

The Colin de Verdière parameter, excluded minors, and the spectral radius

Michael Tait*

Abstract

In this paper we characterize graphs which maximize the spectral radius of their adjacency matrix over all graphs of Colin de Verdière parameter at most m . We also characterize graphs of maximum spectral radius with no H as a minor when H is either K_r or $K_{s,t}$. Interestingly, the extremal graphs match those which maximize the number of edges over all graphs with no H as a minor when r and s are small, but not when they are larger.

1 Introduction

Let H and G be simple graphs. H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges. Properties of graphs with excluded minors have been studied extensively. In particular, Mader [14] proved that for every graph H there is a constant C such that if G does not contain H as a minor, then $|E(G)| \leq C|V(G)|$. Determining this constant C seems to be a very difficult question for general H .

When H is either a complete graph or a complete bipartite graph, there are natural constructions of graphs G which do not contain H as a minor. A complete graph on $r - 2$ vertices joined to an independent set of size $n - r + 2$ is an n vertex graph with $(r - 2)(n - r + 2) + \frac{1}{2}(r - 2)(r - 3)$ edges which does not contain K_r as a minor. A complete graph on $s - 1$ vertices joined to $(n - s + 1)/t$ disjoint copies of K_t is an n -vertex graph with $\frac{1}{2}(t + 2s - 3)(n - s + 1) + \frac{1}{2}(s - 1)(s - 2)$ edges which does not contain $K_{s,t}$ as a minor.

Mader [15] showed that this construction yields the maximum number of edges over all n -vertex graphs with no K_r minor when $r \leq 7$. Surprisingly, this natural construction is not best possible when $r > 7$. Indeed, Kostochka and Thomason [10, 11, 25, 26] showed that the maximum number of edges in a K_r minor free graph is $\Theta(r(\log r)^{1/2}n)$ for large r . Similarly, Chudnovsky, Reed, and Seymour [3] showed that the graph on n vertices with no $K_{2,t}$ minor with the maximum number of edges is given by this natural construction (see also [16]). Kostochka and Prince [12] showed that the same is true when $s = 3$ and t is large enough. It is unknown when $s \in \{4, 5\}$

*mtait@cmu.edu. Research supported by NSF grant DMS-1606350.

and Kostochka and Prince [12] showed that this construction is not best possible when $s \geq 6$.

In this paper we discuss a related question. If G is an n -vertex graph with no H minor, and λ is the spectral radius of its adjacency matrix, how large can λ be? We show that the natural K_r and $K_{s,t}$ minor free graphs described above are extremal for all values of r and $s \leq t$. It is interesting that the extremal graphs for maximizing number of edges and spectral radius are the same for small values of r and s and then differ significantly. We also consider maximizing the spectral radius over the family of n -vertex graphs of Colin de Verdière parameter at most m . Our main theorems are the following.

Theorem 1.1. *Let $m \geq 2$ be an integer. For n large enough, the n -vertex graph of maximum spectral radius with Colin de Verdière parameter at most m is the join of K_{m-1} and a path of length $n - m + 1$.*

Theorem 1.2. *Let $r \geq 3$. For n large enough, the n -vertex graph with no K_r minor of maximum spectral radius is the join of K_{r-2} and an independent set of size $n - r + 2$.*

Theorem 1.3. *Let $t \geq s \geq 2$. For n large enough, if G is an n -vertex graph with no $K_{s,t}$ minor and λ is the spectral radius of its adjacency matrix, then*

$$\lambda \leq \frac{s + t - 3 + \sqrt{(s + t - 3)^2 + 4((s - 1)(n - s + 1) - (s - 2)(t - 1))}}{2}$$

with equality if and only if $n \equiv s - 1 \pmod{t}$ and G is the join of K_{s-1} with a disjoint union of copies of K_t .

The Colin de Verdière parameter of a graph, denoted by $\mu(G)$, was introduced in 1990 [4] motivated by applications in differential geometry. This parameter turned out to have many nice graph-theoretic properties, eg it is minor-monotone. Further, much work has been done relating μ to other graph parameters (for a nice survey, see [2]), and it can be used to give an algebraic description of certain topological properties of a graph (proved by Colin de Verdière [4], Robertson, Seymour, and Thomas [22], and Lovász and Schrijver [13]):

- (i) $\mu(G) \leq 1$ if and only if G is the disjoint union of paths.
- (ii) $\mu(G) \leq 2$ if and only if G is outerplanar.
- (iii) $\mu(G) \leq 3$ if and only if G is planar.
- (iv) $\mu(G) \leq 4$ if and only if G is linklessly embeddable.

Our theorems build on work of Tait and Tobin [24], of Nikiforov [20], and of Hong [8]. In [24], planar and outerplanar graphs of maximum spectral radius were characterized for n large enough. By the above characterization, Theorem 1.1 is a far-reaching generalization of the main results of [24]. In [20] graphs with no $K_{2,t}$ minor of maximum spectral radius were characterized and in [8] graphs with no K_5 minor of maximum spectral radius were characterized. Theorems 1.2 and 1.3 extend these results to all $r \geq 3$ and $2 \leq s \leq t$. In [8] the maximum spectral radius of all graphs with a fixed

tree-width was determined, and Theorem 1.2 shows that the extremal graph is the same as for the family of graphs with no K_r minor. We also mention the following open problem, which would be implied by a solution to a conjecture of Nevo [17]:

Problem 1.4. *Show that if $\mu(G) \leq m$, then $e(G) \leq mn - \binom{m+1}{2}$.*

Finally, we note that finding the graph in a given family of graphs maximizing some function of its eigenvalues has a long history in extremal graph theory (eg Stanley's Bound [23], the Alon-Boppana Theorem [21], the Hoffman ratio bound [6]). In particular, in some cases theorems of this type can strengthen classical extremal results, for example Turán's theorem [19] or the Kővari-Sós-Turán theorem [1, 18].

1.1 Notation, definitions, and outline

If G is a graph, $e(G)$ will denote the number of edges in G . $A(G)$ will denote the adjacency matrix of G and $\lambda_1(A(G))$ or $\lambda_1(G)$ will denote the largest eigenvalue of this matrix. For two sets $S, L \subset V(G)$, $E(S)$ will denote the edges with both endpoints in S and $E(S, L)$ will denote edges with one endpoint in S and one in L . We will often use that n is large enough and that if G is a connected graph, then the eigenvector corresponding to $\lambda_1(G)$ has all positive entries. This fact and the Rayleigh quotient imply that if H is a strict subgraph of G , then $\lambda_1(H) < \lambda_1(G)$.

Given a matrix M , define the *corank* of M to be the dimension of its kernel. If G is an n -vertex graph, then the *Colin de Verdière parameter* of G , denoted by $\mu(G)$, is defined to be the largest corank of any $n \times n$ matrix M such that:

- M1 If $i \neq j$ then $M_{ij} < 0$ if $i \sim j$ and $M_{ij} = 0$ if $i \not\sim j$.
- M2 M has exactly one negative eigenvalue of multiplicity 1.
- M3 There is no nonzero matrix X such that $MX = 0$ and $X_{ij} = 0$ whenever $i = j$ or $M_{ij} \neq 0$.

A nice discussion of where these seemingly *ad hoc* conditions come from is given in [7]. Two key properties that we will use (see [7]) are that if $\mu(G) \leq m$ then there is a fixed finite family of minors that G may not contain and there is a constant c_m such that $e(G) \leq c_m n$.

In section 2 we prove structural results about graphs with excluded minors which we will need during the proof of the main theorems. These results are relatively specific, but may be of independent interest. In section 5 we prove Theorem 1.1. In section 3 we prove Theorem 1.2 and in section 4 we prove Theorem 1.3.

2 Structural results for graphs with excluded minors

Lemma 2.1. *Let G be a bipartite K_r minor free graph on n vertices with partite sets A and B . Let $|A| = k$ and $|B| = n - k$. Then there is an absolute constant C depending only on r such that*

$$e(G) \leq Ck + (r - 2)n.$$

In particular, if $|A| = o(n)$, then $e(G) \leq (r - 2 + o(1))n$.

Proof. Choose c such that any graph on N vertices with no K_r minor has at most cN edges. Choose $B_1 \subset B$ maximal such that for each $v \in B$, there is a pair of vertices in A both adjacent to v , where no distinct vertices in B are assigned the same pair in A . Then $|B_1| \leq ck$, otherwise G contains a K_r minor.

Create an auxiliary graph H with $V(H) = A$ and edges are given by the pairs chosen by vertices in B . Since B_1 was chosen to be maximal, for each $v \in B \setminus B_1$, the neighborhood of v induces a clique in A . Therefore, each $v \in B \setminus B_1$ may have degree at most $r - 2$, otherwise there is a K_r minor in G . Since $B_1 \cup A$ induces a graph on at most $ck + k$ vertices with no K_r minor, $e(G[B_1 \cup A]) \leq c(ck + k)$. Thus

$$e(G) \leq c(c + 1)k + (r - 2)(n - k).$$

□

Lemma 2.2. *Let G be a bipartite n vertex graph with no $K_{s,t}$ minor and partite sets A and B . Let $|A| = k$ and $|B| = n - k$. Then there is a constant C depending only on s and t such that*

$$e(G) \leq Ck + (s - 1)n.$$

In particular, if $|A| = o(n)$, then $e(G) \leq (s - 1 + o(1))n$.

Proof. Choose c so that any N vertex graph with no $K_{s,t}$ minor has at most cN edges. We need the following claim:

Claim 2.3. *If G has N vertices and no $K_{s,t}$ minor, then G has at most $2^{2c}N$ cliques.*

The claim follows by applying induction and noting that there is a vertex of degree at most $2c$.

Now, as in the proof of the previous lemma, choose $B_1 \subset B$ maximal such that for each $v \in B$, there is a pair of vertices in A both adjacent to v , where no distinct vertices in B are assigned the same pair in A . Then $|B_1| \leq ck$, otherwise G contains a $K_{s,t}$ minor.

Create an auxiliary graph H with $V(H) = A$ and edges are given by the pairs chosen by vertices in B . Since B_1 was chosen to be maximal, for each $v \in B \setminus B_1$, the neighborhood of v induces a clique in A . Now let B_2 be the set of vertices in $B \setminus B_1$ that have degree at least s . Each vertex in B_2 dominates a set of size s in A which induces a clique in H . Each such set of size s can be dominated by at most t vertices in B_2 , otherwise G contains a $K_{s,t}$. But H contains only at most $2^{2c}k$ cliques total. Therefore $|B_2| \leq 2^{2c}kt$.

Now, the graph induced by $A \cup B_1 \cup B_2$ is a $K_{s,t}$ minor free graph on at most $k + ck + 2^{2c}kt$, and so $e(G[A \cup B_1 \cup B_2]) \leq c(k + ck + 2^{2c}kt)$. The vertices in $B \setminus (B_1 \cup B_2)$ have degree at most $s - 1$, and so

$$e(G) \leq c(1 + c + 2^{2c}t)k + (s - 1)(n - k).$$

□

If a graph is linklessly embeddable, it does not contain $K_{4,4}$ as a minor. Hence, Lemma 2.2 gives that a bipartite linklessly embeddable graph with $o(n)$ vertices in one partite set has at most $(3 + o(1))n$ edges. The following problem is open, and would be implied by a solution to Conjecture 4.5 in [9]:

Problem 2.4. *If G is a bipartite linklessly embeddable graph, show that $e(G) \leq 3n - 9$.*

Theorem 2.5. *Let G be a graph on n vertices with no K_r minor. Assume that $(1 - 2\delta)n > r$, and $(1 - \delta)n > \binom{r-2}{2} + 2$, and that there is a set A with $|A| = r - 2$ and a set B with $|B| = (1 - \delta)n$ such that every vertex in A is adjacent to every vertex in B . Then we may add edges to A to make it a clique and the resulting graph will still have no K_r minor.*

Proof.

Claim 2.6. *B induces an independent set.*

To see this, suppose that there are $u, v \in B$ that are adjacent. For each pair $x, y \in A$, choose $b_{xy} \in B$ all distinct and different from u and v (this can be done as $|B| > \binom{r-2}{2} + 2$ by assumption). Then the paths $x - b_{xy} - y$ along with u and v form a subdivision of K_r , a contradiction.

Claim 2.7. *Let C be a component of $G \setminus (A \cup B)$. Then there is at most one vertex in B with a neighbor in C .*

Suppose $u, v \in B$ each have a neighbor in C . Then there is a path from u to v where all the interior vertices are in C . Choose b_{xy} as before. Then this path along with the paths $x - b_{xy} - y$ form a subdivision of K_r . This is a contradiction, proving the claim.

Now let D be the set of vertices in B that have degree exactly $r - 2$ in G . Since there are at most δn components in C , we have that $|D| \geq (1 - 2\delta)n$. Now add edges to A so that A induces a clique. Assume that there is now a K_r minor in G . Consequently, there are r disjoint sets X_1, \dots, X_r of vertices such that

- (a) For each X_i , either $G[X_i]$ is connected or every component of $G[X_i]$ intersects A .
- (b) For all distinct i, j either there is an edge of G between X_i and X_j , or both X_i and X_j have nonempty intersection with A .

Choose X_1, \dots, X_r such that as many of them as possible have nonempty intersection with D . Since the vertices of D all have the same neighborhood, we may choose X_1, \dots, X_r such that each X_i contains at most one vertex from D . Since we assumed $(1 - 2\delta)n > r$ there is a vertex $v \in D$ which is not in any of the sets X_1, \dots, X_r .

Claim 2.8. *For $1 \leq i \leq r$, if X_i is disjoint from D then X_i is disjoint from A .*

If there exists X_i that is disjoint from D but intersects A , then we may add v to X_i giving a better choice of X_1, \dots, X_r .

Claim 2.9. *$G[X_i]$ is connected for all i .*

Suppose not. Then by (a), we have that each component of $G[X_i]$ intersects A . Claim 2.8 implies that X_i intersects D . But if each component intersects A and X_i contains a vertex in D , then $G[X_i]$ is connected.

Claim 2.10. *For distinct i, j , there is an edge of G between X_i and X_j .*

Suppose not. Then (b) gives that both X_i and X_j have nonempty intersection with A and Claim 2.8 implies that they both intersect D . Therefore there is an edge (between A and D , ie in $E(G)$) between them.

Now X_1, \dots, X_r form a K_r minor in G , a contradiction. \square

Theorem 2.11. *Let \mathcal{H} be a fixed family of graphs and let G be an n -vertex graph with no minor from \mathcal{H} . Assume further that there is a set A of size $s - 1$ and a set B with $|B| = (1 - \delta)n > (s - 1)(s - 2)/2$ and that every vertex in A is adjacent to every vertex in B . Let c be such that every graph of average degree c has some $H \in \mathcal{H}$ as a minor. Then there is a set of at most*

$$\frac{c(s - 1)(s - 2)}{2(1 - \delta)}$$

edges such that after deleting these edges we can make A a clique without introducing a minor from \mathcal{H} .

Proof. Let d be the average degree of vertices in B . G has at most $cn/2$ edges, so $d|B|/2 \leq c/2$ which implies that

$$d \leq \frac{c}{1 - \delta}.$$

Let $m = (s - 1)(s - 2)/2$ and let M be a set of m vertices in B with sum of degrees as small as possible. By the first moment method, there are at most

$$\frac{c(s - 1)(s - 2)}{2(1 - \delta)}$$

edges in G that are incident with M . For each pair xy of vertices in A , choose a vertex $b_{xy} \in B$ such that all b_{xy} are distinct. For each x, y , delete all edges adjacent to b_{xy} except for the edges to x and to y . Let G' be the subgraph of G produced, and note that G' does not have a minor from \mathcal{H} . Now, x and y are nonadjacent vertices in A , then they are the neighbors of a vertex of degree 2 in G' . Thus making x adjacent to y cannot introduce a minor in G' unless it was already present. \square

3 Graphs with no K_r minor

Let G_r be an n vertex graph with no K_r minor which has maximum spectral radius of its adjacency matrix among all such graphs. In this section, we will prove Theorem 1.2, that G_r is the join of K_{r-2} and an independent set of size $n - r + 2$ for sufficiently large n . Let A be the adjacency matrix of G_r and let λ be the largest eigenvalue of A . Let \mathbf{x} be an eigenvector for λ . Without loss of generality, we may assume the G_r is connected and so \mathbf{x} is well-defined. We will assume throughout this section that \mathbf{x} is normalized to have maximum entry equal to 1, and that z is a vertex such that $\mathbf{x}_z = 1$

(if there is more than one such vertex, choose z arbitrarily). We will use throughout the section that $e(G_r) = O(n)$ since G_r has no K_r minor. The outline of our proof is as follows:

1. First we show that if a vertex has eigenvector entry close to 1, then it has degree close to n (Lemma 3.5).
2. We show that there are $r - 2$ vertices of eigenvector entry close to 1, and hence degree close to n .
3. We use Theorem 2.5 to show that these $r - 2$ vertices induce a clique.
4. We show that each of the $r - 2$ vertices in the clique actually have degree $n - 1$.

First, we split the vertex set into vertices with “large” eigenvector entry and those with “small”. Let

$$L = \{v \in V(G) : \mathbf{x}_v > \epsilon\},$$

and

$$S = \{v \in V(G) : \mathbf{x}_v \leq \epsilon\}$$

where ϵ will be chosen later.

Lemma 3.1. $\sqrt{(r-2)(n-r+2)} \leq \lambda = O(\sqrt{n})$.

Proof. $K_{r-2, n-r+2}$ has no K_r minor. Since G_r is extremal, $\lambda > \lambda_1(A(K_{r-2, n-r+2})) = \sqrt{(r-2)(n-r+2)}$. For the upper bound, since $e(G_r) = O(n)$, Stanley’s bound [23] implies that $\lambda \leq \sqrt{2e(G_r)} = O(\sqrt{n})$. \square

The next lemma shows that L is not too large.

Lemma 3.2.

$$|L| = O(\sqrt{n}).$$

Proof. We sum the eigenvector eigenvalue equation over all vertices in L :

$$\lambda \sum_{u \in L} \mathbf{x}_u = \sum_{u \in L} \sum_{v \sim u} \mathbf{x}_v \leq 2e(G_r),$$

where the last inequality is because we have normalized so that each eigenvector entry is at most 1. Now using that $e(G_r) = O(n)$ and $\lambda = \Omega(\sqrt{n})$ gives the result. \square

Lemma 3.2 and Lemma 2.1 imply that for n large enough, we have

$$e(S, L) \leq (r - 2 + \epsilon)n. \tag{1}$$

Next we use (1) and the eigenvector-eigenvalue equation to give a bound on the sum over all eigenvector entries from S and L .

Lemma 3.3. *There is an absolute constant C_1 such that*

$$\sum_{u \in S} \mathbf{x}_u \leq (1 + C_1\epsilon)\sqrt{(r-2)(n-r+2)}$$

and

$$\sum_{u \in L} \mathbf{x}_u \leq C_1\epsilon\sqrt{(r-2)(n-r+2)}.$$

Proof. For the first inequality, using the eigenvector-eigenvalue equation and summing over vertices in S gives

$$\begin{aligned}\lambda \sum_{u \in S} \mathbf{x}_u &= \sum_{u \in S} \sum_{v \sim u} \mathbf{x}_v = \sum_{u \in S} \sum_{\substack{v \sim u \\ v \in S}} \mathbf{x}_v + \sum_{u \in S} \sum_{\substack{v \sim u \\ v \in L}} \mathbf{x}_v \\ &\leq \sum_{u \in S} \sum_{\substack{v \sim u \\ v \in S}} \epsilon + \sum_{u \in S} \sum_{\substack{v \sim u \\ v \in L}} 1 \leq \epsilon \cdot 2e(S) + e(S, L).\end{aligned}$$

Using (3.3), that $e(S) = O(n)$, and that Lemma 3.1 proves the inequality.

The second inequality is similar:

$$\lambda \sum_{u \in L} \mathbf{x}_u = \sum_{u \in L} \sum_{v \sim u} \mathbf{x}_v = \sum_{u \in L} \sum_{\substack{v \sim u \\ v \in S}} \mathbf{x}_v + \sum_{u \in L} \sum_{\substack{v \sim u \\ v \in L}} \mathbf{x}_v \leq \epsilon e(S, L) + 2e(L).$$

Lemma 3.2 gives that $e(L) = O(\sqrt{n})$ and noting that $\lambda = \Omega(\sqrt{n})$ completes the proof. \square

Now we would like to show that if a vertex has eigenvector entry close to 1, then it must be adjacent to most of the vertices in S . Let $u \in L$. Then

$$\begin{aligned}\sqrt{(r-2)(n-r+2)}\mathbf{x}_u &\leq \lambda \mathbf{x}_u = \sum_{v \sim u} \mathbf{x}_v \leq \sum_{v \in L} \mathbf{x}_v + \sum_{\substack{v \sim u \\ v \in S}} \mathbf{x}_v \\ &= \sum_{v \in L} \mathbf{x}_v + \left(\sum_{v \in S} \mathbf{x}_v - \sum_{\substack{v \not\sim u \\ v \in S}} \mathbf{x}_v \right) \\ &\leq C_1 \epsilon \sqrt{(r-2)(n-r+2)} + \left((1 + C_1 \epsilon) \sqrt{(r-2)(n-r+2)} - \sum_{\substack{v \not\sim u \\ v \in S}} \mathbf{x}_v \right).\end{aligned}$$

That is,

$$\sum_{\substack{v \not\sim u \\ v \in S}} \mathbf{x}_v \leq (1 + 2C_1 \epsilon - \mathbf{x}_u) \sqrt{(r-2)(n-r+2)}. \quad (2)$$

This equation says that if \mathbf{x}_u is close to 1, then the sum of eigenvector entries over all vertices in S not adjacent to u is not too big. In order to show that this implies u is adjacent to most of the vertices in S , we need an easy lower bound on the eigenvector entries of the vertices in $V(G_r)$.

Claim 3.4. *There is an absolute constant C_2 such that for all $u \in V(G_r)$, $\mathbf{x}_u \geq \frac{1}{C_2 \sqrt{(r-2)(n-r+2)}}$.*

Proof. Let $u \in V(G_r)$ be any vertex that is not z (we already know that z satisfies this inequality). If $u \sim z$, then $\lambda \mathbf{x}_u = \sum_{v \sim u} \mathbf{x}_v \geq \mathbf{x}_z$ and the inequality is satisfied.

If not, assume that $\mathbf{x}_u \leq \frac{1}{C_2 \sqrt{(r-2)(n-r+2)}}$. Then

$$\sum_{v \sim u} \mathbf{x}_v = \lambda \mathbf{x}_u \leq \frac{O(1)}{C_2}$$

where the last inequality is by the upper bound in Lemma 3.1. Let H be the graph obtained by removing all edges incident with u and creating one new edge uz . Let the adjacency matrix of H be B and let μ be the spectral radius of B . Note that adding a leaf to a graph with no K_r minor cannot produce a K_r minor, and so H is K_r minor free. Now since $\mu = \max_{\mathbf{z} \neq 0} \frac{\mathbf{z}^T B \mathbf{z}}{\mathbf{z}^T \mathbf{z}}$, we have

$$\mu - \lambda \geq \frac{\mathbf{x}^T B \mathbf{x}}{\mathbf{x}^T \mathbf{x}} - \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \frac{2}{\mathbf{x}^T \mathbf{x}} \left(\mathbf{x}_u \mathbf{x}_z - \mathbf{x}_u \sum_{v \sim u} \mathbf{x}_v \right) = \frac{2\mathbf{x}_u}{\mathbf{x}^T \mathbf{x}} \left(1 - \frac{O(1)}{C_2} \right).$$

Since G_r is extremal, $\mu - \lambda \leq 0$, a contradiction for a large enough constant C_2 . \square

Now we may give a bound on the number of vertices in S not adjacent to u .

Lemma 3.5. *Let $A_u = \{v \in S : v \not\sim u\}$ and assume that $\mathbf{x}_u = 1 - \delta$. Then there is an absolute constant C_3 such that*

$$|A_u| \leq C_3(\delta + \epsilon)n$$

Proof. Applying (2) and Claim 3.4 yields

$$|A_u| \leq C_2(1 + 2C_1\epsilon - \mathbf{x}_u)(r-2)(n-r+2).$$

\square

Lemma 3.5 shows that the number of neighbors of z tends to n as ϵ goes to 0. We now show that there are actually $r-2$ vertices of degree close to n .

Lemma 3.6. *Assume that $1 \leq k < r-2$ and that $\{v_1, \dots, v_k\}$ are a set of vertices each with degree at least $(1-\eta)n$ and with eigenvector entry at least $1-\eta$. Then there is an absolute constant C_4 and a vertex $v_{k+1} \notin \{v_1, \dots, v_k\}$ such that the degree of v_{k+1} is at least $(1-C_4(\eta+\epsilon))n$ and the eigenvector entry for v_{k+1} is at least $1-C_4(\eta+\epsilon)$.*

Proof. Let $A = \{v_1, \dots, v_k\}$. Then the eigenvector-eigenvalue equation for A^2 gives

$$\begin{aligned} (r-2)(n-r+2) &\leq \lambda^2 = \lambda^2 \mathbf{x}_z = \sum_{v \sim z} \sum_{w \sim v} \mathbf{x}_w \leq \sum_{vw \in E(G)} \mathbf{x}_v + \mathbf{x}_w \\ &= \sum_{vw \in E(S)} \mathbf{x}_v + \mathbf{x}_w + \sum_{uv \in E(S,L)} \mathbf{x}_v + \mathbf{x}_w + \sum_{vw \in E(L)} \mathbf{x}_v + \mathbf{x}_w \\ &\leq 2\epsilon O(n) + \sum_{vw \in E(S,L)} \mathbf{x}_v + \mathbf{x}_w + O(\sqrt{n}). \end{aligned}$$

This implies that

$$\sum_{\substack{vw \in E(S,L) \\ v,w \notin A}} \mathbf{x}_v + \mathbf{x}_w \geq (r-2)(n-r+2) - 2\epsilon O(n) - O(\sqrt{n}) - k(1+\epsilon)n.$$

The definition of A and Lemma 2.1 give that the number of edges with one endpoint in S and one endpoint in L which is not in A is at most $(r-2+o(1))n - k(1-\eta)n$. Noting that each term in the sum is at most $\mathbf{x}_v + \epsilon$ and averaging gives that there is a vertex $v \in L \setminus A$ with the requisite eigenvector entry. Applying Lemma 3.5 gives the degree condition. \square

Starting with $z = v_1$ and iteratively applying Lemma 3.6 gives that for any $\delta > 0$, we may choose ϵ small enough that G_r contains a set of $r-2$ vertices with common neighborhood of size at least $(1-\delta)n$ and with each eigenvector entry at least $1-\delta$. Since adding edges to a graph strictly increases its spectral radius, Theorem 2.5 and G_r extremal implies that these $r-2$ vertices must form a clique. From now on, we will refer to this clique of size $r-2$ as K .

To complete the proof, we must show that the vertices in K have degree $n-1$. Once this is proved, it implies that G_r is a subgraph of the join of K_{r-2} and an independent set of size $n-r+2$. But since adding any edge to this graph creates a K_r minor, we will have that G_r is exactly K_{r-2} join an independent set of size $n-r+2$. Let A be the common neighborhood of K and let $B = V(G_r) \setminus (A \cup K)$.

Lemma 3.7. *Let c be a constant such that any graph of average degree c has a K_r minor. Then ϵ may be chosen small enough so that if $v \in V(G_r) \setminus K$, then $\mathbf{x}_v < \frac{1}{2c}$.*

Proof. Note that any vertex in B may be adjacent to at most one vertex in A , otherwise there is a K_r minor. By definition of B , each vertex in B may be adjacent to at most $r-3$ vertices in K . First we give a bound on the sum of the eigenvector entries in B and we use this to give a bound on each eigenvector entry.

$$\lambda \sum_{u \in B} \mathbf{x}_u = \sum_{u \in B} \sum_{v \sim u} \mathbf{x}_v \leq 2e(B) + (r-2)|A| = \delta O(n).$$

That is

$$\sum_{u \in B} \mathbf{x}_u = \delta O(\sqrt{n}).$$

Now let $v \in V(G_r) \setminus K$. Again, note that v may have at most $r-2$ neighbors in $V(G) \setminus B$. Therefore

$$\lambda \mathbf{x}_v = \sum_{w \sim v} \mathbf{x}_w \leq r-2 + \sum_{w \in B} \mathbf{x}_w = r-2 + \delta O(\sqrt{n}).$$

Dividing by λ and choosing ϵ small enough to make δ small enough gives the result. \square

Finally, we complete the proof of Theorem 1.2:

Lemma 3.8. *B is empty.*

Proof. Assume not. Then there is a vertex v in B with at most c neighbors in B . v may be adjacent to at most 1 vertex in A and at most $r - 3$ vertices in K . Let u be a vertex in K which is not adjacent to v . Now let H be the graph obtained from G_r by removing all edges incident with v and then connecting v to each vertex in K . Since K induces a clique, the graph H has no K_r minor. Let B be the adjacency matrix of H and let μ be its spectral radius. Now

$$\mu - \lambda \geq \frac{\mathbf{x}^T B \mathbf{x}}{\mathbf{x}^T \mathbf{x}} - \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \geq \frac{2\mathbf{x}_v}{\mathbf{x}^T \mathbf{x}} \left(\mathbf{x}_u - \sum_{\substack{vw \in E(G_r) \\ w \notin K}} \mathbf{x}_w \right) \geq \frac{2\mathbf{x}_v}{\mathbf{x}^T \mathbf{x}} \left(\mathbf{x}_u - \frac{c+1}{2c} \right),$$

where the last inequality is by Lemma 3.7. Choosing ϵ small enough so that $1 - \delta > \frac{c+1}{c}$ gives that H is a K_r minor free graph with larger spectral radius than G_r , a contradiction. \square

4 Graphs with no $K_{s,t}$ minor

Let $2 \leq s \leq t$ and $G_{s,t}$ be a graph on n vertices with no $K_{s,t}$ minor such that the spectral radius of its adjacency matrix is at least as large as the spectral radius of the adjacency matrix of any other n -vertex graph with no $K_{s,t}$ minor. Throughout this section, A will denote the adjacency matrix of $G_{s,t}$ and λ its spectral radius. \mathbf{x} will be the eigenvector for λ normalized to have infinity norm equal to 1 and z will be a vertex such that $\mathbf{x}_z = 1$. First we will show that for n large enough, $G_{s,t}$ is a subgraph of the join of K_{s-1} and an independent set of size $n - s + 1$. We omit the proof of the following proposition as it is similar to the proofs of Lemmas 3.1–3.7.

Proposition 4.1. *For any $\delta > 0$, if n is large enough, then $G_{s,t}$ contains a set K of $s - 1$ vertices which have common neighborhood of size at least $(1 - \delta)n$ and each of which has eigenvector entry at least $1 - \delta$. Further, for any vertex $u \in V(G_{s,t}) \setminus K$, we have*

$$\mathbf{x}_u < \frac{(1 - \delta)}{c(s - 1)(s - 2)}$$

where c is chosen so that any graph of average degree c has a $K_{s,t}$ minor.

Let A be the common neighborhood of K and $B = V(G_{s,t}) \setminus (A \cup K)$. First we show that K induces a clique and then we show that B is empty.

Lemma 4.2. *K induces a clique.*

Proof. If K induces a clique we are done, so assume that there are vertices $u, v \in K$ such that $u \not\sim v$. Now, Theorem 2.11 guarantees that there is a set of at most $C := \frac{c(s-1)(s-2)}{2(1-\delta)}$ edges such that we may delete these edges, make K a clique, and the resulting graph will have no $K_{s,t}$ minor. Call this set of at most C edges E_1 and call the resulting graph H . Let B be the adjacency matrix of H and μ the spectral radius of B . Note

that all edges in E_1 have at least one endpoint with eigenvector entry less than $\frac{1}{2C}$ by Proposition 4.1. Then

$$\mu - \lambda \geq \frac{2}{\mathbf{x}^T \mathbf{x}} \left(\mathbf{x}_u \mathbf{x}_v - \sum_{wy \in E_1} \mathbf{x}_w \mathbf{x}_y \right) \geq \frac{2}{\mathbf{x}^T \mathbf{x}} \left((1 - \delta)^2 - C \cdot \frac{1}{2C} \right).$$

Choosing δ small enough that $(1 - \delta)^2 > 1/2$ yields that $\mu > \lambda$, a contradiction. So K must induce a clique. □

Lemma 4.3. *B is empty.*

Proof. The proof is similar to the proof of Lemma 3.8, noting that adding a vertex adjacent to a clique of size $s - 1$ to a graph with no $K_{s,t}$ minor cannot introduce a $K_{s,t}$ minor. □

So we have that the vertices of K have degree $n - 1$ in $G_{s,t}$. We now consider the graph induced by $V(G_{s,t}) \setminus K$. Note that if any vertex in this induced graph has t neighbors, this creates a $K_{s,t}$ in $G_{s,t}$. Therefore the graph induced by $V(G_{s,t} \setminus K)$ has maximum degree at most $t - 1$.

We need an interlacing result. We comment that many times interlacing theorems are used to give a lower bound on the spectral radius of a graph via eigenvalues of either a subgraph of the graph or a quotient matrix formed from the graph. This theorem gives an *upper* bound on the spectral radius of a graph based on the eigenvalues of a quotient-like matrix.

Theorem 4.4. *Let H_1 be a d -regular graph on n_1 vertices and H_2 be a graph with maximum degree k on n_2 vertices. Let H be the join of H_1 and H_2 . Define*

$$B := \begin{bmatrix} d & n_2 \\ n_1 & k \end{bmatrix}$$

Then $\lambda_1(H) \leq \lambda_1(B)$ with equality if and only if H_2 is k -regular.

Proof. Let $A(H)$ be the adjacency matrix of H . Let

$$A(H) \begin{bmatrix} x_1 \\ \vdots \\ x_{n_1} \\ y_1 \\ \vdots \\ y_{n_2} \end{bmatrix} = \lambda_1(H) \begin{bmatrix} x_1 \\ \vdots \\ x_{n_1} \\ y_1 \\ \vdots \\ y_{n_2} \end{bmatrix}$$

where the eigenvector entries labeled by x 's correspond to vertices in H_1 and those by y 's to vertices in H_2 . Assume that $[x_1 \dots x_{n_1} y_1 \dots y_{n_2}]^T$ is normalized to have 2-norm equal to 1. Then

$$\lambda_1(H) = 2 \sum_{ij \in E(H_1)} x_i x_j + 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} x_i y_j + 2 \sum_{ij \in E(H_2)} y_i y_j.$$

Now note that

$$2 \sum_{ij \in E(H_1)} x_i x_j \leq \lambda_1(H_1) \left(\sum_{i=1}^{n_1} x_i^2 \right) = d \left(\sum_{i=1}^{n_1} x_i^2 \right)$$

and

$$2 \sum_{ij \in E(H_2)} y_i y_j \leq \lambda_1(H_2) \left(\sum_{j=1}^{n_2} y_j^2 \right) \leq k \left(\sum_{j=1}^{n_2} y_j^2 \right).$$

Two applications of Cauchy-Schwarz and one application of the AM-GM inequality give

$$2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} x_i y_j \leq (n_1 + n_2) \sqrt{\left(\sum_{i=1}^{n_1} x_i^2 \right)} \sqrt{\left(\sum_{j=1}^{n_2} y_j^2 \right)}$$

Let $x^2 = \sum_{i=1}^{n_1} x_i^2$ and $y^2 = \sum_{j=1}^{n_2} y_j^2$. So

$$\lambda_1(H) \leq dx^2 + (n_1 + n_2)xy + ky^2.$$

On the other hand

$$\lambda_1(B) \geq \begin{bmatrix} x & y \end{bmatrix} B \begin{bmatrix} x \\ y \end{bmatrix} = dx^2 + (n_1 + n_2)xy + ky^2.$$

Note that if H_2 is not k -regular, then $\lambda_1(H_2) < k$, and equality cannot occur. On the other hand, if H_2 is k -regular, then the partition of $V(H)$ into $V(H_1)$ and $V(H_2)$ forms an equitable partition with quotient matrix B , implying that B and $A(H)$ have the same spectral radius (cf [5]). \square

Now we may finish the proof of Theorem 1.3.

Proof of Theorem 1.3. We now know that $G_{s,t}$ is a subgraph of the join of a clique of size $s - 1$ (namely K) and an independent set of size $n - s + 1$ and that the graph induced by $V(G_{s,t}) \setminus K$ has maximum degree at most $t - 1$. Theorem 4.4 then yields that

$$\lambda \leq \frac{s + t - 3 + \sqrt{(s + t - 3)^2 + 4((s - 1)(n - s + 1) - (s - 2)(t - 1))}}{2},$$

with equality if and only if the graph induced by $V(G_{s,t}) \setminus K$ is $(t - 1)$ -regular. It remains to show that equality may hold if and only if $V(G_{s,t}) \setminus K$ induces a disjoint union of copies of K_t . To accomplish this, we use a trick of Nikiforov [20]. Assume that H is a connected component of the graph induced by $V(G_{s,t} \setminus K)$ on h vertices. We may assume that this component is $t - 1$ regular and we must show that $h = t$. If $h = t + 1$, then any pair of nonadjacent vertices in H have $t - 1$ common neighbors. These vertices along with K then form a $K_{s,t}$.

Now assume that $h \geq t + 2$. Since H is dominated by K and $G_{s,t}$ has no $K_{s,t}$ minor, H does not have a $K_{1,t}$ minor. By [3], since H is connected we have that $E(H) \leq h + \frac{1}{2}t(t - 3)$, contradicting that H is $(t - 1)$ -regular. Therefore H must be a K_t , and so equality occurs if and only if $V(G_{s,t}) \setminus K$ induces the disjoint union of copies of K_t (implying that $n \equiv s - 1 \pmod{t}$), completing the proof. \square

We note that if t does not divide $n - s + 1$, then our proof only gives that the extremal graph is a subgraph of K_{s-1} join an independent set of size $n - s + 1$, and that the subgraph induced by the set of size $n - s + 1$ has maximum degree $t - 1$. We conjecture a similar construction is extremal when t does not divide $n - s + 1$.

Conjecture 4.5. *Let $2 \leq s \leq t$, and let $0 \leq p < t$. Let $n = s - 1 + kt + p$. For n large enough, the n -vertex graph of maximum spectral radius which does not contain $K_{s,t}$ as a minor is the join of K_{s-1} and $(kK_t + K_p)$.*

5 Graphs with $\mu(G) \leq m$

Let m be a positive integer. Let G_m be a graph on n vertices, with $\mu(G_m) \leq m$, which has the largest spectral radius of its adjacency matrix over all n -vertex graphs with Colin de Verdière parameter at most m . Throughout this section, A will denote the adjacency matrix of G_m , which will have spectral radius λ . \mathbf{x} will be an eigenvector for λ with infinity norm 1. We will use the following theorem of van der Holst, Lovász, and Schrijver [7].

Theorem 5.1. *Let $G = (V, E)$ be a graph and let $v \in V$. Then*

$$\mu(G) \leq \mu(G - v) + 1.$$

If v is connected to all other nodes, and G has at least one edge, then equality holds.

The main results of [24] show that for n large enough, the outerplanar graph of maximum spectral radius is K_1 join P_{n-1} and the planar graph of maximum spectral radius is K_2 join P_{n-2} . Since a graph G is outerplanar if and only if $\mu(G) \leq 2$ and planar if and only if $\mu(G) \leq 3$, Theorem 1.1 is proved if $m \in \{2, 3\}$, and we will from now on assume $m \geq 4$. Since K_2 join P_{n-2} is planar, Theorem 5.1 implies that the join of K_{m-1} and P_{n-m+1} has Colin de Verdière parameter equal to m for any m .

We also note that since $\mu(K_{m,m}) = m$ (cf [7]), our graph G_m may not contain $K_{m,m}$ as a minor. We omit the proof of the following Proposition, as it is similar to the proofs Lemmas 3.1–3.7.

Proposition 5.2. *For any $\delta > 0$, if n is large enough, then G_m contains a set K of $m - 1$ vertices which have a common neighborhood of size at least $(1 - \delta)n$ and each of which has eigenvector entry at least $1 - \delta$. Further, for any vertex $u \in V(G_m) \setminus K$, we have*

$$\mathbf{x}_u < \frac{1 - \delta}{c(s - 1)(s - 2)}$$

where c is chosen so that any graph of average degree c has Colin de Verdière parameter at least $m + 1$.

Let A be the common neighborhood of K and $B = V(G_m) \setminus (A \cup K)$. We show next that K induces a clique and that B is empty.

Lemma 5.3. *K induces a clique.*

Proof. The proof is similar to the proof of Lemma 4.2 once we note that for any integer m the property that $\mu(G) \leq m$ can be characterized by a finite family of excluded minors [7]. \square

Lemma 5.4. *B is empty.*

Proof. The proof is similar to the proof of Lemma 3.8, we must only check that if H is a graph with $\mu(H) = m$, then adding a new vertex adjacent to a clique of size $m - 1$ does not increase the Colin de Verdière parameter. But this follows since adding a new vertex adjacent to a clique of size $m - 1$ is an $(m - 1)$ -clique sum (cf [7]). \square

We may now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We now know that G_m is a subgraph of the join of $K = K_{m-1}$ and an independent set of size $n - m + 1$. Let H be the graph induced by $V(G_m) \setminus K$.

First we claim that H has maximum degree 2. In order to see this, we note that $\mu(K_{1,3}) = 2$. This and Theorem 5.1 imply that the join of $K_{1,3}$ and K_{m-1} has Colin de Verdière parameter $m + 1$. Therefore, this graph cannot be a subgraph of G_m , and so H may not have a vertex of degree 3 or more.

Therefore, H is the disjoint union of paths and cycles. But we now claim that H may not contain any cycles, as a cycle is a K_3 minor. A K_3 minor joined to a K_{m-1} is a K_{m+2} minor, which violates $\mu(G_m) \leq m$.

So now H induces a disjoint union of paths, which means that G_m is a subgraph of the join of K_{m-1} and P_{n-m+1} . By the Perron-Frobenius Theorem and maximality of λ , G_m must be exactly equal to the join of K_{m-1} and P_{n-m+1} . \square

Acknowledgements

We would like to thank Maria Chudnovsky and Paul Seymour; most of the results in section 2 were found in collaboration with them.

References

- [1] L. Babai and B. Guiduli, Spectral extrema for graphs: the Zarankiewicz problem, *Electronic J. Combin.* **16** (2009) R123.
- [2] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst, Parameters related to tree-width, zero forcing, and maximum nullity of a graph, *Journal of Graph Theory* **72** (2013) 146–177.
- [3] M. Chudnovsky, B. Reed, and P. Seymour, The edge-density for $K_{2,t}$ minors, *J. Combin. Theory Ser. B* **101** (2011), 18–46.
- [4] Y. Colin de Verdière, Sur un nouvel invariant des graphes et un critère de planarité, *J. Combin. Theory Ser. B* **50** (1990) 11–21.
- [5] C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York (1993).

- [6] A. J. Hoffman, On eigenvalues and colorings of graphs, *Graph Theory and Its Applications* (Proc. Advanced Sem., Math Research Center, Univ. of Wisconsin, Madison, Wis., 1969), New York.
- [7] H. van der Holst, L. Lovász, and A. Schrijver, The Colin de Verdière graph parameter, *Bolyai Soc. Math. Stud.* **7**, Janós Bolyai Math. Soc. Budapest (1999) 29–85.
- [8] Y. Hong, Tree-width, clique-minors, and eigenvalues, *Discrete Math.* **274** (2004), 281–287.
- [9] G. Kalai, E. Nevo, and I. Novik, Bipartite rigidity, *Trans. Amer. Math. Soc.* **368** (2016) 5515–5545.
- [10] A. V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, *Metody Diskret. Anal.* **38** (1982), 37–58.
- [11] A. V. Kostochka, Lower bound for the Hadwiger number for graphs by their average degree, *Combinatorica* **4** (1984), 307–316.
- [12] A. V. Kostochka and N. Prince, On $K_{s,t}$ minors in graphs of given average degree, *Discrete Math.* **308** (2008), 4435–4445.
- [13] L. Lovász and A. Schrijver, A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs, *Proc. Amer. Math. Soc.* **126** (1998) 1275–1285.
- [14] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, *Math. Ann.* **174** (1967), 265–268.
- [15] Homomorphiesätze für Graphen, *Math. Ann.* **178** (1968), 154–168.
- [16] J. S. Myers, The extremal function for unbalanced bipartite minors, *Discrete Math.* **271** (2003), 209–222.
- [17] E. Nevo, Embeddability and stresses of graphs, *Combinatorica* **27** (2007) 465–472.
- [18] V. Nikiforov, A contribution to the Zarankiewicz problem, *Linear Algebra Appl.* **432** (2010) 1405–1411.
- [19] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, *Combinatorics, Probability and Computing* **11** (2002) 179–189.
- [20] V. Nikiforov, The spectral radius of graphs with no $K_{2,t}$ minor, preprint (arXiv:1703.01839).
- [21] A. Nilli, On the second eigenvalue of a graph, *Discrete Math.* **91** (1991) 207–210.
- [22] N. Robertson, P. Seymour, and R. Thomas, A survey of linkless embeddings, *Graph Structure Theory* (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island (1993) 125–136.
- [23] R. P. Stanley, A bound on the spectral radius of graphs with e edges, *Linear Algebra Appl.* **87** (1987), 267–269.
- [24] M. Tait and J. Tobin, Three conjectures in extremal spectral graph theory, to appear in *J. Combin. Theory Ser. B*.

- [25] A. Thomason, The extremal function for contractions of graphs, *Math. Proc. Cambridge Phil. Soc.* **95** (1984), 261–265.
- [26] A. Thomason, The extremal function for complete minors, *J. Combin. Theory Ser. B* **81** (2001), 318–338.