A Nontrivial Lower Bound on the Shannon Capacities of the Complements of Odd Cycles

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***Abstract*— This paper contains a construction for independent sets in the powers of the complements of odd cycles. In particular, we show that $\alpha\left(\overline{C}_{2n+3}^{2^n}\right) \geq 2^{2^n}+1$. It follows that for $n \geq 0$ we have $\Theta(\overline{C}_{2n+3}) > 2$, where $\Theta(G)$ denotes the Shannon capacity of graph G.**

***Index Terms*—graph powers, independence number, odd cycles, Shannon capacity.**

I. Introduction

Let $G = (V, E)$ be a graph. For $d \geq 1$ let G^d be the graph with vertex set V^d and an edge between distinct vertices $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ if and only if $x_i = y_i$ or $\{x_i, y_i\} \in E$ for $i = 1, \ldots, d$. It is obvious that $\alpha(G^d) \geq (\alpha(G))^d$, where $\alpha(H)$ is the size of the largest independent set in graph H. The Shannon capacity of G is

$$\Theta(G) = \sup_d \left(\alpha(G^d)\right)^{1/d}.$$

This graph invariant was introduced by Shannon in 1956 and gives a measure of the optimal zero-error performance of an associated memoryless communication channel [11] (for a survey of zero-error information theory see [8]).

Motivated in part by the problem of determining the Shannon capacities of graphs, Berge introduced the notion of a perfect graph [3]. A graph G is perfect if $\omega(H) = \chi(H)$ for every induced subgraph H of G, where $\omega(H)$ is the size of the largest clique in H and $\chi(H)$ is the chromatic number of H. Perfect graphs are known to satisfy $\Theta(G) = \alpha(G)$. Berge conjectured that a graph is perfect if and only if it does not contain an odd cycle of length five or more, or the complement of such a graph, as an induced subgraph. This long standing conjecture, known as the strong perfect graph conjecture, was recently proved in a sequence of results by Chudnovsky, Robertson, Seymour and Thomas [5] (for an overview of the proof, see [6]). It follows from this result that the odd cycles of length five or more and their complements are the minimal graphs for which the determination of the Shannon capacity is nontrivial.

The best known upper bounds on the Shannon capacities of these graphs are given by the Lovász theta function [9]:

$$\Theta(C_{2n+1}) \leq \vartheta(C_{2n+1}) = \frac{(2n+1)\cos(\pi/(2n+1))}{1+\cos(\pi/(2n+1))} = n + \frac{1}{2} - O(1/n),$$

$$\Theta(\overline{C}_{2n+1}) \leq \vartheta(\overline{C}_{2n+1}) = \frac{1+\cos(\pi/(2n+1))}{\cos(\pi/(2n+1))} = 2 + O(1/n^2).$$

Since $\alpha(C_5^2) = 5$, this upper bound suffices to establish the Shannon capacity of C_5, which is self complementary: $\Theta(C_5) = \Theta(\overline{C}_5) = \sqrt{5}$. However, the Shannon capacities of odd cycles and the complements of odd cycles on seven or more vertices remain unknown. While there are nontrivial lower bounds on the Shannon capacities of odd cycles (see [2], [4], [7], [12]), the best known lower bound on the Shannon capacities of the complements of the odd cycles on seven or more vertices is the trivial

$$\Theta\left(\overline{C}_{2n+1}\right) \geq \alpha\left(\overline{C}_{2n+1}\right) = 2.$$

The question whether this lower bound is actually the truth is mentioned, e.g., in [1], [10]. Here we show, by means of a construction, that it is not.

Theorem 1. *For $n \geq 0$ we have*

$$\alpha\left(\overline{C}_{2n+3}^{2^n}\right) \geq 2^{2^n}+1.$$

Corollary 2. *For $n \geq 0$ we have*

$$\Theta\left(\overline{C}_{2n+3}\right) \geq \left(2^{2^n}+1\right)^{1/2^n} = 2 + \Omega\left(\frac{1}{2^{2^n+n}}\right).$$

II. Construction

For ease of notation, we identify the vertices of $\overline{C}_{2n+3}^d$ and the elements of the abelian group $\mathbb{Z}_{2n+3}^d$ in the natural way. Adjacency in $\overline{C}_{2n+3}$ can be expressed in terms of the group operation: distinct $u, v \in \mathbb{Z}_{2n+3}$ are adjacent in $\overline{C}_{2n+3}$ if and only if $u - v \notin \{-1, 1\}$. Thus, a vertex set $X \subseteq \mathbb{Z}_{2n+3}^d$ is an independent set if for distinct $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in X$ there exists i such that $x_i - y_i \in \{-1, 1\}$. For example, $\{0, 1\}^d$ is an independent set in $\overline{C}_{2n+3}^d$ (this independent set is called the *standard* independent set below).

We illustrate the spirit of the construction on the $n = 1$ case of the theorem (i.e., by showing $\alpha(\overline{C}_5^2) \geq 5$). We start with the

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standard independent set $\{0,1\}^2$. We would like to modify this independent set in such a way that we can add one more vertex. We do this by 'moving' the vertices in $\{0,1\}^2$ as follows

$$\begin{array}{ll}(0,1)\to(4,1) & (1,1)\to(1,2)\\ (0,0)\to(0,4) & (1,0)\to(2,0).\end{array}$$

Note that the new set, $\{(0,4),(4,1),(1,2),(2,0)\}$, is an independent set, and we can add the vertex $(3,3)$ to this set in order to achieve an independent set of the desired size.

Now, let $n \geq 0$ and $d = 2^n$. We begin with the standard independent set $\{0,1\}^d$ in $\mathbb{Z}^d_{2n+3}$. As in the previous example, we 'move' these vertices in such a way that the new set is still an independent set and one additional vertex can be added. In the following claim we construct a function $f : \{0,1\}^d \to \{0,1,\dots,n\}^d$ that will help us control the movement. For $1 \leq i \leq d$ and $x \in \{0,1\}^d$ we let $f_i(x)$ denote the i^{th} coordinate of $f(x)$.

Claim 3. *Let $n \geq 0$ and $d = 2^n$. There exists a function $f : \{0,1\}^d \to \{0,1,\dots,n\}^d$ such that*

(a) *for all $x \in \{0,1\}^d$ there exists $1 \leq i \leq d$ such that $f_i(x) = n$.*

(b) *for all distinct $x, y \in \{0,1\}^d$ there exists $1 \leq i \leq d$ such that either*

$$\begin{array}{l} x_i \neq y_i \text{ and } f_i(x) = f_i(y) = 0 \quad \text{or} \\ x_i = y_i \text{ and } |f_i(x) - f_i(y)| = 1. \end{array} \tag{1}$$

Proof: We go by induction on n. For $n = 0$ we simply let f be identically equal to zero. Let $n \geq 1$ and suppose $g : \{0,1\}^{2^{n-1}} \to \{0,1,\dots,n-1\}^{2^{n-1}}$ satisfies the $n-1$ case of the claim. We use g to construct $f : \{0,1\}^{2^n} \to \{0,1,\dots,n\}^{2^n}$ by considering the coordinates of $\{0,1\}^{2^n}$ in pairs. For $x \in \{0,1\}^{2^n}$ and $1 \leq j \leq 2^{n-1}$ define

$$z_j(x) = \begin{cases} 0 & \text{if } x_{2j-1} \neq x_{2j} \\ 1 & \text{if } x_{2j-1} = x_{2j}. \end{cases}$$

This produces $z(x)$ in $\{0,1\}^{2^{n-1}}$. Now, for $j = 1,\dots,2^{n-1}$ define

$$f_{2j-1}(x) = \begin{cases} g_j(z(x)) + 1 & \text{if } z_j(x) = 0 \\ 0 & \text{if } z_j(x) = 1, \end{cases}$$

$$f_{2j}(x) = \begin{cases} 0 & \text{if } z_j(x) = 0 \\ g_j(z(x)) + 1 & \text{if } z_j(x) = 1. \end{cases}$$

It remains to check that f has the desired properties. Consider $x \in \{0,1\}^d$. Since there exists a coordinate j such that $g_j(z(x)) = n-1$, there exists a coordinate i, either $2j-1$ or $2j$, such that $f_i(x) = n$. Thus, f satisfies (a).

Consider distinct $x, y \in \{0,1\}^d$. Assume first that $z(x) = z(y)$. Then there exists a pair of coordinates $2j-1, 2j$ such that x and y differ in both of these coordinates. In one of these coordinates, say $k \in \{2j-1, 2j\}$, we have $f_k(x) = f_k(y) = 0$. Thus the first part of (1) holds.

Next, assume that $z(x) \neq z(y)$. Applying property (b) of g to $z(x), z(y)$, we find a coordinate j for which one of the two parts of (1) holds. Suppose first that $z_j(x) \neq z_j(y)$ and $g_j(z(x)) = g_j(z(y)) = 0$. There exists a unique $k \in \{2j-1, 2j\}$ such that $x_k = y_k$. Either $f_k(x) = g_j(z(x)) + 1 = 1$ and $f_k(y) = 0$ or $f_k(x) = 0$ and $f_k(y) = g_j(z(y)) + 1 = 1$. In either case, the second part of (1) holds.

Suppose, on the other hand, that j is a coordinate such that $z_j(x) = z_j(y)$ and $|g_j(z(x)) - g_j(z(y))| = 1$. Either $x_{2j-1} = y_{2j-1}$ and $x_{2j} = y_{2j}$ or $x_{2j-1} \neq y_{2j-1}$ and $x_{2j} \neq y_{2j}$. In the first case, note that there exists a unique $k \in \{2j-1, 2j\}$ such that $f_k(x) = g_j(z(x)) + 1$ and $f_k(y) = g_j(z(y)) + 1$. It follows that

$$|f_k(x) - f_k(y)| = |g_j(z(x)) - g_j(z(y))| = 1,$$

and the second part of (1) holds. In the second case, note that there exists a unique $k \in \{2j-1, 2j\}$ such that $f_k(x) = f_k(y) = 0$. Now the first part of (1) holds. We have verified that f satisfies (b).

□

We replace each $x \in \{0,1\}^d$ with x^* defined by

$$x^*_i = \begin{cases} x_i + f_i(x) & \text{if } x_i = 1 \\ x_i - f_i(x) & \text{if } x_i = 0, \end{cases}$$

where the operations are taken in $\mathbb{Z}_{2n+3}$. It follows from condition (b) on the function f that the set $\{x^* : x \in \{0,1\}^d\}$ is an independent set. It follows from condition (a) that

$$\{x^* : x \in \{0,1\}^d\} \cup \{(n+2, n+2, \dots, n+2)\}$$

is an independent set.

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