Risk measure pricing and hedging in incomplete markets *

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Summary: This article attempts to extend the complete market option pricing theory to incomplete markets. Instead of eliminating the risk by a perfect hedging portfolio, partial hedging will be adopted and some residual risk at expiration will be tolerated. The risk measure (or risk indifference) prices charged for buying or selling an option are associated to the capital required for dynamic hedging so that the risk exposure will not increase. The associated optimal hedging portfolio is decided by minimizing a convex measure of risk. I will give the definition of risk-efficient options and confirm that options evaluated by risk measure pricing rules are indeed risk-efficient. Relationships to utility indifference pricing and pricing by valuation and stress measures will be discussed. Examples using the shortfall risk measure and average VaR will be shown.

Keywords and Phrases: Derivative Pricing, Valuation and Hedging, Incomplete Markets, Dynamic Shortfall Risk, Average Value-at-Risk, Utility Indifference Pricing, Convex Measure of Risk, Coherent Risk Measure, Risk-Efficient Options, Semimartingale Models, Risk Indifference Pricing.

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1 Introduction

The Black-Scholes model is the most celebrated example of option pricing and hedging in a complete market using no-arbitrage theory and martingale methods. Harrison and Kreps [27] studied the problem in the discrete-time case, and Harrison and Pliska [28] and [29] extended the results to the continuous-time case. However, if a financial intermediary (or an individual trader) would like to take a more realistic view of the financial world where she sees jumps in prices or stochastic volatility effects (this paper does not take into account market frictions), she has to think again about some fundamental questions in pricing and hedging facing the incomplete market. The no-arbitrage assumption provides a set of equivalent martingale measures and an interval of arbitrage-free prices. There is no exact replication to provide a unique price. If the trader decides to charge a super-replication (super-hedging) price for selling an option so that she can trade to eliminate all risks, as studied in El Karoui and Quenez [15], Kramkov [38] and Föllmer and Kabanov [18], the price is usually forbiddingly high. Eberlein and Jacod [14] investigated this issue in pure-jump models, and Bellamy and Jeanblanc [5] studied it in jump diffusion models. In both cases, the super-replication price for a European option is the trivial upper bound of the no-arbitrage interval.

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In the most common example of a call option, the super-hedging strategy is to buy and hold, and therefore the price of the call is equal to the initial stock price which is excessively expensive. Since super-hedging is not a realistic solution under such circumstance, the trader is restricted to charging a reasonable price, finding a partial hedging strategy according to some optimal criterion, and bearing some risks in the end.

There are two major approaches that have been developed in searching for solutions of pricing and hedging in incomplete markets. One is to pick a specific martingale measure for pricing according to some optimal criterion, and the other is utility based derivative pricing.

The incompleteness of the market usually gives rise to infinitely many martingale measures, each of which produces a no-arbitrage price. There have been multiple attempts to theoretically pick one for pricing according to different optimal criteria, of which some are related to utility maximization. For instance, we have seen the minimal martingale measure by Föllmer and Schweizer [22]; the minimal entropy martingale measure by Miyahara [41] and Frittelli [23]; the minimax measure by Bellini and Frittelli [6]; the minimal distance martingale measure by Goll and Rüschendorf [26]; and the Esscher transform by Gerber and Shiu [25]. In practice, financial intermediaries tend to first decide a specific jump or mixed diffusion model with certain control parameters for the underlying price process. Then they use the market data (for example, prices of vanilla options) to calibrate those parameters which determine a probability measure they believe is picked by the market, and use it for other exotic derivative pricing. The problem is that some methods do not provide a hedging strategy at all, while some provide one that is not very reasonable financially (for example, the Föllmer-Schweizer minimal martingale measure based on local variance minimization provides a strategy that penalizes over-hedging).

The idea of utility based derivative pricing is as follows: the derivative security is priced so that the utility remains the same whether the optimal trading portfolio includes a marginal amount of the derivative security or not. An incomplete list of references includes Hodges and Neuberger [32], Davis [9], Karatzas and Kou [35], Frittelli [24], Foldes [16], Rouge and El Karoui [47], Kallsen [34], Henderson [30], Hugonnier, Kramkov and Schachermayer [33], Mania and Schweizer [40], and Henderson and Hobson [31]. The disadvantage to this approach is that, in practice, it is quite unusual for the trader to explicitly write down her utility function for derivative pricing.

There are many other approaches that try to extend the arbitrage pricing theory to incomplete markets. See Carr et al. [7], Musiela and Zariphopoulou [43], [44] and [45], Barrieu and El Karoui [3] and [4], and Klöppel and Schweizer [37]. Carr et al. [7] introduced two sets of valuation and stress measures and the corresponding definition of no strictly acceptable opportunities (NSAO) in a static model. Under the NSAO condition, the pricing measure was a linear combination of valuation measures. Larsen et al. [39] adapted this framework to a dynamic setting and found the option price that is associated with a trading strategy to acceptability. We will see in Section 3.2 that this approach is quite closely related to the risk measure pricing method presented in this paper. Barrieu and El Karoui [3] and [4] set up a minimization problem for risk measure subject to dynamic hedging similar to this paper. However, their technique involved modelling both risk measures of the buyer and the seller, and solved for a pair of optimal derivative payoffs as well as the price. I will derive the price for a given option H from modelling the risk measure of the trader only. Meanwhile, Musiela and Zariphopoulou [43], [44] and [45] mainly delt with incompleteness coming from non-traded assets and their pricing scheme is a novel mixture of utility indifference and the martingale approach. An independent work by Klöppel and Schweizer [37] discussed risk measure pricing (they named it 'dynamic monetary concave utility function') for bounded payoffs in a dynamic time setting, with a backward stochastic differential equations representation that induces time-consistency.

Motivated by the idea of utility based derivative pricing, here I would like to keep the central importance of optimal hedging to the theory of pricing, and replace the criterion of maximizing utility by minimizing risk exposure because the latter is more often used in practice. Another reason to adopt risk measure is that it is quite a natural extension to the idea of pricing and hedging in complete markets. In the Black-Scholes world, there is a unique risk-neutral measure, the trader quotes a unique price, does the delta-hedging, and presumably, ends up with no risk and no profit. An extension of this idea into an incomplete market would be that the trader buys or sells the option for an amount such that with active hedging, her risk exposure will not increase at expiration. This would be the minimal condition for her to be willing to enter the deal in the first place. Comparing to utility indifference pricing, we should call risk measure pricing, in essence, risk indifference pricing.

I will set up our problem in the same background as in Hodges and Neuberger [32]. The trader is assumed to start with a portfolio and an optimal hedging. If she buys or sells an additional option, she would have to readjust the hedging strategy. Depending on the distribution of her current portfolio and her risk exposure, it will not be surprising that she might be more willing to buy or sell certain types of options, and this will be fully reflected in the prices she charges for these options. This result is in contrast with the assumptions in Hodges and Neuberger [32] where they argued that "To obtain a more tractable problem, we wish to ignore the interactions that would normally exist between the new opportunity and the rest of the portfolio, and between the success of the replicating strategy and the agent's risk aversion". By this reasoning, they adopted an exponential utility function. However, financial intermediaries usually hedge a portfolio of risky liabilities for the benefit of scale. So there is significant interaction going on between the new options issued and the existing ones. The advance of computing power over the past years and the explosion of active risk management have also made the approach more tractable and practical. Therefore, it is desirable to consider a more sophisticated model that takes these interactions into consideration.

I will implement this option pricing method with an abstract convex risk measure, define the risk measure prices, and derive their properties in Section 2. In practice, the definitions can be adapted to specific choices of risk measures depending on the reality within individual institutions. I will also propose the idea of risk-efficient options and demonstrate its relationship to risk measure pricing. In Section 3, I will show that utility based derivative pricing and pricing methods proposed in Carr et al. [7] are special realizations of risk measure pricing. Two examples using the shortfall risk measure and the average Value-At-Risk are given in Section 4. Section 5 concludes the paper with cautionary remarks regarding the application of the risk measure pricing approach and some discussions of future research.

2 Risk Measure Pricing

2.1 Model Setup

We start with a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. As in usual financial models, there is a money market account B and a risky asset S to invest in, and the market is frictionless. To simplify notations, let the interest rate be zero, therefore $B \equiv 1$. The risky asset S is assumed to be a locally bounded real-valued semimartingale process. This means that S is composed of a finite variation part, a continuous local martingale part and a jump part. Intuitively, the finite variation part controls the growth rate, the continuous local martingale part gives the quadratic variation, and the model incorporates jumps of various sizes and intensity. Geometric Brownian motion, locally bounded Lévy processes, and many jump-diffusion and pure-jump models fall into this category. The reason we do not extend the model to more general processes than semimartingales is that

Delbaen and Schachermayer [11] proved that if there is no arbitrage for simple strategies in the market, then the risky asset has to be a semimartingale.

We would like to price an option that has a payoff $H \geq 0$[†] with expiration date T. Mathematically, this means H is a nonnegative $\mathcal{F}_T$-measurable random variable. A self-financing hedging strategy can be written as a stochastic integral with respect to the risky asset when interest rate is assumed to be zero: $X_t = x + \int_0^t \xi_s dS_s$. We define the *Admissible Set* $\mathcal{X}(x)$ that collects all the processes from admissible self-financing strategies with initial capital x in the same way as Delbaen and Schachermayer [11]:

$$\mathcal{X}(x) = \left\{ X \;\middle|\; X_t = x + \int_0^t \xi_s dS_s \geq c \quad \text{for some finite constant } c \text{ and for all } 0 \leq t \leq T \right\}.$$

Note the admissibility of any strategy does not depend on the initial capital (since c could vary for different strategies). To exclude arbitrage opportunities, we assume that the set of equivalent local martingale measures $\mathcal{M}$ is non-empty[‡]:

Assumption 2.1 $\mathcal{M} = \{ \mathbb{Q} \sim \mathbb{P} \mid S \text{ is a local martingale under } \mathbb{Q} \} \neq \emptyset$.

Suppose the trader starts with liability L which is an $\mathcal{F}_T$-measurable random variable uniformly bounded below by some constant c_L, and an admissible self-financing hedging portfolio $X^{x_0} \in \mathcal{X}(x_0)$ with initial capital x_0. Let her risk measure ρ map any[§] random variable into some real number or $+\infty$. Then her initial risk is $\rho(L - X_T^{x_0})$ where $X_T^{x_0} = x_0 + \int_0^T \xi_t^0 dS_t$. We can define *Minimal Risk* to be the risk associated with optimal hedging:

$$\rho^x(L) = \inf_{X \in \mathcal{X}(x)} \rho(L - X_T). \tag{2.1}$$

Coherent Risk Measure is an axiomatic approach to define the measure of risk proposed by Artzner et al. [2]. It requires the risk measure to satisfy a few fundamental properties. When X and Y are in terms of liabilities, we can rewrite these properties as:[¶]

- *Subadditivity:* $\rho(X+Y) \leq \rho(X) + \rho(Y)$;
- *Positive Homogeneity:* If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$;
- *Monotonicity:* If $X \leq Y$, then $\rho(X) \leq \rho(Y)$;
- *Translation Invariance:* If $m \in \mathbb{R}$, then $\rho(X+m) = \rho(X) + m$.

Föllmer and Schied [19] relaxed the first two conditions to require convexity:

- *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.

They call a risk measure which satisfies convexity, monotonicity and translation invariance a *Convex Risk Measure*. For our pricing purpose, we will make some assumptions about the trader's risk measure. Assumption 2.2 introduces some restrictions on the properties of the risk measure. Assumption 2.3 will guarantee the finiteness of the risk measure prices defined later.

[†] As the associate editor points out, this method works equally well as long as H is uniformly bounded below by a constant. More importantly, he/she points out that the introduction of options trading will in general affect the dynamics of the underlying asset. Therefore, we should think of S as the equilibrium price process in a competitive market where both S and H are traded.

[‡] See Delbaen and Schachermayer [11] for the equivalence between Assumption 2.1 and a variant of the no-arbitrage condition called No Free Lunch With Vanishing Risk (NFLVR) when S is a locally bounded semimartingale.

[§] For the restriction on the set of random variables that have a well-defined risk measure, see the discussion after Assumption 2.2.

[¶] Take $\rho(X) = \psi(-X)$ for some coherent risk measure $\psi(\cdot)$ to derive these properties.

Assumption 2.2 *The trader has a convex risk measure* $\rho(\cdot)$ *that satisfies the Fatou property: If* X_n *is a sequence of random variables which are uniformly bounded below by a constant, and* X_n *converges to* X *a.s., then*

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

Moreover, $\rho(X) = \rho(Y)$ *if* $X = Y$ *a.s.*

There is usually no problem to find a well-defined risk measure that is either coherent or convex on the set of bounded random variables. However, we have to be careful when we are dealing with unbounded random variables (see Delbaen [10]). For a random variable X that is uniformly bounded from below, but unbounded from above, we define $X^m = X \wedge m = \min\{X, m\}$ for some constant m. Obviously, X^m is a uniformly bounded sequence and $X^m \nearrow X$ a.s. as $m \nearrow \infty$. For a sequence Y_n uniformly bounded from below and $Y_n \nearrow Y$ a.s., Fatou property and the monotonicity of $\rho(\cdot)$ imply that $\rho(Y_n) \nearrow \rho(Y)$ whenever $\rho(Y)$ is defined. Thus it is natural to define $\rho(X)$ as the limit

$$\rho(X) = \lim_{m\nearrow\infty} \rho(X^m) = \lim_{m\nearrow\infty} \rho(X \wedge m), \tag{2.2}$$

whether it is finite or $+\infty$. For a general unbounded random variable X, we define $X_m = X \vee (-m) = \max\{X, -m\}$. By the monotonicity property, $\rho(X_m)$ is a decreasing sequence as $m \nearrow \infty$. In the case the limit is finite, we define

$$\rho(X) = \lim_{m\nearrow\infty} \rho(X_m) = \lim_{m\nearrow\infty} \rho(X \vee -m). \tag{2.3}$$

Otherwise, we will leave $\rho(X)$ undefined.

The key in this paper is to define the risk measure on all hedged positions for the minimal risk, and the following assumption would guarantee that.

Assumption 2.3 $\rho(L + H) < \infty$ *and* $\rho^0(0) > -\infty$*.*

Note that $\rho^0(0) > -\infty$ means $\rho(-X_T) = \lim_{n\to\infty} \rho(-(X_T \wedge n)) > -\infty$, for any $X \in \mathcal{X}(0)$. A risk measure without this property will be quite unreasonable because one could trade from zero initial capital to lower risk infinitely. Furthermore, $\rho(L - X_T) \geq \rho(c_L - X_T) = c_L + \rho(-X_T) > -\infty$, so $\rho(L - X_T)$ is well-defined for any $X \in \mathcal{X}(0)$. We will see from the following lemma that the minimal risk $\rho^x(L)$ inherits the properties of the original risk measure $\rho(L)$ for any fixed number x, and therefore the assumption of $\rho^0(0)$ being finite is the same as $\rho^x(0)$ being finite for all finite numbers x.

Lemma 2.4 *Under Assumptions 2.2 and 2.3, the minimal risk defined in (2.1) is a convex risk measure. In particular, the translation invariance property can be written as*

$$\rho^{x_1}(X - x_2) = \rho^{x_1+x_2}(X) = \rho^{x_1}(X) - x_2. \tag{2.4}$$

Proof. Without loss of generality, assume $\rho^x(X)$ and $\rho^x(Y)$ are both finite. Fix any $\epsilon > 0$ and $0 \leq \lambda \leq 1$, find $Z^1 \in \mathcal{X}(x)$ and $Z^2 \in \mathcal{X}(x)$ such that $\rho(X - Z^1_T) \leq \rho^x(X) + \epsilon$ and $\rho(Y - Z^2_T) \leq \rho^x(Y) + \epsilon$. We know $Z^3 = \lambda Z^1 + (1-\lambda)Z^2 \in \mathcal{X}(x)$. By the convexity of $\rho(\cdot)$, we have

$$\begin{aligned}\rho^x(\lambda X + (1-\lambda)Y) &\leq \rho(\lambda X + (1-\lambda)Y - Z^3_T) \\ &= \rho(\lambda(X - Z^1_T) + (1-\lambda)(Y - Z^2_T)) \\ &\leq \lambda\rho(X - Z^1_T) + (1-\lambda)\rho(Y - Z^2_T) \\ &\leq \lambda\rho^x(X) + (1-\lambda)\rho^x(Y) + \epsilon.\end{aligned}$$

It follows that $\rho^x(\cdot)$ has the convexity property. The other properties are easy to prove. ◇

Lemma 2.5 *Under Assumptions 2.2 and 2.3,* $-\infty < \rho(L-H) \leq \rho(L) \leq \rho(L+H) < \infty$ *and* $-\infty < \rho^x(L-H) \leq \rho^x(L) \leq \rho^x(L+H) < \infty$ *for any finite number* x.

Proof. Since $H \geq 0$, we only need to prove $\rho(L-H) > -\infty$ for the first part to hold. We have assumed that the initial liability L is bounded below by a constant c_L. Therefore, $\rho(L-H) \geq \rho(c_L - H) = c_L + \rho(-H)$. So we need to prove that $\rho(-H) > -\infty$, i.e., $\rho(-(H \wedge n))$ approaches a finite limit when $n \to \infty$. By convexity, $\rho(0) \leq \frac{1}{2}\rho(H \wedge n) + \frac{1}{2}\rho(-(H \wedge n))$. Or equivalently, $\rho(-(H \wedge n)) \geq 2\rho(0) - \rho(H \wedge n)$. We know that $c_L + \rho(H) = \rho(c_L + H) \leq \rho(L+H) < \infty$. Therefore, $\rho(H \wedge n) \nearrow \rho(H) < \infty$ and $-\rho(H \wedge n) \searrow -\rho(H) > -\infty$. Also from the definition of the minimal risk and Assumption 2.3, we know $\rho(0) \geq \rho^0(0) > -\infty$. Putting together the last two inequalities, we can conclude the finiteness of the limit we needed to prove. For the second part, notice that $\rho^x(L+H) \leq \rho(L+H-x) = \rho(L+H) - x < \infty$. By Lemma 2.4, we know that $\rho^x(\cdot)$ is a convex risk measure. So we can show $\rho^x(L-H) > -\infty$ in a similar fashion as before, with the help of $-\infty < \rho^x(0) \leq \rho^x(H \wedge n) \leq \rho^x(0) + n \leq \rho(0) + n \leq \rho(H) + n \leq \rho(L+H) - c_L + n < \infty$. ⋄

The next theorem will present a proof of the existence of optimal strategies for obtaining the minimal risk defined in (2.1) under slightly more restrictive admissible sets. Define

$$\mathcal{X}(x,b) = \left\{ X \;\middle|\; X \in \mathcal{X}(x) \text{ and } X_T \geq x - b \right\}, \quad \text{for} \quad b \in \mathbb{R}^+.$$

This means that if $X_t = x + \int_0^t \xi_s dS_s \in \mathcal{X}(x,b)$, then $\int_0^T \xi_s dS_s \geq -b$. Obviously,

$$\mathcal{X}(x) = \bigcup_{b \in \mathbb{R}^+} \mathcal{X}(x,b).$$

Theorem 2.6 *Suppose Assumption 2.1, Assumption 2.2 and Assumption 2.3 hold. Then for every* $b \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$*, there exists an optimal admissible hedging portfolio* $X^* \in \mathcal{X}(x,b)$ *that solves the minimal risk problem:*

$$\rho_b^x(L) = \inf_{X \in \mathcal{X}(x,b)} \rho(L - X_T) = \rho(L - X_T^*). \tag{2.5}$$

Proof. Note that $\rho^x(L) \leq \rho_b^x(L) \leq \rho(L-x) = \rho(L) - x$. From Lemma 2.5, we know that $\rho_b^x(L)$ is finite. Choose $X^n \in \mathcal{X}(x,b)$ such that

$$\rho(L - X_T^n) \searrow \rho_b^x(L).$$

Denote by $conv(X_T^n, X_T^{n+1}, ...)$ a collection of finite convex combinations of elements in the set $\{X_T^n, X_T^{n+1}, ...\}$. Then we can find a sequence $Y^n \in conv(X_T^n, X_T^{n+1}, ...)$ such that $Y^n \to Y^*$ a.s.[‖]. Since $X_T^n \geq x - b$, $Y^n \geq x - b$ and $Y^* \geq x - b$ a.s. We can apply Fatou property to the truncated sequence: $\rho((L - Y^*) \vee (-m)) \leq \liminf_{n\to\infty} \rho((L - Y^n) \vee (-m))$ for any constant m. By monotonicity, we have $\rho(L - Y^*) \leq \rho((L - Y^*) \vee (-m))$ on one side. On the other side, we will prove $\liminf_{n\to\infty} \rho((L - Y^n) \vee (-m)) \leq \liminf_{n\to\infty} \rho(L - Y^n) + \epsilon$ for any $\epsilon > 0$ and any large m. Letting $\epsilon \to 0$, we will get the desired inequality $\rho(L - Y^*) \leq \liminf_{n\to\infty} \rho(L - Y^n)$. Let n_k be the subsequence such that $\lim_{n_k\to\infty} \rho(L - Y^{n_k}) = \liminf_{n\to\infty} \rho(L - Y^n)$. Note that for any fixed n, $\rho((L - Y^n) \vee (-m)) \searrow \rho(L - Y^n) > -\infty$, as $m \nearrow \infty$. The finiteness of $\rho(L - Y^n)$ results from the fact that Y^n is the time T wealth of some admissible self-financing strategy in the set $\mathcal{X}(x,b)$. Then for any $\epsilon > 0$, there exists some large number M such that for all $m \geq M$, we have $\rho((L - Y^{n_k}) \vee (-m)) \leq \rho(L - Y^{n_k}) + \epsilon$. Obviously, $\liminf_{n\to\infty} \rho((L - Y^n) \vee (-m)) \leq \liminf_{n_k\to\infty} \rho((L - Y^{n_k}) \vee (-m)) \leq \lim_{n_k\to\infty} \rho(L - Y^{n_k}) + \epsilon = \liminf_{n\to\infty} \rho(L - Y^n) + \epsilon$. Denote

$$Y^n = \sum_{i=k_1}^{k_m} \lambda_i^n X_T^i,$$

[‖] A proof of this result is given in Delbaen and Schachermayer [11].

where $n \le k_1 < k_2 < ... < k_m$, $0 \le \lambda_i^n \le 1$ and $\sum_{i=k_1}^{k_m} \lambda_i^n = 1$. Since $\rho(\cdot)$ is a convex risk measure,

$$\rho(L - Y^n) = \rho(L - \sum_{i=k_1}^{k_m} \lambda_i^n X_T^i) = \rho(\sum_{i=k_1}^{k_m} \lambda_i^n (L - X_T^i))$$
$$\le \sum_{i=k_1}^{k_m} \lambda_i^n \rho(L - X_T^i) \le \sum_{i=k_1}^{k_m} \lambda_i^n \rho(L - X_T^n) = \rho(L - X_T^n).$$

We conclude that

$$\rho(L - Y^*) \le \liminf_{n\to\infty} \rho(L - Y^n) \le \lim_{n\to\infty} \rho(L - X_T^n) = \rho_b^x(L).$$

If we can show that $Y^* \le X_T^*$ for some $X^* \in \mathcal{X}(x,b)$, then $\rho(L - X^*) \le \rho(L - Y^*)$ and we will be done. Define

$$K_0 = \{ X_T \mid X \in \mathcal{X}(x) \} \quad \text{and} \quad C_0 = K_0 - L_+^0,$$

where L_+^0 is the set of all nonnegative random variables. Assumption 2.1 implies NFLVR condition and Theorem 4.2 in Delbaen and Schachermayer [11] showed that when S is bounded, NFLVR guarantees that C_0 is Fatou closed: if for every sequence $(f_n)_{n\ge1} \in C_0$ uniformly bounded from below and such that $f_n \to f$ a.s, we have $f \in C_0$. Actually, the boundedness condition on S can be dropped, see Delbaen and Schachermayer [12]. Since $Y^n \in K_0 \subseteq C_0$ and Y^n is obviously Fatou convergent to Y^*, we conclude $Y^* \in C_0$. This means we can find $X^* \in \mathcal{X}(x)$ so that $Y^* = X_T^* - B$, where $B \in L_+^0$ is a non-negative random variable. Since $X_T^* \ge Y^* \ge x - b$, $X^* \in \mathcal{X}(x,b)$. ◇

In practice, we assume that the trader will fix a number b and that her initial hedging portfolio $X^{x_0,b} \in \mathcal{X}(x_0,b)$ is optimal in the sense of Theorem 2.6. In general,

$$(2.6)\quad \rho^{x_0}(L) = \inf_{X\in\mathcal{X}(x_0)} \rho(L - X_T) = \lim_{b\nearrow\infty} \min_{X\in\mathcal{X}(x_0,b)} \rho(L - X_T) = \lim_{b\nearrow\infty} \rho_b^{x_0}(L) = \lim_{b\nearrow\infty} \rho(L - X_T^{x_0,b}).$$

2.2 Definition of Risk Measure Pricing

Recall the trader starts with initial liability L, initial capital x_0, and initial minimal risk $\rho^{x_0}(L)$. Let us define the selling and buying prices of an option H according to the following principle: the trader will charge the minimal amount so that the total risk of her portfolio (after re-balancing the hedging) will not increase from selling the option; and she will buy the option with the maximal amount which will also keep the total risk of her portfolio stable.

Therefore, the *Selling Price* is defined as

$$(2.7)\qquad SP = \inf\{ x \,:\, \rho^{x_0+x}(L+H) \le \rho^{x_0}(L) \},$$

and the *Buying Price* is defined as

$$(2.8)\qquad BP = \sup\{ x \,:\, \rho^{x_0-x}(L-H) \le \rho^{x_0}(L) \}.$$

For the rest of Section 2, we will assume 2.1, 2.2 and 2.3. By the translation invariance property of the minimal risk (2.4), the prices defined above can be written as

$$(2.9)\qquad SP = \rho^{x_0}(L+H) - \rho^{x_0}(L),$$
$$(2.10)\qquad BP = \rho^{x_0}(L) - \rho^{x_0}(L-H).$$

Before we move on to derive some general properties of these prices, let me give some justifications for the risk measure pricing method I have just proposed. First of all, the definition of risk measure prices reduce to that of utility based prices as we will see in more detail in Section 3.1, when we define the risk measure as expected utility. Since this approach generalizes from expected utility functions to risk measures with the indifference pricing scheme intact, it is clear why we shall call SP and BP *Risk Indifference Prices* as well.

Secondly, this concept extends the complete market risk-neutral pricing theory to the incomplete case in the sense of risk preservation. In a complete market under the risk neutral pricing framework, the trader starts with zero liability $L = 0$, zero capital $x_0 = 0$, and zero risk $\rho = 0$. If she sells an option with payoff H, then she charges the risk-neutral price $E^{\mathbb{Q}}[H]$, and uses that capital to set up an admissible self-financing hedging portfolio X_t with $X_0 = E^{\mathbb{Q}}[H]$. Since the market is complete, the risk-neutral probability $\mathbb{Q}$ is unique, and by the martingale representation theorem $X_T = H$. Therefore, after selling the option and executing the hedge, the trader ends up with zero risk as she had started with. In an incomplete market under the risk measure pricing framework, the trader starts with liability L, capital x_0, and risk $\rho^{x_0}(L)$. If she sells an option with payoff H, then she charges the risk measure selling price SP, and partially hedges her risk exposure with an admissible self-financing portfolio X_t with total capital $x_0 + SP$. After the transaction and executing the optimal hedging, the trader ends up with the same risk as she started with, namely, $\rho^{x_0}(L)$. As expected, Theorem 2.7 in Section 2.3 shows that the risk measure prices coincide with the risk-neutral price in a complete market. This result is true even when the initial liability L and capital x_0 are nonzero in a complete market.

Last but not least, the risk measure prices are defined here in a way consistent with the principle of optimal design of derivatives.** Suppose the trader has some existing pricing function $p(\tilde{H})$ which decides the selling price for any option $\tilde{H}$. Let us define the *Risk-Efficient Options* to be those payoffs, with the same selling price, that minimize the risk. They are the solutions to the following minimization problem:

$$\inf_{\tilde{H}} \rho^{x_0+x}(L+\tilde{H}) \quad \text{s.t.} \quad p(\tilde{H}) = x. \tag{2.11}$$

Note that $\rho^{x_0+x}(L+\tilde{H})$ is the minimal risk obtained by optimal hedging with increased capital x_0+x and additional liability $\tilde{H}$. When a trader has some flexibility to decide what types of options to sell, it is favorable to sell risk-efficient options so that the trader's risk exposure is minimized. We claim that if the trader sells all options for their risk measure prices, i.e., $p(\cdot) = SP(\cdot)$, then every option she sells is a risk-efficient option. The reason is simple: By translation invariance (2.4) and the equation (2.9) for SP, we have $SP(H_1) = SP(H_2) = x$ implies $\rho^{x_0+x}(L+H_1) = \rho^{x_0+x}(L+H_2)$.

2.3 Properties of Risk Measure Pricing

We now discuss some basic properties that we expect the risk measure prices (or risk indifference prices) to satisfy under Assumptions 2.1, 2.2 and 2.3.

Property 1: Both the buying and the selling prices are finite, and the buying price is bounded above by the selling price $BP \leq SP$.

Proof. The finiteness of the prices follows from Lemma 2.5, and equations (2.9) and (2.10). By Lemma 2.4,

$$\tfrac{1}{2}\rho^{x_0}(L+H) + \tfrac{1}{2}\rho^{x_0}(L-H) \geq \rho^{x_0}(L).$$

This is equivalent to $SP \geq BP$ by inspecting equations (2.9) and (2.10). ◇

**Barrieu and El Karoui [3] and [4] proposed an optimal derivative design problem that involved both risk measures of the buyer and the seller.

Property 2: The selling price is bounded above by the super-hedging price and the buying price is bounded below by the sub-hedging price. Since the sub- and super-hedging prices span the no-arbitrage price interval, the selling and buying prices are arbitrage-free.

PROOF. Since $H \geq 0$, the super-hedging price and the sub-hedging price satisfy

$$\bar{x} = \sup_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[H] \geq \underline{x} = \inf_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[H] \geq 0,$$

where $\mathcal{M}$ is the set of equivalent local martingale measures. Without loss of generality, we suppose $\bar{x} < \infty$ and $\bar{X} \in \mathcal{X}(\bar{x})$ is the super-hedging portfolio, i.e., $\bar{X}_T \geq H$. Let $X^{x_0,b} \in \mathcal{X}(x_0, b) \subseteq \mathcal{X}(x_0)$ be an optimal solution as in (2.5), i.e.,

$$\rho_b^{x_0}(L) = \rho(L - X_T^{x_0,b}) = \min_{X \in \mathcal{X}(x_0,b)} \rho(L - X_T).$$

Note that $X^{x_0,b} + \bar{X} - \bar{x} \in \mathcal{X}(x_0)$. The translation invariance and the monotonicity of the risk measure $\rho(\cdot)$ give

$$\begin{aligned}
\rho^{x_0}(L + H) &\leq \rho(L + H - (X_T^{x_0,b} + \bar{X}_T - \bar{x})) \\
&= \rho(L - X_T^{x_0,b} + H - \bar{X}_T) + \bar{x} \\
&\leq \rho(L - X_T^{x_0,b}) + \bar{x} \\
&= \rho_b^{x_0}(L) + \bar{x}.
\end{aligned}$$

From equation (2.6), we have that $\rho_b^{x_0}(L) \searrow \rho^{x_0}(L)$ as $b \nearrow \infty$. Consequently, $SP = \rho^{x_0}(L + H) - \rho^{x_0}(L) \leq \bar{x}$. Suppose the sub-hedging price $\underline{x} < \infty$ and $\underline{X}^n \in \mathcal{X}(\underline{x}^n)$ is the sub-hedging portfolio for $H \wedge n$. Then we have $\underline{X}_T^n \leq H \wedge n$ and $\underline{X}_t^n = \inf_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[H \wedge n | \mathcal{F}_t] \leq n$. Note that $X^{x_0,b} - \underline{X}^n + \underline{x}^n \in \mathcal{X}(x_0)$. Then by the monotonicity and the translation invariance properties of the risk measure $\rho(\cdot)$, we have

$$\begin{aligned}
\rho^{x_0}(L - H) &\leq \rho^{x_0}(L - H \wedge n) \\
&\leq \rho(L - H \wedge n - (X_T^{x_0,b} - \underline{X}_T^n + \underline{x}^n)) \\
&= \rho(L - X_T^{x_0,b} + \underline{X}_T^n - H \wedge n) - \underline{x}^n \\
&\leq \rho(L - X_T^{x_0,b}) - \underline{x}^n \\
&= \rho_b^{x_0}(L) - \underline{x}^n.
\end{aligned}$$

Since $\rho_b^{x_0}(L) \searrow \rho^{x_0}(L)$ as $b \nearrow \infty$, and $\underline{x}^n \nearrow \underline{x}$ as $n \to \infty$, we have $BP = \rho^{x_0}(L) - \rho^{x_0}(L - H) \geq \underline{x}$. Now suppose $\underline{x} = \infty$. Then for any finite number x, there exists $X \in \mathcal{X}(x)$ such that $X_T \leq H$. Repeating the above argument, we get $BP \geq x$. Letting $x \to \infty$, we arrive at a contradiction to the fact that $BP < \infty$. ⋄

Corollary to Property 2: The price of a bond with constant payoff c is equal to c.

PROOF. The super- and sub-hedging price are

$$\bar{x} = \sup_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[c] = c \quad \text{and} \quad \underline{x} = \inf_{\mathbb{Q} \in \mathcal{M}} E^{\mathbb{Q}}[c] = c.$$

By Property 2, $BP = SP = c$. This result can also be proved directly from the translation invariance property of the minimal risk measure (2.4) and pricing equations (2.9) and (2.10). ⋄

Note that Property 2 and its corollary are worked out for general incomplete markets. Limiting them to a complete market, we can reconcile our pricing and hedging theory with Risk Neutral Pricing (or Arbitrage Pricing) Theory:

Theorem 2.7 (Complete Market) *In a complete market where $\mathcal{M} = \{\mathbb{Q}\}$, both the buying and the selling prices coincide with the risk neutral price, i.e.,*

$$SP = BP = E^{\mathbb{Q}}[H].$$

Proof. In this case, the super- and sub-hedging prices are unique and equal to the risk-neutral price:

$$\bar{x} = \underline{x} = E^{\mathbb{Q}}[H].$$

The theorem then follows easily from Property 2. $\diamond$

Next, let us discuss the dependence of pricing and hedging on the initial capital x_0, the distribution of the initial liability L, and the option payoff H. From (2.4), (2.9) and (2.10), we can easily conclude:

Property 3: Neither the selling price SP nor the buying price BP depends on the initial capital x_0 allocated for hedging. But both of them depend on the distribution of the initial liability L.

In fact, in the case of selling price, (2.9) can be written as $SP = \rho^x(L + H) - \rho^x(L)$ for any finite number x. The same holds true for the buying price. By definition (2.1) and equation (2.4), we can further conclude:

Property 4: The optimal hedging strategies for selling/buying the option H are also initial-wealth-x_0 independent.

In Property 5, we will investigate how the existing liability interacts with the new issuing of options.

Property 5: When the initial liability is zero ($L = 0$),

$$SP = \rho^0(H) - \rho^0(0) \quad \text{and} \quad BP = \rho^0(0) - \rho^0(-H).$$

When the initial liability is the same as the option payoff ($L = H$),

$$BP = \rho^0(H) - \rho^0(0) \leq SP = \rho^0(2H) - \rho^0(H).$$

Proof. By (2.9) and (2.4), when $L = 0$, $SP = \rho^0(H) - \rho^0(0) = \rho^{\rho^0(0)}(H)$. Similarly, we can prove the result in the case of BP. It is also straightforward to check them when $L = H$. $\diamond$

When $L = 0$ and $x_0 = 0$, we see that $\rho^{SP}(H) = \rho^0(H) - SP = \rho^0(0)$. Not surprisingly, when the trader uses SP for hedging the option H she has just sold, her minimal risk remains the same as if she had kept the zero position. Here is the risk consistency principle of her pricing model: in case the trader buys back the same option H she has just sold, she will not be exposed to any additional risk and she will pay exactly the same amount that she has sold the option for; however, if the trader takes on more concentrated risk by selling another option of the same type, she might want to charge more for this unfavorable risk skew. Therefore, the risk measure pricing rule is in general, not linear. For simplicity, we stated Property 5 by assuming $L = 0$. The result is true even if we do not restrict ourselves in such a way.

3 Relationships to Existing Pricing Methods

3.1 Relationship to Utility Based Derivative Pricing

A natural way to include the utility based derivative pricing theory as a special case is to make the following definition:

$$\rho(L) = -E[U(-L)], \tag{3.1}$$

where the utility function $U(\cdot)$ satisfies the usual conditions. The initial minimal risk can be written as

$$\rho^{x_0}(L) = \inf_{X \in \mathcal{X}(x_0)} \rho(L - X_T) = - \sup_{X \in \mathcal{X}(x_0)} E[\, U(X_T - L)\,].$$

By definition, the selling price is

$$\begin{aligned} SP &= \inf \left\{\, x \,:\, \rho^{x_0+x}(L+H) \leq \rho^{x_0}(L) \,\right\} \\ &= \inf \left\{ x \,:\, \sup_{X \in \mathcal{X}(x_0+x)} E[\, U(X_T - L - H)\,] \geq \sup_{X \in \mathcal{X}(x_0)} E[\, U(X_T - L)\,] \right\}. \end{aligned} \tag{3.2}$$

This is precisely the price based on utility maximization first explored in Hodges and Neuberger [32]. A similar argument holds for the buying price. However, the risk measure defined in (3.1) is not translation invariant, and therefore is not a convex risk measure as we require in Assumption 2.2. Some of the properties related to risk measure pricing derived in Section 2.3 will not hold as a consequence. In particular, this explains why the utility indifference prices are initial-capital-x_0 dependent, while the risk indifference prices are not. To satisfy the translation invariance, we can adopt the approach of deriving the risk measure from the acceptance set as in Section 4.1.

3.2 Relationship to Pricing by Valuation and Stress Measures

Carr et al. [7] proposed an approach of pricing options in incomplete markets by a specific set of equivalent probability measures which are not necessarily local martingale measures. Define the set of *Valuation Measures* to be

$$\mathcal{Q}_0 = \{\, \mathbb{Q}_1, \mathbb{Q}_2, ..., \mathbb{Q}_n \,\}, \quad \text{where } \mathbb{Q}_i \sim \mathbb{P} \text{ for } i = 1, ..., n.$$

Define the set of *Stress Measures* to be

$$\mathcal{Q}_1 = \{\, \mathbb{Q}_{n+1}, ..., \mathbb{Q}_m \,\}, \quad \text{where } \mathbb{Q}_i \sim \mathbb{P} \text{ for } i = n+1, ..., m.$$

The *Acceptance Set* is a collection of liabilities that satisfy the following condition:

$$\mathcal{A} = \{\, \tilde{L} \,:\, E^{\mathbb{Q}}[\tilde{L}] \leq \gamma^{\mathbb{Q}}, \, \forall \, \mathbb{Q} \in \mathcal{Q} \,\}, \tag{3.3}$$

where $\mathcal{Q} = \mathcal{Q}_0 \cup \mathcal{Q}_1$, and each probability measure $\mathbb{Q} \in \mathcal{Q}$ is associated with a floor that satisfies $\gamma^{\mathbb{Q}} = 0$ for $\mathbb{Q} \in \mathcal{Q}_0$, and $\gamma^{\mathbb{Q}} > 0$ for $\mathbb{Q} \in \mathcal{Q}_1$. Their *Selling Price* is defined as the minimal capital required for some partial hedging so that the outcome becomes acceptable:

$$p\,(H) = \inf\{\, x \,:\, \exists X \in \mathcal{X}(x) \; s.t. \; H - X_T \in \mathcal{A} \,\}. \tag{3.4}$$

We can always define the risk measure induced from an acceptance set as the smallest amount of capital injection to reduce the liability for it to be acceptable:

$$\begin{aligned} \rho_{\mathcal{A}}(\tilde{L}) &= \inf\{\, y \,:\, \tilde{L} - y \in \mathcal{A} \,\} \\ &= \inf\{\, y \,:\, E^{\mathbb{Q}}[\tilde{L} - y] \leq \gamma^{\mathbb{Q}}, \, \forall \, \mathbb{Q} \in \mathcal{Q} \,\} \\ &= \max_{\mathbb{Q} \in \mathcal{Q}} \left(E^{\mathbb{Q}}[\tilde{L}] - \gamma^{\mathbb{Q}} \right). \end{aligned} \tag{3.5}$$

In contrast, the acceptance set induced from a risk measure ρ is usually defined as

$$\mathcal{A}_\rho = \{ \tilde{L} \,:\, \rho(\tilde{L}) \le 0 \}.$$

Lemma 3.1 *For $\mathcal{A}$ defined in (3.3) and $\rho_\mathcal{A}$ defined in (3.5), we have*

$$\mathcal{A}_{\rho_\mathcal{A}} = \{ \tilde{L} \,:\, \rho_\mathcal{A}(\tilde{L}) \le 0 \} = \mathcal{A}.$$

Proof. It is easy to show $\mathcal{A} \subseteq \mathcal{A}_{\rho_\mathcal{A}}$. To prove the other inclusion, we have to show that $\tilde{L} \notin \mathcal{A}$ implies $\rho_\mathcal{A}(\tilde{L}) > 0$. $\tilde{L} \notin \mathcal{A}$ implies there exists some $\mathbb{Q}_k \in \mathcal{Q}$ such that $E^{\mathbb{Q}_k}[\tilde{L}] = \gamma^{\mathbb{Q}_k} + \epsilon$ for some $\epsilon > 0$. Therefore, $E^{\mathbb{Q}_k}[\tilde{L} - \frac{\epsilon}{2}] = \gamma^{\mathbb{Q}_k} + \frac{\epsilon}{2} > \gamma^{\mathbb{Q}_k}$ and $\rho_\mathcal{A}(\tilde{L}) = \inf\{ y \,:\, \tilde{L} - y \in \mathcal{A} \} \ge \frac{\epsilon}{2} > 0$. $\diamond$

In the risk measure pricing framework, let the initial liability $L = 0$ and the minimal risk be normalized to $\rho^0(0) = 0$. Then the risk measure prices coincide with the price defined in (3.4):

$$\begin{aligned} SP &= \inf\{ x \,:\, \rho^{x_0+x}(L+H) \le \rho^{x_0}(L) \} \\ &= \inf\{ x \,:\, \rho^{x}(H) \le 0 \} \\ &= \inf\{ x \,:\, \exists X \in \mathcal{X}(x) \ s.t.\ \rho(H - X_T) \le 0 \} \\ &= \inf\{ x \,:\, \exists X \in \mathcal{X}(x) \ s.t.\ H - X_T \in \mathcal{A}_{\rho_\mathcal{A}} \} \\ &= \inf\{ x \,:\, \exists X \in \mathcal{X}(x) \ s.t.\ H - X_T \in \mathcal{A} \} \\ &= p(H). \end{aligned}$$

We will check now whether the induced risk measure $\rho_\mathcal{A}(\cdot)$ defined in (3.5) satisfy the assumptions imposed on risk indifference pricing measures.

Lemma 3.2 (Valuation and Stress Risk Measure) *Suppose*

$$E^{\mathbb{Q}}[L+H] < \infty \quad \textit{and} \quad \sup_{X \in \mathcal{X}(0)} E^{\mathbb{Q}} X_T < \infty^{\dagger\dagger}, \quad \forall \mathbb{Q} \in \mathcal{Q}.$$

Then the risk measure defined in (3.5) is a convex risk measure and satisfies Assumptions 2.2 and 2.3. Therefore, it can be used as a pricing risk measure as in (2.7) and (2.8). In particular, the existence result (Theorem 2.6) and the properties of the risk measure prices derived in section 2.3 are valid under additional Assumption 2.1.

Proof. Since $\mathcal{A}$ defined in (3.3) is a convex set which satisfies the property that if $X \in \mathcal{A}$ and $Y \le X$, then $Y \in \mathcal{A}$, we can easily show that the induced risk measure $\rho_\mathcal{A}$ is a convex risk measure. Let X_n be a sequence of random variables that are uniformly bounded from below by a constant and $X_n \to X$ a.s. Fatou's Lemma implies $E^{\mathbb{Q}}[X] \le \liminf_{n\to\infty} E^{\mathbb{Q}}[X_n]$ for any $\mathbb{Q} \in \mathcal{Q}$. It follows that $\rho(X) \le \liminf_{n\to\infty} \rho(X_n)$. It is also obvious that $X = Y$ a.s. implies $\rho(X) = \rho(Y)$ because $\mathcal{Q}$ only includes equivalent probability measures. For the assumption on finiteness:

$$\begin{aligned} \rho(L+H) &= \max_{\mathbb{Q}\in\mathcal{Q}} \left(E^{\mathbb{Q}}[L+H] - \gamma^{\mathbb{Q}} \right) < \infty; \\ \rho^0(0) &= \inf_{X\in\mathcal{X}(0)} \max_{\mathbb{Q}\in\mathcal{Q}} \left(-E^{\mathbb{Q}} X_T - \gamma^{\mathbb{Q}} \right) \\ &\ge \max_{\mathbb{Q}\in\mathcal{Q}} \inf_{X\in\mathcal{X}(0)} \left(-E^{\mathbb{Q}} X_T - \gamma^{\mathbb{Q}} \right) = \max_{\mathbb{Q}\in\mathcal{Q}} \Big(- \sup_{X\in\mathcal{X}(0)} E^{\mathbb{Q}} X_T - \gamma^{\mathbb{Q}} \Big) > -\infty. \end{aligned}$$ $\diamond$

†† The second inequality is satisfied if $\mathcal{Q}$ only contains equivalent local martingale measures, under which case all admissible self-financing portfolio processes are supermartingales and $\sup_{X\in\mathcal{X}(0)} E^{\mathbb{Q}} X_T = 0$. As a result, the minimal risk is normalized to $\rho^0(0) = 0$.

4 Some Examples of Risk Measure Pricing

4.1 An Example of Pricing with Shortfall Risk Measure

Cvitanić and Karatzas [8] defined the dynamic measure of shortfall risk that took into consideration a dynamic hedging portfolio. Föllmer and Leukert [17] generalized the model to be based on semimartingale processes. Föllmer and Schied [19] proposed a static version of shortfall risk measure that was a convex risk measure. We extend these concepts to define a dynamic version of shortfall risk measure that is convex, so it can serve the purpose of the pricing theory developed in section 2.

Recall that the trader starts with initial liability L which is uniformly bounded below by a constant c_L, and initial capital x_0 for hedging. Define the *Shortfall Acceptance Set* to be a collection of liabilities that satisfy the following condition:

$$\mathcal{A} = \{ \tilde{L} \,:\, E\,[\, l(\tilde{L}^+)\,] \leq \tilde{x} \,\}, \tag{4.1}$$

where the loss function $l(\cdot)$ is convex and increasing on $[0, \infty)$, and is not a constant function. Assume $\tilde{x} > l(0)$ is a finite number. We will discuss the special case $\tilde{x} = l(0)$ seperately. Define the *Shortfall Risk Measure* from the acceptance set:

$$\rho(\tilde{L}) = \inf\{\, a \,:\, \tilde{L} - a \in \mathcal{A}\,\} = \inf\{\, a \,:\, E\,[\, l((\tilde{L} - a)^+)\,] \leq \tilde{x}\,\}. \tag{4.2}$$

If $E\,[l(\tilde{L}^+)]$ is finite, then $\rho(\tilde{L})$ is finite and by the monotone convergence theorem,

$$\rho(\tilde{L}) = \min\{\, a \,:\, E\,[\, l((\tilde{L} - a)^+)\,] \leq \tilde{x}\,\}.$$

The shortfall risk measure is equal to the smallest amount of capital injection to reduce the liability for it to be acceptable. As we will see from (4.3), $(L - X_T)^+$ is the shortfall after the trader hedges the liability L with a self-financing admissible strategy X. $\tilde{x}$ is the amount of expected shortfall we are willing to tolerate in the end. $\rho^{x_0}(L)$ turns out to be the minimal amount of additional capital to be used for hedging so that the expected shortfall is bounded above by $\tilde{x}$. Observe that the higher the tolerance of the expected shortfall $\tilde{x}$, the lower the risk measure $\rho^{x_0}(L)$. This implies the riskier the trader, the less the capital she will use for the purpose of hedging.

Lemma 4.1 *The minimal risk defined in (2.1) becomes*

$$\rho^{x_0}(L) = \inf_{X \in \mathcal{X}(x_0)} \rho(L - X_T) = \min\{\, a \,:\, \inf_{X \in \mathcal{X}(x_0+a)} E\,[\, l((L - X_T)^+)\,] \leq \tilde{x}\,\}, \tag{4.3}$$

when applied to the shortfall risk measure defined in (4.2).

Proof.

$$\begin{aligned} \rho^{x_0}(L) &= \inf_{X \in \mathcal{X}(x_0)} \rho(L - X_T) \\ &= \inf_{X \in \mathcal{X}(x_0)} \min\{\, a \,:\, E\,[\, l((L - a - X_T)^+)\,] \leq \tilde{x}\,\} \\ &= \min\{\, a \,:\, \inf_{X \in \mathcal{X}(x_0)} E\,[\, l((L - a - X_T)^+)\,] \leq \tilde{x}\,\} \\ &= \min\{\, a \,:\, \inf_{X \in \mathcal{X}(x_0+a)} E\,[\, l((L - X_T)^+)\,] \leq \tilde{x}\,\}. \qquad (4.4) \end{aligned}$$

◇

Lemma 4.2 (Shortfall Risk Measure) *Assume $E\,[l((L + H)^+)]$ is finite. Then the shortfall risk measure defined in (4.2) is a convex risk measure and satisfies Assumptions 2.2 and 2.3 under Assumption 2.1. Therefore, it can be used as a pricing risk measure as in (2.7) and (2.8). In particular, the existence result (Theorem 2.6) and the properties of the risk measure prices derived in section 2.3 are valid.*

Remark 4.3 *Similar to Lemma 4.1, we can show that*

$$\rho_b^{x_0}(L) = \inf_{X\in\mathcal{X}(x_0,b)} \rho(L - X_T) = \min\{\, a \,:\, \inf_{X\in\mathcal{X}(x_0+a,b)} E\,[\, l((L-X_T)^+)\,] \le \tilde{x}\,\}.$$

Define

$$u(a) = \inf_{X\in\mathcal{X}(x_0+a,b)} E\,[\, l((L-X_T)^+)\,].$$

Then $u(a) \ge l(0)$ is a decreasing and convex function. Applying Theorem 2.6, we know the optimal solution $X^* \in \mathcal{X}(x_b, b)$ to

$$\rho_b^{x_0}(L) = \min_{X\in\mathcal{X}(x_0,b)} \rho(L - X_T)$$

exists for any $b \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}$. Then $\rho_b^{x_0}(L) + X^*$ is the optimal solution to

$$u(\rho_b^{x_0}(L)) = \min_{X\in\mathcal{X}(x_0+\rho_b^{x_0}(L),b)} E\,[\, l((L-X_T)^+)\,] = \min_{X\in\mathcal{X}(\rho_b^{0}(L),b)} E\,[\, l((L-X_T)^+)\,],$$

and vice versa. In particular, $u(\rho_b^{x_0}(L)) = \tilde{x}$. We may conclude the optimal solution to the minimization problem

$$\min_{X\in\mathcal{X}(x,b)} E\,[\, l((L-X_T)^+)\,]$$

exists. This is an enlargement of the admissible set of the existence Theorem 3.2 in Föllmer and Leukert [17] where they required the admissible portfolios to be nonnegative.

Proof. As $a_n \nearrow \infty$, $(L+H-a_n)^+ \searrow 0$ a.s. By the Bounded Convergence Theorem, $E\,[l((L+H-a_n)^+)] \searrow l(0)$. Because we chose $\tilde{x} > l(0)$, we can thus be sure that $\rho(L+H) < \infty$. We will prove $\rho^0(0) > -\infty$ by showing Assumption 2.1 is otherwise violated. Suppose $\rho^0(0) = -\infty$. This implies that for any large number n, we can find some $X^n \in \mathcal{X}(0)$ such that $\rho(-X_T^n) < -n$ or $E\,[\, l((n - X_T^n)^+)\,] \le \tilde{x}$. Since $l(\cdot)$ is a convex and increasing function on $[0,\infty)$ which is not a constant, it is continuous and satisfies $l(x) \to \infty$ as $x \to \infty$. Then there exists a subsequence of X_T^n which converges to ∞ a.s. This creates an arbitrage opportunity and contradicts Assumption 2.1. It is obvious that if $X = Y$ a.s., then $\rho(X) = \rho(Y)$. Since $l(x^+)$ is a convex and increasing function, the shortfall acceptance set $\mathcal{A}$ defined in (4.1) is a convex set. It can be easily verified that if $X \in \mathcal{A}$ and $Y \le X$, then $Y \in \mathcal{A}$. Therefore the induced shortfall risk measure $\rho(\cdot)$ defined in (4.2) is a convex risk measure. We will finish the proof by showing that $\rho(\cdot)$ satisfies the Fatou property. Let $\tilde{L}_n$ be a sequence of random variables that are uniformly bounded below by a constant and converge to $\tilde{L}$ *a.s.* By Fatou's Lemma,

$$E[\, l((\tilde{L}-a)^+)\,] \le \liminf_{n\to\infty} E[\, l((\tilde{L}_n - a)^+)\,].$$

This implies $\rho(\tilde{L}) \le \liminf_{n\to\infty} \rho(\tilde{L}_n)$. ◇

Let us study the special case when $\tilde{x} = l(0)$. Without loss of generality, let the shortfall tolerance be zero ($\tilde{x} = 0$), and the loss function $l(x)$ be strictly increasing near $x = 0$ and satisfies $l(0) = 0$. The shortfall acceptance set becomes

$$\mathcal{A} = \{\, \tilde{L} \,:\, E\,[\, l(\tilde{L}^+)\,] \le 0\,\} = \{\, \tilde{L} \,:\, \tilde{L} \le 0 \quad a.s.\,\}.$$

The shortfall risk measure can be computed from the acceptance set:

$$\rho(\tilde{L}) = \inf\{\, a \,:\, \tilde{L} - a \in \mathcal{A}\,\} = \inf\{\, a \,:\, \tilde{L} \le a \quad a.s.\,\}.$$

By Lemma 4.1, the initial minimal risk with initial capital $x_0 = 0$ is

$$\begin{aligned}
\rho^0(L) &= \min\{\, a \,:\, \inf_{X\in\mathcal{X}(a)} E\,[\, l((L-X_T)^+)\,] \le 0\,\} \\
&= \min\{\, a \,:\, \exists X \in \mathcal{X}(a) \ s.t.\ L \le X_T \quad a.s.\,\} \\
&= \bar{x}(L) \quad \text{(the super-hedging price for } L).
\end{aligned}$$

By translation invariance, the buying and selling prices are

$$SP = \bar{x}(L+H) - \bar{x}(L) \quad \text{and} \quad BP = \bar{x}(L) - \bar{x}(L-H),$$

where they are reduced to the differences between the super-hedging prices. In particular, when the initial liability $L = 0$ a.s., the risk measure prices coincide with super-hedging prices $SP = BP = \bar{x}(H)$ which (not surprisingly because $\tilde{x} = 0$) is a very conservative choice.

4.2 An Example of Pricing with Average VaR

Recall from Section 2.1 that our probability space is $(\Omega, \mathcal{F}, \mathbb{P})$. An industry-standard risk measure is VaR (Value-at-Risk) defined by

$$VaR_\lambda(L) = \inf\{\, m \mid \mathbb{P}[\, L > m\,] \le \lambda \,\}.$$

For an overview of VaR, see Duffie and Pan [13]. Artzer et al. [2] looked for an axiomatic approach to define coherent risk measures with desirable properties which VaR lack. They provided an example called *Worst Conditional Expectation*:

$$WCE_\lambda(L) = \sup\{\, E[\, L \,|\, A\,] \,|\, A \in \mathcal{F}, \mathbb{P}[A] > \lambda \,\}.$$

A variant of VaR which is a coherent risk measure is called Average Value at Risk[‡‡]

$$AVaR_\lambda(L) = \frac{1}{\lambda}\int_0^\lambda VaR_\gamma(L) d\gamma.$$

Notice that AVaR takes into consideration the size of the loss as well as its probability. When the probability space Ω is atomless, AVaR coincides with the Worst Conditional Expectation. In general, we have the following relationship between these concepts:

$$AVaR_\lambda(L) \ge WCE_\lambda(L) \ \ge E[\, L \,|\, L \ge VaR_\lambda(L)\,] \ \ge VaR_\lambda(L).$$

Lemma 4.4 (Average Value at Risk) *Assume $AVaR_\lambda(L+H) < \infty$. Then the Average Value-at-Risk defined above is a coherent risk measure, and therefore a convex risk measure, and satisfies Assumptions 2.2 and 2.3 under Assumption 2.1. Therefore, it can be used as a pricing risk measure as in (2.7) and (2.8). In particular, the existence result (Theorem 2.6) and the properties of the risk measure prices derived in section 2.3 are valid.*

Proof. Since almost sure convergence imply convergence in distribution, the Fatou property is easily satisfied. It is also obvious that if $X = Y$ a.s., then $AVaR_\lambda(X) = AVaR_\lambda(Y)$. Note

$$\rho^0(0) = \inf_{X \in \mathcal{X}(0)} AVaR_\lambda(-X_T) \ge \inf_{X \in \mathcal{X}(0)} VaR_\lambda(-X_T).$$

Suppose $\inf_{X \in \mathcal{X}(0)} VaR_\lambda(-X_T) = -\infty$. Then for any large number n, we can find $X^n \in \mathcal{X}(0)$ such that $VaR_\lambda(-X^n_T) \le -n$. Thus $P(X^n_T < n) \le \lambda$. This is a contradiction to Assumption 2.1. Therefore, $\inf_{X \in \mathcal{X}(0)} VaR_\lambda(-X_T) > -\infty$ and $\rho^0(0) > -\infty$. ⋄

5 Conclusion

The Arbitrage Pricing Theory in complete markets has been widely adopted in financial industries. The lack of such a transparent, strong and applicable pricing and hedging theory has limited the use of incomplete market models in practice. This article attempts to extend the pricing theory to

[‡‡]See Föllmer, Schield [21] for references. Average Value-at-Risk is sometimes called Conditional Value at Risk or Expected Shortfall, and they are studied in Acerbi, Tasche [1], and Rockafellar, Uryasev [46].

incomplete markets based on a more practical choice of risk measure (which is mostly a mandatory banking practice), and attempts to associate prices with optimal hedging strategies as in utility based pricing. All the definitions and properties of pricing and hedging strategies derived are related to the general properties of the risk measure and the admissibility of the hedging processes, and are quite independent of the specifics of the underlying processes. Therefore, only general pricing properties are given which can serve as a guideline for building specific implementations. This approach is still quite subjective and specific to each institution's financial position and risk control, therefore it should be applied only where appropriate. Looking at the problem more closely, I think traders trade options for different reasons. A trader might actively buy or sell options to make profits. In this case, the trader is trying to maximize return (presumably positively associated to her bonus) over the risk taken, and there might not be a better alternative than existing approaches, for example, some sort of utility maximization subject to risk limits. Another reason a trader trades is to provide market with liquidity. She has to quote either a buying or selling price when a customer shows interest in a product. In this case, risk measure pricing can provide a bottom line for the pricing and hedging requirement.

We are only concerned with European type of options in this paper. An excellent earlier work which covers the American option pricing case is Merton [42]. Further research can be done to extend risk measure pricing to American type of options. As for the shortfall risk measure defined in Section 4.1, more work can be done in the direction of a robust version. Specifically, the acceptance set (4.1) can be modified to

$$\mathcal{A} = \{\, L \,:\, \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\,[\, l(L^+)\,] \le \tilde{x}\,\},$$

where $\mathcal{P}$ is a set of test measures. Related topics are studied in Kirch [36], Sekine [48] and Föllmer and Schied [20].

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