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# Construction of Lyapunov functionals for stochastic difference equations with continuous time

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**Abstract**

One general method of Lyapunov functionals construction which was used earlier both for stochastic differential equations with aftereffect and for stochastic difference equations with discrete time here is applied for stochastic difference equations with continuous time.


*Keywords:* Difference equations; Stochastic; Stability; Lyapunov functionals construction method

## 0. Introduction

Stability investigation of hereditary systems [2–4] is connected often with construction of Lyapunov functionals. One general method of Lyapunov functionals construction was proposed and developed in [5–9,14] both for stochastic differential equations with aftereffect and for stochastic difference equations with discrete time. Here it is shown that after some modification of the basic Lyapunov type theorem this method can be used also for stochastic difference equations with continuous time, which are enough popular with researches [1,10–13].

## 1. Stability theorem

Let $\{\Omega, \mathcal{F}, \mathbf{P}\}$ be a probability space and $\{f_t, t \geq t_0\}$ be a nondecreasing family of sub-$\sigma$-algebras of $\mathcal{F}$, i.e. $f_{t_1} \subset f_{t_2}$ for $t_1 < t_2$. Consider a stochastic difference equation

$$\begin{aligned} x(t+h_0) = {} & a_1(t, x(t), x(t-h_1), x(t-h_2), \dots) \\ & + a_2(t, x(t), x(t-h_1), x(t-h_2), \dots)\xi(t+h_0), \quad t > t_0 - h_0 \end{aligned} \tag{1.1}$$

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with the initial condition

$$x(\theta) = \phi(\theta), \qquad \theta \in \Theta = \left[t_0 - h_0 - \max_{j\geq 1} h_j, t_0\right]. \tag{1.2}$$

Here $x \in \mathbf{R}^n$, $h_0, h_1, \ldots$ are positive constants, the functionals $a_1 \in \mathbf{R}^n$ and $a_2 \in \mathbf{R}^{n\times m}$ satisfy the condition

$$|a_l(t, x_0, x_1, x_2, \ldots)|^2 \leq \sum_{j=0}^{\infty} a_{lj}|x_j|^2, \qquad A = \sum_{l=1}^{2}\sum_{j=0}^{\infty} a_{lj} < \infty, \tag{1.3}$$

$\phi(\theta)$, $\theta \in \Theta$, is a $f_{t_0}$-measurable function, the perturbation $\xi(t) \in \mathbf{R}^m$ is a $f_t$-measurable stationary stochastic process with conditions

$$\mathbf{E}\xi(t) = 0, \qquad \mathbf{E}\xi(t)\xi'(t) = I. \tag{1.4}$$

A solution of problem (1.1), (1.2) is a $f_t$-measurable process $x(t) = x(t; t_0, \phi)$, which is equal to the initial function $\phi(t)$ from (1.2) for $t \leq t_0$ and with probability 1 is defined by Eq. (1.1) for $t > t_0$.

**Definition 1.1.** The trivial solution of Eq. (1.1), (1.2) is called $p$-stable, $p > 0$, if for any $\epsilon > 0$ and $t_0 \geq 0$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\mathbf{E}|x(t; t_0, \phi)|^p < \epsilon$ for all $t \geq t_0$ if $\|\phi\|^p = \sup_{\theta\in\Theta} \mathbf{E}|\phi(\theta)|^p < \delta$.

**Definition 1.2.** The trivial solution of Eq. (1.1), (1.2) is called asymptotically $p$-stable, $p > 0$, if it is $p$-stable and for all initial functions $\phi$

$$\lim_{t\to\infty} \mathbf{E}|x(t; t_0, \phi)|^p = 0. \tag{1.5}$$

**Definition 1.3.** The trivial solution of Eq. (1.1), (1.2) is called asymptotically $p$-quasistable, $p > 0$, if it is $p$-stable and for each $t \in [t_0, t_0 + h_0)$ and all initial functions $\phi$

$$\lim_{j\to\infty} \mathbf{E}|x(t + jh_0; t_0, \phi)|^p = 0. \tag{1.6}$$

**Definition 1.4.** The solution of Eq. (1.1) with initial condition (1.2) is called $p$-integrable, $p > 0$, if for all initial functions $\phi$

$$\int_{t_0}^{\infty} \mathbf{E}|x(t; t_0, \phi)|^p \,\mathrm{d}t < \infty. \tag{1.7}$$

If in Definitions 1.1–1.4 $p = 2$ then the solution is called correspondingly mean square stable, asymptotically mean square stable, asymptotically mean square quasistable, mean square integrable.

**Remark 1.1.** It is easy to see that condition (1.6) follows from (1.5) but the inverse statement is not true.

**Theorem 1.1.** *Let there exist a nonnegative functional* $V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \ldots)$ *and positive numbers* $c_1, c_2$, *such that*

$$\mathbf{E}V(t) \leq c_1 \sup_{s\leq t} \mathbf{E}|x(s)|^2, \quad t \in [t_0, t_0 + h_0), \tag{1.8}$$

$$\mathbf{E}\Delta V(t) \le -c_2\mathbf{E}|x(t)|^2, \quad t \ge t_0, \tag{1.9}$$

*where*

$$\Delta V(t) = V(t+h_0) - V(t). \tag{1.10}$$

*Then the trivial solution of* Eq. (1.1), (1.2) *is asymptotically mean square quasistable.*

**Proof.** Rewrite condition (1.9) in the form $\mathbf{E}\Delta V(t+jh_0) \le -c_2\mathbf{E}|x(t+jh_0)|^2$, $t \ge t_0$, $j = 0, 1, \dots$. Summing this inequality from $j=0$ to $j=i$, by virtue of (1.10) we obtain

$$\mathbf{E}V(t+(i+1)h_0) - \mathbf{E}V(t) \le -c_2\sum_{j=0}^{i}\mathbf{E}|x(t+jh_0)|^2.$$

Therefore,

$$c_2\sum_{j=0}^{\infty}\mathbf{E}|x(t+jh_0)|^2 \le \mathbf{E}V(t), \quad t \ge t_0. \tag{1.11}$$

From here it follows also that

$$c_2\mathbf{E}|x(t)|^2 \le \mathbf{E}V(t), \qquad t \ge t_0. \tag{1.12}$$

Using (1.9) and (1.10), we have

$$\mathbf{E}V(t) \le \mathbf{E}V(t-h_0) \le \mathbf{E}V(t-2h_0) \le \cdots \le \mathbf{E}V(s), \quad t \ge t_0, \tag{1.13}$$

where $s = t - [(t-t_0)/h_0]h_0 \in [t_0, t_0+h_0)$, $[t]$ is the integer part of a number $t$. From (1.8) it follows

$$\sup_{s\in[t_0,t_0+h_0)}\mathbf{E}V(s) \le c_1\sup_{t\le t_0+h_0}\mathbf{E}|x(t)|^2. \tag{1.14}$$

Using (1.1)–(1.4), for $t \le t_0+h_0$ we obtain

$$\begin{aligned}\mathbf{E}|x(t)|^2 &= \sum_{l=1}^{2}\mathbf{E}|a_l(t-h_0, x(t-h_0), x(t-h_0-h_1), x(t-h_0-h_2), \dots)|^2\\ &\le \sum_{l=1}^{2}\left(a_{l0}\mathbf{E}|\phi(t-h_0)|^2 + \sum_{j=1}^{\infty}a_{lj}\mathbf{E}|\phi(t-h_0-h_j)|^2\right) \le A\|\phi\|^2.\end{aligned} \tag{1.15}$$

From (1.11)–(1.15) we have

$$c_2\sum_{j=0}^{\infty}\mathbf{E}|x(t+jh_0)|^2 \le c_1A\|\phi\|^2, \quad t \ge t_0, \tag{1.16}$$

and also

$$c_2\mathbf{E}|x(t)|^2 \le c_1A\|\phi\|^2, \quad t \ge t_0. \tag{1.17}$$

From (1.17) we get that the trivial solution of Eq. (1.1), (1.2) is mean square stable. From (1.16) it follows that for each $t \ge t_0$ $\lim_{j\to\infty}\mathbf{E}|x(t+jh_0)|^2 = 0$. Therefore, the trivial solution of Eq. (1.1), (1.2) is asymptotically mean square quasistable. Theorem is proven. □

**Remark 1.2.** If the conditions of Theorem 1.1 hold and $A < 1$ ($A$ is defined by (1.3)) then the trivial solution of Eq. (1.1), (1.2) is asymptotically mean square stable. Really, similar to (1.15) one can get $\mathbf{E}|x(t)|^2 \leq A^{[(t-t_0)/h_0]+1}\|\phi\|^2$, $t \geq t_0$. Therefore, $\lim_{t\to\infty} \mathbf{E}|x(t)|^2 = 0$ for all initial functions $\phi$.

**Remark 1.3.** If the conditions of Theorem 1.1 hold then the solution of Eq. (1.1) for each initial function (1.2) is mean square integrable. Really, integrating (1.9) from $t = t_0$ to $t = T$, by virtue of (1.10) we have

$$\int_T^{T+h_0} \mathbf{E}V(t)\,\mathrm{d}t - \int_{t_0}^{t_0+h_0} \mathbf{E}V(t)\,\mathrm{d}t \leq -c_2 \int_{t_0}^{T} \mathbf{E}|x(t)|^2\,\mathrm{d}t.$$

From here and (1.14) and (1.15) it follows

$$c_2 \int_{t_0}^{T} \mathbf{E}|x(t)|^2\,\mathrm{d}t \leq \int_{t_0}^{t_0+h_0} \mathbf{E}V(t)\,\mathrm{d}t \leq c_1 A\|\phi\|^2 h_0 < \infty,$$

and by $T \to \infty$ we obtain (1.7).

**Corollary 1.1.** *Let there exist a functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \dots)$ *and positive numbers* $c_1, c_2, p$, *such that conditions* (1.8) *and* (1.12) *and* $\mathbf{E}\Delta V(t) \leq 0$ *hold. Then the trivial solution of* Eq. (1.1) *is mean square stable.*

From Theorem 1.1, Remarks 1.2 and 1.3 and Corollary 1.1 it follows that an investigation of stability of the trivial solution of Eq. (1.1) can be reduced to construction of appropriate Lyapunov functionals. Below some formal procedure of Lyapunov functionals construction for equation of type (1.1) is described.

## 2. Formal procedure of Lyapunov functionals construction

The proposed procedure of Lyapunov functionals construction consists of four steps.

- *Step* 1. Represent the functionals $a_1$ and $a_2$ at the right-hand side of Eq. (1.1) in the form

$$\begin{aligned} a_1(t, x(t), x(t-h_1), x(t-h_2), \dots) &= F_1(t) + F_2(t) + \Delta F_3(t), \\ a_2(t, x(t), x(t-h_1), x(t-h_2), \dots) &= G_1(t) + G_2(t), \end{aligned} \tag{2.1}$$

  where $F_1(t) = F_1(t, x(t), x(t-h_1), \dots, x(t-h_k))$, $G_1(t) = G_1(t, x(t), x(t-h_1), \dots, x(t-h_k))$, $k \geq 0$ is a given integer, $F_j(t) = F_j(t, x(t), x(t-h_1), x(t-h_2), \dots)$, $j = 2, 3$, $G_2(t) = G_2(t, x(t), x(t-h_1), x(t-h_2), \dots)$, $F_1(t, 0, \dots, 0) \equiv F_2(t, 0, 0, \dots) \equiv F_3(t, 0, 0, \dots) \equiv G_1(t, 0, \dots, 0) \equiv G_2(t, 0, 0, \dots) \equiv 0$, $\Delta F_3(t) = F_3(t+h_0) - F_3(t)$.
- *Step* 2. Suppose that for the auxiliary equation

$$\begin{aligned} y(t+h_0) = {} & F_1(t, x(t), x(t-h_1), \dots, x(t-h_k)) \\ & + G_1(t, x(t), x(t-h_1), \dots, x(t-h_k))\xi(t+h_0), \quad t > t_0 - h_0, \end{aligned} \tag{2.2}$$

  there exists a Lyapunov functional $v(t) = v(t, y(t), y(t-h_1), \dots, y(t-h_k))$, which satisfies the conditions of Theorem 1.1.

- *Step* 3. Consider Lyapunov functional $V(t)$ for Eq. (1.1) in the form $V(t) = V_1(t) + V_2(t)$, where the main component is $V_1(t) = v(t, x(t) - F_3(t), x(t-h_1), \dots, x(t-h_k))$. Calculate $\mathbf{E}\Delta V_1(t)$ and in a reasonable way estimate it.
- *Step* 4. In order to satisfy the conditions of Theorem 1.1 the additional component $V_2(t)$ is chosen by some standard way.

## 3. Linear Volterra equations with constant coefficients

Let us demonstrate the formal procedure of Lyapunov functionals construction described above for stability investigation of the scalar equation

$$x(t+1) = \sum_{j=0}^{[t]+r} a_j x(t-j) + \sum_{j=0}^{[t]+r} \sigma_j x(t-j)\xi(t+1), \quad t > -1, \tag{3.1}$$

$$x(s) = \phi(s), \qquad s \in [-(r+1), 0],$$

where $r \geq 0$ is a given integer, $a_j$ and $\sigma_j$ are known constants.

### *3.1. The first way of Lyapunov functional construction*

Following Step 1 of the procedure represent Eq. (3.1) in form (2.1) with $F_3(t) = 0$, $G_1(t) = 0$, $k \geq 0$,

$$F_1(t) = \sum_{j=0}^{k} a_j x(t-j), \qquad F_2(t) = \sum_{j=k+1}^{[t]+r} a_j x(t-j), \qquad G_2(t) = \sum_{j=0}^{[t]+r} \sigma_j x(t-j), \tag{3.2}$$

and consider (Step 2) the auxiliary equation

$$y(t+1) = \sum_{j=0}^{k} a_j y(t-j), \quad t > -1, \ \ k \geq 0, \tag{3.3}$$

$$y(s) = \begin{cases} \phi(s), & s \in [-(r+1), 0], \\ 0, & s < -(r+1). \end{cases}$$

Introduce into consideration the vector $Y(t) = (y(t-k), \dots, y(t-1), y(t))'$ and represent the auxiliary equation (3.3) in the form

$$Y(t+1) = AY(t), \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_1 & a_0 \end{pmatrix}. \tag{3.4}$$

Consider the matrix equation

$$A'DA - D = -U, \quad U = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{3.5}$$

and suppose that the solution $D$ of this equation is a positive semidefinite symmetric matrix of dimension $k+1$ with $d_{k+1,k+1} > 0$. In this case the function $v(t) = Y'(t)DY(t)$ is Lyapunov function for Eq. (3.4), i.e. it satisfies the conditions of Theorem 1.1, in particular, condition (1.9). Really, using (3.4) and (3.5), we have $\Delta v(t) = -y^2(t)$.

Following Step 3 of the procedure, we will construct Lyapunov functional $V(t)$ for Eq. (3.1) in the form $V(t) = V_1(t) + V_2(t)$, where

$$V_1(t) = X'(t)DX(t), \quad X(t) = (x(t-k), \dots, x(t-1), x(t))'. \tag{3.6}$$

Using representation (3.2) rewrite now Eq. (3.1) as follows

$$X(t+1) = AX(t) + B(t), \quad B(t) = (0, \dots, 0, b(t))', \quad b(t) = F_2(t) + G_2(t)\xi(t+1), \tag{3.7}$$

where the matrix $A$ is defined by (3.4). Calculating $\Delta V_1(t)$, by virtue of Eq. (3.7) we have

$$\Delta V_1(t) = (AX(t) + B(t))'D(AX(t) + B(t)) - X'(t)DX(t) = -x^2(t) + B'(t)DB(t) + 2B'(t)DAX(t). \tag{3.8}$$

Put

$$\alpha_l = \sum_{j=l}^{\infty} |a_j|, \qquad \delta_l = \sum_{j=l}^{\infty} |\sigma_j|, \quad l = 0, 1, \dots. \tag{3.9}$$

Using (3.7), (3.2) and (3.9), we obtain

$$\begin{aligned} \mathbf{E}B'(t)DB(t) &= d_{k+1,k+1}[\mathbf{E}F_2^2(t) + \mathbf{E}G_2^2(t)] \\ &\le d_{k+1,k+1}\left[\alpha_{k+1} \sum_{j=k+1}^{[t]+r} |a_j|\mathbf{E}x^2(t-j) + \delta_0 \sum_{j=0}^{[t]+r} |\sigma_j|\mathbf{E}x^2(t-j)\right], \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \mathbf{E}B'(t)DAX(t) &= \mathbf{E}b(t)\left[\sum_{l=1}^{k} d_{l,k+1}x(t-k+l) + d_{k+1,k+1}\sum_{m=0}^{k} a_m x(t-m)\right] \\ &= \mathbf{E}b(t)\left[\sum_{m=0}^{k-1}(a_m d_{k+1,k+1} + d_{k-m,k+1})x(t-m) + a_k d_{k+1,k+1}x(t-k)\right] \\ &= d_{k+1,k+1}\mathbf{E}F_2(t)\sum_{m=0}^{k} Q_{km}x(t-m), \end{aligned} \tag{3.11}$$

where

$$Q_{km} = a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}}, \quad m = 0, \dots, k-1, \qquad Q_{kk} = a_k. \tag{3.12}$$

Putting

$$\beta_k = \sum_{m=0}^{k} |Q_{km}| = |a_k| + \sum_{m=0}^{k-1} \left| a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}} \right| \tag{3.13}$$

and using (3.11), (3.2), (3.9) and (3.13), we have

$$\begin{aligned} 2\mathbf{E}B'(t)DAX(t) &= 2d_{k+1,k+1} \sum_{m=0}^{k} \sum_{j=k+1}^{[t]+r} Q_{km} a_j \mathbf{E}x(t-m)x(t-j) \\ &\le d_{k+1,k+1} \left( \alpha_{k+1} \sum_{m=0}^{k} |Q_{km}| \mathbf{E}x^2(t-m) + \beta_k \sum_{j=k+1}^{[t]+r} |a_j| \mathbf{E}x^2(t-j) \right). \end{aligned} \tag{3.14}$$

Put now

$$R_{km} = \begin{cases} \alpha_{k+1}|Q_{km}| + \delta_0|\sigma_m|, & 0 \le m \le k, \\ (\alpha_{k+1} + \beta_k)|a_m| + \delta_0|\sigma_m|, & m > k. \end{cases} \tag{3.15}$$

Then from (3.8), (3.10) and (3.14) it follows

$$\mathbf{E}\Delta V_1(t) \le -\mathbf{E}x^2(t) + d_{k+1,k+1} \sum_{m=0}^{[t]+r} R_{km} \mathbf{E}x^2(t-m). \tag{3.16}$$

Choosing (Step 4) the functional $V_2(t)$ in the form

$$V_2(t) = d_{k+1,k+1} \sum_{m=1}^{[t]+r} \gamma_m x^2(t-m), \quad \gamma_m = \sum_{j=m}^{\infty} R_{kj}, \tag{3.17}$$

we obtain

$$\Delta V_2(t) = d_{k+1,k+1} \left( \gamma_1 x^2(t) - \sum_{m=1}^{[t]+r} R_{km} x^2(t-m) \right). \tag{3.18}$$

Put $V(t) = V_1(t) + V_2(t)$. From (3.16) and (3.18) we have $\mathbf{E}\Delta V(t) \le -(1 - \gamma_0 d_{k+1,k+1})\mathbf{E}x^2(t)$. If $\gamma_0 d_{k+1,k+1} < 1$ then the functional $V(t)$ satisfies condition (1.9) of Theorem 1.1. It is easy to check that condition (1.8) holds too. Using (3.17), (3.15) and (3.13), one can show that $\gamma_0 = \alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k + \delta_0^2$. Thus, if

$$\alpha_{k+1} < \sqrt{\beta_k^2 + d_{k+1,k+1}^{-1} - \delta_0^2} - \beta_k, \tag{3.19}$$

then the trivial solution of Eq. (3.1) is asymptotically mean square quasistable.

**Remark 3.1.** If $a_j = 0$ for $j > k$ and matrix equation (3.5) has a positive semidefinite solution $D$ with condition $\delta_0^2 < d_{k+1,k+1}^{-1}$ then the trivial solution of Eq. (3.1) is asymptotically mean square quasistable.

**Remark 3.2.** Suppose that in Eq. (3.1) $a_j = 0$ for $j > k$ and $\sigma_j = 0$ if $j \neq m$ for some $m \geq 0$. In this case $\alpha_{k+1} = 0$, $\delta_0^2 = \sigma_m^2$ and from (3.8), (3.10) and (3.14) it follows that $\mathbf{E}\Delta V_1(t) = -\mathbf{E}x^2(t) + d_{k+1,k+1}\sigma_m^2\mathbf{E}x^2(t-m)$. Putting $V_2(t) = d_{k+1,k+1}\sigma_m^2 \sum_{j=1}^{m} x^2(t-j)$, for the functional $V(t) = V_1(t)+V_2(t)$ we obtain $\mathbf{E}\Delta V(t) = (d_{k+1,k+1}\sigma_m^2 - 1)\mathbf{E}x^2(t)$. So, if $d_{k+1,k+1}\sigma_m^2 \geq 1$ then $\mathbf{E}V(t) \geq \mathbf{E}V(0) > 0$. But from the other hand it is easy to see that if $\lim_{t\to\infty} \mathbf{E}x^2(t) = 0$ then $\lim_{t\to\infty} \mathbf{E}V(t) = 0$ too. From this contradiction it follows that the condition $d_{k+1,k+1}\sigma_m^2 < 1$ is [14] the necessary and sufficient condition for asymptotic mean square quasistability.

**Remark 3.3.** In the case $k = 0$ condition (3.19) takes the form $\alpha_0^2+\delta_0^2 < 1$. Note that under this condition the trivial solution of Eq. (3.1) is not asymptotically mean square quasistable only but asymptotically mean square stable too. Using Remark 1.2 it is enough to show that for Eq. (3.1) the constant $A$ defined by (1.3) is $A = \alpha_0^2 + \delta_0^2 < 1$. In the case $k = 1$ condition (3.19) is a condition of asymptotic mean square quasistability only and can be written in the form

$$\alpha_0^2 + \delta_0^2 < 1 + \frac{2|a_0|}{1-a_1}(|a_1| - \alpha_0 a_1), \quad |a_1| < 1.$$

It is easy to see that this condition is not worse than previous one. One can show that for each $k = 1, 2, \ldots$ an obtained condition is not worse than the condition obtained for previous $k$.

### *3.2. The second way of Lyapunov functional construction*

Let us get another stability condition. Eq. (3.1) can be represented (Step 1) in form (2.1) with $F_1(t) = \beta x(t)$, $F_2(t) = G_1(t) = 0$, $k = 0$,

$$\beta = \sum_{j=0}^{\infty} a_j, \qquad F_3(t) = -\sum_{m=1}^{[t]+r} x(t-m)\sum_{j=m}^{\infty} a_j, \qquad G_2(t) = \sum_{j=0}^{[t]+r} \sigma_j x(t-j), \tag{3.20}$$

i.e.

$$x(t+1) = \beta x(t) + \Delta F_3(t) + G_2(t)\xi(t+1). \tag{3.21}$$

In this case the auxiliary equation (Step 2) is $y(t+1) = \beta y(t)$. The function $v(t) = y^2(t)$ is Lyapunov function for this equation if $|\beta| < 1$. We will construct (Step 3) Lyapunov functional $V(t)$ for Eq. (3.1) in the form $V(t) = V_1(t)+V_2(t)$, where $V_1(t) = (x(t)-F_3(t))^2$. Calculating $\mathbf{E}\Delta V_1(t)$, by virtue of representation (3.21) we obtain $\mathbf{E}\Delta V_1(t) = (\beta^2 - 1)\mathbf{E}x^2(t) + Q(t)$, where $Q(t) = -2(\beta - 1)\mathbf{E}x(t)F_3(t) + \mathbf{E}G_2^2(t)$. Putting

$$\alpha = \sum_{m=1}^{\infty}\left|\sum_{j=m}^{\infty} a_j\right|, \qquad B_m = |\beta - 1|\left|\sum_{j=m}^{\infty} a_j\right| + \delta_0\sigma_m, \tag{3.22}$$

and using (3.20) and (3.9), one can show $|Q(t)| \leq (\alpha|\beta - 1| + \delta_0\sigma_0)\mathbf{E}x^2(t) + \sum_{m=1}^{[t]+r} B_m\mathbf{E}x^2(t-m)$. As a result we have $\mathbf{E}\Delta V_1(t) \leq (\beta^2 - 1 + \alpha|\beta - 1| + \delta_0\sigma_0)\mathbf{E}x^2(t) + \sum_{m=1}^{[t]+r} B_m\mathbf{E}x^2(t-m)$.

Put now (Step 4) $V_2(t) = \sum_{m=1}^{[t]+r} \gamma_m x^2(t-m)$, $\gamma_m = \sum_{j=m}^{\infty} B_j$. Then similar to (3.18) for the functional $V(t) = V_1(t) + V_2(t)$ we have $\mathbf{E}\Delta V(t) \le (\beta^2 - 1 + 2\alpha|\beta - 1| + \delta_0^2)\mathbf{E}x^2(t)$. Thus, if $\beta^2 + 2\alpha|\beta-1| + \delta_0^2 < 1$ or

$$\delta_0^2 < (1-\beta)(1+\beta-2\alpha), \qquad |\beta| < 1. \tag{3.23}$$

then the trivial solution of Eq. (3.1) is asymptotically mean square quasistable.

## 4. Example

Consider the difference equation

$$x(t+1) = ax(t) + \sum_{j=1}^{[t]+r} b^j x(t-j) + \sigma x(t-r)\xi(t+1), \quad t > -1, \tag{4.1}$$

$$x(\theta) = \phi(\theta), \quad \theta \in [-(r+1), 0], \quad r \ge 0.$$

From (3.9) and (3.22) it follows that by virtue of conditions (3.19) and (3.23) stability regions for Eq. (4.1) can be obtained for $|b| < 1$ only. To obtain another type of condition for asymptotic mean square quasistability of the trivial solution of Eq. (4.1) let us transform the sum from the right hand side of Eq. (4.1) for $t > 0$ by the following way

$$\begin{aligned}\sum_{j=1}^{[t]+r} b^j x(t-j) &= b\left(x(t-1) + \sum_{j=1}^{[t]-1+r} b^j x(t-1-j)\right)\\ &= b[(1-a)x(t-1) + x(t) - \sigma x(t-1-r)\xi(t)].\end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1) we obtain Eq. (4.1) in the form

$$x(t+1) = ax(t) + \sum_{j=1}^{r-1} b^j x(t-j) + \sigma x(t-r)\xi(t+1), \quad t \in (-1, 0],$$

$$x(t+1) = (a+b)x(t) + b(1-a)x(t-1) - b\sigma x(t-1-r)\xi(t) + \sigma x(t-r)\xi(t+1), \quad t > 0. \tag{4.3}$$

Consider now the functional $V_1(t)$ in form (3.6) where $k = 1$ and the matrix $D$ is the solution of Eq. (3.5) with the elements

$$d_{11} = a_1^2 d_{22}, \qquad d_{12} = \frac{a_0 a_1}{1-a_1} d_{22}, \qquad d_{22} = \frac{1-a_1}{(1+a_1)[(1-a_1)^2 - a_0^2]},$$
$$a_0 = a+b, \quad a_1 = b(1-a). \tag{4.4}$$

Note that the matrix $D$ with the elements (4.4) is a positive semidefinite one if and only if

$$|b(1-a)| < 1, \qquad |a+b| < 1 - b(1-a). \tag{4.5}$$

Here $\Delta V_1(t)$ is defined by (3.8) with $A$ and $X(t)$ defined by (3.4) and (3.6) for $k = 1$ with $B(t) = (0, b(t))'$, $b(t) = \sigma x(t-r)\xi(t+1) - b\sigma x(t-1-r)\xi(t)$. Calculating $\mathbf{E}\Delta V_1(t)$ similar to (3.8) and (3.10)–(3.16) one can get $\mathbf{E}\Delta V_1(t) = -\mathbf{E}x^2(t) + \sigma^2 d_{22}[\mathbf{E}x^2(t-r) + \gamma\mathbf{E}x^2(t-1-r)]$, where

$$\gamma = b^2 - 2b\frac{a+b}{1-b(1-a)}.$$

Note that by condition (4.5) $\gamma > -1$. Really, $\gamma + 1 > b^2 - 2|b| + 1 = (|b|-1)^2 \geq 0$.

Put now $\gamma_0 = \max(\gamma, 0)$ and

$$V_2(t) = \sigma^2 d_{22}\left[(1+\gamma_0)\sum_{m=1}^{r} x^2(t-m) + \gamma_0 x^2(t-1-r)\right].$$

It is easy to show that $\Delta V_2(t) = \sigma^2 d_{22}[(1+\gamma_0)x^2(t) - x^2(t-r) - \gamma_0 x^2(t-1-r)]$. So, for the functional $V(t) = V_1(t) + V_2(t)$ we have

$$\mathbf{E}\Delta V(t) = -(1-\sigma^2 d_{22}(1+\gamma_0))\mathbf{E}x^2(t) + \sigma^2 d_{22}(\gamma-\gamma_0)\mathbf{E}x^2(t-1-r). \tag{4.6}$$

If $\gamma \geq 0$ then $\gamma_0 = \gamma$ and $\mathbf{E}\Delta V(t) = -(1-\sigma^2 d_{22}(1+\gamma))\mathbf{E}x^2(t)$. So, similar to Remark 3.2 the inequality

$$\sigma^2 d_{22}(1+\gamma) < 1 \tag{4.7}$$

is [14] the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of Eq. (4.3) (or (4.1)).

If $\gamma < 0$, i.e. $\gamma \in (-1, 0)$, then $\gamma_0 = 0$ and from (4.6) it follows

$$\mathbf{E}\Delta V(t) = -(1-\sigma^2 d_{22})\mathbf{E}x^2(t) + \sigma^2 d_{22}\gamma\mathbf{E}x^2(t-1-r). \tag{4.8}$$

Since $\gamma < 0$ then $\mathbf{E}\Delta V(t) \leq -(1-\sigma^2 d_{22})\mathbf{E}x^2(t)$ and from Theorem 1.1 it follows that the inequality $\sigma^2 d_{22} < 1$ is a sufficient condition for asymptotic mean square quasistability of the trivial solution of Eq. (4.3) (or (4.1)).

Let us suppose that $\sigma^2 d_{22} \geq 1$ but condition (4.7) holds. In this case each mean square bounded solution of Eq. (4.3), i.e. $\mathbf{E}x^2(t) \leq C$, is asymptotically mean square quasitrivial, i.e. $\lim_{j\to\infty}\mathbf{E}x^2(t+j) = 0$. Really, putting in (4.8) $t+j$ instead of $t$ and summing from $j = 0$ to $j = i$ we obtain

$$\mathbf{E}V(t+i+1) - \mathbf{E}V(t) = -(1-\sigma^2 d_{22})\sum_{j=0}^{i}\mathbf{E}x^2(t+j) + \sigma^2 d_{22}\gamma\left(\sum_{j=0}^{i-1-r}\mathbf{E}x^2(t+j) + \sum_{j=-1-r}^{-1}\mathbf{E}x^2(t+j)\right).$$

From here, using $V(t+i+1) \geq 0$ and $\gamma < 0$, we have

$$(1-\sigma^2 d_{22})\sum_{j=0}^{i}\mathbf{E}x^2(t+j) - \sigma^2 d_{22}\gamma\sum_{j=0}^{i-1-r}\mathbf{E}x^2(t+j) \leq \mathbf{E}V(t),$$

or

$$(1-\sigma^2 d_{22}(1+\gamma))\sum_{j=0}^{i}\mathbf{E}x^2(t+j)\le \mathbf{E}V(t)+\sigma^2 d_{22}|\gamma|\sum_{j=i-r}^{i}\mathbf{E}x^2(t+j).$$

If the solution of Eq. (4.3) is mean square bounded, i.e. $\mathbf{E}x^2(t)\le C$, then

$$(1-\sigma^2 d_{22}(1+\gamma))\sum_{j=0}^{\infty}\mathbf{E}x^2(t+j)\le \mathbf{E}V(t)+\sigma^2 d_{22}|\gamma|(r+1)C,$$

and therefore $\lim_{j\to\infty}\mathbf{E}x^2(t+j)=0$. So, by condition (4.7) in the regions $\{\gamma\ge 0\}$ and $\{\gamma<0,\ \sigma^2 d_{22}<1\}$ the trivial solution of Eq. (4.1) is asymptotically mean square quasistable. In the region $\{\gamma<0,\ \sigma^2 d_{22}\ge 1\}$ we can conclude only that each mean square bounded solution of Eq. (4.1) is asymptotically mean square quasitrivial.

In reality in the region $\{\gamma<0,\ \sigma^2 d_{22}\ge 1\}$ the trivial solution of Eq. (4.1) can be asymptotically mean square quasistable too. Really, in Fig. 1 the region given by condition (4.7) for $\sigma^2=0.2$ and also the following different parts of this region: (1) $\{\gamma\ge 0\}$, (2) $\{\gamma<0,\ \sigma^2 d_{22}<1\}$, (3) $\{\gamma<0,\ \sigma^2 d_{22}\ge 1\}$, are shown. Solving matrix equation (3.5) for $k=0$, $k=1$, $k=2$ and by virtue of program "Mathematica" for $k=3$ and $k=4$ stability regions for asymptotic mean square quasistability of the trivial solution of Eq. (4.1) given by condition (3.19) were obtained. In Fig. 2 the regions of asymptotic mean square quasistability of the trivial solution of Eq. (4.1) for $\sigma^2=0.2$ obtained by condition (3.19) for $k=0$ (the

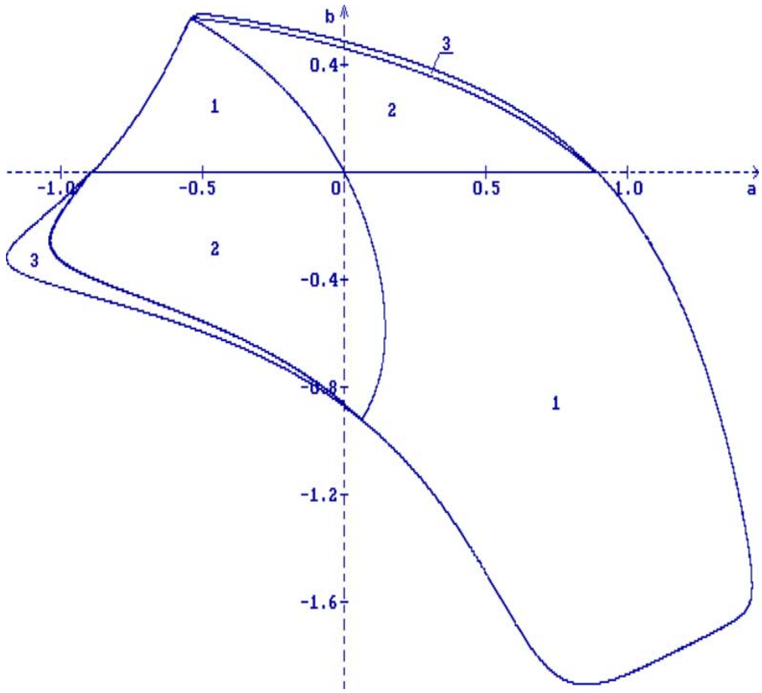


Fig. 1. Different parts of the stability region.

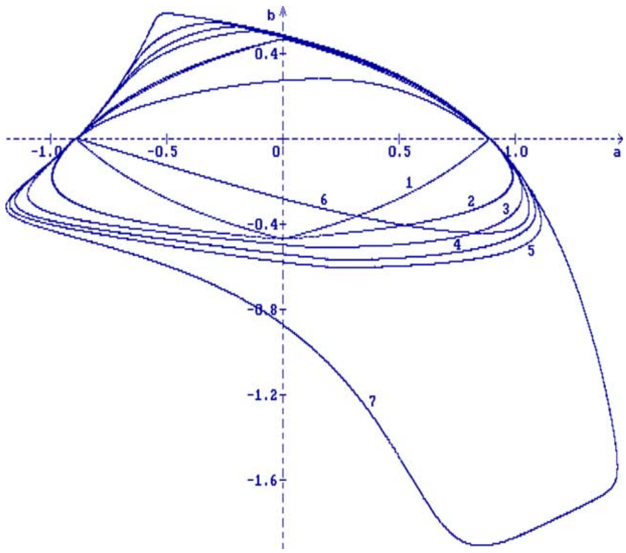


Fig. 2. Stability regions given by different stability conditions.

curve number 1), for $k = 1$ (the curve number 2), for $k = 2$ (the curve number 3), for $k = 3$ (the curve number 4), for $k = 4$ (the curve number 5), by condition (3.23) (the curve number 6) and the region given by condition (4.7) (the curve number 7) are shown. It is easy to see that some part of the region $\{\gamma < 0, \sigma^2 d_{22} \geq 1\}$ belongs to the regions given by condition (3.19) and therefore the trivial solution of Eq. (4.1) is there asymptotically mean square quasistable.

According to Remark 3.3 in Fig. 2 one can see also that the region of asymptotic mean square quasistability $Q_k$ of the trivial solution of Eq. (4.1), obtained by condition (3.19), expands if $k$ increases, i.e. $Q_0 \subset Q_1 \subset Q_2 \subset Q_3 \subset Q_4$.

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