Constitutive equations of elastic polycrystalline materials

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1. INTRODUCTION

In continuum mechanics we describe an elastic material element by a constitutive equation of the form

$$\boldsymbol{T} = \mathfrak{F}_{\boldsymbol{\kappa}}(\boldsymbol{F}), \tag{1}$$

where $\boldsymbol{\kappa}$ is the reference (local) placement of the material element, $\boldsymbol{F}$ is the gradient of some transplacement which brings the material element from placement $\boldsymbol{\kappa}$ to another placement $\boldsymbol{\chi}$, and $\boldsymbol{T}$ is the Cauchy stress in the configuration $\boldsymbol{\chi}$. If $\boldsymbol{\kappa}$ is an undistorted placement, the material symmetries of an elastic element are formulated by means of members of the rotation group Orth^+; precisely, we say that $\boldsymbol{Q} \in \mathrm{Orth}^+$ is a symmetry transformation relative to the placement $\boldsymbol{\kappa}$ if

$$\mathfrak{F}_{\boldsymbol{\kappa}}(\boldsymbol{F}\boldsymbol{Q}) = \mathfrak{F}_{\boldsymbol{\kappa}}(\boldsymbol{F}) \tag{2}$$

for all transplacement gradients $\boldsymbol{F}$ in the domain of $\mathfrak{F}_{\boldsymbol{\kappa}}$. If we denote by $\boldsymbol{Q}\boldsymbol{\kappa}$ the placement obtained from $\boldsymbol{\kappa}$ with a transplacement whose gradient is $\boldsymbol{Q}$, we can rewrite Eq. (2) as:

$$\mathfrak{F}_{\boldsymbol{Q}\boldsymbol{\kappa}}(\boldsymbol{F}) = \mathfrak{F}_{\boldsymbol{\kappa}}(\boldsymbol{F}) \tag{3}$$

for all possible transplacement gradients $\boldsymbol{F}$. Commenting on Eq. (3), Truesdell ([1], p. 238) writes: "... the response of the given body-point X is just the same in deformations with respect to two distinct reference placements. That is, in terms of the ideal experiments we sometimes invoke so as to visualize the assertions of the theory, no measurement of stress on the part of [the body] near X can distinguish $[\boldsymbol{Q\kappa}]$ from $[\boldsymbol{\kappa}]$". Accordingly, to determine directly if $\boldsymbol{Q}$ is a symmetry transformation relative to the placement $\boldsymbol{\kappa}$, we should conduct two stress-strain experiments, one on the placement $\boldsymbol{\kappa}$ and one on the placement $\boldsymbol{Q\kappa}$, and compare the results. The question arises whether we could obtain information on material symmetries indirectly by other means. For instance, suppose the material point X in question is that of a single crystal. In this regard we may invoke Neumann's principle (1885), the statement of which we quote from Love ([2], pp. 155–156):

> F. Neumann propounded a fundamental principle in regard to the physical behaviour of crystalline materials. It may be stated as follows:—Any kind of symmetry, which is possessed by the crystallographic form of a material, is possessed by the material in respect of every physical quality. ... The law is an induction from experience, and the evidence for it consists partly in *a posteriori* verifications.

Assuming the validity of this principle, by specifying the crystal class we can deduce at least partial information on the symmetries of the material point X in a single crystal (at a natural placement) without any measurement of stress.

Other examples abound. In some micromechanical models of elastic polycrystals (see Sayers [3], Hirao et al. [4]; cf. also Man and Paroni [5], Paroni and Man [6], and the references therein), the constitutive equation depends on the crystallographic texture of the polycrystal, i.e., the preferred orientations of the crystallites comprising the polycrystalline aggregate. Crystallographic texture is described mathematically by the orientation distribution function (ODF), which has the rotation group as its domain. It is the ODF which determines the elastic symmetries of the polycrystal in the micromechanical models cited above. Another example comes from the biomechanics of cancellous bones tissues. The cancellous structure is a lattice of narrow rods and plates of calcified bone tissue called trabeculae. The trabeculae are surrounded by marrow. A quantitative stereological measure of the microstructural arrangement of trabeculae and pores in cancellous bone is given in the theory of Cowin [7] by a second-order positive-definite tensor called the fabric tensor. The fabric tensor quantitatively characterizes the trabecular architecture. Cowin in his work on adaptive anisotropy assumes that all the anisotropy of cancellous bone is due to the anisotropy of its trabecular structure.

The examples above suggest that we should consider a constitutive equation of the form

$$\boldsymbol{T} = \mathfrak{F}_{\boldsymbol{\kappa}}(\boldsymbol{F}, \boldsymbol{\xi}), \tag{4}$$

where $\boldsymbol{\xi}$ describes the "significant" properties of the material in the given configuration. For example, in the case of the single crystal, we take $\boldsymbol{\xi} = (\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)$, where $\boldsymbol{d}_i$ are the lattice vectors of a unit cell of the crystallite. In the case of a polycrystalline material we equate $\boldsymbol{\xi}$ to the ODF of the aggregate, while in the case of cancellous bone tissues we put $\boldsymbol{\xi}$ as the fabric tensor. The idea is to introduce into the constitutive equation a new variable $\boldsymbol{\xi}$ which describes not only the symmetries of the material but also the nature of it. We call $\boldsymbol{\xi}$ the material descriptor and think of $\boldsymbol{\kappa}$ mainly as a geometrical variable: it specifies how the body element is geometrically placed in space, not how the material contents and structural details are arranged within the body element. For example, let us consider a material element in a given placement $\boldsymbol{\kappa}$ which undergoes a cycle of elasto-plastic deformation. Suppose the process takes the material element from the placement $\boldsymbol{\kappa}$ back to $\boldsymbol{\kappa}$ itself, but in such a way that the "significant" microstructure, whatever it is, is altered by the cycle of elasto-plastic deformation. In this case we have two identical geometrical placements "filled out by the same material" but in two different ways, and hence the material element is described by different material descriptors before and after the elasto-plastic deformation. The material symmetry of the element will also change as a result of the elasto-plastic deformation. Indeed we shall henceforth adopt a formulation in which material symmetry is completely characterized by the material descriptor. Thus we require $\mathfrak{F}_{\boldsymbol{\kappa}}$ in Eq. (4) to obey the identity

$$\mathfrak{F}_{\boldsymbol{Q}\boldsymbol{\kappa}}(\boldsymbol{F}, \boldsymbol{\xi}) = \mathfrak{F}_{\boldsymbol{\kappa}}(\boldsymbol{F}, \boldsymbol{\xi}) \tag{5}$$

for every $\boldsymbol{Q} \in \mathrm{Orth}^+$ and for all transplacement gradients $\boldsymbol{F}$. This last equation may be interpreted as follows: as far as the mechanical response is concerned, any two configurations related by a rigid displacement and with the same material descriptor are indistinguishable. (Cf. Man [8] for another discussion which motivates the adoption of Eq. (5) as a postulate for the special case where the material descriptor $\boldsymbol{\xi}$ is the ODF.)

In Section 2 we present a general and abstract setting in which the preceding ideas can be formulated precisely. There we study also some of the consequences. In particular we show that all rotations which leave the material descriptor $\boldsymbol{\xi}$ invariant belong to the symmetry group of the response function $\mathfrak{F}_{\boldsymbol{\kappa}}$. Thus we confirm the validity of Neumann's principle in our theory.

In Section 3 we specialize the theory developed in Section 2 to the the case of a polycrystalline material in linear elasticity with initial stress. To derive a constitutive equation in this context was the original motivation that led us to undertake the research reported in this paper. Man and Paroni [5, 9] and Paroni and Man [6] developed simple micromechanical models for prestressed and textured polycrystals which consist of crystallites of the same chemical composition. According to these models, given the ODF of the polycrystal and the second- and third-order elastic constants of the crystallites comprising the aggregate, the elasticity tensor $\mathbb{C}$ and the acoustoelastic tensor $\mathbb{D}$ of the polycrystalline aggregate can be evaluated by orientational averaging through appropriate integrations over the rotation group. Should we proceed directly, for each model hundreds of very time-consuming integrals would have to be evaluated by symbolic integration. Fortunately, not all the elastic constants of the polycrystal are independent if the crystallites comprising the aggregate and the aggregate itself should display crystal and texture symmetries. Should there be crystal and texture symmetries for us to exploit, identifying the independent components of the elasticity and acoustoelastic tensors and finding the relations which deliver the dependent ones would greatly cut down the number of integrals required to be evaluated. Seeking such relations is the goal of Section 3, where we shall carry out explicit computations for prestressed orthorhombic aggregates of cubic crystallites. In Subsection 3.1, we specialize the theory developed in Section 2 to the case in which the material descriptor is the pair $(\overset{\circ}{\boldsymbol{T}}, w)$, where $\overset{\circ}{\boldsymbol{T}}$ is the initial stress tensor and w the orientation distribution function. To be consistent with the notation we used in our previous work [5, 6, 9], we recast the equations of Section 2 so that the stress and deformation measures are the second Piola-Kirchhoff stress tensor and the Lagrangian strain, respectively. This choice has also the advantage that the response function will then automatically satisfy material frame-indifference. In Subsection 3.2 we further suppose that the deformation be small and the polycrystal be weakly-textured. Under these assumptions we linearize the problem and obtain a generalized version of Hartig's law [10, 11]. The linearized constitutive equation has the second Piola-Kirchhoff stress tensor written as the sum of three terms, namely (see Eq. (45) below): (i) the initial stress tensor $\overset{\circ}{\boldsymbol{T}}$; (ii) the fourth-order elasticity tensor $\mathbb{C}(w)$ as applied to the infinitesimal strain tensor; (iii) the sixth-order acoustoelastic tensor $\mathbb{D}(w)$ as applied to the initial stress tensor and to the infinitesimal strain tensor. In this subsection we deduce also the implications that Eq. (5), which postulates that the response function be indifferent to rotation of reference placement, would impose on the elasticity and acoustoelastic tensors. In particular, we derive two equations which should be satisfied by the fourth- and the sixth-order tensors $\mathbb{C}(w)$ and $\mathbb{D}(w)$. The equation concerning the fourth-order elasticity tensor $\mathbb{C}(w)$ was studied by Man [8]. In Subsection 3.3 we study the consequences of the equation concerning the sixth-order acoustoelastic tensor $\mathbb{D}(w)$ in the case of an orthorhombic aggregate of cubic crystallites. We obtain a model-independent formula (see Theorem 3.3.1 below) for the sixth-order acoustoelastic tensor $\mathbb{D}(w)$, which contains seven texture coefficients and nine material constants: four constants for the isotropic part and another five for the anisotropic part. Our formula shows explicitly the effects of texture on elastic response. The texture coefficients in our formula can be evaluated by X-ray analysis; the nine material constants can either be determined from experiments or, as the number of constants is sufficiently small, be estimated through micromechanical modeling.

The present paper generalizes, complements, and partly supercedes Section 5.2 of Man [8], where orthorhombic aggregates of cubic crystallites with $\overset{\circ}{\boldsymbol{T}} = \boldsymbol{0}$ are discussed. As such, it may be regarded as a sequel to and a further development of the earlier paper.

In what follows we adopt the Schoenflies notation for point groups and follow Gurtin ([12], p. 7) for notation of tensor spaces. In particular Sym, Orth$^+$, and Lin$^+$ denote the set of all symmetric second-order tensors, all rotations, and all second-order tensors with positive determinant, respectively.

2. ABSTRACT SETTING

Let $\mathcal{B}$ be a continuous body, X a point in $\mathcal{B}$, and $T_X\mathcal{B}$ the tangent space of $\mathcal{B}$ at X. In the theory of simple materials, $T_X\mathcal{B}$ is the body element at X. By abuse of language, we shall often refer to the body element $T_X\mathcal{B}$ simply as material point X. In what follows, whether X stands for a geometric point in $\mathcal{B}$ or the body element $T_X\mathcal{B}$ should be clear from the context.

Let V be the translation space of the three-dimensional Euclidean space E^3. A placement of the body element $T_X\mathcal{B}$ is a bijective linear mapping $\boldsymbol{\kappa} : T_X\mathcal{B} \to V$. In the classical constitutive theory of elastic materials, each placement $\boldsymbol{\kappa}$ defines a state of the material point X. Here we assume that the material point X has additional structure and enjoys the following properties:

1. In addition to $\boldsymbol{\kappa}$, a material descriptor $\boldsymbol{\xi}$ must be specified to define a state of X. For the abstract setting we need not make any special assumption on the nature of $\boldsymbol{\xi}$. We simply take

$$\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \boldsymbol{\xi}^2, \cdots, \boldsymbol{\xi}^n) \in \mathfrak{D}^1 \times \mathfrak{D}^2 \times \cdots \times \mathfrak{D}^n = \mathfrak{D}, \tag{6}$$

   where each set $\mathfrak{D}^i$ denotes the domain of some physical quantity $\boldsymbol{\xi}^i$. We call the set of all possible $(\boldsymbol{\kappa}, \boldsymbol{\xi})$ the state space Σ of the material point X.

2. Pertaining to each initial state $(\boldsymbol{\kappa}, \boldsymbol{\xi}) \in \Sigma$, there is a set $\mathcal{E}(\boldsymbol{\kappa}, \boldsymbol{\xi})$ of elastic transplacements open to X, each of which is completely characterized by some $\boldsymbol{F} \in \mathrm{Lin}^+$ and brings X to another state in Σ.

In general there are, of course, also inelastic transplacements open to X at each initial state in Σ, which will result in definite final states in Σ. For instance, consider the cold-rolling of aluminum sheets, where the material descriptor in question is crystallographic texture. On the passage of a superimposed ultrasonic wave, a material element with a given texture in the "hot band" (i.e., the end-product after hot-rolling from an ingot) will undergo time-dependent elastic deformations. The same material element, however, will end up with some "deformation texture" after the plate undergoes a further process of cold-rolling, which can often be specified simply by the thickness reduction, i.e., the ratio in percent when the final gauge of the cold-rolled sheet is compared with the thickness of the hot band. The transplacement due to the cold-rolling is inelastic. Assumption 2 above asserts that among all transplacements open to X at $(\boldsymbol{\kappa}, \boldsymbol{\xi})$, we can distinguish one single class whose members are elastic. Implicitly, however, we have assumed that there are inelastic processes (e.g., recrystallization and plastic deformation in the case of sheet metals) which will lead to change of state in Σ, for it is because of such possibilities that render it necessary for us to go beyond the classical constitutive theory of elastic materials here. Should we restrict our discussion to some $(\boldsymbol{\kappa}, \boldsymbol{\xi}) \in \Sigma$ as initial state of X and to the class of elastic transplacements that the material point may proceed from there, the classical formulation of elasticity would have been adequate.

Let $\boldsymbol{F}, \boldsymbol{G} \in \mathrm{Lin}^+$. For a given $\boldsymbol{F} \in \mathcal{E}(\boldsymbol{\kappa}, \boldsymbol{\xi})$, let us denote by $\mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa}, \boldsymbol{\xi})$ the state that results from the initial state $(\boldsymbol{\kappa}, \boldsymbol{\xi})$ after the elastic transplacement characterized by $\boldsymbol{F}$. We may regard $\mathbb{T}_{\boldsymbol{F}}$ as a one-to-one mapping defined on a subset $\mathcal{D}(\mathbb{T}_{\boldsymbol{F}})$ of Σ. In general, $\mathcal{D}(\mathbb{T}_{\boldsymbol{F}}) \neq \Sigma$. Let $\mathcal{R}(\mathbb{T}_{\boldsymbol{F}})$ be the range of $\mathbb{T}_{\boldsymbol{F}}$. As $\boldsymbol{F}$ is elastic, we assume that $(\mathbb{T}_{\boldsymbol{F}})^{-1} = \mathbb{T}_{\boldsymbol{F}^{-1}}$ and $\mathcal{D}(\mathbb{T}_{\boldsymbol{F}^{-1}}) = \mathcal{R}(\mathbb{T}_{\boldsymbol{F}})$. If $(\boldsymbol{\kappa}^*, \boldsymbol{\xi}^*) = \mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa}, \boldsymbol{\xi})$, clearly we have $\boldsymbol{\kappa}^* = \boldsymbol{F}\boldsymbol{\kappa}$, where $\boldsymbol{F}\boldsymbol{\kappa}$ denotes $\boldsymbol{F} \circ \boldsymbol{\kappa}$, and we write $\boldsymbol{\xi}^* = \mathrm{pr}_2 \circ \mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa}, \boldsymbol{\xi})$. Obviously, if $(\boldsymbol{\kappa}, \boldsymbol{\xi}) \in \mathcal{D}(\mathbb{T}_{\boldsymbol{F}})$ and $\mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa}, \boldsymbol{\xi}) \in \mathcal{D}(\mathbb{T}_{\boldsymbol{G}})$, then $(\boldsymbol{\kappa}, \boldsymbol{\xi}) \in \mathcal{D}(\mathbb{T}_{\boldsymbol{GF}})$ and $\mathbb{T}_{\boldsymbol{GF}}(\boldsymbol{\kappa}, \boldsymbol{\xi}) = \mathbb{T}_{\boldsymbol{G}}(\mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa}, \boldsymbol{\xi}))$.

We make the following assumptions on the state space Σ and on the mapping $\mathbb{T}_{\boldsymbol{Q}}$ when $\boldsymbol{Q}$ is a rotation:

1. For each $\boldsymbol{Q} \in \mathrm{Orth}^+$, $\mathcal{D}(\mathbb{T}_{\boldsymbol{Q}}) = \Sigma$, and $\mathrm{pr}_2 \circ \mathbb{T}_{\boldsymbol{Q}}$ is independent of $\boldsymbol{\kappa}$. Henceforth we write $\mathrm{pr}_2 \circ \mathbb{T}_{\boldsymbol{Q}}(\boldsymbol{\kappa}, \boldsymbol{\xi}) = \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi} = (\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi}^1, ..., \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi}^n)$.

2. If $(\boldsymbol{\kappa}, \boldsymbol{\xi}) \in \Sigma$, then $(\boldsymbol{Q}\boldsymbol{\kappa}, \boldsymbol{\xi}) \in \Sigma$ and $(\boldsymbol{\kappa}, \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi}) \in \Sigma$ for each $\boldsymbol{Q} \in \mathrm{Orth}^+$.

Definition 2.1. We say that a state $(\boldsymbol{\kappa}^*, \boldsymbol{\xi}^*) \in \Sigma$ is *elastically accessible* from the state $(\boldsymbol{\kappa}, \boldsymbol{\xi})$ if there is an elastic transplacement of $\mathcal{B}$ with gradient $\boldsymbol{F}$ at X such that

$$(\boldsymbol{\kappa}^*, \boldsymbol{\xi}^*) = \mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa}, \boldsymbol{\xi}). \tag{7}$$

$\square$

The relation "is elastically accessible from", which we denote by $\mathcal{E}_A$, is clearly an equivalence relation on Σ. Each of the equivalence classes of $\mathcal{E}_A$ would define a different "elastic material" in the classical constitutive theory.

The Cauchy stress $\boldsymbol{T}$ is a function of state on Σ, i.e., $\boldsymbol{T} : \Sigma \to \mathrm{Sym}$. Let us consider the restriction of $\boldsymbol{T}$ on the states in the equivalence class $(\boldsymbol{\kappa}, \boldsymbol{\xi})/\mathcal{E}_A$. Let $(\boldsymbol{\kappa}^\#, \boldsymbol{\xi}^\#) = \mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa}, \boldsymbol{\xi}) \in (\boldsymbol{\kappa}, \boldsymbol{\xi})/\mathcal{E}_A$. We write

$$\boldsymbol{T}(\boldsymbol{\kappa}^\#, \boldsymbol{\xi}^\#) = \boldsymbol{T}(\mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa}, \boldsymbol{\xi})) = \mathfrak{F}(\boldsymbol{F}; \boldsymbol{\kappa}, \boldsymbol{\xi}); \tag{8}$$

here $\mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\xi})$ is the constitutive function of the classical elastic material with state space $(\boldsymbol{\kappa}, \boldsymbol{\xi})/\mathcal{E}_A$, whose domain includes all $\boldsymbol{F} \in \mathrm{Lin}^+$ for which $\mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa}, \boldsymbol{\xi})$ is defined. Let $(\boldsymbol{\kappa}^*, \boldsymbol{\xi}^*)$ be another state in $(\boldsymbol{\kappa}, \boldsymbol{\xi})/\mathcal{E}_A$. As both $(\boldsymbol{\kappa}^\#, \boldsymbol{\xi}^\#)$ and $(\boldsymbol{\kappa}^*, \boldsymbol{\xi}^*)$ are elastically accessible from $(\boldsymbol{\kappa}, \boldsymbol{\xi})$, $(\boldsymbol{\kappa}^\#, \boldsymbol{\xi}^\#)$ is elastically accessible from $(\boldsymbol{\kappa}^*, \boldsymbol{\xi}^*)$. Let $(\boldsymbol{\kappa}^*, \boldsymbol{\xi}^*) = \mathbb{T}_{\boldsymbol{G}}(\boldsymbol{\kappa}, \boldsymbol{\xi})$ and $(\boldsymbol{\kappa}^\#, \boldsymbol{\xi}^\#) = \mathbb{T}_{\boldsymbol{F}^*}(\boldsymbol{\kappa}^*, \boldsymbol{\xi}^*)$. Then $\boldsymbol{\kappa}^* = \boldsymbol{G}\boldsymbol{\kappa}$, $\boldsymbol{F}^* = \boldsymbol{F}\boldsymbol{G}^{-1}$, and

$$\boldsymbol{T}(\boldsymbol{\kappa}^\#, \boldsymbol{\xi}^\#) = \mathfrak{F}(\boldsymbol{F}; \boldsymbol{\kappa}, \boldsymbol{\xi}) = \mathfrak{F}(\boldsymbol{F}^*; \boldsymbol{\kappa}^*, \boldsymbol{\xi}^*) = \mathfrak{F}(\boldsymbol{F}\boldsymbol{G}^{-1}; \mathbb{T}_{\boldsymbol{G}}(\boldsymbol{\kappa}, \boldsymbol{\xi})). \tag{9}$$

In particular, for $\boldsymbol{G} = \boldsymbol{Q} \in \mathrm{Orth}^+$, we have

$$\mathfrak{F}(\boldsymbol{F}; \boldsymbol{\kappa}, \boldsymbol{\xi}) = \mathfrak{F}(\boldsymbol{F}\boldsymbol{Q}^T; \boldsymbol{Q}\boldsymbol{\kappa}, \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi}). \tag{10}$$

Material frame-indifference further dictates that $\mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\xi})$ satisfies

$$\boldsymbol{Q}\mathfrak{F}(\boldsymbol{F}; \boldsymbol{\kappa}, \boldsymbol{\xi})\boldsymbol{Q}^T = \mathfrak{F}(\boldsymbol{Q}\boldsymbol{F}; \boldsymbol{\kappa}, \boldsymbol{\xi}) \qquad \forall \boldsymbol{Q} \in \mathrm{Orth}^+. \tag{11}$$

Combining the preceding two equations, we obtain

$$\boldsymbol{Q}\mathfrak{F}(\boldsymbol{F};\boldsymbol{\kappa},\boldsymbol{\xi})\boldsymbol{Q}^T = \mathfrak{F}(\boldsymbol{Q}\boldsymbol{F}\boldsymbol{Q}^T;\boldsymbol{Q}\boldsymbol{\kappa},\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi}) \tag{12}$$

for each $\boldsymbol{Q} \in \mathrm{Orth}^+$ and each triple $(\boldsymbol{F},\boldsymbol{\kappa},\boldsymbol{\xi})$, where $(\boldsymbol{\kappa},\boldsymbol{\xi}) \in \Sigma$ and $\boldsymbol{F}$ runs over all tensors in Lin^+ for which $\mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa},\boldsymbol{\xi})$ is defined.

Now we regard the response function $\mathfrak{F}$ as a function of three variables.

Definition 2.2. We say that the response function $\mathfrak{F}$ is *indifferent to rotation of reference placement* if

$$\mathfrak{F}(\cdot;\boldsymbol{Q}\boldsymbol{\kappa},\boldsymbol{\xi}) = \mathfrak{F}(\cdot;\boldsymbol{\kappa},\boldsymbol{\xi}) \tag{13}$$

holds true for every $\boldsymbol{Q} \in \mathrm{Orth}^+$ and $(\boldsymbol{\kappa},\boldsymbol{\xi}) \in \Sigma$. □

Remark 2.1. In the "classical" theory of constitutive equations the set of material descriptors $\mathfrak{D}$ is assumed to be empty. Hence only response functions associated with isotropic materials are indifferent to rotation of reference placement. For a "classical" anisotropic material it is possible to find a constitutive equation which satisfies Eq. (13) by considering, as material descriptors, a set of auxiliary tensors. For instance, consider a material point which is orthotropic relative to the placement $\boldsymbol{\kappa}$. If the planes orthogonal to $\boldsymbol{f}_i$, for $i = 1,2,3$, are the planes of orthotropic symmetry, we can take (cf. Boehler [13]) $\mathfrak{D} = \mathrm{Sym}\times\mathrm{Sym}\times\mathrm{Sym}$ and $\boldsymbol{\xi}^i = \boldsymbol{M}_i$, where $\boldsymbol{M}_i = \boldsymbol{f}_i \otimes \boldsymbol{f}_i$ (no summation on i). □

Hereafter we consider only response functions which are indifferent to rotation of reference placement. For such response functions $\mathfrak{F}$, we immediately obtain from Eq. (12) the equality

$$\boldsymbol{Q}\mathfrak{F}(\boldsymbol{F};\boldsymbol{\kappa},\boldsymbol{\xi})\boldsymbol{Q}^T = \mathfrak{F}(\boldsymbol{Q}\boldsymbol{F}\boldsymbol{Q}^T;\boldsymbol{\kappa},\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi}), \tag{14}$$

which is valid for each $\boldsymbol{Q} \in \mathrm{Orth}^+$ and each triple $(\boldsymbol{F},\boldsymbol{\kappa},\boldsymbol{\xi})$, where $(\boldsymbol{\kappa},\boldsymbol{\xi}) \in \Sigma$ and $\boldsymbol{F}$ runs over all tensors in Lin^+ for which $\mathbb{T}_{\boldsymbol{F}}(\boldsymbol{\kappa},\boldsymbol{\xi})$ is defined.

We now study the symmetries of the elastic material point X as the state space is restricted to $(\boldsymbol{\kappa}, \boldsymbol{\xi})/\mathcal{E}_A$. We call the subgroup

$$g^i_{\boldsymbol{\xi}} = \{\boldsymbol{Q} \in \mathrm{Orth}^+ : \boldsymbol{\xi}^i = \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}} \boldsymbol{\xi}^i\} \tag{15}$$

of Orth^+ the symmetry group of the i-th component of the material descriptor for the state $(\boldsymbol{\kappa}, \boldsymbol{\xi})$, and

$$g_{\boldsymbol{\xi}} = \{\boldsymbol{Q} \in \mathrm{Orth}^+ : \boldsymbol{\xi} = \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}} \boldsymbol{\xi}\} \tag{16}$$

the symmetry group of the material descriptor of the state $(\boldsymbol{\kappa}, \boldsymbol{\xi})$. Of course we have

$$g_{\boldsymbol{\xi}} = \bigcap_{i=1}^{n} g^i_{\boldsymbol{\xi}}. \tag{17}$$

We then define the symmetry group $g(\boldsymbol{\kappa}, \boldsymbol{\xi})$ of the response function $\mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\xi})$ as

$$g(\boldsymbol{\kappa}, \boldsymbol{\xi}) = \{\boldsymbol{Q} \in \mathrm{Orth}^+ : \mathfrak{F}(\cdot; \boldsymbol{Q}\boldsymbol{\kappa}, \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}} \boldsymbol{\xi}) = \mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\xi})\}. \tag{18}$$

The next proposition puts in relation the symmetry group of the response function $\mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\xi})$ and the symmetry group of the material descriptor $\boldsymbol{\xi}$ of $(\boldsymbol{\kappa}, \boldsymbol{\xi})$.

Proposition 2.1 The following inclusion holds:

$$g_{\boldsymbol{\xi}} \subset g(\boldsymbol{\kappa}, \boldsymbol{\xi}).$$

Moreover, for a given $\boldsymbol{\kappa}$, if the map $\boldsymbol{\xi} \mapsto \mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\xi})$ is one-to-one, we have

$$g(\boldsymbol{\kappa}, \boldsymbol{\xi}) = g_{\boldsymbol{\xi}}. \tag{19}$$

Proof. If $\boldsymbol{Q} \in g_{\boldsymbol{\xi}}$, then by the definition of $g_{\boldsymbol{\xi}}$ we have that $\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}} \boldsymbol{\xi} = \boldsymbol{\xi}$. Hence for all $\boldsymbol{F}$ in the domain of $\mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\xi})$ we have

$$\mathfrak{F}(\boldsymbol{F}; \boldsymbol{\kappa}, \boldsymbol{\xi}) = \mathfrak{F}(\boldsymbol{F}; \boldsymbol{\kappa}, \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}} \boldsymbol{\xi}) = \mathfrak{F}(\boldsymbol{F}; \boldsymbol{Q}\boldsymbol{\kappa}, \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}} \boldsymbol{\xi}),$$

where we have appealed to the assumption that $\mathfrak{F}$ is indifferent to rotation of reference placement (see Eq. (13)). The preceding equality tells us that $\boldsymbol{Q} \in g(\boldsymbol{\kappa}, \boldsymbol{\xi})$ and therefore $g_{\boldsymbol{\xi}} \subset g(\boldsymbol{\kappa}, \boldsymbol{\xi})$.

Now, for a given $\boldsymbol{\kappa}$, let us suppose that the map $\boldsymbol{\xi} \mapsto \mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\xi})$ is one-to-one. Let $\boldsymbol{Q} \in g(\boldsymbol{\kappa}, \boldsymbol{\xi})$. Since $\mathfrak{F}$ is indifferent to rotation of reference placement, we obtain from the definition of the symmetry group $g(\boldsymbol{\kappa}, \boldsymbol{\xi})$ the identity

$$\mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\xi}) = \mathfrak{F}(\cdot; \boldsymbol{Q}\boldsymbol{\kappa}, \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi}) = \mathfrak{F}(\cdot; \boldsymbol{\kappa}, \boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi}).$$

Now using the injectivity assumption we immediately find $\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}\boldsymbol{\xi} = \boldsymbol{\xi}$, and thence $g(\boldsymbol{\kappa}, \boldsymbol{\xi}) \subset g_{\boldsymbol{\xi}}$. $\square$

Remark 2.2. In the classical theory of linear elasticity of crystals, under natural placements the elasticity tensor of crystals with C_3, C_4, and T symmetry has D_3, D_4, and O symmetry, respectively ([14]; see also [15], Section 47). These furnish examples where the inclusion $g_{\boldsymbol{\xi}} \subset g(\boldsymbol{\kappa}, \boldsymbol{\xi})$ is strict. $\square$

3. ELASTIC POLYCRYSTALLINE MATERIALS

3.1 Preliminaries

In this section we assume that material point X is given in a placement $\boldsymbol{\kappa}$ with prestress $\overset{\circ}{\boldsymbol{T}}$, and that it is microscopically an aggregate of numerous crystallites $\mathcal{B}_\alpha$ of the same chemical composition. Hereafter we take $\boldsymbol{\kappa}$ as the reference placement and assume that the orientation of each crystallite $\mathcal{B}_\alpha$ at the given configuration is defined by a rotation $\boldsymbol{R}_\alpha$ from the orientation of a fixed reference crystallite (for more details see Man & Paroni [5]). Hence the orientations of the crystallites pertaining to X are represented by appropriate elements of the rotation group $\mathcal{G} = \mathrm{Orth}^+$.

In what follows we shall appeal to some standard notions and theorems in measure theory, as applied to probability measures and the Haar measure defined on the rotation group. All that we shall need can be found easily in texts on analysis [16, 17].

We describe the texture of the material point X (i.e., the totality of orientations of crystallites at X) by a probability (Borel) measure $\wp$ defined on the rotation group $\mathcal{G}$. We call $\wp$ the orientation measure of X at the configuration in question. Configurations with different arrangements of crystallites will generally have different orientation measures. For a Borel subset A of Orth^+, $\wp(A)$ gives the probability that we find, at the given configuration, a specific crystallite having an orientation represented by an element in A. Clearly, we have

$$\wp(\mathcal{G}) = 1. \tag{20}$$

In the literature [18, 19] on texture analysis, the texture of a macroscopic material point in a polycrystalline aggregate is usually described by the orientation distribution function (ODF) w. If the orientation measure $\wp$ is absolutely continuous with respect to the Haar measure $\wp_H$, w is given in terms of $\wp$ and $\wp_H$ by

$$w = \frac{1}{8\pi^2}\,\frac{d\wp}{d\wp_H}\,, \tag{21}$$

where $d\wp/d\wp_H$ denotes the Radon-Nikodym derivative of $\wp$ with respect to $\wp_H$.

Let the rotation group $\mathcal{G}$ be equipped with the measure $m = 8\pi^2\wp_H$, and let $L^2(\mathcal{G}, m)$ be the space of square-integrable functions on $\mathcal{G}$. Let

$$\begin{aligned} \mathcal{H}_0 &= \{f \in L^2(\mathcal{G}, m) : \int_{\mathcal{G}} f dm = 0\}, \\ \mathcal{H} &= \{w \in L^2(\mathcal{G}, m) : w = w_{\text{iso}} + f, \text{where} f \in \mathcal{H}_0\}, \end{aligned} \tag{22}$$

where $w_{\text{iso}} = 1/(8\pi^2)$ is the ODF of an aggregate whose crystallites have no preferred orientations.

The crystallites comprising the polycrystalline aggregate may possess a specific crystal symmetry. Let g_{cr} be the subgroup of rotations which render the reference crystallite invariant. Let $\hat{w}_{\boldsymbol{Q}} \in \mathcal{H}$ be defined by

$$\hat{w}_{\boldsymbol{Q}}(\boldsymbol{R}) = w(\boldsymbol{R}\boldsymbol{Q}) \tag{23}$$

for each $\boldsymbol{R} \in \mathcal{G}$, and let

$$\mathcal{H}_{\text{cr}} = \{w \in \mathcal{H} : w = \hat{w}_{\boldsymbol{Q}} \text{ for each } \boldsymbol{Q} \in g_{\text{cr}}\}, \tag{24}$$

which is clearly a closed subspace of $\mathcal{H}$. In what follows we take the material descriptor of X to be

$$\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \boldsymbol{\xi}^2) = (\overset{\circ}{\boldsymbol{T}}, w), \tag{25}$$

with domain

$$\mathfrak{D} = \text{Sym} \times \mathcal{H}_{\text{cr}}, \tag{26}$$

and we define, for each $\boldsymbol{Q} \in \text{Orth}^+$,

$$\mathcal{T}_{\boldsymbol{Q}} w(\boldsymbol{R}) = w(\boldsymbol{Q}^T\boldsymbol{R}), \qquad \forall\, \boldsymbol{R} \in \mathcal{G}, \tag{27}$$

$$\mathcal{T}_{\boldsymbol{Q}}\overset{\circ}{\boldsymbol{T}} = \boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T. \tag{28}$$

Remark 3.1.1 In the theoretical setting of the present paper (see Section 2 above), the choice of the material descriptor is of utmost importance. This choice will completely characterize the theory to be developed. Here, by choosing the pair $(\overset{\circ}{\boldsymbol{T}}, w)$ as the material descriptor, we have elected not to consider explicitly the effects that some other factors and microstructures (e.g., grain shape, grain size, etc.) might have on elastic response. In particular, for linear elasticity with initial stress, these other factors, thus set aside, can affect the constitutive equation only by their possible influence on the material constants. When $\overset{\circ}{\boldsymbol{T}} = \mathbf{0}$, the present theory reduces to one for which Paroni [20] has conducted a foundational study in the framework of stochastic homogenization. There the choice of w as material descriptor is linked to a major assumption, namely that the orientation field of the crystallites in the polycrystal be statistically independent.

Remark 3.1.2. The symmetry group $g^1_{\boldsymbol{\xi}}$ (see Eq. (15)) is, according to Eqs. (25) and (28),

$$g^1_{\boldsymbol{\xi}} = \{\boldsymbol{Q} \in \mathcal{G} : \overset{\circ}{\boldsymbol{T}} = \boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T\}. \tag{29}$$

From the knowledge of the group $g^1_{\boldsymbol{\xi}}$ it is possible to find (see Coleman & Noll [21]) the most general form that $\overset{\circ}{\boldsymbol{T}}$ may assume; for instance, if $g^1_{\boldsymbol{\xi}} = \mathcal{G}$, then $\overset{\circ}{\boldsymbol{T}}$ must be a multiple of the identity. The symmetry group $g^2_{\boldsymbol{\xi}}$ is the group of texture symmetry of X, namely

$$g^2_{\boldsymbol{\xi}} = g_{\text{tex}} = \{\boldsymbol{Q} \in \mathcal{G} : w(\boldsymbol{R}) = w(\boldsymbol{Q}^T\boldsymbol{R}), \quad \forall\, \boldsymbol{R} \in \mathcal{G}\}. \tag{30}$$

From our knowledge of g_{cr} and g_{tex}, it is possible to simplify the representation of w. In fact, if the group $\mathcal{G}$ is parametrized by means of the Euler angles (ψ, θ, ϕ), the definition of which we follow the convention adopted by Roe [19], it is well known (see Bunge [18, 22] or Roe [19]) that w can be expanded as an infinite series of the generalized spherical functions

$$w(\psi, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} W_{lmn} Z_{lmn}(\cos\theta) e^{-im\psi} e^{-in\phi}, \tag{31}$$

where Z_{lmn} are the normalized Jacobi polynomials [19]. We call W_{lmn} $(l \geq 1)$ the texture coefficients. Under an appropriate choice of spatial coordinate system and of reference crystallite, a nontrivial $g_{\rm cr}$ or $g_{\rm tex}$ will render many of the texture coefficients W_{lmn} zero. For instance, if $g_{\rm cr} = \mathcal{G}$ or $g_{\rm tex} = \mathcal{G}$, then only W_{000} will be different from zero and w will reduce to $w_{\rm iso}$. □

Remark 3.1.3. If W_{lmn} are the coefficients of the orientation distribution function w, and if $\widetilde{W}_{lmn}$ are the coefficients of $\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}} w$, then we have the following relation [8, 19]:

$$\widetilde{W}_{lmn} = \sqrt{\frac{2}{2l+1}} \sum_{p=-l}^{l} W_{lpn} Z_{lpm}(\cos\beta) e^{-ip\alpha} e^{-im\gamma}, \tag{32}$$

where (α, β, γ) are the Euler angles pertaining to $\boldsymbol{Q}^T$. □

To describe the constitutive equation of a polycrystalline material we find it convenient to fix the reference placement $\boldsymbol{\kappa}$ and use the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}$ and the Lagrangian strain $\boldsymbol{E}$, which are related to the Cauchy stress tensor $\boldsymbol{T}$ and the deformation gradient $\boldsymbol{F}$ by

$$\begin{aligned} \boldsymbol{\sigma} &= (\det \boldsymbol{F})\, \boldsymbol{F}^{-1} \boldsymbol{T} \boldsymbol{F}^{-T}, \\ \boldsymbol{E} &= \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}), \end{aligned} \tag{33}$$

respectively.

We consider a constitutive equation of the form

$$\boldsymbol{\sigma} = \mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{E}; \overset{\circ}{\boldsymbol{T}}, w) = \mathfrak{G}(\boldsymbol{E}; \boldsymbol{\kappa}, \overset{\circ}{\boldsymbol{T}}, w), \tag{34}$$

along with the condition

$$\mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{0}; \overset{\circ}{\boldsymbol{T}}, w) = \overset{\circ}{\boldsymbol{T}}. \tag{35}$$

From Eqs. (8) and (34), we easily find

$$\mathfrak{F}_{\boldsymbol{\kappa}}(\boldsymbol{F}; \overset{\circ}{\boldsymbol{T}}, w) = (\det \boldsymbol{F})^{-1}\, \boldsymbol{F} \mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{E}; \overset{\circ}{\boldsymbol{T}}, w) \boldsymbol{F}^T, \tag{36}$$

which can be used to translate the equations in Section 2 to their counterparts in the present context. Indeed Eq. (10), which governs rotation of reference placement, becomes

$$\mathfrak{G}_{\boldsymbol{Q\kappa}}(\boldsymbol{QEQ}^T; \boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T, \mathcal{T}_{\boldsymbol{Q}}w) = \boldsymbol{Q}\mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{E}; \overset{\circ}{\boldsymbol{T}}, w)\boldsymbol{Q}^T. \tag{37}$$

Material frame-indifference is automatically satisfied by the response function $\mathfrak{G}_{\boldsymbol{\kappa}}$. Indifference to rotation of reference placement is now embodied in the requirement that

$$\mathfrak{G}_{\boldsymbol{Q\kappa}}(\boldsymbol{E}; \overset{\circ}{\boldsymbol{T}}, w) = \mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{E}; \overset{\circ}{\boldsymbol{T}}, w), \qquad \forall\, \boldsymbol{Q} \in \mathrm{Orth}^+. \tag{38}$$

By appealing to Eqs. (37) and (38), we can easily rewrite Eq. (14) as follows:

$$\mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{QEQ}^T; \boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T, \mathcal{T}_{\boldsymbol{Q}}w) = \boldsymbol{Q}\mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{E}; \overset{\circ}{\boldsymbol{T}}, w)\boldsymbol{Q}^T, \qquad \forall\, \boldsymbol{Q} \in \mathrm{Orth}^+. \tag{39}$$

The preceding equation will be useful for later development.

3.2 Hartig's law for small elastic deformations

Henceforth we consider *small* elastic deformations superimposed on the reference placement $\boldsymbol{\kappa}$ of the material point X with prestress $\overset{\circ}{\boldsymbol{T}}$. For many materials, the empirical uniaxial stress-strain relation in question is given by Hartig's law (see Bell [10]). To introduce a three-dimensional generalization (see Man [11]) of this empirical law, we begin by applying Taylor's theorem to the constitutive function $\mathfrak{G}_{\boldsymbol{\kappa}}(\cdot; \cdot, \cdot)$.

We assume that $\mathfrak{G}_{\boldsymbol{\kappa}}(\cdot; \cdot, \cdot)$ be of class C^2 in the first two variables and of class C^1 in the last variable. Let $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}$ be the displacement gradient, and let $\boldsymbol{e} = (\boldsymbol{H} + \boldsymbol{H}^T)/2$ be the infinitesimal strain tensor. It is well known that

$$\boldsymbol{E} = \boldsymbol{e} + o(\|\boldsymbol{H}\|). \tag{40}$$

By applying Taylor's theorem to Eq. (34) and using Eqs. (35) and (40), we obtain

$$\boldsymbol{\sigma} = \mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{E}; \overset{\circ}{\boldsymbol{T}}, w) = \overset{\circ}{\boldsymbol{T}} + D_1\mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{0}; \overset{\circ}{\boldsymbol{T}}, w)[\boldsymbol{e}] + o(\|\boldsymbol{H}\|), \tag{41}$$

where $D_1\mathfrak{G}_{\boldsymbol{\kappa}}$ denotes the Frechét derivative of $\mathfrak{G}_{\boldsymbol{\kappa}}(\cdot;\cdot,\cdot)$ with respect to the first variable $\boldsymbol{E}$. Appealing once more to Taylor's theorem, we have

$$D_1\mathfrak{G}_{\boldsymbol{\kappa}}(\mathbf{0};\overset{\circ}{\boldsymbol{T}},w)[\cdot] = D_1\mathfrak{G}_{\boldsymbol{\kappa}}(\mathbf{0};\mathbf{0},w)[\cdot] + D_2D_1\mathfrak{G}_{\boldsymbol{\kappa}}(\mathbf{0};\mathbf{0},w)[\overset{\circ}{\boldsymbol{T}},\cdot] + o(\|\overset{\circ}{\boldsymbol{T}}\|), \tag{42}$$

where $D_2D_1\mathfrak{G}_{\boldsymbol{\kappa}}$ denotes the Frechét derivative of $D_1\mathfrak{G}_{\boldsymbol{\kappa}}(\cdot;\cdot,\cdot)$ with respect to the second variable $\overset{\circ}{\boldsymbol{T}}$. Letting

$$\begin{aligned}\mathbb{C}(w) &= D_1\mathfrak{G}_{\boldsymbol{\kappa}}(\mathbf{0};\mathbf{0},w),\\ \mathbb{D}(w) &= D_2D_1\mathfrak{G}_{\boldsymbol{\kappa}}(\mathbf{0};\mathbf{0},w),\end{aligned} \tag{43}$$

and using Eq. (42), we can rewrite Eq. (41) as

$$\boldsymbol{\sigma} = \mathfrak{G}_{\boldsymbol{\kappa}}(\boldsymbol{E};\overset{\circ}{\boldsymbol{T}},w) = \overset{\circ}{\boldsymbol{T}} + \mathbb{C}(w)[\boldsymbol{e}] + \mathbb{D}(w)[\overset{\circ}{\boldsymbol{T}},\boldsymbol{e}] + o(\|\boldsymbol{H}\|) + o(\|\overset{\circ}{\boldsymbol{T}}\|). \tag{44}$$

Comparing the preceding equation with Hartig's law, we discard [11] all terms which are of order $o(\|\boldsymbol{H}\|)$ and $o(\|\overset{\circ}{\boldsymbol{T}}\|)$ in Eq. (44). Thus the resulting constitutive equation reads

$$\boldsymbol{\sigma} = \overset{\circ}{\boldsymbol{T}} + \mathbb{C}(w)[\boldsymbol{e}] + \mathbb{D}(w)[\overset{\circ}{\boldsymbol{T}},\boldsymbol{e}]. \tag{45}$$

Predictions from this constitutive equation have been corroborated by experiments in acoustoelasticity (see [5, 9, 23] and the references therein). Substituting Eq. (45) into Eq. (39), we easily obtain

$$\mathbb{C}(\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}w)[\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T] = \boldsymbol{Q}\mathbb{C}(w)[\boldsymbol{e}]\boldsymbol{Q}^T \qquad \forall\,\boldsymbol{Q}\in\mathrm{Orth}^+, \tag{46}$$

$$\mathbb{D}(\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}w)[\boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T,\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T] = \boldsymbol{Q}\mathbb{D}(w)[\overset{\circ}{\boldsymbol{T}},\boldsymbol{e}]\boldsymbol{Q}^T \qquad \forall\,\boldsymbol{Q}\in\mathrm{Orth}^+, \tag{47}$$

which are consequences of the fact that the response function $\mathfrak{G}$ is indifferent to rotation of reference placement.

Remark 3.2.1. Paroni [20] examined the validity of Eq. (46) from the viewpoint of stochastic homogenization. In that framework properties of macroscopic response functions such as indifference to rotation of reference placement should follow as theorems, not as *a priori* assumptions. He proved Eq. (46) as a mathematical conclusion by making the assumption that the orientation field of the crystallites in the polycrystalline aggregate be statistically independent. $\square$

Remark 3.2.2. It follows immediately from Eqs. (46) and (47) that

$$\mathbb{C}(w)[\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T] = \boldsymbol{Q}\mathbb{C}(w)[\boldsymbol{e}]\boldsymbol{Q}^T \qquad \forall\, \boldsymbol{Q} \in g^2_{\boldsymbol{\xi}}, \tag{48}$$

$$\mathbb{D}(w)[\boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T] = \boldsymbol{Q}\mathbb{D}(w)[\overset{\circ}{\boldsymbol{T}}, \boldsymbol{e}]\boldsymbol{Q}^T \qquad \forall\, \boldsymbol{Q} \in g^2_{\boldsymbol{\xi}}. \tag{49}$$

Thus both the symmetry groups of $\mathbb{C}(w)$ and $\mathbb{D}(w)$ contain the texture symmetry group $g^2_{\boldsymbol{\xi}}$ as a subgroup. $\square$

Remark 3.2.3. Clearly $\boldsymbol{Q} = -\boldsymbol{I}$ also satisfies Eq. (49). Thus the sixth-order tensor $\mathbb{D}(w)$ is invariant under the subgroup of O(3) generated by $g^2_{\boldsymbol{\xi}}$ and $-\boldsymbol{I}$. In particular, if w is orthorhombic (i.e., $g^2_{\boldsymbol{\xi}} = D_2$ in the Schoenflies notation), then $\mathbb{D}(w)$ is orthotropic with D_{2h} as its group of symmetry. $\square$

Remark 3.2.4 In order for the response function $\mathfrak{G}_{\boldsymbol{\kappa}}(\cdot; \overset{\circ}{\boldsymbol{T}}, w)$ to be orthotropic, not only do we require the texture symmetry to be orthorhombic, but also the axes of orthorhombic symmetry of the texture to be the principal directions of $\overset{\circ}{\boldsymbol{T}}$. $\square$

By Taylor's theorem, we have

$$\mathbb{C}(w) = \mathbb{C}(w_{\text{iso}}) + D\mathbb{C}(w_{\text{iso}})[w - w_{\text{iso}}] + o(\|w - w_{\text{iso}}\|), \tag{50}$$

$$\mathbb{D}(w) = \mathbb{D}(w_{\text{iso}}) + D\mathbb{D}(w_{\text{iso}})[w - w_{\text{iso}}] + o(\|w - w_{\text{iso}}\|), \tag{51}$$

where $D\mathbb{C}(\cdot)$ and $D\mathbb{D}(\cdot)$ denote the Frechét derivative of $\mathbb{C}(\cdot)$ and $\mathbb{D}(\cdot)$, respectively. We say that a polycrystalline aggregate is *weakly-textured* for the material tensors $\mathbb{C}$ and $\mathbb{D}$ if discarding the $o(\|w - w_{\text{iso}}\|)$ term in Eqs. (50) and (51) still gives an adequate approximation to $\mathbb{C}(w)$ and $\mathbb{D}(w)$, respectively. For weakly-textured polycrystals, Eq. (46) was studied in detail by Man [8]. Here we shall concentrate on Eq. (47).

Setting $\mathbb{D}^{'}(w - w_{\text{iso}}) = D\mathbb{D}(w_{\text{iso}})[w - w_{\text{iso}}]$, we can simply write

$$\mathbb{D}(w) = \mathbb{D}(w_{\text{iso}}) + \mathbb{D}^{'}(w - w_{\text{iso}}) \tag{52}$$

for weakly-textured polycrystals. Using this equation and Eq. (47), we obtain (cf. Man [8] for details of similar arguments in the case of $\mathbb{C}$)

$$\begin{aligned} \mathbb{D}(w_{\text{iso}})[\boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T] &= \boldsymbol{Q}\mathbb{D}(w_{\text{iso}})[\overset{\circ}{\boldsymbol{T}}, \boldsymbol{e}]\boldsymbol{Q}^T, & (53) \\ \mathbb{D}^{'}(\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}(w - w_{\text{iso}}))[\boldsymbol{Q}\overset{\circ}{\boldsymbol{T}}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T] &= \boldsymbol{Q}\mathbb{D}^{'}(w - w_{\text{iso}})[\overset{\circ}{\boldsymbol{T}}, \boldsymbol{e}]\boldsymbol{Q}^T, & (54) \end{aligned}$$

for each $\boldsymbol{Q} \in \mathcal{G}$.

Remark 3.2.5. From Eq. (53) we deduce that $\mathbb{D}(w_{\text{iso}})$ is an isotropic sixth-order tensor. It follows immediately that $\mathbb{D}(w_{\text{iso}})$ has the following representation (cf. Man [11], Spencer [24])

$$\mathbb{D}(w_{\text{iso}})[\overset{\circ}{\boldsymbol{T}}, \boldsymbol{e}] = \beta_1 \mathrm{tr}\,(\overset{\circ}{\boldsymbol{T}})\mathrm{tr}\,(\boldsymbol{e})\boldsymbol{I} + \beta_2 \mathrm{tr}\,(\overset{\circ}{\boldsymbol{T}})\boldsymbol{e} + \beta_3(\mathrm{tr}\,(\boldsymbol{e})\overset{\circ}{\boldsymbol{T}} + \mathrm{tr}\,(\overset{\circ}{\boldsymbol{T}}\boldsymbol{e})\boldsymbol{I}) + \beta_4(\overset{\circ}{\boldsymbol{T}}\boldsymbol{e} + \boldsymbol{e}\overset{\circ}{\boldsymbol{T}}), \tag{55}$$

where the β_i, $i = 1, \ldots, 4$, are material constants. $\square$

For later reference, we conclude this subsection by displaying the statement of a theorem, originally due to Man [8], on the sixth-order tensor $\mathbb{D}^{'}(w - w_{\text{iso}})$.

Theorem 3.2.1 Let $w \in \mathcal{H}$, and let W_{lmn} be the texture coefficients pertaining to w. Suppose $\mathbb{D}^{'}(w - w_{\text{iso}})$ satisfies Eq. (54). Then the sixth-order tensor $\mathbb{D}^{'}(w - w_{\text{iso}})$ can depend only on those W_{lmn} with $1 \leq l \leq 6$.

3.3 A representation theorem

In this subsection we shall derive a representation theorem on $\mathbb{D}^{'}(w - w_{\text{iso}})$ for orthorhombic aggregates of cubic crystallites. For simplicity, we suppose that the material in question be hyperelastic, i.e., there exists a stored-energy function

$\Phi(\boldsymbol{E}; \overset{\circ}{\boldsymbol{T}}, w)$ such that $\boldsymbol{\sigma} = D_1\Phi(\boldsymbol{E}; \overset{\circ}{\boldsymbol{T}}, w)$. It follows that

$$\mathbb{D}(w)[\overset{\circ}{\boldsymbol{T}}, \boldsymbol{A}] \cdot \boldsymbol{B} = \mathbb{D}(w)[\overset{\circ}{\boldsymbol{T}}, \boldsymbol{B}] \cdot \boldsymbol{A} \qquad \forall \boldsymbol{A}, \boldsymbol{B} \in \mathrm{Sym}, \tag{56}$$

and for all $w \in \mathcal{H}_{\mathrm{cr}}$ and $\overset{\circ}{\boldsymbol{T}} \in \mathrm{Sym}$. By using Eqs. (52) and (56), we immediately find

$$\mathbb{D}^{'}(w - w_{\mathrm{iso}})[\overset{\circ}{\boldsymbol{T}}, \boldsymbol{A}] \cdot \boldsymbol{B} = \mathbb{D}^{'}(w - w_{\mathrm{iso}})[\overset{\circ}{\boldsymbol{T}}, \boldsymbol{B}] \cdot \boldsymbol{A} \qquad \forall \boldsymbol{A}, \boldsymbol{B} \in \mathrm{Sym}, \tag{57}$$

and for all $w \in \mathcal{H}_{\mathrm{cr}}$ and $\overset{\circ}{\boldsymbol{T}} \in \mathrm{Sym}$.

Consider a material point X which consists of cubic crystallites and is in a state defined by placement $\boldsymbol{\kappa}$, prestress $\overset{\circ}{\boldsymbol{T}}$, and an orthorhombic texture. As mentioned in Remark 3.2.3, the sixth-order tensor $\mathbb{D}^{'}(w - w_{\mathrm{iso}})$ in question is orthotropic, and it satisfies Eq. (57). It follows that $\mathbb{D}^{'}(w - w_{\mathrm{iso}})$ will have at most 39 independent components. We choose a Cartesian coordinate system whose coordinate planes coincide with the planes of orthotropic symmetry of $\mathbb{D}^{'}(w - w_{\mathrm{iso}})$. Let $\boldsymbol{i}_j$ $(j = 1, 2, 3,)$ be the coordinate basis vectors of the chosen coordinate system, and let $D^{'}{}_{ijklmn} = \mathbb{D}^{'}(w - w_{\mathrm{iso}})[\boldsymbol{i}_m \odot \boldsymbol{i}_n, \boldsymbol{i}_k \odot \boldsymbol{i}_l] \cdot \boldsymbol{i}_i \odot \boldsymbol{i}_j$ be the components of the sixth-order tensor $\mathbb{D}^{'}(w - w_{\mathrm{iso}})$, where $\boldsymbol{i}_m \odot \boldsymbol{i}_n$ is the symmetric part of $\boldsymbol{i}_m \otimes \boldsymbol{i}_n$. In what follows we shall sometimes find it convenient to use the Voigt notation and to write the components of the tensor $\mathbb{D}^{'}(w - w_{\mathrm{iso}})$ in short as $D^{'}{}_{ijk}$. Under this convention we may take the 39 independents components to be $D^{'}{}_{iij}$ for $i = 1, 2, \cdots, 6$ and $j = 1, 2, 3$, $D^{'}{}_{ijj}$ for $i = 1, 2, 3$ and $j = 4, 5, 6$, $D^{'}{}_{i3j}$ for $i = 1, 2$ and $j = 1, 2, 3$, $D^{'}{}_{12i}$ for $i = 1, 2, 3$, $D^{'}{}_{456}$, $D^{'}{}_{465}$ and $D^{'}{}_{564}$. We pick a reference orientation for the crystallites so that the three four-fold axes of cubic symmetry in the reference orientation agree with the coordinate axes of the chosen coordinate system.

Let w be the orientation distribution function pertaining to the texture at the given configuration, and let W_{lmn} be the corresponding texture coefficients. Then it is well known that the W_{lmn} are real and that $W_{lmn} = 0$ for $l = 1, 2, 3, 5$, and there are only three independent W_{4mn} and four independent W_{6mn}, which can be chosen to be W_{400}, W_{420}, W_{440}, W_{600}, W_{620}, W_{640} and W_{660}.

From Eq. (31), Theorem 3.2.1, and the fact that $\mathbb{D}^{'}(w-w_{\text{iso}})$ is linear in $w-w_{\text{iso}}$, we deduce that $\mathbb{D}^{'}(w-w_{\text{iso}})$ can be written as a linear combination of the texture coefficients as follows:

$$\begin{aligned}\mathbb{D}^{'}(w-w_{\text{iso}}) &= \mathbb{D}^{(40)}W_{400}+\mathbb{D}^{(42)}W_{420}+\mathbb{D}^{(44)}W_{440}\\ &\quad +\mathbb{D}^{(60)}W_{600}+\mathbb{D}^{(62)}W_{620}+\mathbb{D}^{(64)}W_{640}+\mathbb{D}^{(66)}W_{660},\end{aligned} \tag{58}$$

which we write schematically as

$$\mathbb{D}^{'}(w-w_{\text{iso}})=\sum_{l,m}\mathbb{D}^{(lm)}W_{lm0}. \tag{59}$$

Denoting by $\widetilde{W}_{lmn}$ the texture coefficients pertaining to $\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}w$, we have

$$\mathbb{D}^{'}(\boldsymbol{\mathcal{T}}_{\boldsymbol{Q}}(w-w_{\text{iso}}))=\sum_{l,m}\mathbb{D}^{(lm)}\widetilde{W}_{lm0}, \tag{60}$$

because the sixth-order tensor $\mathbb{D}^{'}=D\mathbb{D}(w_{\text{iso}})$ remains the same even if the texture has changed. From Eqs. (54), (59) and (60), we find

$$\boldsymbol{Q}\mathbb{D}^{'}(w-w_{\text{iso}})[\boldsymbol{Q}^T\mathring{\boldsymbol{T}}\boldsymbol{Q},\boldsymbol{Q}^T\boldsymbol{e}\boldsymbol{Q}]\boldsymbol{Q}^T=\sum_{l,m}\mathbb{D}^{(lm)}[\mathring{\boldsymbol{T}},\boldsymbol{e}]\widetilde{W}_{lm0}, \tag{61}$$

or

$$\sum_{l,m}\boldsymbol{Q}\mathbb{D}^{(lm)}[\boldsymbol{Q}^T\mathring{\boldsymbol{T}}\boldsymbol{Q},\boldsymbol{Q}^T\boldsymbol{e}\boldsymbol{Q}]\boldsymbol{Q}^TW_{lm0}=\sum_{l,m}\mathbb{D}^{(lm)}[\mathring{\boldsymbol{T}},\boldsymbol{e}]\widetilde{W}_{lm0}, \tag{62}$$

which should hold for all $\mathring{\boldsymbol{T}}$, $\boldsymbol{e}\in\text{Sym}$, $w\in\mathcal{H}_{\text{cr}}$ and all $\boldsymbol{Q}\in\text{Orth}^+$. Using Eq. (32), we recast Eq. (62) as

$$\begin{aligned}&\sum_{l,m}\boldsymbol{Q}\mathbb{D}^{(lm)}[\boldsymbol{Q}^T\mathring{\boldsymbol{T}}\boldsymbol{Q},\boldsymbol{Q}^T\boldsymbol{e}\boldsymbol{Q}]\boldsymbol{Q}^TW_{lm0}\\ &=\sum_{l,m}\sum_{p=-l}^{l}\sqrt{\frac{2}{2l+1}}\mathbb{D}^{(lm)}[\mathring{\boldsymbol{T}},\boldsymbol{e}]W_{lp0}Z_{lpm}(\cos\beta)e^{-ip\alpha}e^{-im\gamma}.\end{aligned} \tag{63}$$

Since the preceding identity holds for every $w\in\mathcal{H}_{\text{cr}}$, we conclude that

$$\begin{aligned}&\boldsymbol{Q}\mathbb{D}^{(lm)}[\boldsymbol{Q}^T\mathring{\boldsymbol{T}}\boldsymbol{Q},\boldsymbol{Q}^T\boldsymbol{e}\boldsymbol{Q}]\boldsymbol{Q}^T\\ &=\sum_{p}\sqrt{\frac{2}{2l+1}}\mathbb{D}^{(lp)}[\mathring{\boldsymbol{T}},\boldsymbol{e}]e^{-ip\gamma}(Z_{lmp}(\cos\beta)e^{-im\alpha}+Z_{l\bar{m}p}(\cos\beta)e^{im\alpha}),\end{aligned} \tag{64}$$

which holds for all $\mathring{\boldsymbol{T}}$, $\boldsymbol{e} \in$ Sym and all $\boldsymbol{Q} \in \mathrm{Orth}^{+}$. In Eq. (64), $\bar{m} = -m$, and the sum is over $p = 0, 2, 4$ if $l = 4$ and over $p = 0, 2, 4, 6$ if $l = 6$.

For each l, Eq. (64) is a constraint that the tensors $\mathbb{D}^{(lm)}$ must satisfy, and we may use this constraint to obtain information about these tensors. To this end, we notice first from Eq. (64) that tensors with different l satisfy equations which are independent from each other. Hence we may consider the problems of delineating the tensors $\mathbb{D}^{(4m)}$ and $\mathbb{D}^{(6m)}$ separately. For $l = 4$, there are three sixth-order tensors at issue, namely $\mathbb{D}^{(4m)}$ with $m = 0, 2, 4$, each of which has at most 39 independent components; hence we shall have 117 undetermined coefficients to contend with. For $l = 6$, there are four sixth-order tensors with a total of 156 undetermined coefficients.

As the number of undetermined coefficients is rather large, we try to simplify the problem by dividing the 39 independent components of $\mathbb{D}^{'}(w - w_{\text{iso}})$ into the following 9 groups:

$D^{'}_{111}$	$D^{'}_{112}$	$D^{'}_{123}$	$D^{'}_{441}$	$D^{'}_{442}$	$D^{'}_{155}$	$D^{'}_{144}$	$D^{'}_{456}$	$D^{'}_{121}$
$D^{'}_{222}$	$D^{'}_{113}$	$D^{'}_{132}$	$D^{'}_{552}$	$D^{'}_{443}$	$D^{'}_{166}$	$D^{'}_{255}$	$D^{'}_{465}$	$D^{'}_{122}$
$D^{'}_{333}$	$D^{'}_{221}$	$D^{'}_{231}$	$D^{'}_{663}$	$D^{'}_{551}$	$D^{'}_{244}$	$D^{'}_{366}$	$D^{'}_{564}$	$D^{'}_{131}$
	$D^{'}_{223}$			$D^{'}_{553}$	$D^{'}_{266}$			$D^{'}_{133}$
	$D^{'}_{331}$			$D^{'}_{661}$	$D^{'}_{344}$			$D^{'}_{232}$
	$D^{'}_{332}$			$D^{'}_{662}$	$D^{'}_{355}$			$D^{'}_{233}$

These groups are chosen in such a way that from the knowledge of one term in each group it is possible to deduce all the other terms in the same group by using Eq. (64) with $\boldsymbol{Q}$ restricted to the class of rotations with the Euler angles equal to integral multiples of $\pi/2$.

To delineate $\mathbb{D}^{(4m)}$, we select one component from each group above and express it as a linear combination of the texture coefficients W_{400}, W_{420}, $W_{4\bar{2}0}$ and W_{440}. While $W_{4\bar{2}0} = W_{420}$ for the given texture under the chosen coordinate system, we include the texture coefficient $W_{4\bar{2}0}$ here because later we shall use a rotation of

45° about the 3-axis, and the texture coefficients W_{420} and $W_{4\bar{2}0}$ are not necessarily equal for the texture thus rotated. In the following computations we shall find it more convenient to use Eq. (61) or (62) instead of Eq. (64). To simplify notation we write, for a given $\boldsymbol{Q} \in \mathrm{Orth}^+$,

$$\begin{aligned} \widetilde{D'}_{ijklmn} &= Q_{ip}Q_{jq}Q_{kr}Q_{ls}Q_{mt}Q_{nu}D'_{pqrstu}, & (65)\\ \widetilde{D'}^{(lm)}_{ijklmn} &= Q_{ip}Q_{jq}Q_{kr}Q_{ls}Q_{mt}Q_{nu}D'^{(lm)}_{pqrstu} & (66) \end{aligned}$$

whenever no confusion should arise, and we rewrite Eq. (62) as

$$\sum_{l,m} \widetilde{D'}^{(lm)}_{ijkpqn} W_{lm0} = \sum_{l,m} D'^{(lm)}_{ijkpqn} \widetilde{W}_{lm0}. \tag{67}$$

We pick in the first group the term D'_{333} and write it as

$$D'_{333} = aW_{400} + bW_{420} + cW_{4\bar{2}0} + dW_{440}. \tag{68}$$

Before we proceed further, we emphasize once again that in the chosen coordinate system $W_{420} = W_{4\bar{2}0}$ for the given texture. Consider a rotation $\boldsymbol{Q}$ with $\boldsymbol{Q}^T$ defined by the Euler angles $(\alpha, \beta, \gamma) = (\pi/2, 0, 0)$. Using Eq. (32), we find

$$\widetilde{W}_{400} = W_{400}, \qquad \widetilde{W}_{420} = -W_{420}, \qquad \widetilde{W}_{4\bar{2}0} = -W_{4\bar{2}0}, \qquad \widetilde{W}_{440} = W_{440}, \tag{69}$$

while from Eq. (65) we obtain $\widetilde{D'}_{333} = D'_{333}$. Substituting Eqs. (68), (69) into Eq. (67), we obtain the equation

$$aW_{400} - bW_{420} - cW_{4\bar{2}0} + dW_{440} = aW_{400} + bW_{420} + cW_{4\bar{2}0} + dW_{440}, \tag{70}$$

which immediately implies $b = -c$. Thus D'_{333} depends neither on W_{420} nor on $W_{4\bar{2}0}$. Similarly we can see that $D'_{123}, D'_{663}, D'_{366}$ and D'_{456} also do not depend on W_{420} and $W_{4\bar{2}0}$.

Now consider a rotation $\boldsymbol{Q}$ with $\boldsymbol{Q}^T$ defined by the Euler angles $(\alpha, \beta, \gamma) = (\pi/4, 0, 0)$. In this case we have

$$\widetilde{W}_{400} = W_{400}, \qquad \widetilde{W}_{420} = -iW_{420}, \qquad \widetilde{W}_{4\bar{2}0} = iW_{4\bar{2}0}, \qquad \widetilde{W}_{440} = -W_{440}. \tag{71}$$

Note that for this rotated texture $\widetilde{W}_{420}$ is different from $\widetilde{W}_{4\bar{2}0}$; this explain why we keep track of $W_{4\bar{2}0}$ even if it is equal to W_{420} for the given texture under the chosen coordinate system. For this rotation we have $\widetilde{D'}_{333} = D'_{333}$, so from Eq. (67) we find $d = 0$. Therefore $D'_{333} = aW_{400}$.

We claim that $D'_{331}, D'_{133}, D'_{355}$ and D'_{443} also do not depend on W_{440}. We shall prove our claim for D'_{331} only, as a similar treatment will work for the other components mentioned. We write

$$D'_{331} = eW_{400} + fW_{420} + gW_{4\bar{2}0} + hW_{440}, \tag{72}$$

and consider a rotation $\boldsymbol{Q}$ with $\boldsymbol{Q}^T$ defined by the Euler angles $(\alpha, \beta, \gamma) = (\pi/2, 0, 0)$. For this rotation we find $\widetilde{D'}_{331} = D'_{332}$. By using Eqs. (67) and (69), we obtain

$$D'_{332} = eW_{400} - fW_{420} - gW_{4\bar{2}0} + hW_{440}. \tag{73}$$

Now using a rotation $\boldsymbol{Q}^T$ defined by $(\alpha, \beta, \gamma) = (\pi/4, 0, 0)$, we find $\widetilde{D'}_{331} = (D'_{331} + D'_{332})/2$. Hence, by using Eqs. (67), (71), (72) and (73), we deduce not only that $h = 0$, but also that $f = g$.

In summary, from the discussion above we can write:

$$\begin{aligned}
D'_{333} &= a_1 W_{400}, \\
D'_{331} &= b_1 W_{400} + b_2 W_{420} + b_2 W_{4\bar{2}0}, \\
D'_{123} &= c_1 W_{400} + c_2 W_{440}, \\
D'_{663} &= d_1 W_{400} + d_2 W_{440}, \\
D'_{443} &= e_1 W_{400} + e_2 W_{420} + e_2 W_{4\bar{2}0}, \\
D'_{355} &= f_1 W_{400} + f_2 W_{420} + f_2 W_{4\bar{2}0}, \\
D'_{366} &= g_1 W_{400} + g_2 W_{440}, \\
D'_{456} &= h_1 W_{400} + h_2 W_{440}, \\
D'_{133} &= l_1 W_{400} + l_2 W_{420} + l_2 W_{4\bar{2}0},
\end{aligned} \tag{74}$$

where the parameters $a_1, b_1, ...,$ and l_2 need not all be independent.

It is possible to use Eqs. (67), (74), and 90° rotations to express all the remaining independent components of $\mathbb{D}^{'}(w - w_{\text{iso}})$ in terms of the 17 parameters $a_1, b_1, ..., l_2$ above, which we will now show. Using the rotation $\boldsymbol{Q}$ with $\boldsymbol{Q}^T$ defined by $(\alpha, \beta, \gamma) = (3\pi/2, \pi/2, \pi/2)$, we find from Eq. (32) that

$$\begin{aligned}
\widetilde{W}_{400} &= \frac{\sqrt{2}}{8}\left(\frac{3\sqrt{2}}{2}W_{400} + \sqrt{5}\,W_{420} + \sqrt{5}\,W_{4\bar{2}0} + \sqrt{35}\,W_{440}\right),\\
\widetilde{W}_{420} &= \frac{\sqrt{2}}{8}\left(\sqrt{5}\,W_{400} + \sqrt{2}\,W_{420} + \sqrt{2}\,W_{4\bar{2}0} - \sqrt{14}\,W_{440}\right),\\
\widetilde{W}_{4\bar{2}0} &= \frac{\sqrt{2}}{8}\left(\sqrt{5}\,W_{400} + \sqrt{2}\,W_{420} + \sqrt{2}\,W_{4\bar{2}0} - \sqrt{14}\,W_{440}\right),\\
\widetilde{W}_{440} &= \frac{\sqrt{2}}{16}\left(\sqrt{35}\,W_{400} - \sqrt{14}\,W_{420} - \sqrt{14}\,W_{4\bar{2}0} + \sqrt{2}\,W_{440}\right).
\end{aligned} \tag{75}$$

Since $\widetilde{D^{'}}_{333} = D^{'}_{222}$, we deduce from Eqs. (67) and (75) that

$$D^{'}_{222} = \frac{3}{8}a_1 W_{400} + \frac{\sqrt{10}}{8}a_1 W_{420} + \frac{\sqrt{10}}{8}a_1 W_{4\bar{2}0} + \frac{\sqrt{70}}{8}a_1 W_{440}. \tag{76}$$

Similarly, using the identities

$$\begin{aligned}
&\widetilde{D^{'}}_{331} = D^{'}_{221}, \quad && \widetilde{D^{'}}_{123} = D^{'}_{132}, \quad && \widetilde{D^{'}}_{663} = D^{'}_{552}, \quad && \widetilde{D^{'}}_{443} = D^{'}_{442},\\
&\widetilde{D^{'}}_{355} = D^{'}_{266}, && \widetilde{D^{'}}_{366} = D^{'}_{255}, && \widetilde{D^{'}}_{456} = D^{'}_{465}, && \widetilde{D^{'}}_{133} = D^{'}_{122},
\end{aligned}$$

we obtain expressions for $D^{'}_{221}, D^{'}_{132}, D^{'}_{552}, D^{'}_{442}, D^{'}_{266}, D^{'}_{255}, D^{'}_{465}$ and $D^{'}_{122}$. By using the rotation $\boldsymbol{Q}$ with $\boldsymbol{Q}^T$ defined by $(\alpha, \beta, \gamma) = (\pi/2, 0, 0)$ (see Eq. (69)), we can find $D^{'}_{111}, D^{'}_{112}, D^{'}_{231}, D^{'}_{441}, D^{'}_{551}, D^{'}_{166}, D^{'}_{144}, D^{'}_{564}$, and $D^{'}_{122}$ from $D^{'}_{222}, D^{'}_{221}, D^{'}_{132}, D^{'}_{552}, D^{'}_{442}, D^{'}_{266}, D^{'}_{255}, D^{'}_{465}$, and $D^{'}_{122}$, respectively.

Consider now the rotation $\boldsymbol{Q}$ with $\boldsymbol{Q}^T$ defined by $(\alpha, \beta, \gamma) = (0, \pi/2, 0)$. From Eq. (32) we obtain

$$\begin{aligned}
\widetilde{W}_{400} &= \frac{\sqrt{2}}{8}\left(\frac{3\sqrt{2}}{2}W_{400} - \sqrt{5}\,W_{420} - \sqrt{5}\,W_{4\bar{2}0} + \sqrt{35}\,W_{440}\right),\\
\widetilde{W}_{420} &= \frac{\sqrt{2}}{8}\left(-\sqrt{5}\,W_{400} + \sqrt{2}\,W_{420} + \sqrt{2}\,W_{4\bar{2}0} + \sqrt{14}\,W_{440}\right),\\
\widetilde{W}_{4\bar{2}0} &= \frac{\sqrt{2}}{8}\left(-\sqrt{5}\,W_{400} + \sqrt{2}\,W_{420} + \sqrt{2}\,W_{4\bar{2}0} + \sqrt{14}\,W_{440}\right),
\end{aligned}$$

$$\widetilde{W}_{440} = \frac{\sqrt{2}}{16}\left(\sqrt{35}\,W_{400} + \sqrt{14}\,W_{420} + \sqrt{14}\,W_{4\bar{2}0} + \sqrt{2}\,W_{440}\right). \tag{77}$$

From Eq. (67) and from our knowledge of $D'_{331}, D'_{443}, D'_{355}$ and D'_{133}, we find $D'_{113}, D'_{661}, D'_{155}$ and D'_{131}, respectively. Finally, to get the four components left, we use once more the rotation $\boldsymbol{Q}$ with $\boldsymbol{Q}^T$ defined by $(\alpha,\beta,\gamma) = (\pi/2, 0, 0)$ to obtain $D'_{223}, D'_{662}, D'_{244}$, and D'_{232} from our knowledge of D'_{113}, D'_{661}, D'_{155} and D'_{131}, respectively.

To proceed further, we use the rotation $\boldsymbol{Q}$ with $\boldsymbol{Q}^T$ defined by $(\alpha,\beta,\gamma) = (\pi/4, 0, 0)$ to find relations among the 17 parameters. Using Eq. (65) we find

$$\widetilde{D'}_{113} = \frac{1}{4}D'_{113} + \frac{1}{2}D'_{123} + D'_{663} + \frac{1}{4}D'_{223}. \tag{78}$$

The expressions for D'_{123} and D'_{663} in terms of the 17 parameters are given in Eq. (74). Here let us display those for D'_{113} and D'_{223}, which we have already found:

$$\begin{aligned} D'_{113} &= \left(\frac{3}{8}b_1 - \frac{\sqrt{10}}{4}b_2\right)W_{400} - \frac{\sqrt{10}}{40}\left(5b_1 - 2b_2\sqrt{10}\right)W_{420} \\ &\quad -\frac{\sqrt{10}}{40}\left(5b_1 - 2b_2\sqrt{10}\right)W_{4\bar{2}0} + \frac{\sqrt{70}}{280}\left(35b_1 + 14\sqrt{10}b_2\right)W_{440} \qquad (79) \\ D'_{223} &= \left(\frac{3}{8}b_1 - \frac{\sqrt{10}}{4}b_2\right)W_{400} + \frac{\sqrt{10}}{40}\left(5b_1 - 2b_2\sqrt{10}\right)W_{420} \\ &\quad +\frac{\sqrt{10}}{40}\left(5b_1 - 2b_2\sqrt{10}\right)W_{4\bar{2}0} + \frac{\sqrt{70}}{280}\left(35b_1 + 14\sqrt{10}b_2\right)W_{440}. \qquad (80) \end{aligned}$$

Substituting these four expressions into Eq. (78) and appealing to Eqs. (67) and (71), we arrive at the equation

$$\left(-\frac{3}{16}b_1 + \frac{\sqrt{10}}{8}b_2 + \frac{1}{2}c_1 + d_1\right)W_{400} + \frac{1}{8}\left(\frac{3\sqrt{70}}{2}b_1 + 6\sqrt{7}b_2 + 4c_2 + 8d_2\right)W_{440} = 0. \tag{81}$$

Since this equation holds for all $w \in \mathcal{H}_{\mathrm{cr}}$, we deduce that

$$\begin{aligned} -\frac{3}{16}b_1 + \frac{\sqrt{10}}{8}b_2 + \frac{1}{2}c_1 + d_1 &= 0, \\ \frac{3\sqrt{70}}{2}b_1 + 6\sqrt{7}b_2 + 4c_2 + 8d_2 &= 0, \qquad (82) \end{aligned}$$

which can be used to express, for instance, c_1 and c_2 in terms of b_1, b_2, d_1 and d_2. Now using the analog of Eq. (78) for the component $\widetilde{D'}_{131}$ and proceeding as we just did, we can find l_1 and g_2 as a function of b_1, b_2, d_1, d_2, g_1 and l_2. Similarly we can use: the equation for $\widetilde{D'}_{552}$ to write h_1, h_2 as linear combinations of e_1, e_2, d_1 and d_2; the equation for $\widetilde{D'}_{155}$ to express g_1, b_1 in terms of b_2, d_1, d_2, e_1, e_2, f_1, f_2 and l_2. In the same way, from the equation for $\widetilde{D'}_{111}$ we find b_2, d_2 as functions of a_1, d_1, e_1, e_2, f_2 and l_2; from the equation for $\widetilde{D'}_{663}$ we write f_1 in terms of a_1, d_1, e_1, e_2, f_2 and l_2; from the equation for $\widetilde{D'}_{166}$ we find f_2 as a function of a_1, d_1, e_1, e_2 and l_2; and finally from the equation for $\widetilde{D'}_{662}$ we express l_2 as a linear combination of a_1, d_1, e_1 and e_2.

Thence of the seventeen parameters at most four can be independent. The relations among the parameters that we found are:

$$
\begin{aligned}
b_1 &= -\frac{1}{2}a_1 + 4d_1 + 3e_1 - 2\sqrt{10}\,e_2, \\
b_2 &= \frac{\sqrt{10}}{8}\,a_1 + \sqrt{10}\,d_1 + \frac{5\sqrt{10}}{4}\,e_1 - 3e_2, \\
c_1 &= -\frac{1}{2}a_1 - 3d_1 - 2e_1, \\
c_2 &= -\sqrt{70}\,d_1 - \sqrt{70}\,e_1 + 4\sqrt{7}\,e_2, \\
d_2 &= -\sqrt{70}\,d_1 - \sqrt{70}\,e_1 + 4\sqrt{7}\,e_2, \\
f_1 &= -\frac{1}{4}\,a_1 - d_1 - \frac{11}{8}\,e_1 + \frac{\sqrt{10}}{4}\,e_2, \\
f_2 &= \frac{\sqrt{10}}{16}\,a_1 - \frac{\sqrt{10}}{4}\,d_1 - \frac{7\sqrt{10}}{32}\,e_1 + \frac{9}{8}\,e_2, \\
g_1 &= \frac{1}{16}\,a_1 + \frac{5}{2}\,d_1 + \frac{21}{8}\,e_1 - \frac{3\sqrt{10}}{4}\,e_2, \\
g_2 &= -\frac{\sqrt{70}}{16}\,a_1 - \frac{\sqrt{70}}{2}\,d_1 - \frac{3\sqrt{70}}{8}\,e_1 + \frac{\sqrt{7}}{2}\,e_2, \\
h_1 &= 2\,d_1 + \frac{19}{8}\,e_1 - \frac{3\sqrt{10}}{4}\,e_2, \\
h_2 &= -\frac{\sqrt{70}}{8}\,e_1 + \frac{\sqrt{7}}{2}\,e_2, \\
l_1 &= \frac{1}{4}\,a_1 + 2\,d_1 + 2\,e_1,
\end{aligned}
$$

$$l_2 = \frac{\sqrt{10}}{16}a_1 + \frac{\sqrt{10}}{2}d_1 + \frac{\sqrt{10}}{4}e_1 - e_2. \tag{83}$$

Our analysis above does not guarantee that the four parameters a_1, d_1, e_1 and e_2 are independent. It will be difficult to prove by brute force the independence of these four parameters by letting $\boldsymbol{Q}$ in Eq. (61) run over a complete set of generators of the rotation group. Instead we can answer this question easily by using group-theoretic methods (see Man [25] for an outline). A group-theoretic analysis, which will be presented elsewhere, indeed shows that there should be four independent parameters here. The expressions for the 39 independent components of $\mathbb{D}^{(4m)}$ $(m = 0, 2, 4)$ as linear combinations of four parameters are listed in the Appendix at the end of this paper. The same expressions can be derived alternatively by group-theoretic techniques [25], which give an independent corroboration of the correctness of the computations above.

Remark 3.3.1. The choice of a_1, d_1, e_1, and e_2 as the independent parameters is rather arbitrary, as a glance at Eq. (83) clearly reveals. This arbitrariness can be understood as follows. For a given texture, we find that the tensor $\mathbb{D}^{(40)}W_{400} + \mathbb{D}^{(42)}W_{420} + \mathbb{D}^{(44)}W_{440}$ must lie in a certain 4-dimensional subspace of the space of sixth-order tensors. The choice of four independent parameters is tantamount to selecting a basis in this subspace. □

Now we mimic what we did to find relations among the coefficients of $\mathbb{D}^{(6m)}$. By arguing as in the previous case, we see that we can write:

$$\begin{aligned}
D'_{333} &= a_2 W_{600}, \\
D'_{331} &= b_3 W_{600} + b_4 W_{620} + b_4 W_{6\bar{2}0}, \\
D'_{123} &= c_3 W_{600} + c_4 W_{640}, \\
D'_{663} &= d_3 W_{600} + d_4 W_{640}, \\
D'_{443} &= e_3 W_{600} + e_4 W_{620} + e_4 W_{6\bar{2}0}, \\
D'_{355} &= f_3 W_{600} + f_4 W_{620} + f_4 W_{6\bar{2}0},
\end{aligned}$$

$$
\begin{aligned}
D'_{366} &= g_3 W_{600} + g_4 W_{640}, \\
D'_{456} &= h_3 W_{600} + h_4 W_{640}, \\
D'_{133} &= l_3 W_{600} + l_4 W_{620} + l_4 W_{6\bar{2}0}.
\end{aligned} \tag{84}
$$

Note that the components above do not depend on W_{660}, but there are other components which do. Proceeding as in the previous case, we can express all the 39 independent components in terms of the 17 parameters above. Then, using the rotation $\boldsymbol{Q}$ with $\boldsymbol{Q}^T$ defined by $(\alpha, \beta, \gamma) = (\pi/4, 0, 0)$, we find the relations among these parameters. In this step we use different equations than before, so let us describe very briefly how we actually do the computations. First we use the equation for $\widetilde{D'}_{113}$ to find c_3 and c_4, after which we appeal to the equation for $\widetilde{D'}_{131}$ to determine l_3 and g_4, and we obtain h_3 and h_4 from the equation for $\widetilde{D'}_{552}$. Then by using the equation for $\widetilde{D'}_{155}$ we get g_3 and b_3, while b_4 and d_4 are deduced from the equation for $\widetilde{D'}_{111}$. At this point we obtain f_3 from the equation for $\widetilde{D'}_{663}$. From the equation for $\widetilde{D'}_{166}$ we find f_4 and e_3, and we deduce l_4 from the $\widetilde{D'}_{662}$ term. Finally, we obtain e_4 and d_3 from the equations for $\widetilde{D'}_{465}$ and $\widetilde{D'}_{132}$, respectively.

In this case, of the seventeen parameters only one is independent. The relations among the parameters are found to be:

$$
\begin{aligned}
b_3 = e_3 = f_3 = l_3 &= -\frac{1}{2} a_2, \\
c_3 = d_3 = g_3 = h_3 &= \frac{1}{8} a_2, \\
b_4 = -e_4 = f_4 = l_4 &= \frac{\sqrt{105}}{30} a_2, \\
c_4 = d_4 = g_4 = h_4 &= -\frac{\sqrt{14}}{8} a_2.
\end{aligned} \tag{85}
$$

In summary we have proved the following representation theorem:

Theorem 3.3.1 Let w be the orientation distribution function of an orthorhombic aggregate of cubic crystallites. Any sixth-order tensor $\mathbb{D}(w)$ which satisfies Eqs. (47)

and (56) can be written, to within terms linear in the texture coefficients, as

$$\mathbb{D}(w) = \mathbb{D}(w_{\mathrm{iso}}) + \mathbb{D}^{'}(w - w_{\mathrm{iso}}), \tag{86}$$

with $\mathbb{D}(w_{\mathrm{iso}})$ given by Eq. (55) and

$$\mathbb{D}^{'}(w - w_{\mathrm{iso}}) = \sum_{p=1}^{4} \gamma_p \mathbf{\Phi}^{(p)}(W_{4mn}) + \gamma_5 \mathbf{\Phi}^{(5)}(W_{6mn}), \tag{87}$$

where γ_i $(i = 1, ..., 5)$ are material constants and $\mathbf{\Phi}^{(j)}$ $(j = 1, ..., 5)$ are explicitly determined sixth-order tensors which are linear in the indicated texture coefficients. The components of $\mathbf{\Phi}^{(j)}$ assume their simplest form if we choose a spatial coordinate system whose coordinate planes coincide with the planes of orthotropic symmetry of $\mathbb{D}^{'}(w - w_{\mathrm{iso}})$ and we pick a reference orientation for the crystallites so that the three four-fold axes of cubic symmetry in the reference orientation agree with the coordinate axes of the chosen spatial coordinate system. The nontrivial components of $\mathbf{\Phi}^{(j)}$, under the aforementioned choice of spatial coordinate system and of reference orientation, are displayed explicitly in the Appendix. □

Remark 3.3.2 In the case of orthorhombic texture, Eq. (49) and Remark 3.2.3 together assert that $\mathbb{D}(w)$ is an orthotropic tensor; hence in general it has 39 independent components, which are functions of w. By the decomposition $\mathbb{D}(w) = \mathbb{D}(w_{\mathrm{iso}}) + \mathbb{D}^{'}(w - w_{\mathrm{iso}})$, we observe that $\mathbb{D}(w_{\mathrm{iso}})$, which is isotropic, involves 4 material constants (see Eq. (55)), and that $\mathbb{D}^{'}(w - w_{\mathrm{iso}})$ is an orthorhombic tensor with 39 constants. When the components of $\mathbb{D}^{'}(w - w_{\mathrm{iso}})$ are each expressed as a linear combination of the seven independent texture coefficients (see Eq. (58)), the number of undetermined coefficients becomes $39 \times 7 = 273$, and there remain 7 texture coefficients. Hence, to specify $\mathbb{D}(w)$ we need 277 constants and 7 texture coefficients. Now, with Theorem 3.3.1 in hand, this number drops down to 9 material constants (4 for the isotropic part and 5 for the anisotropic part) and 7 texture coefficients. There are at least two micromechanical models [6] in linear elasticity with initial stress where determination of one material constant will involve a number of time-consuming symbolic integrations. For these models Theorem 3.3.1 and its counterpart for $\mathbb{C}(w)$ (see Man [8]) are very helpful, for they greatly reduce the labor required in deriving the constitutive equation for prestressed orthorhombic aggregates of cubic crystallites. $\square$

To recapitulate, the present paper together with the earlier paper by Man [8] furnish an explicit representation formula to the constitutive equation

$$\boldsymbol{\sigma} = \mathring{\boldsymbol{T}} + \mathbb{C}(w)[\boldsymbol{e}] + \mathbb{D}(w)[\mathring{\boldsymbol{T}}, \boldsymbol{e}]. \tag{88}$$

for orthorhombic aggregates of cubic crystallites in linear elasticity with initial stress. See Theorem 3.3 of Man [8] and Theorem 3.3.1 above for the explicit expression for $\mathbb{C}(w)$ and $\mathbb{D}(w)$, respectively.

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Appendix

We express the 39 components of the sixth-order tensor $\mathbb{D}^{'}(w - w_{iso})$ as

$$D^{'}{}_{ijk} = \sum_{p=1}^{3} \left(\gamma_1 \frac{\alpha_{ijk}^p}{16} + \gamma_2 \frac{\phi_{ijk}^p}{32} + \gamma_3 \frac{\delta_{ijk}^p}{8\sqrt{10}} + \gamma_4 \frac{\xi_{ijk}^p}{4} \right) A_p + \gamma_5 \sum_{p=1}^{4} \frac{\eta_{ijk}^p}{16} F_p, \tag{89}$$

where $\gamma_1 = a_1, \gamma_2 = e_1, \gamma_3 = e_2, \gamma_4 = d_1, \gamma_5 = a_2$,

$$A_1 = W_{400}, \quad A_2 = 2\sqrt{10}\, W_{420}, \quad A_3 = \sqrt{70}\, W_{440}, \tag{90}$$

and

$$F_1 = W_{600}, \quad F_2 = \sqrt{105}\, W_{620}/15, \quad F_3 = \sqrt{14}\, W_{640}, \quad F_4 = \sqrt{231}\, W_{660}. \tag{91}$$

The coefficients $\alpha_{ijk}^p, \xi_{ijk}^p, \phi_{ijk}^p, \delta_{ijk}^p$, and η_{ijk}^p are listed in the table below. The sub-indices ijk are listed in the left column. Thus we have $\alpha_{111}^1 = 6$, $\xi_{221}^2 = 4$, etc.

$D^{'}$	α^1	α^2	α^3	ξ^1	ξ^2	ξ^3	ϕ^1	ϕ^2	ϕ^3	δ^1	δ^2	δ^3	η^1	η^2	η^3	η^4
111	6	-2	2	0	0	0	0	0	0	0	0	0	-5	15	-3	1
112	2	0	-2	16	-4	0	136	-32	-8	-120	32	-8	-1	1	1	-1
113	-8	2	0	-4	0	4	-64	8	32	0	8	-32	6	-16	2	0
221	2	0	-2	16	4	0	136	32	-8	-120	-32	-8	-1	-1	1	1
222	6	2	2	0	0	0	0	0	0	0	0	0	-5	-15	-3	-1
223	-8	-2	0	-4	0	4	-64	-8	32	0	-8	-32	6	16	2	0
331	-8	2	0	16	4	0	96	40	0	-160	-24	0	-8	16	0	0
332	-8	-2	0	16	-4	0	96	-40	0	-160	24	0	-8	-16	0	0
333	16	0	0	0	0	0	0	0	0	0	0	0	16	0	0	0
123	-8	0	0	-12	0	-4	-64	0	-32	0	0	32	2	0	-2	0
144	-4	-1	0	-5	-3	1	-21	-21	9	-5	11	-7	2	0	-2	0
155	-4	1	0	1	0	-1	1	2	-9	-15	2	7	6	-16	2	0
166	1	0	-1	-4	1	0	-34	9	-2	30	-7	-2	-1	1	1	-1
244	-4	-1	0	1	0	-1	1	-2	-9	-15	-2	7	6	16	2	0
255	-4	1	0	-5	3	1	-21	21	9	-5	-11	-7	2	0	-2	0
266	1	0	-1	-4	-1	0	-34	-9	-2	30	7	-2	-1	-1	1	1
344	-4	-1	0	-4	1	0	-44	7	0	20	-9	0	-8	-16	0	0
355	-4	1	0	-4	-1	0	-44	-7	0	20	9	0	-8	16	0	0
366	1	0	-1	10	0	-2	84	0	-12	-60	0	4	2	0	-2	0
456	0	0	0	8	0	0	76	0	-4	-60	0	4	2	0	-2	0
121	4	-1	0	8	-2	0	44	-12	4	-20	4	4	-1	1	1	-1
122	4	1	0	8	2	0	44	12	4	-20	-4	4	-1	-1	1	1
131	-1	0	1	-2	0	2	4	-4	12	20	-4	-4	6	-16	2	0
132	-3	-1	-1	-22	2	-2	-164	20	-12	140	-28	4	2	0	-2	0
133	4	1	0	8	2	0	64	8	0	0	-8	0	-8	16	0	0
231	-3	1	-1	-22	-2	-2	-164	-20	-12	140	28	4	2	0	-2	0
232	-1	0	1	-2	0	2	4	4	12	20	4	-4	6	16	2	0
233	4	-1	0	8	-2	0	64	-8	0	0	8	0	-8	-16	0	0
441	0	0	0	-16	-4	0	-140	-28	-4	140	28	4	2	0	-2	0
442	0	0	0	0	0	0	12	4	4	20	4	-4	6	16	2	0
443	0	0	0	0	0	0	32	0	0	0	8	0	-8	-16	0	0
551	0	0	0	0	0	0	12	-4	4	20	-4	-4	6	-16	2	0
552	0	0	0	-16	4	0	-140	28	-4	140	-28	4	2	0	-2	0
553	0	0	0	0	0	0	32	0	0	0	-8	0	-8	16	0	0
661	0	0	0	0	0	0	12	-4	4	-20	4	4	-1	1	1	-1
662	0	0	0	0	0	0	12	4	4	-20	-4	4	-1	-1	1	1
663	0	0	0	4	0	-4	0	0	-32	0	0	32	2	0	-2	0
465	0	0	0	3	1	1	11	13	9	-5	-11	-7	2	0	-2	0
564	0	0	0	3	-1	1	11	-13	9	-5	11	-7	2	0	-2	0

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