Trigonometric Functions

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Unless otherwise stated, all measurements are in degrees, not radians.

Things that you must know

All basic trigonometric identities can be derived from the following facts:

- $\tan(x) = \frac{\sin(x)}{\cos(x)}$, $\sec(x) = \frac{1}{\cos(x)}$, $\csc(x) = \frac{1}{\sin(x)}$, $\cot(x) = \frac{\cos(x)}{\sin(x)}$.
- $\sin^2(x) + \cos^2(x) = 1$, $\sin(x) = \cos(90 - x)$.
- $\sin(90) = 1$, $\sin(-x) = -\sin(x)$.
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

You can also start with the formula for $\cos(x + y)$ instead of $\sin(x + y)$. For a nice proof of the $\sin(x+y)$ formula, see Khan Academy's video "Proof of the sine angle addition identity."

Things that you should know

All of these facts should be in your toolbag for any question about trigonometry. If you forget them, you can derive them from the above section, but at a minimum, it is helpful to be aware that these facts exist.

- $\cos(-x) = \cos(x)$, $\tan(-x) = -\tan(x)$.
- $\sin(180 - x) = \sin(x)$, $\cos(180 - x) = -\cos(x)$.
- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.
- $\tan(x + y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$.
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
- $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$
- $\sin(\frac{1}{2}x) = \sqrt{\frac{1-\cos(x)}{2}}$, $\cos(\frac{1}{2}x) = \sqrt{\frac{1+\cos(x)}{2}}$, $\tan(\frac{1}{2}x) = \sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$.
- $\sec^2(x) = \tan^2(x) + 1$, $\csc^2(x) = \cot^2(x) + 1$.

Things that are good to know

- $\cos(x)\cos(y) = \frac{1}{2}(\cos(x - y) + \cos(x + y))$.
- $\sin(x)\sin(y) = \frac{1}{2}(\cos(x - y) - \cos(x + y))$.
- $\sin(x)\cos(y) = \frac{1}{2}(\sin(x + y) + \sin(x - y))$.
- $a\sin(x) + b\cos(x) = \sqrt{a^2 + b^2}\sin(x + \tan^{-1}(\frac{b}{a}))$.

Practice with identities

1. Derive the half-angle identities and the the product-to-sum identities from the formulas for $\cos(x+y)$ and $\sin(x+y)$.
2. Find formulas for $\sin(x-y)$, $\cos(x-y)$, $\cos(x+y+z)$.
3. Express $\cos(3x), \cos(4x)$ in terms of $\cos(x)$. Looking at $\cos(1x), \cos(2x), \cos(3x), \cos(4x)$, do you see any patterns?
4. Compute $\sin(15)$, $\cos(15)$. Then memorize!
5. Compute $\tan(\sin^{-1}(\frac{8}{17}))$, $\cos(\cot^{-1}(\frac{1}{5}))$.

Problems

1. (ARML 81) Compute $\sin^2(10)+\sin^2(20)+\cdots+\sin^2(90)$.
2. (ARML 79) Find all x such that $\sin(4\tan^{-1}(x)) = \frac{24}{25}$.
3. (ARML 88) Compute the smallest $x > 0$ such that $8\sin(x)\cos^5(x) - 8\sin^5(x)\cos(x) = 1$.
4. (SMT 12) Find all solutions α with $0 < \alpha < 90$ to the equation $1+\sqrt{3}\tan(60-\alpha) = \frac{1}{\sin(\alpha)}$
5. (AIME 84) Compute $\cot(\cot^{-1}(3)+\cot^{-1}(7)+\cot^{-1}(13)+\cot^{-1}(21))$.
6. Compute $\cos(72)$ and $\cos(36)$. (Hint: Consider isoceles triangle ABC with $\angle A = \angle C$ and $\angle B = 36°$).
7. (AIME 08) Let $\alpha = \frac{\pi}{2008}$ (in radians). Find the least positive integer n such that $\sum_{k=1}^{n} 2\cos(k^2\alpha)\sin(k\alpha)$ is an integer.
8. (ARML 79) Find the smallest positive x such that $\cos(5) = \sin(25) + \sin(x)$.
9. (NYCIML) Compute $\sin(10)\sin(50)\sin(70)$.
10. (GCTM 06) Compute $\sum_{n=0}^{\infty} \cot^{-1}(n^2+n+1)$.
11. (AIME 00) Find the least positive integer n such that
$$\frac{1}{\sin(n)} = \frac{1}{\sin(45)\sin(46)} + \frac{1}{\sin(47)\sin(48)} + \cdots + \frac{1}{\sin(133)\sin(134)}$$
12. (Putnam 03) Find the minimum value of
$$|\sin(x)+\cos(x)+\tan(x)+\cot(x)+\sec(x)+\csc(x)|$$
for $x \in \mathbb{R}$.
13. (USAMO 98) Let $a_0, \ldots, a_n$ be numbers in $(0, \frac{\pi}{2})$ such that
$$\sum_{i=0}^{n} \tan(a_i - \tfrac{\pi}{4}) \geq n-1$$
Prove that $\prod_{i=0}^{n} \tan(a_i) \geq n^{n+1}$.