Writing proofs

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ARML Practice 3/2/2014

Warm-up / Review

1. (From my research) If $x_n = 2 - \frac{1}{x_{n-1}}$ for $n \geq 2$, solve for x_n in terms of x_1. (For a more concrete problem, set $x_1 = 2$.)
2. (From this year's HMMT) Find the sum of the real roots of $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 0$.

Warm-up

Solutions

1. Let $p_n = \prod_{k=1}^{n} x_k$. Multiplying through by p_{n-1}, we get

$$p_{n-1}x_n = 2p_{n-1} - \frac{p_{n-1}}{x_{n-1}} \quad \Leftrightarrow \quad p_n = 2p_{n-1} - p_{n-2}.$$

Solving this two-term recurrence, we get $p_n = a + bn$ for constants a, b. We also know p_n satisfies $p_1 = x_1$ and $p_2 = 2x_1 - 1$, so $p_n = (x_1 - 1)n + 1$.

Finally, we can solve for $x_n = \frac{p_n}{p_{n-1}}$, getting

$$x_n = \frac{1 + (x_1 - 1)n}{1 + (x_1 - 1)(n - 1)}.$$

When $x_1 = 2$, this gives the relatively nice $x_n = 1 + \frac{1}{n}$.

Warm-up
Solutions

2. First note that we can't answer $-\left(\frac{-10}{5}\right) = 2$ because this gives the sum of all roots, some of which may not be real.

   This is the official solution:

   Rearrange the equation $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 0$ to get $x^5 + (1-x)^5 - 12 = 0$ by spotting that four out of five coefficients look like the binomial coefficients $\binom{5}{k}$.

   The graph of $f(x) = x^5 + (1-x)^5 - 12$ is symmetric about $x = \frac{1}{2}$. It can have up to 4 sign changes total; however, these are placed symmetrically about $\frac{1}{2}$, so it can have 0, 1, or 2 on either side. But $f(\frac{1}{2}) = -12 < 0$ while $f(x) \sim 5x^4 > 0$ for large x. Therefore there is one sign change on either side of $\frac{1}{2}$.

   So there are 2 real roots, which sum to 1 by symmetry.

Warm-up
Solutions

2. A useful trick for exposing funny business in disguise is to make the substitution $x = y + \frac{1}{2}$ (in this case; in general, add $\frac{1}{n}$ of the sum of the roots) to make the equation *depressed* (sum of roots is zero).

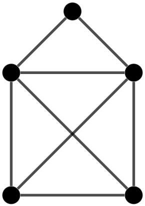
   Simplifying, we get
$$5y^4 + \frac{5}{2}y^2 - \frac{191}{16} = 0.$$

   If $z = y^2$, z has a positive and a negative root, which we will call z^+ and z^-. So the solutions to y are two real roots $(\pm\sqrt{z^+})$ and two imaginary roots $(\pm\sqrt{z^-})$.

   Finally, the two real roots for x are $\frac{1}{2} \pm \sqrt{z^+}$, whose sum is 1.

Background

A *graph* consists of a set of *vertices* and a set of *edges* connecting some pairs of vertices. If vertices v, w are connected by an edge, we say they are *adjacent*. The *degree* of a vertex is the number of vertices adjacent to it.



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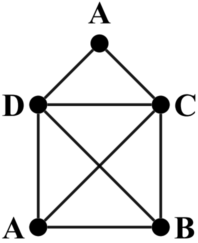


A *coloring* of a graph assigns a color to each vertex. A coloring is *good* if no two adjacent vertices are the same color. The *chromatic number* of a graph is the minimum number of colors needed in any good coloring of that graph.

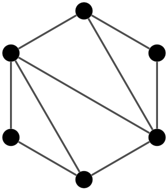
Proof warning

PROOF WARNING!

The solutions to problems #1 through #3 on the next few slides have holes in them. Don't just read the solutions and believe them. (Although the ideas are mostly correct.)

Problem #1

1. Given a convex n-gon (for $n \geq 3$), we can add $n - 3$ non-intersecting diagonals to partition the polygon into triangles. We can view this as a graph in which the vertices are the corners of the polygon, and the edges are the sides and the added diagonals.

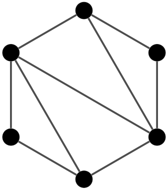


Prove that all such graphs have chromatic number 3.

Problem #1

Solution (make sure you are aware of the PROOF WARNING)

Proof. We induct on n. For $n = 3$, we just have a triangle, and can give each vertex its own color.



For $n > 3$, pick a vertex with no diagonals out of it. If that vertex is removed, what remains is a triangulation of an $(n-1)$-gon, which we can color using only 3 colors.

Now add the removed vertex back in. We can always assign it a color, because it only has 2 neighbors, which cannot eliminate all 3 options for the color.

Problem #2

2a. Let P be a graph with one vertex v_n for each positive integer n. If $a < b$, then an edge connects vertices v_a and v_b if and only if $\frac{b}{a}$ is a prime number. What is the chromatic number of P? Prove your answer.

2b. Let T be a graph with one vertex v_n for each positive integer n. An edge connects v_a and v_b if $|a - b|$ is a power of two. What is the chromatic number of T? Prove your answer.

Problem #2

Solutions (make sure you are aware of the PROOF WARNING)

2a. To color the powers of a prime, we alternate colors: $1, 2, 4, 8, \ldots$ are colored red, blue, red, blue, ... Otherwise, to color n, we look at its prime factors: for example, 5 is blue, $15 = 3 \cdot 5$ is red, and $30 = 2 \cdot 3 \cdot 5$ is blue. Then if $\frac{a}{b}$ is prime, v_a and v_b are opposite colors, so the chromatic number is 2.

2b. Color each vertex v_n by the remainder $n \bmod 3$. Whenever v_a and v_b have the same color, we have $a \equiv b \pmod 3$, so $a - b$ is divisible by 3, and therefore $|a - b|$ is not a power of 2, so there is no edge between v_a and v_b. This uses 3 colors, so the chromatic number of T is 3.

Problem #3

A graph is *finite* if it has a finite number of vertices.

3a. Let G be a finite graph in which every vertex has degree k. Prove that the chromatic number of G is at most $k+1$.

3b. In terms of n, what is the minimum number of edges a finite graph with chromatic number n could have? Prove your answer.

Problem #3

Solutions (make sure you are aware of the PROOF WARNING)

3a. Go through the vertices in arbitrary order. For each vertex, give it the first color not already given to any of its neighbors. This eliminates at most k choices, so if there are $k+1$ colors we will always be able to color a vertex.

3b. By part (a), to have chromatic number at least n, we need all the vertices to have degree $n-1$. This requires at least n vertices; if we have n vertices with $n-1$ edges out of each, there are $\frac{n(n-1)}{2}$ edges total (since each edge counts for two vertices). Therefore a graph with chromatic number n needs at least $\frac{n(n-1)}{2}$ edges.

Since the graph with n vertices which are all connected has $\frac{n(n-1)}{2}$ edges and chromatic number n, this is enough.

Problem #4

A k-clique of a graph is a set of k vertices such that all pairs of vertices in the clique are adjacent.

4a. Find a graph with chromatic number 3 that does not contain any 3-cliques. (This is not a proof problem.)

4b. Prove that, for all $n > 3$, there exists a graph with chromatic number n that does not contain any n-cliques.

Problem #4

A k-clique of a graph is a set of k vertices such that all pairs of vertices in the clique are adjacent.

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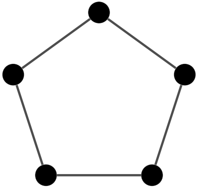

4b. Prove that, for all $n > 3$, there exists a graph with chromatic number n that does not contain any n-cliques.

Problem #4
Solutions

4b. We induct on n. For $n = 3$, the answer to part (a) is our base case.

Given a graph G with chromatic number $n-1$ and no $(n-1)$-cliques, construct a new graph H with an additional vertex v adjacent to every vertex of G. Then:

Problem #4

Solutions

4b. We induct on n. For $n = 3$, the answer to part (a) is our base case.

Given a graph G with chromatic number $n - 1$ and no $(n - 1)$-cliques, construct a new graph H with an additional vertex v adjacent to every vertex of G. Then:

- H has chromatic number at least n because G requires $n - 1$ colors, and v is forced to have an n-th color since it can't use any of those.
- H has chromatic number at most n because we can color G by $n - 1$ colors and then give v the last color.
- H has no n-cliques: if $\{w_1, \ldots, w_n\}$ is an n-clique of H, remove either v, if v is one of $w_1, \ldots, w_n$, or else an arbitrary vertex, to get an $(n - 1)$-clique of G, which cannot exist.

Problem #5

The *size* of a finite graph is the number of vertices in the graph.

5a. Show that, for any $n > 2$, and any positive integer N, there are finite graphs with size at least N and with chromatic number n such that removing any vertex (and all its incident edges) from the graph decreases its chromatic number.

5b. Show that, for any positive integers n and r, there exists a positive integer N such that for any finite graph having size at least N and chromatic number equal to n, it is possible to remove r vertices (and all their incident edges) in such a way that the remaining vertices form a graph with chromatic number at least $n - 1$.

Meta-exercise: negate the claims in each of the problems.

Problem #5

Solution to the meta-exercise

$\neg$5a. There is some $n > 2$ and some positive integer N such that all finite graphs with size at least N and chromatic number n contain a vertex which can be deleted without decreasing the chromatic number.

$\neg$5b. There are some positive integers n and r such that for all N, there are finite graphs with size at least N and chromatic number n such that removing any r vertices leaves a graph with chromatic number at most $n - 2$.

Problem #5
Solution to (a)

Show that, for any $n > 2$, and any positive integer N, there are finite graphs with size at least N and with chromatic number n such that removing any vertex (and all its incident edges) from the graph decreases its chromatic number.

Take a cycle on $2N + 1$ vertices, a clique on $n - 3$ vertices, and join every vertex of the clique to every vertex of the cycle. As in 4(b), this has chromatic number n: the cycle has chromatic number 3, and every vertex of the clique needs its own color.

If a vertex of the clique is deleted, we can color the rest of the clique with only $n - 4$ colors. If a vertex of the cycle is deleted, we can color the rest of the cycle with only 2 colors. Thus, the chromatic number always decreases.

Problem #5

Solution to (b)

Show that, for any positive integers n and r, there exists a positive integer N such that for any finite graph having size at least N and chromatic number equal to n, it is possible to remove r vertices (and all their incident edges) in such a way that the remaining vertices form a graph with chromatic number at least $n-1$.

Choose $N = nr$. If G is a graph with at least N vertices and chromatic number equal to n, pick some good coloring of G with n colors. At least one of the colors is used for at least r vertices, so remove some r vertices that all have the same color.

If the resulting graph H had chromatic number $n-2$ or less, then we could take a good coloring of H with $n-2$ colors, and extend it to a good coloring of G by giving the deleted vertices a new color. This is a contradiction: G: cannot be colored with $n-1$ colors.

So H has chromatic number at least $n-1$, and we are done.

Problem #8

Two colorings are *distinct* if there is no way to the relabel the colors to transform one into the other. (They're not just the same coloring with different names for the colors.) Equivalently, they are distinct if and only if there is some pair of vertices which are the same color in one coloring but different colors in the other.

For what pairs (n, k) of positive integers does there exist a finite graph with chromatic number n which has exactly k distinct good colorings using n colors?