Combinatorics | Misha Lavrov

Markov Chains in Problem Solving

Western PA ARML Practice | *September 14, 2014*

Warm-up

1. (2006 AMC 10) A player pays \$5 to play a game. A die is rolled. If the number on the die is odd, the game is lost. If the number on the die is even, the die is rolled again. In this case the player wins if the second number matches the first and loses otherwise. How much should the player win if the game is fair? (In a fair game the probability of winning times the amount won is what the player should pay.)
2. (You've seen this already, but just to make sure) Two cows stand on opposite faces of a cube and take turns moving to a random adjacent face. A cow moving into the same face as another cow knocks the other cow over. What is the probability that the first cow to move eventually knocks over the second cow?

Problems

1. (Mathcounts) Aiden, Brayden, and Chloe[1] take turns flipping a coin in alphabetical order. The first to flip heads wins (if none of them flip heads, the game restarts from Aiden). What is the probability that Aiden wins?
2. Suppose that the two cows decide to make their game more exciting and play it on the vertices of the cube, starting on opposite vertices. What is the probability that the first cow to move eventually knocks over the second cow?
3. (1995 AIME) If you flip a fair coin repeatedly, what is the probability you see a run of 5 consecutive heads before seeing a run of 2 consecutive tails?
4. In a gambling game, you start with \$1, and every time you play, you are equally likely to win or lose \$1. What is the probability that you reach \$10 before you go broke?
5. What are your expected winnings in this game, assuming you decide to play until you reach \$10 or you go broke?
6. Suppose you are quite good at the gambling game, and you win \$1 with a probability of $\frac{2}{3}$ (and lose \$1 otherwise). You still start with \$1, and keep playing as long as you can. What is the probability that you *never* run out of money?
7. (2001 AMC 10) A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

[1] These names are the most popular names given to babies in 2013 with those initial letters. So get used to seeing them.

8. In the St. Petersburg Lottery, you flip coins until you get tails, and then you win 2^n rubles, where n is the number of heads you flipped. How many rubles should the lottery cost in order to be fair?

9. A monkey sitting at a typewriter types a single letter every second, which is chosen randomly from the set $\{A, B, C\}$. (All three of these letters are equally likely.)

   (a) What is the average time before the monkey types "C"?

   (b) What is the average time before the monkey types "AAA"?

   (c) What is the average time before the monkey types "ABACAB"?

10. (2009 AIME) Dave rolls a fair six-sided die until a six appears for the first time. Independently, Linda also rolls a fair six-sided die until a six appears for the first time. What is the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die?

11. The US government releases 100 collectible trading cards with the 100 US Senators. You can buy a random card for a dollar, or you can buy the entire collection for \$400. If you want to collect all 100 cards, and duplicates are worthless to you, which is the better deal?