

IMPROVED CONDITION FOR STABILIZATION OF CONTROLLED INVERTED PENDULUM UNDER STOCHASTIC PERTURBATIONS

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ABSTRACT. Known sufficient condition for stabilization of the controlled inverted pendulum under stochastic perturbations is improved via V.Kolmanovskii and L.Shaikhet general method of Lyapunov functionals construction.

1. Introduction. Statement of the problem. The problem of stabilization for the controlled inverted pendulum during many years is very popular among the researches (see, for instance [1], [2], [3], [4], [5], [7], [8], [13], [14], [15], [16], [17], [19], [22], [23]). The linearized mathematical model of the controlled inverted pendulum can be described by the linear differential equation of the second order

$$\ddot{x}(t) - ax(t) = u(t), \quad a > 0, \quad t \geq 0. \quad (1)$$

The classical way of stabilization [8] uses the control $u(t) = -b_1x(t) - b_2\dot{x}(t)$, $b_1 > a$, $b_2 > 0$. But this type of control which represents an instantaneous feedback is quite difficult to realize because usually we need some finite time to make measurements of the coordinates and velocities, to treat the results of the measurements and to implement them in the control action.

In [4], [5] the control $u(t)$ was proposed that does not depend on a velocity but depends on the previous values of the trajectory $x(s)$, $s \leq t$, and has the form

$$u(t) = \int_0^\infty dK(\tau)x(t - \tau). \quad (2)$$

The kernel $K(\tau)$ in (2) is a function of bounded variation on $[0, \infty]$ and the integral is understood in the Stieltjes sense. It means in particular that both distributed and discrete delays can be used depending on the concrete choice of the kernel $K(\tau)$.

The initial condition for system (1), (2) has the form

$$x(s) = \varphi(s), \quad \dot{x}(s) = \dot{\varphi}(s), \quad s \leq 0, \quad (3)$$

where $\varphi(s)$ is a given continuously differentiable function.

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It is supposed also that system (1) is under influence of stochastic perturbations type of

$$\ddot{x}(t) - (a + \sigma \dot{\xi}(t))x(t) = u(t), \quad (4)$$

where $\xi(t)$ is a standard Wiener process, σ is a constant.

Put $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$. Then equation (2)-(4) can be represented in the form of the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= ax_1(t) + \int_0^\infty dK(\tau)x_1(t-\tau) + \sigma x_1(t)\dot{\xi}(t), \end{aligned} \quad (5)$$

with the initial condition $x_1(s) = \varphi(s)$, $x_2(s) = \dot{\varphi}(s)$, $s \leq 0$.

Definition 1.1. The zero solution of system (5) is called mean square stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbb{E}|x_1(t)|^2 + \mathbb{E}|x_2(t)|^2 < \epsilon$ for any $t \geq 0$ provided that $\sup_{s \leq 0} \{\mathbb{E}|\varphi(s)|^2 + \mathbb{E}|\dot{\varphi}(s)|^2\} < \delta$. If, besides, $\lim_{t \rightarrow \infty} \{\mathbb{E}|x_1(t)|^2 + \mathbb{E}|x_2(t)|^2\} = 0$ for every initial function φ , then the zero solution of equation (5) is called asymptotically mean square stable.

Put

$$\begin{aligned} k_i &= \int_0^\infty \tau^i dK(\tau), \quad i = 0, 1, & k_2 &= \int_0^\infty \tau^2 |dK(\tau)|, \\ a_1 &= -(a + k_0), & p &= \frac{a_1 + 1}{k_1}. \end{aligned} \quad (6)$$

Theorem 1.2. [5]. Let $a_1 > 0$, $k_1 > 0$,

$$\sigma^2 < \frac{2a_1}{p} \left(1 - \frac{k_2}{4} \left(1 + \sqrt{1 + p^2} \right) \right). \quad (7)$$

Then the zero solution of system (5) is asymptotically mean square stable.

2. Improved stability condition. The following theorem gives improved stability condition for system (5).

Theorem 2.1. Let $a_1 > 0$, $k_1 > 0$,

$$\sigma^2 < 2a_1 \left(k_1 - k_2 \sqrt{\frac{a_1}{2(2 - k_2)}} \right). \quad (8)$$

Then the zero solution of system (5) is asymptotically mean square stable.

Proof. To prove asymptotic mean square stability of system (5) it is enough [9] to construct Lyapunov functional $V(t, x_t)$ satisfying the condition $LV(t, x_t) \leq -c|x(t)|^2$, where L is the generate operator of system (5). To construct corresponding Lyapunov functional we will use the general method of Lyapunov functionals construction (see [10], [11], [12], [20]). This method consists of four steps. Corresponding to the first step of the method transform system (5) in the following way. Since

$$\int_{t-\tau}^t x_2(s) ds = \int_{t-\tau}^t \dot{x}_1(s) ds = x_1(t) - x_1(t-\tau)$$

then using (6) we have

$$\int_0^\infty dK(\tau)x_1(t-\tau) = k_0x_1(t) - \int_0^\infty dK(\tau) \int_{t-\tau}^t x_2(s) ds.$$

Therefore from the second equation of system (5) via (6) it follows

$$\dot{x}_2(t) = -a_1x_1(t) - \int_0^\infty dK(\tau) \int_{t-\tau}^t x_2(s)ds + \sigma x_1(t)\dot{\xi}(t). \tag{9}$$

Using (6) we obtain

$$\frac{d}{dt} \int_0^\infty dK(\tau) \int_{t-\tau}^t (s-t+\tau)x_2(s)ds = k_1x_2(t) - \int_0^\infty dK(\tau) \int_{t-\tau}^t x_2(s)ds. \tag{10}$$

Subtracting (10) from (9) we transform system (5) to the form

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{z}(t) &= -a_1x_1(t) - k_1x_2(t) + \sigma x_1(t)\dot{\xi}(t), \end{aligned} \tag{11}$$

where

$$z(t) = x_2(t) - \int_0^\infty dK(\tau) \int_{t-\tau}^t (s-t+\tau)x_2(s)ds. \tag{12}$$

Following the second step of the general method of Lyapunov functionals construction we consider the auxiliary system of ordinary differential equations

$$\dot{y}_1(t) = y_2(t), \quad \dot{y}_2(t) = -a_1y_1(t) - k_1y_2(t). \tag{13}$$

The inequalities $a_1 > 0, k_1 > 0$ are the necessary and sufficient conditions for asymptotic stability of system (13). It means that for square form $w(y_1, y_2) = d_1y_1^2 + d_2y_2^2$ with positive coefficients d_1, d_2 there exists a positive definite Lyapunov function $v = p_{11}y_1^2 + 2p_{12}y_1y_2 + p_{22}y_2^2$ such that $\dot{v} = -w$. The system of equations for p_{11}, p_{12}, p_{22}

$$2a_1p_{12} = d_1, \quad 2(k_1p_{22} - p_{12}) = d_2, \quad p_{11} - k_1p_{12} - a_1p_{22} = 0, \tag{14}$$

has the solution:

$$p_{11} = \left(\frac{k_1}{2a_1} + \frac{1}{2k_1} \right) d_1 + \frac{a_1d_2}{2k_1}, \quad p_{12} = \frac{d_1}{2a_1}, \quad p_{22} = \frac{d_1 + a_1d_2}{2a_1k_1}. \tag{15}$$

Following the third step of the general method of Lyapunov functionals construction we will construct Lyapunov functional for system (11) in the form $V = V_1 + V_2$, where

$$V_1 = p_{11}x_1^2(t) + 2p_{12}x_1(t)z(t) + p_{22}z^2(t), \tag{16}$$

$z(t)$ and p_{11}, p_{12}, p_{22} are defined by (12), (15).

Let L be the generate operator of system (11). Calculating LV_1 via (11), (14), (16) we have

$$\begin{aligned} LV_1 &= 2(p_{11}x_1(t) + p_{12}z(t))x_2(t) + \sigma^2p_{22}x_1^2(t) \\ &\quad + 2(p_{12}x_1(t) + p_{22}z(t))(-a_1x_1(t) - k_1x_2(t)) \\ &= (-d_1 + \sigma^2p_{22})x_1^2(t) - d_2x_2^2(t) \\ &\quad + 2a_1p_{22} \int_0^\infty dK(\tau) \int_{t-\tau}^t (s-t+\tau)x_1(t)x_2(s)ds \\ &\quad + d_2 \int_0^\infty dK(\tau) \int_{t-\tau}^t (s-t+\tau)x_2(t)x_2(s)ds. \end{aligned}$$

For arbitrary $\gamma > 0$ we obtain

$$\begin{aligned}
 LV_1 &\leq (-d_1 + \sigma^2 p_{22})x_1^2(t) - d_2 x_2^2(t) \\
 &\quad + a_1 p_{22} \int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s-t+\tau) \left(\gamma x_1^2(t) + \frac{1}{\gamma} x_2^2(s) \right) ds \\
 &\quad + \frac{d_2}{2} \int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s-t+\tau) (x_2^2(t) + x_2^2(s)) ds \\
 &= \left(-d_1 + \sigma^2 p_{22} + \frac{k_2}{2} \gamma a_1 p_{22} \right) x_1^2(t) + \left(-d_2 + \frac{d_2 k_2}{4} \right) x_2^2(t) \\
 &\quad + \alpha \int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s-t+\tau) x_2^2(s) ds,
 \end{aligned} \tag{17}$$

where

$$\alpha = \frac{a_1 p_{22}}{\gamma} + \frac{d_2}{2}. \tag{18}$$

Following the fourth step of the general method of Lyapunov functionals construction we choose the functional V_2 in the form

$$V_2 = \frac{\alpha}{2} \int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s-t+\tau)^2 x_2^2(s) ds.$$

Then

$$LV_2 = \frac{\alpha k_2}{2} x_2^2(t) - \alpha \int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s-t+\tau) x_2^2(s) ds. \tag{19}$$

Via (17), (19) for the functional $V = V_1 + V_2$ we have

$$LV \leq \left(-d_1 + \sigma^2 p_{22} + \frac{k_2}{2} \gamma a_1 p_{22} \right) x_1^2(t) + \left(-d_2 + \frac{d_2 k_2}{4} + \frac{\alpha k_2}{2} \right) x_2^2(t). \tag{20}$$

If the expressions in the brackets in (20) are negative, i.e.,

$$+\sigma^2 p_{22} + \frac{k_2}{2} \gamma a_1 p_{22} < d_1, \quad \frac{d_2 k_2}{4} + \frac{\alpha k_2}{2} < d_2, \tag{21}$$

then the zero solution of equation (5) is asymptotically mean square stable.

Via (18), (21) we have

$$\frac{a_1 k_2 p_{22}}{d_2(2-k_2)} < \gamma < \frac{2(d_1 - \sigma^2 p_{22})}{a_1 k_2 p_{22}}, \quad k_2 < 2. \tag{22}$$

So, if the inequality

$$\frac{a_1 k_2 p_{22}}{d_2(2-k_2)} < \frac{2(d_1 - \sigma^2 p_{22})}{a_1 k_2 p_{22}} \tag{23}$$

holds then there exists $\gamma > 0$ such that (22) (and therefore (21)) holds too.

From (23) it follows

$$\sigma^2 < \frac{1}{p_{22}} \left(d_1 - \frac{(a_1 k_2 p_{22})^2}{2d_2(2-k_2)} \right).$$

Putting $d_1 = qd_2$ and using representation (15) for p_{22} we have

$$\sigma^2 < 2a_1 k_1 \left(\frac{q}{q+a_1} - A(q+a_1) \right), \quad A = \frac{k_2^2}{8k_1^2(2-k_2)}. \tag{24}$$

The right hand part of inequality (24) reaches its maximum by $q = \sqrt{a_1 A^{-1}} - a_1$. Substituting this q into (24) we obtain (8). The proof is completed. \square

To show that condition (8) is better than (7) it is enough to note that via (6)

$$\begin{aligned} & p \left(k_1 - k_2 \sqrt{\frac{a_1}{2(2-k_2)}} \right) - \left(1 - \frac{k_2}{4} \left(1 + \sqrt{1+p^2} \right) \right) \\ &= a_1 - pk_2 \sqrt{\frac{a_1}{2(2-k_2)}} + \frac{k_2}{4} \left(1 + \sqrt{1+p^2} \right) \\ &= \left(\sqrt{a_1} - \frac{pk_2}{2\sqrt{2(2-k_2)}} \right)^2 + \frac{k_2}{4} \left(1 + \sqrt{1+p^2} - \frac{p^2 k_2}{2(2-k_2)} \right) \\ &> 0. \end{aligned} \tag{25}$$

A positivity of the second summand in (25) easy follows from the condition (that is evidently supposed in (7)) $k_2 \left(1 + \sqrt{1+p^2} \right) < 4$.

Example 1. Consider control (2) with $dK(\tau) = (b_1\delta(\tau - h_1) + b_2\delta(\tau - h_2))d\tau$, where $\delta(\tau)$ is Dirac’s function. In this case equation (4), (2) takes the form

$$\ddot{x}(t) - ax(t) = b_1x(t - h_1) + b_2x(t - h_2) + \sigma x(t)\dot{\xi}(t). \tag{26}$$

Put here $a = 1, b_1 = 1, b_2 = -3, \sigma = 0.3$. In Figure 1 one can see that stability region in the space of the parameters (h_1, h_2) given by condition (8) (number 1) is appreciably more than stability region given by condition (7) (number 2).

To investigate how far the bound of stability region given by condition (8) is situated from the bound of the exact stability region numerical simulation of the solution of equation (26) was made. Similar to [22] the difference analogue of equation (26) was used in the form

$$x_{i+1} = x_i + \tau y_i, \quad y_{i+1} = y_i + \tau \left(ax_i + \sum_{l=1}^2 b_l x_{i-m_l} \right) + \sigma x_i \Delta \xi_{i+1}, \quad i \geq 0,$$

where $\tau > 0$ is the step of discretization, $x_i = x(t_i), y_i = \dot{x}(t_i), t_i = i\tau, h_1 = m_1\tau, h_2 = m_2\tau, \Delta \xi_{i+1} = \xi(t_{i+1}) - \xi(t_i)$. For numerical simulation of Wiener process trajectories was used a special algorithm from [18]. Earlier this algorithm was realized also in [6].

Numerical simulation of system (26) solution with $\tau = 0.01$ gives (see Figure 1) the points $A(0.07, 0.01), B(0.2, 0.06), C(0.4, 0.12), D(0.6, 0.17), E(0.8, 0.22), F(1.0, 0.24), G(1.2, 0.23), H(1.4, 0.15), I(1.6, 0.07), J(1.75, 0.01)$, with the following property: the curve passing through these points approximately shows the bound of the exact stability region. In all these points the solution of equation (26) is unstable. But inside of this region the solution of equation (26) is stable. For example, in Figure 2 fifty trajectories of the solution of equation (26) are shown that were obtained in the point $I(1.6, 0.07)$ by the initial condition $x(s) = 1, s \in [-1.6, 0]$. One can see that these trajectories fill whole space. In Figure 3 fifty trajectories of the solution of equation (26) are shown that were obtained in the point $I_0(1.58, 0.07)$ (that is situated enough close to the point $I(1.6, 0.07)$ but inside of the stability region) by the initial condition $x(s) = 7, s \in [-1.6, 0]$. One can see that all these trajectories go to zero. The same situation is observed in all other considered points.

In Figure 4 one can see the picture that is similar to Figure 1 but for $\sigma = 0.5$ with the points $A(0.13, 0.01), B(0.27, 0.06), C(0.4, 0.1), D(0.59, 0.15), E(0.78, 0.20), F(0.98, 0.22), G(1.21, 0.21), H(1.36, 0.16), I(1.52, 0.09), I_0(1.5, 0.09), J(1.66, 0.03)$.

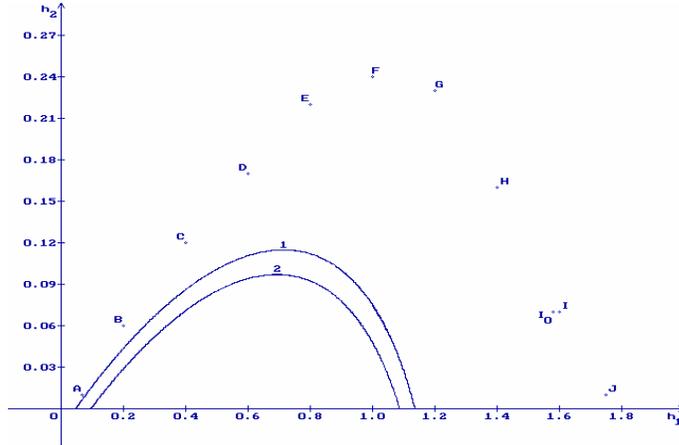


Figure.1. The bounds of stability regions given by condition (8) (number 1), condition (7) (number 2), exact stability region (A, B, C, D, E, F, G, H, I, J), $\sigma = 0.3$.

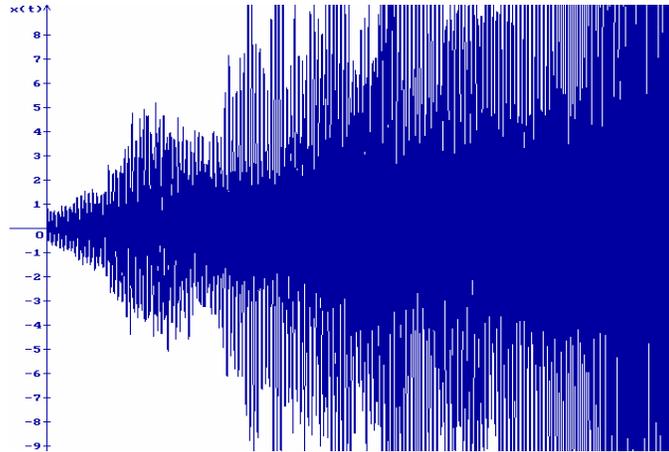


Figure 2. Fifty trajectories of the unstable solution of equation (26) in the point $I(1.60,0.07)$ by the initial condition $x(s) = 1, s \in [-1.6, 0]$.

Note that here (as well as in [6]) it is shown that numerical simulation allows to construct (approximately) the exact bound of stability region of considered model. The numerical simulation shows also (in Figures 1 and 4) that improved condition (8) is not only better, than old condition (7), but for enough small values of the delays h_1 and h_2 (the points A and B) it gives the bound that is enough close to the bound of the exact stability region. On the other hand for big values of delay

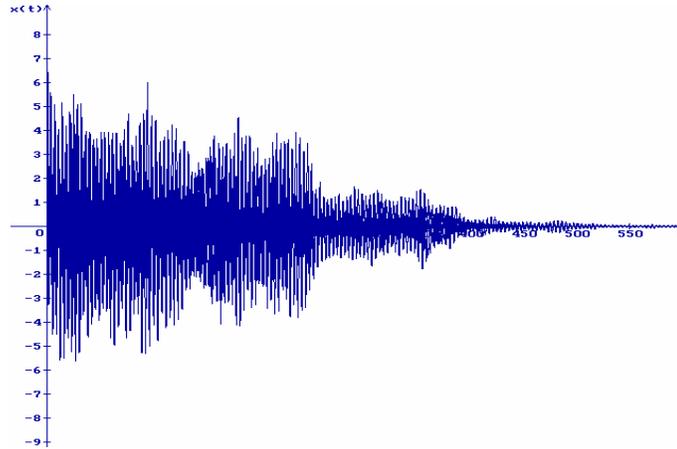


Figure 3. Fifty trajectories of the stable solution of equation (26) in the point $J_0(1.58, 0.07)$ by the initial condition $x(s)=7, s \in [-1.6, 0]$.

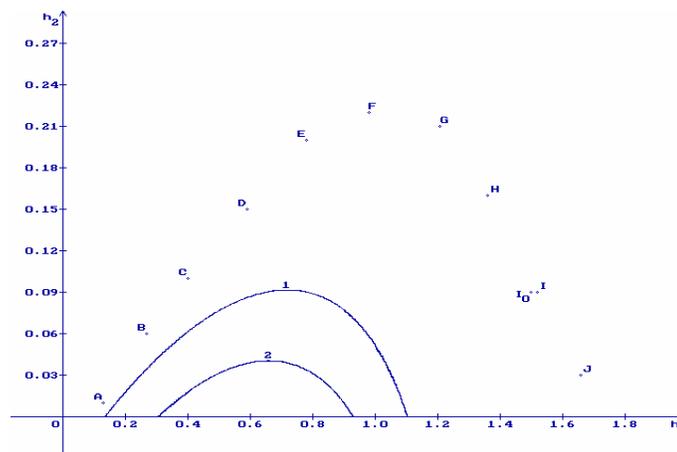


Figure 4. The bounds of stability regions given by condition (8) (number 1), condition (7) (number 2), exact stability region (A, B, C, D, E, F, G, H, I, J), $\sigma=0.5$.

h_1 (for example, from 0.7 to 1.1) the improved condition (8) gives the bound that is enough far from the bound of the exact stability region. Moreover, for larger values of h_1 (greater than 1.2) condition (8) does not give improvement at all.

The enough big distance between the bound of stability region given by condition (8) and the bound of the exact stability region shows in particular (in Figures 1 and 4) that there is a good stimulus for the further researches to improve present analytical methods of construction of stability conditions.

3. Nonlinear model. Consider the problem of stabilization for the nonlinear model of the inverted pendulum

$$\ddot{x}(t) - (a + \sigma \dot{\xi}(t)) \sin x(t) = u(t), \quad t \geq 0, \quad (27)$$

with control (2) and the initial condition (3). Similarly to (11) equation (27), (2) can be transformed to the system of nonlinear stochastic differential equations of neutral type

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{z}(t) &= -a_1 x_1(t) - k_1 x_2(t) + af(x_1(t)) + \sigma(x_1(t) + f(x_1(t)))\dot{\xi}(t), \end{aligned} \quad (28)$$

where $f(x) = \sin x - x$, a_1 , k_1 and $z(t)$ are defined by (6), (12).

Definition 3.1. The zero solution of system (28) is called stable in probability if for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists $\delta > 0$ such that the solution $(x_1(t), x_2(t)) = (x_1(t, \varphi), x_2(t, \varphi))$ of system (28) satisfies the condition $\mathbb{P}\{|x_1(t, \varphi)| + |x_2(t, \varphi)| > \epsilon_1\} < \epsilon_2$ for any initial function $\varphi(s)$ such that $\mathbb{P}\{\sup_{s \leq 0} (|\varphi(s)| + |\dot{\varphi}(s)|) \leq \delta\} = 1$.

Note that system (11) is the linear part of system (28) and since $|f(x)| \leq \frac{1}{6}|x|^3$ then the order of nonlinearity of system (28) equals 3. As it follows from [21] if the order of nonlinearity of the system under consideration is more than one then the condition which is a sufficient for asymptotic mean square stability of the linear part of this system is also a sufficient condition for stability in probability of the initial nonlinear system. So, if the conditions of Theorem 2.1 hold then the zero solution of system (28) is stable in probability.

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