Simplifying finite sums

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Warm-up

1. Find $1 + 2 + 3 + \cdots + 100$. (The story goes that Gauss was given this problem by his teacher in elementary school to keep him busy so he'd quit asking hard questions. But he figured it out in his head in about ten seconds.)
2. (2003 AIME I.) One hundred concentric circles with radii $1, 2, 3, \ldots, 100$ are drawn in a plane. The smallest circle is colored red, the strip around it green, and from there the colors alternate. What fraction of the total area of the largest circle is colored green?

1. The classic argument goes like this:

$$\begin{array}{cccccccccccc} & 1 & + & 2 & + & 3 & + & \cdots & + & 100 \\ + & 100 & + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline & 101 & + & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

   If S is the sum, $2S = 100 \cdot 101 = 10100$, so $S = 5050$.

2. The area of a strip between the circle of radius r and the circle of radius $r+1$ is $\pi(r+1)^2 - \pi r^2 = (2r+1)\pi$, which we can rewrite as $(r+1)\pi + r\pi$. Then

$$\begin{aligned} x &= \frac{100^2\pi - 99^2\pi + 98^2\pi - 97^2\pi + \cdots + 2^2\pi - 1^2\pi}{10000\pi} \\ &= \frac{100\pi + 99\pi + 98\pi + 97\pi + \cdots + 2\pi + \pi}{10000\pi} \\ &= \frac{5050\pi}{10000\pi} = 0.505. \end{aligned}$$

Basic summations

1. Arithmetic series:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

   In general, given an arithmetic progression that starts at a, ends at z, and has n terms, its sum is $n \cdot \frac{a+z}{2}$.

2. Geometric series: for $r \neq 1$,

$$\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

   As a special case, $\sum_{k=0}^{n-1} 2^k = 2^n - 1$.

Exchanging double sums

Consider the sum $S = \sum_{k=0}^{n-1} k2^k$. We will evaluate this sum as follows:

$$\sum_{k=0}^{n-1} k2^k = \sum_{k=0}^{n-1} \sum_{\ell=0}^{k-1} 2^k = \sum_{\ell=0}^{n-1} \sum_{k=\ell+1}^{n-1} 2^k.$$

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Having reordered the two sums, we first evaluate the inner one:

$$\sum_{k=\ell+1}^{n-1} 2^k = \sum_{k=0}^{n-1} 2^k - \sum_{k=0}^{\ell} 2^k = (2^n - 1) - (2^{\ell+1} - 1) = 2^n - 2^{\ell+1}.$$

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Now the outer sum is also easy:

$$\sum_{\ell=0}^{n-1}(2^n - 2^{\ell+1}) = n2^n - 2\sum_{\ell=0}^{n-1} 2^{\ell} = (n-2)2^n + 2.$$

Practice with exchanging double sums

1. We define the n-th harmonic number H_n by

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

   Express the sum $\sum_{k=1}^{n} H_k$ in terms of H_n.

2. Things will get trickier when you do this to $\sum_{k=1}^{n} k^2$.

   (Recall that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.)

Exchanging double sums

Solutions

1. $\sum_{k=1}^{n} H_k = (n+1)H_n - n.$
2. Let $\square_n = \sum_{k=0}^{n} k^2$. When we expand this out into two sums, switch the sums, and simplify, we get back

$$\square_n = \sum_{\ell=1}^{n} \left(\binom{n+1}{2} - \binom{\ell}{2} \right) = \frac{2n^3 + 3n^2 + n}{4} - \frac{1}{2} \sum_{\ell=1}^{n} \ell^2.$$

We don't yet know how to simplify the last sum, but since it is just $\frac{1}{2}\square_n$, we can solve the equation for $\square_n$ to get

$$\square_n = \frac{n(n+1)(2n+1)}{6}.$$

Review of binomial coefficients

Recall that $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(\cdots)(n-r+1)}{r!}$. These show up in Pascal's triangle:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array} \qquad \begin{array}{c} \binom{0}{0} \\ \binom{1}{0}\binom{1}{1} \\ \binom{2}{0}\binom{2}{1}\binom{2}{2} \\ \binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3} \\ \binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4} \end{array}$$

The key point today is the identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, or $\binom{n}{r} = \binom{n+1}{r+1} - \binom{n}{r+1}$. This lets us make many sums telescope: e.g.,

$$\sum_{k=0}^{n-1} k = \sum_{k=0}^{n-1} \binom{k}{1} = \sum_{k=0}^{n-1} \left(\binom{k+1}{2} - \binom{k}{2} \right) = \binom{n}{2}.$$

Sums of binomial coefficients

In general, we have:

$$\sum_{k=0}^{n-1} \binom{k}{r} = \sum_{k=0}^{n-1} \left(\binom{k+1}{r+1} - \binom{k}{r+1} \right) = \binom{n}{r+1} - \cancel{\binom{0}{r+1}}.$$

We can use this to evaluate sums of arbitrary polynomials.

1. Write your degree-d polynomial in k as a sum of $\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{d}$. For example, $k^2 = 2\binom{k}{2} + \binom{k}{1}$.
2. Now we can evaluate the sum:

$$\sum_{k=1}^{n} k^2 = 2\sum_{k=1}^{n} \binom{k}{2} + \sum_{k=1}^{n} \binom{k}{1} = 2\binom{n+1}{3} + \binom{n+1}{2}.$$

3. Then simplify: $2\binom{n+1}{3} + \binom{n+1}{2} = \frac{1}{6}(2n^3 + 3n^2 + n)$.

Practice with binomial coefficients

1. Use this technique to find a formula for $\sum_{k=1}^{n} k^3$.
2. Here is a systematic method of writing a polynomial $P(x)$ as $a_0\binom{x}{0} + a_1\binom{x}{1} + \cdots + a_d\binom{x}{d}$:
   2.1 If the polynomial is $P(x)$ and has degree d, write $P(0), P(1), \ldots, P(d)$ in a row.
   2.2 On the next line, write down the differences: between and below two numbers a and b write $b - a$.
   2.3 Repeat step 2 to the row of differences, and keep going until a row with only one number is left.
   2.4 The coefficients of $\binom{x}{0}, \ldots, \binom{x}{d}$ are the first entries in each row.

   Use this method to find $\sum_{k=1}^{n} k^4$.
3. Why does the method above work?

Practice with binomial coefficients

Solutions

1. Writing k^3 as $6\binom{k}{3} + 6\binom{k}{2} + \binom{k}{1}$, we get

$$\sum_{k=1}^{n} k^3 = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} = \frac{n^2(n+1)^2}{4}.$$

2. The rows of differences we get are:

$$\begin{array}{ccccccccc} 0 & & 1 & & 16 & & 81 & & 256 \\ & 1 & & 15 & & 65 & & 175 & \\ & & 14 & & 50 & & 110 & & \\ & & & 36 & & 60 & & & \\ & & & & 24 & & & & \end{array}$$

So $k^4 = 24\binom{k}{4} + 36\binom{k}{3} + 14\binom{k}{2} + \binom{k}{1}$, and

$$\sum_{k=1}^{n} k^4 = 24\binom{n+1}{5} + 36\binom{n+1}{4} + 14\binom{n+1}{3} + \binom{n+1}{2}.$$

The difference operator

Given any function f, Δf is another function defined by $\Delta f(x) = f(x+1) - f(x)$.

(A thing that turns functions into other functions is called an *operator*. Δ is called the *difference operator*.)

The basic identity we rely on is this:

$$\sum_{k=a}^{b-1} \Delta f(k) = f(b) - f(a).$$

To simplify any finite sum whatsoever, all we need to do is find a function f such that Δf is the function we're summing.

This is what we did in the previous section: we discovered that $\Delta_x \binom{x}{r} = \binom{x}{r-1}$, which let us solve any sum with binomial coefficients in it.

Difference operator problems

1. Compute $\Delta(2^x)$, $\Delta(x2^x)$, and $\Delta(x^2 2^x)$.
2. Use #1, and the basic rule that $\Delta(f+g) = \Delta f + \Delta g$, to find
$$\sum_{k=0}^{n-1} 2^k, \quad \sum_{k=0}^{n-1} k2^k, \text{ and } \sum_{k=0}^{n-1} k^2 2^k.$$
3. Let F_n denote the n-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Find a formula for
$$\sum_{k=0}^{n-1} F_k.$$
4. Prove that for any function f,
$$\sum_{k=0}^{n-1} k\Delta f(k) = nf(n) - \sum_{k=1}^{n} f(k).$$

Difference operator problems

Solutions

1. $\Delta(2^x) = 2^x$, $\Delta(x2^x) = (x+2)2^x$, and $\Delta(x^2 2^x) = (x^2+4x+2)2^x$.

2. From #1, we can deduce that $\Delta((x^2-4x+6)2^x) = x^2 2^x$. Therefore
$$\sum_{k=0}^{n-1} k^2 2^k = (k^2-4k+6)2^k - 6.$$
The other two sums are similar, but easier.

3. We have $\Delta F_n = F_{n+1} - F_n = F_{n-1}$, so $F_n = \Delta F_{n+1}$, and therefore
$$\sum_{k=0}^{n-1} F_n = F_{n+1} - F_1 = F_{n+1} - 1.$$

4. Sum both sides of $\Delta(kf(k)) = f(k+1) - k\Delta f(k)$.