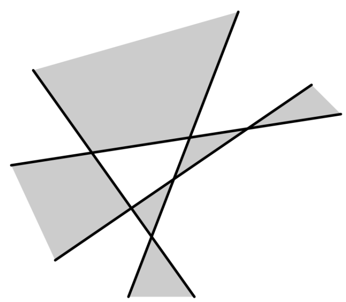
Writing Proofs | Misha Lavrov

Induction

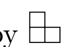
Western PA ARML Practice | *March 6, 2016*

Warm-up

Some number of straight lines are drawn in the plane, as shown below. Prove that the resulting regions of the plane can always be colored black and white so that any two adjacent regions receive a different color.



Problems

1. Prove that a $2^n \times 2^n$ chessboard with any one square removed can always be covered by  shaped tiles.

2. Chicken McNuggets come in boxes of 6, 9, and 20 nuggets. Prove that for any integer $n > 43$, it is possible to buy exactly n nuggets with a combination of these boxes.

   If we also allow 4-nugget boxes[1], then what is the largest integer n such that it's impossible to buy exactly n nuggets?

3. Find the mistake in the following proofs:

   (a) **False Theorem 1:** All integers are equal.

   We prove by induction that any two adjacent integers are equal: $n = n + 1$ for any n.

   By the induction hypothesis, we have $n - 1 = n$ for any n. Add 1 to both sides: then $n = n + 1$. This completes the induction step. Therefore $n = n + 1$ for all n, so $1 = 2 = 3 = 4 = \cdots$.

   (b) **False Theorem 2:** Any n points $A_1, A_2, \ldots, A_n$ are collinear.

   We proceed by induction on n. When $n = 1$ or $n = 2$, the statement is trivial.

[1] Historically, these were introduced after this problem—a well-known induction problem—was first posed.

Assume the induction hypothesis for n and suppose we have a collection of points $A_1, A_2, \ldots, A_{n+1}$. By the induction hypothesis, $A_1, A_2, \ldots, A_n$ all lie on a common line ℓ_1. Also by the induction hypothesis, $A_2, A_3, \ldots, A_{n+1}$ all lie on a common line ℓ_2. But ℓ_1 and ℓ_2 both pass through A_2 and A_n, so they must be the same line $\ell_1 = \ell_2 = \ell$. Therefore all $n+1$ points lie on ℓ, and we are done.

4. Prove that for all $n \geq 1$, the sum of the first n odd numbers is a perfect square.

   (Hint: you'll need a stronger induction statement.)

5. Prove that for all $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$.

   (Hint: you'll need a stronger induction statement.)

6. In a chess tournament, each pair of players played exactly one game, and incredibly, none of them ended in draws.

   Prove that there is a participant in the tournament (call him Bobby) such that every other player either lost their game with Bobby, or lost a game with someone else that lost to Bobby.

7. (VTRMC 2012) Define a sequence (a_n) for n a positive integer inductively by $a_1 = 1$ and

$$a_n = \frac{n}{\prod_{d|n, d<n} a_d}, \quad \text{where the product ranges over all proper divisors } d \text{ of } n.$$

   Thus $a_2 = 2$, $a_3 = 3$, $a_4 = 2$, etc. Find a_{999000}.

8. A binary sequence such as 0011010 is written on a blackboard. In a step, you are allowed to change the first number (from 0 to 1 or vice versa) or the number after the first occurrence of 1. (Starting with 0011010, you could get to 1011010 or to 0010010.)

   Prove that you can change any sequence to any other sequence of the same length.

9. (From my research) We say that a *Hanoi* integer is one in which no two adjacent digits are equal (such as 123456, or 271828, but not 144702). We define two operations on Hanoi integers:

   - A *tweak* changes the last digit to anything else that's not the same as the previous digit: we may change 123456 to 123454, or 123450.
   - A *twiddle* takes the longest suffix of two digits alternating, as $\ldots xyxy$, and switches the two digits: we may change 123456 to 123465, or 271828 to 271282, or 363636 to 636363. Twiddles are forbidden if they would put a 0 at the beginning of the integer.

   Prove that:

   (a) You can change any n-digit Hanoi integer to any other in $2^n - 1$ tweaks and twiddles.

   (b) All $2^n - 1$ tweaks and twiddles are required if you are going from x to y, and x and y have no digits in common. (That is, if x contains the digit 2, y cannot contain a 2 anywhere.)