Recursions

1 Introduction

Recursion is when a value is defined based on previously defined values. For example, we could have a sequence defined by

$$F_0 = F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2}$$

This sequence is recursive because each term (after a certain point) is defined as the sum of the two previous terms. This sequence is the well-known Fibonacci sequence.

Calculating something based on previous values is sometimes useful for solving combinatorics problems. If it is hard to calculate a value explicitly, there may be a way to calculate it as the nth term of a recursive sequence. Sometimes, you will need to define your own quantities for a recursive strategy to succeed.

2 Warm-up

(AIME 2006) A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \le k \le 8$. A tower is to be built using all 8 cubes according to the rules:

- Any cube may be the bottom cube in the tower.

- The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

3 Problems

1. How many ways are there to divide a 1×8 strip into blocks of size 1×1 or 1×2?

2. How many ways are there to divide a 1×8 strip into blocks of size 1×1 or 1×2 or 1×3?

3. How many ways are there to cover a 2×5 rectangle with 1×2 dominoes and 1×1 squares? Dominoes may be rotated.

4. How many 100 character strings with the letters A, B, C are there with the property that there are never two same letters adjacent to each other or with only one letter in between? (For example, BABCAB is acceptable but ABCBBA is not).

5. (AIME 2001) A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

6. (AMC 12A 2007) Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

   (A) 121   (B) 123   (C) 125   (D) 127   (E) 129

7. Find a recursive formula for the number of ways to divide n distinguishable objects into m indistiguishable boxes. None of the boxes can be empty. (These numbers are called Stirling numbers of the second kind).

8. (http://fivethirtyeight.com/features/whats-the-best-way-to-drop-a-smartphone/)

   By the way, this is a great website that does a lot of applied math and statistics.

   You work for a tech firm developing the newest smartphone that supposedly can survive falls from great heights. Your firm wants to advertise the maximum height from which the phone can be dropped without breaking.

   You are given two of the smartphones and access to a 100-story tower from which you can drop either phone from whatever story you want. If it doesnt break when it falls, you can retrieve it and use it for future drops. But if it breaks, you dont get a replacement phone.

   Using the two phones, what is the minimum number of drops you need to ensure that you can determine exactly the highest story from which a dropped phone does not break? (Assume you know that it breaks when dropped from the very top.) What if, instead, the tower were 1,000 stories high?

9. (Putnam 2015) Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that

$$|i - j| = 1 \text{ implies } |\pi(i) - \pi(j)| \leq 2$$

   for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

   does not depend on n, and find its value.