E^{\mathbb{P}}[\hat{g}\hat{h}] = xy, \tag{4.14}$$

where $0 \leq \delta_1 \leq 1$ is an $\mathcal{F}_T$-measurable random variable.
(ii) If $\hat{h} \in \mathcal{D}(y)$ is an optimal solution to (Dual-Shortfall), then there is an optimal solution to (Primal-Shortfall) $\hat{g} \in \mathcal{C}(x)$ such that

$$\hat{g} = H 1_{\{0 \leq \hat{h} < 1\}} + \delta_2 1_{\{\hat{h} = 1\}}, \quad \textit{and} \quad E^{\mathbb{P}}[\hat{g}\hat{h}] = xy, \tag{4.15}$$

where $0 \leq \delta_2 \leq H$ is an $\mathcal{F}_T$-measurable random variable.

Remark 1.20. *There exist optimal solutions that do not have overshoots, i.e., $\hat{g} \leq H$ and $\hat{h} \leq 1$.*

Remark 1.21. *Rewrite equation (4.14) as*

$$\hat{g} = \tilde{\phi} H = (1_{\{0 \le \hat{h} < 1\}} + \gamma 1_{\{\hat{h}=1\}})H, \tag{4.16}$$

where $0 \le \gamma \le 1$ *is an* $\mathcal{F}_T$*-measurable random variable. This gives the structure of the optimal test* $\tilde{\phi}$ *whose existence is derived in Proposition 3.1 and Theorem 3.2 of Föllmer and Leukert (2000) in the case when the loss function* $l(x) = x$*. In the notation of our paper, they proved that there exists an* $\mathcal{F}_T$*-measurable random variable* $\tilde{\phi} \in [0, 1]$ *that solves the problem*

$$\min_{\phi \in [0,1]} E^{\mathbb{P}}[(1-\phi)H] \quad s.t. \quad \sup_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[\phi H] \le x, \tag{4.17}$$

and $\tilde{\phi} H$ *is the optimal solution to the (Primal-Shortfall). This result also generalizes Proposition 4.1 in Föllmer and Leukert (2000) from the complete market to the incomplete market case where they give the same form of equation (4.16), only in the complete market.*

Proof. (i): Theorem 1.13 and the following equation

$$\partial U(\hat{g}) = \partial(\hat{g} \wedge H) = \begin{cases} 0, & \text{if } \hat{g} > H; \\ 1, & \text{if } 0 < \hat{g} < H; \\ [0,1], & \text{if } \hat{g} = H; \\ [1, \infty), & \text{if } \hat{g} = 0; \end{cases}$$

gives that $\tilde{h} \in \mathcal{D}(y)$ is an optimal solution to (Dual-Shortfall) if and only if

$$\tilde{h} = \tilde{h} 1_{\{\hat{g}=0\}} + 1_{\{0<\hat{g}<H\}} + \delta_1 1_{\{\hat{g}=H\}}, \quad \text{and} \quad E^{\mathbb{P}}[\hat{g}\tilde{h}] = xy,$$

where $0 \le \delta_1 \le 1$ is a random variable and $\tilde{h} \ge 1$ on the set $\hat{g} = 0$. Obviously, $\hat{h} = \tilde{h} \wedge 1$ is also an optimal solution to (Dual-Shortfall) since $\mathcal{D}(y)$ is solid.
(ii) This is similarly proved with the following equation

$$-\partial V(\hat{h}) = -\partial(1-\hat{h})^+ H = \begin{cases} 0, & \text{if } \hat{h} > 1; \\ H, & \text{if } 0 < \hat{h} < 1; \\ [0, H], & \text{if } \hat{h} = 1 \\ [H, \infty), & \text{if } \hat{h} = 0. \end{cases}$$

◇

1.5 Discrete case

We will write a necessary and sufficient condition for optimality in a finite dimensional case where we replace the condition of expectation with maximality. Define the set of maximal elements to be

$$\mathcal{C}^*(x) = \{\, g \in \mathcal{C}(x) \,:\, f \geq g, f \in \mathcal{C}(x) \Rightarrow f = g \,\}.$$

The model is set up as follows. There are two times $t = 0$ and $t = 1$. The probability space is $\Omega = \{\omega_1, \omega_2, ...\omega_n\}$ with probability distribution $(p_1, p_2, ..., p_n)$. There are m risky securities $S^1, S^2, ..., S^m$ and a derivative security H. The interest rate is zero. Assume the model to be arbitrage free. Define

$$s_{ij} = S_1^i(\omega_j) - S_0^i \quad \text{for} \quad i = 1, 2, ..., m \quad \text{and} \quad j = 1, 2, ..., n.$$

Let

$$(h_1, h_2, ..., h_n) = (H(\omega_1), H(\omega_2), ..., H(\omega_n)).$$

Suppose we invest $(\theta_1, \theta_2, ..., \theta_m)$ shares in the stocks. A self-financing strategy X is $X_0 = x$ and $X_1(\omega_j) = x + \sum_{i=1}^m \theta_i s_{ij}$ for $j = 1, 2, ..., n$. Define $\mathcal{X}_1(x)$ to be the set of the wealths of admissible strategies at time 1:

$$\mathcal{X}_1(x) = \{\, X_1 \,:\, X \in \mathcal{X}(x) \,\}.$$

The primal sets are

$$\begin{aligned} &\mathcal{X}_1(x) = \mathcal{C}^*(x) \\ &= \left\{ \left(x + \sum_{i=1}^m \theta_i s_{ij} \right)_{j=1,\ldots,n} : (\theta_j)_{j=1,\ldots,n} \in \mathbb{R}^n \text{ and } x + \sum_{i=1}^m \theta_i s_{ij} \geq 0 \ \forall j \right\}. \end{aligned}$$

Define

$$\mathcal{Y}_1(y) = \{\, Y_1 \,:\, Y \in \mathcal{Y}(y) \,\}.$$

The dual sets are

$$\mathcal{Y}_1(y) = \left\{ (y_1, y_2, ..., y_n) \in \mathbb{R}_+^n \,:\, \sum_{j=1}^n p_j x_j y_j \leq xy \quad \forall (x_1, x_2, ..., x_n) \in \mathcal{X}_1(x) \right\};$$

$$\mathcal{D}^*(y) = \left\{ (y_1, y_2, ..., y_n) \in \mathbb{R}_+^n \,:\, \sum_{j=1}^n p_j y_j = y, \quad \sum_{j=1}^n p_j y_j s_{ij} = 0 \quad \forall i \right\}.$$

The primal problem is

$$u(x) = \sup_{(x_1, x_2, \ldots, x_n) \in \mathcal{X}_1(x)} \sum_{j=1}^n p_j \left(h_j \wedge x_j \right).$$

The dual problem is

$$\begin{aligned} v(y) &= \inf_{(y_1,y_2,\ldots,y_n)\in\mathcal{Y}_1(y)} \sum_{j=1}^{n} p_j h_j (1-y_j)^+ \\ &= \inf_{(y_1,y_2,\ldots,y_n)\in\mathcal{D}^*(y)} \sum_{j=1}^{n} p_j h_j (1-y_j)^+ . \end{aligned}$$

Theorem 1.22. *Suppose $y \in \partial u(x)$ where $x \geq 0$ and $y \geq 0$. Then for $V(y,\omega_j) = h_j(1-y)^+$ and an optimal solution to the dual problem $\hat{Y} = (\hat{y}_1, \hat{y}_2, \ldots \hat{y}_n)$, $\mathcal{C}^*(x) \cap -\partial V(\hat{Y})$ is the set of optimal solutions to the primal problem.*

PROOF. In our finite dimensional discrete model

$$(x_1, x_2, \ldots, x_n) \in \mathcal{C}^*(x) \quad \Leftrightarrow \quad \sum_{j=1}^{n} p_j x_j y_j = xy \quad \forall (y_1, y_2, \ldots, y_n) \in \mathcal{D}^*.$$

Applying Theorem 1.13, we get the desired result. ◇

Here we give a very simple incomplete market example in a discrete time setting with finite state space to illustrate the duality theory we have just developed for the case of minimizing shortfall risk.

Example 1.23 (Trinomial Tree). *There are two times $t = 0$ and $t = 1$. The probability space is $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with probability distribution $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \mathbb{P}(\omega_3) = \frac{1}{3}$. The stock price follow the process*

$$S_0 = 1,\ S_1(\omega_1) = \tfrac{1}{2},\ S_1(\omega_2) = 1,\ S_1(\omega) = \tfrac{3}{2}.$$

The interest rate is assumed to be zero. The model is obviously arbitrage-free. If we purchase ξ shares of stock at time $t = 0$ with initial capital $X_0 = x$, then at time $t = 1$ we have

$$X_1(\omega_1) = x - \tfrac{1}{2}\xi,\ X_1(\omega_2) = x,\ X_1(\omega_3) = x + \tfrac{1}{2}\xi.$$

For this self-financing strategy to be admissible, we must have

$$X_1(\omega) \geq 0 \quad \forall \omega \in \Omega \quad \Leftrightarrow \quad -2x \leq \xi \leq 2x.$$

If we let $(x_1, x_2, x_3) = (X_1(\omega_1), X_1(\omega_2), X_1(\omega_3))$, we can write the primal sets as

$$\begin{aligned} \mathcal{X}_1(x) &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}_3^+ \ : \ x_1 + x_3 = 2x,\ x_2 = x \right\}; \\ \mathcal{C}(x) &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}_3^+ \ : \ x_1 + x_3 \leq 2x,\ x_2 \leq x \right\}. \end{aligned}$$

It is easy to compute that the dual sets are

$$\begin{aligned}\mathcal{Y}_1(y) &= \left\{ (y_1, y_2, y_3) \in \mathbb{R}_3^+ \;:\; \tfrac{2}{3}y_1 + \tfrac{1}{3}y_2 \le y,\ \tfrac{1}{3}y_2 + \tfrac{2}{3}y_3 \le y \right\} \\ &= \mathcal{D}(y),\end{aligned}$$

where $(y_1, y_2, y_3) = (Y_1(\omega_1), Y_1(\omega_2), Y_1(\omega_3))$. *We can check that the bipolar relationship in Proposition 1.6 is satisfied*

- *Let* $(y_1, y_2, y_3) \in \mathbb{R}_3^+$ *be given.*
  Then $(y_1, y_2, y_3) \in \mathcal{D}(1) \;\Leftrightarrow\; \frac{1}{3}\sum_{i=1}^3 x_i y_i \le 1, \quad \forall\, (x_1, x_2, x_3) \in \mathcal{C}(1).$
- *Let* $(x_1, x_2, x_3) \in \mathbb{R}_3^+$ *be given.*
  Then $(x_1, x_2, x_3) \in \mathcal{C}(1) \;\Leftrightarrow\; \frac{1}{3}\sum_{i=1}^3 x_i y_i \le 1, \quad \forall\, (y_1, y_2, y_3) \in \mathcal{D}(1).$

Suppose the option payoff is

$$H(\omega_1) = \tfrac{1}{4},\ H(\omega_2) = 0,\ H(\omega_3) = \tfrac{3}{4}.$$

Then the state-dependent utility function is

$$U(x, \omega_1) = x \wedge \tfrac{1}{4},\ U(x, \omega_2) = 0,\ U(x, \omega_3) = x \wedge \tfrac{3}{4},$$

and the stochastic conjugate function is

$$V(y, \omega_1) = \tfrac{1}{4}(1 - y)^+,\ V(y, \omega_2) = 0,\ V(y, \omega_3) = \tfrac{3}{4}(1 - y)^+.$$

The primal problem can be stated as

$$\begin{aligned}u(x) &= \sup_{g \in \mathcal{C}(x)} E^{\mathbb{P}}[U(g)] \\ &= \sup_{|\xi| \le 2x} \tfrac{1}{3}\left[\left((x - \tfrac{1}{2}\xi) \wedge \tfrac{1}{4}\right) + \left((x + \tfrac{1}{2}\xi) \wedge \tfrac{3}{4}\right)\right] \\ &= \sup_{0 \le x_1 \le 2x} \tfrac{1}{3}\left[\left(x_1 \wedge \tfrac{1}{4}\right) + \left((2x - x_1) \wedge \tfrac{3}{4}\right)\right], \quad \textit{for} \quad x \ge 0.\end{aligned}$$

The optimal solution $\hat{g} \in \mathcal{C}(x)$ *is*

$$\hat{g}(\omega_1) = x - \tfrac{1}{2}\xi,\ \hat{g}(\omega_2) \in [0, x],\ \hat{g}(\omega_3) = x + \tfrac{1}{2}\xi,$$

where $\xi \in \left[(-2x) \vee (2x - \frac{1}{2}), (2x) \wedge (-2x + \frac{3}{2})\right]$. *Obviously, the solution is not unique on the set* $H > 0$ *except for* $x = 0$ *and* $x = \frac{1}{2}$. *The primal value function is*

$$u(x) = \tfrac{2}{3}(x \wedge \tfrac{1}{2}), \quad \textit{for} \quad x \ge 0.$$

Obviously, $u(x)$ *is concave, continuous, increasing, piecewise linear, and equal to a constant on* $[\frac{1}{2}, \infty)$ *where* $\bar{x} = \frac{1}{2}$ *is the super-hedging price. Similarly, we* *can compute the optimal strategy $\hat{h} \in \mathcal{D}(y)$ and the value function $v(y)$ for the dual problem*

$$\begin{aligned} v(y) &= \inf_{h \in \mathcal{D}(y)} E^{\mathbb{P}}\left[V(h)\right] \\ &= \left(\tfrac{1}{3} - \tfrac{1}{2}y\right) 1_{\{0 \le y \le \frac{2}{3}\}}, \quad \text{for} \quad y \ge 0. \end{aligned}$$

Obviously, $v(y)$ is convex, continuous, decreasing, piecewise linear, and equal to a constant on $[\frac{2}{3}, \infty)$ where $u^r(0) = \frac{2}{3}$. Note that $v^r(0) = -\frac{1}{2} = -\bar{x}$.

- $y \in [0, \frac{2}{3}]$*:* $\hat{h}(\omega_1) = \frac{3}{2}y,\ \hat{h}(\omega_2) = 0,\ \hat{h}(\omega_3) = \frac{3}{2}y$*;*
- $y \in (\frac{2}{3}, \infty)$*:* $\hat{h}(\omega_1) \in [1, \frac{1}{2}(3y - \eta)],\ \hat{h}(\omega_2) = \eta,\ \hat{h}(\omega_3) \in [1, \frac{1}{2}(3y - \eta)]$*, where $\eta \in [0, 3y - 2]$.*

The optimal solution is not unique on the set $H > 0$ unless we take $\hat{h} \wedge 1$. When $x \in [0, \bar{x}) = [0, \frac{1}{2})$, then $y = \partial u(x) = \frac{2}{3}$ and when $x \in (\bar{x}, \infty)$, then $y = \partial u(x) = 0$. Therefore, the relevant interval for dual optimal solution is $y \in [0, \frac{2}{3}]$ and $\frac{1}{y}\hat{h}$ obviously defines a martingale measure that is absolutely continuous with respect to the original measure $\mathbb{P}$. It is not difficult to check that

$$v(y) = \max_{x \ge 0}\,[u(x) - xy], \qquad u(x) = \max_{y \ge 0}\,[v(y) + xy],$$

and for $y \in \partial u(x)$, we have

$$\hat{h} \in \partial U(\hat{g}), \qquad \hat{g} \in -\partial V(\hat{h}).$$

In particular, we can always find a pair of optimal solutions that have the form (4.15) and (4.14). Also, $E^{\mathbb{P}}[\hat{g}(x)\hat{h}(y)] = xy$ is always true. When $x = \frac{1}{2}$, the maximal set is

$$\mathcal{C}^*(\tfrac{1}{2}) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}_3^+ \ :\ x_1 + x_3 = 1,\ x_2 = \tfrac{1}{2} \right\}.$$

For $y \in \partial u(\frac{1}{2}) = [0, \frac{2}{3}]$, $\hat{h}(y) = (\frac{3}{2}y, 0, \frac{3}{2}y)$. Therefore, when $y = \frac{2}{3}$,

$$\partial V(\hat{h}(y), \omega_1) = [-\tfrac{1}{4}, 0], \quad \partial V(\hat{h}(y), \omega_2) = [-\tfrac{1}{2}, \infty), \quad \partial V(\hat{h}(y), \omega_3) = [-\tfrac{3}{4}, 0],$$

and when $y \in [0, \frac{2}{3})$,

$$\partial V(\hat{h}(y), \omega_1) = -\tfrac{1}{4}, \quad \partial V(\hat{h}(y), \omega_2) = [-\tfrac{1}{2}, \infty), \quad \partial V(\hat{h}(y), \omega_3) = -\tfrac{3}{4}.$$

Obviously,

$$\hat{g}(\tfrac{1}{2}) = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}) = \mathcal{C}^*(\tfrac{1}{2}) \cap -\partial V(\hat{h}(y)), \quad \forall\, y \in \partial u(\tfrac{1}{2}).$$

◇

Next we give an example of an infinite dimensional tree where the dual optimal solution does not define a martingale measure. More discussion in a continuous time model is given later in Section 3.3.

Example 1.24 (Infinite Dimensional Tree). *There are two times $t = 0$ and $t = 1$. The probability space is $\Omega = \{\omega_0, \omega_1, ...\}$ with probability distribution $\mathbb{P}(\omega_0) = 1 - \epsilon$, $\mathbb{P}(\omega_n) = \frac{\epsilon}{2^n}$ for $n \geq 1$. The stock price follow the process*

$$S_0 = 1, \quad S_1(\omega_0) = 2, \quad S_1(\omega_n) = \tfrac{1}{n}, \quad \textit{for} \quad n \geq 1.$$

The interest rate is assumed to be zero. The model is obviously arbitrage-free. If we purchase ξ shares of stock at time $t = 0$ with initial capital $X_0 = x$, then at time $t = 1$ we have $X_1(\omega) = x + \xi(S_1(\omega) - S_0)\ \forall \omega \in \Omega$. For this self-financing strategy to be admissible, we must have

$$X_1(\omega) \geq 0 \quad \forall \omega \in \Omega \quad \Leftrightarrow \quad -x \leq \xi \leq x.$$

We can write the primal set as

$$\mathcal{X}_1(x) = \{\, x + \xi(S_1 - S_0) \,:\, \xi \in [-x, x] \,\}, \quad \textit{for} \quad x \geq 0.$$

It is easy to compute that the dual set is

$$\mathcal{Y}_1(y) = \left\{\, Y \geq 0 \,:\, E^{\mathbb{P}}[Y] \leq y, 2E^{\mathbb{P}}[Y] - y \leq E^{\mathbb{P}}[S_1 Y] \leq y \,\right\}, \quad \textit{for} \quad y \geq 0.$$

Suppose the option payoff is

$$H(\omega_0) = 4, \quad H(\omega_n) = 0, \quad \textit{for} \quad n \geq 1.$$

Then the super-hedging price is $\bar{x} = 2$. The state-dependent utility function is

$$U(x, \omega_0) = x \wedge 4, \quad U(x, \omega_n) = 0, \quad \textit{for} \quad n \geq 1,$$

and the stochastic conjugate function is

$$V(y, \omega_0) = 4(1 - y)^+, \quad V(y, \omega_n) = 0, \quad \textit{for} \quad n \geq 1.$$

The primal problem can be stated as

$$\begin{aligned} u(x) &= \sup_{X_1 \in \mathcal{X}_1(x)} E^{\mathbb{P}}\left[U(X_1)\right] \\ &= \sup_{|\xi| \leq x} (1 - \epsilon)\left[(x + \xi) \wedge 4\right], \end{aligned}$$

When $x \in [0,2]$, the optimal strategy is unique, $\xi = x$, and the optimal solution to the primal problem is $\hat{X}_1(\omega) = xS_1(\omega)\ \forall \omega \in \Omega$. When $x \in (2,\infty)$, the optimal strategy is not unique anymore $\xi \in [4-x, x]$. The primal value function is

$$u(x) = 2x(1-\epsilon) \wedge 4(1-\epsilon), \quad \textit{for} \quad x \geq 0.$$

Obviously, $u(x)$ is concave, continuous, increasing, piecewise linear, and equal to a constant on $[\bar{x},\infty) = [2,\infty)$. Similarly, we can compute the optimal strategy $\hat{Y} \in \mathcal{Y}_1(y)$ and the value function $v(y)$ for the dual problem

$$\begin{aligned} v(y) &= \inf_{Y \in \mathcal{Y}_1(y)} E^{\mathbb{P}}[V(Y)] \\ &= \inf_{Y \in \mathcal{Y}_1(y)} 4(1-\epsilon)(1-Y(\omega_0))^+, \quad \textit{for} \quad y \geq 0. \end{aligned}$$

Obviously, $y = 2(1-\epsilon) \in \partial u(x)$ for any $x \geq 0$. It is easy to check that

$$\hat{Y}(\omega_0) = 1, \quad \hat{Y}(\omega_n) = 0, \quad \textit{for} \quad n \geq 1,$$

is the unique optimal solution to the dual problem in the set $\mathcal{Y}_1(2(1-\epsilon))$. However,

$$E^{\mathbb{P}}\left[\frac{\hat{Y}}{y}\right] = \frac{1}{2} < 1,$$

and $\frac{\hat{Y}}{y}$ does not define a martingale measure. In general, the dual value function is

$$v(y) = [\,4(1-\epsilon) - 2x\,] \vee 0, \quad \textit{for} \quad y \geq 0.$$

It is convex, continuous, decreasing, piecewise linear, and equal to a constant on $[2(1-\epsilon),\infty)$ where $u^r(0) = 2(1-\epsilon)$. Note that $v^r(0) = -2 = -\bar{x}$. It is not difficult to check that

$$v(y) = \max_{x \geq 0}[u(x) - xy], \qquad u(x) = \max_{y \geq 0}[v(y) + xy],$$

and for $y \in \partial u(x)$, we have

$$\hat{Y} \in \partial U(\hat{X}_T), \qquad \hat{X}_T \in -\partial V(\hat{Y}),$$

such that $E^{\mathbb{P}}[\hat{X}_T(x)\hat{Y}(y)] = xy$. The maximal set is the same as the admissible set

$$\mathcal{C}^*(x) = \mathcal{X}_1(x).$$

When $x \in (0,2)$, and for the scenario $\omega = \omega_0$, we have $\hat{X}_T(x) = 2x$. However, $\mathcal{X}_1(x) = [0,2x]$ and $-\partial V(\hat{Y}) = [0,4]$. Therefore, $X_T(x) \in \mathcal{C}^*(x) \cup (-\partial V(\hat{Y}))$ is not sufficient for $X_T(x)$ to be an optimal solution to the primal problem. $\diamond$

Chapter 2

Complete Market Models

In a complete market, we have a unique martingale measure, i.e., $\mathcal{M} = \{\mathbb{Q}^*\}$. Let the Radon-Nikodym derivative be

$$Z = \frac{d\mathbb{Q}^*}{d\mathbb{P}}.$$

Lemma 4.3 in Kramkov and Schachermayer (1999) proved that Z dominates all elements in the set $\mathcal{Y}(1)$ as well as $\mathcal{D} \triangleq \mathcal{D}(1)$ defined in Section 1.2. Therefore the optimal solution to (Dual-Shortfall), $\hat{h} \in \mathcal{D}(y)$, is unique and can be written as

$$\hat{h} = yZ.$$

In light of Remark 1.21, there is an optimal solution to (Primal-Shortfall), $\hat{g} \in \mathcal{C}(x)$, that can be written as

$$\hat{g} = \left(1_{\{Z<\frac{1}{y}\}} + \gamma 1_{\{Z=\frac{1}{y}\}}\right) H = \left(1_{\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}>y\}} + \gamma 1_{\{\frac{d\mathbb{P}}{d\mathbb{Q}^*}=y\}}\right) H, \tag{0.1}$$

where $y \in \partial u(x)$.

Remark 2.1. *Notice that when we fix the initial capital x for hedging the option H, it is hard to compute the derivatives of the primal value function to get the dual variable that satisfies $y \in \partial u(x)$. However, in the case of a complete market, it is intuitively clear what the dual variable y represents. Recall (4.17) of Chapter 1. When the local martingale measure set is a singleton, it is more cost effective to reduce the shortfall risk when $\frac{d\mathbb{P}}{d\mathbb{Q}^*}$ is large. So y is the cut-off level of the favorable scenario set where we should perfectly hedge the option payoff. This result of course holds for the incomplete market as well, although in which case it is more difficult to find the optimal dual solution which might not define a local martingale measure as shown in Example 1.24 of Chapter 1.*

For the rest of this chapter, we will present three complete market models where optimal solutions can be explicitly computed. We will check both the duality and the HJB conditions in each of these cases.

2.1 Poisson jump model

Suppose the dynamics of the discounted[1] asset price process under $\mathbb{P}$ are

$$\begin{aligned} dS_t &= S_{t-}\left[\,\mu dt - (1-\alpha)(dN_t - \lambda dt)\,\right] \\ &= S_{t-}\left[\,cdt - (1-\alpha)dN_t\,\right], \end{aligned}$$

where N_t is a standard Poisson process with intensity λ, and $c = \mu + (1-\alpha)\lambda$. Assume the constants satisfy $\mu > 0, 0 < \alpha < 1, \lambda > 0$. The Doléans-Dade formula gives the solution to the above stochastic differential equation

$$S_t = S_0 e^{\,ct}\alpha^{N_t}.$$

Note that the price goes up exponentially with parameter c when there is no Poisson jump. In the case when a jump occurs, the price jumps down to a fraction of itself αS.

2.1.1 Optimal Strategy

The intensity for the Poisson process under the risk-neutral measure $\mathbb{Q}^*$ is $\lambda^* = \frac{c}{1-\alpha} > \lambda$ and the price process can be written as

$$dS_t = S_{t-}\left[\,-(1-\alpha)(dN_t - \lambda^* dt)\,\right].$$

The Radon-Nikodym derivatives are

$$Z_T \triangleq \frac{d\mathbb{Q}^*}{d\mathbb{P}} = \left(\frac{\lambda^*}{\lambda}\right)^{N_T} e^{-(\lambda^*-\lambda)T}, \qquad \frac{1}{Z_T} = \frac{d\mathbb{P}}{d\mathbb{Q}^*} = \left(\frac{\lambda}{\lambda^*}\right)^{N_T} e^{-(\lambda-\lambda^*)T}.$$

Suppose the option payoff is a function of the underlying $H = H(S_T)$, and define H_k to be the payoff when there are k jumps in the underlying

$$H_k = H(S_0 e^{\,cT}\alpha^k), \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{1.2}$$

[1] For simplicity of notation, we start with discounted prices to get rid of the interest rate.

Define

$$\begin{aligned} x_k(T) &= \mathbb{Q}^*(N_T = k) \\ &= \mathbb{Q}^*(S_T = S_0 e^{\,cT}\alpha^k) \\ &= \mathbb{Q}^*\left(Z_T = \left(\tfrac{\lambda^*}{\lambda}\right)^k e^{-(\lambda^*-\lambda)T}\right) \\ &= e^{-\lambda^* T}\tfrac{(\lambda^* T)^k}{k!}, \quad \text{for} \quad k = 0, 1, 2, \ldots \end{aligned}$$

Proposition 2.2. *Suppose the initial capital x satisfies $0 < x < E^{\mathbb{Q}^*}[H]$. There exists a nonnegative integer n and a $\gamma \in [0, 1)$ such that*

$$x = \sum_{k=0}^{n-1} x_k(T) H_k + \gamma\, x_n(T) H_n.$$

The wealth of the optimal strategy at time $0 \le t \le T$ is

$$\begin{aligned} \hat{X}_t &= E^{\mathbb{Q}^*}[\,\hat{X}_T \wedge H \,|\, \mathcal{F}_t\,] = E^{\mathbb{Q}^*}[\,\hat{g}\,|\,\mathcal{F}_t\,] \\ &= \begin{cases} 0, & \text{if } N_t > n; \\ \gamma\, x_0(\tau) H_n, & \text{if } N_t = n; \\ \sum_{k=0}^{n-1-N_t} x_k(\tau) H_{N_t+k} + \gamma\, x_{n-N_t}(\tau) H_n, & \text{if } N_t < n; \end{cases} \end{aligned}$$

where $\tau = T - t$ is the time to maturity. In particular, when $\tau = 0$,

$$\hat{X}_T = \begin{cases} 0, & \text{if } N_T > n; \\ \gamma\, H_n, & \text{if } N_T = n; \\ H_{N_T}, & \text{if } N_T < n. \end{cases}$$

The optimal strategy is to invest $\hat{\Delta}_t$ shares in the underlying, where

$$\begin{aligned} \hat{\Delta}_t &= \frac{\hat{X}(t, \alpha S_{t-}) - \hat{X}(t, S_{t-})}{\alpha S_{t-} - S_{t-}} \\ &= \begin{cases} 0, & \text{if } N_{t-} > n; \\ \frac{\gamma\, x_0(\tau) H_n}{(1-\alpha) S_{t-}}, & \text{if } N_{t-} = n; \\ \frac{1}{(1-\alpha) S_{t-}} \left\{ \sum_{k=0}^{n-2-N_t} x_k(\tau)(H_{N_{t-}+k} - H_{N_{t-}+k+1}) \right. & \\ \left. + x_{n-1-N_{t-}}(\tau)(H_{n-1} - \gamma H_n) + \gamma\, x_{n-N_{t-}}(\tau) H_n \right\}, & \text{if } N_{t-} < n. \end{cases} \end{aligned}$$

Recall $S_{t-} = S_0 e^{\,ct} \alpha^{N_{t-}}$.

Remark 2.3. *For the wealth process to remain nonnegative, we have the following admissibility constraint on the trading strategy*

$$\Delta_t \le \frac{X_{t-}}{(1-\alpha) S_{t-}}.$$

*When there have already been n downward jumps in the price process by time t, the value process of the optimal strategy is $\hat{X}_t = \gamma x_0(\tau) H_n = e^{-\lambda^* \tau} \gamma H_n$, and the optimal strategy is to use the maximal admissible strategy so that the wealth approaches γH_n at maturity as an exponential function if there would be no more jumps; otherwise, the wealth hits zero. When there have been $i < n$ downward jumps in the price process by time t, the wealth approaches H_i for a perfect replication if no more jumps occur in the future. If one more jump occurs, we come to the case of aiming for H_{i-1}. We can name this as '*dynamic aiming for the corner strategy*'. This strategy also satisfies the dynamic programming principle developed in Theorem 6.4 of Kirch (2002) when we set $a = -b$, $l(x) = x$ and $F(x) = 1$.*

Proof. In general, we know the optimal solution to (Primal-Shortfall) is

$$\hat{g} = \left(1_{\{Z < \frac{1}{y}\}} + \gamma 1_{\{Z = \frac{1}{y}\}} \right) H.$$

By the Arbitrage Pricing Theory, the wealth process is a martingale under the risk-neutral measure

$$X_t = E^{\mathbb{Q}^*}[\, \hat{g} \,|\, \mathcal{F}_t \,]$$

such that the budget constraint is satisfied,

$$x = X_0 = E^{\mathbb{Q}^*}[\hat{g}].$$

Obviously we can find $0 \leq n < \infty$ and $0 \leq \gamma < 1$ such that

$$x = \sum_{k=0}^{n-1} x_k(T) H_k + \gamma\, x_n(T) H_n,$$

because of the assumption $0 < x < E^{\mathbb{Q}^*}[H]$. The formula for $\hat{X}_t$ can be computed using conditional expectation. If $N_t < n$, we have

$$\begin{aligned} \hat{X}_t &= E^{\mathbb{Q}^*}[\, \hat{g} \,|\, \mathcal{F}_t \,] \\ &= E^{\mathbb{Q}^*}[\, H_{N_T} 1_{\{N_T < n\}} + \gamma H_n 1_{\{N_T = n\}} \,|\, \mathcal{F}_t \,] \\ &= E^{\mathbb{Q}^*}[\, H_{N_t + N_T - N_t} 1_{\{N_T - N_t < n - N_t\}} + \gamma H_n 1_{\{N_T - N_t = n - N_t\}} \,|\, \mathcal{F}_t \,] \\ &= \sum_{k=0}^{n-1-N_t} x_k(\tau) H_{N_t + k} + \gamma x_{n - N_t}(\tau) H_n. \end{aligned}$$

It is simpler to compute the case when $N_t \geq n$. By the Markovian structure of the process $\hat{X}(t, S_t)$, the martingale property of the value function $\hat{X}_t$ and the

Itô-Doeblin formula, we can compute the optimal strategy $\hat{\Delta}_t$ from the SDE

$$d\hat{X}(t, S_t) = (\hat{X}(t, \alpha S_{t-}) - \hat{X}(t, S_{t-}))(dN_t - \lambda^* dt),$$

and the self-financing condition

$$d\hat{X}(t, S_t) = \hat{\Delta}_t dS_t = -\hat{\Delta}_t S_{t-}(1-\alpha)(dN_t - \lambda^* dt).$$

◇

Corollary 2.4. *In the simplest case when the option payoff is a constant, which we assume without loss of generality is $H \equiv 1$, suppose the initial capital x satisfies $0 < x < 1$. There is a nonnegative integer n and a $\gamma \in [0, 1)$ such that*

$$x = \sum_{k=0}^{n-1} x_k(T) + \gamma\, x_n(T).$$

The wealth of the optimal strategy at time $0 \le t \le T$ is

$$\hat{X}_t = \begin{cases} 0, & \text{if } N_t > n; \\ \gamma\, x_0(\tau), & \text{if } N_t = n; \\ \sum_{k=0}^{n-1-N_t} x_k(\tau) + \gamma\, x_{n-N_t}(\tau), & \text{if } N_t < n; \end{cases}$$

where $\tau = T - t$ is the time to maturity. The optimal strategy is to invest $\hat{\Delta}_t$ shares in the underlying, where

$$\hat{\Delta}_t = \begin{cases} 0, & \text{if } N_{t-} > n; \\ \frac{\gamma\, x_0(\tau)}{(-\alpha) S_{t-}}, & \text{if } N_{t-} = n; \\ \frac{(1-\gamma)\, x_{n-1-N_{t-}}(\tau) + \gamma\, x_{n-N_{t-}}(\tau)}{(-\alpha) S_{t-}}, & \text{if } N_{t-} < n. \end{cases}$$

We can also write the wealth in the integral form

$$\hat{X}_t = \begin{cases} 0, & \text{if } N_t > n; \\ \gamma\, x_0(\tau), & \text{if } N_t = n; \\ 1 - \lambda^* \int_t^T x_{n-1-N_t}(T-u)du + \gamma\, x_{n-N_t}(\tau), & \text{if } N_t < n. \end{cases}$$

and

$$\hat{X}_t = \begin{cases} 0, & \text{if } N_t > n; \\ \gamma\, x_0(\tau), & \text{if } N_t = n; \\ 1 - \lambda^* \int_t^T x_{n-N_t}(T-u)du - (1-\gamma)\, x_{n-N_t}(\tau), & \text{if } N_t < n. \end{cases}$$

Proof.The optimal wealth process and strategy are obtained by a direct application of Proposition 2.2. To get the integral forms, we need the equalities

$$\lambda^* \int_t^s x_k(T-u)du = \sum_{i=0}^{k} x_i(T-u)|_t^s,$$
$$\lambda^* \int_t^T x_k(T-u)du = 1 - \sum_{i=0}^{k} x_i(\tau),$$
$$dx_k(T-u) = \lambda^*[x_k(T-u) - x_{k-1}(T-u)],$$
$$\lambda^* \int_t^T x_{k-1}(T-u)du = x_k(T-t) + \lambda^* \int_t^T x_k(T-u)du.$$

◇

2.1.2 HJB Approach

Another standard approach to solve the optimal control problem is to check the HJB equation. Rewrite the self-financing strategy as

$$dX_t = \Delta_t dS_t = \Delta_t S_{t-}(cdt - (1-\alpha)dN_t) = \pi_t(cdt - (1-\alpha)dN_t),$$

where $\pi_t = \Delta_t S_{t-}$ is the monetary amount invested in the underlying. As defined in (1.2), let the option payoff be a function of the value process X_T through its dependence on the number of Poisson jumps. Our optimal control problem can be written as

$$\max_{\pi_t \le \frac{X_{t-}}{1-\alpha}} E\,[X_T \wedge H] \quad s.t. \quad dX_t = \pi_t(cdt - (1-\alpha)dN_t).$$

Define the value process as

$$u(t, \tilde{X}_t) = \max_{\pi_t \le \frac{X_{t-}}{1-\alpha}} E\,[X_T \wedge H|\mathcal{F}_t],$$

where $\tilde{X} \in \mathcal{X}(x)$ is the optimal solution. Suppose $u_t(t,x)$ and $u_x(t,x)$ exist. Using the usual verification lemma argument, we derive the HJB equation:

$$\max_{\pi_t \le \frac{x}{1-\alpha}} (u_t(t,x) + c\pi_t u_x(t,x) + \lambda[u(t, x-(1-\alpha)\pi_t) - u(t,x)]) = 0,$$

with the boundary condition $u(T,x) = x \wedge H$. Define

$$\hat{u}(t, \hat{X}_t) = E\,[\hat{X}_T \wedge H \mid \mathcal{F}_t],$$

where $\hat{X}_t$ is the wealth process defined in Proposition 2.2, associated with the strategy $\hat{\pi}_t$. We will check their optimality in the following lemma.

Lemma 2.5. *The value function $\hat{u}(t,x)$ satisfies*

$$\hat{u}_t(t,x) + c\hat{\pi}_t\hat{u}_x(t,x) + \lambda[\hat{u}(t,x-(1-\alpha)\hat{\pi}_t) - \hat{u}(t,x)] = 0,$$

with $\hat{u}(T,x) = x \wedge H$.

Proof. Suppose $\hat{X}_t \in \Big[\sum_{k=0}^{i-1} x_k(\tau)H_k, \sum_{k=0}^{i} x_k(\tau)H_k\Big)$, or equivalently,

$$\hat{X}_t = \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k} + \gamma\, x_{n-i}(\tau)H_n, \quad 0 \le \gamma < 1, \quad i \ge 1.$$

We can write

$$\gamma = \frac{\hat{X}_t - \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k}}{x_{n-i}(\tau)H_n}.$$

We know the optimal strategy is

$$\begin{aligned}
\hat{\pi}(t,\hat{X}_t) &= \hat{\Delta}_t S_{t-} \\
&= \frac{1}{(1-\alpha)}\Bigg\{\sum_{k=0}^{n-i-2} x_k(\tau)(H_{i+k} - H_{i+k+1}) \\
&\qquad + x_{n-i-1}(\tau)(H_{n-1} - \gamma H_n) + \gamma\, x_{n-i}(\tau)H_n\Bigg\} \\
&= \frac{1}{(1-\alpha)}\Bigg\{\sum_{k=0}^{n-i-2} x_k(\tau)(H_{i+k} - H_{i+k+1}) + \hat{X}_t - \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k} \\
&\qquad + x_{n-i-1}(\tau)\left(H_{n-1} - \frac{\hat{X}_t - \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k}}{x_{n-i}(\tau)}\right)\Bigg\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\pi}(t,x) = \frac{1}{(1-\alpha)}\Bigg\{&\sum_{k=0}^{n-i-2} x_k(\tau)(H_{i+k} - H_{i+k+1}) + x - \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k} \\
&+ x_{n-i-1}(\tau)\left(H_{n-1} - \frac{x - \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k}}{x_{n-i}(\tau)}\right)\Bigg\}.
\end{aligned}$$

Define

$$y_k(T) = \mathbb{P}(N_T = k) = e^{-\lambda T}\tfrac{(\lambda T)^k}{k!}, \quad \text{for} \quad k = 0, 1, 2... \quad .$$

Then the value function $\hat{u}(t,x)$ can be computed

$$\begin{aligned}
\hat{u}(t,\hat{X}_t) &= E[\,\hat{X}_T \wedge H \,|\, \mathcal{F}_t\,] \\
&= E[\,H_{i+N_T-N_t}1_{\{N_T-N_t<n-i\}} + \gamma H_n 1_{\{N_T-N_t=n-i\}} \,|\, \mathcal{F}_t\,] \\
&= \sum_{k=0}^{n-i-1} y_k(\tau)H_{i+k} + \gamma y_{n-i}(\tau)H_n \\
&= \sum_{k=0}^{n-i-1} y_k(\tau)H_{i+k} + \frac{\hat{X}_t - \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k}}{x_{n-i}(\tau)} y_{n-i}(\tau).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{u}(t,x) &= \sum_{k=0}^{n-i-1} y_k(\tau)H_{i+k} + \frac{x - \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k}}{x_{n-i}(\tau)} y_{n-i}(\tau), \\
\hat{u}_x(t,x) &= \frac{y_{n-i}(\tau)}{x_{n-i}(\tau)}, \\
\hat{u}(t,x-(1-\alpha)\hat{\pi}) &= \sum_{k=0}^{n-i-2} y_k(\tau)H_{i+1+k} + \frac{x - \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k}}{x_{n-i}(\tau)} y_{n-i-1}(\tau).
\end{aligned}$$

To find $\hat{u}_t(t,x)$, we need

$$\frac{dx_k(\tau)}{dt} = \lambda^*(x_k(\tau)-x_{k-1}(\tau)), \quad \frac{dy_k(\tau)}{dt} = \lambda(y_k(\tau)-y_{k-1}(\tau)), \quad \text{for} \quad k = 0,1,2...,$$

where we define $x_{-1}(\tau) = 0$, and $y_{-1}(\tau) = 0$. Then

$$\begin{aligned}
\hat{u}_t(t,x) = &\sum_{k=0}^{n-i-1} \lambda(y_k(\tau) - y_{k-1}(\tau))H_{i+k} \\
&+ \left\{ -\sum_{k=0}^{n-i-1} \lambda^*(x_k(\tau) - x_{k-1}(\tau))H_{i+k} \right\} \frac{y_{n-i}(\tau)}{x_{n-i}(\tau)} \\
&+ \left\{ x - \sum_{k=0}^{n-i-1} x_k(\tau)H_{i+k} \right\} \\
&\cdot \frac{\lambda(y_{n-i}(\tau) - y_{n-i-1}(\tau))x_{n-i}(\tau) - \lambda^*(x_{n-i}(\tau) - x_{n-i-1}(\tau))y_{n-i}(\tau)}{x_{n-i}(\tau)^2}
\end{aligned}$$

It is routine to check the value function $\hat{u}(t,x)$ and strategy $\hat{\pi}(t,x)$ satisfy the HJB equality

$$\hat{u}_t(t,x) + c\hat{\pi}_t\hat{u}_x(t,x) + \lambda[\hat{u}(t,x-(1-\alpha)\hat{\pi}_t) - \hat{u}(t,x)] = 0.$$

The case

$$\hat{X}_t = \gamma_t x_0(\tau)H_n, \quad 0 \le \gamma_t < 1$$

can be similarly checked. ◇

2.1.3 Duality

Recall in the proof of Lemma 2.5, we have computed the value function to the (Primal-Shortfall) in the case $x \in \left[\sum_{k=0}^{n-1} x_k(\tau)H_k, \sum_{k=0}^{n} x_k(\tau)H_k\right)$:

$$\begin{aligned} u(x) = \hat{u}(0,x) &= \sum_{k=0}^{n-1} y_k(T)H_k + \gamma y_n(T)H_n, \text{ where } \gamma = \frac{x - \sum_{k=0}^{n-1} x_k(T)H_k}{x_n(T)H_n}, \\ u'(x) = \hat{u}_x(0,x) &= \frac{y_n(T)}{x_n(T)}. \end{aligned}$$

At the beginning of Chapter 2, we have shown the optimal value function to the (Dual-Shortfall) is

$$v(y) = E[(1 - yZ_T)^+ H].$$

We would like to check the duality results we have derived in Section 1.4.

Lemma 2.6. *The optimal value functions computed in the previous two subsections satisfy the duality equality*

$$v(y) = u(x) - xy, \quad \textit{when} \quad y = u'(x).$$

Proof. Note that when $y = u'(x) = \frac{y_n(T)}{x_n(T)}$ as computed in Lemma 2.5,

$$\begin{aligned} v(y) &= E[(1 - yZ_T)^+ H] \\ &= E\left[\left(1 - \tfrac{y_n(T)}{x_n(T)} \left(\tfrac{\lambda^*}{\lambda}\right)^{N_T} e^{-(\lambda^* - \lambda)T}\right)^+ H\right] \\ &= E\left[\left(1 - \left(\tfrac{\lambda^*}{\lambda}\right)^{N_T - n}\right)^+ H\right] \\ &= \sum_{k=0}^{n-1} \left(1 - \left(\tfrac{\lambda^*}{\lambda}\right)^{k-n}\right) e^{-\lambda T} \tfrac{(\lambda T)^k}{k!} H_k \\ &= \sum_{k=0}^{n-1} y_k(T)H_k - \tfrac{y_n(T)}{x_n(T)} \sum_{k=0}^{n-1} x_k(T)H_k \\ &= u(x) - xu'(x) \\ &= u(x) - xy. \end{aligned}$$

◇

2.2 Geometric Brownian motion model

Suppose the dynamics of the discounted[2] asset price process under $\mathbb{P}$ follows the Black-Scholes model

$$dS_t = S_t[\,\mu dt + \sigma dW_t\,], \quad S_t = S_0 e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t}$$

where W_t is a standard Brownian motion. Assume the constant drift and volatility satisfy $\mu > 0, \sigma > 0$.

2.2.1 Optimal Strategy

Under the risk-neutral measure $\mathbb{Q}^*$, $W_t^* = W_t + \theta t$ is a Brownian motion, where $\theta = \frac{\mu}{\sigma}$. The price process can be written as

$$dS_t = S_t \sigma dW_t^*, \quad S_t = S_0 e^{\sigma W_t^* - \frac{1}{2}\sigma^2 t}.$$

The Radon-Nikodym derivatives are

$$\begin{aligned} Z_T &\triangleq \frac{d\mathbb{Q}^*}{d\mathbb{P}} = e^{-\theta W_T - \frac{1}{2}\theta^2 T} = e^{-\theta W_T^* + \frac{1}{2}\theta^2 T}, \\ \frac{1}{Z_T} &= \frac{dP_T}{dP_T^*} = e^{\theta W_T + \frac{1}{2}\theta^2 T} = e^{\theta W_T^* - \frac{1}{2}\theta^2 T}. \end{aligned}$$

Proposition 2.7. *Suppose the initial capital $0 < x < E^{\mathbb{Q}^*}[H]$. There exists a y such that*

$$x = E^{\mathbb{Q}^*}[1_{\{Z_T < \frac{1}{y}\}} H],$$

and the wealth of the optimal strategy at time $0 \le t \le T$ is

$$\hat{X}_t = E^{\mathbb{Q}^*}[\,1_{\{Z_T < \frac{1}{y}\}} H \,|\, \mathcal{F}_t\,].$$

Suppose the payoff is a function of the underlying price process $H = H(S_T)$. Then the usual delta hedge is optimal

$$\hat{\Delta}_t = \frac{\partial}{\partial s}\hat{X}(t, S_t).$$

Proof. The distribution of the Radon-Nikodym derivative is diffuse. Therefore the optimal solution to (Primal-Shortfall) is $\hat{g} = 1_{\{Z_T < \frac{1}{y}\}} H$. Nothing else is new in this proposition. ◇

Föllmer and Leukert (2000) computed the optimal strategy for a call option under this model. We will give an example of a bond.

[2]For simplicity of notation, we start with discounted prices to get rid of the interest rate.

Corollary 2.8. *In the simplest case when the option payoff is a constant, which we assume without loss of generality is $H \equiv 1$, suppose the initial capital x satisfies $0 < x < 1$. Let y be the solution to*

$$x = 1 - N(d_+(0, y)),$$

where $N(\cdot)$ is the c.d.f. of a standard normal distribution and

$$d_+(t, y) = \frac{\ln y + \frac{1}{2}\theta^2(T-t)}{\theta\sqrt{T-t}}.$$

The wealth of the optimal strategy at time $0 \le t \le T$ is

$$\hat{X}_t = 1 - N(d_+(t, yZ_t)) = 1 - N\left(\frac{\ln y + \frac{1}{2}\theta^2 T - \frac{\theta}{\sigma}\left(\ln S_t - \ln S_0 + \frac{1}{2}\sigma^2 t\right)}{\theta\sqrt{T-t}}\right).$$

The optimal strategy is

$$\hat{\pi}_t = \hat{\Delta}_t S_t = \frac{1}{\sigma\sqrt{2\pi(T-t)}}\, e^{-\frac{d_+(t, yZ_t)^2}{2}}.$$

Proof. The existence of y follows easily from the fact $0 < x < 1$.

$$\hat{X}_t = E^{\mathbb{Q}^*}[\, 1_{\{Z_T < \frac{1}{y}\}} \,|\, \mathcal{F}_t\,] = E^{\mathbb{Q}^*}[\, 1_{\{Z_{T-t} < \frac{1}{yZ_t}\}} \,|\, \mathcal{F}_t\,] = 1 - N(d_+(t, yZ_t)).$$

Noticing that

$$Z_t = e^{-\frac{\theta}{\sigma}(\ln S_t - \ln S_0) - \frac{1}{2}\theta\sigma t + \frac{1}{2}\theta^2 t},$$

we get the two expressions for $\hat{X}_t$. The delta hedge is derived by simple differentiation. ◇

2.2.2 HJB Approach

Another standard approach to solve the optimal control problem is to check the HJB equation. Rewrite the self-financing strategy as

$$dX_t = \Delta_t dS_t = \Delta_t S_t(\mu dt + \sigma dW_t) = \pi_t(\mu dt + \sigma dW_t),$$

where $\pi_t = \Delta_t S_t$ is the monetary amount invested in the underlying. In the simplest case when the payoff function is a constant $H \equiv 1$, our optimal control problem can be written as

$$\max_{\pi_t} E[X_T \wedge 1] \quad s.t. \quad dX_t = \pi_t(\mu dt + \sigma dW_t), \quad X_t \ge 0, \quad 0 \le t \le 1.$$

Define the value process as

$$u(t, \tilde{X}_t) = \max_{\pi_t} E[X_T \wedge 1 | \mathcal{F}_t],$$

where $\tilde{X} \in \mathcal{X}(x)$ is the optimal solution. Suppose $u_t(t,x)$ and $u_x(t,x)$ exist. Using the usual verification lemma argument, we derive the HJB equation:

$$\max_{\text{admissible } \pi_t} (u_t(t,x) + \mu \pi_t u_x(t,x) + \tfrac{1}{2}\sigma^2 \pi_t^2 u_{xx}(t,x)) = 0,$$

with the boundary condition $u(T,x) = x \wedge 1$. Define

$$\hat{u}(t, \hat{X}_t) = E\,[\hat{X}_T \wedge 1 | \mathcal{F}_t],$$

where $\hat{X}_t$ is the wealth process defined in Corollary 2.8 associated with the strategy $\hat{\pi}_t$. We will check their optimality in the following lemma.

Lemma 2.9. *The value function $\hat{u}(t,x)$ satisfies*

$$\hat{u}_t(t,x) + \mu \hat{\pi}_t \hat{u}_x(t,x) + \tfrac{1}{2}\sigma^2 \hat{\pi}_t^2 \hat{u}_{xx}(t,x) = 0,$$

with the boundary condition $\hat{u}(T,x) = x \wedge 1$.

Proof. The value function $\hat{u}(t,x)$ can be computed as follows:

$$\begin{aligned}
\hat{u}(t, \hat{X}_t) &= E^{\mathbb{P}}[\hat{X}_T \wedge 1 | \mathcal{F}_t] \\
&= 1 - N\left(d_+(t, yZ_t) - \theta\sqrt{T-t}\right) \\
&= 1 - N\left(N^{-1}(1 - \hat{X}_t) - \theta\sqrt{T-t}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{u}(t,x) &= 1 - N\left(N^{-1}(1-x) - \theta\sqrt{T-t}\right) \\
\hat{u}_t(t,x) &= -N'\left(N^{-1}(1-x) - \theta\sqrt{T-t}\right)\frac{\theta}{2\sqrt{T-t}} \\
\hat{u}_x(t,x) &= \frac{N'\left(N^{-1}(1-x) - \theta\sqrt{T-t}\right)}{N'(N^{-1}(1-x))} \\
&= e^{\theta\sqrt{T-t}\,N^{-1}(1-x) - \frac{1}{2}\theta^2(T-t)} \\
\hat{u}_{xx}(t,x) &= \hat{u}_x(t,x)\frac{\theta\sqrt{T-t}}{N'(N^{-1}(1-x))}(-1) \\
&= -\theta\sqrt{T-t}\frac{N'\left(N^{-1}(1-x) - \theta\sqrt{T-t}\right)}{\left(N'(N^{-1}(1-x))\right)^2}.
\end{aligned}$$

Recall

$$\hat{\pi}(t,x) = \frac{1}{\sigma\sqrt{2\pi(T-t)}}\, e^{-\frac{\left(N^{-1}(1-x)\right)^2}{2}} = \frac{1}{\sigma\sqrt{T-t}} N'(N^{-1}(1-x)),$$

then

$$\mu\hat{\pi}\hat{u}_x(t,x) = \frac{\theta}{\sqrt{T-t}} N'\left(N^{-1}(1-x) - \theta\sqrt{T-t}\right).$$

It is routine to check the value function $\hat{u}$ and strategy $\hat{\pi}_t$ satisfy the HJB equality

$$\hat{u}_t(t,x) + \mu\hat{\pi}\hat{u}_x(t,x) + \tfrac{1}{2}\sigma^2\hat{\pi}^2\hat{u}_{xx}(t,x) = 0.$$

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2.2.3 Duality

Recall in the proof of Lemma 2.9, we have computed the value function to the (Primal-Shortfall)

$$u(x) = \hat{u}(0,x) = 1 - N\left(N^{-1}(1-x) - \theta\sqrt{T}\right)$$

$$u'(x) = \hat{u}_x(0,x) = \frac{N'\left(N^{-1}(1-x) - \theta\sqrt{T}\right)}{N'(N^{-1}(1-x))} = e^{\theta\sqrt{T}\,N^{-1}(1-x) - \frac{1}{2}\theta^2 T}.$$

At the beginning of Chapter 2, we have explained the optimal value function to the (Dual-Shortfall), when $H \equiv 1$, is

$$v(y) = E[(1 - yZ_T)^+].$$

We would like to check the duality results we have derived in Section 1.4.

Lemma 2.10. *The optimal value functions in the case of $H \equiv 1$ satisfy the duality equality*

$$v(y) = u(x) - xy, \quad \textit{when} \quad y = u'(x).$$

Proof. Note that when $y = u'(x) = e^{\theta\sqrt{T}\,N^{-1}(1-x) - \frac{1}{2}\theta^2 T}$,

$$\begin{aligned} v(y) &= E[(1 - yZ_T)^+] \\ &= E\left[\left(1 - e^{\theta\sqrt{T}\,N^{-1}(1-x) - \frac{1}{2}\theta^2 T} e^{-\theta W_T - \frac{1}{2}\theta^2 T}\right)^+\right] \\ &= 1 - N\left(N^{-1}(1-x) - \theta\sqrt{T}\right) - xe^{\theta\sqrt{T}\,N^{-1}(1-x) - \frac{1}{2}\theta^2 T} \\ &= u(x) - xu'(x) = u(x) - xy. \end{aligned}$$

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2.3 Geometric Brownian motion with Poisson jump model

Suppose the dynamics of the discounted[3] asset price processes under $\mathbb{P}$ are

$$dS_t^1 = S_{t-}^1[\mu_1 dt + \sigma_1 dW_t - (1-\alpha_1)(dN_t - \lambda dt)],$$
$$dS_t^2 = S_{t-}^2[\mu_2 dt + \sigma_2 dW_t - (1-\alpha_2)(dN_t - \lambda dt)],$$

where N_t is a standard Poisson process with intensity λ, and W_t a standard Brownian motion. Assume the constants satisfy $\mu_i > 0, \sigma_i > 0, 0 < \alpha_i < 1, \lambda > 0$ for $i = 1, 2$. The Doléans-Dade formula gives the solutions to the above stochastic differential equations

$$S_t^1 = S_0^1 e^{\sigma_1 W_t + (\mu_1 - \frac{1}{2}\sigma_1^2 + (1-\alpha_1)\lambda)t} \alpha_1^{N_t},$$
$$S_t^2 = S_0^2 e^{\sigma_2 W_t + (\mu_2 - \frac{1}{2}\sigma_2^2 + (1-\alpha_2)\lambda)t} \alpha_2^{N_t}.$$

Note that the prices follow geometric Brownian motions when there is no Poisson jump. In the case when a jump occurs, the prices jumps down to a fraction of themselves $\alpha_i S^i$ for $i = 1, 2$.

Kirch et al. (2002) and Nakano (2004) computed the shortfall risk minimizing strategy when the loss function is $l(x) = \frac{x^p}{p}$ where $p \in (1, \infty)$. We present, in this section, the results for $l(x) = x$.

2.3.1 Optimal Strategy

Assumption 2.11. *Suppose (ψ, ν) are given as the solutions to the following equations*

$$\begin{cases} \mu_1 + \sigma_1 \psi_t - (1-\alpha_1)\lambda\nu = 0 \\ \mu_2 + \sigma_2 \psi_t - (1-\alpha_2)\lambda\nu = 0 \end{cases} \Rightarrow \begin{cases} \psi = -\frac{\mu_1(1-\alpha_2) - \mu_2(1-\alpha_1)}{(1-\alpha_2)\sigma_1 - (1-\alpha_1)\sigma_2} \\ \nu = -\frac{\mu_1\sigma_2 - \mu_2\sigma_1}{\lambda((1-\alpha_2)\sigma_1 - (1-\alpha_1)\sigma_2)}. \end{cases}$$

We assume $\psi < 0$ and $\nu > 0$ for computational reasons.[4]

Under the risk-neutral measure $\mathbb{Q}^*$, $W_t^* = W_t - \psi t$ is a standard Brownian motion, and $N_t - \lambda^* t$ is a compensated Poisson process, where $\lambda^* = (1+\nu)\lambda > \lambda$.

[3]For simplicity of notation, we start with discounted prices to get rid of the interest rate.
[4]These conditions can be relaxed and the corresponding optimal solutions can be computed.

The price processes can be written as

$$dS_t^1 = S_{t-}^1[\sigma_1 dW_t^* - (1-\alpha_1)(dN_t - \lambda^* dt)],\ S_t^1 = S_0^1 e^{\sigma_1 W_t^* - (\frac{1}{2}\sigma_1^2 - (1-\alpha_1)\lambda^*)t}\alpha_1^{N_t};$$
$$dS_t^2 = S_{t-}^2[\sigma_2 dW_t^* - (1-\alpha_2)(dN_t - \lambda^* dt)],\ S_t^2 = S_0^2 e^{\sigma_2 W_t^* - (\frac{1}{2}\sigma_2^2 - (1-\alpha_2)\lambda^*)t}\alpha_2^{N_t}.$$

The Radon-Nikodym derivatives are

$$\begin{aligned}
Z_T \triangleq \frac{d\mathbb{Q}^*}{d\mathbb{P}} &= \left(\frac{\lambda^*}{\lambda}\right)^{N_T} e^{-(\lambda^*-\lambda)T} e^{\psi W_T - \frac{1}{2}\psi^2 T} = (1+\nu)^{N_T} e^{-\lambda\nu T} e^{\psi W_T - \frac{1}{2}\psi^2 T}, \\
&= \left(\frac{\lambda^*}{\lambda}\right)^{N_T} e^{-(\lambda^*-\lambda)T} e^{\psi W_T^* + \frac{1}{2}\psi^2 T} = (1+\nu)^{N_T} e^{-\lambda\nu T} e^{\psi W_T^* + \frac{1}{2}\psi^2 T}, \\
\frac{1}{Z_T} = \frac{d\mathbb{P}}{d\mathbb{Q}^*} &= \left(\frac{\lambda}{\lambda^*}\right)^{N_T} e^{-(\lambda-\lambda^*)T} e^{-\psi W_T + \frac{1}{2}\psi^2 T} = \left(\frac{1}{1+\nu}\right)^{N_T} e^{\lambda\nu T} e^{-\psi W_T + \frac{1}{2}\psi^2 T} \\
&= \left(\frac{\lambda}{\lambda^*}\right)^{N_T} e^{-(\lambda-\lambda^*)T} e^{-\psi W_T^* - \frac{1}{2}\psi^2 T} = \left(\frac{1}{1+\nu}\right)^{N_T} e^{\lambda\nu T} e^{-\psi W_T^* - \frac{1}{2}\psi^2 T}.
\end{aligned}$$

The Radon-Nikodym derivative process satisfies the SDE

$$dZ_t = Z_{t-}[\psi dW_t + \nu(dN_t - \lambda dt)].$$

Proposition 2.12. *Suppose the initial capital x satisfies $0 < x < E^{\mathbb{Q}^*}[H]$. Then there exists a y such that*

$$x = E^{\mathbb{Q}^*}[1_{\{Z_T < \frac{1}{y}\}} H],$$

and the wealth of the optimal strategy at time $0 \le t \le T$ is

$$\hat{X}_t = E^{\mathbb{Q}^*}[\,1_{\{Z_T < \frac{1}{y}\}} H \,|\, \mathcal{F}_t\,].$$

Suppose the option payoff is a function of the underlying price process $H = H(S_T^1, S_T^2)$. Then the optimal strategy $(\hat{\Delta}_t^1, \hat{\Delta}_t^2)$ is defined by

$$\begin{cases} \hat{\Delta}_t^1 \sigma_1 S_{t-}^1 + \hat{\Delta}_t^2 \sigma_2 S_{t-}^2 = \hat{X}_{s^1}(t, S_{t-}^1, S_{t-}^2)\sigma_1 S_{t-}^1 + \hat{X}_{s^2}(t, S_{t-}^1, S_{t-}^2)\sigma_2 S_{t-}^2, \\ \hat{\Delta}_t^1 (1-\alpha_1) S_{t-}^1 + \hat{\Delta}_t^2 (1-\alpha_2) S_{t-}^2 = \hat{X}(t, S_{t-}^1, S_{t-}^2) - \hat{X}(t, \alpha_1 S_{t-}^1, \alpha_2 S_{t-}^2), \end{cases}$$

where $\hat{X}_{s^1}(t, S_{t-}^1, S_{t-}^2)$ and $\hat{X}_{s^1}(t, S_{t-}^1, S_{t-}^2)$ denote the partial derivatives, and

$$d\hat{X}(t, S_t^1, S_t^2) = \hat{\Delta}_t^1 dS_t^1 + \hat{\Delta}_t^2 dS_t^2.$$

The solution of the optimal strategy can be written as

$$\begin{cases} \hat{\Delta}_t^1 = \frac{(1-\alpha_2)\left(\hat{X}_{s^1}(t,S_{t-}^1,S_{t-}^2)\sigma_1 S_{t-}^1 + \hat{X}_{s^2}(t,S_{t-}^1,S_{t-}^2)\sigma_2 S_{t-}^2\right)}{S_{t-}^1((1-\alpha_2)\sigma_1 - (1-\alpha_1)\sigma_2)} \\ \qquad\qquad + \frac{\sigma_2\left(\hat{X}(t,\alpha_1 S_{t-}^1,\alpha_2 S_{t-}^2) - \hat{X}(t,S_{t-}^1,S_{t-}^2)\right)}{S_{t-}^1((1-\alpha_2)\sigma_1 - (1-\alpha_1)\sigma_2)}, \\ \hat{\Delta}_t^2 = \frac{(1-\alpha_1)\left(\hat{X}_{s^1}(t,S_{t-}^1,S_{t-}^2)\sigma_1 S_{t-}^1 + \hat{X}_{s^2}(t,S_{t-}^1,S_{t-}^2)\sigma_2 S_{t-}^2\right)}{S_{t-}^2((1-\alpha_1)\sigma_2 - (1-\alpha_2)\sigma_1)} \\ \qquad\qquad + \frac{\sigma_1\left(\hat{X}(t,\alpha_1 S_{t-}^1,\alpha_2 S_{t-}^2) - \hat{X}(t,S_{t-}^1,S_{t-}^2)\right)}{S_{t-}^2((1-\alpha_1)\sigma_2 - (1-\alpha_2)\sigma_1)}. \end{cases}$$

Proof. The distribution of the Radon-Nikodym derivative is diffuse ($\psi \neq 0$ by Assumption 2.11). Therefore the optimal solution to (Primal-Shortfall) is $\hat{g} = 1_{\{Z_T < \frac{1}{y}\}} H$. By theorem 1.19, we know there is a $y \in \partial u(x)$ such that $E^{\mathbb{P}}[\hat{g}\hat{h}] = xy$ where $\hat{h}$ is the optimal solution to (Dual-Shortfall). By the discussion at the beginning of Chapter 2, we know $\hat{h} = yZ_T$. Therefore there exists a y such that

$$x = E^{\mathbb{P}}[\hat{g} Z_T] = E^{\mathbb{Q}^*}[\hat{g}] = E^{\mathbb{Q}^*}[1_{\{Z_T < \frac{1}{y}\}} H].$$

Since

$$\begin{cases} S_t^1 = S_0^1 e^{\sigma_1 W_t^* - (\frac{1}{2}\sigma_1^2 - (1-\alpha_1)\lambda^*)t} \alpha_1^{N_t} \\ S_t^2 = S_0^2 e^{\sigma_2 W_t^* - (\frac{1}{2}\sigma_2^2 - (1-\alpha_2)\lambda^*)t} \alpha_2^{N_t} \end{cases}$$

$$\Rightarrow \begin{cases} W_t^* = \frac{\ln \alpha_2 \left(\ln S_t^1 - \ln S_0^1 + (\frac{1}{2}\sigma_1^2 - (1-\alpha_1)\lambda^*)t\right) - \ln \alpha_1 \left(\ln S_t^2 - \ln S_0^2 + (\frac{1}{2}\sigma_2^2 - (1-\alpha_2)\lambda^*)t\right)}{\sigma_1 \ln \alpha_2 - \sigma_2 \ln \alpha_1} \\ N_t = \frac{\sigma_2 \left(\ln S_t^1 - \ln S_0^1 + (\frac{1}{2}\sigma_1^2 - (1-\alpha_1)\lambda^*)t\right) - \sigma_1 \left(\ln S_t^2 - \ln S_0^2 + (\frac{1}{2}\sigma_2^2 - (1-\alpha_2)\lambda^*)t\right)}{\sigma_2 \ln \alpha_1 - \sigma_1 \ln \alpha_2}, \end{cases}$$

$$Z_t = (1+\nu)^{N_t} e^{-\lambda \nu t} e^{\psi W_t^* + \frac{1}{2}\psi^2 t} = e^{\psi W_t^* + \ln(1+\nu) N_t + (\frac{1}{2}\psi^2 - \lambda\nu)t},$$

we can see that Z_t is a function of (t, S_t^1, S_t^2). By the assumption $H = H(S_T^1, S_T^2)$, we know the process $\hat{X}$ possess the Markovian property and can be written as a function of (t, S_t^1, S_t^2). Since we have a complete market, the optimal value function can be replicated by trading in the stocks,

$$\begin{aligned} d\hat{X}(t, S_t^1, S_t^2) &= \hat{\Delta}_t^1 dS_t^1 + \hat{\Delta}_t^2 dS_t^2 \\ &= (\hat{\Delta}_t^1 \sigma_1 S_{t-}^1 + \hat{\Delta}_t^2 \sigma_2 S_{t-}^2) dW_t^* \\ &\quad - (\hat{\Delta}_t^1 (1-\alpha_1) S_{t-}^1 + \hat{\Delta}_t^2 (1-\alpha_2) S_{t-}^2) d(N_t - \lambda^* t). \end{aligned}$$

On the other side, Itô-Doeblin formula and the martingale property of $\hat{X}_t$ give

$$\begin{aligned} d\hat{X}(t, S_t^1, S_t^2) &= \hat{X}_t(t, S_{t-}^1, S_{t-}^2)dt + \hat{X}_{s^1}(t, S_{t-}^1, S_{t-}^2) dS_t^1 + \hat{X}_{s^2}(t, S_{t-}^1, S_{t-}^2) dS_t^2 \\ &\quad + \tfrac{1}{2}\hat{X}_{s^1 s^1}(t, S_{t-}^1, S_{t-}^2) d\langle S^{1c}\rangle_t + \hat{X}_{s^1 s^2}(t, S_{t-}^1, S_{t-}^2) d\langle S^{1c}, S^{2c}\rangle_t \\ &\quad + \tfrac{1}{2}\hat{X}_{s^2 s^2}(t, S_{t-}^1, S_{t-}^2) d\langle S^{2c}\rangle_t \\ &\quad + \left(\hat{X}(t, \alpha_1 S_t^1, \alpha_2 S_t^2) - \hat{X}(t, S_{t-}^1, S_{t-}^2)\right) dN_t \\ &\quad - \hat{X}_{s^1}(t, S_{t-}^1, S_{t-}^2)\Delta S_t^1 - \hat{X}_{s^2}(t, S_{t-}^1, S_{t-}^2)\Delta S_t^2 \\ &= \left(\hat{X}_{s^1}(t, S_{t-}^1, S_{t-}^2)\sigma_1 S_{t-}^1 + \hat{X}_{s^2}(t, S_{t-}^1, S_{t-}^2)\sigma_2 S_{t-}^2\right) dW_t^* \\ &\quad + \left(\hat{X}(t, \alpha_1 S_{t-}^1, \alpha_2 S_{t-}^2) - \hat{X}(t, S_{t-}^1, S_{t-}^2)\right)(dN_t - \lambda^* dt). \end{aligned}$$

Therefore, the optimal strategy $(\hat{\Delta}_t^1, \hat{\Delta}_t^2)$ satisfy the given equations. ◇

Remark 2.13. *If*

$$\frac{1-\alpha_1}{\sigma_1} \equiv \frac{\hat{X}(t, S^1_{t-}, S^2_{t-}) - \hat{X}(t, \alpha_1 S^1_{t-}, \alpha_2 S^2_{t-})}{\hat{X}_{s^1}(t, S^1_{t-}, S^2_{t-})\sigma_1 S^1_{t-} + \hat{X}_{s^2}(t, S^1_{t-}, S^2_{t-})\sigma_2 S^2_{t-}},$$

then

$$\begin{cases} \hat{\Delta}^1_t = \frac{\hat{X}_{s^1}(t, S^1_{t-}, S^2_{t-})\sigma_1 S^1_{t-} + \hat{X}_{s^2}(t, S^1_{t-}, S^2_{t-})\sigma_2 S^2_{t-}}{\sigma_1 S^1_{t-}}; \\ \hat{\Delta}^2_t \equiv 0. \end{cases}$$

In this case, the first stock has the same ratio of quadratic variation between jumps and the continuous variation part as the option does, and is the only instrument necessary for hedging. However, the price movement of the second stock still affects the hedging ratio.

Define

$$\begin{aligned} x_k(T) &= P^*(N_T = k) \\ &= P^*\left(Z_T = \left(\tfrac{\lambda^*}{\lambda}\right)^k e^{-(\lambda^*-\lambda)T} e^{\psi W^*_T + \frac{1}{2}\psi^2 T}\right) \\ &= e^{-\lambda^* T}\tfrac{(\lambda^* T)^k}{k!}, \quad \text{for} \quad n = 0, 1, 2, ... \end{aligned}$$

Corollary 2.14. *In the simplest case when the option payoff is a constant, which we may assume without loss of generality is $H \equiv 1$, suppose the initial capital x satisfies $0 < x < 1$. Let y be the solution to*

$$x = \sum_{k=0}^{\infty} x_k(T)[1 - N(d_k(y, T))],$$

where $\tau = T - t$ is the time to maturity, $N(\cdot)$ is the c.d.f. of a standard normal distribution, and

$$d_k(y, \tau) = \frac{\ln y + k\ln(1+\nu) + (\frac{1}{2}\psi^2 - \lambda\nu)\tau}{-\psi\sqrt{\tau}}.$$

The wealth of the optimal strategy at time $0 \le t \le T$ is

$$\hat{X}_t = \sum_{k=0}^{\infty} x_k(\tau)[1 - N(d_k(yZ_t, \tau))].$$

The optimal strategy $(\hat{\Delta}^1_t, \hat{\Delta}^2_t)$ can be explicitly computed from the formulae in Proposition 2.12 with the equations

$$\hat{X}_{s^1}(t, S^1_{t-}, S^2_{t-})\sigma_1 S^1_{t-} + \hat{X}_{s^2}(t, S^1_{t-}, S^2_{t-})\sigma_2 S^2_{t-} = \sum_{k=0}^{\infty} x_k(\tau) N'(d_k(yZ_{t-}, \tau))\frac{1}{\sqrt{\tau}};$$

$$
\begin{aligned}
&\hat{X}(t, \alpha_1 S^1_{t-}, \alpha_2 S^2_{t-}) - \hat{X}_t(t, S^1_{t-}, S^2_{t-}) \\
&\quad = -\sum_{k=0}^{\infty} x_k(\tau) N(d_{k+1}(yZ_{t-}, \tau)) + \sum_{k=0}^{\infty} x_k(\tau) N(d_k(yZ_{t-}, \tau)) \\
&\quad = \sum_{k=1}^{\infty} \left(1 - \frac{k}{\lambda^* T}\right) x_k(\tau) N(d_k(yZ_{t-}, \tau)) + x_0(\tau) N(d_0(yZ_{t-}, \tau)).
\end{aligned}
$$

Proof. The existence of y follows easily from the fact $0 < x < 1$. The value function of the optimal strategy is the expected value of (discounted) payoff in a complete market,

$$
\begin{aligned}
\hat{X}(t, S^1_t, S^2_t) &= E^{\mathbb{Q}^*}[\, 1_{\{Z_T < \frac{1}{y}\}} \,|\, \mathcal{F}_t \,] \\
&= E^{\mathbb{Q}^*}[\, 1_{\{Z_\tau < \frac{1}{yZ_t}\}} \,|\, \mathcal{F}_t \,] \\
&= \sum_{k=0}^{\infty} x_k(\tau)[1 - N(d_k(yZ_t, \tau))].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\hat{X}_{s^1}(t, S^1_{t-}, S^2_{t-}) \\
&\quad = \sum_{k=0}^{\infty} x_k(\tau)\, [-N'(d_k(yZ_{t-}, \tau))]\, \frac{1}{-\psi\sqrt{\tau} Z_{t-}} \\
&\qquad \cdot Z_{t-} \left(\psi \frac{\ln \alpha_2}{\sigma_1 \ln \alpha_2 - \sigma_2 \ln \alpha_1} \frac{1}{S^1_{t-}} - \ln(1+\nu) \frac{\sigma_2}{\sigma_1 \ln \alpha_2 - \sigma_2 \ln \alpha_1} \frac{1}{S^1_{t-}}\right) \\
&\quad = \frac{1}{S^1_{t-}} \left(\frac{\psi \ln \alpha_2 - \sigma_2 \ln(1+\nu)}{\sigma_1 \ln \alpha_2 - \sigma_2 \ln \alpha_1}\right) \sum_{k=0}^{\infty} x_k(\tau) N'(d_k(yZ_{t-}, \tau)) \frac{1}{\psi\sqrt{\tau}},
\end{aligned}
$$

$$
\begin{aligned}
&\hat{X}_{s^2}(t, S^1_{t-}, S^2_{t-}) \\
&\quad = \sum_{k=0}^{\infty} x_k(\tau)\, [-N'(d_k(yZ_{t-}, \tau))]\, \frac{1}{-\psi\sqrt{\tau} Z_{t-}} \\
&\qquad Z_{t-} \left(\psi \frac{-\ln \alpha_1}{\sigma_1 \ln \alpha_2 - \sigma_2 \ln \alpha_1} \frac{1}{S^2_{t-}} - \ln(1+\nu) \frac{-\sigma_1}{\sigma_1 \ln \alpha_2 - \sigma_2 \ln \alpha_1} \frac{1}{S^2_{t-}}\right) \\
&\quad = \frac{1}{S^2_{t-}} \left(\frac{-\psi \ln \alpha_1 + \sigma_1 \ln(1+\nu)}{\sigma_1 \ln \alpha_2 - \sigma_2 \ln \alpha_1}\right) \sum_{k=0}^{\infty} x_k(\tau) N'(d_k(yZ_{t-}, \tau)) \frac{1}{\psi\sqrt{\tau}}.
\end{aligned}
$$

We get

$$
\hat{X}_{s^1}(t, S^1_{t-}, S^2_{t-}) \sigma_1 S^1_{t-} + \hat{X}_{s^2}(t, S^1_{t-}, S^2_{t-}) \sigma_2 S^2_{t-} = \sum_{k=0}^{\infty} x_k(\tau) N'(d_k(yZ_{t-}, \tau)) \frac{1}{\sqrt{\tau}}.
$$

Since

$$Z(t, \alpha_1 S^1_{t-}, \alpha_2 S^2_{t-}) = Z(t, S^1_{t-}, S^2_{t-})(1+\nu),$$

$$\begin{aligned}
\hat{X}(t,&\alpha_1 S^1_{t-}, \alpha_2 S^2_{t-}) - \hat{X}_t(t, S^1_{t-}, S^2_{t-}) \\
&= \sum_{k=0}^{\infty} x_k(\tau)[1 - N(d_k(yZ_{t-}(1+\nu), \tau))] - \sum_{k=0}^{\infty} x_k(\tau)[1 - N(d_k(yZ_{t-}, \tau))] \\
&= -\sum_{k=0}^{\infty} x_k(\tau) N(d_k(Z_{t-}(1+\nu), \tau)) + \sum_{k=0}^{\infty} x_k(\tau) N(d_k(yZ_{t-}, \tau)) \\
&= -\sum_{k=0}^{\infty} x_k(\tau) N(d_{k+1}(yZ_{t-}, \tau)) + \sum_{k=0}^{\infty} x_k(\tau) N(d_k(yZ_{t-}, \tau)) \\
&= -\sum_{k=1}^{\infty} \frac{k}{\lambda^* T} x_k(\tau) N(d_k(yZ_{t-}, \tau)) + \sum_{k=0}^{\infty} x_k(\tau) N(d_k(yZ_{t-}, \tau)) \\
&= \sum_{k=1}^{\infty} \left(1 - \frac{k}{\lambda^* T}\right) x_k(\tau) N(d_k(yZ_{t-}, \tau)) + x_0(\tau) N(d_0(yZ_{t-}, \tau))
\end{aligned}$$

◇

2.3.2 HJB Approach

Another standard approach to solve the optimal control problem is to check the HJB eqaution. Rewrite the self-financing strategy as

$$\begin{aligned}
dX_t &= \hat{\Delta}^1_t dS^1_t + \hat{\Delta}^2_t dS^2_t \\
&= \hat{\Delta}^1_t S^1_{t-}[\mu_1 dt + \sigma_1 dW_t - (1-\alpha_1) dM_t] \\
&\quad + \hat{\Delta}^2_t S^2_{t-}[\mu_2 dt + \sigma_2 dW_t - (1-\alpha_2) dM_t] \\
&= \pi^1_t[\mu_1 dt + \sigma_1 dW_t - (1-\alpha_1) dM_t] + \pi^2_t[\mu_2 dt + \sigma_2 dW_t - (1-\alpha_2) dM_t],
\end{aligned}$$

where $dM_t = dN_t - \lambda dt$ is the compensated Poisson process under $\mathbb{P}$ and $\pi^i_t = \hat{\Delta}^i_t S_t$, $i = 1, 2$ are the monetary amounts invested in each underlying. In the simplest case when the payoff function is a constant $H \equiv 1$, our optimal control problem can be written as

$$\max_{\pi^1_t, \pi^2_t} E[X_T \wedge 1],$$

where $dX_t = \pi^1_t[\mu_1 dt + \sigma_1 dW_t - (1-\alpha_1) dM_t] + \pi^2_t[\mu_2 dt + \sigma_2 dW_t - (1-\alpha_2) dM_t]$,

$$X_t \geq 0, \quad \text{for all} \quad 0 \leq t \leq T.$$

Define the value process

$$u(t, \tilde{X}_t) = \max_{\pi^1_t, \pi^2_t} E[X_T \wedge 1 | \mathcal{F}_t],$$

where $\tilde{X}_t \in \mathcal{X}(x)$ is the optimal solution. Notice that $\tilde{X}_t$ possesses the Markovian property. Suppose $u_t(t,x)$ and $u_x(t,x)$ exist. Using the usual verification lemma argument, we derive the HJB equation:

$$\max_{\text{admissible}\,\pi_t^1,\pi_t^2} u_t(t,x) + [\pi_t^1(\mu_1 + (1-\alpha_1)\lambda) + \pi_t^2(\mu_2 + (1-\alpha_2)\lambda)]u_x(t,x)$$
$$+ \tfrac{1}{2}(\pi_t^1\sigma_1 + \pi_t^2\sigma_2)^2 u_{xx}(t,x) + \lambda[u(t, x - (1-\alpha_1)\pi_t^1 - (1-\alpha_2)\pi_t^2) - u(t,x)] = 0,$$

with the boundary condition $u(T,x) = x \wedge 1$. Define

$$\hat{u}(t,\hat{X}_t) = E\,[\hat{X}_T \wedge 1|\mathcal{F}_t],$$

where $\hat{X}_t$ is the wealth process defined in Corollary 2.14 associated with the strategies $\hat{\pi}_t^1$ and $\hat{\pi}_t^2$. We will check their optimality in the following lemma.

Lemma 2.15. *The value function $\hat{u}(t,x)$ satisfies*

$$\hat{u}_t(t,x) + [\hat{\pi}_t^1(\mu_1 + (1-\alpha_1)\lambda) + \hat{\pi}_t^2(\mu_2 + (1-\alpha_2)\lambda)]\hat{u}_x(t,x)$$
$$+ \tfrac{1}{2}(\hat{\pi}_t^1\sigma_1 + \hat{\pi}_t^2\sigma_2)^2 \hat{u}_{xx}(t,x) + \lambda[\hat{u}(t, x - (1-\alpha_1)\hat{\pi}_t^1 - (1-\alpha_2)\hat{\pi}_t^2) - \hat{u}(t,x)] = 0,$$

with the boundary condition $\hat{u}(T,x) = x \wedge 1$.

Proof.Define

$$y_k(T) = \mathbb{P}(N_T = k) = e^{-\lambda T}\tfrac{(\lambda T)^k}{k!}, \quad \text{for} \quad k = 0, 1, 2...$$

The value function derived from $\hat{X}_t$ in Corollary 2.14 can be computed as follows:

$$\begin{aligned}\hat{u}(t,\hat{X}_t) &= E[\hat{X}_T \wedge 1|\mathcal{F}_t] \\ &= \mathbb{P}\left(\frac{Z_T}{Z_t} > yZ_t\right) \\ &= \sum_{k=0}^{\infty} y_k(\tau)[1 - N(d_k(yZ_t,\tau) + \psi\sqrt{\tau})].\end{aligned}$$

Recall from Corollary 2.14 that $\hat{X}_t$ and Z_t are related by

$$\hat{X}(t,Z_t) = \sum_{k=0}^{\infty} x_k(\tau)[1 - N(d_k(yZ_t,\tau))].$$

We can see that both $\hat{u}$ and $\hat{X}$ are functions of Z. Therefore,

$$\hat{u}(t,x) = \sum_{k=0}^{\infty} y_k(\tau)[1 - N(d_k(yz,\tau) + \psi\sqrt{\tau})],$$

where $\quad x = \sum_{k=0}^{\infty} x_k(\tau)[1 - N(d_k(yz,\tau))],$

$$\begin{aligned}
\hat{u}_x(t,x) = \frac{\hat{u}_z}{x_z} &= \frac{\sum_{k=0}^{\infty} y_k(\tau)[-N'(d_k(yz,\tau) + \psi\sqrt{\tau})]\frac{1}{-\psi\sqrt{\tau}z}}{\sum_{k=0}^{\infty} x_k(\tau)[-N'(d_k(yz,\tau))]\frac{1}{-\psi\sqrt{\tau}z}} \\
&= \frac{\sum_{k=0}^{\infty} y_k(\tau)N'(d_k(yz,\tau) + \psi\sqrt{\tau})}{\sum_{k=0}^{\infty} x_k(\tau)N'(d_k(yz,\tau))},
\end{aligned}$$

$$\begin{aligned}
\hat{u}(t, x - (1-\alpha_1)\pi_t^1 - (1-\alpha_2)\pi_t^2) &= \sum_{k=0}^{\infty} y_k(\tau)[1 - N(d_k(yz\frac{1}{1+\nu},\tau) + \psi\sqrt{\tau})] \\
&= \sum_{k=0}^{\infty} y_k(\tau)[1 - N(d_{k+1}(yz,\tau) + \psi\sqrt{\tau})],
\end{aligned}$$

where $x - (1-\alpha_1)\pi_t^1 - (1-\alpha_2)\pi_t^2 = \sum_{k=0}^{\infty} x_k(\tau)[1 - N(d_k(yz\frac{1}{1+\nu},\tau))]$

$$= \sum_{k=0}^{\infty} x_k(\tau)[1 - N(d_{k+1}(yz,\tau))],$$

$$\begin{aligned}
\hat{u}_x(t, x - (1-\alpha_1)\pi_t^1 - (1-\alpha_2)\pi_t^2) &= \frac{\sum_{k=0}^{\infty} y_k(\tau)N'(d_k(yz\frac{1}{1+\nu},\tau) + \psi\sqrt{\tau})}{\sum_{k=0}^{\infty} x_k(\tau)N'(d_k(yz\frac{1}{1+\nu},\tau))} \\
&= \frac{\sum_{k=0}^{\infty} y_k(\tau)N'(d_{k+1}(yz,\tau) + \psi\sqrt{\tau})}{\sum_{k=0}^{\infty} x_k(\tau)N'(d_{k+1}(yz,\tau))},
\end{aligned}$$

$$\begin{aligned}
\hat{u}_{xx}(t,x) = \tfrac{\hat{u}_{xz}(t,x)}{x_z} &= \frac{1}{\left(\sum_{k=0}^{\infty} x_k(\tau)N'(d_k(yz,\tau))\right)^2}\frac{1}{\sum_{k=0}^{\infty} x_k(\tau)[-N'(d_k(yz,\tau))]\frac{1}{-\psi\sqrt{\tau}z}} \\
&\cdot\left\{\sum_{k=0}^{\infty} y_k(\tau)N'(d_k(yz,\tau) + \psi\sqrt{\tau})(-d_k(yz,\tau) - \psi\sqrt{\tau})\tfrac{1}{-\psi\sqrt{\tau}z}\sum_{k=0}^{\infty} x_k(\tau)N'(d_k(yz,\tau))\right. \\
&\left. - \sum_{k=0}^{\infty} y_k(\tau)N'(d_k(yz,\tau) + \psi\sqrt{\tau})\sum_{k=0}^{\infty} x_k(\tau)N'(d_k(yz,\tau))(-d_k(yz,\tau))\tfrac{1}{-\psi\sqrt{\tau}z}\right\} \\
&= \frac{1}{\left(\sum_{k=0}^{\infty} x_k(\tau)N'(d_k(yz,\tau))\right)^3} \\
&\cdot\left\{\sum_{k=0}^{\infty} y_k(\tau)N'(d_k(yz,\tau) + \psi\sqrt{\tau})(d_k(yz,\tau) + \psi\sqrt{\tau})\sum_{k=0}^{\infty} x_k(\tau)N'(d_k(yz,\tau))\right. \\
&\left. - \sum_{k=0}^{\infty} y_k(\tau)N'(d_k(yz,\tau) + \psi\sqrt{\tau})\sum_{k=0}^{\infty} x_k(\tau)N'(d_k(yz,\tau))d_k(yz,\tau))\right\}.
\end{aligned}$$

To find $\hat{u}_t(t,x)$ we need

$$\frac{dx_k(\tau)}{dt} = \lambda^*(x_k(\tau) - x_{k-1}(\tau)), \quad \frac{dy_k(\tau)}{dt} = \lambda(y_k(\tau) - y_{k-1}(\tau)), \text{ for } k = 0, 1, 2...,$$

where we define $x_{-1}(\tau) = y_{-1}(\tau) = 0$, and

$$\begin{aligned} \frac{\partial\, d_k(yz,\tau)}{\partial t} &= -\frac{(\frac{1}{2}\psi^2 - \lambda\nu)(-\psi\sqrt{\tau}) - (\ln yz + k\ln(1+\nu) + (\frac{1}{2}\psi^2 - \lambda\nu)\tau)\frac{-\psi}{2\sqrt{\tau}}}{\psi^2\tau} \\ &= \frac{d_k(yz,\tau)}{2\tau} - \frac{\frac{1}{2}\psi^2 - \lambda\nu}{-\psi\sqrt{\tau}}. \end{aligned}$$

Then

$$\begin{aligned} \hat{u}_t(t,x) &= \hat{u}_t + \hat{u}_z \cdot z_t = \hat{u}_t + \hat{u}_z \cdot -\frac{x_t}{x_z} = \hat{u}_t - \hat{u}_x \cdot x_t \\ &= \sum_{k=0}^{\infty} \lambda(y_k(\tau) - y_{k-1}(\tau))[1 - N(d_k(yz,\tau) + \psi\sqrt{\tau})] \\ &\quad + \sum_{k=0}^{\infty} y_k(\tau)[-N'(d_k(yz,\tau) + \psi\sqrt{\tau})] \left(\frac{d_k(yz,\tau)}{2\tau} - \frac{\frac{1}{2}\psi^2 - \lambda\nu}{-\psi\sqrt{\tau}} - \frac{\psi}{2\sqrt{\tau}} \right) \\ &\quad - \frac{\sum_{k=0}^{\infty} y_k(\tau) N'(d_k(yz,\tau) + \psi\sqrt{\tau})}{\sum_{k=0}^{\infty} x_k(\tau) N'(d_k(yz,\tau))} \\ &\quad \cdot \left\{ \sum_{k=0}^{\infty} \lambda^*(x_k(\tau) - x_{k-1}(\tau))[1 - N(d_k(yz,\tau))] \right. \\ &\quad \left. + \sum_{k=0}^{\infty} x_k(\tau)[-N'(d_k(yz,\tau))] \left(\frac{d_k(yz,\tau)}{2\tau} - \frac{\frac{1}{2}\psi^2 - \lambda\nu}{-\psi\sqrt{\tau}} \right) \right\} \\ &= -\lambda \sum_{k=0}^{\infty} y_k(\tau) N(d_k(yz,\tau) + \psi\sqrt{\tau}) + \lambda \sum_{k=0}^{\infty} y_k(\tau) N(d_{k+1}(yz,\tau) + \psi\sqrt{\tau}) \\ &\quad - \sum_{k=0}^{\infty} y_k(\tau) N'(d_k(yz,\tau) + \psi\sqrt{\tau}) \left(\frac{d_k(yz,\tau)}{2\tau} \right) \\ &\quad + \frac{\sum_{k=0}^{\infty} y_k(\tau) N'(d_k(yz,\tau) + \psi\sqrt{\tau})}{\sum_{k=0}^{\infty} x_k(\tau) N'(d_k(yz,\tau))} \cdot \left\{ \lambda^* \sum_{k=0}^{\infty} x_k(\tau) N(d_k(yz,\tau)) \right. \\ &\quad - \lambda^* \sum_{k=0}^{\infty} x_k(\tau) N(d_{k+1}(yz,\tau)) \\ &\quad \left. + \sum_{k=0}^{\infty} x_k(\tau) N'(d_k(yz,\tau)) \left(\frac{d_k(yz,\tau)}{2\tau} + \frac{\psi}{2\sqrt{\tau}} \right) \right\}. \end{aligned}$$

Recall

$$
\begin{aligned}
\hat{\pi}^1(t,x) &= \frac{1}{(1-\alpha_2)\sigma_1-(1-\alpha_1)\sigma_2}\left\{\sum_{k=0}^{\infty}x_k(\tau)N'(d_k(yz,\tau))\frac{(1-\alpha_2)}{\sqrt{\tau}}\right.\\
&\quad\left.+\sigma_2\left(-\sum_{k=0}^{\infty}x_k(\tau)N(d_{k+1}(yz,\tau))+\sum_{k=0}^{\infty}x_k(\tau)N(d_k(yz,\tau))\right)\right\},\\
\hat{\pi}^2(t,x) &= -\frac{1}{(1-\alpha_2)\sigma_1-(1-\alpha_1)\sigma_2}\left\{\sum_{k=0}^{\infty}x_k(\tau)N'(d_k(yz,\tau))\frac{(1-\alpha_1)}{\sqrt{\tau}}\right.\\
&\quad\left.+\sigma_1\left(-\sum_{k=0}^{\infty}x_k(\tau)N(d_{k+1}(yz,\tau))+\sum_{k=0}^{\infty}x_k(\tau)N(d_k(yz,\tau))\right)\right\}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\hat{\pi}^1(t,x)\sigma_1+\hat{\pi}^2(t,x)\sigma_2=\sum_{k=0}^{\infty}x_k(\tau)N'(d_k(yz,\tau))\frac{1}{\sqrt{\tau}},\\
&\hat{\pi}^1(t,x)(\mu_1+(1-\alpha_1)\lambda)+\hat{\pi}^2(t,x)(\mu_2+(1-\alpha_2)\lambda)\\
&\qquad=-\frac{\psi}{\sqrt{\tau}}\sum_{k=0}^{\infty}x_k(\tau)N'(d_k(yz,\tau))\\
&\qquad\quad+\lambda^*\left(\sum_{k=0}^{\infty}x_k(\tau)N(d_{k+1}(yz,\tau))-\sum_{k=0}^{\infty}x_k(\tau)N(d_k(yz,\tau))\right).
\end{aligned}
$$

It is routine to check the value function $\hat{u}$ and strategy $\hat{\pi}_t$ satisfy the HJB equality

$$
\begin{aligned}
&\hat{u}_t(t,x)+[\hat{\pi}_t^1(\mu_1+(1-\alpha_1)\lambda)+\hat{\pi}_t^2(\mu_2+(1-\alpha_2)\lambda)]\hat{u}_x(t,x)\\
&+\tfrac{1}{2}(\hat{\pi}_t^1\sigma_1+\hat{\pi}_t^2\sigma_2)^2\hat{u}_{xx}(t,x)+\lambda[\hat{u}(t,x-(1-\alpha_1)\hat{\pi}_t^1-(1-\alpha_2)\hat{\pi}_t^2)-\hat{u}(t,x)]=0.
\end{aligned}
$$

◇

2.3.3 Duality

Recall in the proof of Lemma 2.15, we have computed the value function to the (Primal-Shortfall)

$$
u(x)=\hat{u}(0,x)=\sum_{k=0}^{\infty}y_k(T)[1-N(d_k(y,T)+\psi\sqrt{T})],
$$

where $\quad x=\sum_{k=0}^{\infty}x_k(T)[1-N(d_k(y,T))],\quad$ and

$$
u'(x)=\hat{u}_x(0,x)=\frac{\sum_{k=0}^{\infty}y_k(T)N'(d_k(y,T)+\psi\sqrt{T})}{\sum_{k=0}^{\infty}x_k(T)N'(d_k(y,T))}.
$$

At the beginning of Chapter 2, we have shown that the optimal value function to the (Dual-Shortfall) is

$$v(y) = E[(1 - yZ_T)^+],$$

when $H \equiv 1$. We would once again like to check the duality results we have derived in Section 1.4.

Lemma 2.16. *The optimal value functions in the case of $H \equiv 1$ satisfy the duality equality*

$$v(y) = u(x) - xy, \quad \textit{when} \quad y = u'(x).$$

PROOF. Note that when $y = u'(x)$,

$$\begin{aligned}
E[(1 - yZ_T)^+] &= E\left[\left(1 - y\,(1+\nu)^{N_T}\,e^{-\lambda\nu T}e^{\psi W_T - \frac{1}{2}\psi^2 T}\right)^+\right] \\
&= E\left[E\left[\left(1 - y\,(1+\nu)^{k}\,e^{-\lambda\nu T}e^{\psi W_T - \frac{1}{2}\psi^2 T}\right)\right.\right. \\
&\quad \left.\left. 1_{\{\frac{W_T}{\sqrt{T}} > d_k(y,T) + \psi\sqrt{T}\}} \,\Big|\, N_T = k\right]\right] \\
&= \sum_{k=0}^{\infty} y_k(T)[1 - N(d_k(y,T) + \psi\sqrt{T})] \\
&\quad - y\sum_{k=0}^{\infty} x_k(\tau)[1 - N(d_k(y,T))] \\
&= u(x) - xu'(x) \\
&= u(x) - xy.
\end{aligned}$$

◇

Chapter 3

An Incomplete Market Model

3.1 Set up of the mixed diffusion model

Suppose the dynamics of the discounted[1] asset price process under $\mathbb{P}$ are

$$dS_t = S_{t-}\left[\,\mu_t dt + \sigma_t dW_t - (1-\alpha_t)dM_t\,\right], \tag{1.1}$$

where $M_t = N_t - \int_0^t \lambda_s ds$ is a compensated Poisson process, and W_t a standard Brownian motion. Assume $\mu_t > 0, \sigma_t > 0, 0 < \alpha_t < 1, \lambda_t > 0$ to be predictable processes. The Doléans-Dade formula gives the solution to the above stochastic differential equation

$$S_t = S(0)e^{\int_0^t \ln\alpha_s dN_s + \int_0^t \sigma_s dW_s + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2 + (1-\alpha_s)\lambda_s)ds}.$$

Note that the price follows an Itô process when there is no Poisson jump. In the case when a jump occurs, the price jumps down to a fraction of itself αS.

Remark 3.1. *Since we have two sources of uncertainty and only one price process for hedging, the market is incomplete.*

Proposition 3.1 in Bellamy and Jeanblanc (2000) uses the martingale representation theorem to characterize the Radon-Nikodym derivatives of the local

[1]For simplicity of notation, we start with discounted prices to get rid of the interest rate.

martingale measures with respect to $\mathbb{P}$

$$\mathcal{M} \triangleq \Big\{ \mathbb{Q} \,:\, Z_t = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\Big|\, \mathcal{F}_t\right] = \exp\left[\int_0^t \psi_s dW_s - \tfrac{1}{2}\int_0^t \psi_s^2 ds \right.$$
$$\left. + \int_0^t \ln(1+\nu_s) dN_s - \lambda_t \int_0^t \nu_s ds\right] \text{ is a martingale,}$$
$$\text{and the predictable processes } \psi_t \text{ and } \nu_t \text{ satisfy}$$
$$\mu_t + \sigma_t \psi_t - \lambda_t (1-\alpha_t)\nu_t = 0 \quad \text{and} \quad \nu_t > -1 \quad d\mathbb{P}\otimes dt \ a.s. \Big\}.$$

Obviously, the Radon-Nikodym derivative satisfies the SDE

$$dZ_t = Z_{t-}\,[\,\psi_t dW_t + \nu_t dM_t\,]\,.$$

3.2 Characterization of primal and dual sets

Recall the set of admissible self-financing portfolios starting at x defined by (1.1) in Chapter 1 is

$$\mathcal{X}(x) = \left\{ X \,\middle|\, X_t = x + \int_0^t \xi_s dS_s \ \geq 0 \ \ \mathbb{P} - a.s., \text{ for } \ \ 0 \leq t \leq T \right\},$$

where ξ is a predictable process. The dual set defined by (1.3) in Chapter 1 is

$$\mathcal{Y}(y) = \{\, Y \geq 0 \mid Y_0 = y \text{ and } XY \text{ is a } \mathbb{P}\text{-supermartingale for any } X \in \mathcal{X}(1)\,\}\,.$$

We will derive some general results characterizing these sets in the setting of chapter 1 based on the assumption of the price process dynamics (1.1) which are independent of the problem of minimizing shortfall risk raised in Section 1.4.

Lemma 3.2. *Assume (1.1).* $X_t = x + \int_0^t \xi_s dS_s$ *is an element of* $\mathcal{X}(x)$ *if and only if*

$$\xi_t \leq \tfrac{X_{t-}}{(1-\alpha_t)S_{t-}} \quad \textit{on the set} \quad \{X_{t-} > 0\}, \quad \textit{and}$$
$$\xi_t = 0 \qquad \textit{on the set} \quad \{X_{t-} \leq 0\}.$$

Proof. When $X_{t-} > 0$ and $\triangle N_t = 1$,

$$X_t = X_{t-} + \triangle X_t \geq 0 \quad \Leftrightarrow \quad \triangle X_t \geq -X_{t-} \quad \Leftrightarrow$$
$$-(1-\alpha_t)\xi_t S_{t-} \geq -X_{t-} \quad \Leftrightarrow \quad \xi_t \leq \tfrac{X_{t-}}{(1-\alpha_t)S_{t-}}.$$

◇

In this particular model, we can characterize the dual set $\mathcal{Y}(y)$ by the following lemma and proposition.

Lemma 3.3. *Assume (1.1).* $Y_t \in \mathcal{Y}(y)$ *if and only if* $Y_t \geq 0$ *has the decomposition*

$$Y_t = Z_t - A_t,$$

where Z_t *is a local martingale under* $\mathbb{P}$ *starting at* $Z_0 = y$ *with the representation*

$$dZ_t = Z_{t-}\left[\,\psi_t dW_t + \nu_t dM_t\,\right],$$

where ψ_t *and* ν_t *are predictable processes, and* A_t *is a predictable and increasing process with* $A_0 = 0$ *which can be decomposed into the part that is absolutely continuous with respect to the Lebesgue measure, the singularly continuous part and the jump part, i.e.,*

$$dA_t = a_t dt + dA_t^s + \Delta A_t,$$

so that the following equation holds true

$$0 \leq \theta_t \triangleq \mu_t Y_{t-} + Z_{t-}\left\{\,\sigma_t\psi_t - \lambda_t(1-\alpha_t)\nu_t\,\right\} \leq (1-\alpha_t)a_t, \quad d\mathbb{P}\otimes dt - a.s. \tag{2.2}$$

Proof. Let $Y \in \mathcal{Y}(y)$. Since $1 \in \mathcal{X}(1)$, Y is a supermartingle. Using the Doob-Meyer decomposition we can write

$$Y_t = Z_t - A_t,$$

where Z is a local martingale with $Z_0 = y$ and A is a predictable and increasing process with $A_0 = 0$. The martingale representation theorem gives

$$dZ_t = Z_{t-}\left[\,\psi_t dW_t + \nu_t dM_t\,\right],$$

where ψ_t and ν_t are predictable processes. Obviously, A_t can also be decomposed into three parts

$$dA_t = a_t dt + dA_t^s + \Delta A_t,$$

where the first part is absolutely continuous with respect to the Lebesgue measure, the second is the singularly continuous part, and the last is the pure jump part. Notice that since A is predictable, N and A do not jump at the same time

$\mathbb{P}-a.s.$. Therefore for any $X \in \mathcal{X}(1)$ and $Y \in \mathcal{Y}(y)$, Ito's lemma gives

$$\begin{aligned}
X_tY_t &= \int_0^t X_{s-}dY_s^c + \int_0^t Y_{s-}dX_s^c + \int_0^t d\langle X^c, Y^c\rangle_s + \sum_{0<s\le t}(X_sY_s - X_{s-}Y_{s-}) \\
&= \int_0^t X_{s-}Z_{s-}(\psi_s dW_s - \lambda_s\nu_s ds) - \int_0^t X_{s-}(a_s ds + dA_s^s) \\
&\quad + \int_0^t Y_{s-}\xi_s S_{s-}[\mu_s ds + \sigma_s dW_s + \lambda_s(1-\alpha_s)ds] + \int_0^t \sigma_s Z_{s-}\psi_s S_{s-}\xi_s ds \\
&\quad \int_0^t \left[(X_{s-} - (1-\alpha_s)\xi_s S_{s-})(Y_{s-} + Z_{s-}\nu_s) - X_{s-}Y_{s-}\right] dN_s \\
&\quad + \sum_{0<s\le t}\left[X_{s-}(Y_{s-} - \triangle A_s) - X_{s-}Y_{s-}\right] \\
&= \text{“local martingale part ”} - \int_0^t X_{s-}dA_s^s - \sum_{0<\le t} X_{s-}\triangle A_s \\
&+ \int_0^t \left[-X_{s-}a_s + \mu_s Y_{s-}\xi_s S_{s-} + \sigma_s Z_{s-}\psi_s S_{s-}\xi_s - \lambda_s(1-\alpha_s)\xi_s S_{s-}Z_{s-}\nu_s\right] ds.
\end{aligned}$$

X_tY_t is a supermartingale if and only if

$$-X_{s-}a_s + \xi_s S_{s-}\left\{\, \mu_s Y_{s-} + Z_{s-}\left[\sigma_s\psi_s - \lambda_s(1-\alpha_s)\nu_s\right] \right\} \le 0 \tag{2.3}$$

holds $d\mathbb{P}\otimes ds - a.s.$ Therefore, $Y \in \mathcal{Y}(y)$ if and only if (2.3) holds for all $X \in \mathcal{X}(1)$. Let

$$\theta_t \triangleq \mu_t Y_{t-} + Z_{t-}\left\{\, \sigma_t\psi_t - \lambda_t(1-\alpha_t)\nu_t \,\right\}.$$

We now show how to use Lemma 3.2 to prove that the inequality (2.3) which can be written as

$$-X_{t-}a_t + \xi_t S_{t-}\theta_t \le 0, \quad d\mathbb{P}\otimes dt - a.s. \tag{2.4}$$

is equivalent to

$$0 \le \theta_t \le (1-\alpha_t)a_t, \quad d\mathbb{P}\otimes dt - a.s., \tag{2.5}$$

and therefore finish the proof of this lemma. Let $\tau = \inf\{t : \theta_t < 0\}$ and $\tilde{\tau} = \inf\{t > \tau : \theta \ge 0\} \wedge T$. Suppose $\mathbb{P}(\tilde{\tau} - \tau > 0) > 0$. Define

$$\xi_t = \left(\frac{a_t X_{t-}}{\theta_t S_{t-}} - 1\right) 1_{\{X_{t-}>0\}} 1_{\{\tau < t \le \tilde{\tau}\}}.$$

Note that ξ_t is a predictable processes and $X_t = 1 + \int_0^t \xi_s dS_t$ is well-defined. Since $\xi_t < 0 < \frac{X_{t-}}{(1-\alpha_t)S_{t-}}$ on the set $\{X_{t-} > 0\}$, ξ_t is an admissible strategy by Lemma 3.2, and $X \in \mathcal{X}(1)$. Let $\tau' = \inf\{t : X_t \leq 0\}$. Because $\xi = 0$ on $[0, \tau]$, $X_\tau = X_0 = 1$. Since $\xi_t \leq 0$, X_t does not jump to 0, and therefore $\mathbb{P}(\tau' - \tau > 0) = 1$ and $\mathbb{P}(\tau' \wedge \tilde{\tau} - \tau > 0) > 0$. On the set $[\tau, \tau' \wedge \tilde{\tau}]$,

$$\xi_t < \frac{a_t X_{t-}}{\theta_t S_{t-}}, \qquad -X_{t-}a_t + \xi_t S_{t-}\theta_t > 0,$$

a contradiction to (2.4). Therefore, $\mathbb{P}(\tilde{\tau} - \tau > 0) = 0$ and $\theta \geq 0$ $d\mathbb{P} \otimes dt - a.s.$ Let $B = \{(t, \omega) : \theta_t > (1 - \alpha_t)a_t\}$. Define

$$\xi_t = \frac{X_{t-}}{(1 - \alpha_t)S_{t-}} 1_{\{X_{t-} > 0\}} 1_B.$$

Again by Lemma 3.2, ξ_t is admissible and $X \in \mathcal{X}(1)$. Then on set $B \cap \{X_{t-} > 0\}$,

$$-X_{t-}a_t + \xi_t S_{t-}\theta_t = -X_{t-}a_t + \frac{X_{t-}\theta_t}{1 - \alpha_t} = X_{t-}\left(-a_t + \frac{\theta_t}{1 - \alpha_t}\right) > 0,$$

a contradiction to (2.4). Therefore, $\mathbb{P}(B \cap \{X_{t-} > 0\}) = 0$. Since the first jump of the Poisson process is not predictable, $\mathbb{P}(B) = 0$. We have (2.4) $\Rightarrow$ (2.5). For the reverse direction,

$$\begin{aligned} -X_{t-}a_t + \xi_t S_{t-}\theta_t &\leq -X_{t-}a_t + \frac{\theta_t X_{t-}}{1 - \alpha_t} \\ &= X_{t-}\left(\frac{\theta_t}{1 - \alpha_t} - a_t\right) \leq 0, \end{aligned}$$

by Lemma 3.2, and (2.5). Thus we have (2.5) $\Rightarrow$ (2.4). ◇

Proposition 3.4. *Assume (1.1). Let $Y_t = Z_t - A_t \in \mathcal{Y}(y)$ be given. Then there exists a local martingale $\tilde{Z}_t \in \mathcal{Y}(y)$ such that $\tilde{Z}_t \geq Y_t$. $\tilde{Z}_t$ has the decomposition*

$$\begin{aligned} &d\tilde{Z}_t = \tilde{Z}_{t-}[\,\tilde{\psi}_t dW_t + \tilde{\nu}_t dM_t\,] \quad \textit{on} \quad [0, \tau),\ \textit{s.t.}\ \mu_t + \sigma_t\tilde{\psi}_t - \lambda_t(1 - \alpha_t)\tilde{\nu}_t = 0; \\ &\tilde{Z}_t \equiv 0 \quad \textit{on} \quad [\tau, T], \end{aligned}$$

where the stopping time τ is defined as $\tau = \inf\{t \geq 0 : \tilde{Z} \leq 0\}$.

Remark 3.5. *Recall from Theorem 1.18 in Section 1.4 that the (Dual-Shortfall) problem has an optimal solution. A consequence of this Proposition when applied to the shortfall minimization problem is that the optimal solutions are obtained by local martingales of the above form in this mixed diffusion model.*

PROOF. Define the stopping time

$$\tau_y = \inf\{\, t \geq 0 \,:\, Y_t = 0 \,\}.$$

Since Y_t is a nonnegative supermartingale, we have $Y_t \equiv 0$ on $[\tau, T]$. Define $\tilde{Z}_t$ to be a process with $\tilde{Z}(0) = y$ satisfying

$$d\tilde{Z}_t = \tilde{Z}_{t-}[\, \tilde{\psi}_t dW_t + \tilde{\nu}_t dM_t \,] \quad \text{on the set} \quad [0, \tau_z),$$

$$\tilde{Z}_t \equiv 0 \quad \text{on the set} \quad [\tau_z, T],$$

where

$$\begin{aligned}
\tau_z &= \inf\{t \geq 0 : Z \leq 0\}, \\
\tilde{\psi}_t &= \frac{\psi_t Z_{t-}}{Y_{t-}} 1_{[0,\tau_y)}, \\
\tilde{\nu}_t &= \left\{ \frac{\nu_t Z_{t-}}{Y_{t-}} + \frac{\theta_t}{\lambda_t(1-\alpha_t)Y_{t-}} \right\} 1_{[0,\tau_y)} + \frac{\mu_t}{\lambda_t(1-\alpha_t)} 1_{[\tau_y,\tau_z)},
\end{aligned}$$

"$\tilde{Z}_t \geq Y_t$": On $[0, \tau_y)$:

$$\begin{aligned}
\log \tilde{Z}_t &= \log y + \int_0^t \frac{1}{\tilde{Z}_{s-}} d\tilde{Z}^c_s - \frac{1}{2}\int_0^t \frac{1}{\tilde{Z}^2_{s-}} d\langle \tilde{Z}^c \rangle_s + \sum_{0<s\leq t} \left(\log \tilde{Z}_s - \log \tilde{Z}_{s-} \right) \\
&= \int_0^t \tilde{\psi}_s dW_s - \int_0^t \lambda_s \tilde{\nu}_s ds - \frac{1}{2}\int_0^t \tilde{\psi}^2_s ds + \int_0^t \log \frac{\tilde{Z}_{s-} + \triangle \tilde{Z}_s}{\tilde{Z}_{s-}} dN_s \\
&= \int_0^t \frac{\psi_s Z_{s-}}{Y_{s-}} dW_s - \int_0^t \frac{\lambda_s \nu_s Z_{s-}}{Y_{s-}} ds - \int_0^t \frac{\theta_s}{(1-\alpha_s)Y_{s-}} ds \\
&\quad - \frac{1}{2}\int_0^t \frac{\psi^2_s Z^2_{s-}}{Y^2_{s-}} ds + \int_0^t \log(1 + \tilde{\nu}_s) dN_s
\end{aligned}$$

$$\begin{aligned}
\log Y_t &= \log y + \int_0^t \frac{1}{Y_{s-}} dY^c_s - \frac{1}{2}\int_0^t \frac{1}{Y^2_{s-}} d\langle Y^c \rangle_s + \sum_{0<s\leq t} (\log Y_s - \log Y_{s-}) \\
&= \int_0^t \frac{\psi_s Z_{s-}}{Y_{s-}} dW_s - \int_0^t \frac{1}{Y_{s-}} (a_s ds + dA^s_s) - \int_0^t \frac{\lambda_s \nu_s Z_{s-}}{Y_{s-}} ds \\
&\quad - \frac{1}{2}\int_0^t \frac{\psi^2_s Z^2_{s-}}{Y^2_{s-}} ds + \int_0^t \log\left(1 + \frac{\triangle Y_s}{Y_{s-}}\right) dN_s + \sum_{0<s\leq t} \log\left(1 + \frac{\triangle Y_s}{Y_{s-}}\right) \triangle A_s
\end{aligned}$$

Notice that at the jump times of A, $\triangle Y_s = -\triangle A_s < 0$, $\mathbb{P} - a.s.$ because A is predictable and thus A and N do not jump at the same time. By Lemma 3.3,

$\theta_s \leq a_s(1-\alpha_s),\ \mathbb{P}-a.s.$. Therefore,

$$\begin{aligned}\log\frac{\tilde{Z}_t}{Y_t} &= \int_0^t \frac{1}{Y_s}\left[a_s ds + dA_s^s - \frac{\theta_s}{1-\alpha_s}ds\right] \\ &\quad + \int_0^t \left[\log(1+\tilde{\nu}_s) - \log\left(1+\frac{\triangle Y_s}{Y_{s-}}\right)\right] dN_s - \sum_{0<s\leq t}\log\left(1+\frac{\triangle Y_s}{Y_{s-}}\right)\triangle A_s \\ &\geq \int_0^t \left[\log(1+\tilde{\nu}_s) - \log\left(1+\frac{\triangle Y_s}{Y_{s-}}\right)\right] dN_s\end{aligned}$$

But at the jump times of N,

$$\frac{\triangle Y_s}{Y_{s-}} = \frac{\triangle Z_s}{Y_{s-}} = \frac{\nu_s Z_{s-}}{Y_{s-}} \leq \tilde{\nu}_s,$$

because Lemma 3.3 gives $\theta_s \geq 0,\ \mathbb{P}-a.s.$. Therefore, $\log\frac{\tilde{Z}_t}{Y_t} \geq 0$, i.e., $\tilde{Z}_t \geq Y_t$ on $[0,\tau_y)$.

On $[\tau_y, T]$:

$$\tilde{Z}_t \geq 0 = Y_t.$$

"$\tilde{Z}_t \in \mathcal{Y}_t$": $\tilde{Z}_t \geq Y_t \geq 0$. On the set $[0,\tau_y)$,

$$\begin{aligned}\mu_t + \sigma_t\tilde{\psi}_t - \lambda_t(1-\alpha_t)\tilde{\nu}_t &= \mu_t + \frac{\sigma_t\psi_t Z_{t-}}{Y_{t-}} - \frac{\lambda_t(1-\alpha_t)\nu_t Z_{t-}}{Y_{t-}} - \frac{\theta_t}{Y_{t-}} \\ &= \frac{1}{Y_{t-}}\left[\,\mu_t Y_{t-} + \sigma_t\psi_t Z_{t-} - \lambda_t(1-\alpha_t)\nu_t Z_{t-} - \theta_t\,\right] \\ &= 0.\end{aligned}$$

On the set $[\tau_y,\tau_z)$, $\mu_t + \sigma_t\tilde{\psi}_t - \lambda_t(1-\alpha_t)\tilde{\nu}_t = 0$ by definition. By Lemma 3.3 we conclude that $\tilde{Z}_t \in \mathcal{Y}(y)$.

"Local martingale property ": Since a stopped local martingale is a local martingale, by construction, $\tilde{Z}_t \geq 0$ is a local martingale. ◇

3.3 An open question - Is the solution to (Dual-Shortfall) a martingale?

Recall the (Dual-Shortfall) problem from Section 1.4.

$$v(y) = \inf_{Y\in\mathcal{Y}(y)} E^{\mathbb{P}}\left[\,(1-Y)^+H\,\right] = \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[\,(1-h)^+H\,\right].$$

From Remark 3.5, we know that there is an optimal solution to (Dual-Shortfall) that is a local martingale $\tilde{Z} \in \mathcal{Y}(y)$ with the decomposition

$$
\begin{aligned}
(3.6)\quad & d\tilde{Z}_t = \tilde{Z}_{t-}[\,\tilde{\psi}_t dW_t + \tilde{\nu}_t dM_t\,] \text{ on } [0,\tau),\ s.t.\ \mu_t + \sigma_t\tilde{\psi}_t - \lambda_t(1-\alpha_t)\tilde{\nu}_t = 0; \\
& \tilde{Z}_t \equiv 0 \quad \text{on} \quad [\tau, T],
\end{aligned}
$$

where the stopping time τ is defined as $\tau = \inf\{t \geq 0 : \tilde{Z} \leq 0\}$. If $\tilde{\psi}$ and $\tilde{\nu}$ are bounded processes, then $\tilde{Z}_t$ will be a martingale[2]. Since we are maximizing a convex function (when $H \equiv 1$), this reminds us of Hajek's mean comparison theorem from Hajek (1985) in the Brownian case:

Theorem 3.6 (Hajek's Mean Comparison Theorem). *Let x be a continuous martingale with representation $x_t = x_0 + \int_0^t \sigma_s dW_s$ such that for some Lipschitz continuous function ρ on $\mathbb{R}$, $|\sigma_s| \leq \rho(x_s)$. Let y be the unique solution to the stochastic differential equation $y_s = x_0 + \int_0^t \rho(y_s) dW_s$. Then for any convex function Φ and any $t \geq 0$*

$$E[\Phi(x_t)] \leq E[\Phi(y_t)].$$

If we could extend the comparison theorem to our case, then the following lemma would be sufficient to prove that there indeed exists an optimal $\hat{Z} \in \mathcal{Y}(y)$ satisfying $d\hat{Z}_t = \hat{Z}_{t-}[\,\hat{\psi}_t dW_t + \hat{\nu}_t dM_t\,]$ when $Z_{t-} > 0$, for some bounded predictable processes $\hat{\psi}$ and $\hat{\nu}$ as long as $-\frac{\mu_t}{\sigma_t}$ and $\frac{\mu_t}{(1-\alpha_t)\lambda_t}$ are bounded processes. This $\hat{Z}_t$ would be a martingale.

Lemma 3.7. *Suppose an optimal solution $\tilde{Z} \in \mathcal{Y}(y)$ to (Dual-Shortfall) has the decomposition $(\tilde{\psi}_t, \tilde{\nu}_t)$ of (3.6). Define $(\hat{\psi}_t, \hat{\nu}_t)$ to be*

$$\hat{\psi}_t = (\tilde{\psi}_t \wedge 0) \vee -\frac{\mu_t}{\sigma_t}, \qquad \hat{\nu}_t = \frac{\mu_t + \sigma_t\hat{\psi}_t}{(1-\alpha_t)\lambda_t}$$

Then $|\hat{\psi}_t| \leq |\tilde{\psi}_t|$ and $|\hat{\nu}_t| \leq |\tilde{\nu}_t|$. Obviously, $(\hat{\psi}_t, \hat{\nu}_t)$ lies on the line between the two points $(-\frac{\mu_t}{\sigma_t}, 0)$ and $\left(0, \frac{\mu_t}{(1-\alpha_t)\lambda_t}\right)$.

However, counter-examples can be constructed to show that Hajek's mean comparison theorem only holds in very specific cases and cannot be extended to jump processes in general, see Večeř and Xu (2004).

[2]See corollary 3 in section 6, Chapter II of Protter (1995).

3.4 Upper and lower bounds of the value function

From Remark 3.5, we know that there exists a dual optimal solution that is a local martingale. Define the following set of interest

$$\begin{aligned}\mathcal{L} = \{\, Z \,:\, Z_0 = 1 \text{ and for the stopping time } \tau = \inf\{t \geq 0 : Z \leq 0\} \\ Z \text{ has the representation } dZ_t = Z_{t-}[\,\psi_t dW_t + \nu_t dM_t\,] \text{ on } [0,\tau), \\ s.t.\ \mu_t + \sigma_t\psi_t - \lambda_t(1-\alpha_t)\nu_t = 0, \text{ and } Z_t \equiv 0 \quad \text{on } [\tau, T].\,\}\end{aligned}$$

We know that $\mathcal{L} \subseteq \mathcal{Y}(1)$ from Lemma 3.3 and there exists an optimal solution $\hat{Y} \in \mathcal{Y}(y)$ to (Dual-Shortfall) satisfying $\frac{\hat{Y}}{y} \in \mathcal{L}$, for $y \geq 0$, from Proposition 3.4.

Theorem 3.8. *Let $x < \bar{x}$ where $\bar{x}$ is the super-hedging price defined in Assumption 1.14. There exists some y and an $\mathcal{F}_T$-measurable random variable $\hat{\gamma}$ such that there is an optimal solution $\hat{X} \in \mathcal{X}(x)$ to (Primal-Shortfall) that can be written as*

$$\hat{X}_T \wedge H = \hat{\phi} H = (1_{\{0 \leq y\hat{Z}_T < 1\}} + \hat{\gamma} 1_{\{y\hat{Z}_T = 1\}})H,$$

where $\hat{Z} \in \mathcal{L}$, and $y\hat{Z} \in \mathcal{Y}(y)$ is an optimal solution to (Dual-Shortfall). Furthermore,

$$E^{\mathbb{P}}[\hat{X}_T \hat{Z}_T] = x.$$

Proof.It is a direct application of Theorem 1.18, 1.19 and Proposition 3.4. ⋄

Recall the primal and dual value functions defined in Section 1.4 for (Primal-Shortfall) and (Dual-Shortfall) in this mixed diffusion model are

$$\begin{aligned}u^{BP}(x) &= \sup_{X \in \mathcal{X}(x)} E^{\mathbb{P}}\,[\,H \wedge X_T\,] = \sup_{g \in \mathcal{C}(x)} E^{\mathbb{P}}\,[\,H \wedge g\,]\,; \\ v^{BP}(y) &= \inf_{Y \in \mathcal{Y}(y)} E^{\mathbb{P}}\,\left[\,(1-Y)^+ H\,\right] = \inf_{h \in \mathcal{D}(y)} E^{\mathbb{P}}\,\left[\,(1-h)^+ H\,\right].\end{aligned}$$

Here we use superscripts BP to indicate that the price process involve both Brownian motion and Poisson process. We know from theorem 1.18 that they satisfy the duality equality

$$\begin{aligned}v^{BP}(y) &= \max_{x \geq 0}[\,u^{BP}(x) - xy\,] \quad \text{for any} \quad y \geq 0, \\ u^{BP}(x) &= \min_{y \geq 0}[\,v^{BP}(y) + xy\,] \quad \text{for any} \quad x \geq 0, \\ v^{BP}(y^{BP}) &= u^{BP}(x^{BP}) - x^{BP}y^{BP} \quad \text{when} \quad y^{BP} \in \partial u^{BP}(x^{BP}), \\ &\qquad \text{or equivalently}, x^{BP} \in -\partial v^{BP}(y^{BP}).\end{aligned}$$

3.4.1 Upper bounds

For any $Z \in \mathcal{L}$, we can define

$$v^Z(y) = E^{\mathbb{P}}\left[(1-yZ)^+H\right], \tag{4.7}$$

$$u^Z(x) = \inf_{y>0}\left[v^Z(y) + xy\right]. \tag{4.8}$$

Since $v^Z(y)$ is convex, decreasing with the right limits for the right-hand derivatives, similar results can be derived as in Appendix B so we know $u^Z(x)$ is concave and increasing such that:

$u^Z(y^Z) = v^Z(x^Z) + x^Z y^Z$ when $y^Z \in \partial u^Z(x^Z)$, or equivalently, $x^Z \in -\partial v^Z(y^Z)$.

Lemma 3.9. $u^Z(x) = E^{\mathbb{P}}[H]$ *for* $x \in [\,E^{\mathbb{P}}[ZH]\,,\,\infty\,)$.

Proof. From (4.7), $-v^Z_y(0+) = E^{\mathbb{P}}[ZH]$. Therefore

$$u^Z(E^{\mathbb{P}}[ZH]) = v^Z(0) = E^{\mathbb{P}}[H].$$

From (4.8), $u^Z(x) \le v^Z(0)$. Since $u^Z(x)$ is an increasing function, we get the desired result. $\diamond$

Proposition 3.10. *For any* $y \ge 0$, *we have*

$$v^{BP}(y) \le v^Z(y).$$

Proof. By Remark 3.5, we know

$$v^{BP}(y) = \inf_{Y\in\mathcal{L}} E^{\mathbb{P}}\left[\,(1-yY)^+H\,\right].$$

Since $Z \in \mathcal{L}$, we easily conclude $v^{BP}(y) \le v^Z(y)$. $\diamond$

Lemma 3.11. *Suppose* u_1 *and* u_2 *are concave functions on* $[a,b]$ *and there exist some* $x^* \in (a,b)$ *such that*

$$u_1(x) - u_2(x) \le u_1(x^*) - u_2(x^*), \forall x \in [a,b], \tag{4.9}$$

then $\partial u_1(x^*) \supseteq \partial u_2(x^*) \neq \emptyset$.

PROOF. For a concave function $u_2(x)$, and an interior point $x^* \in (a,b)$

$$\partial u_2(x^*) = \{\, y \,|\, u_2(x) \leq u_2(x^*) + y(x - x^*), \forall x \in [a,b] \,\}.$$

For any $y \in \partial u_2(x^*)$, (4.9) implies

$$u_1(x) \leq u_1(x^*) - u_2(x^*) + u_2(x) \leq u_1(x^*) + y(x - x^*), \forall x \in [a,b],$$

thus $y \in \partial u_1(x^*)$. $\diamond$

Recall from Assumption 1.14, the super-hedging price $\bar{x}$ is assumed to be finite

$$\bar{x} \triangleq \sup_{\mathbb{Q}\in\mathcal{M}} E^{\mathbb{Q}}[H] < \infty.$$

Theorem 3.12. *Suppose* $0 \leq x \leq \bar{x}$. *Then* $u^{BP}(x) \leq u^Z(x)$.

PROOF. Since u^{BP} and u^Z are continuous functions on $[0,\bar{x}]$ by Theorem 1.18, we can define

$$x^* \triangleq \operatorname*{argmax}_{x\in[0,\bar{x}]} \{u^{BP}(x) - u^Z(x)\}.$$

Suppose $x^* \in (0,\bar{x})$. By Lemma 3.11, we can pick $y^* \in \partial u^{BP}(x^*) \cap \partial u^Z(x^*)$. From the duality relationship, we know $u^{BP}(x^*) = v^{BP}(y^*)+x^*y^*$, and $u^Z(x^*) = v^Z(y^*) + x^*y^*$. Then for any $x \in [0,\bar{x}]$, we have

$$\begin{aligned} u^{BP}(x) - u^Z(x) &\leq u^{BP}(x^*) - u^Z(x^*) \\ &= v^{BP}(y^*) + x^*y^* - (v^Z(y^*) + x^*y^*) \leq v^{BP}(y^*) - v^Z(y^*) \leq 0, \end{aligned}$$

by Proposition 3.10. Suppose $x^* = 0$.

$$u^{BP}(x) - u^Z(x) \leq u^{BP}(0) - u^Z(0) = 0 - \inf_{y>0} v^Z(y) = 0.$$

Suppose $x^* = \bar{x}$. Lemma 3.9 gives $u^Z(\bar{x}) = E^{\mathbb{P}}[H]$ and

$$u^{BP}(x) - u^Z(x) \leq u^{BP}(\bar{x}) - u^Z(\bar{x}) = E^{\mathbb{P}}[H] - E^{\mathbb{P}}[H] = 0. \qquad \diamond$$

Remark 3.13. *Notice that we did not use any specific property from the jump-diffusion price process in the above proofs. Therefore, this theorem can easily be extended to the semimartingale models.*

Remark 3.14. *By Lemma 3.9 and Theorem 3.12, we know the upper bound is effective on the interval* $x \in [\,0\,,\, E^{\mathbb{P}}[ZH]\,]$.

We can specialize the above results to two familiar cases: the exponential Brownian motion model and the exponential Poisson model. For the first one, the stock price process is

$$dS_t = S_t[\,\mu_t dt + \sigma_t dW_t\,].$$

There exists a unique Radon-Nikodym derivative process

$$Z_t^B = e^{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds}, \quad \text{where} \quad \theta_s = \frac{\mu_s}{\sigma_s}.$$

For the second model, the stock price process is

$$dS_t = S_t[\,\mu_t dt - (1-\alpha_t)dM_t\,].$$

There exists a unique Radon-Nikodym derivative process

$$Z_T^N = e^{\int_0^t (\ln \lambda_s^* - \ln \lambda_s)dN_s - \int_0^t (\lambda_s^* - \lambda_s)ds}, \quad \text{where} \quad \lambda_s^* = \frac{\mu_s + (1-\alpha_s)\lambda_s}{1-\alpha_s}.$$

Obviously, $Z_T^B \in \mathcal{L}$ and $Z_T^N \in \mathcal{L}$. We can define the corresponding value functions

$$v^B(y) = E^{\mathbb{P}}[(1-yZ_T^B)^+ H], \qquad v^N(y) = E^{\mathbb{P}}[(1-yZ_T^N)^+ H],$$
$$u^B(x) = \inf_{y>0}[v^B(y) + xy], \qquad u^N(x) = \inf_{y>0}[v^N(y) + xy].$$

Corollary 3.15. *For any $y \geq 0$, we have*

$$v^{BP}(y) \leq \min\{v^B(y), v^P(y)\}.$$

Suppose $0 \leq x \leq \min\{\,E^{\mathbb{P}}[Z^B H],\, E^{\mathbb{P}}[Z^N H]\,\} \leq \bar{x}$. *Then*

$$u^{BP}(x) \leq \min\{u^B(x), u^P(x)\}.$$

3.4.2 Lower bounds

Recall from Lemma 3.2, $X_t = x + \int_0^t \xi_s dS_s$ is an admissible strategy if and only if

$$\xi_t \leq \frac{X_{t-}}{(1-\alpha_t)S_{t-}} \text{ on the set } \{X_{t-} > 0\}, \quad \text{and } \xi_t = 0 \text{ on the set } \{X_{t-} \leq 0\}.$$

This is to make sure that X_t does not jump below 0, and once it hits 0, it stays there. In the case of option payoff H, the primal problem is

$$u(x) = \max_{\text{admissible } X} E^{\mathbb{P}}[H \wedge X_T].$$

Therefore, each particular strategy gives a lower bound for the value function $u(x)$. In the next section, we choose the strategy to invest the maximal amount of capital in the stock to take advantage of the upward drift which, we call the 'Bold Strategy'.[3] As in Section 1.4, define the super-hedging wealth process to be

$$W_t = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{M}} E^{\mathbb{Q}}\,[\,H\,|\,\mathcal{F}_t\,], \quad W_0 = \bar{x}, \quad W_T = H.$$

Define

$$\xi_t^{Bold} \triangleq \frac{X_{t-}}{(1-\alpha_t)S_{t-}} \text{ on the set } \{0 < X_{t-} < W_t\};$$
$$\xi_t^{Bold} \triangleq 0 \text{ on the set } \{X_{t-} = 0\};$$
$$\xi_t^{Bold} \triangleq \text{super-hedging strategy on the set } \{X_{t-} \geq W_t\};$$

where $\bar{x}$ is the super-hedging price, and

$$u^{Bold}(x) \triangleq E^{\mathbb{P}}\,[H \wedge X_T^{Bold}], \text{ where } X_t^{Bold} \triangleq x + \int_0^t \xi_s^{Bold} dS_s.$$

Then $u^{Bold}(x) \leq u^{BP}(x)$ for all $x \geq 0$.

3.5 Numerical results

3.5.1 Call option case

Let $H = (S_T - K)^+$. For the purpose of obtaining closed form solutions for our numerical example, we let the parameters of the mixed-diffusion process be constants. The stock price process therefore is

$$\begin{aligned} dS_t &= S_{t-}[\mu dt + \sigma dW_t - (1-\alpha)dM_t] \\ &= S_{t-}[\sigma dW_t^* - (1-\alpha)dM_t] \\ &= S_{t-}[\sigma dW_t - (1-\alpha)dM_t^*], \\ S_T &= S_0 e^{[\mu - \frac{1}{2}\sigma^2 + \lambda(1-\alpha)]T + \sigma W_T + N_T \ln\alpha} \\ &= S_0 e^{[-\frac{1}{2}\sigma^2 + \lambda(1-\alpha)]T + \sigma W_T^* + N_T \ln\alpha}, \end{aligned}$$

where $\mu > 0, \sigma > 0, 0 < \alpha < 1, \lambda > 0$. $W_t^* = W_t + \theta t$ is a Brownian motion under $E^B[\cdot]$ when the Radon-Nikodym derivative is

$$Z_T^B = e^{-\theta W_T - \frac{1}{2}\theta^2 T}, \quad \theta = \frac{\mu}{\sigma}.$$

[3] Intuitively, a more conservative strategy should work better because we don't need to take extra risk of bankruptcy for overshooting our goal too early.

$M_t^* = N_t - \lambda^* t$ is a compensated Poisson process under $E^N[\cdot]$ when the Radon-Nikodym derivative is

$$Z_T^N = \left(\frac{\lambda^*}{\lambda}\right)^{N_T} e^{-(\lambda^*-\lambda)T}, \quad \lambda^* = \frac{\mu + (1-\alpha)\lambda}{1-\alpha}.$$

Define

$$d_1(n) \triangleq \frac{\ln\frac{S_0}{K} + [\mu - \frac{1}{2}\sigma^2 + \lambda(1-\alpha)]T + n\ln\alpha}{\sigma\sqrt{T}}, \quad d_2(y) \triangleq \frac{-\ln y + \frac{1}{2}\theta^2 T}{\theta\sqrt{T}};$$

$$d_1^*(n) \triangleq d_1(n) - \theta\sqrt{T}, \quad d_2^*(y) \triangleq d_2(y) - \theta\sqrt{T};$$

$$d_3(y) \triangleq \frac{-\ln y + (\lambda^* - \lambda)T}{\ln\lambda^* - \ln\lambda}.$$

We will compute, in the next two lemmas, upper bounds for our value function from Z^B and Z^N.

Lemma 3.16. *Suppose $0 < x \le E^B[(S_T - K)^+]$. There exists an N^* and a y^B such that $d_1^*(N^*+1) < d_2^*(y^B) \le d_1^*(N^*)$ and*

$$\begin{aligned} x &= -\partial v^B(y^B) = E^B[(S_T-K)^+ 1_{\{1\ge y^B Z^B\}}] \\ &= \sum_{n=0}^{N^*} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\left[S_0 e^{\lambda(1-\alpha)T + n\ln\alpha} N\left(d_2^*(y^B) + \sigma\sqrt{T}\right) - KN\left(d_2^*(y^B)\right)\right] \\ &\quad + \sum_{n=N^*+1}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\left[S_0 e^{\lambda(1-\alpha)T + n\ln\alpha} N\left(d_1^*(n) + \sigma\sqrt{T}\right) - KN\left(d_1^*(n)\right)\right]. \end{aligned}$$

The corresponding value function can be computed

$$\begin{aligned} u^B(x) &= v^B(y^B) + xy^B = E[(S_T-K)^+ 1_{\{1\ge y^B Z^B\}}] \\ &= \sum_{n=0}^{N^*} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\left[S_0 e^{[\mu+\lambda(1-\alpha)]T + n\ln\alpha} N\left(d_2(y^B) + \sigma\sqrt{T}\right) - KN\left(d_2(y^B)\right)\right] \\ &\quad + \sum_{n=N^*+1}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\left[S_0 e^{[\mu+\lambda(1-\alpha)]T + n\ln\alpha} N\left(d_1(n) + \sigma\sqrt{T}\right) - KN\left(d_1(n)\right)\right]. \end{aligned}$$

Proof. We need to solve for y^B such that

$$\begin{aligned} x &= -\partial v^B(y^B) \\ &= -\partial E[(1 - yZ^B)^+ H] \\ &= E[Z^B H 1_{\{1\ge y^B Z^B\}}] \\ &= E[Z^B (S_T-K)^+ 1_{\{1\ge y^B Z^B\}}] \\ &= E^B[(S_T-K)^+ 1_{\{1\ge y^B Z^B\}}]. \end{aligned}$$

Note that as $y \to \infty$, $E^B[(S_T-K)^+ 1_{\{1 \ge yZ^B\}}] \to 0$, as $y \to 0$, $E^B[(S_T - K)^+ 1_{\{1\ge yZ^B\}}] \to E^B[(S_T-K)^+]$, and $E^B[(S_T-K)^+1_{\{1\ge yZ^B\}}]$ is a decreasing function of y. Therefore, if $0 < x \le E^B[(S_T-K)^+]$, we can find a y^B that satisfies the above equation. To be more specific, note that, conditioned on $N_t = n$,

$$S_T \ge K \Leftrightarrow -\frac{W_T}{\sqrt{T}} \le d_1(n) \Leftrightarrow -\frac{W_T^*}{\sqrt{T}} \le d_1^*(n),$$
$$1 \ge y^B Z^B \Leftrightarrow -\frac{W_T}{\sqrt{T}} \le d_2(y^B) \Leftrightarrow -\frac{W_T^*}{\sqrt{T}} \le d_2^*(y^B).$$

Then we can compute the expectation under the change of measure by Z^B:

$$\begin{aligned}
x &= E^B[(S_T-K)^+1_{\{1\ge y^B Z^B\}}] \\
&= \sum_{n=0}^{\infty} e^{-\lambda T}\tfrac{(\lambda T)^n}{n!} E^B\Bigg[\left(S_0 e^{[-\frac{1}{2}\sigma^2+\lambda(1-\alpha)]T+\sigma W_T^*+n\ln\alpha} - K\right) \\
&\qquad\qquad 1_{\{-\frac{W_T^*}{\sqrt{T}}\le d_1^*(n), -\frac{W_T^*}{\sqrt{T}}\le d_2^*(y^B)\}}\Bigg] \\
&= \sum_{n=0}^{N^*} e^{-\lambda T}\tfrac{(\lambda T)^n}{n!} E^B\left[\left(S_0 e^{[-\frac{1}{2}\sigma^2+\lambda(1-\alpha)]T+\sigma W_T^*+n\ln\alpha} - K\right)1_{\{-\frac{W_T^*}{\sqrt{T}}\le d_2^*(y^B)\}}\right] \\
&+ \sum_{n=N^*+1}^{\infty} e^{-\lambda T}\tfrac{(\lambda T)^n}{n!} E^B\left[\left(S_0 e^{[-\frac{1}{2}\sigma^2+\lambda(1-\alpha)]T+\sigma W_T^*+n\ln\alpha} - K\right)1_{\{-\frac{W_T^*}{\sqrt{T}}\le d_1^*(n)\}}\right] \\
&= \sum_{n=0}^{N^*} e^{-\lambda T}\tfrac{(\lambda T)^n}{n!}\left[S_0 e^{\lambda(1-\alpha)T+n\ln\alpha} N\left(d_2^*(y^B)+\sigma\sqrt{T}\right) - KN\left(d_2^*(y^B)\right)\right] \\
&\quad + \sum_{n=N^*+1}^{\infty} e^{-\lambda T}\tfrac{(\lambda T)^n}{n!}\left[S_0 e^{\lambda(1-\alpha)T+n\ln\alpha} N\left(d_1^*(n)+\sigma\sqrt{T}\right) - KN\left(d_1^*(n)\right)\right].
\end{aligned}$$

$u^B(x)$ can be computed from the following formula

$$\begin{aligned}
u^B(x) &= v^B(y^B) + xy^B \\
&= E[(1-y^BZ^B)^+H] + y^B E[Z^B H 1_{\{1\ge y^BZ^B\}}] \\
&= E[(1-y^BZ^B)H1_{\{1\ge y^BZ^B\}}] + E[y^BZ^BH1_{\{1\ge y^BZ^B\}}] \\
&= E[H1_{\{1\ge y^BZ^B\}}] \\
&= E[(S_T-K)^+1_{\{1\ge y^BZ^B\}}] \\
&= \sum_{n=0}^{N^*} e^{-\lambda T}\tfrac{(\lambda T)^n}{n!}\left[S_0 e^{[\mu+\lambda(1-\alpha)]T+n\ln\alpha} N\left(d_2(y^B)+\sigma\sqrt{T}\right) - KN\left(d_2(y^B)\right)\right] \\
&+ \sum_{n=N^*+1}^{\infty} e^{-\lambda T}\tfrac{(\lambda T)^n}{n!}\left[S_0 e^{[\mu+\lambda(1-\alpha)]T+n\ln\alpha} N\left(d_1(n)+\sigma\sqrt{T}\right) - KN\left(d_1(n)\right)\right]. \diamond
\end{aligned}$$

Lemma 3.17. *Suppose $0 < x \le E^N[(S_T - K)^+]$. There exists an N^* and a y^N where $d_3(y^N) = N^*$, and $0 \le \gamma < 1$ such that $x \in -\partial v^N(y^N)$ and*

$$x = E^N[(S_T - K)^+ 1_{\{1>y^N Z^N\}}] + \gamma^N E^N[(S_T - K)^+ 1_{\{1=y^N Z^N\}}]$$
$$= \sum_{n=0}^{N^*-1} e^{-\lambda^* T} \frac{(\lambda^* T)^n}{n!} \left[S_0 e^{[\mu+\lambda(1-\alpha)]T + n\ln\alpha} N\left(d_1(n) + \sigma\sqrt{T}\right) - KN\left(d_1(n)\right)\right]$$
$$+ \gamma^N e^{-\lambda^* T} \frac{(\lambda^* T)^{N^*}}{N^*!} \left[S_0 e^{[\mu+\lambda(1-\alpha)]T + N^*\ln\alpha} N\left(d_1(N^*) + \sigma\sqrt{T}\right) - KN\left(d_1(N^*)\right)\right].$$

The corresponding value function can be computed

$$u^N(x) = v^N(y^N) + xy^N$$
$$= E[(S_T - K)^+ 1_{\{1>y^N Z^N\}}] + \gamma^N E[(S_T - K)^+ 1_{\{1=y^N Z^N\}}]$$
$$= \sum_{n=0}^{N^*-1} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \left[S_0 e^{[\mu+\lambda(1-\alpha)]T + n\ln\alpha} N\left(d_1(n) + \sigma\sqrt{T}\right) - KN\left(d_1(n)\right)\right]$$
$$+ \gamma^N e^{-\lambda T} \frac{(\lambda T)^{N^*}}{N^*!} \left[S_0 e^{[\mu+\lambda(1-\alpha)]T + N^*\ln\alpha} N\left(d_1(N^*) + \sigma\sqrt{T}\right) - KN\left(d_1(N^*)\right)\right].$$

PROOF. Note that

$$-\partial v^N(y) = [\, E\,[Z^N H 1_{\{1>yZ^N\}}]\,,\, E\,[Z^N H 1_{\{1\ge yZ^N\}}]\,].$$

We would like to find a y^N such that $x \in -\partial v^N(y^N)$. Then there exists $0 \le \gamma^N < 1$ such that

$$\begin{aligned} x &= E[Z^N H 1_{\{1>y^N Z^N\}}] + \gamma^N E[Z^N H 1_{\{1=y^N Z^N\}}] \\ &= E[Z^N (S_T - K)^+ 1_{\{1>y^N Z^N\}}] + \gamma^N E[Z^N (S_T - K)^+ 1_{\{1=y^N Z^N\}}] \\ &= E^N[(S_T - K)^+ 1_{\{1>y^N Z^N\}}] + \gamma^N E^N[(S_T - K)^+ 1_{\{1=y^N Z^N\}}]. \end{aligned}$$

Note that $Z^N \ge e^{-(\lambda^*-\lambda)T}$. Therefore

- When $y > e^{(\lambda^*-\lambda)T}$, $E^N[(S_T - K)^+ 1_{\{1\ge yZ^N\}}] = 0$.
- As $y \to 0$, $E^N[(S_T - K)^+ 1_{\{1\ge yZ^N\}}] \to E^N[(S_T - K)^+]$.

Also, $E^N[(S_T - K)^+ 1_{\{1\ge yZ^N\}}]$ is a decreasing function of y. Therefore, if $0 < x \le E^N[(S_T - K)^+]$, we can find a y^N and a γ^N that satisfy the above equation. To be more specific, note that, conditioned on $N_T = n$,

$$\begin{aligned} &S_T \ge K \Leftrightarrow -\frac{W_T}{\sqrt{T}} \le d_1(n) \Leftrightarrow -\frac{W_T^*}{\sqrt{T}} \le d_1^*(n), \\ &1 \ge y^N Z^N \Leftrightarrow N_T \le d_3(y^N). \end{aligned}$$

Then we can compute the expectation under the change of measure by Z^N:

$$
\begin{aligned}
x &= E^N[(S_T-K)^+1_{\{1>y^NZ^N\}}] + \gamma^N E^N[(S_T-K)^+1_{\{1=y^NZ^N\}}] \\
&= \sum_{n=0}^{\infty} e^{-\lambda^* T}\frac{(\lambda^* T)^n}{n!}E^N\left[\left(S_0 e^{[\mu-\frac{1}{2}\sigma^2+\lambda(1-\alpha)]T+\sigma W_T+n\ln\alpha}-K\right)\right. \\
&\qquad\qquad \left. 1_{\{-\frac{W_T}{\sqrt{T}}\le d_1(n), N_T<d_3(y^N)\}}\right] \\
&\quad + \gamma^N e^{-\lambda^* T}\frac{(\lambda^* T)^n}{n!}E^N\left[\left(S_0 e^{[\mu-\frac{1}{2}\sigma^2+\lambda(1-\alpha)]T+\sigma W_T+n\ln\alpha}-K\right)\right. \\
&\qquad\qquad \left. 1_{\{-\frac{W_T}{\sqrt{T}}\le d_1(n), N_T=d_3(y^N)\}}\right] \\
&= \sum_{n=0}^{N^*-1} e^{-\lambda^* T}\frac{(\lambda^* T)^n}{n!}E^N\left[\left(S_0 e^{[\mu-\frac{1}{2}\sigma^2+\lambda(1-\alpha)]T+\sigma W_T+n\ln\alpha}-K\right)\right. \\
&\qquad\qquad \left. 1_{\{-\frac{W_T}{\sqrt{T}}\le d_1(n)\}}\right] \\
&\quad + \gamma^N e^{-\lambda^* T}\frac{(\lambda^* T)^{N*}}{N^*!}E^N\left[\left(S_0 e^{[\mu-\frac{1}{2}\sigma^2+\lambda(1-\alpha)]T+\sigma W_T+N^*\ln\alpha}-K\right)\right. \\
&\qquad\qquad \left. 1_{\{-\frac{W_T}{\sqrt{T}}\le d_1(N^*)\}}\right] \\
&= \sum_{n=0}^{N^*-1} e^{-\lambda^* T}\frac{(\lambda^* T)^n}{n!}\left[S_0 e^{[\mu+\lambda(1-\alpha)]T+n\ln\alpha}N\left(d_1(n)+\sigma\sqrt{T}\right)-KN\left(d_1(n)\right)\right] \\
&+ \gamma^N e^{-\lambda^* T}\frac{(\lambda^* T)^{N^*}}{N^*!}\left[S_0 e^{[\mu+\lambda(1-\alpha)]T+N^*\ln\alpha}N\left(d_1(N^*)+\sigma\sqrt{T}\right)-KN\left(d_1(N^*)\right)\right].
\end{aligned}
$$

$u^B(x)$ can be computed from the following formula:

$$
\begin{aligned}
u^B(x) &= v^N(y^N) + xy^N \\
&= E[(1-y^NZ^N)^+H] + y^N E[Z^N H 1_{\{1>y^NZ^N\}}] + y^N\gamma^N E[Z^N H 1_{\{1=y^NZ^N\}}] \\
&= E[(1-y^NZ^N)H1_{\{1>y^NZ^N\}}] + E[y^NZ^NH1_{\{1>y^NZ^N\}}] \\
&\quad + y^N\gamma^N E[Z^NH1_{\{1=y^NZ^N\}}] \\
&= E[H1_{\{1>y^NZ^N\}}] + y^N\gamma^N E[Z^NH1_{\{1=y^NZ^N\}}] \\
&= E[H1_{\{1>y^NZ^N\}}] + \gamma^N E[H1_{\{1=y^NZ^N\}}] \\
&= E[(S_T-K)^+1_{\{1>y^NZ^N\}}] + \gamma^N E[(S_T-K)^+1_{\{1=y^NZ^N\}}] \\
&= \sum_{n=0}^{N^*-1} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\left[S_0 e^{[\mu+\lambda(1-\alpha)]T+n\ln\alpha}N\left(d_1(n)+\sigma\sqrt{T}\right)-KN\left(d_1(n)\right)\right] \\
&+ \gamma^N e^{-\lambda T}\frac{(\lambda T)^{N^*}}{N^*!}\left[S_0 e^{[\mu+\lambda(1-\alpha)]T+N^*\ln\alpha}N\left(d_1(N^*)+\sigma\sqrt{T}\right)-KN\left(d_1(N^*)\right)\right]. \diamond
\end{aligned}
$$

Now let us turn to the Bold Strategy for lower bounds. Note that the super-hedging price for a call option in the mixed diffusion model is the current stock price, see Bellamy and Jeanblanc (2000). Define a predictable time τ_1 when the value of the portfolio approaches zero continuously; a stopping time τ_2 when it jumps to zero; and a stopping time τ_3 when the wealth reaches the super-hedging price of the call option:

$$\begin{aligned}
\tau_1 &= \inf\{t > 0 \,:\, X_{t-} = 0\};\\
\tau_2 &= \inf\{t > 0 \,:\, X_{t-} > 0,\, X_t = 0\};\\
\tau_3 &= \inf\{t > 0 \,:\, X_{t-} = S_{t-}\};\\
\tau &= \tau_1 \wedge \tau_2 \wedge \tau_3.
\end{aligned}$$

Then on the set $t < \tau$,

$$\begin{aligned}
\xi_t^{Bold} &= \frac{X_{t-}}{(1-\alpha)S_{t-}}\\
dX_t^{Bold} &= X_{t-}^{Bold}(\tfrac{\mu}{1-\alpha}dt + \tfrac{\sigma}{1-\alpha}dW_t - dM_t).
\end{aligned}$$

In our case, $\tau_1 = \infty$, and $\tau = \tau_2 \wedge \tau_3$. The final wealth of the Bold Strategy is

$$\begin{aligned}
X_T^{Bold} &= x\, e^{\frac{\sigma}{1-\alpha}W_T + \left(\frac{\mu}{1-\alpha}+\lambda-\frac{\sigma^2}{2(1-\alpha)^2}\right)T} \quad \text{on the set} \quad \{\tau > T\};\\
X_T^{Bold} &= 0, \quad \text{on the set} \quad \{\tau_2 < \tau_3, \tau_2 \le T\};\\
X_T^{Bold} &= S_T, \quad \text{on the set} \quad \{\tau_3 < \tau_2, \tau_3 \le T\}.
\end{aligned}$$

By the independence of the Poisson process and the Brownian motion, we can compute the corresponding value function,

$$\begin{aligned}
&u^{Bold}(x) = E^{\mathbb{P}}\left[(S_T-K)^+ \wedge X_T^{Bold}\right]\\
&= E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[(S_T-K)^+ \wedge X_T^{Bold}\cdot 1_{\{\tau\le T\}} + (S_T-K)^+ \wedge X_T^{Bold}\cdot 1_{\{\tau>T\}} | N_T\right]\right]\\
&= e^{-\lambda T}\, E^{\mathbb{P}}\left[(S_T-K)^+ \wedge X_T^{Bold}\cdot 1_{\{\tau\le T\}} + (S_T-K)^+ \wedge X_T^{Bold}\cdot 1_{\{\tau>T\}} | N_T = 0\right]\\
&+ (1-e^{-\lambda T})\, E^{\mathbb{P}}\left[(S_T-K)^+ \wedge X_T^{Bold}\cdot 1_{\{\tau\le T\}}\right.\\
&\qquad\qquad \left. + (S_T-K)^+ \wedge X_T^{Bold}\cdot 1_{\{\tau>T\}} | N_T \ge 1\right]\\
&= e^{-\lambda T}\, E^{\mathbb{P}}\left[\left(S_0 e^{\sigma W_T + \left(\mu+\lambda(1-\alpha)-\frac{\sigma^2}{2}\right)T} - K\right)^+ \cdot 1_{\{\tau_3\le T\}}\right.\\
&+ \left.\left(S_0 e^{\sigma W_T + \left(\mu+\lambda(1-\alpha)-\frac{\sigma^2}{2}\right)T} - K\right)^+ \wedge x\, e^{\frac{\sigma}{1-\alpha}W_T + \left(\frac{\mu}{1-\alpha}+\lambda-\frac{\sigma^2}{2(1-\alpha)^2}\right)T}\cdot 1_{\{\tau_3>T\}}\right]\\
&+ (1-e^{-\lambda T})\, E^{\mathbb{P}}\left[\left(S_0 e^{\sigma W_T + \left(\mu+\lambda(1-\alpha)-\frac{\sigma^2}{2}\right)T + N_T\ln\alpha} - K\right)^+ \cdot 1_{\{\tau_3<\tau_2\}} \,|\, \tau_2 \le T\right].
\end{aligned}$$

For notational simplicity, let's define

$$\theta^{Bold} = \frac{\mu + \lambda(1-\alpha) - \frac{\sigma^2}{2}\frac{2-\alpha}{1-\alpha}}{\sigma},$$
$$b(x) = \frac{\frac{1-\alpha}{\alpha}\ln\frac{S_0}{x}}{\sigma},$$
$$a = \frac{\ln\frac{K}{S_0} - \frac{\sigma^2}{2(1-\alpha)}T}{\sigma},$$
$$a_n(x) = \frac{\ln\frac{K}{S_0} - \frac{\sigma^2}{2(1-\alpha)}T - n\ln\alpha - \sigma b(x)}{\sigma},$$
$$f_1(w) = S_0 e^{\sigma w + \frac{\sigma^2}{2(1-\alpha)}T} - K,$$
$$f_2(w) = x e^{\frac{\sigma}{1-\alpha}w + \frac{\sigma^2}{2(1-\alpha)}T}.$$

Notice that $f_1(a) = 0 < f_2(a)$, both functions are convex, and they intersect at most at two points. It will become clear in the proof of the following lemma that we can compute the value function of the Bold Strategy under three different cases:

- Case 1: $f_1(w) \le f_2(w)$ on $w \in [a, b(x)]$.
- Case 2: $f_1(w) \le f_2(w)$ on $w \in [a, c]$, and $f_1(w) \ge f_2(w)$ on $w \in [c, b(x)]$.
- Case 3: $f_1(w) \le f_2(w)$ on $w \in [a, c_1]$, $f_1(w) \ge f_2(w)$ on $w \in [c_1, c_2]$, and $f_1(w) \le f_2(w)$ on $w \in [c_2, b(x)]$.

Lemma 3.18. *Assume* $0 < x < \bar{x} = S_0$ *and* $a \le 0 < b(x)$. *Then the value function of the Bold Strategy is*

- *Case 1:* $u^{Bold}(x) = I_1(a,\infty,T) - I_2(a,\infty,T) + I_7(x) - I_8(x)$.
- *Case 2:* $u^{Bold}(x) = I_1(b(x),\infty,T) - I_2(b(x),\infty,T) + I_3(x,a,b(x),T) - I_4(x,a,b(x),T) + I_1(a,c,T) - I_3(x,a,c,T) - I_2(a,c,T) + I_4(x,a,c,T) + I_5(x,c,b(x),T) - I_6(x,c,b(x),T) + I_7(x) - I_8(x)$.
- *Case 3:* $u^{Bold}(x) = I_1(b(x),\infty,T) - I_2(b(x),\infty,T) + I_3(x,a,b(x),T) - I_4(x,a,b(x),T) + I_1(a,c_1,T) - I_3(x,a,c_1,T) - I_2(a,c_1,T) + I_4(x,a,c_1,T) + I_1(c_2,b(x),T) - I_3(x,c_2,b(x),T) - I_2(c_2,b(x),T) + I_4(x,c_2,b(x),T) + I_5(x,c_1,c_2,T) - I_6(x,c_1,c_2,T) + I_7(x) - I_8(x)$.

where $N(\cdot)$ is the cumulative distribution function of a standard normal random variable, and

$$
\begin{aligned}
I_1(l,u,\tau) &= S_0 e^{(\mu-\alpha\lambda)\tau}\left[N\left(\tfrac{u-(\sigma+\theta^{Bold})\tau}{\sqrt{\tau}}\right)-N\left(\tfrac{l-(\sigma+\theta^{Bold})\tau}{\sqrt{\tau}}\right)\right],\\
I_2(l,u,\tau) &= K e^{-\lambda\tau}\left[N\left(\tfrac{u-\theta^{Bold}\tau}{\sqrt{\tau}}\right)-N\left(\tfrac{l-\theta^{Bold}\tau}{\sqrt{\tau}}\right)\right],\\
I_3(x,l,u,\tau) &= S_0 e^{(\mu-\alpha\lambda)\tau+2b(x)(\sigma+\theta^{Bold})}\\
&\quad\cdot\left[N\left(\tfrac{u-2b(x)-(\sigma+\theta^{Bold})\tau}{\sqrt{\tau}}\right)-N\left(\tfrac{l-2b(x)-(\sigma+\theta^{Bold})\tau}{\sqrt{\tau}}\right)\right]\\
&= e^{2b(x)(\sigma+\theta^{Bold})} I_1(l-2b(x),u-2b(x),\tau),\\
I_4(x,l,u,\tau) &= K e^{-\lambda\tau+2b(x)\theta^{Bold}}\left[N\left(\tfrac{u-2b(x)-\theta^{Bold}\tau}{\sqrt{\tau}}\right)-N\left(\tfrac{l-2b(x)-\theta^{Bold}\tau}{\sqrt{\tau}}\right)\right]\\
&= e^{2b(x)\theta^{Bold}} I_2(l-2b(x),u-2b(x),\tau),\\
I_5(x,l,u,\tau) &= x e^{\frac{\mu}{1-\alpha}\tau}\left[N\left(\frac{u-(\theta^{Bold}+\frac{\sigma}{1-\alpha})\tau}{\sqrt{\tau}}\right)-N\left(\frac{l-(\theta^{Bold}+\frac{\sigma}{1-\alpha})\tau}{\sqrt{\tau}}\right)\right],\\
I_6(x,l,u,\tau) &= x e^{\frac{\mu}{1-\alpha}\tau+2b(x)(\theta^{Bold}+\frac{\sigma}{1-\alpha})}\\
&\quad\cdot\left[N\left(\frac{u-2b(x)-(\theta^{Bold}+\frac{\sigma}{1-\alpha})\tau}{\sqrt{\tau}}\right)-N\left(\frac{l-2b(x)-(\theta^{Bold}+\frac{\sigma}{1-\alpha})\tau}{\sqrt{\tau}}\right)\right]\\
&= e^{2b(x)(\theta^{Bold}+\frac{\sigma}{1-\alpha})} I_5(x,l-2b(x),u-2b(x),\tau),\\
I_7(x) &= \sum_{n=1}^{\infty} S_0 e^{n\ln\alpha+(\mu-\alpha\lambda)T}\int_0^T\left(1-N\left(\tfrac{a_n(x)-(\sigma+\theta^{Bold})(T-s)}{\sqrt{T-s}}\right)\right)\\
&\quad\cdot\frac{b(x)}{\sqrt{2\pi s^3}}e^{-\frac{(b(x)-(\theta^{Bold}+\sigma)s)^2}{2s}}\frac{(\lambda(T-s))^n}{n!}\,ds,\\
I_8(x) &= \sum_{n=1}^{\infty} K e^{-\lambda T}\int_0^T\left(1-N\left(\tfrac{a_n(x)-\theta^{Bold}(T-s)}{\sqrt{T-s}}\right)\right)\\
&\quad\cdot\frac{b(x)}{\sqrt{2\pi s^3}}e^{-\frac{(b(x)-\theta^{Bold}s)^2}{2s}}\frac{(\lambda(T-s))^n}{n!}\,ds.
\end{aligned}
$$

Proof.

$$X_T^{Bold} < S_T \Longleftrightarrow \widetilde{W}_T \triangleq W_T + \theta^{Bold}T < b(x) \quad \text{on the set} \quad \{\tau_2 > T\},$$

and

$$
\begin{aligned}
S_T - K &= S_0 e^{\sigma\widetilde{W}_T+\frac{\sigma^2}{2(1-\alpha)}T} - K = f_1(\widetilde{W}_T),\\
X_T^{Bold} &= x e^{\frac{\sigma}{1-\alpha}\widetilde{W}_T+\frac{\sigma^2}{2(1-\alpha)}T} = f_2(\widetilde{W}_T).
\end{aligned}
$$

We know for $\tilde{w} \le \tilde{m}, \tilde{m} > 0$,

$$P(\widetilde{M}_s \in d\tilde{m}, \widetilde{W}_s \in d\tilde{w}) = \frac{2(2\tilde{m} - \tilde{w})}{\sqrt{2\pi s^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2s}} e^{\theta^{Bold}\tilde{w} - \frac{1}{2}(\theta^{Bold})^2 s} d\tilde{w} d\tilde{m},$$

where $\widetilde{M}$ is the running maximum of the drifted Brownian motion $\widetilde{W}$. Note that $b(x) > 0$ for $0 < x < S_0$, and

$$P(\tau_3 \in ds) = \frac{|b(x)|}{\sqrt{2\pi s^3}} e^{-\frac{(b(x)-\theta^{Bold}s)^2}{2s}} ds = \frac{b(x)}{\sqrt{2\pi s^3}} e^{-\frac{(b(x)-\theta^{Bold}s)^2}{2s}} ds, \quad s > 0$$

on the set $\{s < \tau_2\}$. We can also compute the conditional probability

$$P(\tau_2 \in du \,|\, \tau_2 \le T) = \frac{P(\tau_2 \in du, \tau_2 \le T)}{P(\tau_2 \le T)} = \frac{P(\tau_2 \in du)}{P(\tau_2 \le T)} = \frac{\lambda e^{-\lambda u}}{1 - e^{-\lambda T}} du,$$

for $0 \le u \le T$. Note that $S_T > K$ iff $\widetilde{W}_T > a$, on the set $\{\tau_2 > T\}$, and the value function can be written as

$$u^{Bold}(x) = \text{term } 1 + \text{term } 2 + \text{term } 3,$$

where

$$\text{term } 1 = e^{-\lambda T} E\left[f_1(\widetilde{W}_T) \cdot 1_{\{\widetilde{M}_T > b(x), \widetilde{W}_T > a\}} \right],$$

$$\text{term } 2 = e^{-\lambda T} E\left[\left(f_1(\widetilde{W}_T) \wedge f_2(\widetilde{W}_T) \right) \cdot 1_{\{\widetilde{M}_T < b(x), \widetilde{W}_T > a\}} \right],$$

$$\text{term } 3 = (1 - e^{-\lambda T}) \int_0^T \int_0^u E\left[\left(S_0 e^{\sigma(b(x) + \widetilde{W}_{T-s}) + \frac{\sigma^2}{2(1-\alpha)}T + (1+N_{T-u})\ln\alpha} - K \right)^+ \right]$$
$$\cdot P(\tau_3 \in ds) P(\tau_2 \in du \,|\, \tau_2 \le T).$$

Note the condition $a \le 0 < b(x)$, we have

$$\begin{aligned}
\text{term } 1 &= e^{-\lambda T} \int_{b(x)}^{\infty} \int_a^{\tilde{m}} S_0 e^{\sigma\tilde{w} + \frac{\sigma^2}{2(1-\alpha)}T} \frac{2(2\tilde{m} - \tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} e^{\theta^{Bold}\tilde{w} - \frac{1}{2}(\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
&\quad - e^{-\lambda T} \int_{b(x)}^{\infty} \int_a^{\tilde{m}} K \frac{2(2\tilde{m} - \tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} e^{\theta^{Bold}\tilde{w} - \frac{1}{2}(\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
&= e^{-\lambda T} \int_a^{b(x)} S_0 e^{(\sigma + \theta^{Bold})\tilde{w} + \frac{\sigma^2}{2(1-\alpha)}T - \frac{1}{2}(\theta^{Bold})^2 T} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} d\tilde{w} \\
&\quad - e^{-\lambda T} \int_a^{b(x)} K e^{\theta^{Bold}\tilde{w} - \frac{1}{2}(\theta^{Bold})^2 T} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} d\tilde{w} \\
&\quad + e^{-\lambda T} \int_{b(x)}^{\infty} S_0 e^{(\sigma + \theta^{Bold})\tilde{w} + \frac{\sigma^2}{2(1-\alpha)}T - \frac{1}{2}(\theta^{Bold})^2 T} \frac{1}{\sqrt{2\pi T}} e^{-\frac{\tilde{w}^2}{2T}} d\tilde{w} \\
&\quad - e^{-\lambda T} \int_{b(x)}^{\infty} K e^{\theta^{Bold}\tilde{w} - \frac{1}{2}(\theta^{Bold})^2 T} \frac{1}{\sqrt{2\pi T}} e^{-\frac{\tilde{w}^2}{2T}} d\tilde{w} \\
&= I_1(b(x), \infty, T) - I_2(b(x), \infty, T) + I_3(x, a, b(x), T) - I_4(x, a, b(x), T).
\end{aligned}$$

where

$$
\begin{aligned}
I_1(l,u,\tau) &= e^{-\lambda\tau}\int_l^u S_0 e^{(\sigma+\theta^{Bold})\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}\tau-\frac{1}{2}(\theta^{Bold})^2\tau}\,\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{\tilde{w}^2}{2\tau}}\,d\tilde{w}\\
&= S_0 e^{(\mu-\alpha\lambda)\tau}\,\left[N\left(\tfrac{u-(\sigma+\theta^{Bold})\tau}{\sqrt{\tau}}\right)-N\left(\tfrac{l-(\sigma+\theta^{Bold})\tau}{\sqrt{\tau}}\right)\right],\\
I_2(l,u,\tau) &= e^{-\lambda\tau}\int_l^u K e^{\theta^{Bold}\tilde{w}-\frac{1}{2}(\theta^{Bold})^2\tau}\,\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{\tilde{w}^2}{2\tau}}\,d\tilde{w}\\
&= K e^{-\lambda\tau}\,\left[N\left(\tfrac{u-\theta^{Bold}\tau}{\sqrt{\tau}}\right)-N\left(\tfrac{l-\theta^{Bold}\tau}{\sqrt{\tau}}\right)\right],\\
I_3(x,l,u,\tau) &= e^{-\lambda\tau}\int_l^u S_0 e^{(\sigma+\theta^{Bold})\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}\tau-\frac{1}{2}(\theta^{Bold})^2\tau}\,\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{(2b(x)-\tilde{w})^2}{2\tau}}\,d\tilde{w}\\
&= S_0 e^{(\mu-\alpha\lambda)\tau+2b(x)(\sigma+\theta^{Bold})}\,\left[N\left(\tfrac{u-2b(x)-(\sigma+\theta^{Bold})\tau}{\sqrt{\tau}}\right)\right.\\
&\qquad \left.-N\left(\tfrac{l-2b(x)-(\sigma+\theta^{Bold})\tau}{\sqrt{\tau}}\right)\right]\\
&= e^{2b(x)(\sigma+\theta^{Bold})}I_1(l-2b(x),u-2b(x),\tau),\\
I_4(x,l,u,\tau) &= e^{-\lambda\tau}\int_l^u K e^{\theta^{Bold}\tilde{w}-\frac{1}{2}(\theta^{Bold})^2\tau}\,\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{(2b(x)-\tilde{w})^2}{2\tau}}\,d\tilde{w}\\
&= K e^{-\lambda\tau+2b(x)\theta^{Bold}}\,\left[N\left(\tfrac{u-2b(x)-\theta^{Bold}\tau}{\sqrt{\tau}}\right)-N\left(\tfrac{l-2b(x)-\theta^{Bold}\tau}{\sqrt{\tau}}\right)\right]\\
&= e^{2b(x)\theta^{Bold}}I_2(l-2b(x),u-2b(x),\tau).
\end{aligned}
$$

Under Case 1:

$$
\begin{aligned}
\text{term } 1 + \text{term } 2 &= e^{-\lambda T}E\left[\left(S_0 e^{\sigma\widetilde{W}_T+\frac{\sigma^2}{2(1-\alpha)}T}-K\right)\cdot 1_{\{\widetilde{M}_T>b(x),\widetilde{W}_T>a\}}\right]\\
&\quad + e^{-\lambda T}E\left[\left(S_0 e^{\sigma\widetilde{W}_T+\frac{\sigma^2}{2(1-\alpha)}T}-K\right)\cdot 1_{\{\widetilde{M}_T<b(x),\widetilde{W}_T>a\}}\right]\\
&= e^{-\lambda T}E\left[\left(S_0 e^{\sigma\widetilde{W}_T+\frac{\sigma^2}{2(1-\alpha)}T}-K\right)\cdot 1_{\{\widetilde{W}_T>a\}}\right]\\
&= e^{-\lambda T}\int_a^\infty\left(S_0 e^{\sigma\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}T}-K\right)\,\frac{1}{\sqrt{2\pi T}}e^{-\frac{(\tilde{w}-\theta^{Bold}T)^2}{2T}}\,d\tilde{w}\\
&= e^{-\lambda T}\int_a^\infty\left(S_0 e^{(\sigma+\theta^{Bold})\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}T-\frac{1}{2}(\theta^{Bold})^2T}-Ke^{\theta^{Bold}\tilde{w}-\frac{1}{2}(\theta^{Bold})^2T}\right)\\
&\qquad\cdot\frac{1}{\sqrt{2\pi T}}e^{-\frac{\tilde{w}^2}{2T}}\,d\tilde{w}\\
&= I_1(a,\infty,T)-I_2(a,\infty,T).
\end{aligned}
$$

Under Case 2: $(a < 0, c > 0)$

$$
\begin{aligned}
\text{term 2} &= e^{-\lambda T} \int_0^{b(x)} \int_a^{c\wedge\tilde{m}} S_0 e^{\sigma\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}T} \frac{2(2\tilde{m}-\tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} \\
&\qquad \cdot e^{\theta^{Bold}\tilde{w}-\frac{1}{2}(\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
&\quad - e^{-\lambda T} \int_0^{b(x)} \int_a^{c\wedge\tilde{m}} K \frac{2(2\tilde{m}-\tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} e^{\theta^{Bold}\tilde{w}-\frac{1}{2}(\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
&\quad + e^{-\lambda T} \int_c^{b(x)} \int_c^{\tilde{m}} x e^{\frac{\sigma}{1-\alpha}\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}T} \frac{2(2\tilde{m}-\tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} \\
&\qquad \cdot e^{\theta^{Bold}\tilde{w}-\frac{1}{2}(\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
&= e^{-\lambda T} \int_a^c S_0 e^{(\sigma+\theta^{Bold})\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}T-\frac{1}{2}(\theta^{Bold})^2 T} \\
&\qquad \cdot \frac{1}{\sqrt{2\pi T}} \left(e^{-\frac{\tilde{w}^2}{2T}} - e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} \right) d\tilde{w} \\
&\quad - e^{-\lambda T} \int_a^c K e^{\theta^{Bold}\tilde{w}-\frac{1}{2}(\theta^{Bold})^2 T} \frac{1}{\sqrt{2\pi T}} \left(e^{-\frac{\tilde{w}^2}{2T}} - e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} \right) d\tilde{w} \\
&\quad + e^{-\lambda T} \int_c^{b(x)} x e^{\left(\frac{\sigma}{1-\alpha}+\theta^{Bold}\right)\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}T-\frac{1}{2}(\theta^{Bold})^2 T} \\
&\qquad \cdot \frac{1}{\sqrt{2\pi T}} \left(e^{-\frac{\tilde{w}^2}{2T}} - e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} \right) d\tilde{w} \\
&= I_1(a,c,T) - I_3(x,a,c,T) - I_2(a,c,T) + I_4(x,a,c,T) \\
&\quad + I_5(x,c,b(x),T) - I_6(x,c,b(x),T),
\end{aligned}
$$

where

$$
\begin{aligned}
I_5(x,l,u,\tau) &= e^{-\lambda\tau} \int_l^u x e^{\left(\frac{\sigma}{1-\alpha}+\theta^{Bold}\right)\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}\tau-\frac{1}{2}(\theta^{Bold})^2\tau} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{\tilde{w}^2}{2\tau}} d\tilde{w} \\
&= x e^{\frac{\mu}{1-\alpha}\tau} \left[N\left(\frac{u-(\theta^{Bold}+\frac{\sigma}{1-\alpha})\tau}{\sqrt{\tau}} \right) - N\left(\frac{l-(\theta^{Bold}+\frac{\sigma}{1-\alpha})\tau}{\sqrt{\tau}} \right) \right] \\
I_6(x,l,u,\tau) &= e^{-\lambda\tau} \int_l^u x e^{\left(\frac{\sigma}{1-\alpha}+\theta^{Bold}\right)\tilde{w}+\frac{\sigma^2}{2(1-\alpha)}\tau-\frac{1}{2}(\theta^{Bold})^2\tau} \\
&\qquad \cdot \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(2b(x)-\tilde{w})^2}{2\tau}} d\tilde{w} \\
&= x e^{\frac{\mu}{1-\alpha}\tau+2b(x)(\theta^{Bold}+\frac{\sigma}{1-\alpha})} \\
&\quad \cdot \left[N\left(\frac{u-2b(x)-(\theta^{Bold}+\frac{\sigma}{1-\alpha})\tau}{\sqrt{\tau}} \right) - N\left(\frac{l-2b(x)-(\theta^{Bold}+\frac{\sigma}{1-\alpha})\tau}{\sqrt{\tau}} \right) \right] \\
&= e^{2b(x)(\theta^{Bold}+\frac{\sigma}{1-\alpha})} I_5(x, l-2b(x), u-2b(x), \tau).
\end{aligned}
$$

The case $(a < 0, c < 0)$ yield the same result.

Under Case 3: $(a < 0, c_1 < 0, c_2 > 0)$

$$
\begin{aligned}
\text{term } 2 = {} & e^{-\lambda T} \int_0^{b(x)} \int_a^{c_1} S_0 e^{\sigma \tilde{w} + \frac{\sigma^2}{2(1-\alpha)} T} \frac{2(2\tilde{m} - \tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} \\
& \cdot e^{\theta^{Bold} \tilde{w} - \frac{1}{2} (\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
& - e^{-\lambda T} \int_0^{b(x)} \int_a^{c_1} K \frac{2(2\tilde{m} - \tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} e^{\theta^{Bold} \tilde{w} - \frac{1}{2} (\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
& + e^{-\lambda T} \int_{c_2}^{b(x)} \int_{c_2}^{\tilde{m}} S_0 e^{\sigma \tilde{w} + \frac{\sigma^2}{2(1-\alpha)} T} \frac{2(2\tilde{m} - \tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} \\
& \cdot e^{\theta^{Bold} \tilde{w} - \frac{1}{2} (\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
& - e^{-\lambda T} \int_{c_2}^{b(x)} \int_{c_2}^{\tilde{m}} K \frac{2(2\tilde{m} - \tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} e^{\theta^{Bold} \tilde{w} - \frac{1}{2} (\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
& + e^{-\lambda T} \int_0^{b(x)} \int_{c_1}^{\tilde{m} \wedge c_2} x e^{\frac{\sigma}{1-\alpha} \tilde{w} + \frac{\sigma^2}{2(1-\alpha)} T} \frac{2(2\tilde{m} - \tilde{w})}{\sqrt{2\pi T^3}} e^{-\frac{(2\tilde{m}-\tilde{w})^2}{2T}} \\
& \cdot e^{\theta^{Bold} \tilde{w} - \frac{1}{2} (\theta^{Bold})^2 T} d\tilde{w} d\tilde{m} \\
= {} & e^{-\lambda T} \int_a^{c_1} S_0 e^{(\sigma + \theta^{Bold}) \tilde{w} + \frac{\sigma^2}{2(1-\alpha)} T - \frac{1}{2} (\theta^{Bold})^2 T} \\
& \cdot \frac{1}{\sqrt{2\pi T}} \left(e^{-\frac{\tilde{w}^2}{2T}} - e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} \right) d\tilde{w} \\
& - e^{-\lambda T} \int_a^{c_1} K e^{\theta^{Bold} \tilde{w} - \frac{1}{2} (\theta^{Bold})^2 T} \frac{1}{\sqrt{2\pi T}} \left(e^{-\frac{\tilde{w}^2}{2T}} - e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} \right) d\tilde{w} \\
& + e^{-\lambda T} \int_{c_2}^{b(x)} S_0 e^{(\sigma + \theta^{Bold}) \tilde{w} + \frac{\sigma^2}{2(1-\alpha)} T - \frac{1}{2} (\theta^{Bold})^2 T} \\
& \cdot \frac{1}{\sqrt{2\pi T}} \left(e^{-\frac{\tilde{w}^2}{2T}} - e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} \right) d\tilde{w} \\
& - e^{-\lambda T} \int_{c_2}^{b(x)} K e^{\theta^{Bold} \tilde{w} - \frac{1}{2} (\theta^{Bold})^2 T} \frac{1}{\sqrt{2\pi T}} \left(e^{-\frac{\tilde{w}^2}{2T}} - e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} \right) d\tilde{w} \\
& + e^{-\lambda T} \int_{c_1}^{c_2} x e^{\left(\frac{\sigma}{1-\alpha} + \theta^{Bold} \right) \tilde{w} + \frac{\sigma^2}{2(1-\alpha)} T - \frac{1}{2} (\theta^{Bold})^2 T} \\
& \cdot \frac{1}{\sqrt{2\pi T}} \left(e^{-\frac{\tilde{w}^2}{2T}} - e^{-\frac{(2b(x)-\tilde{w})^2}{2T}} \right) d\tilde{w} \\
= {} & I_1(a, c_1, T) - I_3(x, a, c_1, T) - I_2(a, c_1, T) + I_4(x, a, c_1, T) \\
& + I_1(c_2, b(x), T) - I_3(x, c_2, b(x), T) - I_2(c_2, b(x), T) \\
& + I_4(x, c_2, b(x), T) + I_5(x, c_1, c_2, T) - I_6(x, c_1, c_2, T)
\end{aligned}
$$

The cases ($a<0, c_1>0, c_2>0$) and ($a<0, c_1<0, c_2<0$) yield the same result. Recall

$$a_n(x) = \frac{\ln\frac{K}{S_0} - \frac{\sigma^2}{2(1-\alpha)}T - n\ln\alpha - \sigma b(x)}{\sigma}.$$

We have

$$\begin{aligned}
&E\left[\left(S_0 e^{\sigma(b(x)+\widetilde{W}_{T-s})+\frac{\sigma^2}{2(1-\alpha)}T+(1+N_{T-u})\ln\alpha} - K\right)^+\right] \\
&= E\left[E\left[\left(S_0 e^{\sigma(b(x)+\widetilde{W}_{T-s})+\frac{\sigma^2}{2(1-\alpha)}T+(1+n)\ln\alpha} - K\right)\cdot 1_{\{\widetilde{W}_{T-s}>a_{n+1}(x)\}}\right.\right. \\
&\qquad\qquad \left.\left.\Big|\, N_{T-u}=n\right]\right] \\
&= \sum_{n=0}^{\infty} e^{-\lambda(T-u)}\frac{(\lambda(T-u))^n}{n!} e^{\sigma b(x)+\frac{\sigma^2}{2(1-\alpha)}s+(1+n)\ln\alpha} \\
&\qquad\qquad \cdot I_1(x, a_{n+1}(x), \infty, T-s)e^{\lambda(T-s)} \\
&\quad - \sum_{n=0}^{\infty} e^{-\lambda(T-u)}\frac{(\lambda(T-u))^n}{n!} I_2(x, a_{n+1}(x), \infty, T-s)e^{\lambda(T-s)}.
\end{aligned}$$

Therefore, term 3 equals

$$\begin{aligned}
&(1-e^{-\lambda T})\int_0^T\int_0^u E\left[\left(S_0 e^{\sigma(b(x)+\widetilde{W}_{T-s})+\frac{\sigma^2}{2(1-\alpha)}T+(1+N_{T-u})\ln\alpha} - K\right)^+\right] \\
&\quad \cdot P(\tau_3\in ds)P(\tau_2\in du\,|\,\tau_2\le T) \\
&= I_7(x) - I_8(x),
\end{aligned}$$

where

$$\begin{aligned}
I_7(x) &= \sum_{n=1}^{\infty}\int_0^T e^{\sigma b(x)+\frac{\sigma^2}{2(1-\alpha)}s+n\ln\alpha} I_1(x, a_n(x), \infty, T-s) \\
&\qquad \cdot \frac{b(x)}{\sqrt{2\pi s^3}} e^{-\frac{(b(x)-\theta^{Bold}s)^2}{2s}-\lambda s}\frac{(\lambda(T-s))^n}{n!}\,ds \\
&= \sum_{n=1}^{\infty} S_0 e^{n\ln\alpha+(\mu-\alpha\lambda)T}\int_0^T \left(1 - N\left(\frac{a_n(x)-(\sigma+\theta^{Bold})(T-s)}{\sqrt{T-s}}\right)\right) \\
&\qquad \cdot \frac{b(x)}{\sqrt{2\pi s^3}} e^{-\frac{(b(x)-(\theta^{Bold}+\sigma)s)^2}{2s}}\frac{(\lambda(T-s))^n}{n!}\,ds,
\end{aligned}$$

$$
\begin{aligned}
I_8(x) &= \sum_{n=1}^{\infty} \int_0^T I_2(x, a_n(x), \infty, T-s) \frac{b(x)}{\sqrt{2\pi s^3}} e^{-\frac{(b(x)-\theta^{Bold}s)^2}{2s} - \lambda s} \frac{(\lambda(T-s))^n}{n!}\, ds \\
&= \sum_{n=1}^{\infty} K e^{-\lambda T} \int_0^T \left(1 - N\left(\tfrac{a_n(x)-\theta^{Bold}(T-s)}{\sqrt{T-s}}\right)\right) \\
&\qquad\qquad \cdot \frac{b(x)}{\sqrt{2\pi s^3}} e^{-\frac{(b(x)-\theta^{Bold}s)^2}{2s}} \frac{(\lambda(T-s))^n}{n!}\, ds.
\end{aligned}
$$

◇

As we can see from Figure 3.1 and Table 3.1, in the case of an at-the-money call option when we expect on average one 20% of drop in the stock price each year, 10% excess return and 20% volatility, the upper and lower bounds tell the story which match our intuition. When the initial capital is small, the risky Bold Strategy works very well, and the upper and lower bounds are very close. As the initial capital gets bigger, the optimal investment strategy should become more conservative, therefore the Bold Strategy is moving away from being optimal and the bounds are not tight anymore.

As we can see from Figure 3.2 and Table 3.2, everything else being the same, when we expect on average one 80% of drop in the stock price each year, the upper bound produced by the change of measure on the Poisson part $u^N(x)$ and the lower bound produced by the Bold Strategy $u^{Bold}(x)$ are nearly indistinguishable.

3.5.2 Bond case

When the option payoff is a constant, which we assume without loss of generality is $H \equiv 1$, we can similarly compute the formulae which are simpler than the ones computed in the call option case. Comparing Figure 3.5.2 and Figure 3.5.2, we observe that when the Brownian motion becomes the dominant part in the stock dynamics, the upper bounds switch positions. Recall from Corollary 3.15, it is better to take the minimum of the two upper bounds. In general, we can try to optimize the way we choose the dual processes to derive better upper bounds.

x	$u^{Bold}(x)$	$u^N(x)$	$u^B(x)$
1	1.63	1.65	3.23
2	3.26	3.30	5.57
3	4.86	4.95	7.57
4	6.39	6.59	9.36
5	7.81	8.24	10.98
6	9.09	9.89	12.45
7	10.22	11.54	13.80
8	11.18	13.17	15.03
9	11.97	14.27	16.15
10	12.61	15.36	17.14
11	13.08	16.46	18.00

Table 3.1: The graph shows the upper and lower bounds in the call option case with initial stock price $S_0 = 100$, strike $K = 100$, maturity $T = 1$, and parameters $\mu = .1$, $\sigma = .2$, $\alpha = .8$ and $\lambda = 1$. $u^B(x)$ is the upper bound produced by the change of measure on the Brownian part; $u^N(x)$ is the upper bound produced by the change of measure on the Poisson part; $u^{Bold}(x)$ is the lower bound produced by the bold strategy.

x	$u^{Bold}(x)$	$u^N(x)$	$u^B(x)$
1	1.1331	1.1331	3.4457
2	2.2663	2.2663	6.0485
3	3.3994	3.3994	8.3546
4	4.5326	4.5326	10.4712
5	5.6657	5.6657	12.4480
6	6.7989	6.7989	14.3138
7	7.9320	7.9320	16.0872
8	9.0651	9.0652	17.7815
9	10.1983	10.1983	19.4063
10	11.3314	11.3315	20.9692
11	12.4645	12.4646	22.4760
12	13.5976	13.5978	23.9316
13	14.7307	14.7309	25.3399
14	15.8638	15.8641	26.7041
15	16.9969	16.9972	28.0271
16	18.1300	18.1304	29.3113
17	19.2630	19.2635	30.5586
18	20.3960	20.3967	31.7709
19	21.5289	21.5298	32.9498
20	22.6618	22.6630	34.0965
21	23.7946	23.7961	35.2122
22	24.9273	24.9293	36.2981
23	26.0599	26.0624	37.3550
24	27.1923	27.1956	38.3836
25	28.3244	28.3287	39.3848
26	29.4562	29.4619	40.3590
27	30.5877	30.5950	41.3068
28	31.7185	31.7282	42.2286
29	32.8487	32.8613	43.1246
30	33.9778	33.9945	43.9952
31	35.1058	35.1276	44.8405
32	36.2320	36.2608	45.6606
33	37.3561	37.3940	46.4553
34	38.4772	38.5270	47.2246
35	39.5945	39.6602	47.9682

Table 3.2: The graph shows the upper and lower bounds in the call option case with initial stock price $S_0 = 100$, strike $K = 100$, maturity $T = 1$, and parameters $\mu = .1$, $\sigma = .2$, $\alpha = .2$ and $\lambda = 1$. $u^B(x)$ is the upper bound produced by the change of measure on the Brownian part; $u^N(x)$ is the upper bound produced by the change of measure on the Poisson part; $u^{Bold}(x)$ is the lower bound produced by the bold strategy.

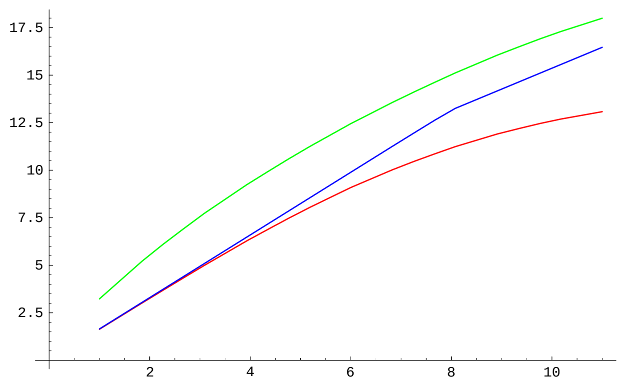

Figure 3.1: The graph shows the upper and lower bounds in the call option case with initial stock price $S_0 = 100$, strike $K = 100$, maturity $T = 1$, and parameters $\mu = .1$, $\sigma = .2$, $\alpha = .8$ and $\lambda = 1$. The green (upper) curve is the upper bound $u^B(x)$ produced by the change of measure on the Brownian part; the blue (middle) curve is the upper bound $u^N(x)$ produced by the change of measure on the Poisson part; the red (lower) curve is the lower bound $u^{Bold}(x)$ produced by the bold strategy.

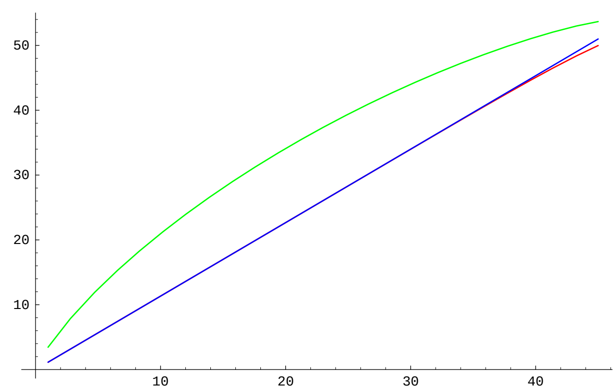

Figure 3.2: The graph shows the upper and lower bounds in the call option case with initial stock price $S_0 = 100$, strike $K = 100$, maturity $T = 1$, and parameters $\mu = .1$, $\sigma = .2$, $\alpha = .2$ and $\lambda = 1$. The green (upper) curve is the upper bound $u^B(x)$ produced by the change of measure on the Brownian part; the blue (middle) curve is the upper bound $u^N(x)$ produced by the change of measure on the Poisson part; the red (lower) curve is the lower bound $u^{Bold}(x)$ produced by the bold strategy. The curves $u^N(x)$ and $u^{Bold}(x)$ are nearly indistinguishable.

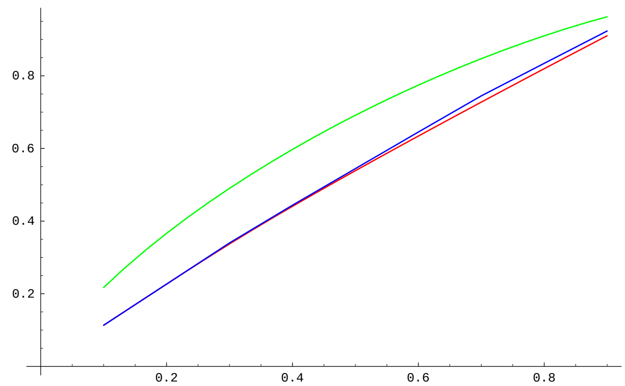

Figure 3.3: The graph shows the upper and lower bounds in the Bond case (payoff $\equiv 1$) with maturity $T = 1$, and parameters $\mu = .1$, $\sigma = .2$, $\alpha = .2$ and $\lambda = 1$. The green line is the upper bound $u^B(x)$ produced by the change of measure on the Brownian part; the blue line is the upper bound $u^N(x)$ produced by the change of measure on the Poisson part; the red line is the lower bound $u^{Bold}(x)$ produced by the bold strategy.

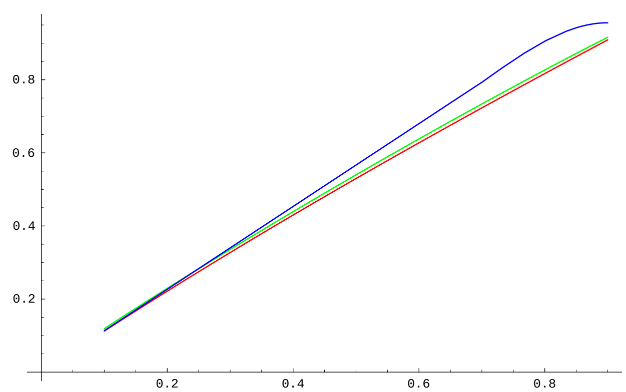

Figure 3.4: The graph shows the upper and lower bounds in the Bond case (payoff $\equiv 1$) with maturity $T = 1$, and parameters $\mu = .1$, $\sigma = 1$, $\alpha = .2$ and $\lambda = .1$. The green line is the upper bound $u^B(x)$ produced by the change of measure on the Brownian part; the blue line is the upper bound $u^N(x)$ produced by the change of measure on the Poisson part; the red line is the lower bound $u^{Bold}(x)$ produced by the bold strategy.

Appendix A

Lemmas for Proposition 1.12 in Chapter 1

Lemma A.1. *Suppose Assumptions 1.1, 1.3 and 1.4 hold, then*

$$\sup_{g\in\mathcal{B}_n}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right]=\inf_{h\in\mathcal{D}(y)}\sup_{g\in\mathcal{B}_n} E^{\mathbb{P}}\left[U(g)-gh\right]$$

Proof. "$\le$": We always have

$$\begin{aligned}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right]&\le E^{\mathbb{P}}\left[U(g)-gh\right],\\ \sup_{g\in\mathcal{B}_n}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right]&\le \sup_{g\in\mathcal{B}_n} E^{\mathbb{P}}\left[U(g)-gh\right],\\ \sup_{g\in\mathcal{B}_n}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right]&\le \inf_{h\in\mathcal{D}(y)}\sup_{g\in\mathcal{B}_n} E^{\mathbb{P}}\left[U(g)-gh\right].\end{aligned}$$

"$\ge$": Let $h\in\mathcal{D}(y)$ be given. Let $(g_k)_{k\ge 1}$ be a sequence in $\mathcal{B}_n$ such that the expectation on the left-hand side increases to the limit:

$$E^{\mathbb{P}}\left[U(g_k)-g_kh\right]\nearrow\sup_{g\in\mathcal{B}_n} E^{\mathbb{P}}\left[U(g)-gh\right]\quad\text{as}\quad n\to\infty.$$

By Lemma 1.8, there exists a sequence

$$\hat{g}_k\in conv(g_k,g_{k+1},\dots),\quad\text{and}\quad\hat{g}_k\to\hat{g}\quad\text{a.s.}$$

Note that $\hat{g}_k\in\mathcal{B}_n$ and $\hat{g}\in\mathcal{B}_n$. Concavity of U implies

$$E^{\mathbb{P}}\left[U(\hat{g}_k)-\hat{g}_kh\right]\ge E^{\mathbb{P}}\left[U(g_k)-g_kh\right].$$

By Assumption 1.4 and the Dominated Convergence Theorem,

$$E^{\mathbb{P}}\left[U(\hat{g}_k)\right] \to E^{\mathbb{P}}\left[U(\hat{g})\right].$$

Furthermore, $|\hat{g}_k - \hat{g}|h \leq 2n|h| \in L_1$, and again the dominated convergence theorem implies

$$|E^{\mathbb{P}}\left[\hat{g}_k h\right] - E^{\mathbb{P}}\left[\hat{g}h\right]| \leq E^{\mathbb{P}}\left[|\hat{g}_k - \hat{g}||h|\right] \to 0.$$

It follows that

$$\begin{aligned} E^{\mathbb{P}}\left[U(\hat{g}) - \hat{g}h\right] &= \lim_{k\to\infty} E^{\mathbb{P}}\left[U(\hat{g}_k) - \hat{g}_k h\right] \\ &= \sup_{g\in\mathcal{B}_n} E^{\mathbb{P}}\left[U(g) - gh\right]. \end{aligned}$$

Because $\hat{g}$ depends on h, we now write it as $\hat{g}(h)$. We have

$$\begin{aligned} E^{\mathbb{P}}\left[U(\hat{g}(h)) - \hat{g}(h)h\right] &= \sup_{g\in\mathcal{B}_n} E^{\mathbb{P}}\left[U(g) - gh\right], \\ \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(\hat{g}(h)) - \hat{g}(h)h\right] &= \inf_{h\in\mathcal{D}(y)} \sup_{g\in\mathcal{B}_n} E^{\mathbb{P}}\left[U(g) - gh\right], \\ \sup_{g\in\mathcal{B}_n} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] &\geq \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(\hat{g}(h)) - \hat{g}(h)h\right] \\ &= \inf_{h\in\mathcal{D}(y)} \sup_{g\in\mathcal{B}_n} E^{\mathbb{P}}\left[U(g) - gh\right]. \end{aligned}$$

$\diamond$

Remark A.2. *We still need to prove equation (3.8):*

$$\lim_{n\to\infty} \sup_{g\in\mathcal{B}_n} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] = \sup_{x>0} \sup_{g\in\mathcal{C}(x)} E^{\mathbb{P}}\left[U(g) - xy\right]$$

for Proposition 1.12 to be valid. It is basic technique to check for a function $f(\cdot)$ the following two equations hold:

$$\lim_{n\to\infty} \sup_{g\in\mathcal{B}_n} f(g) = \sup_{g\in\cup_{n=1}^{\infty}\mathcal{B}_n} f(g) \quad \textit{and} \quad \sup_{x>0} \sup_{g\in\mathcal{C}(x)} f(g) = \sup_{g\in\cup_{x>0}\mathcal{C}(x)} f(g).$$

Since

$$\cup_{n=1}^{\infty}\mathcal{B}_n = L_+^{\infty}(\Omega, \mathcal{F}, \mathbb{P}),$$

equation (3.8) is equivalent to

$$\sup_{g\in L_+^{\infty}} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] = \sup_{g\in\cup_{x>0}\mathcal{C}(x)} E^{\mathbb{P}}\left[U(g) - xy\right],$$

which will be a consequence of Lemma A.3 and Lemma A.5.

Lemma A.3. *Suppose Assumptions 1.1, 1.3 and 1.4 hold, then*

$$\sup_{g\in L_+^\infty}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right] = \sup_{g\in\cup_{x>0}\mathcal{C}(x)}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right]$$

Proof. "$\le$": Note that for any $g\in\mathcal{C}(x)$, by definition, there exists $X\in\mathcal{X}(x)$, i.e.

$$X_t = x+\int_0^t \xi_s dS_s \ge 0 \quad \mathbb{P}-a.s., \quad \text{for} \quad 0\le t\le T,$$

such that $0\le g\le X_T$. For any $x>0$, $X\equiv x\in\mathcal{C}(x)$, and thus

$$L_+^\infty(\Omega,\mathcal{F},\mathbb{P}) \;=\; \cup_{n=1}^\infty \mathcal{B}_n \;\subseteq\; \cup_{x>0}\mathcal{C}(x).$$

Therefore,

$$\sup_{g\in L_+^\infty}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right] \le \sup_{g\in\cup_{x>0}\mathcal{C}(x)}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right].$$

"$\ge$": Given $\epsilon>0$, choose $g_\epsilon\in\cup_{x>0}\mathcal{C}(x)$ such that

$$\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g_\epsilon)-g_\epsilon h\right] \ge \sup_{g\in\cup_{x>0}\mathcal{C}(x)}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right]-\epsilon.$$

Define $g_k=g_\epsilon\wedge k\in L_+^\infty$, we have $g_k\nearrow g_\epsilon$ a.s. Then

$$\begin{aligned}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g_k)-g_k h\right] &= E^{\mathbb{P}}\left[U(g_k)\right]-\sup_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[g_k h\right]\\ &\ge E^{\mathbb{P}}\left[U(g_k)\right]-\sup_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[g_\epsilon h\right].\end{aligned}$$

From Assumption 1.4, we can use the Monotone Convergence Theorem to get

$$\begin{aligned}\liminf_{k\to\infty}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g_k)-g_k h\right] &\ge \liminf_{k\to\infty}\left(E^{\mathbb{P}}\left[U(g_k)\right]-\sup_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[g_\epsilon h\right]\right)\\ &= E^{\mathbb{P}}\left[U(g_\epsilon)\right]-\sup_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[g_\epsilon h\right]\\ &= \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g_\epsilon)-g_\epsilon h\right]\\ &\ge \sup_{g\in\cup_{x>0}\mathcal{C}(x)}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right]-\epsilon.\end{aligned}$$

Therefore, there exists a K such that

$$\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g_K)-g_K h\right] \ge \sup_{g\in\cup_{x>0}\mathcal{C}(x)}\inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g)-gh\right]-2\epsilon.$$

Recall that $g_k \in L_+^\infty$, we have

$$\sup_{g\in L_+^\infty} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] \geq \sup_{g\in\cup_{x>0}\mathcal{C}(x)} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] - 2\epsilon.$$

This gives us the desired inequality. ◇

Example A.4. *A simple example will show that the above lemma is relevant. In a complete market, a call option is priced under the unique risk-neutral measure*

$$x = E^{\mathbb{Q}}\left[(S_T - K)^+\right] < \infty.$$

Starting with the initial capital x computed above, we can find a non-negative self-financing portfolio that exactly replicates the option payoff at expiration time T,

$$X_T = x + \int_0^T \xi_t dS_t = (S_T - K)^+.$$

Therefore by definition

$$X_T \in \mathcal{C}(x).$$

However, since S_T could be unbounded,

$$X_T \notin \mathcal{B}_n, \qquad \text{for any} \quad n > 0.$$

Lemma A.5. *Suppose Assumptions 1.1, 1.3 and 1.4 hold, then*

$$\sup_{g\in\cup_{x>0}\mathcal{C}(x)} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] = \sup_{g\in\cup_{x>0}\mathcal{C}(x)} E^{\mathbb{P}}\left[U(g) - xy\right],$$

or equivalently,

$$\sup_{x>0}\sup_{g\in\mathcal{C}(x)} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] = \sup_{x>0}\sup_{g\in\mathcal{C}(x)} E^{\mathbb{P}}\left[U(g) - xy\right].$$

Proof. "$\geq$": For any $g \in \mathcal{C}(x)$ and $h \in \mathcal{D}(y)$, we have $E^{\mathbb{P}}[gh] \leq xy$ by Proposition 1.6. Therefore

$$\sup_{g\in\mathcal{C}(x)} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] \geq \sup_{g\in\mathcal{C}(x)} E^{\mathbb{P}}\left[U(g) - xy\right].$$

And we have one direction of the inequality:

$$\sup_{x>0}\sup_{g\in\mathcal{C}(x)} \inf_{h\in\mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] \geq \sup_{x>0}\sup_{g\in\mathcal{C}(x)} E^{\mathbb{P}}\left[U(g) - xy\right].$$

"$\leq$": Let $g \in \mathcal{C}(x)$ be given and define

$$zy = \sup_{h\in\mathcal{D}(y)} E^{\mathbb{P}}[gh] \leq xy.$$

This implies

$$E^{\mathbb{P}}[gh] \leq zy \quad \forall h \in \mathcal{D}(y).$$

By (ii) of Theorem 1.6, we cam conclude that $g \in \mathcal{C}(z)$. Then

$$\begin{aligned}
\inf_{h \in \mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] &= E^{\mathbb{P}}\left[U(g)\right] - \sup_{h \in \mathcal{D}(y)} E^{\mathbb{P}}[gh] \\
&= E^{\mathbb{P}}\left[U(g)\right] - zy \\
&\leq \sup_{\tilde{g} \in \mathcal{C}(z)} E^{\mathbb{P}}\left[U(\tilde{g}) - zy\right] \\
&\leq \sup_{0 \leq z \leq x} \sup_{\tilde{g} \in \mathcal{C}(z)} E^{\mathbb{P}}\left[U(\tilde{g}) - zy\right],
\end{aligned}$$

where the right hand side is independent of z. Since the choice of $g \in \mathcal{C}(x)$ is free,

$$\sup_{g \in \mathcal{C}(x)} \inf_{h \in \mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] \leq \sup_{0 \leq z \leq x} \sup_{\tilde{g} \in \mathcal{C}(z)} E^{\mathbb{P}}\left[U(\tilde{g}) - zy\right],$$

and thus

$$\begin{aligned}
&\sup_{x > 0} \sup_{g \in \mathcal{C}(x)} \inf_{h \in \mathcal{D}(y)} E^{\mathbb{P}}\left[U(g) - gh\right] \\
&\qquad \leq \sup_{x > 0} \sup_{0 \leq z \leq x} \sup_{\tilde{g} \in \mathcal{C}(z)} E^{\mathbb{P}}\left[U(\tilde{g}) - zy\right] \\
&\qquad = \sup_{z > 0} \sup_{\tilde{g} \in \mathcal{C}(z)} E^{\mathbb{P}}\left[U(\tilde{g}) - zy\right].
\end{aligned}$$

So the other direction of the inequality is proved and we obtain equality. $\diamond$

Appendix B

Results of convex dual functions in space $\mathbb{R}$

Assumption B.1. *Suppose $u(x) : (0,\infty) \to \mathbb{R}$ is a concave and increasing function, $u^r(0+) > 0$ and $u^r(\infty) = \lim_{x\to\infty} u^r(x) = 0$.*

Definition B.2. $\bar{x} = \inf\{\, x : u$ *is a constant function on* $[x,\infty)\,\}$.

Remark B.3. *To derive results for the purpose of utility maximization, we assume that $u(x)$ is concave instead of being convex. Note that $u(x)$ does not have to be strictly concave.*

Lemma B.4. *Assume B.1. Define*

$$I(y) = \inf\{\, x \,:\, u^r(x) \le y \,\}, \quad \textit{for} \quad y > 0. \tag{1.1}$$

Then $I(y)$ is a decreasing, right continuous function, $I(y) = 0$ for $y \ge u^r(0+)$, and $I(0+) = \bar{x}$. $u^r(x)$ is symmetrically related to $I(y)$ by

$$u^r(x) = \inf\{\, y \,:\, I(y) \le x \,\}, \quad \textit{for} \quad x > 0. \tag{1.2}$$

Proof. Since $u(x)$ is concave, the right-hand derivative $u^r(x)$ exists and decreases to 0 as $x \to \infty$. Therefore, $I(y)$ is well defined for all $y > 0$. It is straight forward to check that $I(y)$ is decreasing, right continuous, $I(y) = 0$ for $y \ge u^r(0+)$ and $I(0+) = \bar{x}$. Fix $\tilde{x}$, by definition (1.1)

$$I(u^r(\tilde{x})) = \inf\{\, x \,:\, u^r(x) \le u^r(\tilde{x}) \,\}.$$

Since $u^r(x)$ is right-continuous and decreasing, $I(u^r(\tilde{x})) = \tilde{x}$ if u^r is strictly decreasing on $(\tilde{x} - \epsilon, \tilde{x}]$ for some $\epsilon > 0$. Also $I(u^r(\tilde{x})) < \tilde{x}$ if u^r is a constant on $(\tilde{x} - \epsilon, \tilde{x}]$ for some $\epsilon > 0$. Therefore, $I(u^r(\tilde{x})) \leq \tilde{x}$, and

$$u^r(\tilde{x}) \geq \inf\{\, y \,:\, I(y) \leq \tilde{x} \,\}.$$

Let $\tilde{y} > 0$ satisfy $I(\tilde{y}) \leq \tilde{x}$. We have $u^r(I(\tilde{y})) \geq u^r(\tilde{x})$. By definition (1.1), $u^r(I(\tilde{y})) \leq \tilde{y}$. Therefore $u^r(\tilde{x}) \leq \tilde{y}$, and we have

$$u^r(\tilde{x}) \leq \inf\{\, y \,:\, I(y) \leq \tilde{x} \,\}.$$

$\diamond$

Remark B.5. *Since $u(x)$ is a concave function, it can only have countably many points where the derivative does not exist, i.e., $UND = \{\, x \,:\, u^l(x) \neq u^r(x) \,\}$ is a countable set. u also have only countably many linear pieces. For everywhere else, u', the derivative of u exists, is strictly decreasing and therefore invertible. Suppose $u(x)$ is linear on $x \in [a, b]$. Then $y = u'(x)$ for some $x \in (a, b)$ implies $I(y-) = b, I(y) = a$. So the linear pieces of u are associated to the jumps in I. Define*

$$IJump = \{\, y \,:\, I(y-) > I(y) \,\}.$$

We know $IJump$ is a countable set. Suppose $u^l(x) > u^r(x)$. Then $y \in [u^r(x), u^l(x))$ implies $I(y) = x$. Therefore a kink in u (a point in UND), corresponds to a constant piece in I. Define

$$IConst = \{\, [u^r(x), u^l(x)) \,:\, u^l(x) > u^r(x) \text{ for some } x > 0 \,\}.$$

Define

$$IInv = (0, u^r(0+))/(IJump \cup IConst).$$

On the set $IInv$, $I(y)$ is strictly decreasing, invertible and $I = (u')^{-1}$.

Theorem B.6. *Assume B.1. Define the Legendre-Fenchel transform*

$$v(y) = \sup_{x>0}[\, u(x) - xy \,], \qquad \text{for each } y > 0.$$

Then
(i) $v(y)$ is a convex and decreasing function, and $v(y) < \infty$ for any $y > 0$.
(ii) $u(x) = \inf_{y>0}[\, v(y) + xy \,]$.
(iii) $-v^r(y) = I(y)$, i.e.,
$-v^r(y) = \inf\{\, x \,:\, u^r(x) \leq y \,\}$ and $u^r(x) = \inf\{\, y \,:\, -v^r(y) \leq x \,\}$.
(iv) When $y \in (0, u^r(0+)]$, $v(y) = u(x) - xy$ if and only if $y \in \partial u(x)$; when

$y \in [u^r(0+), \infty)$, $v(y) \equiv u(0+)$ is a constant. When $x \in (0, \bar{x}]$, $u(x) = v(y) + xy$ if and only if $x \in -\partial v(y)$; when $x \in [\bar{x}, \infty)$, $u(x) = u(\bar{x}) = v(0+)$ is a constant.
(v) $u(x)$ is strictly concave at x if and only if $v^r(y)$ is continuous at $y \in \partial u(x)$; $v(y)$ is strictly convex at y if and only if $u^r(x)$ is continuous at $x \in -\partial v(y)$.

Remark B.7. *Suppose $u(x)$ is right continuous at $x = 0$, and $v(y)$ is right continuous at $y = 0$. Note that $\partial u(0) = [u^r(0+), \infty)$ and $\partial v(0) = [\bar{x}, \infty)$. Then (iv) can be stated in a stronger version:*
(iv)' $v(y) = u(x) - xy$ if and only if $y \in \partial u(x)$, or equivalently, $x \in -\partial v(y)$.

Proof. (iii) Since $u(x)$ is concave, $u(x) - xy$ is concave for any fixed $y > 0$. Note that for $y \in (0, u^r(0+))$, $0 < I(y) < \infty$, and

$$\text{when} \quad x < I(y), \quad u^r(x) > y; \qquad \text{when} \quad x > I(y), \quad u^r(x) \le y.$$

We conclude that

$$v(y) = u(I(y)) - yI(y), \quad \text{for} \quad y \in (0, u^r(0+)). \tag{1.3}$$

Suppose $u^r(0+) < \infty$. Then for $y \ge u^r(0+)$, $u^r(x) - y \le 0$, and thus $u(x) - xy$ is a decreasing function of x and so

$$v(y) = u(0+), \quad \text{when} \quad u^r(0+) < \infty \quad \text{and} \quad y \in [u^r(0+), \infty). \tag{1.4}$$

When $y \ge u^r(0+)$, $v^r(y) = 0 = -I(y)$. When $y \in (0, u^r(0+))$, we will discuss in three cases. By definition and (1.3), for $y \in (0, u^r(0+))$,

$$v^r(y) = \lim_{h \searrow 0} \frac{[\, u(I(y+h)) - (y+h)I(y+h)\,] - [\, u(I(y)) - yI(y)\,]}{h}.$$

For any $y \in IConst$, $I(y+h) = I(y)$ and $u(I(y+h)) = u(I(y))$, therefore

$$v^r(y) = \lim_{h \searrow 0} \frac{-hI(y)}{h} = -I(y).$$

For any $y \in IInv$, $I = (u')^{-1}$ and u' is continuous by the analysis of remark B.5 and the symmetric relationship between I and u^r in lemma B.4, therefore

$$\begin{aligned} v^r(y) &= \lim_{h \searrow 0} \left\{ \frac{u(I(y+h)) - u(I(y))}{h} - \frac{y[I(y+h) - I(y)]}{h} - I(y+h) \right\} \\ &= \lim_{h \searrow 0} \left\{ \frac{u'(\xi)[I(y+h) - I(y)]}{h} - \frac{y[I(y+h) - I(y)]}{h} \right\} - I(y) \\ &\qquad\qquad \text{for some} \quad \xi \in [I(y+h), I(y)] \\ &= -I(y). \end{aligned}$$

Since both v^r and I are right continuous functions, they have to agree on the countable set $IJump$. In conclusion,

$$-v^r(y) = I(y). \tag{1.5}$$

By Lemma B.4, we have

$$u^r(x) = \inf\{\, y \,:\, I(y) \le x \,\} = \inf\{\, y \,:\, -v^r(y) \le x \,\}.$$

(i) (1.3) and (1.4) implies $v(y) < \infty$ for all $y > 0$. By (1.5) and Lemma B.4, we know $v^r(y) \le 0$ and is increasing. Therefore, $v(y)$ is decreasing and convex.
(ii) By the definition of v, we have the inequality

$$v(y) \ge u(x) - xy, \quad \text{or} \quad u(x) \le v(y) + xy, \quad \text{for any} \quad x > 0, y > 0.$$

When $x \in (0, \bar{x})$, we will discuss in two cases. On the set $IInv \cup IConst$, (1.3) gives

$$u(x) = v(y) + xy, \quad \text{where} \quad x = I(y).$$

On the set $IJump$, $u(x)$ is a linear function, and therefore

$$u(x) = v(y) + xy, \quad \text{where} \quad x =\in [I(y), I(y-)].$$

When $x \in [\bar{x}, \infty)$, $u(x) = v(0+)$ is a constant.
(iv) Define

$$\tilde{I}(y) = \{\, x : y \in \partial u(x) \,\}.$$

Following the same analysis in deriving (1.3), we get

$$v(y) = u(x) - xy \quad \text{if and only if} \quad x \in \tilde{I}(y) \quad \text{when} \quad y \in (0, u^r(0+)).$$

Or equivalently,

$$v(y) = u(x) - xy \quad \text{if and only if} \quad y \in \partial u(x) \quad \text{when} \quad y \in (0, u^r(0+)).$$

By continuity, the equality works for $y = u^r(0+)$ as well. By (ii) and the fact $I(0+) = -v^r(0+) = \bar{x}$, we conclude they are equivalent to $x \in -\partial v(y)$.
(v) This is an obvious consequence of the analysis in Remark B.5.

$\diamond$

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