Number Theory | Misha Lavrov

What we learned from the AMC

Western PA ARML Practice | *February 8, 2015*

Factorization

1. Factor $xy + 3x + 2y + 6$.
2. (AMC 12, Problem 10) Integers x and y with $x > y > 0$ satisfy $x + y + xy = 80$. What is x?
3. Find all positive integer solutions to the equation $xy = 10(x + y)$.
4. (AMC 10, Problem 15) Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?
5. A triple of positive integers (x, y, z) is called a *Pythagorean triple* if $x^2 + y^2 = z^2$. How many Pythagorean triples have $x = 60$?
6. (AMC 12, Problem 18) The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?
7. (AMC 10, Problem 20) (Modified) A rectangle with positive integer side lengths in cm has area A cm^2 and perimeter P cm. How many of the numbers between 1 and 100 can equal $A + P$?

   (It might help you to know that the first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, ...)

Super exciting problems

1. (AMC 12, Problem 22) For each positive integer n, let $S(n)$ be the number of sequences of length n consisting solely of the letters A and B, with no more than three As in a row and no more than three Bs in a row. What is the remainder when S(2015) is divided by 12?
2. Compute 14^{14} mod 60. Then, compute $14^{14^{14^{14^{14}}}}$ mod 60.
3. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. If a googolplex is $10^{10^{100}}$, what is the last digit of $F_{\text{googolplex}}$?
4. Show that for any integers $x, y \geq 2$,

$$\Bigg|\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{x} - \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}_{y}\Bigg| \geq 11.$$

   (If you cannot solve this one, try to prove as large a lower bound as possible. A lower bound of 7 is a reasonable one to try for.)