Combinatorics 1

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1 Notes on Basic Counting

$|A|$ means the number of elements in A.

1.1 Binomial Coefficients, Combinations, Permutations

$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$. $n!$ is called "n factorial". This is the number of ways to order (or the number of permutations of) n distinct objects. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of ways to choose a set of k objects out of n distinct objects. It's called a binomial coefficient since it is the coefficient of x^k in $(1+x)^n$.

1.2 Complementary Counting

If $A \subseteq B$, then $|A| = |B| - |B \setminus A|$. When it seems like it's difficult to count something directly, perhaps try counting some sort of complement. Example problem: How many positive integers less than 1000 contain at least one 1 as a digit?

1.3 Casework

If $A \cap B = \varnothing$, then $|A \cup B| = |A| + |B|$. This is used very frequently to break a problem down into easier parts.

1.4 Inclusion-Exclusion

For sets A and B, we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In general, for sets $A_1, \ldots, A_n$, we have

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| = (|A_1| + \cdots + |A_n|) \\ - (|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|) \\ + (|A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|) \\ + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|. \end{aligned}$$

1.5 Stars and Bars

How many ordered pairs of non-negative integers (a, b) satisfy $a + b = 8$?

$$\star\star \mid \star\star\star\star\star\star$$

How many monic cubic integer polynomials with non-negative integer coefficients satisfy $f(1) = 7$?

$$\star \mid \star\star\star\star\star \mid$$

2 Problems

1. Annie has ten monsters and fifteen friends. If a team consists of Annie, five of her monsters, and a friend, how many teams can she make?

2. In Patrick's guild, there are ten distinct members, and each member can be one of five classes. In how many ways can the members choose their classes such that the guild has at most two classes that are not represented?

3. Victor has three red, three blue, and three green heroes (all of whom are distinct), and a team consists of four heroes.

   (a) How many teams can he make if he must have at least one hero of each color?

   (b) How many teams can he make if he can have at most two heroes of any color?

   (c) How many teams can he make if Olivia and Lucina must be on the team together or not at all, and further if Corrin appears he cannot be on the team with either of them?

4. (AMC 10) How many ways are there to choose three distinct vertices such that the plane determined by these three vertices contains points inside the cube?

5. Corrin is at the bottom left square of a seven-by-seven grid and can move two squares per turn. If he is only allowed to move right or up, he will reach the top right square in six turns. In how many ways can he reach that square?

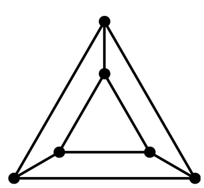
6. (a) Find the number of solutions to $x + y + z = 15$ if x, y, z must be positive integers.

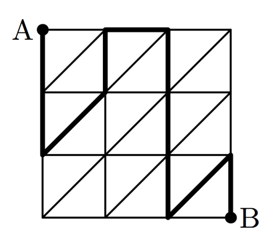
   (b) What if the equation in (a) were $x + y + z \leq 15$ instead?

   (c) What if (a) had the additional condition that the smallest element were at most 3?

7. (AIME) Each of the 2001 students at a high school studies either Spanish or French, and some study both. The number who study Spanish is between 80 percent and 85 percent of the school population, and the number who study French is between 30 percent and 40 percent. Let m be the smallest number of students who could study both languages, and let M be the largest number of students who could study both languages. Find $M - m$.

8. (Mandelbrot) In the diagram shown below left, how many ways are there to color two of the dots red, two of the dots blue, and two of the dots green so that dots of the same color are joined by a segment?





9. (Mandelbrot) How many paths are there from A to B through the network shown if you may only move up, down, right, and up-right? A path also may not traverse any portion of the network more than once. A sample path is shown above and to the right.

10. (AIME) At a certain university, the division of mathematical sciences consists of the departments of mathematics, statistics, and computer science. There are two male and two female professors in each department. A committee of six professors is to contain three men and three women and must also contain two professors from each of the three departments. Find the number of possible committees that can be formed subject to these requirements.

11. (AIME) Let S be the vertices of a regular 12-gon. A subset $Q \subseteq S$ is called *communal* if there exists a circle that contains every point in Q and no point in $S \setminus Q$. Find the number of communal subsets.

12. Victor and twelve friends have a pile of sixty orbs. In how many ways can they distribute the orbs among themselves such that at least two people do not receive anything?

13. (AIME) Find the number of ordered triple (a, b, c) where a, b, and c are positive integers, a is a factor of b, a is a factor of c, and $a + b + c = 100$.

14. (AIME) How many ordered pairs of sets $(\mathcal{A}, \mathcal{B})$ are there such that they partition the set $\{1, 2, \ldots, 12\}$, $|\mathcal{A}| \notin \mathcal{A}$, and $|\mathcal{B}| \notin \mathcal{B}$?

15. (AIME) Find the number of 3x3 matrices that satisfy the following conditions:

    - Five of the entries are 1 and the other four are 0.
    - Out of the three rows, three columns, and two diagonals, at most one has all three entries equal.

16. (AIME) A grid has six rows and four columns. Find the number of ways to shade twelve squares such that each row has two shaded squares and each column has three.

3 More Problems

1. (Canada) Each square of an m by n board is colored red or blue, where m, n are odd. A row is said to be *red-dominated* if there are more red squares than blue squares in the row, and a column is said to be *blue-dominated* if there are more blue squares than red squares in the column. Determine, with proof, the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns.

2. (Russia) In a chess tournament $2n + 3$ players take part. Every two play exactly one match. The schedule is such that no two matches are played at the same time, and each player, after taking part in a match, is free in at least n next (consecutive) matches. Prove that one of the players who play in the opening match will also play in the closing match.