SIAM J. CONTROL OPTIM.
Vol. 48, No. 7, pp. 4481–4499

SOME NEW ASPECTS OF LYAPUNOV-TYPE THEOREMS FOR STOCHASTIC DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE*

L. SHAIKHET†

Abstract. Some new Lyapunov-type theorems for stochastic differential equations of neutral type are proved. It is shown that these theorems simplify an application of Kolmanovskii and Shaikhet's general method of Lyapunov functionals construction for stability investigation of different mathematical models.

Key words. Lyapunov-type theorems, stochastic differential equations, stability, general method of Lyapunov functionals construction

AMS subject classifications. 34D20, 34K20, 93E15, 34K50, 34F05, 60H10

DOI. 10.1137/080744165

1. Introduction.

Investigation of hereditary systems is very important both in theory and applications (see, for instance, [1, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 28, 29, 30, 31, 42]). Investigation of stability properties of hereditary systems is often connected with construction of some appropriate Lyapunov functionals. The general method of Lyapunov functionals construction was proposed and developed by Kolmanovskii and Shaikhet for stochastic functional-differential equations, for stochastic difference equations with discrete time and continuous time, and for partial differential equations (see [7, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 32, 33, 34, 36, 37, 38]). This method was applied for stability investigation of some mathematical models in mechanics and biology (see [2, 3, 4, 5, 35, 39, 41]). Here some new aspect of Lyapunov-type theorems is proposed, which allows us to simplify an application of the general method of Lyapunov functionals construction for stability investigation of different mathematical models that can be described by stochastic differential equations of neutral type. In particular, a stochastic delay differential equation of nth order is considered. Similar results for stochastic difference equations were obtained in [40].

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a probability space, let $\{\mathfrak{F}_t, t \geq 0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, let $\mathbf{E}$ be the expectation with respect to the measure $\mathbf{P}$, and let H be the space of $\mathfrak{F}_0$-adapted functions $\varphi(s)$, $s \leq 0$, such that $\|\varphi\|^2 = \sup_{s\leq 0} \mathbf{E}|\varphi(s)|^2 < \infty$.

Consider the stochastic differential equation of neutral type

$$(1.1)\qquad \begin{gathered} d(x(t) - G(t, x_t)) = a_1(t, x_t)dt + a_2(t, x_t)dw(t), \quad t \geq 0, \\ x(s) = \varphi_0(s), \quad s \leq 0, \quad \varphi_0 \in H. \end{gathered}$$

Here $x(t) \in \mathbf{R}^n$ is a value of the process x in the moment of time t; $x_t = x(t+s)$, $s \leq 0$, is a trajectory of the process x to the moment of time t and for each fixed $t \geq 0$; $x_t = x(t+s) \in H$, $s \leq 0$; $w(t) \in \mathbf{R}^m$ is the standard $\mathfrak{F}_t$-adapted Wiener process; and the functionals $G(t, \varphi)$, $a_1(t, \varphi)$, $a_2(t, \varphi)$ are defined on $[0, \infty) \times H$, $G(t, \varphi) \in \mathbf{R}^n$,

*Received by the editors December 17, 2008; accepted for publication (in revised form) May 3, 2010; published electronically July 22, 2010.
http://www.siam.org/journals/sicon/48-7/74416.html
†Department of Higher Mathematics, Donetsk State University of Management, Chelyuskintsev 163-a, 83015 Donetsk, Ukraine (leonid@dsum.edu.ua, leonid.shaikhet@usa.net).

$a_1(t,\varphi)\in\mathbf{R}^n$, $a_2(t,\varphi)\in\mathbf{R}^{n\times m}$, $a_i(t,0)=0$, $i=1,2$, as

$$|G(t,\varphi)|\le\int_0^\infty|\varphi(-s)|dK(s),\quad \int_0^\infty dK(s)<1, \tag{1.2}$$

where the integral in (1.2) is a Stieltjes integral and $K(s)$ is a nondecreasing function of bounded variation.

DEFINITION 1.1. *The zero solution of* (1.1) *is called mean square stable if for each* $\epsilon>0$ *there exists* $\delta>0$ *such that* $\mathbf{E}|x(t)|^2<\epsilon$, $t\ge 0$, *if* $\|\varphi_0\|^2<\delta$. *If, in addition,* $\lim_{t\to\infty}\mathbf{E}|x(t)|^2=0$ *for every initial function* $\varphi_0\in H$, *then the zero solution of* (1.1) *is called asymptotically mean square stable.*

DEFINITION 1.2. *The zero solution of* (1.1) *is called mean square integrable if* $\int_0^\infty\mathbf{E}|x(t)|^2dt<\infty$.

THEOREM 1.1 (see [16]). *Assume that condition* (1.2) *holds and there exists the functional*

$$V(t,\varphi)=W(t,\varphi)+|\varphi(0)-G(t,\varphi)|^2 \tag{1.3}$$

such that

$$\begin{aligned}0\le\mathbf{E}W(t,x_t)&\le c_1\|x_t\|^2,\\ \mathbf{E}LV(t,x_t)&\le -c_2\mathbf{E}|x(t)|^2,\end{aligned} \tag{1.4}$$

where $c_i>0$, $i=1,2$, *and* L *is the generator of* (1.1). *Then the zero solution of* (1.1) *is asymptotically mean square stable.*

Note that the considered functionals $G(t,\varphi)$, $a_1(t,\varphi)$, $a_2(t,\varphi)$, $V(t,\varphi)$, $W(t,\varphi)$ are deterministic functionals of two arguments t and φ, but after changing φ on the stochastic process $x_t=x(\omega,t+s)$, $\omega\in\Omega$, $s\le 0$, they are stochastic processes $G(t,x_t)$, $a_1(t,x_t)$, $a_2(t,x_t)$, $V(t,x_t)$, $W(t,x_t)$; i.e., each process depends on $\omega\in\Omega$. For example, $W(t,x_t)=W(t,x(\omega,t+s))$, $\omega\in\Omega$, $s\le 0$.

From Theorem 1.1 it follows that for investigation of the asymptotic behavior of the solution of (1.1) it is necessary to construct some appropriate Lyapunov functional.

Below, the formal procedure of Lyapunov functionals construction for (1.1) is described. This procedure consists of four steps.

Step 1. Transform (1.1) into the form

$$dz(t,x_t)=(b_1(t,x(t))+c_1(t,x_t))dt+(b_2(t,x(t))+c_2(t,x_t))dw(t), \tag{1.5}$$

where $z(t,x_t)$, $c_1(t,x_t)$, $c_2(t,x_t)$ are some functionals on x_t, $z(t,0)=0$, $c_i(t,0)=0$, $i=1,2$, and functions $b_i(t,x(t))$, $i=1,2$, depend on t and $x(t)$ only and do not depend on the previous values $x(t+s)$, $s<0$, of the solution, $b_i(t,0)=0$.

Step 2. Assume that the zero solution of the auxiliary equation without memory,

$$dy(t)=b_1(t,y(t))dt+b_2(t,y(t))dw(t), \tag{1.6}$$

is asymptotically mean square stable, and therefore there exists a Lyapunov function $v(t,y)$ for which the condition $L_0v(t,y)\le -c|y|^2$ holds. Here L_0 is the generator of (1.6), $c>0$.

Step 3. A Lyapunov functional $V(t,x_t)$ for (1.1) is constructed in the form $V=V_1+V_2$, where $V_1(t,x_t)=v(t,z(t,x_t))$. Here the argument y of the function $v(t,y)$ is replaced on the functional $z(t,x_t)$ from the left-hand side of (1.5).

Step 4. Usually, the functional V_1 does not satisfy the conditions of Theorem 1.1. In order to satisfy these conditions it is necessary to calculate LV_1 and estimate it. Then the additional component V_2 of the functional V can be easily chosen in a standard way.

Note that representation (1.5) is not unique. This fact allows us via different representations of type (1.5) to construct different Lyapunov functionals and as a result to get different sufficient conditions for asymptotic mean square stability.

Some standard way of constructing the additional functional V_2 allows us to reject the fourth step of the procedure and not use the functional V_2 at all. Below, corresponding auxiliary Lyapunov-type theorems are considered.

2. Lyapunov-type theorems.

The following theorems allow us, in some cases, to use Lyapunov functionals with conditions that are weaker than those in Theorem 1.1.

THEOREM 2.1. *Assume that there exists a functional $V_1(t, x_t)$ of type* (1.3) *such that*

$$\begin{aligned} \mathbf{E}LV_1(t,x_t) \le \mathbf{E}x'(t)P(t)x(t) + \sum_{i=1}^{k} \mathbf{E}x'(t-\tau_i(t))Q_i(t-\tau_i(t))x(t-\tau_i(t)) \\ + \sum_{j=0}^{m} \int_0^\infty d\mu_j(s) \int_{t-s}^{t} (\theta - t + s)^j \mathbf{E}x'(\theta)R_j(\theta)x(\theta)d\theta, \end{aligned} \tag{2.1}$$

where L is the generator of (1.1)*; $P(t)$, $t \ge 0$, is a symmetric negative definite matrix; $Q_i(t)$, $i = 1, \dots, k$, $R_j(t)$, $j = 0, \dots, m$, $t \ge 0$, are symmetric nonnegative definite matrices; $\mu_j(s)$, $j = 0, \dots, m$, $s \ge 0$, are nondecreasing functions of bounded variation such that*

$$r_j = \int_0^\infty \frac{s^{j+1}}{j+1} d\mu_j(s) < \infty; \tag{2.2}$$

$\tau_i(t)$, $i = 1, \dots, k$, $t \ge 0$, are differentiable nonnegative functions with $\dot{\tau}_i(t) \le \hat{\tau}_i < 1$; and $P(t) + Q(t)$ is a matrix which is uniformly nonnegative with respect to $t \ge 0$, i.e.,

$$x'(P(t) + Q(t))x \le -c|x|^2, \qquad c > 0, \qquad x \in \mathbf{R}^n, \tag{2.3}$$

where

$$Q(t) = \sum_{i=1}^{k} \frac{1}{1-\hat{\tau}_i} Q_i(t) + \sum_{j=0}^{m} r_j R_j(t). \tag{2.4}$$

Then the zero solution of (1.1) *is asymptotically mean square stable.*

Proof. Put

$$\begin{aligned} V_2(t,x_t) = \sum_{i=1}^{k} \frac{1}{1-\hat{\tau}_i} \int_{t-\tau_i(t)}^{t} x'(s)Q_i(s)x(s)ds \\ + \sum_{j=0}^{m} \int_0^\infty d\mu_j(s) \int_{t-s}^{t} \frac{(\theta - t + s)^{j+1}}{j+1} x'(\theta)R_j(\theta)x(\theta)d\theta. \end{aligned}$$

Then

$$\mathbf{E}LV_2(t,x_t) = \sum_{i=1}^{k} \frac{1}{1-\hat{\tau}_i}\mathbf{E}x'(t)Q_i(t)x(t)$$

(2.5)
$$-\sum_{i=1}^{k} \frac{1-\dot{\tau}_i(t)}{1-\hat{\tau}_i}\mathbf{E}x'(t-\tau_i(t))Q_i(t-\tau_i(t))x(t-\tau_i(t))$$

$$+\sum_{j=0}^{m} r_j\mathbf{E}x'(t)R_j(t)x(t) - \sum_{j=0}^{m}\int_0^\infty d\mu_j(s)\int_{t-s}^{t}(\theta-t+s)^j\mathbf{E}x'(\theta)R_j(\theta)x(\theta)d\theta.$$

From (2.1), (2.4), (2.5) for the functional $V(t,x_t) = V_1(t,x_t)+V_2(t,x_t)$ it follows that

$$\mathbf{E}LV(t,x_t) \le \mathbf{E}x'(t)(P(t)+Q(t))x(t). \tag{2.6}$$

Via (2.3) this means that there exists a functional $V(t,x_t)$ satisfying the conditions of Theorem 1.1, and therefore the zero solution of (1.1) is asymptotically mean square stable. The proof is completed. ∎

Remark 2.1. From (2.3), (2.6) it follows that $\mathbf{E}LV(t,x_t) \le -c\mathbf{E}|x(t)|^2$, $c > 0$. Therefore,

$$\mathbf{E}V(t,x_t) - \mathbf{E}V(0,\phi) \le -c\int_0^t \mathbf{E}|x(s)|^2 ds,$$

and via $V(t,x_t) \ge 0$ we have $c\int_0^t \mathbf{E}|x(s)|^2 ds \le \mathbf{E}V(0,\phi) < \infty$. This means that by conditions (2.1), (2.3) the solution of (1.1) is also mean square integrable.

Remark 2.2. In the scalar case from Remark 2.1 it follows that if by condition (2.1) the solution of (1.1) is mean square nonintegrable, that is, $\int_0^\infty \mathbf{E}x^2(t)dt = \infty$, then $\sup_{t\ge 0}(P(t)+Q(t)) \ge 0$.

THEOREM 2.2. *Assume that there exists a functional $V_1(t,x_t)$ of type* (1.3) *such that*

(2.7)
$$\mathbf{E}LV_1(t,x_t) \le \mathbf{E}x'(t)P(t)x(t) + \sum_{i=1}^{k}\mathbf{E}x'(t-\tau_i)Q_i(t-\tau_i)x(t-\tau_i)$$
$$+\int_0^\infty d\mu(\tau)\int_{t-\tau}^{t}\mathbf{E}x'(s)R(t,s+\tau)x(s)ds,$$

where L is the generator of (1.1)*; $P(t)$, $t \ge 0$, is a symmetric negative definite matrix; $Q_i(t)$, $i = 1,\dots,k$, $R(t,s)$, $t \ge 0$, $s \ge 0$, are symmetric nonnegative definite matrices; and $\mu(\tau)$, $\tau \ge 0$, is nondecreasing function of bounded variation such that*

$$Q(t) = \sum_{i=1}^{k} Q_i(t) + \int_0^\infty d\mu(\tau)\int_t^{t+\tau} R(\theta,t+\tau)d\theta < \infty, \quad t \ge 0. \tag{2.8}$$

If condition (2.3) *holds, then the zero solution of* (1.1) *is asymptotically mean square stable.*

Proof. Put

$$V_2(t,x_t) = \sum_{i=1}^{k}\int_{t-\tau_i}^{t} x'(s)Q_i(s)x(s)ds + \int_0^\infty d\mu(\tau)\int_{t-\tau}^{t}\int_t^{s+\tau} x'(s)R(\theta,s+\tau)x(s)d\theta ds.$$

Then via (2.8),

$$\mathbf{E}LV_2(t,x_t) = \mathbf{E}x'(t)Q(t)x(t) - \sum_{i=1}^{k} \mathbf{E}x'(t-\tau_i)Q_i(t-\tau_i)\mathbf{E}x(t-\tau_i) - \int_0^\infty d\mu(\tau)\int_{t-\tau}^t \mathbf{E}x'(s)R(t,s+\tau)x(s)ds. \tag{2.9}$$

From (2.7)–(2.9) it follows that the functional $V(t,x_t) = V_1(t,x_t) + V_2(t,x_t)$ satisfies condition (2.6). Via (2.3) this means that there exists a functional $V(t,x_t)$ satisfying the conditions of Theorem 1.1, and therefore the zero solution of (1.1) is asymptotically mean square stable. The proof is completed. ◻

Remark 2.3. Theorems 2.1 and 2.2 give useful development and improvement of the general method of Lyapunov functionals construction. Via these theorems one can get good stability conditions using much simpler Lyapunov functionals than those via Theorem 1.1. The simpler functionals can be used in different applications.

3. Demonstrative examples. Consider three different representations of type (1.5) for the scalar stochastic differential equation

$$\dot{x}(t) = ax(t) + \sum_{i=1}^{p} b_i x(t-h_i(t)) + \sum_{i=1}^{p} c_i \int_{t-h_i(t)}^t x(s)ds + \sigma x(t-\tau(t))\dot{w}(t). \tag{3.1}$$

3.1. Suppose that in (3.1)

$$h_i(t) \le h_i^0, \qquad \dot{h}_i(t) \le \hat{h}_i < 1, \qquad \dot{\tau}(t) \le \hat{\tau} < 1, \tag{3.2}$$

and put

$$B(h) = \sum_{i=1}^{p} \frac{|b_i|}{\sqrt{1-\hat{h}_i}}, \qquad C_0(h) = \sum_{i=1}^{p} |c_i| h_i^0. \tag{3.3}$$

Let us consider (3.1) as a representation of (1.5) with $z(t,x_t) = x(t)$ and the auxiliary equation $\dot{y}(t) = ay(t)$. The zero solution of this equation is asymptotically stable if and only if $a < 0$. Using the corresponding Lyapunov function $v(y) = y^2$, we obtain the functional $V_1(t,x_t)$ in the form $V_1(t,x_t) = x^2(t)$.

Using (3.2), (3.3) and some positive numbers γ_i, $i = 1,\dots,p$, we have

$$\begin{aligned} LV_1 &= 2x(t)\left(ax(t) + \sum_{i=1}^{p} b_i x(t-h_i(t)) + \sum_{i=1}^{p} c_i \int_{t-h_i(t)}^t x(s)ds\right) + \sigma^2 x^2(t-\tau(t)) \\ &\le \left(2a + C_0(h) + \sum_{i=1}^{p}\gamma_i|b_i|\right)x^2(t) + \sum_{i=1}^{p}\gamma_i^{-1}|b_i|x^2(t-h_i(t)) \\ &\quad + \sum_{i=1}^{p}|c_i|\int_{t-h_i^0}^t x^2(s)ds + \sigma^2x^2(t-\tau(t)). \end{aligned}$$

So, we obtain representation (2.1) with

$$P = 2a + C_0(h) + \sum_{i=1}^{p}\gamma_i|b_i|, \quad k = p+1, \quad \tau_k(t) = \tau(t), \quad m = 0, \quad R_0 = 1,$$

$$Q_i = \gamma_i^{-1}|b_i|, \quad \tau_i(t) = h_i(t), \quad i = 1,\dots,p, \quad Q_k = \sigma^2, \quad d\mu_0(s) = \sum_{i=1}^{p}|c_i|\delta(s-h_i^0)ds,$$

and

$$P + Q = 2a + 2C_0(h) + \frac{\sigma^2}{1-\hat{\tau}} + \sum_{i=1}^{p}\left(\gamma_i + \frac{\gamma_i^{-1}}{1-\hat{h}_i}\right)|b_i|.$$

To minimize $P + Q$ put $\gamma_i = \frac{1}{\sqrt{1-\hat{h}_i}}$. From Theorem 2.1 we obtain the following assertion: If

$$\frac{\sigma^2}{2(1-\hat{\tau})} + B(h) + C_0(h) < |a|, \qquad a < 0, \tag{3.4}$$

then the zero solution of (3.1) is asymptotically mean square stable.

3.2. In addition to (3.2) assume that

$$|\dot{h}_i(t)| \le \hat{h}_i^0 \tag{3.5}$$

and put

$$B_0(h) = \sum_{i=1}^{p}|b_i|h_i^0, \qquad B_1(h) = \sum_{i=1}^{p}\frac{|b_i|\hat{h}_i^0}{\sqrt{1-\hat{h}_i}}. \tag{3.6}$$

Consider representation (1.5) of (3.1) in the form of a differential equation of neutral type

$$\dot{z}(t,x_t) = S_0x(t) + \sum_{i=1}^{p}\left(b_i\dot{h}_i(t)x(t-h_i(t)) + c_i\int_{t-h_i(t)}^{t}x(s)ds\right) + \sigma x(t-\tau(t))\dot{w}(t), \tag{3.7}$$

where

$$z(t,x_t) = x(t) + \sum_{j=1}^{p}b_j\int_{t-h_j(t)}^{t}x(s)ds, \qquad S_0 = a + \sum_{i=1}^{p}b_i. \tag{3.8}$$

Condition (1.2) for (3.7) has the form $B_0(h) < 1$.

The auxiliary equation for (3.7) is $\dot{y}(t) = S_0y(t)$, and the zero solution of this equation is asymptotically stable if and only if $S_0 < 0$. Using the corresponding Lyapunov function $v(y) = y^2$ we obtain the functional $V_1(t,x_t)$ in the form $V_1(t,x_t) = z^2(t,x_t)$.

Via (3.2), (3.3), (3.5)–(3.8), and some positive numbers γ_{1i}, γ_{2ij}, we obtain

$$\begin{aligned}LV_1(t,x_t) &\le 2S_0x^2(t) + \sum_{i=1}^{p}|b_i|\hat{h}_i^0(\gamma_{1i}x^2(t) + \gamma_{1i}^{-1}x^2(t-h_i(t))) + \sigma^2x^2(t-\tau(t))\\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p}|b_jb_i|\hat{h}_i^0\int_{t-h_j^0}^{t}(\gamma_{2ij}x^2(s) + \gamma_{2ij}^{-1}x^2(t-h_i(t)))ds\\ &+ \sum_{i=1}^{p}\sum_{j=1}^{p}|b_jc_i|\int_{t-h_i^0}^{t}\int_{t-h_j^0}^{t}(x^2(\theta)+x^2(s))dsd\theta + \sum_{j=1}^{p}|S_0b_j + c_j|\int_{t-h_j^0}^{t}(x^2(t)+x^2(s))ds.\end{aligned}$$

As a result we have representation (2.1),

$$LV_1(t,x_t) \le P(t)x^2(t) + \sigma^2x^2(t-\tau(t)) + \sum_{i=1}^{p}Q_ix^2(t-h_i(t)) + \sum_{j=1}^{p}q_j\int_{t-h_j^0}^{t}x^2(s)ds,$$

where

$$P = 2S_0 + \sum_{i=1}^{p} |b_i|\hat{h}_i^0\gamma_{1i} + \sum_{j=1}^{p} |S_0 b_j + c_j|h_j^0, \quad k = p+1, \quad \tau_k(t) = \tau(t), \quad m = 0,$$

$$Q_i = |b_i|\hat{h}_i^0\gamma_{1i}^{-1} + |b_i|\hat{h}_i^0 \sum_{j=1}^{p} |b_j|h_j^0\gamma_{2ij}^{-1}, \quad \tau_i(t) = h_i(t), \quad i = 1, \dots, p, \quad Q_k = \sigma^2, \quad R_0 = 1,$$

$$d\mu_0(s) = \sum_{j=1}^{p} q_j\delta(s - h_j^0)ds, \quad q_j = |S_0 b_j + c_j| + |b_j| \sum_{i=1}^{p} |b_i|\hat{h}_i^0\gamma_{2ij} + |b_j|C_0(h) + |c_j|B_0(h),$$

and

$$P + Q = 2S_0 + 2\sum_{j=1}^{p} |S_0 b_j + c_j|h_j^0 + 2B_0(h)C_0(h) + \frac{\sigma^2}{1-\hat{\tau}}$$
$$+ \sum_{i=1}^{p} |b_i|\hat{h}_i^0 \left(\gamma_{1i} + \frac{\gamma_{1i}^{-1}}{1-\hat{h}_i}\right) + \sum_{j=1}^{p} |b_j|h_j^0 \sum_{i=1}^{p} |b_i|\hat{h}_i^0 \left(\gamma_{2ij} + \frac{\gamma_{2ij}^{-1}}{1-\hat{h}_i}\right).$$

Choosing the optimal values of $\gamma_{1i} = \gamma_{2ij} = \frac{1}{\sqrt{1-\hat{h}_i}}$, we can minimize $P + Q$ and use Theorem 2.1 to get the following stability condition: If

$$\frac{\sigma^2}{2(1-\hat{\tau})} + \sum_{j=1}^{p} |S_0 b_j + c_j|h_j^0 + B_1(h) + B_0(h)(B_1(h) + C_0(h)) < |S_0|, \quad S_0 < 0, \tag{3.9}$$

then the zero solution of (3.1) is asymptotically mean square stable.

Remark 3.1. It is easy to see that instead of condition (3.9) one can use the rougher, but simpler, condition

$$\frac{\sigma^2}{2(1-\hat{\tau})} + (1 + B_0(h))(B_1(h) + C_0(h)) < |S_0|(1 - B_0(h)). \tag{3.10}$$

3.3. Now put

$$\begin{gathered} C_1(h) = \sum_{j=1}^{p} |c_j|h_j^0\hat{h}_j^0, \qquad C_2(h) = \sum_{i=1}^{p} |c_i|(h_i^0)^2, \\ A_0(h) = B_0(h) + \frac{1}{2}C_2(h), \qquad A_1(h) = B_1(h) + C_1(h), \end{gathered} \tag{3.11}$$

and consider representation (1.5) of (3.1) in the form of a differential equation of neutral type

$$\dot{z}(t, x_t) = S(t)x(t) + \sum_{i=1}^{p} \dot{h}_i(t) \left(b_i x(t - h_i(t)) + c_i \int_{t-h_i(t)}^{t} x(s)ds\right) + \sigma x(t - \tau(t))\dot{w}(t), \tag{3.12}$$

where

$$\begin{gathered} z(t, x_t) = x(t) + \sum_{i=1}^{p} \int_{t-h_i(t)}^{t} (b_i + c_i(s - t + h_i(t)))x(s)ds, \\ S(t) = a + \sum_{i=1}^{p} (b_i + c_i h_i(t)). \end{gathered} \tag{3.13}$$

Condition (1.2) for (3.12) has the form $A_0(h) < 1$.

The auxiliary equation in this case is $\dot{y}(t) = S(t)y(t)$, and if $\sup_{t\geq 0} S(t) < 0$, then the zero solution of this equation is asymptotically stable. Using the corresponding Lyapunov function $v(y) = y^2$ we obtain the functional $V_1(t, x_t)$ in the form $V_1(t, x_t) = z^2(t, x_t)$.

Then via (3.2), (3.5), (3.6), (3.11)–(3.13), and some positive numbers γ_{1i}, γ_{2ij}, we obtain

$$\begin{aligned}LV_1(t,x_t) &\leq 2S(t)x^2(t) + |S(t)|\sum_{i=1}^{p}\int_{t-h_i(t)}^{t}(|b_i|+|c_i|(s-t+h_i(t)))(x^2(t)+x^2(s))ds\\ &+\sum_{i=1}^{p}\hat{h}_i^0\left(|b_i|(\gamma_{1i}x^2(t)+\gamma_{1i}^{-1}x^2(t-h_i(t)))+|c_i|\int_{t-h_i(t)}^{t}(x^2(t)+x^2(s))ds\right)\\ &+\sum_{j=1}^{p}\sum_{i=1}^{p}\hat{h}_i^0|b_j|\int_{t-h_i(t)}^{t}(|b_i|+|c_i|(s-t+h_i(t)))(\gamma_{2ij}x^2(s)+\gamma_{2ij}^{-1}x^2(t-h_j(t)))ds\\ &+\sum_{j=1}^{p}\sum_{i=1}^{p}\hat{h}_i^0|c_j|\int_{t-h_i(t)}^{t}\int_{t-h_j(t)}^{t}(|b_i|+|c_i|(s-t+h_i(t)))(x^2(\theta)+x^2(s))d\theta ds+\sigma^2x^2(t-\tau(t)).\end{aligned}$$

Now put

$$S_m = \inf_{t\geq 0}|S(t)|, \qquad S_M = \sup_{t\geq 0}|S(t)|,$$

$$I_i(h_i(t)) = \int_{t-h_i(t)}^{t}(|b_i|+|c_i|(s-t+h_i(t)))ds, \quad J_{0i}(h_i(t)) = \int_{t-h_i(t)}^{t}x^2(s)ds,$$

$$J_{1i}(h_i(t)) = \int_{t-h_i(t)}^{t}(|b_i|+|c_i|(s-t+h_i(t)))x^2(s)ds, \quad i=1,\dots,p.$$

Via (3.2), (3.6), (3.11) we have

$$I_i(h_i(t)) \leq I_i(h_i^0) = |b_i|h_i^0 + \frac{1}{2}|c_i|(h_i^0)^2, \qquad \sum_{i=1}^{p} I_i(h_i(t)) \leq A_0(h),$$

$$J_{0i}(h_i(t)) \leq J_{0i}(h_i^0), \qquad J_{1i}(h_i(t)) \leq J_{1i}(h_i^0).$$

So, we obtain representation (2.1),

$$\begin{aligned}LV_1(t,x_t) &\leq P(t)x^2(t) + \sigma^2x^2(t-\tau(t)\\ &+\sum_{j=1}^{p}Q_jx^2(t-h_j(t))) + \sum_{i=1}^{p}q_{0i}J_{0i}(h_i^0) + \sum_{i=1}^{p}q_{1i}J_{1i}(h_i^0),\end{aligned}$$

where

$$P(t) = (-2+A_0(h))|S(t)| + C_1(h) + \sum_{i=1}^{p}\hat{h}_i^0|b_i|\gamma_{1i}, \quad k=p+1, \quad \tau_k=\tau, \quad m=1,$$

$$Q_k = \sigma^2, \quad Q_j = |b_j|\hat{h}_j^0\left(\gamma_{1j}^{-1}+\sum_{i=1}^{p}\gamma_{2ij}^{-1}I_i(h_i^0)\right), \quad j=1,\dots,k, \quad R_0=R_1=1,$$

$$d\mu_0(s) = \sum_{i=1}^{p}(q_{0i}+q_{1i}|b_i|)\delta(s-h_i^0)ds, \quad d\mu_1(s) = \sum_{i=1}^{p}q_{1i}|c_i|\delta(s-h_i^0)ds,$$

$$q_{0i} = (1+A_0(h))|c_i|\hat{h}_i^0, \qquad q_{1i} = S_M + C_1(h) + \sum_{j=1}^{p}|b_j|\hat{h}_j^0\gamma_{2ij},$$

and

$$P(t)+Q=(-2+A_0(h))|S(t)|+A_0(h)S_M+\frac{\sigma^2}{1-\hat{\tau}}+2C_1(h)(1+A_0(h))$$
$$+\sum_{j=1}^{p}|b_j|\hat{h}_j^0\left(\gamma_{1j}+\frac{\gamma_{1j}^{-1}}{1-\hat{h}_j}\right)+\sum_{i=1}^{p}\sum_{j=1}^{p}|b_j|\hat{h}_j^0 I_i(h_i^0)\left(\gamma_{2ij}+\frac{\gamma_{2ij}^{-1}}{1-\hat{h}_j}\right).$$

To minimize $P(t)+Q$, put $\gamma_{1j}=\gamma_{2ij}=\frac{1}{\sqrt{1-\hat{h}_j}}$. Then

$$P(t)+Q=(-2+A_0(h))|S(t)|+A_0(h)S_M+2A_1(h)(1+A_0(h))+\frac{\sigma^2}{1-\hat{\tau}}.$$

Via $\sup_{t\geq 0}S(t)<0$, we obtain the following estimation for $P(t)+Q$:
(3.14)

$$\sup_{t\geq 0}(P(t)+Q)\leq(-2+A_0(h))\,S_m+A_0(h)S_M+2A_1(h)(1+A_0(h))+\frac{\sigma^2}{1-\hat{\tau}}.$$

From (3.14) via Theorem 2.1 we obtain the following: If $\sup_{t\geq 0}S(t)<0$ and

$$\frac{\sigma^2}{1-\hat{\tau}}+A_0(h)S_M+2A_1(h)(1+A_0(h))<(2-A_0(h))S_m, \tag{3.15}$$

then the zero solution of (3.1) is asymptotically mean square stable.

3.4. Consider the equation with variable coefficients

$$\dot{x}(t)=a(t)x(t)-b(t)x(t-h)+\sigma(t)x(t-\tau)\dot{w}(t),\qquad t\geq 0, \tag{3.16}$$

where $a(t)$ and $b(t)$ are positive functions, $\sigma(t)$ is an arbitrary function, $h>0$, and $\tau>0$.

Suppose that

$$c(t)=b(t+h)-a(t)\geq c_0>0,\qquad \sup_{t\geq 0}\int_t^{t+h}b(s)ds<1, \tag{3.17}$$

and represent (3.16) in the form

$$\dot{z}(t,x_t)=-c(t)x(t)+\sigma(t)x(t-\tau)\dot{w}(t), \tag{3.18}$$

where $c(t)$ is defined as in (3.17) and

$$z(t,x_t)=x(t)-\int_{t-h}^{t}b(s+h)x(s)ds. \tag{3.19}$$

Note that (3.18), (3.19) is a differential equation of neutral type.

Consider the auxiliary differential equation without delay

$$\dot{y}(t)=-c(t)y(t). \tag{3.20}$$

Using Lyapunov function $v(t)=y^2(t)$, via (3.17) we have $\dot{v}(t)=-2c(t)y^2(t)\leq -2c_0y^2(t)$. So, the zero solution of (3.20) is asymptotically stable.

Following the general method of Lyapunov functionals construction, we will use Lyapunov functional $V_1(t,x_t)$ for (3.18), (3.19) in the $V_1(t,x_t)=z^2(t,x_t)$. Calculating $LV_1(t,x_t)$ via (3.18), (3.19) we obtain representation (2.7),

$$\begin{aligned}LV_1(t,x_t) &= c(t)\left(-2x^2(t)+2\int_{t-h}^{t} b(s+h)x(s)x(t)ds\right)+\sigma^2(t)x^2(t-\tau)\\ &\le c(t)\left(-2x^2(t)+\int_{t-h}^{t} b(s+h)(x^2(s)+x^2(t))ds\right)+\sigma^2(t)x^2(t-\tau)\\ &= P(t)x^2(t)+\sigma^2(t)x^2(t-\tau)+\int_{t-h}^{t} R(t,s+h)x^2(s)ds,\end{aligned}$$

where

$$P(t)=c(t)\left(-2+\int_t^{t+h} b(s)ds\right),\qquad Q_1(t)=\sigma^2(t+\tau),$$
$$k=1,\qquad d\mu(\tau)=\delta(\tau-h)d\tau,\qquad R(t,s)=c(t)b(s),$$

and

$$P(t)+Q(t)=c(t)\left(-2+\int_t^{t+h} b(s)ds+\frac{b(t+h)}{c(t)}\int_t^{t+h} c(\theta)d\theta+\frac{\sigma^2(t+\tau)}{c(t)}\right).$$

So, if

$$\sup_{t\ge 0}\left(\int_t^{t+h} b(s)ds+\frac{b(t+h)}{c(t)}\int_t^{t+h} c(s)ds+\frac{\sigma^2(t+\tau)}{c(t)}\right)<2,$$

then condition (2.3) holds and the zero solution of (3.19) is asymptotically mean square stable.

4. Scalar equation of nth order.

4.1. Case $n>1$. Consider the scalar equation

$$x^{(n)}(t)=\sum_{j=1}^{n}\int_0^\infty x^{(j-1)}(t-s)dK_j(s)+\sigma x(t-\tau)\dot w(t),\qquad t\ge 0, \tag{4.1}$$

where $x^{(j)}(t)=\frac{d^j x(t)}{dt^j}$, $j=1,\dots,n$. Initial conditions for (4.1) have the form

$$x^{(j)}(\theta)=\varphi_0^{(j)}(\theta),\qquad \theta\le 0, \tag{4.2}$$

where $\varphi_0(\theta)$ is a given $n-1$ times continuously differentiable function. The kernels $K_j(s)$ are functions of bounded variation on $[0,\infty)$ such that

$$\alpha_{ij}=\int_0^\infty s^i|dK_j(s)|<\infty,\qquad 0\le i\le n,\qquad 1\le j\le n. \tag{4.3}$$

Put $x_i(t)=x^{(i-1)}(t)$ and rewrite (4.1) as a system

$$\begin{aligned}\dot x_i(t)&=x_{i+1}(t),\qquad i=1,\dots,n-1,\\ \dot x_n(t)&=\sum_{j=1}^{n}\int_0^\infty x_j(t-s)dK_j(s)+\sigma x_1(t-\tau)\dot w(t).\end{aligned} \tag{4.4}$$

Also put

$$\beta_{ij} = \int_0^\infty s^i dK_j(s), \tag{4.5}$$

and note that for $m = 1, \dots, n$ we have

$$\int_0^\infty x_n(t-s)dK_m(s) = x_n(t)\beta_{0m} - \frac{d}{dt}\left[\int_0^\infty dK_m(s)\int_{t-s}^t x_n(s_0)ds_0\right]. \tag{4.6}$$

Similarly, using (4.5) it is easy to check that for $i = 1, \dots, n-1$

$$\begin{aligned} \int_0^\infty x_{n-i}(t-s)dK_m(s) &= \sum_{j=1}^{i+1}(-1)^{j-1}x_{n-i+j-1}(t)\frac{\beta_{j-1,m}}{(j-1)!} \\ &+(-1)^{i+1}\frac{d}{dt}\left[\int_0^\infty dK_m(s)\int_{t-s}^t x_n(\theta)\frac{(\theta-t+s)^i}{i!}d\theta\right]. \end{aligned} \tag{4.7}$$

Putting

$$\begin{aligned} z(x_t) &= \sum_{l=0}^{n-1}(-1)^{l+1}\int_0^\infty dK_{n-l}(s)\int_{t-s}^t x_n(\theta)\frac{(\theta-t+s)^l}{l!}d\theta, \\ a_l &= \sum_{i=l}^{n-1}(-1)^{i-l}\frac{\beta_{i-l,n-i}}{(i-l)!}, \qquad l = 0, 1, \dots, n-1, \end{aligned} \tag{4.8}$$

via (4.7), (4.8) we obtain

$$\begin{aligned} \sum_{j=1}^{n}\int_0^\infty x_j(t-s)dK_j(s) &= \sum_{i=0}^{n-1}\int_0^\infty x_{n-i}(t-s)dK_{n-i}(s) \\ &= \sum_{i=0}^{n-1}\sum_{j=1}^{i+1}(-1)^{j-1}x_{n-i+j-1}(t)\frac{\beta_{j-1,n-i}}{(j-1)!} + \dot{z}(x_t) \\ &= \sum_{i=0}^{n-1}\sum_{l=0}^{i}(-1)^{i-l}x_{n-l}(t)\frac{\beta_{i-l,n-i}}{(i-l)!} + \dot{z}(x_t) = \sum_{l=0}^{n-1}a_l x_{n-l}(t) + \dot{z}(x_t). \end{aligned} \tag{4.9}$$

Following the procedure of Lyapunov functionals construction and using (4.4), (4.9), represent (4.1) in the form

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t), \quad i = 1, \dots, n-1, \\ \frac{d}{dt}[x_n(t) - z(x_t)] &= \sum_{l=0}^{n-1}a_l x_{n-l}(t) + \sigma x_1(t-\tau)\dot{w}(t). \end{aligned} \tag{4.10}$$

Via (4.10) we obtain the auxiliary system

$$\dot{y}_i(t) = y_{i+1}(t), \quad i = 1, \dots, n-1, \qquad \dot{y}_n(t) = \sum_{l=0}^{n-1}a_l y_{n-l}(t). \tag{4.11}$$

Let $y = (y_1, \dots, y_n)'$, and let A be an $(n \times n)$-matrix such that

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix}. \tag{4.12}$$

Via (4.11), (4.12) we have $\dot{y} = Ay$.

Assume that A is the Hurwitz matrix. Then for arbitrary positive definite symmetric matrix D there exists a unique positive definite symmetric matrix B satisfying the Lyapunov matrix equation

$$A'B + BA = -D. \tag{4.13}$$

Consider the Lyapunov function for the auxiliary equation in the form $v(y) = y'By$. Because of (4.13) we have $\dot{v}(y) = -y'Dy$. According to the procedure of Lyapunov functionals construction we consider the functional

$$V_1(t, x_t) = (x_1(t), \dots, x_{n-1}(t), x_n(t) - z(x_t))' B (x_1(t), \dots, x_{n-1}(t), x_n(t) - z(x_t)). \tag{4.14}$$

Let D be a diagonal matrix with positive entries d_l, $l = 1, \dots, n$. From (4.14) it follows that $LV_1(t, x_t)$ with respect to (4.10) equals

$$\begin{aligned} LV_1(t, x_t) &= -\sum_{l=1}^{n} d_l x_l^2(t) - 2\sum_{l=1}^{n} z(x_t)(BA)_{nl} x_l(t) + b_{nn}\sigma^2 x_1^2(t-\tau) \\ &\le -\sum_{l=1}^{n} d_l x_l^2(t) + 2\sum_{l=1}^{n} \beta_l |z(x_t) x_l(t)| + b_{nn}\sigma^2 x_1^2(t-\tau), \end{aligned} \tag{4.15}$$

where $(BA)_{nl}$ is nlth entry of the matrix BA and $\beta_l = |(BA)_{nl}|$, $b_{nn} = (B)_{nn}$.

Also put

$$\alpha = \sum_{j=0}^{n-1} \frac{\alpha_{j+1,n-j}}{(j+1)!}, \qquad W(t, x_t) = \sum_{j=0}^{n-1} \int_0^\infty |dK_{n-j}(s)| \int_{t-s}^{t} x_n^2(\theta) \frac{(\theta - t + s)^j}{j!} d\theta,$$

and suppose that $\alpha > 0$. Then using (4.8) and some positive numbers γ_l, $l = 1, \dots, n$, we have

$$\begin{aligned} 2|z(x_t)x_l(t)| &\le 2\sum_{i=0}^{n-1} \int_0^\infty |dK_{n-i}(s)| \int_{t-s}^{t} |x_l(t) x_n(\theta)| \frac{(\theta - t + s)^i}{i!} d\theta \\ &\le \sum_{i=0}^{n-1} \left[\gamma_l x_l^2(t) \frac{\alpha_{i+1,n-i}}{(i+1)!} + \int_0^\infty |dK_{n-i}(s)| \int_{t-s}^{t} \frac{x_n^2(\theta)}{\gamma_l} \frac{(\theta - t + s)^i}{i!} d\theta \right] \\ &\le \alpha \gamma_l x_l^2(t) + \gamma_l^{-1} W(t, x_t). \end{aligned} \tag{4.16}$$

Thus, we obtain the following representation of type (2.1):

$$LV_1(t, x_t) \le -\sum_{l=1}^{n} d_l x_l^2(t) + \alpha \sum_{l=1}^{n} \beta_l \gamma_l x_l^2(t) + b_{nn}\sigma^2 x_1^2(t-\tau) + \sum_{l=1}^{n} \beta_l \gamma_l^{-1} W(x_t) \tag{4.17}$$

with $k = 1$, $\tau_1 = \tau$, $m = n - 1$, $d\mu_j(s) = |dK_{n-j}(s)|$,

$$P = \begin{pmatrix} \alpha\beta_1\gamma_1 - d_1 & 0 & \dots & 0 \\ 0 & \alpha\beta_2\gamma_2 - d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha\beta_n\gamma_n - d_n \end{pmatrix},$$

$$R_j = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{j!}\sum_{l=1}^{n}\beta_l\gamma_l^{-1} \end{pmatrix},$$

$(Q)_{ij} = 0$ for all i, j with the exception of $(Q)_{11} = (Q_1)_{11} = b_{nn}\sigma^2$, and

$$(Q)_{nn} = \sum_{l=1}^{n}\beta_l\gamma_l^{-1}\sum_{j=0}^{n-1}\frac{1}{j!}\int_0^\infty \frac{s^{j+1}}{j+1}|dK_{n-j}(s)| = \alpha\sum_{l=1}^{n}\frac{\beta_l}{\gamma_l}.$$

So, the matrix $P + Q$ is a diagonal matrix with

$$(P+Q)_{11} = \alpha\beta_1\gamma_1 + b_{nn}\sigma^2 - d_1, \quad (P+Q)_{ll} = \alpha\beta_l\gamma_l - d_l, \quad l = 2, \dots, n-1,$$
$$(P+Q)_{nn} = \alpha\left[\beta_n\left(\gamma_n + \frac{1}{\gamma_n}\right) + \sum_{l=1}^{n-1}\frac{\beta_l}{\gamma_l}\right] - d_n.$$

It is easy to see that $(P + Q)_{nn}$ reaches its minimum with respect to γ_n if $\gamma_n = 1$. In addition, via (4.12), (4.13) we have $\beta_1 = |a_{n-1}|b_{nn}$, $2\beta_n = d_n$. So, we can conclude that if there exist positive numbers $\gamma_1, \gamma_2, \dots, \gamma_{n-1}$ such that

$$\gamma_1 < \frac{1}{\alpha}\left(\frac{d_1}{\beta_1} - \frac{\sigma^2}{|a_{n-1}|}\right), \qquad \gamma_l < \frac{d_l}{\alpha\beta_l}, \quad l = 2, \dots, n-1,$$
$$\sum_{l=1}^{n-1}\frac{\beta_l}{\gamma_l} < \left(\frac{1}{\alpha} - 1\right)d_n, \qquad \alpha < 1, \tag{4.18}$$

then the matrix $P + Q$ is negative definite, and therefore the zero solution of (4.1) is asymptotically mean square stable.

Let us rewrite inequalities (4.18) in the form

$$0 < \alpha\left(\frac{d_1}{\beta_1} - \frac{\sigma^2}{|a_{n-1}|}\right)^{-1} < \frac{1}{\gamma_1}, \qquad \frac{\alpha\beta_l}{d_l} < \frac{1}{\gamma_l}, \quad l = 2, \dots, n-1,$$
$$\frac{\beta_1}{\gamma_1} + \sum_{l=2}^{n-1}\frac{\beta_l}{\gamma_l} < \left(\frac{1}{\alpha} - 1\right)d_n, \qquad \alpha < 1. \tag{4.19}$$

From the system of inequalities (4.19) it follows that

$$\alpha\beta_1\left(\frac{d_1}{\beta_1} - \frac{\sigma^2}{|a_{n-1}|}\right)^{-1} + \alpha\sum_{l=2}^{n-1}\frac{\beta_l^2}{d_l} < \frac{\beta_1}{\gamma_1} + \sum_{l=2}^{n-1}\frac{\beta_l}{\gamma_l} < \left(\frac{1}{\alpha} - 1\right)d_n. \tag{4.20}$$

So, if the condition

$$\alpha\beta_1\left(\frac{d_1}{\beta_1} - \frac{\sigma^2}{|a_{n-1}|}\right)^{-1} + \alpha\sum_{l=2}^{n-1}\frac{\beta_l^2}{d_l} < \left(\frac{1}{\alpha} - 1\right)d_n$$

or

$$\sigma^2 < |a_{n-1}| \left(\frac{d_1}{\beta_1} - \frac{\beta_1}{\Theta d_n - \sum_{l=2}^{n-1} \beta_l^2 d_l^{-1}} \right), \quad \sum_{l=2}^{n-1} \frac{\beta_l^2}{d_l} < \Theta d_n, \quad \Theta = \frac{1}{\alpha^2} - \frac{1}{\alpha}, \tag{4.21}$$

holds, then there exist positive numbers $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ such that (4.20) holds too, and the zero solution of (4.1) is asymptotically mean square stable.

So, we have proved the following.

THEOREM 4.1. *Let there exist some diagonal matrix D with the positive entries $d_1, \ldots, d_n$ such that matrix equation* (4.13) *has a positive definite solution B and that inequalities* (4.21) *hold. Then the zero solution of* (4.1) *is asymptotically mean square stable.*

Remark 4.1. Without loss of generality it is possible to put $d_n = 1$. If it is not so, then (4.13) can be divided on d_n. Thus all entries of the matrices D and B will be divided by d_n.

Remark 4.2. Note that condition (4.21) is correct also without assumption $\alpha > 0$. In fact, if $\alpha = 0$ (which means also that $z(x_t) \equiv 0$), then we have $\sigma^2 < |a_{n-1}| d_1/\beta_1$, which follows immediately from (4.15) and $\beta_1 = |a_{n-1}| b_{nn}$.

Remark 4.3. The stability condition obtained in Theorem 4.1 uses representation (4.7), where integrals in the right-hand side depend only on x_n for all i. Following the same procedure one can try to obtain other stability conditions using the representations where the right-hand side depends on x_m for $m \le n$. For example, for $n = 2$ we have

$$\int_0^\infty x_1(t-s) dK_1(s) = \beta_{01} x_1(t) - \beta_{11} x_2(t) + \frac{d}{dt} \int_0^\infty dK_1(s) \int_{t-s}^t (\tau - t + s) x_2(\tau) d\tau,$$

$$\int_0^\infty x_i(t-s) dK_i(s) = \beta_{0i} x_i(t) - \frac{d}{dt} \int_0^\infty dK_i(s) \int_{t-s}^t x_i(\tau) d\tau, \quad i = 1, 2,$$

$$\int_0^\infty x_2(t-s) dK_2(s) = \frac{d}{dt} \int_0^\infty x_1(t-s) dK_2(s).$$

4.2. Particular cases of condition (4.21). It is easy to see that stability condition (4.21) is the best one for those $d_1, \ldots, d_n$ for which the right-hand side of inequality (4.21) reaches its maximum. Let us consider some particular cases of condition (4.21) when it can be formulated immediately in terms of the parameters of considered equation (4.1).

4.2.1. Case $n = 1$. Equation (4.1) has the form

$$\dot{x}(t) = \int_0^\infty x(t-s) dK(s) + \sigma x(t-\tau) \dot{w}(t), \qquad t \ge 0. \tag{4.22}$$

For the functional $V_1(t, x_t) = x^2(t)$ similar to (4.17) (by conditions $\gamma_1 = 1$, $d_1 = 1$) we have

$$LV_1(t, x_t) \le (-1 + \alpha\beta_1) x^2(t) + b_{11} \sigma^2 x^2(t-\tau) + \beta_1 \int_0^\infty |dK(s)| \int_{t-s}^t x^2(\theta) d\theta,$$

where

$$\alpha = \alpha_{11} = \int_0^\infty s |dK(s)|, \quad a_0 = \beta_{01} = \int_0^\infty dK(s) < 0,$$

$$b_{11} = -\frac{1}{2a_0} = \frac{1}{2|\beta_{01}|} > 0, \quad \beta_1 = |b_{11} a_0| = \frac{1}{2}.$$

Stability condition (4.21) for (4.22) takes the form $\sigma^2 < 2|\beta_{01}|(1-\alpha)$. If, in particular, $dK(s) = -b\delta(s-h)ds$, $b > 0$, then $\alpha = bh$, $\beta_{01} = -b$ and the stability condition takes the form $\sigma^2 < 2b(1-bh)$. Note that the last condition follows also immediately from (3.15) for $\hat{\tau} = 0$, $S_m = S_M = b$, $A_0(h) = bh$, $A_1(h) = 0$.

4.2.2. Case $n = 2$. Equation (4.1) has the form

$$\ddot{x}(t) = \int_0^\infty x(t-s)dK_1(s) + \int_0^\infty \dot{x}(t-s)dK_2(s) + \sigma x(t-\tau)\dot{w}(t), \quad t \geq 0. \tag{4.23}$$

Following Remark 4.2 we will consider the corresponding matrix equation (4.13) with

$$A = \begin{pmatrix} 0 & 1 \\ a_1 & a_0 \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}. \tag{4.24}$$

Here $d > 0$, $a_0 = \beta_{02} - \beta_{11}$, $a_1 = \beta_{01}$, β_{ij} are defined by (4.5), and the entries of the matrix B are defined by (4.13) and are

$$b_{11} = \left(\frac{a_0}{2a_1} - \frac{1}{2a_0}\right)d + \frac{a_1}{2a_0}, \quad b_{12} = -\frac{d}{2a_1}, \quad b_{22} = \frac{d-a_1}{2a_0a_1}. \tag{4.25}$$

Necessary and sufficient conditions for the matrix B to be positive definite are

$$a_0 < 0, \qquad a_1 < 0. \tag{4.26}$$

Stability condition (4.21) takes the form

$$\sigma^2 < |a_1|\left(\frac{d}{\beta_1} - \frac{\alpha^2\beta_1}{1-\alpha}\right), \tag{4.27}$$

where

$$\alpha = \alpha_{12} + \frac{1}{2}\alpha_{21}, \qquad \beta_1 = \frac{d+|a_1|}{2|a_0|}, \tag{4.28}$$

α_{ij} are defined by (4.3).

Via (4.27), (4.28),

$$\sigma^2 < 2a_0a_1\left(\frac{d}{d+|a_1|} - \frac{\alpha^2(d+|a_1|)}{4a_0^2(1-\alpha)}\right). \tag{4.29}$$

The right-hand side of (4.29) reaches its maximum by $d = 2|a_0|\alpha^{-1}\sqrt{(1-\alpha)|a_1|} - |a_1|$. So, as a result we obtain the sufficient condition for asymptotic mean square stability of the zero solution of (4.23) in the form

$$\sigma^2 < 2|a_1|\left(|a_0| - \alpha\sqrt{\frac{|a_1|}{1-\alpha}}\right), \qquad \alpha < 1. \tag{4.30}$$

Example 4.1. Consider the equation

$$\ddot{x}(t) + a\dot{x}(t-h_1) + bx(t-h_2) + \sigma x(t-\tau)\dot{w}(t) = 0 \tag{4.31}$$

with $a > 0$, $b > 0$. Equation (4.31) is obtained from (4.23) if $dK_1(s) = -b\delta(s-h_2)ds$, $dK_2(s) = -a\delta(s-h_1)ds$. In this case $\alpha_{12} = ah_1$, $\alpha_{21} = bh_2^2$, $\alpha = ah_1 + \frac{1}{2}bh_2^2$, $\beta_{01} = -b$, $\beta_{02} = -a$, $\beta_{11} = -bh_2$.

Stability condition (4.30) takes the form

$$\sigma^2 < 2b\left(a - bh_2 - \alpha\sqrt{\frac{b}{1-\alpha}}\right), \qquad \alpha = ah_1 + \frac{1}{2}bh_2^2 < 1.$$

Example 4.2. *Consider the equation*

$$\ddot{x}(t) = ax(t) + b_1x(t-h_1) + b_2x(t-h_2) + \sigma x(t)\dot{w}(t) \tag{4.32}$$

that is obtained from (4.23) *if* $dK_1(s) = (a\delta(s)+b_1\delta(s-h_1)+b_2\delta(s-h_2))ds$, $dK_2(s) = 0$.

Equation (4.32) is a mathematical model of the controlled inverted pendulum by stochastic perturbations. Stability of this model was investigated in [3], where the condition of asymptotic mean square stability was obtained in the form (in the notation of this paper)

$$\sigma^2 < \frac{2|a_1|}{\beta}\left(1 - \frac{\alpha}{2}\left(1 + \sqrt{1+\beta^2}\right)\right), \tag{4.33}$$

where

$$\begin{gathered} \alpha = \frac{1}{2}(|b_1|h_1^2 + |b_2|h_2^2), \qquad \beta = \frac{|a_1|+1}{|a_0|}, \\ a_0 = -(b_1h_1 + b_2h_2) < 0, \qquad a_1 = a + b_1 + b_2 < 0. \end{gathered} \tag{4.34}$$

Let us show that condition (4.30), (4.34) is better than (4.33). It is enough to note that

$$\begin{aligned} &\beta\left(|a_0| - \alpha\sqrt{\frac{|a_1|}{1-\alpha}}\right) - \left(1 - \frac{\alpha}{2}\left(1+\sqrt{1+\beta^2}\right)\right) \\ &= |a_1| - \alpha\beta\sqrt{\frac{|a_1|}{1-\alpha}} + \frac{\alpha}{2}\left(1+\sqrt{1+\beta^2}\right) \\ &= \left(\sqrt{|a_1|} - \frac{\alpha\beta}{2\sqrt{1-\alpha}}\right)^2 + \frac{\alpha}{2}\left(1+\sqrt{1+\beta^2} - \frac{\alpha\beta^2}{2(1-\alpha)}\right) > 0. \end{aligned} \tag{4.35}$$

A positivity of the second summand in (4.35) easily follows from the condition (that is assumed in (4.33)) $\alpha(1+\sqrt{1+\beta^2}) < 2$.

4.2.3. Case $n = 3$. Equation (4.1) has the form

$$\dddot{x}(t) = \int_0^\infty x(t-s)dK_1(s) + \int_0^\infty \dot{x}(t-s)dK_2(s) + \int_0^\infty \ddot{x}(t-s)dK_3(s) + \sigma x(t-\tau)\dot{w}(t). \tag{4.36}$$

Via Remark 4.1 we will consider the corresponding matrix equation (4.13) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_2 & a_1 & a_0 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}. \tag{4.37}$$

Equation (4.13), (4.37) by conditions

$$a_i < 0, \quad i = 0, 1, 2, \qquad A_0 = a_0a_1 + a_2 > 0 \tag{4.38}$$

has a positive definite solution B with the entries

$$
\begin{aligned}
b_{11} &= \frac{1}{2}\left(\frac{a_1}{a_2}+\frac{a_0^2}{A_0}\right)d_1+\frac{a_0a_2d_2+a_2^2}{2A_0}, \qquad b_{12}=\frac{a_0^2a_1}{2a_2A_0}d_1+\frac{|a_2|d_2+a_1a_2}{2A_0},\\
b_{13} &= \frac{d_1}{2|a_2|}, \qquad b_{22}=\frac{a_0^3+a_2}{2a_2A_0}d_1+\frac{(a_0^2+|a_1|)d_2+a_1^2+a_0a_2}{2A_0},\\
b_{23} &= \frac{a_0^2}{2|a_2|A_0}d_1+\frac{|a_0|d_2+|a_2|}{2A_0}, \qquad b_{33}=\frac{a_0}{2a_2A_0}d_1+\frac{d_2+|a_1|}{2A_0}.
\end{aligned} \tag{4.39}
$$

Calculating $\beta_1=|a_2|b_{33}$, $\beta_2=|b_{13}+a_1b_{33}|$, we obtain the representation

$$\beta_l=\rho_{l1}d_1+\rho_{l2}d_2+\rho_{l3}, \qquad l=1,2, \tag{4.40}$$

where

$$\rho_{11}=\frac{|a_0|}{2A_0},\quad \rho_{12}=\frac{|a_2|}{2A_0},\quad \rho_{13}=\frac{a_1a_2}{2A_0},\quad \rho_{21}=\frac{1}{2A_0},\quad \rho_{22}=\frac{|a_1|}{2A_0},\quad \rho_{23}=\frac{a_1^2}{2A_0}. \tag{4.41}$$

So, stability condition (4.21) can be written in the form

$$\sigma^2<|a_2|\sup_{d_1>0,d_2>0,\beta_2^2d_2^{-1}<\Theta} f(d_1,d_2), \qquad f(d_1,d_2)=\frac{d_1}{\beta_1}-\frac{\beta_1}{\Theta-\beta_2^2d_2^{-1}}. \tag{4.42}$$

For the fixed a_i, $i=0,1,2$, using (4.38)–(4.42), the maximum of the function $f(d_1,d_2)$ can be obtained numerically.

Example 4.3. *Consider* (4.36) *with* $dK_j(s)=k_j\delta(s-h_j)ds$, $\alpha_{ij}=|k_j|h_j^i$, $\beta_{ij}=k_jh_j^i$, $j=1,2,3$, $i=0,1,2$. *Then*

$$
\begin{aligned}
a_0 &= \beta_{03}-\beta_{12}+\frac{1}{2}\beta_{21}=k_3-k_2h_2+\frac{1}{2}k_1h_1^2, \qquad a_1=\beta_{02}-\beta_{11}=k_2-k_1h_1,\\
a_2 &= \beta_{01}=k_1, \qquad A_0=\left(k_3-k_2h_2+\frac{1}{2}k_1h_1^2\right)(k_2-k_1h_1)+k_1,\\
\alpha &= \alpha_{13}+\frac{1}{2}\alpha_{22}+\frac{1}{6}\alpha_{31}=|k_3|h_3+\frac{1}{2}|k_2|h_2^2+\frac{1}{6}|k_1|h_1^3, \qquad \Theta=\frac{1}{\alpha^2}-\frac{1}{\alpha}.
\end{aligned}
$$

Put, for example, $h_1=h_2=h_3=0.1$, $k_1=-1$, $k_2=-2$, $k_3=-3$. Then $a_0=-2.805<0$, $a_1=-1.9<0$, $a_2=-1<0$, $A_0=4.3295>0$, $\alpha\approx 0.310<1$, $\Theta\approx 7.171$, $\rho_{11}\approx 0.324$, $\rho_{12}\approx 0.115$, $\rho_{13}\approx 0.219$, $\rho_{21}\approx 0.115$, $\rho_{22}\approx 0.219$, $\rho_{23}\approx 0.417$. Conditions (4.38) hold. The function $f(d_1,d_2)$ reaches its maximum by $d_1\approx 4.49$, $d_2\approx 0.54$. Stability condition (4.42) takes the form $\sigma^2<2.246$.

For $h_3=0.2$ and the same values of all other parameters, the function $f(d_1,d_2)$ reaches its maximum by $d_1\approx 0.75$, $d_2\approx 0.96$, and stability condition (4.42) takes the form $\sigma^2<0.1969$.

REFERENCES

[1] R. Bellman and K. Cooke, *Differential-Difference Equations*, Academic Press, New York, London, 1963.

[2] E. Beretta, V. Kolmanovskii, and L. Shaikhet, *Stability of epidemic model with time delays influenced by stochastic perturbations*, Math. Comput. Simulation, 45 (1998), pp. 269–277.

[3] P. Borne, V. Kolmanovskii, and L. Shaikhet, *Stabilization of inverted pendulum by control with delay*, Dynam. Systems Appl., 9 (2000), pp. 501–514.

[4] P. Borne, V. Kolmanovskii, and L. Shaikhet, *Steady-state solutions of nonlinear model of inverted pendulum*, in Proceedings of The Third Ukrainian-Scandinavian Conference in Probability Theory and Mathematical Statistics (Kiev, 1999), Theory Stoch. Process., 5 (1999), pp. 203–209.

[5] N. Bradul and L. Shaikhet, *Stability of the positive point of equilibrium of Nicholson's blowflies equation with stochastic perturbations: Numerical analysis*, Discrete Dyn. Nat. Soc., 2007 (2007), article 92959.

[6] T. A. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[7] T. Caraballo, J. Real, and L. Shaikhet, *Method of Lyapunov functionals construction in stability of delay evolution equations*, J. Math. Anal. Appl., 334 (2007), pp. 1130–1145.

[8] J. M. Cushing, *Integro-Differential Equations and Delay Models in Population Dynamics*, Lecture Notes in Biomath. 20, Springer-Verlag, Berlin, 1977.

[9] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[10] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, London, 1966.

[11] J. K. Hale, *Oscillations in neutral functional differential equations*, in Nonlinear Mechanics (Centro Internaz. Mat. Estivo (C.I.M.E.), I Ciclo, Bressanone, 1972), Edizioni Cremonese, Rome, 1973, pp. 97–111.

[12] J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[13] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[14] V. B. Kolmanovskii and A. D. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1992.

[15] V. B. Kolmanovskii and A. D. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[16] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations*, Academic Press, New York, 1986.

[17] V. B. Kolmanovskii and L. E. Shaikhet, *Control of Systems with Aftereffect*, Transl. Math. Monogr. 157, AMS, Providence, RI, 1996.

[18] V. B. Kolmanovskii and L. E. Shaikhet, *Stability of stochastic systems with aftereffect*, Avtomat. i Telemekh., no. 7, 1993, pp. 66–85 (in Russian); Automat. Remote Control, 54 (1993), pp. 1087–1107 (in English).

[19] V. B. Kolmanovskii and L. E. Shaikhet, *A method for constructing Lyapunov functionals for stochastic systems with aftereffect*, Differ. Uravn., 29 (1993), pp. 1909–1920, 2022 (in Russian); Differential Equations, 29 (1993), pp. 1657–1666 (in English).

[20] V. B. Kolmanovskii and L. E. Shaikhet, *New results in stability theory for stochastic functional-differential equations (SFDEs) and their applications*, in Proceedings of Dynamic Systems and Applications, Vol. 1 (Atlanta, 1993), Dynamic, Atlanta, GA, 1994, pp. 167–171.

[21] V. B. Kolmanovskii and L. E. Shaikhet, *A method for constructing Lyapunov functionals for stochastic differential equations of neutral type*, Diff. Uravn., 31 (1995), pp. 1851–1857, 1941 (in Russian); Differential Equations, 31 (1995), pp. 1819–1825 (in English).

[22] V. B. Kolmanovskii and L. E. Shaikhet, *General method of Lyapunov functionals construction for stability investigations of stochastic difference equations*, in Dynamical Systems and Applications, World Sci. Ser. Appl. Anal. 4, World Scientific Publishing, River Edge, NJ, 1995, pp. 397–439.

[23] V. B. Kolmanovskii and L. E. Shaikhet, *Asymptotic behavior of some discrete-time systems*, Avtomat. i Telemekh., 1996, no. 12, pp. 58–66 (in Russian); Automat. Remote Control, 57 (1996), pp. 1735–1742 (in English).

[24] V. B. Kolmanovskii and L. E. Shaikhet, *Some peculiarities of the general method of Lyapunov functionals construction*, Appl. Math. Lett., 15 (2002), pp. 355–360.

[25] V. B. Kolmanovskii and L. E. Shaikhet, *Construction of Lyapunov functionals for stochastic hereditary systems: A survey of some recent results*, Math. Comput. Modelling, 36 (2002), pp. 691–716.

[26] V. B. Kolmanovskii and L. E. Shaikhet, *About one application of the general method of Lyapunov functionals construction*, Internat. J. Robust Nonlinear Control, 13 (2003), pp. 805–818.

[27] V. B. Kolmanovskii and L. E. Shaikhet, *About some features of general method of Lyapunov functionals construction*, Stab. Control Theory Appl., 6 (2004), pp. 49–76.

[28] G. S. Ladde, V. Lakshmikantham, and B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987.

[29] M. Malek-Zavarei and M. Jamshidi, *Time Delay Systems: Analysis, Optimization and Applications*, North–Holland Systems Control Ser. 9, North–Holland, Amsterdam, 1988.

[30] S.-E. A. Mohammed, *Stochastic Functional Differential Equations*, Longman Scientific and Technical, Harlow, UK, 1986.

[31] A. D. Myshkis, *General Theory of Differential Equations with Delay*, Transl. Math. Monogr. 55, AMS, Providence, RI, 1951.

[32] B. Paternoster and L. E. Shaikhet, *Application of the general method of Lyapunov functionals construction for difference Volterra equations*, Comput. Math. Appl., 47 (2004), pp. 1165–1176.

[33] L. E. Shaikhet, *Modern state and development perspectives of Lyapunov functionals method in the stability theory of stochastic hereditary systems*, Theory Stoch. Process., 2 (1996), pp. 248–259.

[34] L. E. Shaikhet, *Necessary and sufficient conditions of asymptotic mean square stability for stochastic linear difference equations*, Appl. Math. Lett., 10 (1997), pp. 111–115.

[35] L. E. Shaikhet, *Stability of predator-prey model with aftereffect by stochastic perturbations*, Stab. Control Theory Appl., 1 (1998), pp. 3–13.

[36] L. E. Shaikhet, *Lyapunov functionals construction for stochastic difference second kind Volterra equations with continuous time*, Adv. Difference Equ., 2004 (2004), pp. 67–91.

[37] L. E. Shaikhet, *Construction of Lyapunov functionals for stochastic difference equations with continuous time*, Math. Comput. Simulation, 66 (2004), pp. 509–521.

[38] L. E. Shaikhet, *General method of Lyapunov functionals construction in stability investigations of nonlinear stochastic difference equations with continuous time*, Stoch. Dyn., 5 (2005), pp. 175–188.

[39] L. E. Shaikhet, *Stability of difference analogue of linear mathematical inverted pendulum*, Discrete Dyn. Nat. Soc., 2005 (2005), pp. 215–226.

[40] L. E. Shaikhet, *Some new aspect of Lyapunov type theorems for stochastic difference equations with continuous time*, Asian J. Control, 8 (2006), pp. 76–81.

[41] L. E. Shaikhet, *Stability of a positive point of equilibrium of one nonlinear system with aftereffect and stochastic perturbations*, Dynam. Systems Appl., 17 (2008), pp. 235–253.

[42] V. Volterra, *Lesons sur la theorie mathematique de la lutte pour la vie*, Gauthier-Villars, Paris, 1931.