On a problem of Neumann

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Abstract

A conjecture widely attributed to Neumann is that all finite non-desarguesian projective planes contain a Fano subplane. In this note, we show that any finite projective plane of even order which admits an orthogonal polarity contains a Fano subplane. The number of planes of order less than n previously known to contain a Fano subplane was $O(\log n)$, whereas the number of planes of order less than n that our theorem applies to is not bounded above by any polynomial in n.
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1 Introduction

A fundamental question in incidence geometry is about the subplane structure of projective planes. There are relatively few results concerning when a projective plane of order k is a subplane of a projective plane of order n. Neumann [9] found Fano subplanes in certain Hall planes, which led to the conjecture that every finite non-desarguesian plane contains $PG(2,2)$ as a subplane (this conjecture is widely attributed to Neumann, though it does not appear in her work).

Johnson [7] and Fisher and Johnson [4] showed the existence of Fano subplanes in many translation planes. Petrak [10] showed that Figueroa planes contain $PG(2,2)$ and Caliskan and Petrak [3] showed that Figueroa planes of odd order contain $PG(2,3)$. Caliskan and Moorhouse [2] showed that all Hughes planes contain $PG(2,2)$ and that the Hughes plane of order q^2 contains $PG(2,3)$ if $q \equiv 5 \pmod 6$. We prove the following.

Theorem 1. *Let* Π *be a finite projective plane of even order which admits an orthogonal polarity. Then* Π *contains a Fano subplane.*

Ganley [5] showed that a finite semifield plane admits an orthogonal polarity if and only if it can be coordinatized by a commutative semifield. A result of Kantor [8] implies that the number of nonisomorphic planes of order n a power of 2 that can be coordinatized by a commutative semifield is not bounded above by any polynomial in n. Thus, Theorem 1 applies to many projective planes.

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2 Proof of Theorem 1

The proof of Theorem 1 is graph theoretic, and we collect some definitions and results first. Let $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective plane of order n. We write $p \in l$ or say p is on l if $(p, l) \in \mathcal{I}$. Let π be a polarity of Π. That is, π maps points to lines and lines to points, π^2 is the identity function, and π respects incidence. Then one may construct the polarity graph G_π^o as follows. $V(G_\pi^o) = \mathcal{P}$ and $p \sim q$ if and only if $p \in \pi(q)$. That is, the neighborhood of a vertex p is the line $\pi(p)$ that p gets mapped to under the polarity. If $p \in \pi(p)$, then p is an *absolute point* and the vertex p will have a loop on it. A polarity is *orthogonal* if exactly $n+1$ points are absolute. We note that as neighborhoods in the graph represent lines in the geometry, each vertex in G_π^o has exactly $n+1$ neighbors (if v is an absolute point, it has exactly n neighbors other than itself). We provide proofs of the following preliminary observations for completeness.

Lemma 1. *Let Π be a projective plane with polarity π, and G_π^o be the associated polarity graph.*

(a) For all $u, v \in V(G_\pi^o)$, u and v have exactly 1 common neighbor.

(b) G_π^o is C_4 free.

(c) If u and v are two absolute points of G_π^o, then $u \not\sim v$.

(d) If $v \in V(G_\pi^o)$, then the neighborhood of v induces a graph of maximum degree at most 1.

(e) Let $e = uv$ be an edge of G_π^o such that neither u nor v is an absolute point. Then e lies in a unique triangle in G_π^o.

Proof. To prove (a), let u and v be an arbitrary pair of vertices in $V(G_\pi^o)$. Because Π is a projective plane, $\pi(u)$ and $\pi(v)$ meet in a unique point. This point is the unique vertex in the intersection of the neighborhood of u and the neighborhood of v. (b) and (c) follow from (a).

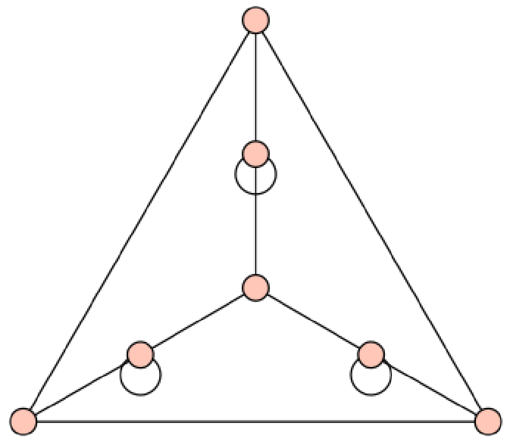
To prove (d), if there is a vertex of degree at least 2 in the graph induced by the neighborhood of v, then G_π^o contains a 4-cycle, a contradiction by (b).

Finally, let $u \sim v$ and neither u nor v an absolute point. Then by (a) there is a unique vertex w adjacent to both u and v. Now uvw is the purported triangle, proving (e). □

Proof of Theorem 1. We will now assume Π is a projective plane of even order n, that π is an orthogonal polarity, and that G_π^o is the corresponding polarity graph (including loops). Since n is even and π is orthogonal, a classical theorem of Baer ([1], see also Theorem 12.6 in [6]) says that the $n+1$ absolute points under π all lie on one line. Let $a_1, \ldots, a_{n+1}$ be the set of absolute points and let l be the line containing them. Then there is some $p \in \mathcal{P}$ such that $\pi(l) = p$. This means that in G_π^o, the neighborhood of p is exactly the set of points $\{a_1, \ldots, a_{n+1}\}$. For $1 \leq i \leq n+1$, let N_i be the neighborhood of a_i. Then by Lemma 1.b, $N_i \cap N_j = \emptyset$ if $i \neq j$. Further, counting gives that

$$V(G_\pi^o) = p \cup \left(\bigcup_{i=1}^{n+1} a_i \right) \cup \left(\bigcup_{i=1}^{n+1} N_i \right). \tag{1}$$

Figure 1: ER_2^o



Let ER_2^o be the graph on 7 points which is the polarity graph (with loops) of $PG(2,2)$ under the orthogonal polarity.

Lemma 2. *If ER_2^o is a subgraph of G_π^o, then Π contains a Fano subplane.*

Proof. Let $v_1, \ldots, v_7$ be the vertices of a subgraph ER_2^o of G_π^o. Let $l_i = \pi(v_i)$ for $1 \leq i \leq 7$. Then the lines $l_1, \ldots, l_7$ in Π restricted to the points $v_1, \ldots, v_7$ form a point-line incidence structure, and one can check directly that it satisfies the axioms of a projective plane. □

Thus, it suffices to find ER_2^o in G_π^o. To find ER_2^o it suffices to find distinct i, j, k such that there are $v_i \in N_i$, $v_j \in N_j$, and $v_k \in N_k$ where $v_i v_j v_k$ forms a triangle in G_π^o, for then the points $p, a_i, a_j, a_k, v_i, v_j, v_k$ yield the subgraph ER_2^o. Now note that for all i, and for $v \in N_i$, v has exactly n neighbors that are not absolute points. There are $n+1$ choices for i and $n-1$ choices for $v \in N_i$. As each edge is counted twice, this yields

$$\frac{n(n-1)(n+1)}{2}$$

edges with neither end an absolute point. By Lemma 1.e, there are at least

$$\frac{n^3 - n}{6}$$

triangles in G_π^o. By Lemma 1.c, there are no triangles incident with p, by Lemma 1.b, there are no triangles that have more than one vertex in N_i for any i, and by Lemma 1.d there are at most $\lfloor \frac{n-1}{2} \rfloor = \frac{n}{2} - 1$ triangles incident with a_i for each i. Therefore, by (1), there are at least

$$\frac{n^3 - n}{6} - (n+1)\left(\frac{n}{2} - 1\right)$$

copies of ER_2^o in G_π^o. This expression is positive for all even natural numbers n. □

3 Concluding Remarks

First, we note that the proof of Theorem 1 actually implies that there are $\Omega\left(n^3\right)$ copies of $PG(2,2)$ in any plane satisfying the hypotheses, and echoing Petrak [10], perhaps one could find subplanes of order 4 for n large enough. We also note that it is crucial in the proof that the absolute points form a line. When n is odd, the proof fails (as it must, since our proof does not detect if Π is desarguesian or not).

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References

[1] Reinhold Baer. Projectivities with fixed points on every line of the plane. *Bulletin of the American Mathematical Society*, 52(4):273–286, 1946.

[2] Cafer Caliskan and G Eric Moorhouse. Subplanes of order 3 in hughes planes. *The Electronic Journal of Combinatorics*, 18(P2):1, 2011.

[3] Cafer Caliskan and Bryan Petrak. Subplanes of order 3 in figueroa planes. *Finite Fields and Their Applications*, 20:24–29, 2013.

[4] J Chris Fisher and Norman L Johnson. Fano configurations in subregular planes. *Note di Matematica*, 28(2):69–98, 2010.

[5] MJ Ganley. Polarities in translation planes. *Geometriae Dedicata*, 1(1):103–116, 1972.

[6] Daniel R Hughes and Frederick Charles Piper. *Projective planes*, volume 6. Springer, 1973.

[7] Norman L Johnson. Fano configurations in translation planes of large dimension. *Note di Matematica*, 27(1):21–38, 2009.

[8] William M Kantor. Commutative semifields and symplectic spreads. *Journal of Algebra*, 270(1):96–114, 2003.

[9] Hanna Neumann. On some finite non-desarguesian planes. *Archiv der Mathematik*, 6(1):36–40, 1954.

[10] Bryan Petrak. Fano subplanes in finite figueroa planes. *Journal of Geometry*, 99(1-2):101–106, 2010.