STABILITY IN ABSTRACT ELEMENTARY CLASSES

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ABSTRACT ELEMENTARY CLASSES

Definition. Let $\mathcal{K}$ be a class of structures all in the same similarity type $L(\mathcal{K})$, and let $\prec_{\mathcal{K}}$ be a partial order on $\mathcal{K}$. The ordered pair $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an *abstract elementary class* iff

A0 (Closure under isomorphism)

(a) For every $M \in \mathcal{K}$ and every $L(\mathcal{K})$-structure N if $M \cong N$ then $N \in \mathcal{K}$.

(b) Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.

A1 For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.

A2 Let M, N, M^* be $L(\mathcal{K})$-structures. If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$.

A3 (Löwenheim-Skolem) There exists a cardinal $LS(\mathcal{K}) \geq +\aleph_0 + |L(\mathcal{K})|$ such that for every $M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \supseteq A$ and $\|N\| \leq |A| + LS(\mathcal{K})$.

A4 (Tarski-Vaught Chain)

(a) For every regular cardinal μ and every $N \in \mathcal{K}$ if $\{M_i \prec_{\mathcal{K}} N \ : \ i < \mu\} \subseteq \mathcal{K}$ is increasing (i.e. $i < j \Longrightarrow M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $\bigcup_{i<\mu} M_i \prec_{\mathcal{K}} N$.

(b) For every regular μ, if $\{M_i \ : \ i < \mu\} \subseteq \mathcal{K}$ is increasing then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i<\mu} M_i$.

In pure model theory, through solutions to Łǒs Conjecture, central concepts such as prime models, minimal types, super-stability, etc. were identified.

Łǒs conjecture.(1956) Let T be a first-order theory. If there exists $\lambda > |T| + \aleph_0$ such that $I(\lambda, T) = 1$ then $I(\mu, T) = 1$ holds for every $\mu > |T| + \aleph_0$.

According to Shelah [Sh702] one of the most important directions of pure model theory is the classification of non-elementary classes. To that end, a good test-question should be proposed.

Shelah's conjecture.(about 1977) Let T be a countable theory in $L_{\omega_1,\omega}$. If there exists $\lambda \geq \beth_{\omega_1}$ such that $I(\lambda, T) = 1$ then $I(\mu, T) = 1$ holds for every $\mu \geq \beth_{\omega_1}$.

Status - very open. Partial results:

1. Keisler (1971), under the additional assumption of existence sequentially homogeneous model.

2. Shelah (1984) [87a] and [87b] for excellent classes.

3. Lessmann (1998) for countable finite diagrams.

4. Makkai and Shelah [Sh285] under the additional assumption that both λ and μ are above a strongly compact cardinal, AND $\lambda = \chi^+$.

* **5.** Kolman-Shelah [Sh362] and Shelah [Sh 472] partial going down results for λ above measurable cardinal.

* **6.** Shelah [Sh394] partial going down result for Abstract Elementary Classes with the Amalgamation Property.

TOWARDS STABILITY THEORY

1. Pillay for existentially closed structures.

2. Hrushovski for Robinson theories.

3. Buechler-Lessmann for sequentially homogeneous structures.

* **4.** Shelah [Sh300] for universal classes.

* **5.** Shelah [Sh600] for good frames.

AMALGAMATION PROPERTY

Notation. For $M, N \in \mathcal{K}$ a monomorphism $f : M \to N$ is called a $\prec_{\mathcal{K}}$*-embedding* iff $f[M] \prec_{\mathcal{K}} N$. Denote this by $f : M \hookrightarrow_{\mathcal{K}} N$.

$$\mathcal{K}_\lambda := \{M \in \mathcal{K} \mid \|M\| = \lambda\}.$$

Definition. Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an abstract elementary class, and let $\lambda \geq \mathrm{LS}(\mathcal{K})$ be a cardinal.

1. We say that $M_0 \in \mathcal{K}_\lambda$ is an *amalgamation base* iff for every $M_1, M_2 \in \mathcal{K}_\lambda$ and every $f_\ell : M_0 \hookrightarrow_{\mathcal{K}} M_\ell$ for $\ell = 1, 2$, there exists $N \in \mathcal{K}_\lambda$ so that there are $g_1 : M_1 \hookrightarrow_{\mathcal{K}} N$ and $g_2 : M_2 \hookrightarrow_{\mathcal{K}} N$ so that $g_1 \circ f_1 \restriction |M_0| = g_2 \circ f_2 \restriction |M_0|$. Namely, the following diagram is commutative:

$$\begin{tikzcd} M_1 \arrow[r, "g_1"] & N \\ M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[u, "g_2"'] \end{tikzcd}$$

2. We say that $\mathcal{K}$ satisfies the λ *amalgamation property* (λ- AP) iff for every $M_0 \in \mathcal{K}_\lambda$, M_0 is an amalgamation base.

3. $\mathcal{K}$ satisfies the *amalgamation property* (AP) iff $\mathcal{K}$ satisfies the λ-AP for every $\lambda > LS(\mathcal{K})$.

THE NOTION OF GALOIS TYPES

Definition.

$\mathcal{K}^3_\lambda := \{(M, N, a) \mid a \in M;\ M \prec_{\mathcal{K}} N;\ M, N \in \mathcal{K}_\lambda\}.$

Definition. Let $M \in \mathcal{K}_\lambda$, We say that (M, N_1, a_1) and $(M, N_2, a_2) \in \mathcal{K}^3_\lambda$, are $\sim$*-related*, written

$$(M, N_1, a_1) \sim (M, N_2, a_2),$$

if there exists $N \in \mathcal{K}_\lambda$ and $\prec_{\mathcal{K}}$-embeddings

$$h_1 : N_1 \to N \quad \text{and} \quad h_2 : N_2 \to N,$$

such that

$$h_1(a_1) = h_2(a_2)$$

and the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[r, "h_1"] & N \\ M \arrow[u, "id"] \arrow[r, "id"'] & N_2 \arrow[u, "h_2"'] \end{tikzcd}$$

Lemma. Suppose that $\mathcal{K}_\lambda$ has the amalgamation property. For $M \in \mathcal{K}_\lambda$ the relation $\sim$ is an equivalence relation on $\mathcal{K}^3_\lambda$.

Remark. Shelah and Villaveces are working towards Shelah's Conjecture with only density of amalgamation bases [Sh635].

TYPES AND STABILITY

Assume that $\mathcal{K}$ has the amalgamation property.

Definition

1. For $(M, N, a) \in \mathcal{K}^3_\lambda$, we let
$$\mathtt{tp}(a/M, N) = (M, N, a)/ \sim .$$
This is the *type of* a *over* M *in* N.

2. For $M \in \mathcal{K}_\lambda$, we let
$$S(M) = \{ \, \mathtt{tp}(a'/M, N') \ : \ (M, N', a') \in \mathcal{K}^3_\lambda \}.$$

Definition. $\mathcal{K}$ is *stable in* λ iff $|S(M)| \leq \lambda$ for every $M \in \mathcal{K}_\lambda$.

DEPENDENCE RELATION: μ-SPLITTING

Definition. A type $p \in S(M)$ *μ-splits over* N iff there exists a $\prec_{\mathcal{K}}$-mapping h and models $N_1, N_2 \in \mathcal{K}_\mu$ such that

1. $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$,
2. $h : N_1 \cong N_2$,
3. $h \restriction N = id_N$ and
4. $p \restriction N_2 \neq h(p \restriction N_1)$.

Theorem. [Sh 394] Suppose that $\mathcal{K}$ is stable in μ and $\langle M_i \mid i \leq \kappa \rangle$ is an $\prec_{\mathcal{K}}$-increasing and continuous chain of models in $\mathcal{K}_\mu$. If $p \in S(M_\kappa)$ and $2^\kappa > \mu$, then there exists $i < \kappa$ such that $p \restriction M_{i+1}$ does not μ-split over M_i.

Definition.[G-V] Define $\kappa_\mu(\mathcal{K})$ to be the minimal regular cardinal κ such that for every $\langle M_i \in \mathcal{K}_\mu \mid i \leq \kappa\rangle$, $\prec_{\mathcal{K}}$-increasing and continuous, and every $p \in S(M_\kappa)$, there exists $i < \kappa$ such that p does not μ-split over M_i. Otherwise, let $\kappa_\mu(\mathcal{K}) = \infty$.

Theorem.(Shelah-Villaveces) Under GCH, categoricity in some $\lambda > \mu > LS(\mathcal{K})$ implies that $\kappa_\mu(\mathcal{K}) = \omega$.

Proposition.[Sh 394] Stability in μ implies that $\kappa_\mu(\mathcal{K}) \leq \mu^+$.

Question. Does stability, or even superstability, imply that $\kappa_\mu(\mathcal{K}) < \mu^+$?

Definition. $\mathcal{K}$ satisfies *E_μ-equivalence* iff for every $M \in \mathcal{K}$ and every $p, q \in S(M)$,

$$p = q \Leftrightarrow [\forall N \in \mathcal{K}_\mu, N \prec_{\mathcal{K}} M](p \upharpoonright N = q \upharpoonright N).$$

Morley Sequences

Definition.[G-V] Let $\mathcal{K}$ be an AEC. The class is called μ-*superstable* iff there exists $\mu > LS(\mathcal{K})$ satisfying

1. $\mathcal{K}$ is stable in μ and
2. $\kappa_\mu(\mathcal{K}) < \mu^+$.

Theorem.[G-V] Suppose $\mathcal{K}$ is μ-superstable for some $\mu \geq LS(\mathcal{K})$ and $\mathcal{K}$ has the amagamation property. Let $M \in \mathcal{K}_{>\mu}$, $A, I \subset M$ be given such that $|I| \geq \mu^+ > |A|$. Then there exists $J \subset I$ of cardinality μ^+, indiscernible over A. Moreover J can be chosen to be a Morley sequence over A.

Where

Definition $\{\mathbf{a}_i \mid i < \mu^+\}$ is a *Morley sequence over* M_0 iff there exists an increasing $\prec_{\mathcal{K}}$-chain $\{M_i \mid i < \mu^+\}$ such that

1. $p_i := \mathtt{tp}(\mathbf{a}_i/M_i, M_{i+1})$ does not μ-split over M_0 and
2. $\langle \mathbf{a}_i \mid i \in \lambda \rangle$ is an indiscernible sequence over M_0.

Outline of Proof. Given $\{\mathbf{a}_i \mid i < \mu^+\rangle \subseteq I$, define $\langle M_i \in \mathcal{K}_\mu \mid i < \mu^+\rangle$, a $\prec_{\mathcal{K}}$-increasing and continuous chain of models such that

1. $A \subseteq M_0$,

2. M_{i+1} is (μ, ω)-limit over M_i (substitute for relatively saturated) and

3. $\mathbf{a}_i \in M_{i+1}$.

Define $p_i := \mathtt{tp}(\mathbf{a}_i/M_i, M_{i+1})$.
Let $f : S^{\mu^+}_{\kappa_\mu(\mathcal{K})} \to \mu^+$ be the mapping

$f(i) := min\{j < i \mid p_i$ does not $\mu -$ split over $M_j\}$.

By Fodor's Lemma, there exists a stationary $S \subseteq S^{\mu^+}_{\kappa_\mu(\mathcal{K})}$ and $j_0 < \mu^+$ such that

$$i \in S \Rightarrow f(i) = j_0.$$

Since $\mathcal{K}$ is stable in μ, we can apply the pigeon-hole principle and find $S^* \subseteq S$ of cardinality μ^+ and p^* such that

$$i \in S^* \Rightarrow p_i \restriction M_{j_0} = p^*.$$

Enumerate and rename S^*. Let $M^* := M_1$. Again, by stability we can find $J \subset S^*$ of cardinality μ^+ such that for every $i \in S^*$, $p^{**} = p_i \restriction M^*$. Enumerate in increasing order and rename the elements in J.

Claim 1. For $i < j \in J$, $p_i = p_j \restriction M_i$.

Claim 2. $\{\mathbf{a}_i \mid i \in J\}$ are indiscernible over M_0.