Euclid and Air Travel

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ARML Practice 9/15/2013

Airlines in Hilberta

Air travel in the Canadian province of Hilberta is a thriving business. The following regulations are enforced:

[I1] For any two airports, there is exactly one airline servicing both.

[I2] Each airline services at least two airports.
(Otherwise, what's the point?)

[I3] There are three airports with no airline serving all three.

Warm-up exercises:

1. If there are four airports in Hilberta, what are all the ways they can be connected by airlines?
2. Each airline provides direct flights between all airports it services. Show that if you decide to boycott one airline, then any trip is still possible and requires at most one stop.

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Airlines in Hilberta

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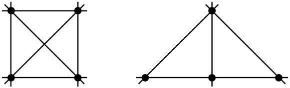

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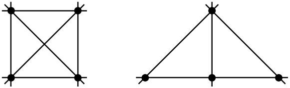


2. Call the endpoints of your trips P and Q, and the airline you're boycotting ℓ. There is an airline ℓ' servicing both P and Q, by [I1].

   - If ℓ' is not the same as ℓ, then we can take a direct flight.
   - Otherwise, by [I3] there is an airport R not on ℓ. By [I1], there are direct flights from P to R and from R to Q.

Redundant regulations

[I1] For any two airports, there is exactly one airline servicing both.
[I2] Each airline services at least two airports.
[I3] There are three airports with no airline serving all three.

We say that a regulation is *redundant* if there is no way to violate it while following all the others.

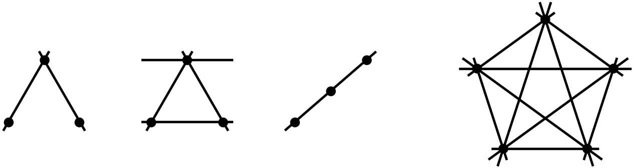
1. Demonstrate that none of the regulations [I1]–[I3] are redundant.
2. The government of Hilberta plans to add the following regulation. Is it redundant?

   [P] If airline ℓ does not service airport P, at most one other airline can service P and share no airports with ℓ.

3. The government of Hilberta plans to add the following regulation. Is it redundant?

   [Q] If there are n airports, there must be at least $\sqrt{n}$ airlines.

1,2. The following pictures demonstrate that none of [I1], [I2], [I3], or [P] are redundant:



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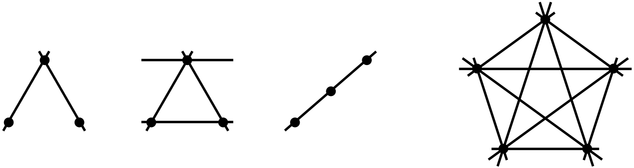

3. Suppose the airline servicing the most airports serves k.

- Let ℓ such an airline, and P an airport not serviced by ℓ. There are k different airlines connecting P to ℓ's k airports, for a total of $k+1$.

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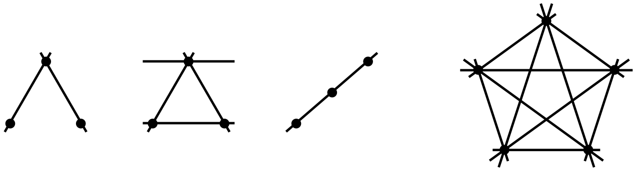

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   - Let P be an airport. It is connected to $n-1$ other airports, and at least $\frac{n-1}{k}$ airlines are necessary to do so.

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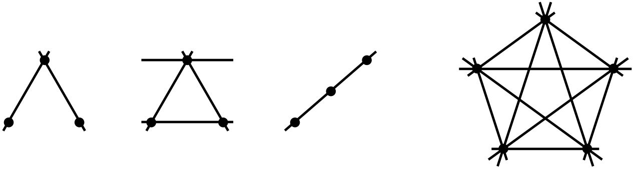

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There are $\max\left\{k+1, \frac{n-1}{k}\right\} \geq \sqrt{n}$ airlines: [Q] is redundant.

Geometry

- If we replace "airports" by points and "airlines" by lines, what we get is called an *incidence geometry*:

  [I1] Through any two points, there is exactly one line.
  [I2] Each line contains at least two points.
  [I3] There are three points that do not all lie on one line.

- [I1]–[I3] and [P] (the parallel axiom) are the beginning of Hilbert's axioms for Euclidean geometry.

  [P] Given a line ℓ and a point P not on ℓ, there is at most one line through P parallel to ℓ.

- One could add approximately 12 other axioms to get a system for which Euclidean plane geometry is the *only* model.

The projective plane

A *projective plane* is characterized by the following four axioms:

[P1] Through any two points, there is exactly one line.

[P2] Any two lines meet in exactly one point.

[P3] Every line contains at least three points.

[P4] There are three points that do not all lie on one line.

Projective plane warm-up:

1. Find a projective plane with seven points.
2. Verify that the axioms [P1]–[P4] are *independent* (i.e. none of them are redundant given the other three).

Facts about projective planes

1. Suppose we know that some line ℓ in the projective plane contains n points.
    1.1 Prove that every point not on ℓ must lie on n lines.
    1.2 Prove that every line, not just ℓ, must contain n points.
    1.3 Prove that there are $n^2 - n + 1$ points and $n^2 - n + 1$ lines.
2. Let $\mathcal{P}$ be a projective plane. Define a new geometry $\mathcal{Q}$ as follows:
    - The points of $\mathcal{Q}$ are the lines of $\mathcal{P}$.
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    - A line ℓ of $\mathcal{Q}$ contains a point P whenever the line of $\mathcal{P}$ corresponding to P contains the point of $\mathcal{P}$ corresponding to ℓ.

    Check that $\mathcal{Q}$ is also a projective plane.

1.1. Let P be a point not on ℓ. For every point Q on ℓ, there is exactly one line through P and Q; for every line through P, there is exactly one point where ℓ meets that line. So there are exactly as many lines through P as there are points on ℓ.

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1.2. Let ℓ_1 and ℓ_2 be two lines. We can find a point P neither on ℓ_1 nor on ℓ_2.

By using 1.1, we can conclude that the number of lines through P is the same as the number of points on ℓ_1, and also that it's the same as the number of points on ℓ_2. Therefore ℓ_1 and ℓ_2 have the same number of points. So all lines contain n points.

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1.3. Given a point P, there are n lines through P; each has $n-1$ other points on it, and each point is on exactly one of these lines. So there are $1+n(n-1)=n^2-n+1$ total points.

If you switch points and lines in this proof, you get a proof that there are n^2-n+1 lines.

An infinite projective plane

Not all projective planes are finite. One infinite projective plane can be constructed as follows:

- Start with the surface of the sphere, called S^2.
- A point of the projective plane $\mathcal{P}$ is a pair of antipodal points of S^2 (e.g. the north and south poles, together, are a single point).
- A line of of $\mathcal{P}$ is a "great circle" of S^2 (the intersection of S^2 with a plane through the center of the sphere).

Check that $\mathcal{P}$ satisfies axioms [P1]–[P4].

The De Bruijn–Erdős theorem

Theorem (De Bruijn, Erdős, 1948)

In any incidence geometry, there are at least as many lines as points.

Proof.

If you're feeling ambitious, you can find de Bruijn and Erdős's original proof at:

`http://www.renyi.hu/~p_erdos/1948-01.pdf`

Otherwise, you can find my own write-up of the same proof at:

`http://www.math.cmu.edu/~mlavrov/other/db-e.pdf`

It doesn't lend itself well to slides, but if there's interest (and time) I can explain it at the board. □