THE EQUALITY $S1 = D = R$

R. GROSSBERG, A. KOLESNIKOV, I. TOMAŠIĆ, AND M. VANDIEREN

ABSTRACT. The new result of this paper is that for $\theta(\mathbf{x};\mathbf{a})$-stable (a weakening of "T is stable") we have

$$S1[\theta(\mathbf{x};\mathbf{a})] = D[\theta(\mathbf{x};\mathbf{a}), L, \infty] = R[\theta(\mathbf{x};\mathbf{a}), L, \infty].$$

$S1$ is Hrushovski's rank. This is an improvement of a result of Kim and Pillay, who for simple theories under the (strong) assumption that either of the ranks be finite obtained the same identity. Only the first equality is new, the second equality is a result of Shelah from the seventies.

We derive it by studying localizations of several rank functions, we get:

Theorem 0.1 (Main Theorem). *Suppose that μ is regular satisfying $\mu \geq |T|^+$, p is a finite type, and Δ is a set of formulas closed under Boolean operations. If*

(1) *$R[p, \Delta, \mu^+] < \infty$ or*
(2) *p is Δ-stable and μ satisfies: for every sequence of cardinals $\{\mu_i < \mu \mid i < |\Delta| + \aleph_0\}$ we have that $\prod_{i<|\Delta|+\aleph_0} \mu_i < \mu$ holds,*

then $S[p, \Delta, \mu^+] = D[p, \Delta, \mu^+] = R[p, \Delta, \mu^+]$.

The S rank above is a localized version of Hrushovski's $S1$ rank. This rank, as well as our systematic use of local stability, allows us to get a more conceptual proof of the equality of D and R, which is an old result of Shelah. A particular (asymptotic) case of the theorem offers a new sufficient condition for the equality of $S1$ and $D[\cdot, L, \infty]$. We also manage, due to a more general approach, to avoid some combinatorial difficulties present in Shelah's original exposition.

INTRODUCTION

Historically the introduction and implementation of several rank functions have led to significant developments in the classification of theories. In 1965, Morley published a partial solution to Łŏs' conjecture [Lo], requiring that the similarity type of the theory be countable [Mo]. Later in the seventies, Saharon Shelah solved Łŏs' conjecture in full generality [Sh31]. Although Morley and Shelah's approaches are structurally different, they implement a common tool, a rank function. Morley introduced a rank function which is now referred to as the *Morley rank* and is often denoted by $RM(p)$, where p is set of formulas ($RM(p) := R[p, L, \aleph_0]$ see Definition 1.1).

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For the general solution to Łŏs' conjecture, Shelah identified an important class of theories, the superstable theories. To study these theories, he introduced several tools in [Sh 10], including a rank called $Deg[p]$ ($Deg[p] := Deg[p, L, |T|^{++}]$ see Definition 1.11), and, to characterize stable theories, the function $R[p, \Delta, \aleph_0]$ for p a set of formulas and Δ a finite set of formulas.

The introduction of these rank functions caught the attention of model theorists. Daniel Lascar in his review of [Sha] for the *Mathematical Reviews* wrote: "Ranks are the main combinatorial device in stability." John Baldwin investigated the Morley rank, and in his Ph.D. thesis Baldwin proved that an $\aleph_1$-categorical theory has finite Morely rank (see [Ba1]). For his solution, Baldwin introduced a rank function which was later named $D[\mathbf{x} = \mathbf{x}, L, \infty]$ by Shelah. Among other results Baldwin showed that $D[p, L, \infty]$ and $RM[p]$ coincide and are finite for $\aleph_1$-categorical theories (this was a predecessor of Theorem 0.1).

Daniel Lascar in [La2] introduced the *Lascar rank*, $U(p)$ (defined for complete types p), suitable for the study of superstable theories (it is denoted by $L(p)$ in [Sha]). Lascar's rank is different than the ranks we discuss here (even for differentially closed fields, see [HS]).

In 1974, Baldwin and Blass introduced the basic axiomatic properties of rank functions in [BaBl]; the connected property appears in Lascar [La2]. Later (in his book [Sha] from 1978), Shelah presented two other families of rank functions $D[p, \Delta, \lambda]$ and $R[p, \Delta, \lambda]$. The first is a generalization of $Deg[p]$ (as $Deg[p] = D[p, L, |T|^{++}]$) and the latter is a generalization of the Morley rank (as $RM[p] = R[p, L, \aleph_0]$). The functions $D[p, L, \infty]$ and $R[p, L, \infty]$ (the latter often denoted by ∞-rank$[p]$) are well understood and their basic properties appear in several monographs (e.g. John Baldwin's book [Ba2], Steve Buechler's book [Bu] and Anand Pillay's [Pi]). Shelah's [Sh 10] ends with the question about the relation of $R[p, \Delta, \infty]$ and $Deg[p]$. The question was answered by him in his book. Unfortunately, the deep results in Section 3 of Chapter II [Sha] relating the rank functions $R[p, \Delta, \lambda]$ and $D[p, \Delta, \lambda]$ were not covered by any of the subsequent expositions known to us. (E. g. *if T is superstable, then* $R[\mathbf{x} = \mathbf{x}, L, \infty] < |T|^+$.)

In the early nineties, Ehud Hrushovski [Hr] introduced the function $S1$ that he used to study simple unstable groups. Hrushovski's $S1$ was defined only for finite values. In the unpublished paper [Hr], he introduced and proved the independence theorem for theories with finite $S1$ rank. This theorem was later generalized by Kim and Pillay [KP] for all simple theories. Some of Hrushovski's early results concerning this rank appeared in a restricted form in his papers with Pillay [HP1] and later with Chatzidakis [CH].

This paper started by our attempt to understand some of the more difficult results in Section 3 of Chapter II [Sha]. To our surprise, after writing up complete proofs to Shelah's theorems, we realized that the main combinatorial dividing line is an implicit use of a rank function similar to the rank $S1$; and the latter rank function was introduced explicitly only more than

20 years later in a geometric context by Ehud Hrushovski. More precisely, we introduce and study a new rank function $S[p, \Delta, \lambda]$. It turns out that $S[p, L, \lambda] = S1[p]$ for sufficiently large λ. This gives a more conceptual proof of $R = D$ than Shelah had.

Equality of $D[p, L, \infty]$ and $S1[p, L]$ was previously shown to be true by Kim and Pillay [KP] under the assumption that T is simple when either $D[p, L, \infty]$ or $S1[p]$ are finite. Here we get equality also for infinite valued rank, but we assume local stability.

From Shelah's proofs we extract relativized notions of Δ-stability and Δ-superstability that clarify the arguments significantly (compare with Harnik and Harrington [HH], Bouscaren [Bo] and with Grossberg and Lessmann [GrLe]). The use of local stability has paid off as we can avoid the combinatorial difficulties in the proof of the asymptotic behavior of R, see our Proposition 3.4 and Shelah's Theorem II.3.13 in [Sha].

To lower the complexity of Shelah's original treatment, we introduce the rank function Deg (which generalizes Shelah's function from [Sh 10] and is closely related to D). We show that under local stability the D-rank has a simpler equivalent definition that is "almost like" the definition of Deg.

Lastly, a function $S2$ is introduced, it relates to the $S1$-rank in a similar way as $Deg[p, \Delta, \lambda]$ relates to $R[p, \Delta, \lambda]$. It is shown that when $S1[p] = R[p, L, \infty]$ then also $S1[p] = S2[p]$.

The notation is standard. Throughout the paper, T denotes a complete first-order theory without finite models. The language of T is denoted $L(T)$. The monster model is denoted by $\mathfrak{C}$.

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1. Local stability and the R-rank

We start by recalling the definitions of existing rank functions. We quote their basic properties and connections with (local) stability.

Recall from [Sha]:

Definition 1.1. Let p be a set of formulas in $\mathbf{x}$, $\Delta \subseteq \{\varphi(\mathbf{x}; \mathbf{y}) \mid \varphi \in Fml(L(T))\}$ and λ a cardinality (can be finite) or ∞.

(1) $R[p, \Delta, \lambda] \geq \alpha$ is defined by induction on α:
   (a) $R[p, \Delta, \lambda] \geq 0$, if p is consistent;
   (b) for α limit, $R[p, \Delta, \lambda] \geq \alpha$ if $R[p, \Delta, \lambda] \geq \beta$ for every $\beta < \alpha$
   (c) $R[p, \Delta, \lambda] \geq \alpha + 1$ if for every finite $q \subseteq p$ and every $\mu < \lambda$ there are $\{q_i \mid i \leq \mu\}$ *explicitly contradictory* Δ*-types* such that $R[q \cup q_i, \Delta, \lambda] \geq \alpha$ for every $i \leq \mu$. (When $\lambda = \infty$ we interpret it as no restriction on μ).

(2) For an ordinal α denote by $R[p, \Delta, \lambda] = \alpha$ the statement

$$R[p, \Delta, \lambda] \geq \alpha \quad \text{and} \quad R[p, \Delta, \lambda] \ngeq \alpha + 1.$$

(3) We write $R[p, \Delta, \lambda] = \infty$ if $R[p, \Delta, \lambda] \geq \alpha$ for every ordinal α.

Remarks 1.2. (1) $R[p, L, \aleph_0]$ is *Morley's rank* often denoted by $RM[p]$.
(2) $R[p, L, \infty]$ is often called *infinity rank*, Buechler (in [Bu]) denotes it by $R^\infty[p]$.
(3) $R[p, \Delta, \aleph_0]$ is denoted in Pillay ([Pi]) by $R^\Delta_{\aleph_0}[p]$.
(4) A word on Δ: While many results apply for arbitrary Δ and in developing the basics of stability theory the case when $\Delta = \{\varphi(\mathbf{x}; \mathbf{y})\}$ is quite important. In this paper our focus is in a fairly large Δ. Especially the case when $\Delta = L$ or Δ is a countable fragment of L containg a given formula.

The following are the basic well known properties of the R-rank:

Fact 1.3. *For sets of formulas, p and Δ, and a cardinal λ*

Invariance: *For $f \in \mathrm{Aut}(\mathfrak{C})$, $R[p, \Delta, \lambda] = R[f(p), \Delta, \lambda]$.*

Monotonicity: *For sets of formulas, p and q, whenever $p \vdash q$, $\mu \leq \lambda$ and $\Delta_1 \subseteq \Delta_2$, we have that $R[p, \Delta_1, \lambda] \leq R[q, \Delta_2, \mu]$.*

Finite character: *There is a finite subset $q \subseteq p$, such that $R[p, \Delta, \lambda] = R[q, \Delta, \lambda]$.*

Ultrametric property: *For $\lambda \geq \aleph_0$*

$$R[p \cup \{\bigvee_{1 \leq l \leq n} \psi_l\}, \Delta, \lambda] = \mathrm{Max}_{1 \leq l \leq n} R[p \cup \{\psi_l\}, \Delta, \lambda].$$

Extension property: *For $\lambda \geq \aleph_0$ and a set $A \supseteq \mathrm{dom}(p)$, there exists a complete type $q \supseteq p$ with domain A such that $R[p, \Delta, \lambda] = R[q, \Delta, \lambda]$.*

Connected: *Let p be a finite set of formulas in $\mathbf{x}$. If $R[p, \Delta, \lambda] = \alpha < \infty$ then for every $\beta < \alpha$ there exists a Δ-type q such that $R[p \cup q, \Delta, \lambda] = \beta$.*

As we point out below, the R-rank can be used to capture stability-theoretic properties of theories, including local stability and superstability. In order to establish the framework, recall:

Definition 1.4. Let p be a type in $\mathbf{x}$ and let Δ be a set of formulas such that for all $\varphi \in \Delta$, $\varphi = \varphi(\mathbf{x}; \mathbf{y}_\varphi)$.

(1) A type $q \in S^{\ell(\mathbf{x})}_\Delta(A)$ is called a (p, Δ)-*type* if $p \cup q$ is consistent. The set of all (p, Δ)-types is denoted by $S_{p,\Delta}(A)$;
(2) p is (Δ, λ)-*stable* if for all A, $|A| \leq \lambda$, we have $|S_{p,\Delta}(A)| \leq \lambda$;
(3) p is Δ-*stable* if there is λ such that p is (Δ, λ)-stable;
(4) p is Δ-*superstable* if there is λ such that p is (Δ, μ)-stable for all $\mu \geq \lambda$;
(5) $\varphi(\mathbf{x}; \mathbf{y})$ *has the order property* over p if there is a set $\{\mathbf{a}_n \mid n < \omega\}$ with $\ell(\mathbf{a}_n) = \ell(\mathbf{y})$ such that $p \cup \{\varphi(\mathbf{x}; \mathbf{a}_n)^{\text{if } k \leq n} \mid n < \omega\}$ is consistent for all $k < \omega$;
(6) Δ *has the order property over* p if there is $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ that has the order property over p.

It turns out that such a definition of local stability is well-behaved, generalizes the usual definitions (as $\mathbf{x} = \mathbf{x}$ being L-(super)stable coincides with T being (super)stable), and allows us to develop the theory in a standard way; see e. g. [Sha] I.2.10, II.2.14, II.3.13. We can get:

Theorem 1.5. *Let p be a type in $\mathbf{x}$ and let Δ be a set of formulas. The following are equivalent:*

(1) *For all $\lambda \geq \aleph_0$ such that $\lambda = \lambda^{|\Delta|}$, p is (Δ, λ)-stable;*
(2) *p is Δ-stable;*
(3) *Δ does not have the order property over p.*

By using the standard tree characterization for the R-rank as in [Sha] II.2.2, it is possible to characterize local stability of types in the following way:

Theorem 1.6. *Suppose p is a set of formulas in $\mathbf{x}$ and $\varphi(\mathbf{x}; \mathbf{y})$ is a formula. The following are equivalent*

(1) *$R[p, \varphi, 2] < \omega$,*
(2) *$R[p, \varphi, 2] < \infty$,*
(3) *$R[p, \varphi, \aleph_0] < \infty$,*
(4) *$R[p, \varphi, \infty] < \infty$,*
(5) *the formula φ does not have the order property over p,*
(6) *p is $\{\varphi\}$-stable.*

Corollary 1.7. *Suppose p and Δ are sets of formulas in $\mathbf{x}$. If $R[p, \Delta, \infty] < \infty$, then p is Δ-stable.*

Proof. Suppose $R[p, \Delta, \infty] < \infty$. By monotonicity, $R[p, \varphi, \infty] < \infty$ for every $\varphi \in \Delta$. Theorem 1.6 implies that for all $\varphi \in \Delta$, φ does not have the order property over p. Thus Δ does not have the order property. By Theorem 1.5 we get that p is Δ-stable. $\dashv$

Remark 1.8. In fact $R[p, \Delta, \infty] < \infty$ is equivalent to Δ-superstability of p, for Δ closed under Boolean operations (see Theorem 2.16).

The following was introduced by Shelah (in [Sha]) to study superstable theories

Definition 1.9. $D[p, \Delta, \lambda] \geq 0$ if p is consistent.
$D[p, \Delta, \lambda] \geq \alpha$, for α limit, if for every $\beta < \alpha$, $D[p, \Delta, \lambda] \geq \beta$.
$D[p, \Delta, \lambda] \geq \alpha + 1$ if for every finite $q \subseteq p$, for every $\mu < \lambda$, there are a finite $r \supseteq q$, $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$, $n < \omega$, and $\{\mathbf{a}_i : i \leq \mu\}$ such that:

(1) set $\{\varphi(\mathbf{x}; \mathbf{a}_i) : i \leq \mu\}$ is n-contradictory over r;
(2) for every $i \leq \mu$, $D[r \cup \varphi(\mathbf{x}; \mathbf{a}_i), \Delta, \lambda] \geq \alpha$.

Remarks 1.10. (1) Buechler [Bu] denotes by $D^\infty[p]$ the function $D[p, L, \infty]$ from Definition 1.9.
(2) One of the reasons to require the formulas $\{\varphi(\mathbf{x}; \mathbf{a}_i) \mid i \leq \mu\}$ to be contradictory over some *extension* of q, is to make sure the rank has

the monotonicity property. Clearly, when $\Delta = L$ this requirement can be dropped. In the next section we prove that we don't need to consider the arbitrary finite extension in the definition of D if p is finite, Δ is closed under Boolean operations and p is Δ-stable.

(3) It is well-known that the D-rank is not equal to R-rank in some situations.
(4) The D rank obeys most of the basic properties of the R-rank from Fact 1.3.

We introduce the Deg-rank, which is a simplification of the D-rank. The motivation is to eliminate the extra requirement of having to look at all possible finite extensions of q in the definition of D. We succeed in using Deg, instead of D, to characterize local superstability of types, to find bounds for the D-rank and to eventually prove the equality of $S1$, R, and D under certain conditions. The price to pay for using Deg instead of D is the extra requirement: $\theta(\mathbf{x};\mathbf{y}) \in \Delta$. At the end of Section 2 we show how to remove it and still have the above mentioned equalities.

The Deg-rank is an interpolant of Shelah's ranks: $D[p,\Delta,\lambda]$ (from [Sha]) and $Deg[p]$ (from [Sh 10]).

Definition 1.11. $Deg[p,\Delta,\lambda] \geq 0$ if p is consistent.
$Deg[p,\Delta,\lambda] \geq \delta$, for δ limit, if for every $\alpha < \delta$, $Deg[p,\Delta,\lambda] \geq \alpha$.
$Deg[p,\Delta,\lambda] \geq \alpha+1$ if for every finite $q \subseteq p$, for every $\mu < \lambda$, there exist $\varphi(\mathbf{x};\mathbf{y}) \in \Delta$, $n < \omega$ and $\{\mathbf{a}_i : i \leq \mu\}$ such that:

(1) set $\{\varphi(\mathbf{x};\mathbf{a}_i) : i \leq \mu\}$ is *n-contradictory*, namely, for all $i_1 < \cdots < i_n \leq \mu$ we have that $\models \neg\exists \mathbf{x} \bigwedge_{k=1}^{n} \varphi(\mathbf{x};\mathbf{a}_{i_k})$;
(2) for every $i \leq \mu$, $Deg[q \cup \{\varphi(\mathbf{x};\mathbf{a}_i)\},\Delta,\lambda] \geq \alpha$.

Remarks 1.12. (1) The rank $Deg[p,L,|T|^{++}]$ is $Deg[p]$ in [Sh 10].
(2) When $\Delta = L$, then trivially $Deg[p,\Delta,\lambda] = D[p,\Delta,\lambda]$. When Δ is a proper subset of the set of formulas of L, then the ranks D and Deg are different even for some totally categorical theories.
(3) $Deg[p,L,(2^{|T|})^{++}]$ is what Baldwin in [Ba2] calls *the continuous rank*, denoted by $R_C[p]$. The same rank is denoted by Pillay (in page 72 of [Pi]) as $D(p)$. Notice that by the Normalization lemma, Lemma 1.16 and Proposition 1.15, $Deg[p,L,(2^{|T|})^{++}] = Deg[p,L,|T|^{++}]$.

It is not difficult to verify that the analogs to properties in Fact 1.3 also hold for Deg.

The following is a routine exercise (the first inequality being trivial):

Proposition 1.13. *For every p, Δ and $\lambda \geq \aleph_1$,*
$$Deg[p,\Delta,\lambda] \leq D[p,\Delta,\lambda] \leq R[p,\Delta,\lambda].$$

The Deg-rank can be used to characterize (local) superstability, under the assumption of (local) stability. To show that, we need the tree-characterization and normalization lemmas for Deg-rank. First we define the trees.

The following is a special case of Definition II.3.3 of [Sh 10]:

Definition 1.14. Let α be an ordinal and μ a cardinal. We call h *is* (Δ, α)*-function for* $\mathbf{x}$ iff $h : ds(\alpha) \to \Delta \times \omega$, where

$ds(\alpha) := \{\eta \in {}^{\omega>}\alpha \ : \ \forall \ell < \ell(\eta) - 1 \ (\eta[\ell] > \eta[\ell+1])\}$ and $h(\eta) = \langle \psi_\eta(\mathbf{x}; \mathbf{y}), n_\eta \rangle$.

For $U \subseteq ds(\alpha)$ define

$$\Gamma^U_\mu(\theta(\mathbf{x}; \mathbf{a}), h) :=$$
$$\{\exists \mathbf{x}[\theta(\mathbf{x}; \mathbf{a}) \wedge \bigwedge_{0<\ell\leq\ell(\eta)} \psi_{\eta\restriction\ell}(\mathbf{x}; \mathbf{y}_{\eta\restriction\ell, \nu\restriction\ell})] \mid \eta \in U \cup \{\langle\rangle\}, \nu \in {}^{\ell(\eta)}\mu\} \cup$$
$$\{\neg\exists \mathbf{x}[\bigwedge_{i \in w} \psi_\eta(\mathbf{x}; \mathbf{y}_{\eta, \nu\char`^ i}) \mid \eta \in U, w \subseteq \mu, |w| = n_\eta, \nu \in {}^{\ell(\eta)-1}\mu\}.$$

The following is an important technical property of the previous trees, obtained by using Compactness:

Proposition 1.15. *Let* α, μ, $\theta(\mathbf{x}; \mathbf{a})$, Δ *and* h *be as in the previous definition.*

$\Gamma^{ds(\alpha)}_\mu(\theta(\mathbf{x}; \mathbf{a}), h)$ *is consistent* $\iff$ $\Gamma^{ds(\alpha)}_\lambda(\theta(\mathbf{x}; \mathbf{a}), h)$ *is consistent* $\forall \lambda \geq \aleph_0$.

A proof similar to the one of Theorem II.3.6 in [Sh 10] gives:

Lemma 1.16 (Tree characterization lemma for *Deg*)**.** *Let* $\theta(\mathbf{x}; \mathbf{a})$ *be given and let* $\mathrm{cf}\,\mu > |\alpha| + |T|$*. The following are equivalent:*

(1) $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \alpha$*;*
(2) *There exists a* (Δ, α)*-function* h *in* $\mathbf{x}$ *such that* $\Gamma^{ds(\alpha)}_\mu(\theta(\mathbf{x}; \mathbf{a}), h)$ *is consistent.*

Corollary 1.17. *Let* $\mu > |T| + \aleph_0$ *be a regular cardinal.*
If $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \geq \mu$ *then there exists a* (Δ, μ)*-function* h *such that* $\Gamma^{ds(\mu)}_\mu(\theta(\mathbf{x}; \mathbf{a}), h)$ *is consistent.*

The following lemma asserts that when $\alpha \geq |T|^+$, the tree of formulas given by h in the Lemma 1.16 depends only on the length of η.

Lemma 1.18 (Normalization Lemma)**.** *Let* $\theta(\mathbf{x}; \mathbf{a})$ *be given.*
$Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \geq |T|^+$ *iff there exists* $\{\psi_k, n_k \mid 0 < k < \omega\}$ *such that for every* α *the set* $\Gamma^{ds(\alpha)}_{|T|^+}(\theta(\mathbf{x}; \mathbf{a}), h)$ *is consistent for a* (Δ, α)*-function* h *satisfying* $h(\eta) = \langle \psi_{\ell(\eta)}(\mathbf{x}; \mathbf{y}), n_{\ell(\eta)} \rangle \quad \forall \eta \in ds(\alpha)$.

We omit the proof since it is similar to the proof of Lemma II 3.7 in [Sh 10].

Corollary 1.19. *Let* $\theta(\mathbf{x}; \mathbf{a})$ *be given.*

(1) $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \geq |T|^+$ *iff* $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] = \infty$.
(2) $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] = \infty$ *iff*
there exists $\{\langle \psi_k, n_k \rangle \mid 0 < k < \omega\} \subseteq \Delta \times \omega$ *and there are* $\{\mathbf{a}_\eta \mid \eta \in {}^{\omega>}|T|^+\}$ *such that*

$$\forall \eta \in {}^{\omega}|T|^+ \quad \{\theta(\mathbf{x}; \mathbf{a})\} \cup \{\psi_k(\mathbf{x}; \mathbf{a}_{\eta\restriction k}) \mid k < \omega\}$$

is consistent and for every $0 < k < \omega$ *and for every* $\eta \in {}^{k-1}|T|^+$ *we have that*

$$\{\psi_k(\mathbf{x}; \mathbf{a}_{\eta\hat{}i}) \mid i < |T|^+\} \quad \textit{is } n_k\textit{-contradictory.}$$

Proof. (1) Suppose that $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \geq |T|^+$. By the Normalization Lemma there is h such that for every α the set $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$ is consistent. Using now the Compactness Theorem and Lemma 1.16 we get that
$Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \geq \alpha$ for every α, namely $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] = \infty$. The converse is trivial.

(2) Apply the Normalization Lemma to $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \geq |T|^+$ to produce a $(\Delta, |T|^+)$-function h as there. Let $\{\mathbf{a}_\eta\} \models \Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$.

For the converse, use $h(\eta) := \langle \psi_{\ell(\eta)}, n_{\ell(\eta)} \rangle$ to show that $\Gamma_\mu^{ds(\alpha)}(\theta(\mathbf{x}; \mathbf{a}), h)$ is consistent for every infinite α and apply Lemma 1.16. $\dashv$

Definition 1.20. Let p be a type. We say that p has *the λ-weak tree property over* Δ if there are $\{\varphi_n(\mathbf{x}; \mathbf{y}_n) \in \Delta \mid n < \omega\} \subseteq L(T)$ and $\{\mathbf{a}_\eta \mid \eta \in {}^{\omega>}\lambda\}$ such that the set $p \cup \{\varphi_n(\mathbf{x}; \mathbf{a}_{\eta\restriction n}) \mid n < \omega\}$ is consistent for all $\eta \in {}^{\omega}\lambda$, and for all $\eta \in {}^{\omega>}\lambda$ and all infinite $S \subseteq \lambda$ the set $\{\varphi_{\ell(\eta)+1}(\mathbf{x}; \mathbf{a}_{\eta\hat{}\alpha}) \mid \alpha \in S\}$ is inconsistent.

Using the compactness theorem from Corollary 1.19 we get:

Corollary 1.21. *Let* $\theta(\mathbf{x}; \mathbf{a})$ *be given. If* $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \geq |T|^+$ *then* $\theta(\mathbf{x}; \mathbf{a})$ *has the* λ*-weak tree property over* Δ *for every* $\lambda \geq \aleph_0$.

Corollary 1.22. (1) *If* $Deg[\mathbf{x} = \mathbf{x}, L, |T|^{++}] \geq |T|^+$ *then* T *is not superstable.*
(2) *If* $Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, |T|^{++}] \geq |T|^+$ *then* $\theta(\mathbf{x}; \mathbf{a})$ *is not* Δ*-superstable.*

Proof. Let μ be given. By the previous corollary applied to $\lambda := \beth_\omega(\mu)$ we get that the theory is not stable in λ. $\dashv$

2. The main theorem

To prove Theorem 0.1 it suffices (by Proposition 1.13) to show that $R[p, \Delta, \lambda] \leq D[p, \Delta, \lambda]$. To do this we introduce the S-rank and show that under certain conditions, $R[p, \Delta, \lambda] \leq S[p, \Delta, \lambda]$ and $S[p, \Delta, \lambda] \leq D[p, \Delta, \lambda]$. The S-rank is a close relative of the rank function $S1$ introduced by Hrushovski in the early nineties, see [Hr], [HP1] and [CH]. The definition of $S1$-rank (see Section 3) is motivated by algebraic dimension theory. We find it surprising that in the quest to settle a combinatorial problem – the Main Theorem – one naturally discovers such a geometric object.

Definition 2.1. Let p be a set of formulas, not necessarily finite. Let Δ be a set of formulas, and let λ be a cardinal. In most of what follows, λ is an infinite uncountable cardinal; however, the definition makes sense for finite and countable λ as well.

$S[p, \Delta, \lambda] \geq 0$ if p is consistent.
$S[p, \Delta, \lambda] \geq \alpha$, for α limit, if for every $\beta < \alpha$, $S[p, \Delta, \lambda] \geq \beta$.
$S[p, \Delta, \lambda] \geq \alpha + 1$ if for every finite $p_0 \subset p$ and for every $\mu < \lambda$ there exist $\psi(\mathbf{x}; \mathbf{y}) \in \Delta$ and $\{\mathbf{b}_i \mid i < \mu\}$ such that

(1) $S[p_0 \cup \{\psi(\mathbf{x}; \mathbf{b}_i)\}, \Delta, \lambda] \geq \alpha$ for every $i < \mu$ and
(2) $S[p_0 \cup \{\psi(\mathbf{x}; \mathbf{b}_i) \wedge \psi(\mathbf{x}; \mathbf{b}_j)\}, \Delta, \lambda] < \alpha$ for $i \neq j < \mu$.

(When $\lambda = \infty$ we interpret it as no restriction on μ).

Similarly to Fact 1.3 we get Invariance, Monotonicity, Finite Character, Ultrametric and Extension properties for S-rank, provided we use Δ-types only.

Lemma 2.2. *Given sets of formulas p, q, and Δ; Δ closed under Boolean operations:*

Invariance of S rank: *For $f \in \mathrm{Aut}(\mathfrak{C})$, $S[f(p), \Delta_1, \lambda] = S[p, \Delta_1, \lambda]$.*
Monotonicity of S rank: *If q is a Δ-type, then $S[p \cup q, \Delta, \lambda] \leq S[p, \Delta, \lambda]$.*
Finite Character for S rank: *There exists $p_0 \subseteq_{finite} p$ such that $S[p, \Delta, \lambda] = S[p_0, \Delta, \lambda]$.*
Ultrametric property for S rank: *Let λ be infinite uncountable. For $n < \omega$ and $\{\psi_l \mid l \leq n\} \subset \Delta$,*

$$S[p \cup \{\bigvee_{1 \leq l \leq n} \psi_l\}, \Delta, \lambda] = \mathrm{Max}_{1 \leq l \leq n} S[p \cup \{\psi_l\}, \Delta, \lambda].$$

Extension property for S rank: *Suppose Δ is closed under disjunctions. Given a set $A \supseteq \mathrm{dom}(p)$, there exists $q \in S_\Delta(A)$ such that $S[p, \Delta, \lambda] = S[p \cup q, \Delta, \lambda]$.*

Theorem 2.3. *Let Δ be a set of formulas in $\mathbf{x}$ closed under Boolean operations, $\theta(\mathbf{x}; \mathbf{y}) \in \Delta$; and let $\mu \geq |T|^+$ be a regular cardinal. If $\theta(\mathbf{x}; \mathbf{a})$ is Δ-stable, then $S[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+] \leq Deg[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu^+]$.*

Proof. By induction on ordinals α, we show for all finite p that

$$S[p, \Delta, \mu] \geq \alpha \Rightarrow Deg[p, \Delta, \mu] \geq \alpha. \qquad (*)_\alpha$$

We show that it suffices to prove $(*)_\alpha$ just for $\alpha < |T|^+$:

Claim 2.4. *Suppose that $(*)_\alpha$ holds for every $\alpha < |T|^+$.*
If $S[p, \Delta, \mu^+] \geq |T|^+$ then $Deg[p, \Delta, \mu^+] = \infty$

Proof. Since $S[p, \Delta, \mu^+] \geq |T|^+$ we have by the definition of S that $S[p, \Delta, \mu^+] \geq \alpha$ for every $\alpha < |T|^+$. By $(*)_\alpha$ and the definition of Deg this gives $Deg[p, \Delta, \mu^+] \geq |T|^+$. By an application of Corollary 1.22(1) we are done. $\dashv$

If $\alpha = 0$, then $(*)_\alpha$ holds by the definitions of the ranks. For α a limit ordinal, $(*)_\alpha$ follows from the continuity of the ranks and the induction hypothesis. Let $\alpha = \beta + 1$ be such that $S[\theta(\mathbf{x}; \mathbf{a}), \Delta, \mu] \geq \alpha$ and $(*)_\beta$ is true. By the definition of the S rank, there are $\{\mathbf{b}_i \mid i < \mu\}$ and a formula $\varphi(\mathbf{x}; \mathbf{y}) \in \Delta$ such that

(1) $S[\theta(\mathbf{x};\mathbf{a})\wedge\varphi(\mathbf{x};\mathbf{b}_i),\Delta,\mu]\geq\beta$ for every $i<\mu$ and
(2) $S[\theta(\mathbf{x};\mathbf{a})\wedge\varphi(\mathbf{x};\mathbf{b}_i)\wedge\varphi(\mathbf{x};\mathbf{b}_j),\Delta,\mu]<\beta$ for $i\neq j<\mu$.

Since $\theta(\mathbf{x};\mathbf{a})$ is Δ-stable, by Theorem 1.5 no formula in the set Δ has the order property over $\theta(\mathbf{x};\mathbf{a})$. Therefore, there is a number $n<\omega$ witnessing the failure of the order property for $\varphi(\mathbf{x};\mathbf{y})$, i. e. , for no $\{\mathbf{b}_k \mid k<n\}$

$$\bigwedge_{l<n}\exists\mathbf{x}\left[\theta(\mathbf{x};\mathbf{a})\wedge\bigwedge_{k<n}\varphi(\mathbf{x};\mathbf{b}_k)^{\text{if }(k\geq l)}\right]$$

holds.

For every $\bar{\imath}=\langle\bar{\imath}[0],\ldots,\bar{\imath}[n]\rangle\in[\mu]^{n+1}$ define:

$$\psi(\mathbf{x};\mathbf{c}^{\bar{\imath}}):=\bigwedge_{l<n}\neg\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[l]})\wedge\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[n]}).$$

Since $S[\theta(\mathbf{x};\mathbf{a})\wedge\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[n]}),\Delta,\mu^+]\geq\beta$ and $\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[n]})$ is logically equivalent to

$$\psi(\mathbf{x};\mathbf{c}^{\bar{\imath}})\vee\left(\left[\bigvee_{l<n}\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[l]})\right]\wedge\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[n]})\right),$$

by the ultrametric property of S rank, we obtain

$$\begin{aligned}\beta\ &\leq S[\theta(\mathbf{x};\mathbf{a})\wedge\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[n]}),\Delta,\mu^+]\\&=\max\{S[\theta(\mathbf{x};\mathbf{a})\wedge\psi(\mathbf{x};\mathbf{c}^{\bar{\imath}}),\Delta,\mu^+],\\&\qquad S[\theta(\mathbf{x};\mathbf{a})\wedge\left[\textstyle\bigvee_{l<n}\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[l]})\right]\wedge\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[n]}),\Delta,\mu^+]\}.\end{aligned}$$

Notice that the second argument in the above maximum is less than β. Indeed, applying the ultrametric property again, by the choice of $\varphi(\mathbf{x};\mathbf{y})$ and parameters $\mathbf{b}_i$, we get

$$\begin{aligned}&S[\theta(\mathbf{x};\mathbf{a})\wedge\left[\textstyle\bigvee_{l<n}\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[l]})\right]\wedge\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[n]}),\Delta,\mu^+]\\&\quad=\max\{S[\theta(\mathbf{x};\mathbf{a})\wedge\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[l]})\wedge\varphi(\mathbf{x};\mathbf{b}_{\bar{\imath}[n]}),\Delta,\mu^+]\mid l<n\}<\beta.\end{aligned}$$

Therefore, $S[\theta(\mathbf{x};\mathbf{a})\wedge\psi(\mathbf{x};\mathbf{c}^{\bar{\imath}}),\Delta,\mu^+]\geq\beta$ for every $\bar{\imath}\in[\mu]^{n+1}$. Clearly, all the formulas $\theta(\mathbf{x};\mathbf{a})\wedge\psi(\mathbf{x};\mathbf{c}^{\bar{\imath}})$ are Δ-stable. Applying the induction hypothesis to $\theta(\mathbf{x};\mathbf{a})\wedge\psi(\mathbf{x};\mathbf{c}^{\bar{\imath}})$, we get

$$\forall\bar{\imath}\in[\mu]^{n+1}\quad Deg[\theta(\mathbf{x};\mathbf{a})\wedge\psi(\mathbf{x};\mathbf{c}^{\bar{\imath}}),\Delta,\mu^+]\geq\beta.$$

By the Tree Characterization Lemma for Deg, for every $\bar{\imath}$ there is a (Δ,β)-function $h_{\bar{\imath}}$ such that $\Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x};\mathbf{a})\wedge\psi(\mathbf{x};\mathbf{c}^{\bar{\imath}}),h_{\bar{\imath}})$ is consistent.

Also, since $S[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+]\geq\beta$, by the induction hypothesis, $Deg[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+]\geq\beta$. So there is a (Δ,β)-function h_* such that $\Gamma_\mu^{ds(\beta)}(\theta(\mathbf{x};\mathbf{a}),h_*)$ is consistent.

Now for every $\bar{\imath}$ define a (Δ,α)-function $h^{\bar{\imath}}$:

$$\text{for }\eta\in ds(\alpha)\quad h^{\bar{\imath}}(\eta):=\begin{cases}h_*(\eta) & \text{if }\eta\in ds(\beta);\\ h_{\bar{\imath}}(\nu) & \text{if }\eta=\langle\beta\rangle\hat{\ }\nu;\\ \langle\psi,n\rangle & \text{if }\eta=\langle\beta\rangle.\end{cases}$$

It is enough to show that there is an $\bar{\imath} \in [\mu]^{n+1}$ such that $\Gamma_{\mu}^{ds(\alpha)}(\theta(\mathbf{x};\mathbf{a}), h^{\bar{\imath}})$ is consistent. Moreover, by compactness, it is enough to show $\Gamma_{\aleph_0}^{ds(\alpha)}(\theta(\mathbf{x};\mathbf{a}), h^{\bar{\imath}})$ is consistent.

Suppose $\Gamma_{\aleph_0}^{ds(\alpha)}(\theta(\mathbf{x};\mathbf{a}), h^{\bar{\imath}})$ is not consistent; then for each $\bar{\imath}$ there is a finite $u(\bar{\imath}) \subseteq ds(\alpha)$ such that
$\Gamma_{\aleph_0}^{u(\bar{\imath})}(\theta(\mathbf{x};\mathbf{a}), h^{\bar{\imath}})$ is inconsistent. Define, for $\bar{\imath} \in [\mu]^{n+1}$,

$$F(\bar{\imath}) := \langle u(\bar{\imath}), h^{\bar{\imath}} \restriction u(\bar{\imath})\rangle.$$

Now applying the Combinatorial Lemma from Appendix B to the function F, where $|\operatorname{dom}(F)| = \mu$ (regular $\geq |T|^+$) and $|\operatorname{rge}(F)| \leq |T|$ (since $|\alpha| \leq |T|$), we get $\delta < \mu$ and an increasing sequence $\{\gamma(k) < \delta : k < \omega\}$ such that for every $l_0 < \cdots < l_n < \omega$,

$$\langle u, h\rangle := F(\gamma(l_0), \dots, \gamma(l_{n-1}), \gamma(l_n)) = F(\gamma(l_0), \dots, \gamma(l_{n-1}), \delta).$$

Claim 2.5. *For $l < \omega$, let $\mathbf{d}_l := \mathbf{c}^{\langle \gamma(0), \dots, \gamma(n-1), \gamma(n+l)\rangle}$. The set $\{\psi(\mathbf{x};\mathbf{d}_l) \mid l < \omega\}$ is n-contradictory over $\theta(\mathbf{x};\mathbf{a})$.*

Proof. By Ramsey's Theorem we may assume that the sequence $\{\mathbf{b}_{\gamma(l)} \mid l < \omega\}$ is such that for all $i(0) < \cdots < i(n) < \omega$ and for all $k \leq n$

$$\exists \mathbf{x}\left[\theta(\mathbf{x};\mathbf{a}) \wedge \bigwedge_{l \leq n} \varphi(\mathbf{x};\mathbf{b}_{\gamma(l)})^{\text{if } (l \geq k)}\right] \Leftrightarrow \exists \mathbf{x}\left[\theta(\mathbf{x};\mathbf{a}) \wedge \bigwedge_{l \leq n} \varphi(\mathbf{x};\mathbf{b}_{\gamma(i(l))})^{\text{if } (l \geq k)}\right] \quad (*)$$

If $\{\psi(\mathbf{x};\mathbf{d}_l) \mid l < \omega\}$ is not n-contradictory over $\theta(\mathbf{x};\mathbf{a})$, then there are $\{l(k) \mid k < n\}$ such that

$$\models \exists \mathbf{x}\left[\theta(\mathbf{x};\mathbf{a}) \wedge \bigwedge_{k<n} \psi(\mathbf{x};\mathbf{d}_{l(k)})\right],$$

i. e. , by the definition of ψ and $\mathbf{d}_{l(k)}$,

$$\models \exists \mathbf{x}\left[\theta(\mathbf{x};\mathbf{a}) \wedge \bigwedge_{k<n} \neg\varphi(\mathbf{x};\mathbf{b}_{\gamma(k)}) \wedge \bigwedge_{k<n} \varphi(\mathbf{x};\mathbf{b}_{\gamma(n+l(k))})\right].$$

In particular, for every $m \leq n$,

$$\models \exists \mathbf{x}\left[\theta(\mathbf{x};\mathbf{a}) \wedge \bigwedge_{k<m} \neg\varphi(\mathbf{x};\mathbf{b}_{\gamma(k)}) \wedge \bigwedge_{k=m}^{n-1} \varphi(\mathbf{x};\mathbf{b}_{\gamma(n+l(k))})\right].$$

Now by $(*)$ we get

$$\models \exists \mathbf{x}\left[\theta(\mathbf{x};\mathbf{a}) \wedge \bigwedge_{k<m} \neg\varphi(\mathbf{x};\mathbf{b}_{\gamma(k)}) \wedge \bigwedge_{k=m}^{n-1} \varphi(\mathbf{x};\mathbf{b}_{\gamma(k)})\right],$$

which is a contradiction to the choice of n, which was chosen to witness the failure of the order property over $\theta(\mathbf{x};\mathbf{a})$ for φ. $\dashv$

We may assume that u is closed under taking initial segments and $\Gamma^u_{\aleph_0}(\theta(\mathbf{x};\mathbf{a}),h)$ is consistent: putting $\bar{\imath}_l := \langle\gamma(0),\dots,\gamma(n_1-1),\gamma(n_1+l)\rangle$, $\mathbf{d}_l := \mathbf{c}^{\bar{\imath}_l}$ for $l<\omega$, and

$$\mathbf{b}_{\eta,\nu} := \begin{cases} \mathbf{d}_l & \text{if } \eta=\langle\beta\rangle, \nu=\langle l\rangle;\\ \mathbf{b}^{\bar{\imath}_l}_{\xi,\zeta} & \text{if } \eta=\langle\beta\rangle\hat{\ }\xi, \nu=\langle l\rangle\hat{\ }\zeta;\\ \mathbf{b}^*_{\eta,\nu} & \text{if } \eta\in ds(\beta),\end{cases}$$

for $\eta\in u$, $\nu\in {}^{\ell(\eta)}\aleph_0$, we have $\{\mathbf{b}_{\eta,\nu}\}\models\Gamma^u_{\aleph_0}(\theta(\mathbf{x};\mathbf{a}),h)$. This contradicts the choice of u and we are done.

$\dashv$

The following fact now trivially follows from Theorems 2.3 and 1.13.

Corollary 2.6. *Let Δ be a set of formulas in $\mathbf{x}$ closed under Boolean operations, $\theta(\mathbf{x};\mathbf{y})\in\Delta$; and let $\mu\geq|T|^+$ be regular. If $\theta(\mathbf{x};\mathbf{a})$ is Δ-stable, then*

$$S[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+]\leq Deg[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+]\leq D[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+]\leq R[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+].$$

In order to prove the main theorem, we need to show that $R[\theta(\mathbf{x};\mathbf{a}),\Delta,\lambda]\leq S[\theta(\mathbf{x};\mathbf{a}),\Delta,\lambda]$ under appropriate assumptions. The following lemma will provide the link between the R-rank and the S-rank. It fuels the successor stage of an induction proof of Theorem 2.12.

Lemma 2.7. *Suppose that μ is a regular cardinal, $\mu\geq|T|^+$, and Δ is a set of formulas which is closed under Boolean operations. If $R[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+]\geq\beta+1$ and*

1. *$R[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+]<\infty$ or*
2. *$\theta(\mathbf{x};\mathbf{a})$ is Δ-stable and for every set $\{\mu_i\mid i<|\Delta|+\aleph_0\}$ of cardinalities all less than μ, we have $\prod_{i<|\Delta|+\aleph_0}\mu_i<\mu$ (e.g. $\mu=(2^{|T|})^+$ is such a cardinality),*

then there is a formula $\varphi^*(\mathbf{x},\mathbf{y})\in\Delta$ and a set $\{\mathbf{c}_i\mid i<\mu\}$ such that

1. *$R[\theta(\mathbf{x};\mathbf{a})\wedge\varphi^*(\mathbf{x},\mathbf{c}_i),\Delta,\mu^+]\geq\beta$ for all $i<\mu$;*
2. *$R[\theta(\mathbf{x};\mathbf{a})\wedge\varphi^*(\mathbf{x},\mathbf{c}_i)\wedge\varphi^*(\mathbf{x},\mathbf{c}_j),\Delta,\mu^+]<\beta$ for $i\neq j<\mu$.*

Proof. Since $R[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+]\geq\beta+1$, there are $\{p_i\mid i\leq\mu\}$ explicitly contradictory Δ-types such that $R[\{\theta(\mathbf{x};\mathbf{a})\}\cup p_i,\Delta,\mu^+]\geq\beta$ for all $i\leq\mu$. Using the hypothesis of the lemma, we get

Claim 2.8. *There are a set A, a formula $\varphi(\mathbf{x};\mathbf{y})\in\Delta$ and $\{p_i\mid i<\mu\}\subseteq S_\Delta(A)$ such that $p_i\restriction\varphi\neq p_j\restriction\varphi$ for every $i\neq j$ and $R[\{\theta(\mathbf{x};\mathbf{a})\}\cup p_i,\Delta,\mu^+]\geq\beta$ holds for every $i<\mu$.*

Proof. The argument is by cases corresponding to the hypotheses of the lemma.

(1) Suppose that $R[\theta(\mathbf{x};\mathbf{a}),\Delta,\mu^+]<\infty$. Using Theorem 1.3 (by replacing the p_i's by extensions) we may assume that $R[\{\theta(\mathbf{x};\mathbf{a})\}\cup p_i,\Delta,\mu^+]=\beta$ for all $i\leq\mu$. By the finite character there are $q_i\subseteq_{finite}p_i$ such that $R[\{\theta(\mathbf{x};\mathbf{a})\}\cup q_i,\Delta,\mu^+]=\beta$ for all $i\leq\mu$.

Subclaim 2.9. *For every $i < \mu$ the set $\{j < \mu \mid q_i \subseteq p_j\}$ is bounded.*

Proof. Otherwise, there is $S \subseteq \mu$ of cardinality μ such that there exists i_0 satisfying $q_{i_0} \subseteq p_j$ for all $j \in S$. Namely by monotonicity we have that $R[\{\theta(\mathbf{x};\mathbf{a})\} \cup q_{i_0} \cup p_j, \Delta, \mu^+] = \beta$ for every $j \in S$. This, by the definition of R, implies that $R[\{\theta(\mathbf{x};\mathbf{a})\} \cup q_{i_0}, \Delta, \mu^+] = \beta + 1$ in contradiction to the choice of q_i. $\dashv$

By induction on $i < \mu$ we can now define an increasing sequence $\{j(i) < \mu \mid i < \mu\}$ such that $q_\xi \not\subseteq p_{j(i)}$ for all $\xi \leq i$. By renumerating the set $\{p_{j(i)} \mid i < \mu\}$ we may assume that $\{p_i \mid i < \mu\}$ also satisfies $q_i \not\subseteq p_j$ for all $i < j$. Since Δ is closed under Boolean operations, the formula $\varphi_i := \bigwedge q_i$ is a Δ-formula. Consider the mapping $i \mapsto \varphi_i$. Since the domain is a regular cardinal larger than $|T|$, there are a formula $\varphi(\mathbf{x};\mathbf{y}) \in \Delta$ and a set $S \subseteq \mu$ of cardinality μ such that $\varphi_i = \varphi$ for all $i \in S$. Since $q_i \not\subseteq p_j$ for $j > i$, we have that $\varphi_i \notin p_j$, and since the types are complete this entails that $\neg\varphi_i \in p_j$.

We have shown that $i \neq j$ implies that $p_i \restriction \varphi \neq p_j \restriction \varphi$.

(2) For this part we do not use the local stability assumption; we just use that $\prod_{i<|\Delta|+\aleph_0} \mu_i < \mu$ for every $\{\mu_i < \mu \mid i < |\Delta| + \aleph_0\}$. If there is no set of μ many types as required then let

$$\mu_\varphi := |\{p_i \restriction \varphi \ : \ i < \mu\}| < \mu \quad \text{for every} \quad \varphi \in \Delta.$$

However, the mapping $i \mapsto (p_i \restriction \varphi)_{\varphi\in\Delta}$ is an injection from μ into $\prod_{\varphi\in\Delta}\{p_i \restriction \varphi : i < \mu\}$, which contradicts the above cardinal arithmetic assumption.

$\dashv$

Claim 2.10. $R[\theta(\mathbf{x};\mathbf{a}), \varphi(\mathbf{x};\mathbf{y}), 2] < \omega$.

Proof. We have two arguments according to the hypothesis of the lemma:
(1) By monotonicity, $R[\theta(\mathbf{x};\mathbf{a}), \varphi, \infty] \leq R[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+]$, and by Theorem 1.6 (1), $R[\theta(\mathbf{x};\mathbf{a}), \varphi, 2] < \omega$.
(2) Since $\theta(\mathbf{x};\mathbf{a})$ is Δ-stable, it is $\{\varphi\}$-stable for every $\varphi \in \Delta$. By Theorem 1.6 this entails $R[\theta(\mathbf{x};\mathbf{a}), \varphi, 2] < \omega$. $\dashv$

By the above claim, $R[\theta(\mathbf{x};\mathbf{a}), \varphi(\mathbf{x};\mathbf{y}), 2]$ is a natural number. Let $n_0 := R[\theta(\mathbf{x};\mathbf{a}), \varphi, 2]$.

Claim 2.11. *If the set $\Phi \subseteq \{q \in S_\varphi(A) \mid \theta(\mathbf{x};\mathbf{a}) \cup q$ is consistent$\}$ has cardinality $\kappa \geq \aleph_0$ then there exists $\{r_i, p_i \mid i < \kappa, r_i \subseteq p_i, p_i \in \Phi\}$ such that*

(1) $|r_i| = n_0 + 2$,
(2) *for every $i < \kappa$ if $q \in \Phi$ and $r_i \subseteq q$ then $q = p_i$, and*
(3) *$i \neq j$ implies $p_i \neq p_j$.*

We present the proof of Claim 2.11 in the Appendix.

For every $i < \mu$, we have that $R[\theta(\mathbf{x};\mathbf{a}) \cup p_i \restriction \varphi, \Delta, \mu^+] \geq \beta$. Thus for $\Phi := \{q \in S_\varphi(A) \mid R[\theta(\mathbf{x};\mathbf{a}) \cup q, \Delta, \mu^+] \geq \beta\}$, it follows that $|\Phi| \geq \mu$. Applying the previous claim to Φ, let $\{r_i : i \leq \mu\}$ be as in the claim.

For $i < \mu$ define $\varphi^*(\mathbf{x};\mathbf{c}_i) := \bigwedge r_i$ (since $\mu \geq |T|^+$, we may assume that $\varphi^*(\mathbf{x};\mathbf{y})$ does not depend on i).

Since $r_i \subseteq p_i$, for all $i < \mu$, and $R[\theta \cup p_i, \Delta, \mu^+] \geq \beta$, we get

$$R[\theta(\mathbf{x};\mathbf{a}) \wedge \varphi^*(\mathbf{x};\mathbf{c}_i), \Delta, \mu^+] \geq \beta.$$

However, if $i \neq j$, then

$$R[\theta(\mathbf{x};\mathbf{a}) \wedge \varphi^*(\mathbf{x};\mathbf{c}_i) \wedge \varphi^*(\mathbf{x};\mathbf{c}_j), \Delta, \mu^+] < \beta.$$

Otherwise, by the Extension Property there would be a $q \in S_\varphi(A)$ extending $\varphi^*(\mathbf{x};\mathbf{c}_i) \wedge \varphi^*(\mathbf{x};\mathbf{c}_j)$ such that $R[\theta(\mathbf{x};\mathbf{a}) \cup q, \Delta, \mu^+] \geq \beta$, which would contradict the uniqueness clause (2) from the construction of r_i's from Claim 2.11. ⊣

Theorem 2.12. *Suppose that μ is regular cardinal satisfying $\mu \geq |T|^+$ and Δ is a set of formulas which is closed under Boolean operations, with $\theta(\mathbf{x};\mathbf{y}) \in \Delta$. If*

(1) *$R[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] < \infty$ or*
(2) *$\theta(\mathbf{x};\mathbf{a})$ is Δ-stable and for every $\{\mu_i \mid i < |\Delta| + \aleph_0\}$ cardinalities all less than μ, $\prod_{i<|\Delta|+\aleph_0} \mu_i < \mu$*

holds, then $S[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \geq R[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+]$.

Proof. We show that for all ordinals α,

$$(*)_\alpha \qquad R[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \geq \alpha \Rightarrow S[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \geq \alpha.$$

We proceed by induction on α. For $\alpha = 0$, $(*)_\alpha$ holds by the definitions of ranks. For α a limit ordinal, $(*)_\alpha$ follows from the continuity of the ranks and the induction hypothesis. Suppose $R[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu] \geq \alpha = \beta + 1$ and that $(*)_\beta$ holds. Then by Lemma 2.7 we have a formula $\varphi^*(\mathbf{x};\mathbf{y}) \in \Delta$ and a set $\{\mathbf{c}_i \mid i < \mu\}$ such that

(1) $R[\theta(\mathbf{x};\mathbf{a}) \wedge \varphi^*(\mathbf{x};\mathbf{c}_i), \Delta, \mu^+] \geq \beta$ for all $i < \mu$;
(2) $R[\theta(\mathbf{x};\mathbf{a}) \wedge \varphi^*(\mathbf{x};\mathbf{c}_i) \wedge \varphi^*(\mathbf{x},\mathbf{c}_j), \Delta, \mu^+] < \beta$ for $i \neq j < \mu$.

By induction hypothesis (1) gives us $S[\theta(\mathbf{x};\mathbf{a}) \wedge \varphi^*(\mathbf{x};\mathbf{c}_i), \Delta, \mu^+] \geq \beta$ for all $i < \mu$. And (2) together with Corollary 2.6 imply

$$S[\theta(\mathbf{x};\mathbf{a}) \wedge \varphi^*(\mathbf{x};\mathbf{c}_i) \wedge \varphi^*(\mathbf{x};\mathbf{c}_j), \Delta, \mu^+] < \beta \text{ for } i \neq j \in I.$$

By the definition of S rank, we have $S[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \geq \alpha$, completing the induction. ⊣

Corollary 2.6 and Theorem 2.12 give the following

Theorem 2.13. *Suppose that μ is regular satisfying $\mu \geq |T|^+$ and Δ is a set of formulas closed under Boolean operations with $\theta(\mathbf{x};\mathbf{y}) \in \Delta$. If*

(1) *$R[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] < \infty$ or*
(2) *$\theta(\mathbf{x};\mathbf{a})$ is Δ-stable and μ satisfies: for every sequence of cardinals $\{\mu_i < \mu \mid i < |\Delta| + \aleph_0\}$ we have that $\prod_{i<|\Delta|+\aleph_0} \mu_i < \mu$ holds,*

then $S[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] = Deg[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] = D[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] = R[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+]$.

Now we show how to remove the assumption $\theta(\mathbf{x};\mathbf{y}) \in \Delta$ and get Theorem 0.1. First we prove

Lemma 2.14. [1] *Suppose $\theta(\mathbf{x};\mathbf{a})$ is Δ-stable, where Δ is closed under Boolean operations. Suppose $\lambda \geq \aleph_1$. If $D[\theta,\Delta,\lambda] \geq \alpha+1$ for $\alpha \geq 0$, then for every $\mu < \lambda$ there are $\bar{\psi} \in \Delta$, $k < \omega$, and $\{b_i \mid i < \mu\}$ such that*

(1) *the set $\{\bar{\psi}(\mathbf{x};\mathbf{b}_i) \mid i < \mu\}$ is k-contradictory over $\theta(\mathbf{x};\mathbf{a})$;*
(2) *for every $i < \mu$, $D[\theta(\mathbf{x};\mathbf{a}) \wedge \bar{\psi}(\mathbf{x};\mathbf{b}_i),\Delta,\lambda] \geq \alpha$.*

Proof. Suppose $D[\theta(\mathbf{x};\mathbf{a}),\Delta,\lambda] \geq \alpha+1$ for $\alpha \geq 0$. By the definition, for all $\aleph_0 \leq \mu < \lambda$, μ limit, there is a formula $\varphi(\mathbf{x};\mathbf{c})$ with $\models \varphi(\mathbf{x};\mathbf{c}) \to \theta(\mathbf{x};\mathbf{a})$, there are $\psi \in \Delta$ and $\{\mathbf{a}_i \mid i < \mu\}$ such that for some $n < \omega$

(1) $D[\varphi(\mathbf{x};\mathbf{c}) \wedge \psi(\mathbf{x};\mathbf{a}_i),\Delta,\lambda] \geq \alpha$ and
(2) $\{\psi(\mathbf{x};\mathbf{a}_i) \mid i < \mu\}$ is n-contradictory over $\varphi(\mathbf{x};\mathbf{c})$.

Clearly,

$$D[\theta(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{a}_i),\Delta,\lambda] \geq \alpha \quad \text{for all } i < \mu,$$

so if $\{\psi(\mathbf{x};\mathbf{a}_i) \mid i < \mu\}$ is k-contradictory over $\theta(\mathbf{x};\mathbf{a})$ for some $k < \omega$, then we are done. Otherwise, we may assume that the set $\{\theta(\mathbf{x};\mathbf{a})\} \cup \{\psi(\mathbf{x};\mathbf{a}_i) \mid i < \omega\}$ is consistent. Since $\{\psi(\mathbf{x};\mathbf{a}_i) \mid i < \mu\}$ is n-contradictory over $\varphi(\mathbf{x};\mathbf{c})$, we have

$$\models \varphi(\mathbf{x};\mathbf{c}) \to \varphi(\mathbf{x};\mathbf{c}) \wedge \neg \bigwedge_{j<n} \psi(\mathbf{x};\mathbf{a}_j).$$

Hence $D[\varphi(\mathbf{x};\mathbf{c}) \wedge \neg[\bigwedge_{j<n} \psi(\mathbf{x};\mathbf{a}_j)] \wedge \psi(\mathbf{x};\mathbf{a}_i),\Delta,\lambda] \geq \alpha$ for $i < \mu$. By the ultrametric property for the D-rank and the pigeon-hole principle, refining $\{a_i \mid i < \mu\}$ if necessary we may assume that

$$D[\varphi(\mathbf{x};\mathbf{c}) \wedge \neg\psi(\mathbf{x};\mathbf{a}_0) \wedge \psi(\mathbf{x};\mathbf{a}_i),\Delta,\lambda] \geq \alpha \text{ for } 1 \leq i < \mu. \qquad (*_0)$$

By monotonicity

$$D[\theta(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{a}_0) \wedge \psi(\mathbf{x};\mathbf{a}_i),\Delta,\lambda] \geq \alpha \text{ for } 1 \leq i < \mu.$$

If $\{\psi(\mathbf{x};\mathbf{a}_i) \mid 1 \leq i < \mu\}$ is k-contradictory over $\theta(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{a}_0)$ for some $k < \omega$, then we are done.
Otherwise, $\{\theta(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{a}_0)\} \cup \{\psi(\mathbf{x};\mathbf{a}_i) \mid 1 \leq i < \omega\}$ is consistent. Now we replace $\varphi(\mathbf{x};\mathbf{c})$ in $(*_0)$ by $\varphi(\mathbf{x};\mathbf{c}) \wedge \neg \bigwedge_{1\leq i<n+1} \psi(\mathbf{x};\mathbf{a}_i)$, and get as above

$$D[\varphi(\mathbf{x};\mathbf{c}) \wedge \neg\psi(\mathbf{x};\mathbf{a}_0) \wedge \neg\psi(\mathbf{x};\mathbf{a}_1) \wedge \psi(\mathbf{x};\mathbf{a}_i),\Delta,\lambda] \geq \alpha \text{ for } 2 \leq i < \mu. \qquad (*_1)$$

Observe that since we are only *refining* the sequence $\{a_i \mid i < \mu\}$, after the second step we get that both $\{\theta(\mathbf{x};\mathbf{a})\} \cup \{\psi(\mathbf{x};\mathbf{a}_i) \mid i < \omega\}$ and $\{\theta(\mathbf{x};\mathbf{a}) \wedge \neg\psi(\mathbf{x};\mathbf{a}_0)\} \cup \{\psi(\mathbf{x};\mathbf{a}_i) \mid 1 \leq i < \omega\}$ are consistent.

Repeat this process.

[1] The statement of this Lemma appears as Exercise II.3.10 in [Sha]. In a preliminary version of this paper we stated it without proof. One of the readers doubted the correctnes of this, so we have included a complete proof for completness.

Subclaim 2.15. *The above process terminates in finitely many steps, i.e., there is $l < \omega$ and a sequence $\{\mathbf{a}_i \mid i < \mu\}$ such that*

$$D[\varphi(\mathbf{x};\mathbf{c}) \wedge \bigwedge_{j\leq l} \neg\psi(\mathbf{x};\mathbf{a}_j) \wedge \psi(\mathbf{x};\mathbf{a}_i), \Delta, \lambda] \geq \alpha \text{ for } l+1 \leq i < \mu. \qquad (*_l)$$

and $\{\psi(\mathbf{x};\mathbf{a}_i) \mid l+1 \leq i < \mu\}$ are k-contradictory over $\theta(\mathbf{x};\mathbf{a}) \wedge \bigwedge_{j\leq l} \neg\psi(\mathbf{x};\mathbf{a}_j)$ for some k.

Proof. Since θ is Δ-stable, the formula $\psi \in \Delta$ does not have the order property over $\theta(\mathbf{x};\mathbf{a})$. Let $N < \omega$ witnes failure of the order property for ψ. We claim that the above process terminates after at most N steps. Indeed, otherwise we would have a sequence $\{\mathbf{a}_i \mid i < \mu\}$ such that

$$\models \exists \mathbf{x}[\theta(\mathbf{x};\mathbf{a}) \wedge \bigwedge_{j\leq l} \neg\psi(\mathbf{x};\mathbf{a}_j) \wedge \bigwedge_{l<i<N} \psi(\mathbf{x};\mathbf{a}_i)] \text{ for all } l < N,$$

which contradicts the choice of N. $\dashv$

Let $\{\mathbf{a}_i \mid i < \mu\}$, $l < \omega$ and k be as in the subclaim, let

$$\bar{\psi}(\mathbf{x};\mathbf{y}_0,\ldots,\mathbf{y}_l,\mathbf{y}_{l+1}) := \bigwedge_{j\leq l} \neg\psi(\mathbf{x};\mathbf{y}_j) \wedge \psi(\mathbf{x};\mathbf{y}_{l+1}),$$

and let $\mathbf{b}_i := \mathbf{a}_0\hat{\ }\mathbf{a}_1\hat{\ }\ldots\hat{\ }\mathbf{a}_l\hat{\ }\mathbf{a}_{l+1+i}$ for $i < \mu$. Since Δ is closed under Boolean operations, $\bar{\psi} \in \Delta$. By $(*_l)$ and monotonicity $D[\theta(\mathbf{x};\mathbf{a}) \wedge \bar{\psi}(\mathbf{x};\mathbf{b}_i), \Delta, \lambda] \geq \alpha$, and clearly $\{\bar{\psi}(\mathbf{x};\mathbf{b}_i) \mid i < \mu\}$ is k-contradictory over $\theta(\mathbf{x};\mathbf{a})$. $\dashv$

The above lemma shows that we can don't need to consider arbitrary finite extensions of $\theta(\mathbf{x};\mathbf{a})$ to compute $D[\theta(\mathbf{x};\mathbf{a}), \Delta, \lambda]$. This allows us to get the **Main Theorem** from the abstract.

Proof of the Main Theorem. Let p be a finite type, let $\theta(\mathbf{x};\mathbf{a}) := \wedge p$. Prove

(1) $S[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \leq D[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+]$,
(2) $R[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+] \leq S[\theta(\mathbf{x};\mathbf{a}), \Delta, \mu^+]$.

In view of Lemma 2.14, the proof of (1) repeats almost word-for-word that of Theorem 2.3. Proof of (2) is the same as the proof of Theorem 2.12 except one needs to use (1) and Proposition 1.13 instead of Corollary 2.6. $\dashv$

As a byproduct of our results, we get the following characterization of local superstability (notice the alanogy to Theorem II 3.14 from [Sha]):

Theorem 2.16. *Let p be a finite type and let Δ be a set of formulas closed under Boolean operations. The following conditions are equivalent:*

(1) p is Δ-superstable,
(2) p is Δ-stable and $D[p, \Delta, |T|^{++}] < |T|^+$,
(3) p is Δ-stable and $D[p, \Delta, \infty] < |T|^+$,
(4) $R[p, \Delta, \infty] < |T|^+$,
(5) $R[p, \Delta, \infty] < \infty$,
(6) p is Δ-stable and $Deg[p, \Delta, \infty] < \infty$.
If in addition $\wedge p \in \Delta$, then any of (1)–(6) above is equivalent to
(7) p is Δ-stable and $Deg[p, \Delta, |T|^{++}] < |T|^+$,

Remark 2.17. Even more is true in the above: we can add the condition $R[p, \Delta, (2^{|T|})^{++}] < |T|^+$; see Proposition 3.4.

3. Bounds on ranks

As we mentioned above, the following rank was introduced by Hrushovski in the context of geometric dimension theory. We describe the connection between the ranks S, R, D, and $S1$.

Definition 3.1. $S1[p, \Delta] \geq 0$ if p is consistent.
$S1[p, \Delta] \geq \alpha$, for α limit, if for every $\beta < \alpha$, $S1[p, \Delta] \geq \beta$.
$S1[p, \Delta] \geq \alpha + 1$ if for every finite $p_0 \subset p$ there exists $\psi(\mathbf{x}; \mathbf{y}) \in \Delta$ and $\{\mathbf{b}_n \mid n < \omega\}$ indiscernibles over $\mathrm{dom}(p_0)$ such that

(1) $S1[p_0 \cup \{\psi(\mathbf{x}; \mathbf{b}_n)\}, \Delta] \geq \alpha$ for every $n < \omega$ and
(2) $S1[p_0 \cup \{\psi(\mathbf{x}; \mathbf{b}_n) \wedge \psi(\mathbf{x}; \mathbf{b}_m)\}, \Delta] < \alpha$ for $m \neq n < \omega$.

The $S1$ rank has all the properties of the S rank, and they can be proved in a completely analogous way.

Claim 3.2. *There is a cardinal κ such that for any type p, set of formulas Δ, and $\lambda \geq \kappa^+$ we have $S1[p, \Delta] = S[p, \Delta, \lambda]$. In particular, $S1[p, \Delta] = S[p, \Delta, \infty]$.*

Proof. We use the fact that there exists a cardinal κ such that for any countable set A and any sequence $\{\mathbf{b}_i \mid i < \kappa\}$ there is a sequence $\{c_n \mid n < \omega\}$ of indiscernibles over A such that for all $n < \omega$ there are $i(0) < \cdots < i(n-1) < \kappa$ with $\mathrm{tp}(c_0 \ldots c_{n-1}/A) = \mathrm{tp}(b_{i(0)} \ldots b_{i(n-1)}/A)$. Namely, $\kappa := \beth_{(2^{|T|})^+}$.[2]

We prove by induction

$$S1[p, \Delta] \geq \alpha \Leftrightarrow S[p, \Delta, \lambda] \geq \alpha \qquad (*)_\alpha$$

for every $\lambda \geq \kappa^+$ and every finite type p. Let $\theta(\mathbf{x}; \mathbf{a}) := \bigwedge p$.

If $\alpha = 0$, then $(*)_\alpha$ holds by the definitions of the ranks. For α a limit ordinal, $(*)_\alpha$ follows from the continuity of the ranks and the induction hypothesis.

Let $\alpha = \beta + 1$, $S1[\theta(\mathbf{x}; \mathbf{a}), \Delta] \geq \alpha$ and $(*)_\beta$ is true. By the definition of the $S1$ rank, there are $\psi(\mathbf{x}; \mathbf{y}) \in \Delta$ and $\{\mathbf{b}_n \mid n < \omega\}$ indiscernibles over $\mathbf{a}$ as in the definition. By invariance of the $S1$ rank and Compactness Theorem, there is an indiscernible sequence $\{\mathbf{b}_i \mid i < \lambda\}$ over $\mathbf{a}$, such that

(1) $S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}_i), \Delta] \geq \beta$ for every $i < \lambda$ and
(2) $S1[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}_i) \wedge \psi(\mathbf{x}; \mathbf{b}_j), \Delta] < \beta$ for $i \neq j < \lambda$.

By $(*)_\beta$ we obtain

(1) $S[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}_i), \Delta, \lambda] \geq \beta$ for every $i < \lambda$ and
(2) $S[\theta(\mathbf{x}; \mathbf{a}) \wedge \psi(\mathbf{x}; \mathbf{b}_i) \wedge \psi(\mathbf{x}; \mathbf{b}_j), \Delta, \lambda] < \beta$ for $i \neq j < \lambda$.

[2] This fact appears in [Sh93], Lemma 6.3; for a more detailed account and an improvement of the bound see [GIL], Theorem A1.

Note that we use both directions of $(*)_\beta$ here. By the definition, we see that $S[\theta(\mathbf{x};\mathbf{a}),\Delta,\lambda] \geq \alpha$.

Suppose now that $S[\theta(\mathbf{x};\mathbf{a}),\Delta,\lambda] \geq \alpha$ and fix $\{\mathbf{b}_i \mid i<\kappa\}$ such that

(1) $S[\theta(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{b}_i),\Delta,\lambda] \geq \beta$ for every $i<\kappa$ and
(2) $S[\theta(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{b}_i) \wedge \psi(\mathbf{x};\mathbf{b}_j),\Delta,\lambda] < \beta$ for $i \neq j < \kappa$.

We certainly can find such $\mathbf{b}_i$'s because $\lambda \geq \kappa^+$. By the choice of κ, there is a sequence $\{\mathbf{c}_k \mid k<\omega\}$ of indiscernibles over $\mathbf{a}$ such that for every $n<\omega$ there are $i(0) < i(1) < \cdots < i(n-1) < \kappa$ such that $\mathrm{tp}(\mathbf{c}_0 \ldots \mathbf{c}_{n-1}/\mathbf{a}) = \mathrm{tp}(\mathbf{b}_{i(0)} \ldots \mathbf{b}_{i(n-1)}/\mathbf{a})$. By invariance of the S rank, we have

(1) $S[\theta(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}_n),\Delta,\lambda] \geq \beta$ for every $n<\omega$ and
(2) $S[\theta(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}_n) \wedge \psi(\mathbf{x};\mathbf{c}_m),\Delta,\lambda] < \beta$ for $n \neq m < \omega$.

Using $(*)_\beta$, we get

(1) $S1[\theta(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}_n),\Delta] \geq \beta$ for every $n<\omega$ and
(2) $S1[\theta(\mathbf{x};\mathbf{a}) \wedge \psi(\mathbf{x};\mathbf{c}_n) \wedge \psi(\mathbf{x};\mathbf{c}_m),\Delta] < \beta$ for $m \neq n < \omega$

which shows $(*)_\alpha$.

⊣

The following theorem describes the asymptotic behaviour of the ranks and shows that $S1$ is equal to S, R, and D under the local stability assumption.

Theorem 3.3. *Suppose Δ is closed under Boolean operations. If $\theta(\mathbf{x};\mathbf{a})$ is Δ-stable, then*

$$S1[\theta(\mathbf{x};\mathbf{a}),\Delta] = S[\theta(\mathbf{x};\mathbf{a}),\Delta,\infty] = D[\theta(\mathbf{x};\mathbf{a}),\Delta,\infty] = R[\theta(\mathbf{x};\mathbf{a}),\Delta,\infty].$$

Moreover, for every $\lambda \geq (2^{|T|})^{++}$

$$S1[\theta(\mathbf{x};\mathbf{a}),\Delta] = S[\theta(\mathbf{x};\mathbf{a}),\Delta,\lambda] = D[\theta(\mathbf{x};\mathbf{a}),\Delta,\lambda] = R[\theta(\mathbf{x};\mathbf{a}),\Delta,\lambda].$$

Proof. The first set of equalities and

$$S[\theta(\mathbf{x};\mathbf{a}),\Delta,(2^{|T|})^{++}] = D[\theta(\mathbf{x};\mathbf{a}),\Delta,(2^{|T|})^{++}] = R[\theta(\mathbf{x};\mathbf{a}),\Delta,(2^{|T|})^{++}]$$

follow from Theorem 0.1, clause (2), and Claim 3.2.

The rest follows from monotonicity of the ranks in the third argument and

Proposition 3.4. *For a finite type p and Δ closed under Boolean operations, for every $\lambda \geq (2^{|T|})^{++}$*

$$R[p,\Delta,\lambda] = R[p,\Delta,\infty].$$

Proof. By monotonicity, $R[p,\Delta,\lambda] \geq R[p,\Delta,\infty]$. If $R[p,\Delta,\infty] = \infty$, we have nothing to prove. Otherwise, by Theorem 1.7 we get that p is Δ-stable and we are in the conditions of Theorem 0.1, clause (2). Taking into account the equality $D[p,\Delta,|T|^{++}] = D[p,\Delta,\infty]$ from [Sha], II.3.8, by the choice of λ we get $R[p,\Delta,\lambda] = D[p,\Delta,\lambda] = D[p,\Delta,\infty] = R[p,\Delta,\infty]$. ⊣

⊣

Theorem 3.5. *Suppose Δ is closed under Boolean operations.*

(1) Suppose that $\theta(\mathbf{x};\mathbf{a})$ is Δ-stable. If any of the ranks in Theorem 3.3 is bounded (i.e. is less than ∞), then all the ranks there are less than $|T|^+$.

(2) If p is a type, not necessarily finite, and $R[p,\Delta,\infty] < \infty$, then $R[p,\Delta,\infty] < |T|^+$.

Proof. (1) If any of the ranks in Theorem 3.3 is bounded, then $D[\theta(\mathbf{x};\mathbf{a}),\Delta,\infty] < \infty$. By Theorem II.3.8 in [Sha], we have then $D[\theta(\mathbf{x};\mathbf{a}),\Delta,\infty] < |T|^+$. Applying Theorem 3.3 again, we see that all the ranks must be less than $|T|^+$.

(2) By finite character, there is a finite p_0 with $R[p,\Delta,\infty] = R[p_0,\Delta,\infty] < \infty$. By Theorem 2.16 we get $R[p_0,\Delta,\infty] < |T|^+$ and the assertion follows. $\dashv$

4. $S2$-rank

The following is a generalization of the $S1$ rank.

Definition 4.1. Let p be a set of formulas in $\mathbf{x}$.
$S2[p,\Delta] \geq 0$ if p is consistent.
$S2[p,\Delta] \geq \alpha$, for α limit, if for every $\beta < \alpha$, $S2[p,\Delta] \geq \alpha$.
$S2[p,\Delta] \geq \alpha + 1$ if for every finite $p_0 \subset p$ there exists $\psi(\mathbf{x},\mathbf{y}) \in \Delta$, $l < \omega$ and $\{\mathbf{b}_n \mid n < \omega\}$ indiscernibles over $\mathrm{dom}(p_0)$ such that

(1) $S2[p_0 \cup \{\psi(\mathbf{x},\mathbf{b}_n)\},\Delta] \geq \alpha$ for every $n < \omega$ and
(2) $S2[p_0 \cup \{\bigwedge_{n\in u}\psi(\mathbf{x},\mathbf{b}_n)\},\Delta] < \alpha$ whenever $u \in {}^{l}\omega$.

Remark 4.2. Given $\Delta \subseteq \{\varphi(\mathbf{x};\mathbf{y}) \mid \varphi \in Fml(L(T))\}$ closed under Boolean operations and containing all the formulas of a type p in $\mathbf{x}$, if p is Δ-stable, one can show that $S2[p,\Delta] = S1[p,\Delta]$.

The inequality $S1[p,\Delta] \leq S2[p,\Delta]$ is trivial and holds without any restrictions on Δ and p.

One then can show that $S2[p,\Delta] \leq Deg[p,\Delta,\lambda]$, where λ is infinite cardinal, using a similar argument to that which was used to prove Theorem 2.3.

Appendix A. Proof of Claim 2.11

Proof. (Of the Claim 2.11) Define r_i, p_i by induction on $i < \kappa$. Suppose we have defined $\{r_j, p_j : j < i\}$. Now define an equivalence relation E_i:

$$\text{for } \mathbf{a},\mathbf{b} \in {}^{\ell(\mathbf{y})}\mathrm{A}, \quad \mathbf{a}E_i\mathbf{b} \Leftrightarrow (\forall j < i)[\varphi(\mathbf{x};\mathbf{a}) \in p_j \Leftrightarrow \varphi(\mathbf{x};\mathbf{b}) \in p_j].$$

Subclaim A.1. $|{}^{\ell(\mathbf{y})}\mathrm{A}/E_i| < \kappa$.

Proof. For every $j < i$, let $\mathbf{c}_j \models p_j$. Suppose for the sake of contradiction that $\{\mathbf{a}_k/E_i \mid k < \kappa\}$ are distinct equivalence classes.
Let $q_k := tp_\varphi(\mathbf{a}_k/\cup_{j<i}\mathbf{c}_j)$. Let $k \neq l < \kappa$ be given. Then $\mathbf{a}_k/E_i \neq \mathbf{a}_l/E_i$. By the definition of E_i, there is a $j < i$ such that $\neg(\varphi(\mathbf{x};\mathbf{a}_k) \in p_j \leftrightarrow \varphi(\mathbf{x};\mathbf{a}_l) \in p_j)$. Without loss of generality we may assume that $\varphi(\mathbf{x};\mathbf{a}_k) \in p_j$ and

$\neg\varphi(\mathbf{x};\mathbf{a}_l) \in p_j$. Since $\mathbf{c}_j \models p_j$, we have that $\varphi(\mathbf{c}_j;\mathbf{y}) \in q_k$ and $\neg\varphi(\mathbf{c}_j;\mathbf{y}) \in q_l$. Thus for $k \neq l < \kappa$, $q_k \neq q_l$.

Since $i < \kappa$, we get that $| \cup_{j<i} \mathbf{c}_j| < \kappa$. But $\{q_k : k < \kappa\}$ witnesses that $|S_\varphi(\cup_{j<i}\mathbf{c}_j)| \geq \kappa$, which contradicts our stability assumption. $\dashv$

Subclaim A.2. $|\{q \in \Phi : \mathbf{a}E_i\mathbf{b} \Rightarrow [\varphi(\mathbf{x};\mathbf{a}) \in q \Leftrightarrow \varphi(\mathbf{x};\mathbf{b}) \in q]\}| < \kappa$.

Proof. Let $S := \{q \in \Phi \mid \forall \mathbf{a},\mathbf{b} \in {}^{\ell(\mathbf{y})}\mathrm{A}\ (\mathbf{a}E_i\mathbf{b} \Rightarrow (\varphi(\mathbf{x};\mathbf{a}) \in q \Leftrightarrow \varphi(\mathbf{x};\mathbf{b}) \in q)\}$. For the sake of contradiction, suppose that $|S| = \kappa$. Let $\bar{A} := {}^{\ell(\mathbf{y})}\mathrm{A}\,/E_i$. By subclaim A.1, we have that $|\bar{A}| < \kappa$; but, we have an obvious injection $S \to S_\varphi(\bar{A})$, so $|S_\varphi(\bar{A})| \geq \kappa$, which is a contradiction to the stability assumption. $\dashv$

By induction on $l \leq n_0$ define $p^l \in \Phi$, $\mathbf{a}^i_{l+1}$ and $t(l) \in \{0,1\}$ such that:

- $r^i_l := \{\varphi(\mathbf{x};\mathbf{a}^i_0), \varphi(\mathbf{x};\mathbf{a}^i_1)^{t(0)}, \dots, \varphi(\mathbf{x};\mathbf{a}^i_{l+1})^{t(l)}\} \subseteq p^l$;
- $R[\theta(\mathbf{x};\mathbf{a}) \cup r^i_l, \varphi, 2] \leq n_0 - l$ or $\exists! q \in \Phi$, $q \supseteq r^i_l$.

For $l = 0$, by Subclaims A.1 and A.2 there exists p^0 and $\mathbf{a}^i_0$, $\mathbf{a}^i_1$ such that $\mathbf{a}^i_0 E_i \mathbf{a}^i_1$ and $\varphi(\mathbf{x};\mathbf{a}^i_0) \in p^0$, $\neg\varphi(\mathbf{x};\mathbf{a}^i_1) \in p^0$. Put $t(0) := 1$.

Suppose we have defined everything for l. If $\exists! q \in \Phi$, $q \supseteq r^i_l$, let $\mathbf{a}^i_{l+1} = \mathbf{a}^i_l$, $t(l+1) = t(l)$ and $p^{l+1} = q$. If $|\{q \in \Phi : r^i_l \subseteq q\}| > 1$, then, since $\Phi \subseteq S_\varphi(A)$, for some $\mathbf{a}^i_{l+1} \in {}^{\ell(\mathbf{y})}\mathrm{A}$ there are $q_0, q_1 \in \Phi$, $r^i_l \cup \{\varphi(\mathbf{x};\mathbf{a}^i_{l+1})^t)\} \subseteq q_t$. From definition of R, there is $t(l+1) \in \{0,1\}$ such that

$$R[\{\theta(\mathbf{x};\mathbf{a})\} \cup r^i_l \cup \{\varphi(\mathbf{x};\mathbf{a}^i_{l+1})^{t(l+1)}\}, \varphi, 2] < R[\{\theta(\mathbf{x};\mathbf{a})\} \cup r^i_l, \varphi, 2] \leq n_0 - l$$

and put $p^{l+1} := q_{t(l+1)}$.

Now, it is clear from the construction that $r^i_{n_0+1}$ has a unique extension in Φ and $|r^i_{n_0+1}| \leq n_0 + 2$. Put $r_i := r^i_{n_0+1}$, adding eventually some formulas from the unique extension of $r^i_{n_0+1}$ to satisfy the requirement $|r_i| = n_0 + 2$. $\dashv$

Appendix B. A combinatorial theorem

Claim B.1 (A combinatorial lemma)**.** *Let $\kappa \geq \aleph_0$ and suppose that $\mu > \kappa$ is regular. For every $n < \omega$ and every $F : [\mu]^{n+1} \to \kappa$ there exists a limit ordinal $\delta < \mu$ such that for every $\xi < \mu$ satisfying $\xi \geq \delta$ there exists an increasing $\{\gamma(k) \mid k < \omega\} \subseteq \delta$ such that for any $\ell_0 < \cdots < \ell_n < \omega$ we have that*

$$F(\gamma(\ell_0), \dots, \gamma(\ell_n)) = F(\gamma(\ell_0), \dots, \gamma(\ell_{n-1}), \xi).$$

Proof. Let $\chi > \aleph_0$ be a regular cardinal large enough such that $\{\mu, F\} \subseteq H(\chi)$. Let $\mathfrak{B} := \langle H(\chi), \in, \mu, F, \alpha\rangle_{\alpha<\kappa}$, where μ stands for a unary predicate interpreted by the set of ordinals less than μ, F is interpreted by the function F and α is an individual constant interpreted by the corresponding ordinal.

By the Downward Löwenheim-Skolem-Tarski theorem pick an increasing and continuous elementary chain $\{\mathfrak{B}_i \prec \mathfrak{B} \mid i < \mu\}$ satisfying

(1) $\|\mathfrak{B}_i\| < \mu$ and

(2) $i \subseteq \mu^{\mathfrak{B}_i}$ for all $i < \mu$.

Since we have that $\mu = \bigcup_{i<\mu} \mu^{\mathfrak{B}_i}$, there is a closed unbounded subset C of the set $\{\delta < \mu \mid \delta = \mu^{\mathfrak{B}_\delta} \wedge \delta \text{ is limit}\}$.

The definition of $\{\gamma(k) \mid k < \omega\}$ is by induction on k:
Fix any $\gamma(0) < \gamma(1) < \cdots < \gamma(n-1) < \delta$. Suppose $\{\gamma(j) \mid j \leq k\}$ are defined (for $k \geq n-1$). Let $\psi(x)$ be the following formula (with parameters $\gamma(0), \dots, \gamma(k)$):

$$\bigwedge\{F(\gamma(\ell_0), \dots, \gamma(\ell_{n-1}), x) = \alpha \mid \ell_0 < \cdots < \ell_{n-1} \leq k, \alpha < \kappa,\\ \mathfrak{B} \models F(\gamma(\ell_0), \dots, \gamma(\ell_{n-1}), \xi) = \alpha\}.$$

Since $\mathfrak{B} \models \psi(\xi)$, we have that $\mathfrak{B} \models \exists x \psi(x)$. Since all the parameters of ψ are in $\mathfrak{B}_\delta$, $\mathfrak{B}_\delta \prec \mathfrak{B}$ and $\xi > \gamma(k)$ there exists $\gamma(k+1) < \delta$ such that $\gamma(k+1) > \gamma(k)$ and $\mathfrak{B} \models \psi(\gamma(k+1))$. $\dashv$

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E-mail address, Rami Grossberg: `rami@cmu.edu`

E-mail address, Alexei Kolesnikov: `alexei@cmu.edu`

E-mail address, Ivan Tomašić: `itomasic@maths.ed.ac.uk`

E-mail address, Monica VanDieren: `monicav@cmu.edu`

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213