Algebra	Misha Lavrov

Recurrences

Western PA ARML Practice — *March 1, 2015*

Warm-up

Find closed forms for the following recurrences:

1. $A_0 = 0$ and $A_{n+1} = A_n + 2n - 1$ for $n \geq 0$.
2. $B_0 = 1$ and $B_{n+1} = 2B_n$ for $n \geq 0$.
3. $C_0 = 0$ and $C_{n+1} = B_n + C_n$ for $n \geq 0$.

Problems

1. Solve by homogenizing:
   (a) $w_0 = 0$ and $w_{n+1} = 3w_n + 1$ for $n \geq 0$.
   (b) $x_0 = 0$ and $x_{n+1} = 2x_n + 5$ for $n \geq 0$.
   (c) $y_0 = 0$ and $y_{n+1} = 2y_n + n$ for $n \geq 0$.
   (d) $z_1 = 0$ and $z_{n+1} = (n+1)z_n + (n-1)!$ for $n \geq 1$.
2. Solve by multiplying by a summation factor:
   (a) $w_0 = 0$ and $w_{n+1} = 3w_n + 1$ for $n \geq 0$.
   (b) $x_0 = 0$ and $x_{n+1} = 2x_n + 5$ for $n \geq 0$.
   (c) $y_1 = 1$ and $ny_n = (n-2)y_{n-1} + 2$ for $n \geq 2$.
   (d) $z_0 = 0$ and $z_{n+1} = 2^n z_n + 2^{n(n+1)/2}$ for $n \geq 0$.
3. Try solving problem 1(c) by multiplying by a summation factor, get stuck, and instead find a formula for $\sum_{k=0}^{n} \frac{k}{2^{k+1}}$.
4. (Concrete Math) Multiply by a summation factor to solve the recurrence
$$\begin{aligned} T_0 &= 5 \\ 2T_n &= nT_{n-1} + 3 \cdot n!, \qquad n > 0. \end{aligned}$$
5. The sequence G_n satisfies $G_0 = 0$, $G_1 = 1$, and $G_n = G_{n-1} + G_{n-2} + 2^n$ for $n \geq 2$. Express G_n in terms of the Fibonacci number F_n, which satisfies the recurrence $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

6. (Putnam 1985) Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m$$
$$a_m(j+1) = (a_m(j))^2 + 2a_m(j), \qquad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

7. Recall that the Fibonacci numbers F_n satisfy the recurrence relation $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

   (a) Let ϕ be a solution to $\phi = 1 + \frac{1}{\phi}$. Find a recurrence relation for $F'_n := F_n + \frac{F_{n-1}}{\phi}$, and solve for F'_n.

   (b) Use your formula for F'_n to find a one-term recurrence relation for F_n, and solve for F_n.

More Problems

1. (Concrete Math) Solve the recurrence $Q_0 = \alpha$, $Q_1 = \beta$, and $Q_n = \frac{1+Q_{n-1}}{Q_{n-2}}$ for $n > 1$. You may assume that $Q_n \neq 0$ for all $n \geq 0$.

2. (VTRMC 2011) A sequence (a_n) is defined by $a_0 = -1$, $a_1 = 0$, and

$$a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$$

for all positive integers n. Find a_{100}.

3. Recall that the Fibonacci numbers F_n satisfy the recurrence relation $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Find a recurrence relation for F_n^2.