Probability | Misha Lavrov

Random Variables: Solutions

Western PA ARML Practice | *April 17, 2016*

1 Theoretical Exercises

1. *Prove the following fact: if X is a random variable that takes on nonnegative integer values, then*

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \Pr[X \geq k].$$

Write $\Pr[X \geq k]$ as $\sum_{\ell=k}^{\infty} \Pr[X = \ell]$. Then rearrange the sums:

$$\sum_{k=1}^{\infty} \sum_{\ell=k}^{\infty} \Pr[X = \ell] = \sum_{\ell=1}^{\infty} \sum_{k=1}^{\ell} \Pr[X = \ell] = \sum_{\ell=1}^{\infty} \ell \cdot \Pr[X = \ell] = \mathbb{E}[X].$$

2. *A biased coin lands heads with probability p, and tails otherwise. We flip the coin until it lands heads for the first time; the random variable X is the number of times we flipped the coin.*

   (a) *Write an expression for $\Pr[X = k]$ in terms of p and k.*

   $\Pr[X = k] = p(1 - p)^{k-1}$.

   (b) *Compute $\mathbb{E}[X]$.*

   We have $\Pr[X \geq k] = (1 - p)^{k-1}$, so by Problem 1,

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} (1 - p)^{k-1} = \frac{1}{1 - (1 - p)} = \frac{1}{p}.$$

3. *Now take the same biased coin that lands heads with probability p, and toss it n times. Let Y be the number of times the coin lands heads.*

   (a) *Write an expression for $\Pr[Y = k]$ in terms of n, p, and k.*

   $\Pr[Y = k] = \binom{n}{k} p^k (1 - p)^{n-k}$.

   (b) *Compute $\mathbb{E}[Y]$.*

   Write $Y = Y_1 + Y_2 + \cdots + Y_n$, where Y_i is 1 if the i^{th} coin is heads, and 0 otherwise. Then $\mathbb{E}[Y_i] = p$, so by linearity of expectation $\mathbb{E}[Y] = p + p + \cdots + p = np$.

4. *Let $X_1, X_2, \ldots, X_n$ be the outcomes of n independent rolls of fair six-sided dice. Let $S_k = X_1 + X_2 + \cdots + X_k$. Prove that $S_n \bmod 6$ has the same distribution as a single die roll.*

   Fix a t between 1 and 6; we'll show that $\Pr[S_n \bmod 6 = t] = \frac{1}{6}$.

   For any value of s, we have

$$\Pr[S_n \bmod 6 = t \mid S_{n-1} = s] = \Pr[X_n = (t - s) \bmod 6] = \frac{1}{6}.$$

   Therefore

$$\Pr[S_n \bmod 6 = t] = \sum_{s=0}^{\infty} \Pr[S_n \bmod 6 = t \mid S_{n-1} = s] \cdot \Pr[S_{n-1} = s] = \frac{1}{6} \sum_{s=0}^{\infty} \Pr[S_{n-1} = s] = \frac{1}{6}.$$

2 Other Problems

1. *(ARML 1987) Initial setup: we have 3 jars, called A, B, and C. Jars A and B each contain one white and one black ball; jar C is empty. A random ball is chosen from jar A and placed into jar C. Similarly, a random ball is chosen from jar B and placed into jar C.*

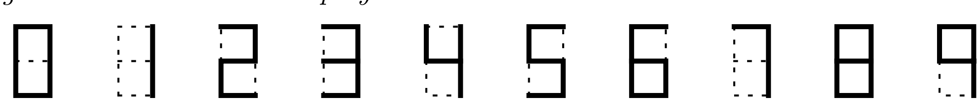
   We now consider jar C. A ball is chosen from jar C at random; it is white. That ball is put back into jar C, which is shaken, and again a ball is chosen at random; again it is white. That ball is put back into jar C, which is shaken, and for a third time a ball is chosen at random; again it is white.

   Compute the probability that the ball still in jar C is black.

   There are two possibilities: either there's one white ball and one black ball in jar C, or else there are two white balls. Initially the odds of these are $2:1$, since there are two ways to end up with balls of different colors in jar C, but only one way to end up with both white balls.

   Drawing a white ball from jar C is twice as likely if both balls in the jar are white. So every time this happens, we multiply the odds by $1:2$. After three such drawings, the odds are $1:4$ in favor of the both-white hypothesis, so the probability that the other ball in jar C is black is $\frac{1}{5}$.

2. *(ARML 1999) A digital watch displays the digits from 0 to 9 as shown below. If one of the seven segments, randomly chosen, fails to light up, compute the expected value of the number of digits which can still be displayed.*



   Let $X_i = 1$ if the digit i can be displayed, and 0 otherwise; our goal is to find $\mathbb{E}[X_0 + X_1 + \cdots + X_9]$. We have

$$\mathbb{E}[X_k] = \Pr[X_k = 1] = \frac{\text{\# of segments in } k}{7}.$$

   So the sum of all of these is the total number of segments in the picture, divided by 7. Counting, we get an answer of $\frac{48}{7}$, unless I miscounted.

3. *The entire surface of a $3 \times 3 \times 3$ cube is painted, and then the cube is cut into 27 $1 \times 1 \times 1$ cubes. The small cubes are reassembled randomly into a $3 \times 3 \times 3$ cube. Compute the expected value of the number of 1×1 squares of paint showing anywhere on the resulting cube's surface.*

   The $1 \times 1 \times 1$ cubes have $27 \cdot 6$ faces, $9 \cdot 6$ of which are painted, so $\frac{1}{3}$ of all faces are painted. Therefore if we choose an arbitrary 1×1 square on the surface of the reassembled cube, it has a $\frac{1}{3}$ chance of being painted.

   Number the 1×1 squares 1 through 54 arbitrarily; let $X_1, X_2, \ldots, X_{54}$ be the indicator variables where $X_i = 1$ if the i^{th} square is painted, and 0 otherwise. Then the expected value we want is

$$\mathbb{E}[X_1 + X_2 + \cdots + X_{54}] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_{54}] = 54 \cdot \frac{1}{3} = 18.$$

4. *(Inspired by AIME 2010) Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After going through security, which is at one end of the airport, Dave walks to his gate, and finds out that there's been a gate change, so Dave walks to his new gate (also assigned at random).*

   Compute the expected value of the distance Dave has to walk.

   The first leg of Dave's trip has expected length

$$\frac{0 + 100 + 200 + \cdots + 1100}{12} = \frac{0 + 1100}{2} = 550.$$

   Next, we find the expected length of the second leg. There are 11 pairs of gates at distance 100, 10 pairs of gates at distance 200, and so on. So the average distance between two random gates is

$$\frac{11 \cdot 100 + 10 \cdot 200 + 9 \cdot 300 + \cdots + 1 \cdot 1100}{11 + 10 + 9 + \cdots + 1}.$$

   We can just compute this, but if we're lazy we can use problem #1 in the previous section. First, divide everything by 100. If X is the distance, in units of 100 feet, then $\Pr[X \geq k] = \frac{\binom{13-k}{2}}{\binom{12}{2}}$, as choosing two gates at least k steps apart is equivalent to choosing two gates out of $13 - k$, then inserting $k - 1$ gates between them. So

$$\mathbb{E}[X] = \frac{1}{\binom{12}{2}} \sum_{k=1}^{11} \binom{13-k}{2} = \frac{1}{\binom{12}{2}} \sum_{\ell=1}^{12} \binom{\ell}{2} = \frac{1}{\binom{12}{2}} \binom{13}{3} = \frac{13}{3}.$$

   Thus, the final answer is $550 + \frac{1300}{3} = \frac{2950}{3}$.

5. *Suppose that 100 fair coins are flipped simultaneously.*

   (a) *What is the expected number of* pairs of coins *that both land heads? (If 5 coins land heads and 95 coins land tails, then there are $\binom{5}{2} = 10$ pairs of coins that land heads.)*

For each pair $\{i, j\}$, let $X_{\{i,j\}}$ be the indicator variable that's 1 if both pairs land heads, and 0 otherwise. Then $\mathbb{E}[X_{\{i,j\}}] = \frac{1}{4}$, and the expected value we want is

$$\sum_{i,j} \mathbb{E}[X_{\{i,j\}}] = \frac{1}{4}\binom{100}{2} = \frac{2475}{2}.$$

(b) *Let X be the number of coins that land heads. What is $\mathbb{E}[X^2]$?*

We have just computed $\mathbb{E}[\binom{X}{2}] = \frac{2475}{2}$. Therefore $\mathbb{E}[X(X-1)] = \mathbb{E}[2\binom{X}{2}] = 2\mathbb{E}[\binom{X}{2}] = 2475$, and $\mathbb{E}[X^2] = \mathbb{E}[X(X-1)+X] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] = 2475 + 50 = 2525$.