Name: ______________________________

Instructions: You have 50 minutes to complete this exam. Show your work and justify all of your responses. No calculators, notes, or other external aids are allowed. You may use the following theorems:

1. (10 points) Show that any coloring of the edges of K_{17} with 3 colors contains a monochromatic triangle.

Solution: Fix a vertex v arbitrarily. Of the 16 edges incident with it, by the pigeonhole principle, one color must have at least 6 edges. Without loss of generality call this Color 1, and let $v_1, \cdots, v_6$ be vertices with edge vv_i Color 1. Then if v_i and v_j are joined by an edge in Color 1 for any $1 \leq i < j \leq 6$, then we have a triangle in Color 1, so we may assume that the vertices $v_1, \cdots, v_6$ induce a K_6 that is only colored with Colors 2 and 3. Now fix a vertex in this K_6 arbitrarily. Of the 5 edges incident with it in the K_6, at least 3 must be of the same color. Without loss of generality assume this is Color 2, and that the 3 edges in Color 2 are connected to vertices u_1, u_2, u_3. If any edge between u_1, u_2, u_3 is of Color 2, then we have a triangle in Color 2, so we may assume that they are all of Color 3. But this creates a triangle in Color 3.

2. (10 points) Use Ramsey's theorem to show that for each k, there exists an n such that any 2 coloring of the integers $[n]$ contains a monochromatic set $\{x_1, x_2, \cdots, x_k, x_1 + x_2 + \cdots + x_k\}$ (when $k = 2$ this is Schur's theorem).

Solution: Let $n = R(k+1, k+1)$ and let $\chi : [n] \to \{\text{red, blue}\}$ be an arbitrary 2-coloring of $[n]$. We 2-color the edges of K_n as follows: let the vertex set be $[n]$ and for $i < j$ color the edge ij with $\chi(j - i)$. Since this is a 2-coloring of the edges of K_n where $n = R(k+1, k+1)$, there must be a monochromatic K_{k+1}. Let this monochromatic K_{k+1} be on the vertices $i_1 < i_2 < \cdots < i_{k+1}$. Then by how the coloring is defined, we know that in the two coloring of the integers $[n]$, the integers

$$\begin{gathered} i_2 - i_1 \\ i_3 - i_2 \\ \vdots \\ i_{k+1} - i_k \\ i_{k+1} - i_1 \end{gathered}$$

must all have the same color. Then we let

$$\begin{aligned} x_1 &= i_2 - i_1 \\ x_2 &= i_3 - i_2 \\ &\vdots \\ x_k &= -k + 1 - i_k. \end{aligned}$$

Then $x_1, x_2, \cdots, x_k, x_1 + x_2 + \cdots + x_k$ all have the same color since $x_1 + \cdots x_k = i_{k+1} - i_1$.

3. (10 points) A graph is called triangle saturated if it does not contain any triangles but changing any non-edge to an edge creates a triangle. The extremal number $\mathrm{ex}(n, K_3)$ is the maximum number of edges in a triangle saturated graph on n vertices. In this problem we will be interested in finding the *minimum* number of edges in a triangle saturated graph. This quantity is denoted

$$\mathrm{sat}(n, K_3).$$

Give an exact formula for $\mathrm{sat}(n, K_3)$ (ie, make a construction giving an upper bound, and then prove that any triangle saturated graph on n vertices must have at least that many edges). It may be useful to know that if you want to solve the following real number optimization problem:

$$\begin{aligned} \text{minimize} \quad & \sum x_i \\ \text{subject to} \quad & \delta \le x_i \le \Delta \\ & \sum x_i^2 \ge k, \end{aligned}$$

Then the solution is given by making as many x_i as possible equal to δ and the rest equal to Δ (with at most one x_i in between these two values).

Solution: First we note that the star $K_{1,n-1}$ is triangle saturated. If any nonedge uv is added, then u, v and the center of the star create a triangle. So $\mathrm{sat}(n, K_3) \le n - 1$.

To show the upper bound, assume that G is a triangle saturated graph. We must show that $e(G) \ge n - 1$. Since G is triangle saturated, if $uv \notin E(G)$ then the addition of uv must create a triangle. This means that there must be a vertex w which is adjacent to both u and v. That is, for any non-edge $u \not\sim v$, we must have

$$d(u, v) = |\Gamma(u) \cap \Gamma(v)| \ge 1.$$

Then

$$\sum_{u \not\sim v} d(u, v) \ge \binom{n}{2} - e(G).$$

However, note that since G is triangle free, it means that for any edge xy, $d(x, y) = 0$. Therefore

$$\sum_{u \not\sim v} d(u, v) = \sum_{u \ne v} d(u, v) = \sum_{v \in V(G)} \binom{d(v)}{2} = \left(\frac{1}{2} \sum_{v} (d(v))^2 \right) - e(G).$$

Combining, we get

$$\sum_v (d(v))^2 \geq \binom{n}{2}.$$

Next we note that a triangle saturated graph cannot have any isolated vertices. If there was an isolated vertex, we could add an edge incident with it and a triangle would not be created. So we are interested in knowing the number of edges in a graph G subject to the constraints

$$\sum_v (d(v))^2 \geq \binom{n}{2}$$

and

$$1 \leq d(v) \leq n-1$$

for all v. Since $e(G) = \frac{1}{2} \sum d_v$, the solution to the optimization problem tells us that

$$\sum d_v \geq (n-1) + 1 + 1 + \cdots + 1.$$

Therefore $e(G) \geq n-1$.