MORLEY SEQUENCES IN ABSTRACT ELEMENTARY CLASSES

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Abstract. In this paper, we explore stability results in the new context of tame abstract elementary classes with the amalgamation property. The main results is:

Theorem 0.1. Let K be a tame abstract elementary class satisfying the amalgamation property without maximal models. There exists a cardinal $\mu_0(K)$ such that for every $\mu \geq \mu_0(K)$ and every $M \in K_{>\mu}$, $A, I \subseteq M$ such that $|I| \geq \mu^+ > |A|$, if K is Galois stable in μ, then there exists $J \subseteq I$ of cardinality μ^+, Galois indiscernible indiscernible over A. Moreover J can be chosen to be a Morley sequence over A.

This result strengthens Claim 4.16 of [Sh394] as we do not assume categoricity. This is also an improvement of a result from [GrLe] concerning the existence of indiscernible sequences.

This is used to get a stability spectrum theorem.

Introduction

Already in the fifties model theorists studied non-elementary classes of structures (e.g. Jónsson [Jo1], [Jo2] and Fraïssé [Fr]). In [Sh88], Shelah introduced the framework of abstract elementary classes and embarked on the ambitious program of developing a classification theory for Abstract Elementary Classes. For a survey of some of the basics, see [Gr2] or Chapter 13 of [Gr1]. While much is known about abstract elementary classes, especially when K is an AEC under the additional assumption that there exists a cardinal $\lambda > \mathrm{Hanf}(K)$ such that K is categorical in λ, little progress has been made towards a fulledged stability theory. One of the open problems from [Sh394] (Remark 4.10 (1)) is to identify of a good (forking-like) notion of independence for abstract elementary classes. This is open even for classes that have the amalgamation property and are categorical above the Hanf number. In [Sh394], several weak notions of independence are introduced under the assumption that the class is categorical. Among these notions is the Galois-theoretic notion of non-splitting. Here we study the notion of

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non-splitting in a more general context than categorical AEC: Tame stable classes. We plan to use Morley sequences for non-splitting as a bootstrap to define a dividing-like concept for these classes.

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1. Background

Definition 1.1. Let K be a class of structures all in the same similarity type $L(K)$, and let $\prec_K$ be a partial order on K. The ordered pair $\langle K, \prec_K\rangle$ is an abstract elementary class, AEC for short iff

- A0 (Closure under isomorphism)
  - (a) For every $M \in K$ and every $L(K)$-structure N if $M \cong N$ then $N \in K$.
  - (b) Let $N_1, N_2 \in K$ and $M_1, M_2 \in K$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_K N_2$ implies that $M_1 \prec_K M_2$.
- A1 For all $M, N \in K$ if $M \prec_K N$ then $M \subseteq N$.
- A2 Let M, N, M^* be $L(K)$-structures. If $M \subseteq N$, $M \prec_K M^*$ and $N \prec_K M^*$ then $M \prec_K N$.
- A3 (Downward Löwenheim-Skolem) There exists a cardinal $LS(K) \geq \aleph_0 + |L(K)|$ such that for every $M \in K$ and for every $A \subseteq |M|$ there exists $N \in K$ such that $N \prec_K M$, $|N| \supseteq A$ and $\|N\| \leq |A| + LS(K)$.
- A4 (Tarski-Vaught Chain)
  - (a) For every regular cardinal μ and every $N \in K$ if $\{M_i \prec_K N : i < \mu\} \subseteq K$ is $\prec_K$-increasing (i.e. $i < j \implies M_i \prec_K M_j$) then $\bigcup_{i<\mu} M_i \in K$ and $\bigcup_{i<\mu} M_i \prec_K N$.
  - (b) For every regular μ, if $\{M_i : i < \mu\} \subseteq K$ is $\prec_K$-increasing then $\bigcup_{i<\mu} M_i \in K$ and $M_0 \prec_K \bigcup_{i<\mu} M_i$.

Notation 1.2. If λ is a cardinal and K is a class of models, K_λ is the subclass of models from K of cardinality λ.

Definition 1.3. For models $M, N \in K$, $f : M \to N$ is an $\prec_K$-embedding iff $f[M] \prec_K N$.

Definition 1.4. (1) We say that a model M is an amalgamation base iff for every $N_i \succ M$ with $\|N_i\| = \|M\|$ $(i = 1, 2)$, there are $\prec_K$-embeddings g_i, $(i = 1, 2)$ and a model N such that the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[r, "g_1"] & N \\ M \arrow[u, "id"] \arrow[r, "id"'] & N_2 \arrow[u, "g_2"'] \end{tikzcd}$$

(2) $K^{am} := \{M \in K \mid M$ is an amalgamation base$\}$.

Definition 1.5. Let $\alpha > 0$ be an ordinal. For triples $(\bar{a}_l, M_l, N_l)$ where $\bar{a}_l \in {}^{\alpha}N_l$ and $M_l \prec_K N_l \in K$ for $l = 0, 1$, we define a binary relation E as follows: $(\bar{a}_0, M_0, N_0) E (\bar{a}_1, M_1, N_1)$ iff $M_0 = M_1$ and there exists $N \in K$ and elementary mappings f_0, f_1 such that $f_l : N_l \to N$ and $f_l \restriction M = id_M$ for $l = 0, 1$ and $f_0(\bar{a}_0) = f_1(\bar{a}_1)$:

$$\begin{tikzcd} N_1 \arrow[r, "f_1"] & N \\ M \arrow[u, "id"] \arrow[r, "id"'] & N_2 \arrow[u, "f_2"'] \end{tikzcd}$$

Remark 1.6. E is an equivalence relation on the class of triples of the form $(\bar{a}, M, N)$ where $M \prec_K N$, $\bar{a} \in N$ and both $M, N \in K^{am}$. When only $M \in K^{am}$, E may fail to be transitive, but the transitive closure of E could be used instead.

While it is standard to use the E relation to define types in abstract elementary classes, we will discuss and make use of stronger relations between triples in section 3 of this paper.

In order to avoid confusing this new notion of "type" with the conventional one (i.e. set of formulas) we will follow [Gr1] and [Gr2] and introduce it below under the name of Galois type.

Definition 1.7. Let α be a positive ordinal (can be one).

(1) For $M, N \in K^{am}$ and $\bar{a} \in {}^{\alpha}N$: The Galois type of $\bar{a}$ in N over M, written ga-tp$(\bar{a}/M, N)$, is defined to be $(\bar{a}, M, N)/E$.
(2) We abbreviate ga-tp$(\bar{a}/M, N)$ by ga-tp$(\bar{a}/M)$.
(3) For $M \in K^{am}$,

$$\text{ga-}S^{\alpha}(M) := \{\text{ga-tp}(\bar{a}/M, N) \mid M \prec N \in K^{am}_{\|M\|}, \bar{a} \in {}^{\alpha}N\}.$$

We write ga-$S(M)$ for ga-$S^1(M)$.
(4) Let $p :=$ ga-tp$(\bar{a}/M', N)$ for $M \prec_K M'$ we denote by $p \restriction M'$ the type ga-tp$(\bar{a}/M, N)$. The domain of p is denoted by dom p and it is by definition M'.
(5) Let $p =$ ga-tp$(\bar{a}/M, N)$, suppose that $M \prec_K N' \prec_K N$ and let $\bar{b} \in {}^{\alpha}N'$ we say that $\bar{b}$ realizes p iff ga-tp$(\bar{b}/M, N') = p \restriction M$.

(6) For types p and q, we write $p \leq q$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and there exists $\bar{a}$ realizing p in some N extending $\mathrm{dom}(p)$ such that $(\bar{a}, \mathrm{dom}(p), N) \in q \restriction \mathrm{dom}(p)$.

Definition 1.8. We say that K is *β-stable in μ* iff for every $M \in K^{am}_\mu$, $|\text{ga-S}^\beta(M)| = \mu$. The class K is *Galois stable in μ* iff K is 1-stable in μ.

Definition 1.9. We say that $M \in K$ is *Galois saturated* iff for every $N \prec_K M$ of cardinality $< \|M\|$, and every $p \in \text{ga-S}(N)$, we have that M realizes p.

Remark 1.10. When $K = \mathrm{Mod}(T)$ for a first-order T, using the compactness theorem one can show (Theorem 2.2.3 of [Gr1]) that for $M \in K$, the model M is Galois saturated iff M is saturated in the first-order sense.

It is interesting to mention

Theorem 1.11 (Shelah [Sh300]). *Let $\lambda > LS(K)$. Suppose that K has the amalgamation property and $N \in K_\lambda$. The following are equivalent*

(1) *N is Galois staurated.*
(2) *N is model-homogenous I.e. if $M \prec_K N$ and $M' \succeq M$ of cardinality less than λ, then there exists a K-embedding over M from M' into N.*

Unfortunately [Sh300] has an incomplete skeleton of a proof, a complete and correct proof appeared in [Sh576]. See also [Gr2].

In first order logic, it is natural to consider saturated models for a stable theory. In this context, saturated models are model homogeneous and hence unique. In abstract elementary classes, the existence and even the uniqueness of saturated models is often difficult to derive. To combat this, Shelah introduced a replacement for saturated models, namely limit-models, which in certain contexts can be shown to exist and be unique.

Definition 1.12. For $M', M \in K_\lambda$, we say that M' is *universal over* M (abbreviated $M \prec^{univ}_K M'$) iff for every $N \in K_\lambda$ with $M \prec N$, there exists a $\prec_K$-embedding g, such that $g : N \to M'$ and $g \restriction M = id_M$.

Definition 1.13. For $M', M \in K_\mu$, we say that M' is a *(μ, θ)-limit over M* iff there exists a $\prec_K$-increasing and continuous sequence of amalgamation bases $\langle M_i \in K_\mu \mid i \leq \theta \rangle$ such that

(1) $M \prec_K M_0$,
(2) $M_\theta = M'$ and
(3) M_{i+1} is universal over M_i.

When $K = \mathrm{Mod}(T)$ for a first-order and stable T then automatically (by Theorem III.3.12 of [Sh:c]):

$$M \in K_\mu \text{ is saturated} \Longrightarrow M \text{ is } (\mu, \theta)\text{-limit for all } \theta < \mu^+ \text{ of cofinality} \geq \kappa(T).$$

When T is countable, stable but not superstable then the saturated model of cardinality μ is $(\mu, \aleph_1)$-limit but not $(\mu, \aleph_0)$-limit.

Claim 1.14 (Claim 1.14.1 from [Sh600]). Suppose K is an abstract elementary class with the amalgamation property. If K is Galois stable in μ, then for every $M \in K_\mu$, there exists $M' \in K_\mu$ such that M' is universal over M. Moreover M' can be chosen to be a (μ, θ)-limit over M for any $\theta < \mu^+$.

The following lemma holds in the context of abstract elementary classes with no maximal models and provides an alternate proof to the existence of universal extensions.

Lemma 1.15 (Theorem 1.3.1 from [ShVi635]). Categoricity in some $\lambda > LS(K)$ implies the existence universal extensions in every μ such that $LS(K) < \mu < \lambda$. Furthermore, this implies the existence of (μ, θ)-limits for $\omega \leq \theta < \mu^+$.

While the existence of limit models is useful, it is desirable that these models be unique. The following Lemma provides one level of uniqueness, namely limit models of the same length are isomorphic:

Lemma 1.16 (Fact 1.3.6 from [ShVi635]). Let μ, $\theta \geq LS(K)$ and $\theta \leq \mu^+$. If M_1 and M_2 are (μ, θ)-limits over M, then there exists an isomorphism $g : M_1 \to M_2$ such that $g \restriction M = id_M$. Moreover if M_1 is a (μ, θ)-limit over M_0, N_1 is a (μ, θ)-limit over N_0 and $g : M_0 \cong N_0$, then there exists a $\prec_K$-mapping $\hat{g}$, extending g such that $\hat{g} : M_1 \cong N_1$.

Showing that two (μ, θ)-limit models are isomorphic for different θs is a central (dichotomy) property even for first-order theories. E.g. Let T be a countable (for simplicity), stable, non-superstable first-order theory. Suppose T is stable in μ. By Harnik's theorem ([Ha]) T has a saturated model of cardinality μ, in fact for $\theta < \mu^+$ the theory has (μ, θ)-limit models. Since T is not superstable there exists a union of an ω-sequence of saturated models which is not saturated (see [AlGr]) thus it is not isomorphic to a (μ, ω_1)-limit model (that was obtained by taking a union of saturated models, see Theorem III.3.12 of [Shc]).

2. Existence of Indiscernibles

Assumption 2.1. For the remainder of this section, we will fix K, an abstract elementary class with the amalgamation property.

Remark 2.2. The focus of this paper are classes with the amalgamation property. Several of the proofs in this section can be adjusted to the context of abstract elementary classes with density of amalgamation bases as in [ShVi635].

The most obvious attempt to generalize Shelah's argument from Lemma I.2.5 of [Shc] does not apply since the notion of type cannot be identified with a set of first order formulas. Moreover, the notion of a type over an arbitrary set does not exist in the context of abstract elementary classes.

From [Sh394]:

Definition 2.3. A type $p \in S^\gamma(N)$ μ-splits over $M \prec_K N$ if and only if $\|M\| \leq \mu$, there exists $N_1, N_2 \in K_\mu$ and h, a K-embedding such that $M \prec_K N_l \prec_K N$ for $l = 1, 2$ and $h : N_1 \to N_2$ such that $h \restriction M = id_M$ and $p \restriction N_2 \neq h(p \restriction N_1)$.

Notice that non splitting is monotonic: I.e. If $p \in \text{ga-S}(N)$ does not split over M (for some $M \prec_K N$) then p does not split over M' for every $M \prec_K M' \prec_K N$.

Similarly to $\kappa(T)$ when T is first-order the following is a natural cardinal invariant of K:

Definition 2.4. Let $\gamma > 0$. We define an invariant $\kappa^\gamma_\mu(K)$ to be the minimal κ such that for every $\langle M_i \in K_\mu \mid i \leq \kappa\rangle$ which satisfies

(1) $\kappa = cf(\kappa) < \mu^+$,
(2) $\langle M_i \mid i \leq \kappa\rangle$ is $\prec_K$-increasing and continuous and
(3) for every $i < \kappa$, M_{i+1} is a (μ, θ)-limit over M_i for some $\theta < \mu^+$,

and for every $p \in \text{ga-S}^\gamma(M_\kappa)$, there exists $i < \kappa$ such that p does not μ-split over M_i. If no such κ exists, we say $\kappa^\gamma_\mu(K) = \infty$.

The following theorem in conjunction with Lemma 1.15 states that categorical abstract elementary classes with no maximal models satisfy $\kappa^1_\mu(K) \leq \omega$, for various μ.

Theorem 2.5 (Theorem 2.2.1 from [ShVi635]). Suppose K is categorical in $\lambda > LS(K)$. Let $\mu < \lambda$. If $\langle M_i \in K_\mu \mid i \leq \kappa\rangle$ and $p \in S(M_\kappa)$ satisfy

(1) $\kappa = cf(\kappa) < \mu^+$,
(2) $\langle M_i \mid i \leq \kappa\rangle$ is $\prec_K$-increasing and continuous and
(3) for every $i < \kappa$, M_{i+1} is a (μ, θ)-limit over M_i for some $\theta < \mu^+$,

then there exists $i < \kappa$ such that p does not μ-split over M_i.

A slight modification of the argument of Claim 3.3 from [Sh394] can be used to prove:

Theorem 2.6. Let $\gamma > 0$. Suppose that K is γ-stable in μ. For every $p \in \text{ga-S}^\gamma(N)$ there exists $M \prec_K N$ of cardinality μ such that p does not μ-split over M. Thus $\kappa^\gamma_\mu(K) \leq \mu$.

In Theorem 4.7 below we present an improvement of Theorem 2.6 for tame AECs: In case K is γ-stable in μ for some μ above its Hanf number then $\kappa^\gamma_\mu(K)$ is bounded by the Hanf number. Notice that the bound does not depend on μ.

The following is a new Galois-theoretic notion of indiscernible sequence.

Definition 2.7. (1) $\langle a_i \mid i < i^*\rangle$ is a Galois indiscernible sequence over M iff for every $i_1 < \cdots < i_n < i^*$ and every $j_1 < \cdots < j_n < i^*$, $\text{ga-tp}(a_{i_1} \ldots a_{i_n}/M) = \text{ga-tp}(a_{j_1} \ldots a_{j_n}/M)$.

(2) $\langle \bar a_i \mid i<\eta\rangle$ is a Galois indiscernible sequence over A iff for every $i_1<\cdots<i_n<\eta$ and every $j_1<\cdots<j_n<\eta$, there exists $M_i, M_j, M^\eta \in K$ and $\prec_K$-mappings f_i, f_j such that
   (a) $A \subseteq M_i, M_j$;
   (b) $f_l : M_l \to M^\eta$, for $l=i,j$;
   (c) $f_i(\bar a_{i_0},\dots,\bar a_{i_n}) = f_j(\bar a_{j_0},\dots,\bar a_{j_n})$ and
   (d) and $f_i \restriction A = f_j \restriction A = id_A$.

Remark 2.8. This is on the surface a weaker notion of indiscernible sequence than is presented in [Sh394]. However, this definition coincides with the first order definition. Additionally, it is suspected that, under some reasonable assumptions, this definition and the definition in [Sh394] are equivalent.

The following lemma provides us with sufficient conditions to find an indiscernible sequence.

Lemma 2.9. Let $\mu \geq LS(K)$, θ, δ be ordinals and γ a positive ordinal. Suppose that $\langle M_i \mid i<\delta\rangle$ and $\langle \bar a_i \mid i<\delta\rangle$ satisfy

(1) $\langle M_i \mid i<\delta\rangle$ are $\preceq_K$-increasing;
(2) $M_i \in K_\mu$;
(3) M_{i+1} is a (μ,θ)-limit over M_i;
(4) $\bar a_i \in {}^{\gamma}M_{i+1}$;
(5) $p_i := \text{ga-tp}(\bar a_i/M_i, M_{i+1})$ does not μ-split over M_0 and
(6) for $i<j<\delta$, $p_i \leq p_j$.

Then, $\langle \bar a_i \mid i<\delta\rangle$ is a Galois indiscernible sequence over M_0.

Definition 2.10. A sequence $\langle \bar a_i, M_i \mid i<\delta\rangle$ satisfying conditions (1) – (6) of Lemma 2.9 is called a Morley sequence.

Remark 2.11. While the statement of the lemma is similar to Shelah's Lemma I.2.5 in [Sh:c], the proof differs, since types are not sets of formulas.

Proof. We prove that for $i_0<\cdots<i_n<\delta$, and $j_0<\cdots<j_n<\delta$, $\text{ga-tp}(\bar a_{i_0},\dots,\bar a_{i_n}/M_0, M_{i_n+1}) = \text{ga-tp}(\bar a_{j_0},\dots,\bar a_{j_n}/M_0, M_{j_n+1})$ by induction on $n<\omega$.

$n=0$: Let $i_0, j_0<\delta$ be given. Condition 6, gives us

$$\text{ga-tp}(\bar a_{i_0}/M_0, M_{i_0+1}) = \text{ga-tp}(\bar a_{j_0}/M_0, M_{j_0+1}).$$

$n>0$: Suppose that the claim holds for all increasing sequences $\bar i$ and $\bar j \in \delta$, of length n. Let $i_0<\cdots<i_n<\delta$, and $j_0<\cdots<j_n<\delta$, be given. Without loss of generality, $i_n \leq j_n$. Define $M^\eta := M_1$. From condition 3 and uniqueness of (μ,ω)-limits, we can find a $\prec_K$-isomorphism, $g: M_{j_n} \to M_{i_n}$ such that $g \restriction M_0 = id_{M_0}$. Moreover we can extend g to $g: M_{j_n+1} \to M_{i_n+1}$. Denote by $\bar b_{j_l} := g(\bar a_{j_l})$ for $l=0,\dots,n$. Notice that $\bar b_{j_l} \in M_{i_n}$ for $l<n$. Since $\text{ga-tp}(\bar b_{j_0},\dots,\bar b_{j_n}/M_0, M_{i_n+1}) = \text{ga-tp}(\bar a_{j_0},\dots,\bar a_{j_n}/M_0, M_{j_n+1})$ it suffices to prove that $\text{ga-tp}(\bar b_{j_0},\dots,\bar b_{j_n}/M_0, M_{i_n+1}) = \text{ga-tp}(\bar a_{i_0},\dots,\bar a_{i_n}/M_0, M_{i_n+1})$.

Also notice that the $\prec_K$-mapping preserves some properties of p_j. Namely, since p_j does not μ-split over M_0, $g(p_j \restriction M_{j_n}) = p_j \restriction M_{i_n}$.
Thus, $\text{ga-tp}(b_{j_n}/M_{i_n}, M_{i_n+1}) = \text{ga-tp}(a_{j_n}/M_{i_n}, M_{i_n+1})$. In particular we have that $\text{ga-tp}(b_{j_n}/M_{i_n}, M_{i_n+1})$ does not μ-split over M_0.

By the induction hypothesis

$$\text{ga-tp}(b_{j_0}, \dots, b_{j_{n-1}}/M_0, M_{i_n}) = \text{ga-tp}(a_{i_0}, \dots, a_{i_{n-1}}/M_0, M_{i_n}).$$

Thus we can find $h_i : M_{i_n+1} \to M'$ and $h_j : M_{i_n+1} \to M'$ such that $h_i(a_{i_0}, \dots, a_{i_{n-1}}) = h_j(b_{j_0}, \dots, b_{j_{n-1}})$. Let us abbreviate $b_{j_0}, \dots, b_{j_{n-1}}$ by $\bar b_j$. Similarly we will write $\bar a_i$ for $a_{i_0}, \dots, a_{i_{n-1}}$.

By appealing to condition 5, we derive several equalities that will be useful in the latter portion of the proof. Since p_j does not μ-split over M_0, we have that $p_j \restriction h_j(M_{i_n}) = h_j(p_j \restriction M_{i_n})$, rewritten as

$$(*)\quad \text{ga-tp}(b_{j_n}/h_j(M_{i_n}), M_{i_n+1}) = \text{ga-tp}(h_j(b_{j_n})/h_j(M_{i_n}), M').$$

Similarly as p_i does not μ-split over M_0, we get
$p_i \restriction h_j(M_{i_n}) = h_j(p_i \restriction M_{i_n})$ and $p_i \restriction h_i(M_{i_n}) = h_i(p_i \restriction M_{i_n})$. These equalities translate to

$$(**)_j\quad \text{ga-tp}(a_{i_n}/h_j(M_{i_n}), M_{i_n+1}) = \text{ga-tp}(h_j(a_{i_n})/h_j(M_{i_n}), M') \text{ and}$$

$$(**)_i\quad \text{ga-tp}(a_{i_n}/h_i(M_{i_n}), M_{i_n+1}) = \text{ga-tp}(h_i(a_{i_n})/h_i(M_{i_n}), M'); \text{ respectively}$$

Finally, from condition 6., notice that

$$(***)\quad \text{ga-tp}(a_{i_n}/M_{i_n}, M_{i_n+1}) = \text{ga-tp}(b_{j_n}/M_{i_n}, M_{i_n+1}).$$

Applying h_j to $(***)$ yields

$$(\dagger)\quad \text{ga-tp}(h_j(b_{j_n})/h_j(M_{i_n}), M') = \text{ga-tp}(h_j(a_{i_n})/h_j(M_{i_n}), M').$$

Since $h_i(\bar a_i) = h_j(\bar b_j) \in h_j(M_{i_n})$, we can draw from $(\dagger)$ the following:

$$(1)\quad \text{ga-tp}(h_j(b_{j_n}){}^\frown h_j(\bar b_j)/M_0, M') = \text{ga-tp}(h_j(a_{j_n}){}^\frown h_i(\bar a_i)/M_0, M').$$

Equality $(**)_i$ allows us to see

$$(2)\quad \text{ga-tp}(a_{i_n}{}^\frown h_i(\bar a_i)/M_0, M') = \text{ga-tp}(h_i(a_{i_n}){}^\frown h_i(\bar a_i)/M_0, M').$$

Since $\text{ga-tp}(h_j(a_{i_n})/h_j(M_{i_n}), M') = \text{ga-tp}(a_{i_n}/h_j(M_{i_n}), M_{i_n+1})$ (equality $(**)_j$) and $h_i(\bar a_i) = h_j(\bar b_j) \in h_j(M_{i_n})$, we get that

$$(3)\quad \text{ga-tp}(h_j(a_{i_n}){}^\frown h_i(\bar a_i)/M_0, M') = \text{ga-tp}(a_{i_n}{}^\frown h_i(\bar a_i)/M_0, M').$$

Combining equalities (1), (2) and (3), we get

$$(\dagger\dagger)\quad \text{ga-tp}(h_i(\bar a_i){}^\frown h_i(a_{i_n})/M_0, M') = \text{ga-tp}(h_j(\bar b_j){}^\frown h_j(b_{j_n})/M_0, M').$$

Recall that $h_i \restriction M_0 = h_j \restriction M_0 = id_{M_0}$. Thus $(\dagger\dagger)$, witnesses that

$$\text{ga-tp}(a_{i_0}, \dots, a_{i_n}/M_0, M_{i_n+1}) = \text{ga-tp}(b_{j_0}, \dots, b_{j_n}/M_0, M_{i_n+1}).$$

$\dashv$

3. Tame Abstract Elementary Classes

By Lindström's Theorem, one obvious feature of non-elementary abstract elementary classes is the absence of the compactness theorem. A method of combating this is to view types as equivalence classes of triples (Definition 1.7) instead of sets of formulas. While this notion of type has led to several profound results in the study of abstract elementary classes, a stronger equivalence relation (denoted E_μ) is eventually utilized in various partial solutions to Shelah's Categoricity Conjecture (see [Sh394] and [Sh576])?.

Shelah identified E_μ as an interesting relation in [Sh394]. Here we recall the definition.

Definition 3.1. Triples $(\bar{a}_1, M, N_1)$ and $(\bar{a}_2, M, N_2)$ are said to be E_μ-related provided that for every $M' \prec_K M$ with $M' \in K_{<\mu}$,

$$(\bar{a}_1, M', N_1) E (\bar{a}_2, M', N_2).$$

Notice that in first order logic, the finite character of consistency implies that two types are equal if and only if they are E_ω-related.

In Main Claim 9.3 of [Sh394], Shelah ultimately proves that, under categoricity in some $\lambda > Hanf(K)$ and under the assumption that K has the amalgamation property, for types over saturated models, E-equivalence is the same as E_μ equivalence for some $\mu < Hanf(K)$.

We now define a context for abstract elementary classes where consistency has small character.

Definition 3.2. Let χ be a cardinal number. We say the abstract elementary class K with the amalgamation property is χ-tame provided that for types, E-equivalence is the same as the E_χ relation. In other words, for $M \in K_{>Hanf(K)}$, $p \neq q \in ga\text{-}S(M)$ implies existence of $N \prec_K M$ of cardinality χ such that $p\upharpoonright N \neq q\upharpoonright N$.

K is tame iff there exists such that K is χ-tame for some $\chi < Hanf(K)$

Remark 3.3. We actually only use that E-equivalence is the same as E_χ-equivalence for types over limit models.

Notice that if K is a finite diagram (i.e. we have amalgamation not only all models but also over subsets of models) then it is a tame AEC.

Example 3.4. There are tame AECs with amalgamation which are not finite diagrams. WILL BE ADDED

Example 3.5. For $\mu_1 < \mu_2 < \beth_{\omega_1}$, there is a class which is μ_2-tame but not μ_1-tame. WILL BE ADDED

4. The order property

Remark 4.1. For the remainder of this section we will assume that K is an abstract elementary class which satisfies the amalgamation property.

The order property, defined next, is an analog of the first order definition of order property using formulas. The order property for non-elementary classes was introduced by Shelah in [Sh394].

Definition 4.2. K is said to have the κ-order property provided that for every α, there exists $\langle \bar a_i \mid i<\alpha\rangle$ where $\bar a_i \in {}^{\kappa}\mathfrak{C}$ such that if $i_0<j_0<\alpha$ and $i_1<j_1<\alpha$,

$$(*) \text{ then for no } f\in \mathrm{Aut}(\mathfrak{C}) \text{ do we have } f(\bar a_{i_0}{}^\frown \bar a_{j_0}) = \bar a_{j_1}{}^\frown \bar a_{i_1}.$$

Remark 4.3 (Trivial monotonicity). Notice that for $\kappa_1<\kappa_2$ if a class has the κ_1-order property then it has the κ_2-order property.

Claim 4.4 (Claim 4.6.3 of [Sh394]). We may replace the phrase every α in Definition 4.2 with every $\alpha<\beth_{(2^{\kappa+LS(K)})^+}$ and get an equivalent definition.

Theorem 4.5 (Claim 4.8.2 of [Sh394]). If K has the κ-order property and $\mu\geq\kappa$, then for some $M\in K_\mu$ we have that $|\text{ga-S}^{\kappa}(M)/E_\kappa|\geq\mu^+$. Moreover, we can conclude that K is not Galois-stable in μ.

Question 4.6. Can we get a version of the stability spectrum theorem for tame stable classes?

Theorem 4.7. Let $\beta>0$. Suppose that K is a κ-tame abstract elementary class. If K is β-stable in μ with $\beth_{(2^{\kappa+LS(K)})^+}\leq\mu$, then $\kappa^{\beta}_{\mu}(K)<\beth_{(2^{\kappa+LS(K)})^+}$.

Proof. Let $\lambda:=\beth_{(2^{\kappa+LS(K)})^+}$. Suppose that the conclusion of the theorem does not hold. Let $\langle M_i\in K_\mu \mid i\leq\lambda\rangle$ and $p\in\text{ga-S}^{\beta}(M_\lambda)$ witness the failure. Namely, the following hold:

(1) $\langle M_i \mid i\leq\lambda\rangle$ is $\prec_K$-increasing and continuous,
(2) for every $i<\lambda$, M_{i+1} is a (μ,θ)-limit over M_i for some $\theta<\mu^+$ and
(3) for every $i<\mu^+$, p μ-splits over M_i.

For every $i<\lambda$ let f_i, N^1_i and N^2_i witness that p μ-splits over M_i. Namely,

$$M_i\prec_K N^1_i, N^2_i\prec_K M;$$
$$f_i: N^1_i\cong N^2_i \text{ with } f_i\restriction M_i=\mathrm{id}_{M_i}$$
$$\text{and } f_i(p\restriction N^1_i)\neq p\restriction N^2_i.$$

By κ-tameness, there exist B_i and $A_i:=f_i^{-1}(B_i)$ of size $<\kappa$ such that

$$f_i(p\restriction A_i)\neq p\restriction B_i.$$

By renumbering our chain of models, we may assume that

(4) $A_i, B_i\subseteq M_{i+1}$.

Since M_{i+1} is a limit model over M_i, we can additionally conclude that

(5) $c_i\in M_{i+1}$ realizes $p\restriction M_i$.

For each $i<\mu$, let $\bar a_i:=A_i{}^\frown B_i{}^\frown c_i$.

Claim 4.8. $\langle \bar a_i \mid i<\lambda\rangle$ witnesses the κ-order property.

Proof. Suppose for the sake of contradiction that there exist $g \in \mathrm{Aut}(\mathfrak{C})$, $i_0 < j_0 < \lambda$ and $i_1 < j_1 < \lambda$ such that

$$g(\bar{d}_{i_0} {}^\frown \bar{d}_{j_0}) = \bar{d}_{j_1} {}^\frown \bar{d}_{i_1}.$$

Notice that since $i_0 < j_0 < \beta$ we have that $\bar{c}_{i_0} \in M_{j_0}$. So $f_{j_0}(\bar{c}_{i_0}) = \bar{c}_{i_0}$. Recall that $f_{j_0}(A_{j_0}) = B_{j_0}$. Thus, f_{j_0} witnesses that

$$(*)\ \text{ga-tp}(\bar{c}_{i_0} {}^\frown A_{j_0}/\emptyset) = \text{ga-tp}(\bar{c}_{i_0} {}^\frown B_{j_0}/\emptyset).$$

Applying g to $(*)$ we get

$$(**)\ \text{ga-tp}(\bar{c}_{j_1} {}^\frown A_{i_1}/\emptyset) = \text{ga-tp}(\bar{c}_{j_1} {}^\frown B_{i_1}/\emptyset).$$

Applying f_{i_1} to the RHS of $(**)$, we notice that

$$(\sharp)\ \text{ga-tp}(f_{i_1}(\bar{c}_{j_1}) {}^\frown B_{i_1}/\emptyset) = \text{ga-tp}(\bar{c}_{j_1} {}^\frown B_{i_1}/\emptyset).$$

Because $i_1 < j_1$, we have that $\bar{c}_{j_1}$ realizes $p \upharpoonright M_{i_1}$. Thus, $(\sharp)$ implies

$$(\sharp\sharp)\ f_{i_1}(p \upharpoonright A_{i_1}) = p \upharpoonright B_{i_1},$$

which contradicts our choice of f_{i_1}, A_{i_1} and B_{i_1}.

$\dashv$

By Claim 4.4 and Theorem 4.5, we have that K is unstable in μ, contradicting our hypothesis.

$\dashv$

5. Morley's sequences

Hypothesis 5.1. For the rest of the paper we make the following assumption: K is a tame abstract elementary class, has no maximal models and satisfies the amalgamation property.

Theorem 5.2. *Suppose $\mu \geq \beth_{(2^{\mathrm{Hanf}(K)})^+}$. Let $M \in K_{>\mu}$, $A, I \subseteq M$ be given such that $|I| \geq \mu^+ > |A|$. If K is Galois stable in μ, then there exists $J \subseteq I$ of cardinality μ^+, Galois indiscernible over A. Moreover J can be chosen to be a Morley sequence over A.*

Proof. Fix $\kappa := \mathrm{cf}(\mu)$. Let $\{\bar{a}_i \mid i < \mu^+\} \subseteq I$ be given. Define $\langle M_i \in K_\mu \mid i < \mu^+ \rangle$ $\prec_K$-increasing and continuous satisfying

(1) $A \subseteq |M_0|$
(2) M_{i+1} is a (μ, κ)-limit over M_i
(3) $\bar{a}_i \in M_{i+1}$

Let $p_i := \text{ga-tp}(\bar{a}_i/M_i; M_{i+1})$ for every $i < \mu^+$. Define $f : S^{\mu^+}_\kappa \to \mu^+$ by

$$f(i) := \min\{j < \mu^+ \mid p_i \text{ does not } \mu\text{-split over } M_j\}.$$

By Theorem 4.7, f is regressive. Thus by Fodor's Lemma, there are a stationary set $S \subseteq S^{\mu^+}_\kappa$ and $j_0 \in I$ such that for every $i \in S$,

$$(\dagger)\quad p_i \text{ does not } \mu\text{-split over } M_{j_0}.$$

By stability and the pigeon-hole principle there exists $p^* \in \text{ga-S}(M_{j_0})$ and $S^* \subseteq S$ of cardinality μ^+ such that for every $i \in S^*$, $p^* = p_i \upharpoonright M_{j_0}$. Enumerate and rename S^*. Let $M^* := M_1$. Again, by stability we can find $J \subseteq S^*$ of cardinality μ^+ such that for every $i \in S^*$, $p^{**} = p_i \upharpoonright M^*$. Enumerate and rename J.

Subclaim 5.3. For $i < j \in J$, $p_i = p_j \upharpoonright M_i$.

Proof. Let $0 < i < j \in J$ be given. Since M_{i+1} and M_{j+1} are (μ,θ)-limits over M_i, there exists an isomorphism $g : M_{j+1} \to M_{i+1}$ such that $g \upharpoonright M_i = id_{M_i}$. Let $\bar b_j := g(\bar a_j)$. Since the type p_j does not μ-split over M_{j_0}, g cannot witness the splitting. Therefore, it must be the case that $\text{ga-tp}(\bar b_j/M_i, M_{i+1}) = p_i \upharpoonright M_i$. Then, it suffices to show that $\text{ga-tp}(\bar b_j/M_i, M_{i+1}) = p_i$.

Since $p_i \upharpoonright M_0 = p_j \upharpoonright M_0$, we can find $\prec_K$-mappings witnessing the equality. Furthermore since M^* is universal over M_0, we can find $h_l : M_{l+1} \to M^*$ such that $h_l \upharpoonright M_0 = id_{M_0}$ for $l = i, j$ and $h_i(\bar a_i) = h_j(\bar b_j)$.

We will use $(\dagger)$ to derive several inequalities. Consider the following possible witness to splitting. Let $N_1 := M_i$ and $N_2 := h_i(M_i)$. Since p_i does not μ-split over M_0, we have that $p_i \upharpoonright N_2 = h_i(p_i \upharpoonright N_1)$, rewritten as

$$(*) \quad \text{ga-tp}(\bar a_i/h_i(M_i), M_{i+1}) = \text{ga-tp}(h_i(\bar a_i)/h_i(M_i), M^*).$$

Similarly we can conclude that

$$(**) \quad \text{ga-tp}(\bar b_j/h_j(M_i), M_{i+1}) = \text{ga-tp}(h_j(\bar b_j)/h_j(M_i), M^*).$$

By choice of J, we know that

$$(***) \quad \text{ga-tp}(\bar b_j/M^*) = \text{ga-tp}(\bar a_i/M^*).$$

Now let us consider another potential witness of splitting: $N_1^* := h_i(M_i)$ and $N_2^* := h_j(M_i)$ with $H^* := h_j \circ h_i^{-1} : N_1^* \to N_2^*$. Since $p_j \upharpoonright M_i$ does not μ-split over M_0, $p_j \upharpoonright N_2^* = H^*(p_j \upharpoonright N_1^*)$. Thus by $(**)$ we have

$$(\sharp) \quad H^*(p_j \upharpoonright N_1^*) = \text{ga-tp}(h_j(\bar b_j)/h_j(M_i), M^*).$$

Now let us translate $H^*(p_j \upharpoonright N_1^*)$. By monotonicity and $(***)$, we have that $p_j \upharpoonright N_1^* = \text{ga-tp}(\bar b_j/h_i(M_i), M_{i+1}) = \text{ga-tp}(\bar a_i/h_i(M_i), M_{i+1})$. We can then conclude by $(*)$ that $p_j \upharpoonright N_1^* = \text{ga-tp}(h_i(\bar a_i)/h_i(M_i), M_{i+1})$. Applying H^* to this equality yields

$$(\sharp\sharp) \quad H^*(p_j \upharpoonright N_1^*) = \text{ga-tp}(h_j(\bar a_i)/h_j(M_i), M^*).$$

By combining the equalities from $(\sharp)$ and $(\sharp\sharp)$ and applying h_j^{-1} we get that

$$\text{ga-tp}(\bar b_j/M_i, M_{i+1}) = \text{ga-tp}(\bar a_i/M_i, M_{i+1}).$$

$\dashv$

Notice that by Subclaim 5.3 and our choice of J, $\langle M_i \mid i \in J\rangle$ and $\langle \bar a_i \mid i \in J\rangle$ satisfy the conditions of Lemma 2.9. Applying Lemma 2.9, we get that $\langle \bar a_i \mid i \in J\rangle$ is a morley sequence over M_0. In particular, since $A \subseteq M_0$, we have that $\langle \bar a_i \mid i \in J\rangle$ is a Morley sequence over A.

a

6. Bounded multiplicity

Definition 6.1. Suppose $M \prec_K N$ and let $\bar{a}, \bar{b} \in |N|$.
$\bar{a} \sim_M \bar{b}$ iff there are $n < \omega$ and there exists a sequence $\langle J_k \mid k \leq 2n \rangle$ satisfying the following conditions:

(1) every J_k is an infinite sequence, Galois indiscernible over M;
(2) $\bar{a} \in J_0$ and $\bar{b} \in J_{2n}$;
(3) For every $k < n$ the sequence $J_{2k} {}^\frown J_{2k+1}$ is Galois indiscernible over M;
(4) For every $k < n$ the sequence $J_{2k+2} {}^\frown J_{2k+1}$ is Galois indiscernible over M.

Theorem 6.2. Let K be as in Hypothesis 5.1. Suppose $\lambda \geq (2^{Hanf(K)})^+$. Let $M \in K_{>\lambda}$, $N \prec_K M$. If K is stable in λ then

$$|\{p \in \text{ga-S}(M) : p \text{ does not split over } N\}| \leq 2^{\mu}.$$

Corollary 6.3 (partial stability spectrum).

7. Dividing

In this section we define dividing and derive some of its properties.

Definition 7.1. Let $p \in \text{ga-S}(M)$ and $N \prec_K M$. We say that p divides over N iff there are $\bar{a} \in M$ non-algebraic over N and a Morley sequence, $\{\bar{a}_n \mid n < \omega\}$ for the ga-tp$(\bar{a}/N; M)$ such that for every collection $\{f_n \in \mathrm{Aut}_M \mathfrak{C} \mid n < \omega\}$ with $f_n(\bar{a}) = \bar{a}_n$ we have

$$\{f_n(p) \mid n < \omega\} \text{ is inconsistent.}$$

Theorem 7.2 (Existence). Suppose that K is stable in λ and κ-tame for some $\kappa < \lambda$. For every $p \in \text{ga-S}(M)$ with $M \in K_{\lambda}$ there exists $N \prec_K M$ of cardinality λ such that p does not divide over N.

Proof. Suppose that p and M form a counter-example. WLOG we may assume that $M = \mathfrak{C}$. Through the proof of Claim 3.3.1 of [Sh394], in order to contradict stability in λ, it suffices to find N_i, N_i^1, N_i^2, h_i for $i < \lambda$ satisfying

(1) $\langle N_i \in K_\lambda \mid i \leq \lambda \rangle$ is a $\prec_K$-increasing and continuous sequence of models;
(2) $N_i \prec_K N_i^l \prec_K N_{i+1}$ for $i < \lambda$ and $l = 1, 2$;
(3) for $i < \lambda$, $h_i : N_i^1 \cong N_i^2$ and $h_i \restriction N_i = \mathrm{id}_{N_i}$ and
(4) $p \restriction N_i^2 \neq h_i(p \restriction N_i^1)$.

Suppose that N_i has been defined. Since p divides over every substructure of cardinality μ, we may find a, $\{a_n \mid n<\omega\}$ and $\{f_n \mid n<\omega\}$ witnessing that p divides over N_i. Namely, we have that $\{f_n(p) \mid n<\omega\}$ is inconsistent. Let $n<\omega$ be such that $f_0(p) \neq f_n(p)$. Then $p \neq f_0^{-1}\circ f_n(p)$. By κ-tameness, we can find $N' \prec_K \mathfrak{C}$ of cardinality μ containing N such that $p\restriction N' \neq (f_0^{-1}\circ f_n(p))\restriction N'$. WLOG $f_0^{-1}\circ f_n \in \mathrm{Aut}_N N'$.

Let $h_i := f_0^{-1}\circ f_n$, $N_i^1 := N'$ and $N_i^2 := N'$. Choose $N_{i+1} \prec_K \mathfrak{C}$ to be an extension of N' of cardinality μ. ⊣

Next step:

(1) extension?

(2) symmetry?

(3) uniqueness (over very saturated models)?

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