Combinatorics

Introduction to graph theory

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ARML Practice 11/03/2013

Warm-up

1. How many (unordered) poker hands contain 3 or more aces?
2. 10-digit ISBN codes end in a "check digit": a number 0–9 or the letter X, calculated from the other 9 digits. If a typo is made in a single digit of the code, this can be detected by checking this calculation.

   Prove that a check digit in the range 1–9 couldn't achieve this goal.
3. There are $\binom{16}{8}$ ways to go from $(0, 0)$ to $(8, 8)$ in steps of $(0, +1)$ or $(+1, 0)$. (Do you remember why?)

   How many of these paths avoid $(2, 2)$, $(4, 4)$, and $(6, 6)$? You don't have to simplify the expression you get.

Warm-up

Solutions

1. There are $\binom{4}{3} \cdot \binom{48}{2}$ hands with 3 aces, and $\binom{4}{4} \cdot \binom{48}{1}$ hands with 4 aces, for a total of 4560.

2. There are 10^9 possible ISBN codes (9 digits, which uniquely determine the check digit). If the last digit were in the range 1–9, there would be $9 \cdot 10^8$ possibilities for the last 9 digits. By pigeonhole, two ISBN codes would agree on the last 9 digits, so they'd only differ in the first digit, and an error in that digit couldn't be detected.

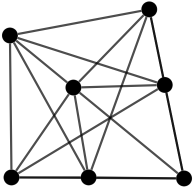
3. We use the inclusion-exclusion principle. (Note that there are, e.g., $\binom{4}{2} \cdot \binom{12}{6}$ paths from $(0,0)$ to $(8,8)$ through $(2,2)$.) This gives us

$$\binom{16}{8} - 2\binom{4}{2}\binom{12}{6} - \binom{8}{4}^2 + 3\binom{4}{2}^2\binom{8}{4} - \binom{4}{2}^4 = 3146.$$

Graphs

Definitions

A *graph* is a set of *vertices*, some of which are joined by *edges*.



A *path* in a graph is a sequence of vertices with an edge from each vertex to the next. A *cycle* is a path whose last vertex is the same as the first.

If there is a path joining any two vertices, the graph is *connected*.

The *degree* of a vertex is the number of edges that connect to it.

1. Prove that if a graph has m edges, the sum of the degrees of all vertices is $2m$.
2. Prove that all graphs have an even number of vertices with odd degree.
3. Prove that all graphs have two vertices with the same degree. (Hint: pigeonhole.)
4. A *tree* is a connected graph with no simple cycles. Prove that a tree has exactly $n - 1$ edges.

1. Adding the degrees counts each edge once from each end.
2. If an odd number of vertices had odd degree, the sum of degrees would be odd; but it's $2m$ by problem 1.
3. If a graph has n vertices, their degrees are in $\{0, 1, \ldots, n-1\}$, and both 0 and $n-1$ can't occur: 0 means "adjacent to nothing" and $n-1$ means "adjacent to everything". So there are $n-1$ possibilities, and by pigeonhole one occurs twice.
4. Suppose we add the edges one by one. Initially we have n vertices and no edges, forming *n connected components*.

   Each edge we add can't join two already-connected vertices: this would create a cycle. So it joins two vertices in different components, bringing the component count down by 1.

   We become connected after $n-1$ edges, at which point any edge creates a cycle.

Hamiltonian and Eulerian paths

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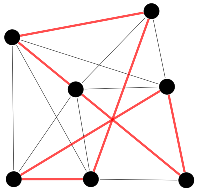


Exercise. There is an easy characterization of when graphs have Eulerian paths or cycles that I won't tell you yet.

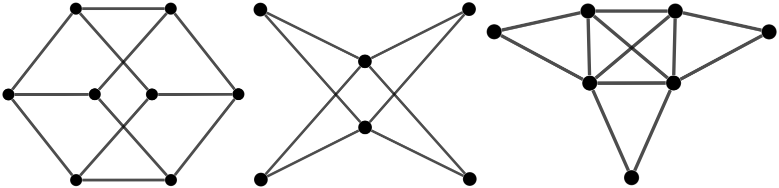
Dirac's Theorem. If a graph G on n vertices is connected and every vertex has degree at least $\frac{1}{2}n$, then it has a Hamiltonian cycle.

I will now prove this!

Hamiltonian and Eulerian paths

Exercises

1. Find Eulerian and Hamiltonian paths and cycles in these graphs when they exist, or prove that they don't.



2. Prove that if every vertex of a graph has degree 2 or more, then it contains a simple cycle.
3. Prove that if every vertex of a graph has degree 10 or more, then it has a simple cycle with at least 11 vertices.
4. When does a graph have an Eulerian cycle?

Hamiltonian and Eulerian paths

Solutions to exercises

2. Start from any vertex and begin a path. At each step, we use an edge different from the one we entered by, which is possible because all degrees are at least 2. Eventually, we'll have to hit a vertex we've already seen, at which point we have a cycle.
3. We do the same thing as in 2; but this time, because we have 10 choices at each edge, we can avoid going back to any of the last 9 vertices, and still have at least one choice remaining. Now, when we hit a vertex we've already seen, the cycle created will have at least 11 vertices.
4. Each vertex must have even degree (every time we enter and leave it, we use up 2 edges). If each vertex does have even degree, then finding the Eulerian cycle is easy.

   (For an Eulerian path, we can have two vertices with odd degree, which must be the endpoints of the path.)