Linear constant coefficient ordinary differential systems

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0 Overview

As the title suggests, these notes concern systems of linear ordinary differential equations with constant, complex coefficients. The reader possessing some familiarity with such problems will likely immediately wonder how such a study can occupy a document of this length. Indeed, one could attempt to summarize the standard theory in a single sentence: rewrite the problem as a first order system, multiply by the appropriate matrix exponential integrating factor, and integrate. The only real subtlety comes in computing the matrix exponential, but this problem is readily dispatched with the Jordan normal form. QED, right?

Wrong! There are three hidden assumptions in this pithy description. The first is that the problem can naturally be rewritten as a first order system, making it amenable to the above attack. The second assumption is that we are only interested in specifying the most basic initial conditions, namely the values of the unknown function and its derivatives up to order one less than the order of the system. The third is that the system is finite dimensional, so the Jordan normal form is available for use. The purpose of these notes is to study what happens when we negate these standard assumptions and consider systems that do not naturally rewrite in first order form, systems with other choices of initial conditions, and systems taking values in infinite dimensional complex Banach spaces. When we begin flipping these switches, the above simple picture falls apart pretty quickly and leaves behind some rather tricky issues.

Why should we care about flipping these switches? In brief: partial differential equations. All of these variants arise naturally when we attempt to use the tools of ordinary differential equations and systems to study partial differential equations and systems. In fact, the author first encountered a discussion of such general ordinary differential systems with general initial conditions (and more!) while reading the seminal PDE paper *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II* by Agmon, Douglis, and Nirenberg [1], in which this theory plays an essential role. In the ADN paper this material is developed quite rapidly, and these notes began as the author's attempt to fill in some of the details and more deeply understand the material. The only references the author could find addressing general systems were the rather old books of Ince [3] and Poole [4], but while they were certainly helpful they did not contain all of the material needed to process ADN. Hence the existence of these notes, which one could think of as a primer on the ODE analysis needed to understand ADN, though there is more here than strictly needed for that purpose.

The reader interested in going further will need a few key tools to make headway. First, it's a good idea to have some basic experience with ordinary differential equations on $\mathbb{R}$; any undergraduate course should suffice. Second, a good grasp of linear algebra over $\mathbb{C}$ is required. Third, the basics of complex analysis will be routinely used: holomorphic functions, path integrals, the Cauchy-Goursat theorem, and the residue theorem will play starring roles. In dealing with systems, we will need to work with holomorphic functions $f : \mathbb{C} \to X$ where X is a complex Banach space. When X is finite dimensional, this is easy, as we can just think of each component being holomorphic, and we define path integrals of such maps in the obvious way, component-wise. However, when X is infinite dimensional some care is needed to work out the properties of holomorphic functions, path integrals, and the versions of Cauchy-Goursat and the residue theorem. The reader who doesn't know or care to learn this material can simply replace all appearances of X with $\mathbb{C}^N$ in Section 1, which is the only place where the infinite dimensional setting is considered. The reader who doesn't know but cares to learn this material is directed either to the author's complex analysis crash course [5] or else to the fantastic book by Dieudonné [2], which does all that is needed and more. The reader who knows this material already is given a gold star. Fourth, an ε worth of Lebesgue integration is used in a couple places to deal with some integration issues, but this is minor and can be easily glossed over or replaced with the corresponding ideas from improper Riemann integrals without a significant loss.

These notes are organized as follows. In Section 1 we study *elliptic* ordinary differential systems taking values in general Banach spaces. Here ellipticity basically means that we can naturally reformulate the problem in terms of an equivalent first order system and follow the above pithy path. This is mostly meant as a reminder of what one learns in a basic ODE course and to point out that very little of that material depends on finite dimensionality or on working over $\mathbb{R}$ or $[0, \infty)$ rather than $\mathbb{C}$.

Section 2 focuses on the scalar case, i.e. $X = \mathbb{C}$, with the goal of producing solutions with more general initial conditions. All nontrivial scalar ordinary differential operators are elliptic, so the theory from Section 1 is in play. The key point is to derive some new (relative to Section 1) representation formulas for solutions using the tools of complex analysis. These representation formulas then allow us to reduce the question of the solvability with general initial conditions to a purely algebraic question, known as the Shapiro-Lopatinsky condition.

In Section 3 we begin the study of general systems (i.e. not necessarily elliptic) with general initial conditions. Some truly bizarre behavior appears in this theory, for example differential operators that can be inverted with other differential operators. The goal is to develop useful representation formulas as above, but the linear algebra is a lot harder, and the representation formulas are more involved. Nevertheless, we derive an analog of Shapiro-Lopatinsky.

In Section 4 we study general systems on intervals $J \subseteq \mathbb{R}$ such that $0 \in J$. In this context we can no longer work with holomorphic data, so we switch to studying solutions in $C^\infty(J; \mathbb{C}^N)$. We prove analogs of the results from Sections 2 and 3. The main utility of this is that it allows us to consider the special case of operators and initial condition operators with real coefficients. We prove that in this case with real data, the solutions we produce are real as well.

In Section 5 we develop a theory that parallels that of Section 3 but with some extra decay conditions imposed on the solutions. This requires shifting from constructing solutions on $\mathbb{C}$ as holomorphic functions to constructing solutions on $[0, \infty) \subset \mathbb{R}$ as smooth functions decaying exponentially as $t \to \infty$. This is the stuff most needed in applications to PDE, where it is combined with the Fourier transform to produce solutions to certain types of boundary value problems on $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$. Once again we derive a version of Shapiro-Lopatinsky that reduces the question of solvability to a purely algebraic condition. The material in Sections 3 and 5 roughly follows the approach of ADN and certainly borrows many of their main tricks.

We conclude the overview with some remarks on notation.

1. It will often (but not always) be convenient to label the indices of a space of dimension d as $0, \ldots, d-1$ rather than the conventional choice of $1, \ldots, d$.

2. We will often abuse notation by using right multiplication by scalars in a vector space.

3. The conventional meaning of an ordinary differential system or equation is a system (or equation) relating an unknown function of one real variable to its various derivatives. When we replace the real variable with a complex one, it is tempting to think of these systems as involving two real variables and thus no longer ordinary. However, the partial derivatives with respect to these two real variables are never treated independently in the equations we consider, so the designation ordinary is still appropriate.

4. If X is a complex Banach space and $f : \mathbb{C} \to X$, then we say f is holomorphic if it is once

differentiable on all of $\mathbb{C}$ (and hence smooth and analytic by complex analysis). We will write

$$H(\mathbb{C}; X) = \{f : \mathbb{C} \to X \mid f \text{ is holomorphic}\}. \tag{0.1}$$

We will use the terminology of [2, 5] when describing complex path integrals. In particular, we refer to "nice" (roughly speaking, almost continuously differentiable) paths as roads and closed paths as loops.

5. Given $f \in H(\mathbb{C}; X)$ we write its zero set as

$$Z(f) = \{z \in \mathbb{C} \mid f(z) = 0\}. \tag{0.2}$$

6. We will only explicitly talk about meromorphic functions with values in $\mathbb{C}$. These can be thought of as functions $f : \mathbb{C}\backslash P(f) \to \mathbb{C}$ that are holomorphic, with each point of $P(f)$ isolated and consisting of an isolated singularity that is at worst a pole of finite order. The set $P(f)$ is the polar set.

7. Given $z_0 \in \mathbb{C}$ and $R > 0$ we write $\partial B(z_0, R)$ to mean both the set $\{w \in \mathbb{C} \mid |z_0 - w| = R\}$ and the simple counter-clockwise loop parameterized by $\gamma : [0, 1] \to \mathbb{C}$ defined via $\gamma(t) = z_0 + Re^{2\pi i t}$. The latter will always appear in path integrals:

$$\int_{\partial B(z_0,R)} f(z) dz. \tag{0.3}$$

8. We write $\mathcal{L}(X, Y)$ for the bounded linear maps between complex Banach spaces X and Y, and we write $\mathcal{L}(X) = \mathcal{L}(X, X)$.

9. We follow the common ODE convention of writing the unknown as $x : \mathbb{C} \to X$. To highlight the connection with ODE on $\mathbb{R}$ we usually write the variable as $\tau \in \mathbb{C}$ rather than $t \in \mathbb{R}$. In this way we can think of τ as a sort of complex time variable.

1 Elliptic ordinary differential systems in arbitrary dimension

We begin our survey of constant coefficient ordinary differential systems by studying the nicest case, in which the system is elliptic. In this case most of the theory works just as well in infinite dimensional complex Banach spaces as it does in $\mathbb{C}$, so we present the Banach framework for the sake of generality.

We begin with a definition.

Definition 1.1. *Let X and Y be complex Banach spaces and consider a polynomial $p : \mathbb{C} \to \mathcal{L}(X, Y)$ with $\deg(p) = n \in \mathbb{N}$ of the form $p(z) = \sum_{k=0}^{n} A_k z^k$, where $A_k \in \mathcal{L}(X, Y)$ for $0 \leq k \leq n$.*

1. *We define the constant coefficient differential operator*

$$p(D) = \sum_{k=0}^{n} A_k D^k, \tag{1.1}$$

where $D^0 = I$ is the identity. More precisely, for $f : \mathbb{C} \to X$ holomorphic, $p(D)f : \mathbb{C} \to Y$ is defined via

$$p(D)f = \sum_{k=0}^{n} A_k D^k f = \sum_{k=0}^{n} A_k f^{(k)}. \tag{1.2}$$

The operators $A_0, \ldots, A_n$ are called the coefficients of $p(D)$.

2. *This induces a linear map $p(D) : H(\mathbb{C}; X) \to H(\mathbb{C}; Y)$ that we call a linear differential operator of order $n = \deg(p)$. We define*

$$\ker(p(D)) = \{x \in H(\mathbb{C}; X) \mid p(D)x = 0\} \tag{1.3}$$

for the kernel of $p(D)$, which is also called the space of homogenous solutions to $p(D)x = 0$ (here homogenous refers to the fact that the right side of the equation is 0).

3. *We say that $p(D)$ is elliptic if $A_n \in \mathcal{L}(X, Y)$ is invertible.*

Some remarks are in order.

Remark 1.2. *If $p(D)$ is elliptic, then the invertibility of $A_n \in \mathcal{L}(X, Y)$ requires that X and Y are isomorphic.*

Remark 1.3. *If $X = Y = \mathbb{C}$, then every constant coefficient differential operator of order $n \geq 0$ is elliptic by the definition of the degree of a polynomial.*

Remark 1.4. *When $\dim(X) \geq 2$ (possibly infinite) an equation of the form $p(D)x = f$ for a given $f \in H(\mathbb{C}; Y)$ is called an ordinary differential system. The term system is used to contrast with the case when $\dim(X) = 1$, in which case the word system is typically replaced with equation.*

Our focus for the moment will be elliptic differential operators. If $p(D)$ is elliptic of order 0, then $p(D) = A_0 \in \mathcal{L}(X, Y)$ is an isomorphism, so there is nothing to study: the unique solution to $p(D)x = f \in H(\mathbb{C}; Y)$ is $x = A_0^{-1} f \in H(\mathbb{C}; X)$. As such, we will restrict our attention to elliptic operators of order $n \geq 1$. Our goal is to find conditions to complement the equation $p(D)x = f$ that lead to unique solvability.

The most important feature of an elliptic differential operator is found in the following lemma, which establishes an equivalence between elliptic differential operators of arbitrary order and first order elliptic operators. In essence, in the elliptic case it suffices to only consider first order systems.

Lemma 1.5. *Let X and Y be complex Banach spaces and let $p(D) : H(\mathbb{C}; X) \to H(\mathbb{C}; Y)$ be an elliptic differential operator of order $n \geq 1$, written $p(D) = \sum_{k=0}^{n} A_k D^k$. Define $\mathbb{A} \in \mathcal{L}(X^n)$ in block-form via*

$$\mathbb{A} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & I \\ -A_n^{-1}A_0 & -A_n^{-1}A_1 & \cdots & \cdots & -A_n^{-1}A_{n-1} \end{pmatrix}. \tag{1.4}$$

Let $f : \mathbb{C} \to Y$ be holomorphic and define the holomorphic map $F : \mathbb{C} \to X^n$ via

$$F = (0, \ldots, 0, A_n^{-1} f). \tag{1.5}$$

Let $\xi_0, \ldots, \xi_{n-1} \in X$. Then the following are equivalent.

1. *There exists a unique holomorphic function $x : \mathbb{C} \to X$ satisfying*

$$\begin{cases} p(D)x = f \\ D^k x(0) = \xi_k \text{ for } 0 \le k \le n-1. \end{cases} \tag{1.6}$$

2. *There exists a unique holomorphic function $\Xi : \mathbb{C} \to X^n$ satisfying*

$$\begin{cases} \Xi' = \mathbb{A}\Xi + F \\ \Xi(0) = (\xi_0, \dots, \xi_{n-1}). \end{cases} \tag{1.7}$$

In either case x and Ξ are related via

$$\Xi = (x, x', \dots, x^{(n-1)}). \tag{1.8}$$

Proof. Suppose the first item holds. Define the holomorphic map $\Xi : \mathbb{C} \to X^n$ via $\Xi(\tau) = (x, x', \dots, x^{(n-1)})$. Clearly $\Xi(0) = (\xi_0, \dots, \xi_{n-1})$. We compute

$$\Xi' = (x', x'', \dots, x^{(n-1)}, x^{(n)}), \tag{1.9}$$

but since $p(D)x = f$, we can solve

$$D^n x = -A_n^{-1} \sum_{k=0}^{n-1} A_k x^{(k)} + A_n^{-1} f, \tag{1.10}$$

and upon plugging this in above we find that

$$\Xi' = (x', x'', \dots, x^{(n-1)}, -A_n^{-1} \sum_{k=0}^{n-1} A_k x^{(k)}) + F = \mathbb{A}\Xi + F. \tag{1.11}$$

It remains only to prove the uniqueness part of the second item. Suppose that $\tilde{\Xi} : \mathbb{C} \to X^n$ is a holomorphic function that solves $\tilde{\Xi}' = \mathbb{A}\tilde{\Xi} + F$ and $\tilde{\Xi}(0) = \Xi(0)$. Define $\Psi = \Xi - \tilde{\Xi}$. Note that for all $\tau \in \mathbb{C}$ we have that $\exp(\tau\mathbb{A})$ is invertible and that $\mathbb{A}\exp(-\tau\mathbb{A}) = \exp(-\tau\mathbb{A})\mathbb{A}$. Using these, we deduce that

$$\begin{aligned} \Psi'(\tau) = \mathbb{A}\Psi(\tau) \Rightarrow \exp(-\tau\mathbb{A})\Psi'(\tau) - \mathbb{A}\exp(-\tau\mathbb{A})\Psi(\tau) = 0 \Rightarrow (\exp(-\tau\mathbb{A})\Psi(\tau))' = 0 \\ \Rightarrow \exp(-\tau\mathbb{A})\Psi(\tau) = \Psi(0) = 0 \Rightarrow \Psi(\tau) = 0. \end{aligned} \tag{1.12}$$

Hence $\Xi - \tilde{\Xi} = \Psi = 0$, and uniqueness is proved. Thus, the second item holds.

Now suppose that the second item holds and write $\Xi = (\Xi_0, \dots, \Xi_{n-1})$. Define the holomorphic function $x : \mathbb{C} \to X$ via $x = \Xi_0$. The equation $\Xi' = \mathbb{A}\Xi + F$ is equivalent to

$$(\Xi_0', \dots, \Xi_{n-1}') = \Xi' = (\Xi_1, \Xi_2, \dots, \Xi_{n-1}, -A_n^{-1} \sum_{k=0}^{n-1} A_k \Xi_k + A_n^{-1} f). \tag{1.13}$$

Since $x = \Xi_0$, we deduce from this that $\Xi_k = D^k \xi$ for $0 \le k \le n-1$, and hence that

$$D^n x = \Xi_{n-1}' = -A_n^{-1} \sum_{k=0}^{n-1} A_k \Xi_k + A_n^{-1} f = -A_n^{-1} \sum_{k=0}^{n-1} A_k D^k x + A_n^{-1} f, \tag{1.14}$$

which in turn implies that $p(D)x = f$. Moreover, the equation $\Xi(0) = (\xi_0, \ldots, \xi_{n-1})$ implies that $D^k x(0) = \xi_k$ for $0 \leq k \leq n-1$.

It remains only to prove the uniqueness assertion of the first item. If $\tilde{x} : \mathbb{C} \to X$ is a holomorphic function such that $p(D)\tilde{x} = f$ and $D^k \tilde{x}(0) = \xi_k$, then we may argue as above to produce $\tilde{\Xi} = (\tilde{x}, \tilde{x}', \ldots, \tilde{x}^{(n-1)}) : \mathbb{C} \to X^n$, a holomorphic function satisfying $\tilde{\Xi}' = \mathbb{A}\tilde{\Xi} + F$ and $\tilde{\Xi}(0) = (\xi_0, \ldots, \xi_{n-1})$. By the uniqueness assertion of the second item we then have that $\Xi = \tilde{\Xi}$ and hence that $x = \tilde{x}$.

□

Remark 1.6. *The conditions $D^k x(0) = \xi_k$ are called initial conditions. This terminology is not completely obvious when we allow any $\tau \in \mathbb{C}$. Its origin lies in applications in which τ is restricted to $[0, \infty) \subset \mathbb{R}$, and is thought of as a time variable parameterizing some evolving process. In this setting the meaning of initial conditions is clear as the time 0 is the initial time in the process.*

The lemma establishes the equivalence of two seemingly different problems. We now show that the first order problem is solvable, which allows us to solve both problems.

Theorem 1.7. *Let X and Y be complex Banach spaces and $p(D) = \sum_{k=0}^n A_k D^k : H(\mathbb{C}; X) \to H(\mathbb{C}; Y)$ be an elliptic differential operator of order $n \geq 1$. Then the following hold.*

1. *Let $\mathbb{A} \in \mathcal{L}(X^n)$ be determined by the coefficients of $p(D)$ as in Lemma 1.5. For every holomorphic map $F : \mathbb{C} \to X^n$ and $(\xi_0, \ldots, \xi_{n-1}) \in X^n$ there exists a unique holomorphic function $\Xi : \mathbb{C} \to X^n$ satisfying*
$$\begin{cases} \Xi' = \mathbb{A}\Xi + F \\ \Xi(0) = (\xi_0, \ldots, \xi_{n-1}). \end{cases} \tag{1.15}$$
Moreover, Ξ is given by the formula
$$\Xi(\tau) = \exp(\tau\mathbb{A})(\xi_0, \ldots, \xi_{n-1}) + \int_{\lambda_\tau} \exp((\tau - z)\mathbb{A})F(z)dz, \tag{1.16}$$
where for any $\tau \in \mathbb{C}$ the road $\lambda_\tau : [0,1] \to \mathbb{C}$ is given by $\lambda_\tau(t) = t\tau$.

2. *For every holomorphic function $f : \mathbb{C} \to Y$ and $\xi_0, \ldots, \xi_{n-1} \in X$ there exists a unique holomorphic function $x : \mathbb{C} \to X$ satisfying*
$$\begin{cases} p(D)x = f \\ D^k x(0) = \xi_k \text{ for } 0 \leq k \leq n-1. \end{cases} \tag{1.17}$$

3. *The map $\mathfrak{F} : H(\mathbb{C}; X^n) \to H(\mathbb{C}; X^n) \times X^n$ given by*
$$\mathfrak{F}(\Xi) = (\Xi' - \mathbb{A}\Xi, \Xi(0)) \tag{1.18}$$
is a linear isomorphism.

4. *The map $\Phi : H(\mathbb{C}; X) \to H(\mathbb{C}; Y) \times X^n$ given by*
$$\Phi(x) = (p(D)x, x(0), x'(0), \ldots, x^{(n-1)}(0)) \tag{1.19}$$
is a linear isomorphism.

Proof. The third item and fourth items are simply linear algebraic restatements of the first and second items, respectively, so it suffices to prove the first and second. In turn, Lemma 1.5 shows that it suffices to prove the first item. We will thus only prove the first item. The uniqueness of such a Ξ follows from the same argument used to prove the first item implies the second in Lemma 1.5, so we may further reduce to proving the existence of a solution, and for this we will show that $\Xi : \mathbb{C} \to X^n$ defined by (1.16) is holomorphic and satisfies (1.15).

Define $\Psi : \mathbb{C} \to X^n$ via $\Psi(\tau) = \int_{\lambda_\tau} \exp(-z\mathbb{A})F(z)dz$. We compute

$$\Psi(\tau) = \int_0^1 \tau \exp(-t\tau\mathbb{A})F(t\tau)dt, \tag{1.20}$$

and hence Ψ is holomorphic with

$$\Psi'(\tau) = \int_0^1 \left[\exp(-t\tau\mathbb{A})F(t\tau) - \tau t\mathbb{A}\exp(-t\tau\mathbb{A})F(t\tau) + \tau t \exp(-t\tau\mathbb{A})F'(t\tau)\right] dt. \tag{1.21}$$

Integrating by parts shows that

$$\begin{aligned}\int_0^1 \tau t \exp(-t\tau\mathbb{A})F'(t\tau)dt &= \int_0^1 t\exp(-t\tau\mathbb{A})\frac{d}{dt}[F(t\tau)]dt \\ &= t\exp(-t\tau\mathbb{A})F(t\tau)\big|_{t=0}^{t=1} - \int_0^1 [\exp(-t\tau\mathbb{A}) - \tau t\mathbb{A}\exp(-t\tau\mathbb{A})]F(t\tau)dt \\ &= \exp(-\tau\mathbb{A})F(\tau) - \int_0^1 [\exp(-t\tau\mathbb{A}) - \tau t\mathbb{A}\exp(-t\tau\mathbb{A})]F(t\tau)dt. \end{aligned} \tag{1.22}$$

Hence,

$$\Psi'(\tau) = \exp(-\tau\mathbb{A})F(\tau), \tag{1.23}$$

and we conclude that Ξ is holomorphic and satisfies

$$\begin{aligned}\Xi'(\tau) &= \mathbb{A}\exp(\tau\mathbb{A})(\xi_0,\dots,\xi_{n-1}) + \mathbb{A}\exp(\tau\mathbb{A})\int_{\lambda_\tau}\exp(-z\mathbb{A})F(z)dz + \exp(\tau\mathbb{A})\Psi'(\tau) \\ &= \mathbb{A}\Xi(\tau) + F(\tau).\end{aligned} \tag{1.24}$$

Moreover,

$$\Xi(0) = (\xi_0,\dots,\xi_{n-1}) + \int_{\lambda_0}\exp(-z\mathbb{A})F(z)dz = (\xi_0,\dots,\xi_{n-1}), \tag{1.25}$$

and existence is proved. ☐

A differential operator $p(D) : H(\mathbb{C};X) \to H(\mathbb{C};Y)$ can be lifted to be viewed as an operator $p(D) : H(\mathbb{C};\mathcal{L}(X)) \to H(\mathbb{C};\mathcal{L}(X,Y))$, which leads to some very useful theoretical tools called propagators. We define these now.

Definition 1.8. *Let X and Y be complex Banach spaces and fix a differential operator $p(D) = \sum_{k=0}^n A_k D^k$ with $A_k \in \mathcal{L}(X,Y)$.*

1. *$p(D)$ induces a linear differential operator $p(D) : H(\mathbb{C};\mathcal{L}(X)) \to H(\mathbb{C};\mathcal{L}(X,Y))$ via*

$$p(D)L = \sum_{k=0}^n A_k D^k L. \tag{1.26}$$

2. *Suppose now that $p(D)$ is elliptic and $n \geq 1$. For each $0 \leq k \leq n-1$ Theorem 1.7 provides us with a unique holomorphic function $L_k : \mathbb{C} \to \mathcal{L}(X)$ such that*

$$\begin{cases} p(D)L_k = 0 \\ D^j L_k(0) = \delta_{kj} I \text{ for } 0 \leq j \leq n-1. \end{cases} \tag{1.27}$$

The maps $\{L_k\}_{k=0}^{n-1}$ are called the propagators of the ODE. Given $\xi_0, \ldots, \xi_{n-1} \in X$ define the holomorphic function $x : \mathbb{C} \to X$ via $x = \sum_{k=0}^{n-1} L_k \xi_k$. Then $p(D)x = \sum_{k=0}^{n-1} [p(D)L_k]\xi_k = 0$ and $D^j x(0) = \sum_{k=0}^{n-1} D^j L_k(0)\xi_k = \sum_{k=0}^{n-1} \delta_{jk} I \xi_k = \xi_j$. This allows us to define the linear map $S : X^n \to \ker(p(D)) \subseteq H(\mathbb{C}; X)$ via

$$S(\xi_0, \ldots, \xi_{n-1}) = \sum_{k=0}^{n-1} L_k \xi_k. \tag{1.28}$$

The map S gives us a way of producing elements of $\ker(p(D))$. We now show that actually S is an isomorphism with a simple inverse.

Theorem 1.9. *Let X and Y be complex Banach spaces and $p(D) : H(\mathbb{C}; X) \to H(\mathbb{C}; Y)$ be an elliptic differential operator of order $n \geq 1$. Then the following hold.*

1. *The map $T : \ker(p(D)) \to X^n$ given by $Tx = (x(0), x'(0), \ldots, x^{(n-1)}(0))$ is a linear isomorphism and $T^{-1} = S$.*

2. *A holomorphic function $x : \mathbb{C} \to X$ satisfies*

$$\begin{cases} p(D)x = 0 \\ D^k x(0) = \xi_k \text{ for } 0 \leq k \leq n-1 \end{cases} \tag{1.29}$$

if and only if $x = S(\xi_0, \ldots, \xi_{n-1})$.

Proof. The linearity of T is obvious. If $Tx = 0$, then $x : \mathbb{C} \to X$ is holomorphic and satisfies

$$\begin{cases} p(D)x = 0 \\ D^k x(0) = 0 \text{ for } 0 \leq k \leq n-1. \end{cases} \tag{1.30}$$

The same is true of the trivial function $0 : \mathbb{C} \to X$, so by the uniqueness assertion of Theorem 1.7 we have that $x = 0$. Thus T is injective.

Given any $(\xi_0, \ldots, \xi_{n-1}) \in X^n$, we may again use Theorem 1.7 to find $x \in \ker(p(D))$ such that $D^k x(0) = \xi_k$ for $0 \leq k \leq n-1$. In turn, this means that $Tx = (\xi_0, \ldots, \xi_{n-1})$, and hence T is surjective. The fact that $T^{-1} = S$ follows from the construction of S. We leave it as an exercise to verify this. The second item then follows directly from the first. □

Next we assemble the propagators $\{L_k\}_{k=0}^{n-1}$ from Definition 1.8 into a higher-order structure and show that the resulting object obeys some remarkable algebraic properties.

Theorem 1.10. *Let X and Y be complex Banach spaces and $p(D) : H(\mathbb{C}; X) \to H(\mathbb{C}; Y)$ be an elliptic differential operator of order $n \geq 1$. For each $\tau \in \mathbb{C}$ define the linear map $\Sigma(\tau) : X^n \to X^n$ via*

$$\Sigma(\tau)(\xi_0, \ldots, \xi_{n-1}) = (x(\tau), x'(\tau), \ldots, x^{(n-1)}(\tau)), \text{ where } \begin{cases} p(D)x = 0 \\ D^k x(0) = \xi_k \text{ for } 0 \leq k \leq n-1. \end{cases} \tag{1.31}$$

Then the following hold.

1. $\Sigma(\tau) \in \mathcal{L}(X^n)$ *for each* $\tau \in \mathbb{C}$*, and in block-form we have that*

$$\Sigma(\tau) = \begin{pmatrix} L_0(\tau) & \cdots & L_{n-1}(\tau) \\ L_0'(\tau) & \cdots & L_{n-1}'(\tau) \\ \vdots & \ddots & \vdots \\ L_0^{(n-1)}(\tau) & \cdots & L_{n-1}^{(n-1)}(\tau) \end{pmatrix}, \tag{1.32}$$

where $L_k : \mathbb{C} \to \mathcal{L}(X)$ *is the holomorphic map from Definition 1.8. In particular, the map* $\Sigma : \mathbb{C} \to \mathcal{L}(X^n)$ *is holomorphic.*

2. $\Sigma(0) = I$*, and for every* $\tau, \omega \in \mathbb{C}$ *we have that* $\Sigma(\omega)\Sigma(\tau) = \Sigma(\omega + \tau)$*. In particular,* $\Sigma(\tau) \in G(\mathcal{L}(X^n))$ *for each* $\tau \in \mathbb{C}$*, where* $G(\mathcal{L}(X^n)) \subset \mathcal{L}(X^n)$ *denotes the set of invertible elements of the Banach algebra* $\mathcal{L}(X^n)$*, and* $\Sigma : \mathbb{C} \to G(\mathcal{L}(X^n))$ *is a group homomorphism.*

3. *We have that* $\Sigma(\tau) = \exp(\tau\mathbb{A})$*, where* $\mathbb{A} \in \mathcal{L}(X^n)$ *is as in Lemma 1.5.*

Proof. The first item follows directly from Theorem 1.9. To prove the second it suffices to prove the third since $\tau\mathbb{A}$ and $\omega\mathbb{A}$ commute, and hence

$$\exp(\tau\mathbb{A})\exp(\omega\mathbb{A}) = \exp((\tau + \omega)\mathbb{A}). \tag{1.33}$$

However, we will give a direct proof of the second item as it is more enlightening.

Let $\omega \in \mathbb{C}$ and let $\Sigma(\tau) = (y_0, \ldots, y_{n-1})$. Then $\Sigma(\omega)(y_0, \ldots, y_{n-1}) = (y(\omega), y'(\omega), \ldots, y^{(n-1)}(\omega))$, where $y = S(y_0, \ldots, y_{n-1})$. In particular, this means that $y : \mathbb{C} \to X$ satisfies

$$\begin{cases} p(D)y = 0 \\ D^k y(0) = y_k = D^k x(\tau), \end{cases} \tag{1.34}$$

where $x = S(\xi_0, \ldots, \xi_{n-1})$. Define $z : \mathbb{C} \to X$ via $z = x(\cdot + \tau)$. Then $p(D)z = 0$ and $D^k z(0) = D^k x(\tau)$, and so by uniqueness we have that $z = y$, i.e. $y = x(\cdot + \tau)$. Hence

$$\begin{aligned} \Sigma(\omega)\Sigma(\tau)(\xi_0, \ldots, \xi_{n-1}) &= \Sigma(\omega)(y_0, \ldots, y_{n-1}) = (y(\omega), y'(\omega), \ldots, y^{(n-1)}(\omega)) \\ &= (x(\omega + \tau), x'(\omega + \tau), \ldots, x^{(n-1)}(\omega + \tau)) = \Sigma(\omega + \tau)(\xi_0, \ldots, \xi_{n-1}). \end{aligned} \tag{1.35}$$

This holds for arbitrary $(\xi_0, \ldots, \xi_{n-1}) \in X^n$, so we conclude that

$$\Sigma(\omega)\Sigma(\tau) = \Sigma(\omega + \tau) \text{ for all } \omega, \tau \in \mathbb{C}. \tag{1.36}$$

This proves the second item.

To prove the third item we simply note that, in the language of Lemma 1.5 the map $\Sigma(\tau)$ is given by $\Sigma(\tau)(\xi_0, \cdots, \xi_{n-1}) = \Xi(\tau)$, where $\Xi' = \mathbb{A}\Xi$ and $\Xi(0) = (\xi_0, \ldots, \xi_{n-1})$. Theorem 1.7 then shows that $\Xi(\tau) = \exp(\tau\mathbb{A})(\xi_0, \ldots, \xi_{n-1})$, and the third item is proved.

□

The propagators from Definition 1.8 now give us the ability to write down explicit formulas for the solutions to elliptic differential systems.

Theorem 1.11. *Let* X *and* Y *be complex Banach spaces and* $p(D) : H(\mathbb{C}; X) \to H(\mathbb{C}; Y)$ *be an elliptic differential operator of order* $n \geq 1$*. Let* $x : \mathbb{C} \to X$ *and* $f : \mathbb{C} \to Y$ *be holomorphic and* $\xi_0, \ldots, \xi_{n-1} \in X$*. Then the following are equivalent.*

1. x satisfies

$$\begin{cases} p(D)x = f \\ D^k x(0) = \xi_k \text{ for } 0 \le k \le n-1. \end{cases} \tag{1.37}$$

2. x is given by

$$\begin{aligned} x(\tau) &= S(\xi_0, \ldots, \xi_{n-1})(\tau) + \int_{\lambda_\tau} L_{n-1}(\tau - z) A_n^{-1} f(z) dz \\ &= \sum_{k=0}^{n-1} L_k(\tau)\xi_k + \int_{\lambda_\tau} L_{n-1}(\tau - z) A_n^{-1} f(z) dz \end{aligned} \tag{1.38}$$

where $\lambda_\tau : [0,1] \to \mathbb{C}$ is the road given by $\lambda_\tau(t) = t\tau$, and L_k and S are as in Definition 1.8.

Proof. Let $\Xi : \mathbb{C} \to X^n$ be the holomorphic map given by

$$\Xi(\tau) = \exp(\tau\mathbb{A})(\xi_0, \ldots, \xi_{n-1}) + \int_{\lambda_\tau} \exp((\tau - z)\mathbb{A}) F(z) dz, \tag{1.39}$$

where $F = (0, \ldots, 0, A_n^{-1} f)$ and $\mathbb{A}$ is determined by the coefficients of $p(D)$ as in Lemma 1.5. Theorem 1.7 shows that Ξ satisfies

$$\begin{cases} \Xi' = \mathbb{A}\Xi + F \\ \Xi(0) = (\xi_0, \ldots, \xi_{n-1}). \end{cases} \tag{1.40}$$

Theorem 1.10 then shows that

$$\Xi(\tau) = \Sigma(\tau)(\xi_0, \ldots, \xi_{n-1}) + \int_{\lambda_\tau} \Sigma(t - z)(0, \ldots, 0, A_n^{-1} f) dz, \tag{1.41}$$

and

$$\Xi_1(\tau) = \sum_{k=0}^{n-1} L_k(\tau)\xi_k + \int_{\lambda_\tau} L_{n-1}((\tau - z)) A_n^{-1} f(z) dz. \tag{1.42}$$

The result then follows immediately from these identities, Lemma 1.5, and the definition of $S : X^n \to \ker(p(D))$.

□

Next we turn our attention to certain linear algebraic questions related to $\ker(p(D))$. Our first result establishes some equivalent conditions to check for linearly independent and spanning sets.

Theorem 1.12. *Let X and Y be complex Banach spaces and $p(D) : H(\mathbb{C}; X) \to H(\mathbb{C}; Y)$ be an elliptic differential operator of order $n \ge 1$. The following hold.*

1. Let $x, y \in \ker(p(D))$. Then the following are equivalent.

(a) $x = y$ in $\ker(p(D))$.

(b) For each $\tau \in \mathbb{C}$ we have that

$$(x(\tau), x'(\tau), \ldots, x^{(n-1)}(\tau)) = (y(\tau), y'(\tau), \ldots, y^{(n-1)}(\tau)). \tag{1.43}$$

(c) *There exists* $\tau \in \mathbb{C}$ *such that*

$$(x(\tau), x'(\tau), \dots, x^{(n-1)}(\tau)) = (y(\tau), y'(\tau), \dots, y^{(n-1)}(\tau)). \tag{1.44}$$

2. *Let* $\varnothing \neq E \subseteq \ker(p(D))$. *Then the following are equivalent.*

   (a) *The set* E *is linearly independent in* $\ker(p(D))$.

   (b) *For each* $\tau \in \mathbb{C}$ *the set* $\{(x(\tau), x'(\tau), \dots, x^{(n-1)}(\tau))\}_{x\in E}$ *is linearly independent in* X^n.

   (c) *There exists* $\tau \in \mathbb{C}$ *such that the set* $\{(x(\tau), x'(\tau), \dots, x^{(n-1)}(\tau))\}_{x\in E}$ *is linearly independent in* X^n.

3. *Let* $\varnothing \neq E \subseteq \ker(p(D))$. *Then the following are equivalent.*

   (a) S *spans* $\ker(p(D))$.

   (b) *For each* $\tau \in \mathbb{C}$ *the set* $\{(x(\tau), x'(\tau), \dots, x^{(n-1)}(\tau))\}_{x\in E}$ *spans* X^n.

   (c) *There exists* $\tau \in \mathbb{C}$ *such that the set* $\{(x(\tau), x'(\tau), \dots, x^{(n-1)}(\tau))\}_{x\in E}$ *spans* X^n.

Proof. We begin with the proof of the first item. If $x = y$, then (1.43) obviously holds for every $\tau \in \mathbb{C}$. The fact that (1.43) implies (1.44) is trivial. If (1.44) holds for some $\tau \in \mathbb{C}$, then we may use Theorem 1.10 to see that

$$\begin{aligned}(x(0), x'(0), \dots, x^{(n-1)}(0)) &= \Sigma(-\tau)(x(\tau), x'(\tau), \dots, x^{(n-1)}(\tau)) \\ &= \Sigma(\tau)(y(\tau), y'(\tau), \dots, y^{(n-1)}(\tau)) = (y(0), y'(0), \dots, y^{(n-1)}(0)),\end{aligned} \tag{1.45}$$

and hence $x = y$ by the uniqueness part of Theorem 1.7. This proves the first item.

We now turn to the proof of the second item. Note that the first item implies that $\{x_1, \dots, x_m\} \subseteq E$ consists of distinct vectors if and only if $\{(x_j(\tau), x_j'(\tau), \dots, x_j^{(n-1)}(\tau))\}_{j=1}^m \subseteq X^n$ consists of distinct vectors for each $\tau \in \mathbb{C}$. Suppose that $\{x_1, \dots, x_m\} \subseteq E$ consists of distinct vectors and $\alpha_1, \dots, \alpha_m \in \mathbb{C}$. According to the first item,

$$\begin{aligned}\sum_{j=1}^m \alpha_j x_j = 0 \text{ in } \ker(p(D)) &\Leftrightarrow \sum_{j=1}^m \alpha_j (x_j(\tau), x_j'(\tau), \dots, x_j^{(n-1)}(\tau)) = 0 \text{ in } X^n \text{ for each } \tau \in \mathbb{C} \\ &\Leftrightarrow \sum_{j=1}^m \alpha_j (x_j(\tau), x_j'(\tau), \dots, x_j^{(n-1)}(\tau)) = 0 \text{ in } X^n \text{ for some } \tau \in \mathbb{C},\end{aligned} \tag{1.46}$$

which immediately implies the second item.

Finally, we prove the third item. Let $x \in \ker(p(D))$, $x_1, \dots, x_m \in E$, and $\alpha_1, \dots, \alpha_m \in \mathbb{C}$. The first item then shows that

$$\begin{aligned}&\sum_{j=1}^m \alpha_j x_j = x \text{ in } \ker(p(D)) \\ &\quad\Leftrightarrow \sum_{j=1}^m \alpha_j (x_j(\tau), x_j'(\tau), \dots, x_j^{(n-1)}(\tau)) = (x(\tau), x'(\tau), \dots, x^{(n-1)}(\tau)) \text{ in } X^n \text{ for each } \tau \in \mathbb{C} \\ &\Leftrightarrow \sum_{j=1}^m \alpha_j (x_j(\tau), x_j'(\tau), \dots, x_j^{(n-1)}(\tau)) = (x(\tau), x'(\tau), \dots, x^{(n-1)}(\tau)) \text{ in } X^n \text{ for some } \tau \in \mathbb{C}.\end{aligned} \tag{1.47}$$

On the other hand, $(y_0, \ldots, y_{n-1}) \in X^n$ if and only if

$$(y_0, \ldots, y_{n-1}) = (x(\tau), x'(\tau), \ldots, x^{(n-1)}(\tau)) \text{ for } x = S\Sigma(-\tau)(y_0, \ldots, y_{n-1}), \tag{1.48}$$

so the previous chain of equivalences proves the third item.

□

An obvious byproduct of this result is that if $p(D)$ has order $n \geq 1$, then $\ker(p(D))$ is infinite dimensional when X is and is finite dimensional when X is. In the finite dimensional case it remains to compute the exact dimension of the space.

Theorem 1.13. *Let X and Y be finite dimensional complex Banach spaces of dimension $N \geq 1$, and let $p(D) : H(\mathbb{C}; X) \to H(\mathbb{C}; Y)$ be an elliptic differential operator of order $n \geq 1$. The following hold.*

1. $\ker(p(D))$ *is finite dimensional, and* $\dim \ker(p(D)) = nN = n \dim(X)$.
2. *Let $\varnothing \neq B \subseteq \ker(p(D))$. Then the following are equivalent.*
   (a) *B is a basis of* $\ker(p(D))$.
   (b) *For each $\tau \in \mathbb{C}$ the set $\{(x(\tau), x'(\tau), \ldots, x^{(n-1)}(\tau))\}_{x \in B}$ is a basis of X^n.*
   (c) *There exists $\tau \in \mathbb{C}$ such that the set $\{(x(\tau), x'(\tau), \ldots, x^{(n-1)}(\tau))\}_{x \in B}$ is a basis of X^n.*

Proof. The first item follows from Theorem 1.9, which shows that $\ker(p(D))$ and X^n are isomorphic. The second item follows by combining the second and third items of Theorem 1.12. □

The following theorem shows how we can use the propagators $\{L_k\}_{k=0}^{n-1} \subset \mathcal{L}(X)$ to produce certain nice bases of $\ker(p(D))$ and, conversely, how we can recover these operators from these nice bases of $\ker(p(D))$.

Theorem 1.14. *Let X and Y be finite dimensional complex Banach spaces of dimension $N \geq 1$ and let $p(D) : H(\mathbb{C}; X) \to H(\mathbb{C}; Y)$ be an elliptic differential operator of order $n \geq 1$. For $0 \leq k \leq n-1$ let $B_k = \{b_{k,1}, \ldots, b_{k,N}\} \subset X$ be a basis, and suppose that $E = \{x_{kj} \mid 0 \leq k \leq n-1 \text{ and } 1 \leq j \leq N\} \subseteq \ker(p(D))$. Then the following are equivalent.*

1. *For $0 \leq k \leq n-1$ and $1 \leq j \leq N$ the function $x_{kj} \in \ker(p(D))$ is given by $x_{kj}(\tau) = L_k(\tau) b_{k,j}$.*
2. *E is a basis of $\ker(p(D))$ and for $0 \leq \ell, k \leq n-1$ and $1 \leq j \leq N$ we have that*

$$D^\ell x_{kj}(0) = \delta_{\ell k} b_{k,j}. \tag{1.49}$$

In either case, for each $0 \leq k \leq n-1$ and $\tau \in \mathbb{C}$ we have that

$$L_k(\tau) = \sum_{j=1}^{N} x_{kj}(\tau) b^*_{k,j}, \tag{1.50}$$

where $\{b^*_{k,1}, \ldots, b^*_{k,N}\} \subset X^*$ is the dual basis associated to B_k.

Proof. The fact that the first item implies the second follows directly from the second item of Theorem 1.13, together with the facts that $D^\ell L_k(0) = \delta_{k\ell} I$, and B_k is a basis of X.

Suppose, then, that the second item holds. For $0 \le k \le n-1$ define $R_K : \mathbb{C} \to \mathcal{L}(X)$ via

$$R_k(\tau) = \sum_{j=1}^N x_{kj}(\tau) b_{k,j}^*. \tag{1.51}$$

i.e. if $y = \sum_{j=1}^N \alpha_j b_{k,j}$ (and hence $\alpha_j = b_{k,j}^*(y) \in \mathbb{C}$), then

$$R_k(\tau) = \sum_{j=1}^N \alpha_j x_{kj}(\tau). \tag{1.52}$$

We then compute

$$p(D) R_k(\tau) = \sum_{j=1}^N p(D) x_{kj}(\tau) b_{k,j}^* = 0. \tag{1.53}$$

On the other hand, for any $y \in X$ we have that

$$D^\ell R_k(0) y = \sum_{j=1}^N b_{k,j}^*(y) D^\ell x_{kj}(0) = \delta_{k\ell} \sum_{j=1}^N b_{k,j}^*(y) b_{k,j} = \delta_{k\ell} y, \tag{1.54}$$

and hence $D^\ell R_k(0) = \delta_{k\ell} I$. However, $L_k : \mathbb{C} \to \mathcal{L}(X)$ is the unique solution to

$$\begin{cases} p(D) L_k = 0 \\ D^\ell L_k(0) = \delta_{k\ell} I, \end{cases} \tag{1.55}$$

so we deduce that $R_k = L_k$. Then

$$L_k(\tau) b_{k,j} = \sum_{m=1}^N x_{km}(\tau) b_{k,m}^*(b_{k,j}) = x_{kj}(\tau) \tag{1.56}$$

and we conclude that the first item holds.

□

Remark 1.15. *The most common use of the first item of Theorem 1.14 occurs when all of the B_k are the same. In practice, though, using the different bases B_k might be convenient in the second item when the basis $\{x_{kj}\}$ of $\ker(p(D))$ is found through some ad hoc means.*

2 Scalar differential equations with general initial conditions

We now turn our attention to the special case $X = \mathbb{C}$, in which case we call the equation $p(D)x = f$ a scalar ODE. Our goals are two-fold. First, we aim to derive some representation formulas for solutions that are more useful than those found in Theorem 1.11. While the formula from the theorem is useful from a theoretical perspective, it is impractical to work with in most situations. The issue is that for the formula to be useful, we first need to know the propagators, but these themselves are solutions to an equation involving $p(D)$, and so computing the propagators presents just as much difficulty as directly solving the original problem. Our new representation formulas will not be given in terms of the propagators. Second, with our new representation formulas in hand, we seek to study more general initial conditions to impose on solutions to $p(D)x = f$ at $\tau = 0$ and to completely characterize when such conditions lead to unique solvability.

We begin with an essential insight that will allow us to achieve these goals. Fix a differential operator $p(D) : H(\mathbb{C};\mathbb{C}) \to H(\mathbb{C};\mathbb{C})$ of order $n \geq 1$ (recall that when $X = \mathbb{C}$ all nontrivial differential operators are automatically elliptic). The insight comes from the simple observation that exponentials behave extremely nicely with respect to differentiation: for any $z \in \mathbb{C}$ we have that

$$\frac{d}{d\tau}e^{\tau z} = ze^{\tau z}. \tag{2.1}$$

Iteratively using this identity then reveals that

$$p(D)e^{\tau z} = p(z)e^{\tau z}, \tag{2.2}$$

and so we have an extremely simple solution to $p(D)x = 0$ for every root $z \in Z(p)$.

Let's suppose for the moment that p has n distinct roots, say $Z(p) = \{z_0, \ldots, z_{n-1}\} \subset B(0, R_0)$. Then for any $c_0, \ldots, c_{n-1} \in \mathbb{C}$ the holomorphic function $x : \mathbb{C} \to \mathbb{C}$ given by

$$x(\tau) = \sum_{k=0}^{n-1} e^{\tau z_k} c_k \tag{2.3}$$

solves $p(D)x = 0$, i.e. $x \in \ker(p(D))$. In fact, if we set $x_k(\tau) = e^{\tau z_k}$, then

$$\begin{pmatrix} x_0(0) & \cdots & x_{n-1}(0) \\ x_0'(0) & \cdots & x_{n-1}'(0) \\ \vdots & \ddots & \vdots \\ x_0^{(n-1)}(0) & \cdots & x_{n-1}^{(n-1)}(0) \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ z_0 & \cdots & z_{n-1} \\ \vdots & \ddots & \vdots \\ z_0^{n-1} & \cdots & z_{n-1}^{n-1} \end{pmatrix} \in \mathbb{C}^{n \times n} \tag{2.4}$$

and the latter matrix is the Vandermonde matrix associated to the set $\{z_0, \ldots, z_{n-1}\}$. An elementary exercise in linear algebra shows that this matrix is invertible precisely when the set of points $\{z_0, \ldots, z_{n-1}\}$ is distinct, and hence Theorem 1.13 tells us that the collection $\{x_0, \ldots, x_{n-1}\} \subset \ker(p(D))$ is a basis. Thus, every solution to $p(D)x = 0$ is of the form (2.3) when p has n distinct roots.

Next we fix some holomorphic function $h : \mathbb{C} \to \mathbb{C}$ and observe that Proposition A.1 implies that

$$\operatorname{Res}(e^{\tau \cdot} h/p, z_k) = e^{\tau z_k} \frac{h(z_k)}{p'(z_k)}. \tag{2.5}$$

Comparing (2.3) and (2.5) suggests that we set $c_k = h(z_k)/p'(z_k)$, which together with the residue theorem implies that for any $R > R_0$ we have

$$x(\tau) = \sum_{k=0}^{n-1} e^{\tau z_k} \frac{h(z_k)}{p'(z_k)} = \sum_{k=0}^{n-1} \operatorname{Res}(e^{\tau \cdot} h/p, z_k) = \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{\tau z} \frac{h(z)}{p(z)} dz. \tag{2.6}$$

We have thus arrived at a rather remarkable representation formula for $x \in \ker(p(D))$, one that does not require computing the propagators in advance.

The above representation formula was derived under the assumption that p had n distinct roots, but the resulting formula is agnostic to this fact, which suggests we might try to use it more generally. Before doing this, we need to make a key observation. The formula produces a solution $x_h \in \ker(p(D))$ for each $h \in H(\mathbb{C};\mathbb{C})$, but $\ker(p(D))$ is of dimension n, while $H(\mathbb{C};\mathbb{C})$ is infinite dimensional. Thus, it's wild overkill to use generic functions $h \in H(\mathbb{C};\mathbb{C})$ in the representation formula. Basic linear algebra suggests that we could reduce to using only h belonging to some subspace of $H(\mathbb{C};\mathbb{C})$ of dimension n, and an obvious choice of such a space is the set of complex polynomials of degree at most $n-1$. This will be our strategy.

To proceed we first need a couple technical tools. The first examines how this representation formula behaves in a more general context.

Proposition 2.1. *Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be meromorphic such that $P(f) \subset B(0,R)$. Then the function $x : \mathbb{C} \to \mathbb{C}$ defined by*

$$x(\tau) = \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{\tau z} f(z) dz \tag{2.7}$$

is holomorphic, and for each $k \in \mathbb{N}$ we have that

$$D^k x(\tau) = \frac{1}{2\pi i} \int_{\partial B(0,R)} z^k e^{\tau z} f(z) dz. \tag{2.8}$$

Moreover, for each $z \in P(f)$ there exists a polynomial $p_z : \mathbb{C} \to \mathbb{C}$ such that $\deg(p_z) \le \operatorname{ord}(f,z) - 1$, and

$$x(\tau) = \sum_{z \in P(f)} e^{\tau z} p_z(\tau) \text{ for all } \tau \in \mathbb{C}. \tag{2.9}$$

Proof. If $P(f) = \varnothing$, then Cauchy-Goursat implies that $x = 0$ and the result follows trivially. Assume then that $P(f) \neq \varnothing$. Since $P(f)$ is bounded, it must be finite, so we can write $P(f) = \{z_1, \dots, z_n\}$ for the n distinct poles of f. Write $n_k = \operatorname{ord}(f, z_k)$ for $1 \le k \le n$. The residue theorem and Proposition A.1 then imply that

$$x(\tau) = \sum_{k=1}^{n} \operatorname{Res}(e^{\tau \cdot} f, z_k) = \sum_{k=1}^{n} \frac{1}{(n_k - 1)!} \lim_{z \to z_k} \left(\frac{d}{dz}\right)^{n_k - 1} \left((z - z_k)^{n_k} e^{\tau z} f(z)\right). \tag{2.10}$$

Define $f_k : \mathbb{C} \to \mathbb{C}$ via $f_k(z) = (z - z_k)^{n_k} f(z)$, which is holomorphic by the definition of the order of a pole. From the Leibniz rule, we compute

$$D^{n_k - 1}\left(e^{\tau z} f_k(z)\right) = \sum_{j=0}^{n_k - 1} \frac{(n_k - 1)!}{j!(n_k - 1 - j)!} \tau^j e^{\tau z} D^{n_k - j - 1} f_k(z), \tag{2.11}$$

and hence

$$x(\tau) = \sum_{k=1}^{n} e^{\tau z_k} \sum_{j=0}^{n_k - 1} c_{jk} \tau^j \tag{2.12}$$

for some constants $c_{jk} \in \mathbb{C}$ for $1 \le k \le n$ and $0 \le j \le n_k - 1$. Hence x is a linear combination of exponentials multiplied by polynomials and is thus holomorphic. In particular, (2.9) is proved.

It remains only to prove (2.8). According to Cauchy-Goursat, we have that

$$x(\tau) = \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{\tau z} f(z) dz = \int_0^1 R \exp(\tau R e^{2\pi i t}) f(R e^{2\pi i t}) dt, \tag{2.13}$$

and hence

$$D^k x(\tau) = \int_0^1 R(Re^{2\pi i t})^k \exp(\tau Re^{2\pi i t}) f(Re^{2\pi i t}) dt = \frac{1}{2\pi i} \int_{\partial B(0,R)} z^k e^{\tau z} f(z) dz, \tag{2.14}$$

which is (2.8). □

The second technical tool associates to a polynomial $p : \mathbb{C} \to \mathbb{C}$ of degree n a collection of polynomials $q_0, \ldots, q_{n-1}$ with some properties that will be extremely useful in working with the linear algebra associated to our representation formula.

Proposition 2.2. *Suppose that $p : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $n \geq 1$ given by $p(z) = \sum_{m=0}^{n} a_m z^{n-m}$. For $0 \leq j \leq n-1$ define the polynomials $q_j : \mathbb{C} \to \mathbb{C}$ via*

$$q_j(z) = \sum_{m=0}^{n-1-j} a_m z^{n-j-1-m}. \tag{2.15}$$

Let $0 < R$ be such that $Z(p) \subseteq B(0,R)$. Then for each $0 \leq j, k \leq n-1$ we have that

$$\frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{z^k q_j(z)}{p(z)} dz = \delta_{jk} \tag{2.16}$$

Proof. First note that the degree of the polynomial $z \mapsto z^k q_j(z)$ is $n-1-j+k$. If $k < j$, then $n-1-j+k \leq n-2$, and so Proposition A.2 implies that

$$\frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{z^k q_j(z)}{p(z)} dz = 0. \tag{2.17}$$

On the other hand, if $j \leq k$, then

$$\begin{aligned} z^k q_j(z) &= z^k(a_0 z^{n-j-1} + \cdots + a_{n-j-1}) = z^{k-j-1}(a_0 z^n + \cdots + a_{n-j-1} z^{j+1}) \\ &= z^{k-j-1}(p(z) - (a_n + a_{n-1} z + \cdots + a_{n-j} z^j)) =: z^{k-j-1} p(z) - z^{-1} r_{j,k}(z), \end{aligned} \tag{2.18}$$

where $\deg(r_{j,k}) \leq k \leq n-1 \leq \deg(zp(z)) - 2$, so Proposition A.2 again implies

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{z^k q_j(z)}{p(z)} dz &= \frac{1}{2\pi i} \int_{\partial B(0,R)} z^{k-j-1} dz + \frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{r_{j,k}(z)}{zp(z)} dz \\ &= \frac{1}{2\pi i} \int_{\partial B(0,R)} z^{k-j-1} dz = \delta_{jk}. \end{aligned} \tag{2.19}$$

□

This suggests some notation.

Definition 2.3. *Suppose that $p : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $n \geq 1$ given by $p(z) = \sum_{m=0}^{n} a_m z^{n-m}$. The polynomials $q_0, \ldots, q_{n-1} : \mathbb{C} \to \mathbb{C}$ given by Proposition 2.2 are called the polynomials associated to p.*

The third technical tool defines a useful linear map from $H(\mathbb{C};\mathbb{C})$ to $C^\infty(\mathbb{C}^2;\mathbb{C})$.

Lemma 2.4. *Let $p : \mathbb{C} \to \mathbb{C}$ be the polynomial of degree $n \geq 1$ given by $p(z) = \sum_{k=0}^{n} A_k z^n$. Let $\{q_k\}_{k=0}^{n-1}$ be the associated polynomials from Definition 2.3. For each holomorphic $f : \mathbb{C} \to \mathbb{C}$ define $\mathcal{T}f : \mathbb{C}^2 \to \mathbb{C}$ via*

$$\mathcal{T}f(\tau, z) = A_n^{-1} \int_{\lambda_\tau} e^{(\tau - w)z} f(w) dw = A_n^{-1} \int_0^1 \tau e^{(1-t)\tau z} f(t\tau) dt, \tag{2.20}$$

where $\lambda_\tau : [0,1] \to \mathbb{C}$ is the road given by $\lambda_\tau(t) = t\tau$. Then the following hold.

1. *For each $z \in \mathbb{C}$ the map $\mathcal{T}f(\cdot, z)$ is holomorphic, and*

$$\partial_1^k \mathcal{T}f(\tau, z) = z^k \mathcal{T}f(\tau, z) + A_n^{-1} \sum_{j=0}^{k-1} D^j f(\tau) z^{k-1-j} \tag{2.21}$$

for all $k \geq 1$ and $\tau, z \in \mathbb{C}$.

2. *For each $\tau \in \mathbb{C}$ the map $\mathcal{T}f(\tau, \cdot)$ is holomorphic, and*

$$\partial_2^k \mathcal{T}f(\tau, z) = A_n^{-1} \int_{\lambda_\tau} (\tau - w)^k e^{(\tau - w)z} f(w) dw \tag{2.22}$$

for all $k \geq 1$ and $\tau, z \in \mathbb{C}$.

3. *$\mathcal{T}f$ is smooth, and the induced map $\mathcal{T} : H(\mathbb{C};\mathbb{C}) \to C^\infty(\mathbb{C}^2;\mathbb{C})$ is linear.*

Proof. Theorem 1.7 shows that for $z \in \mathbb{C}$ fixed, the map $\mathcal{T}f(\cdot, z)$ is holomorphic and satisfies

$$\partial_1 \mathcal{T}f(\tau, z) = A_n^{-1} f(\tau) + z\mathcal{T}f(\tau, z). \tag{2.23}$$

The formula for $\partial_1^k \mathcal{T}f(\tau, z)$ then follows from this and a simple induction argument. The second item follows from a calculation similar to that used to prove the first item of Theorem 1.7; we leave the details as an exercise. Using the first two items, we can readily compute $\partial_1^k \partial_2^j \mathcal{T}f(\tau, z)$ for all $k, j \in \mathbb{N}$, and the resulting expressions are continuous, so $\mathcal{T}f$ is smooth. Linearity is trivial. □

With our technical tools in hand, we can now derive our new representation formulas.

Theorem 2.5. *Let $p : \mathbb{C} \to \mathbb{C}$ be the polynomial of degree $n \geq 1$ given by $p(z) = \sum_{k=0}^{n} A_k z^n$. Let $\{q_k\}_{k=0}^{n-1}$ be the associated polynomials from Definition 2.3. Let $R > 0$ be such that $Z(p) \subset B(0, R)$. Then the following hold.*

1. *For $0 \leq k \leq n-1$ let $L_k : \mathbb{C} \to \mathbb{C}$ be the holomorphic functions given by Definition 1.8. Then*

$$L_k(\tau) = \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{\tau z} \frac{q_k(z)}{p(z)} dz. \tag{2.24}$$

2. *Given holomorphic functions $x, f : \mathbb{C} \to \mathbb{C}$ and $\xi_0, \ldots, \xi_{n-1} \in \mathbb{C}$, the following are equivalent.*

   (a) *x satisfies*

$$\begin{cases} p(D)x = f \\ D^k x(0) = \xi_k \text{ for } 0 \leq k \leq n-1. \end{cases} \tag{2.25}$$

(b) x is given by

$$x(\tau) = \sum_{k=0}^{n-1} \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{\tau z} \frac{\xi_k q_k(z)}{p(z)} dz + \int_{\lambda_\tau} L_{n-1}(\tau - z) A_n^{-1} f(z) dz$$
$$= \frac{1}{2\pi i} \int_{\partial B(0,R)} \left(e^{\tau z} \sum_{k=0}^{n-1} \xi_k q_k(z) + \mathcal{T} f(\tau, z) q_{n-1}(z) \right) \frac{dz}{p(z)}, \tag{2.26}$$

where $\lambda_\tau : [0,1] \to \mathbb{C}$ is the road given by $\lambda_\tau(t) = t\tau$ and $\mathcal{T} f$ is as in Lemma 2.4.

Proof. We begin with the proof of the first item. Define $R_k : \mathbb{C} \to \mathbb{C}$ via

$$R_k(\tau) = \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{\tau z} \frac{q_k(z)}{p(z)} dz. \tag{2.27}$$

Proposition 2.1 shows that R_k is holomorphic and

$$p(D) R_k(\tau) = \frac{1}{2\pi i} \int_{\partial B(0,R)} p(z) e^{\tau z} \frac{q_k(z)}{p(z)} dz = \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{\tau z} q_k(z) dz = 0, \tag{2.28}$$

where the last equality follows from Cauchy-Goursat. On the other hand, the properties of the associated polynomials $\{q_k\}_{k=0}^{n-1}$ imply that

$$D^j R_k(0) = \frac{1}{2\pi i} \int_{\partial B(0,R)} z^j \frac{q_k(z)}{p(z)} dz = \delta_{jk}, \tag{2.29}$$

and so $R_k = L_k$ by uniqueness.

The first item and Theorem 1.11 then imply the equivalence of (2.25) and the first identity in (2.26). It remains only to prove that

$$\int_{\lambda_\tau} L_{n-1}(\tau - z) A_n^{-1} f(z) dz = \frac{1}{2\pi i} \int_{\partial B(0,R)} \mathcal{T} f(\tau, z) \frac{q_{n-1}(z)}{p(z)} dz. \tag{2.30}$$

To this end we first use the first item to write

$$L_{n-1}(z) = \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{zw} \frac{q_{n-1}(w)}{p(w)} dw = \int_0^1 R 2^{2\pi i \theta} \exp(z R e^{2\pi i \theta}) \frac{q_{n-1}(R e^{2\pi i \theta})}{p(R e^{2\pi i \theta})} d\theta. \tag{2.31}$$

Then

$$\int_{\lambda_\tau} L_{n-1}(\tau - z) A_n^{-1} f(z) dz = \int_0^1 \tau L_{n-1}((1-t)\tau) A_n^{-1} f(t\tau) dt$$
$$= \int_0^1 \tau A_n^{-1} f(t\tau) \left(\int_0^1 R e^{2\pi i \theta} \exp((1-t)\tau R e^{2\pi i \theta}) \frac{q_{n-1}(R e^{2\pi i \theta})}{p(R e^{2\pi i \theta})} d\theta \right) dt. \tag{2.32}$$

All of the functions being integrated are smooth functions valued in $\mathbb{C}$. We may then expand into real and imaginary parts and apply Fubini's theorem to compute

$$\int_{\lambda_\tau} L_{n-1}(\tau - z) A_n^{-1} f(z) dz = \int_0^1 R e^{2\pi i \theta} \frac{q_{n-1}(R e^{2\pi i \theta})}{p(R e^{2\pi i \theta})} \left(\int_0^1 \tau \exp((1-t)\tau R e^{2\pi i \theta}) A_n^{-1} f(t\tau) dt \right) d\theta$$
$$= \frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{q_{n-1}(z)}{p(z)} \left(A_n^{-1} \int_{\lambda_\tau} e^{(\tau - w) z} f(w) dw \right) dz = \frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{q_{n-1}(z)}{p(z)} \mathcal{T} f(\tau, z) dz. \tag{2.33}$$

This is (2.30). □

The benefit of our new representation formula is that the solution is solely expressed in terms of the polynomial p and the data $\xi_0, \ldots, \xi_{n-1}$ and f. We don't need to compute the propagators first. The following example illustrates one way this presents an advantage.

Example 2.6. Fix $n \geq 1$ and let Ω be a metric space. Suppose that $a_0, \ldots, a_n : \Omega \to \mathbb{C}$ are continuous and that $a_n(\omega) \neq 0$ for all $\omega \in \Omega$. Define $\pi : \mathbb{C} \times \Omega \to \mathbb{C}$ via

$$\pi(z, \omega) = \sum_{j=0}^{n} a_j(\omega) z^j, \tag{2.34}$$

which in particular means, thanks to our assumption about a_n, that $\pi(\cdot, \omega) : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree n for every $\omega \in \Omega$.

Fix $\xi_0, \ldots, \xi_{n-1} \in \mathbb{C}$ and $f \in H(\mathbb{C}; \mathbb{C})$. We define $x : \mathbb{C} \times \Omega \to \mathbb{C}$ via the condition that $x(\cdot, \omega) \in H(\mathbb{C}; \mathbb{C})$ is the unique solution to

$$\begin{cases} \pi(D, \omega) x(\tau, \omega) = f(\tau) \\ D^k x(0, \omega) = \xi_k \text{ for } 0 \leq k \leq n-1 \end{cases} \tag{2.35}$$

for each $\omega \in \Omega$, where D acts only in the first variable. A natural question arises: how does x change as we change ω?

We can get some very useful information from our representation, but first we need a key observation. The associated polynomials will now also depend on ω since the coefficients a_j do. We write $q_j(z, \omega)$ to emphasize this. The formula for these shows that

$$q_j(z, \omega) = \sum_{m=0}^{n-1-j} a_{n-m}(\omega) z^{n-j-1-m}. \tag{2.36}$$

Fix $\omega_0 \in \Omega$ and let R_0 be such that $Z(\pi(\cdot, \omega_0)) \subset B(0, R_0)$. Since the roots of $\pi(\cdot, \omega)$ vary continuously with ω (see Theorem A.3), we can pick $\varepsilon > 0$ such that $Z(\pi(\cdot, \omega)) \subset B(0, R_0)$ for all $\omega \in B_\Omega(\omega_0, \varepsilon)$. For such ω we can plug into our representation formula to see that

$$x(\tau, \omega) = \frac{1}{2\pi i} \int_{\partial B(0, R_0)} e^{\tau z} \sum_{k=0}^{n-1} \xi_k \frac{q_k(z, \omega)}{\pi(z, \omega)} dz \text{ for all } \tau \in \mathbb{C}. \tag{2.37}$$

Using this and the dominated convergence theorem, we deduce that

$$\lim_{(\tau, \omega) \to (\tau_0, \omega_0)} x(\tau, \omega) = x(\tau_0, \omega_0) \tag{2.38}$$

for every $\tau_0 \in \mathbb{C}$ and $\omega_0 \in \Omega$. Hence, $x \in C^0(\mathbb{C} \times \Omega; \mathbb{C})$.

Suppose now that we have the extra information that Ω is an open subset of a normed vector space and that each a_k is C^m for some $m \geq 1$. Then this argument can be readily modified to deduce that for $1 \leq j \leq m$,

$$\partial_2^j x(\tau, \omega_0) = \frac{1}{2\pi i} \int_{\partial B(0, R_0)} e^{\tau z} \sum_{k=0}^{n-1} \xi_k \, \partial_2^j \left(\frac{q_k(z, \omega)}{\pi(z, \omega)} \right) \bigg|_{\omega = \omega_0} dz, \tag{2.39}$$

which is well-defined since the quotient rule shows the poles of the term in parentheses are a subset of the zeros of $\pi(\cdot, \omega_0)$. In turn, we can also prove that $x \in C^m(\mathbb{C} \times \Omega; \mathbb{C})$.

Of course, these facts can be proved with means other than our new representation formula, but the proof with the formula is rather direct and elegant. It's also worth pointing out that we can replace the constant data $\xi_0, \dots, \xi_{n-1} \in \mathbb{C}$ with continuous (or C^m when Ω is an open subset of a normed vector space) functions $\xi_j : \Omega \to \mathbb{C}$ for $0 \le j \le n-1$ and prove similar results. We leave it as an exercise to formulate and prove these extensions.

△

The flexibility of our new representation formulas also allows us to consider more general initial conditions. Recall from Theorem 1.7 that the map $H(\mathbb{C};\mathbb{C}) \ni x \mapsto (p(D)x, x(0), \dots, D^{n-1}x(0))$ is an isomorphism when $p(D)$ has order $n \ge 1$. This suggests that we consider more general operators

$$B(D) : H(\mathbb{C};\mathbb{C}) \to H(\mathbb{C};\mathbb{C}^n) \tag{2.40}$$

given by

$$B(D)x = (B_0(D)x, \dots, B_{n-1}(D)x) \tag{2.41}$$

where $B_k(D)$ is a differential operator. Note that we don't specify any control on the order of $B_k(D)$. We thus arrive at the problem of solving

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \dots, \xi_{n-1}) \end{cases} \tag{2.42}$$

for given $f \in H(\mathbb{C};\mathbb{C})$ and $(\xi_0, \dots, \xi_{n-1}) \in \mathbb{C}^n$.

In order to attack this problem, we first need to introduce some algebraic notation.

Definition 2.7. *Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial.*

1. *We know from Euclidean division in the set of polynomials over a field that, given any polynomial $q : \mathbb{C} \to \mathbb{C}$ there exist polynomials $d, r : \mathbb{C} \to \mathbb{C}$ such that $q = dp + r$ and either $r = 0$ or else $0 \le \deg(r) < \deg(d)$. If $r = 0$ then we write $q \equiv_p 0$.*

2. *Given polynomials $q_1, q_2 : \mathbb{C} \to \mathbb{C}$ we write $q_1 \equiv_p q_2$ if $q_2 - q_1 \equiv_p 0$. We leave it as an exercise to verify that this is an equivalence relation.*

The previous definition is the analog of modular arithmetic in the space of complex polynomials. It will play a role in our study of differential equations because if we want to specify $p(D)x = f$ and $B_k(D)x(0) = \xi_k$, then we have to worry about how B_k relates to p. Indeed, if $B_k = qp + r$, then, in light of the first equation, the condition $B_k(D)x(0) = \xi_k$ reduces to $r(D)x(0) = \xi_k - q(D)f(0)$. To track this more carefully we introduce the following definition.

Definition 2.8. *Let $1 \le n \in \mathbb{N}$ and let $B : \mathbb{C} \to \mathbb{C}^n$ be a polynomial, written*

$$B(z) = (B_0(z), \dots, B_{n-1}(z)). \tag{2.43}$$

Let $R_k : \mathbb{C} \to \mathbb{C}$ be the polynomial given by $R_k \equiv_p B_k$ and write

$$R_k(z) = \sum_{j=0}^{n-1} R_{k,j} z^j. \tag{2.44}$$

We then define $\mathcal{R}_B \in \mathbb{C}^{n \times n}$ via

$$\mathcal{R}_B = \begin{pmatrix} R_{0,0} & \cdots & R_{0,n-1} \\ \vdots & \ddots & \vdots \\ R_{n-1,0} & \cdots & R_{n-1,n-1} \end{pmatrix}. \tag{2.45}$$

Our next result establishes the basic properties of the map $\mathcal{R}_B$.

Proposition 2.9. *Let $1 \leq n \in \mathbb{N}$ and let $B : \mathbb{C} \to \mathbb{C}^n$ be a polynomial, written $B(z) = (B_0(z), \dots, B_{n-1}(z))$. Let the polynomials $\{R_0, \dots, R_{n-1}\}$ and the matrix $\mathcal{R}_B \in \mathbb{C}^{n \times n}$ be as in Definition 2.8, and let $c_0, \dots, c_{n-1} \in \mathbb{C}^n$ denote the columns of $\mathcal{R}_B$. Then the following hold.*

1. *For $z \in \mathbb{C}$ we have the identities*

$$B(z) \equiv_p \begin{pmatrix} R_0(z) \\ R_1(z) \\ \vdots \\ R_{n-1}(z) \end{pmatrix} = \mathcal{R}_B \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{pmatrix} = c_0 + zc_1 + \cdots + z^{n-1}c_{n-1}. \tag{2.46}$$

2. *The following are equivalent.*
   (a) *$\mathcal{R}_B \in \mathbb{C}^{n \times n}$ is invertible.*
   (b) *The vectors $\{c_0, \dots, c_{n-1}\} \subset \mathbb{C}^n$ are linearly independent.*
   (c) *The polynomials $\{R_0, \dots, R_{n-1}\}$ are linearly independent.*
   (d) *The polynomials $\{B_0, \dots, B_{n-1}\}$ are linearly independent (mod p).*

Proof. The first item follows from direct calculation. We now prove the second item. The equivalence of (a) and (b) is a standard result from linear algebra, and the equivalence of (c) and (d) is trivial. We also know from elementary linear algebra that $\mathcal{R}_B$ is invertible if and only if its rows are linearly independent. We then note that for $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{C}$,

$$\sum_{j=0}^{n-1} \alpha_j R_{j,k} = 0 \text{ for all } 0 \leq k \leq n-1 \Leftrightarrow \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \alpha_j R_{j,k} z^k = 0 \text{ for all } z \in \mathbb{C}$$
$$\Leftrightarrow \sum_{j=0}^{n-1} \alpha_j R_j(z) = 0 \text{ for all } z \in \mathbb{C}, \tag{2.47}$$

and hence the polynomials $\{R_0, \dots, R_{n-1}\}$ are linearly independent if and only if the rows of $\mathcal{R}_B$ are linearly independent, which proves the equivalence of (a) and (c).

□

We now combine our representation formula from Theorem 2.5 with the map $\mathcal{R}_B$ to characterize when the problem

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \dots, \xi_{n-1}) \end{cases} \tag{2.48}$$

is uniquely solvable.

Theorem 2.10. *Let $p(D) : H(\mathbb{C}; \mathbb{C}) \to H(\mathbb{C}; \mathbb{C})$ be a differential operator of order $n \geq 1$, and let $B(D) : H(\mathbb{C}; \mathbb{C}) \to H(\mathbb{C}; \mathbb{C}^n)$ be a differential operator, written $B(D) = (B_0(D), \dots, B_{n-1}(D))$. Let $\mathcal{R}_B \in \mathbb{C}^{n \times n}$ be determined by B as in Definition 2.8. Then the following are equivalent.*

1. *The polynomials $\{B_0, \dots, B_{n-1}\}$ are linearly independent (mod p).*

2. *$\mathcal{R}_B$ is invertible.*

3. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\xi_0, \ldots, \xi_{n-1} \in \mathbb{C}$, then there exists a unique holomorphic $x : \mathbb{C} \to \mathbb{C}$ such that*

$$\begin{cases} p(D)x = f \\ B_k(D)x(0) = \xi_k \text{ for } 0 \le k \le n-1. \end{cases} \tag{2.49}$$

4. *The map $\Phi : H(\mathbb{C};\mathbb{C}) \to H(\mathbb{C};\mathbb{C}) \times \mathbb{C}^n$ given by*

$$\Phi x = (p(D)x, B(D)x(0)) \tag{2.50}$$

is a linear isomorphism.

5. *For every $(\xi_0, \ldots, \xi_{n-1}) \in \mathbb{C}^n$ there exists a holomorphic $x : \mathbb{C} \to \mathbb{C}$ such that*

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = (\xi_0, \ldots, \xi_{n-1}). \end{cases} \tag{2.51}$$

6. *If $x : \mathbb{C} \to \mathbb{C}$ is holomorphic and*

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = 0, \end{cases} \tag{2.52}$$

then $x = 0$.

In any case, the unique solution to (2.49) is given by

$$x(\tau) = y_f(\tau) + \frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{e^{\tau z}}{p(z)} \sum_{k,m=0}^{n-1} q_k(z) (\mathcal{R}_B)^{-1}_{k,m} (\xi_m - B_m(D) y_f(0)) dz, \tag{2.53}$$

where $y_f : \mathbb{C} \to \mathbb{C}$ is the holomorphic function given by

$$y_f(\tau) = \frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{\mathcal{T} f(\tau, z) q_{n-1}(z)}{p(z)} dz \tag{2.54}$$

and $Z(p) \subset B(0,r)$.

Proof. The equivalence of (1) and (2) was established in Proposition 2.9, and the equivalence of (3) and (4) is trivial. The implications (3) $\Rightarrow$ (5) and (3) $\Rightarrow$ (6) are also trivial. To conclude, we will prove (2) $\Leftrightarrow$ (5), (2) $\Leftrightarrow$ (6), and that (5) and (6) $\Rightarrow$ (3).

First, we make some observations used multiple times. For $0 \le k \le n-1$ let the polynomial $R_k : \mathbb{C} \to \mathbb{C}$ be given by $R_k \equiv_p B_k$, which means we can write $B_k = A_k p + R_k$ for some polynomial $A_k : \mathbb{C} \to \mathbb{C}$. Theorem 2.5 shows that a holomorphic function $x : \mathbb{C} \to \mathbb{C}$ satisfies $p(D)x = 0$ if and only if there exist $\alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{C}^n$ such that

$$x(\tau) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{e^{\tau z}}{p(z)} \sum_{k=0}^{n-1} \alpha_k q_k(z) dz, \tag{2.55}$$

where $Z(p) \subset B(0,r)$, in which case $D^k x(0) = \alpha_k$ for $0 \leq k \leq n-1$. For any such x we use Cauchy-Goursat and Proposition 2.2 to compute

$$B_j(D)x(0) = \frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{B_j(z)}{p(z)} \sum_{k=0}^{n-1} \alpha_k q_k(z) dz = \frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{A_j(z)p(z) + R_j(z)}{p(z)} \sum_{k=0}^{n-1} \alpha_k q_k(z) dz$$
$$= \frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{R_j(z)}{p(z)} \sum_{k=0}^{n-1} \alpha_k q_k(z) dz = \frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{1}{p(z)} \sum_{m,k=0}^{n-1} (\mathcal{R}_B)_{j,m} z^m \alpha_k q_k(z) dz$$
$$= \sum_{m,k=0}^{n-1} (\mathcal{R}_B)_{j,m} \alpha_k \delta_{mk} = \sum_{k=0}^{n-1} (\mathcal{R}_B)_{j,k} \alpha_k = (\mathcal{R}_B \alpha)_j, \quad (2.56)$$

and hence

$$B(D)x(0) = \mathcal{R}_B \alpha. \quad (2.57)$$

Proof of (2) $\Leftrightarrow$ (5)**:** The identity (2.57) means that a holomorphic function $x : \mathbb{C} \to \mathbb{C}$ satisfies (2.49) if and only if x is given by (2.55) with $\alpha \in \mathbb{C}^n$ satisfying

$$\mathcal{R}_B \alpha = (\xi_0, \dots, \xi_{n-1}). \quad (2.58)$$

Thus, if $\mathcal{R}_B$ is invertible we can solve for α in terms of $(\xi_0, \dots, \xi_{n-1})$ to produce x solving (2.49), which shows that (2) $\Rightarrow$ (5). Conversely, if (2.49) has a solution for every $(\xi_0, \dots, \xi_{n-1})$, then (2.57) shows that $\mathcal{R}_B$ is surjective and hence invertible, which proves (5) $\Rightarrow$ (2).

Proof of (2) $\Leftrightarrow$ (6)**:** We prove the contrapositive. Assertion (6) is false if and only if there exists $0 \neq x \in \ker(p(D))$ such that $B(D)x(0) = 0$. In light of (2.55) and (2.57), this is equivalent to the existence of $0 \neq \alpha \in \mathbb{C}^n$ such that $\mathcal{R}_B \alpha = 0$. In turn, this is equivalent to $\mathcal{R}_B \in \mathbb{C}^{n \times n}$ not being invertible, i.e. (2) being false.

Proof of (5) **and** (6) $\Rightarrow$ (3)**:** First note that (6) implies that there exists at most one solution to (2.49), so it suffices to prove the existence of a solution. Define the holomorphic function $y : \mathbb{C} \to \mathbb{C}$ via

$$y(\tau) = \frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{\mathcal{T} f(\tau, z) q_{n-1}(z)}{p(z)} dz. \quad (2.59)$$

Theorem 2.5 shows that $p(D)y = 0$ and that $D^k y(0) = 0$ for $0 \leq k \leq n-1$. According to (5), we can then find a holomorphic $h : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{cases} p(D)h = 0 \\ B(D)h(0) = (\xi_0, \dots, \xi_{n-1}) - B(D)y(0). \end{cases} \quad (2.60)$$

Then $x = h + y$ is holomorphic and satisfies (2.49).

□

Remark 2.11. *The condition that the polynomials $\{B_0, \dots, B_{n-1}\}$ are linearly independent (mod p) is called the Shapiro-Lopatinsky condition. What's remarkable about it is that it reduces the question of the solvability of the general problem to checking an algebraic condition.*

Remark 2.12. *In practice it often occurs that* $\deg B_k \leq n-1$ *for* $0 \leq k \leq n-1$*, in which case the polynomials $\{B_0, \dots, B_{n-1}\}$ are linearly independent if and only if they are linearly independent (mod p). In this context we also have that the function y_f from Theorem 2.10 satisfies* $D^k y_f(0) = 0$

for $0 \leq k \leq n-1$, and hence $B(D)y_f(0) = 0$. This means that the unique solution to (2.49) takes the simpler form

$$x(\tau) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \left(e^{\tau z} \sum_{k,m=0}^{n-1} q_k(z)(\mathcal{R}_B)^{-1}_{k,m} \xi_m + \mathcal{T} f(\tau, z) q_{n-1}(z) \right) \frac{dz}{p(z)}. \tag{2.61}$$

3 General finite dimensional ordinary differential systems

We now turn our attention to general finite dimensional ordinary differential systems. From our above analysis we already have a good understanding of these when the operator is elliptic and the initial conditions encode the first few derivatives, so here generality refers to the non-elliptic case and to more general initial conditions. We restrict our attention to the finite dimensional setting because the determinant is going to play a fundamental role in our analysis.

Before proceeding, we need to establish some notation associated to polynomials $q : \mathbb{C} \to \mathbb{C}^{m \times n}$. Write q as $q(z) = \sum_{j=0}^{J} A_j z^j$ for $A_j \in \mathbb{C}^{m\times n}$. It will be essential for us to have a slightly different perspective and view q as a matrix of polynomials via

$$q(z) = \begin{pmatrix} \sum_{j=0}^{J} A_{j,11} z^j & \cdots & \sum_{j=0}^{J} A_{j,1n} z^j \\ \vdots & \ddots & \vdots \\ \sum_{j=0}^{J} A_{j,m1} z^j & \cdots & \sum_{j=0}^{J} A_{j,mn} z^j \end{pmatrix} = \begin{pmatrix} q_{11}(z) & \cdots & q_{1n}(z) \\ \vdots & \ddots & \vdots \\ q_{m1}(z) & \cdots & q_{mn}(z) \end{pmatrix}. \tag{3.1}$$

In other words, we think of q as both a polynomial with matrix-valued coefficients and as a matrix of polynomials. In turn this yields

$$q(D) = \begin{pmatrix} q_{11}(D) & \cdots & q_{1n}(D) \\ \vdots & \ddots & \vdots \\ q_{m1}(D) & \cdots & q_{mn}(D) \end{pmatrix} \tag{3.2}$$

in which we think of $q(D) : H(\mathbb{C}; \mathbb{C}^n) \to H(\mathbb{C}; \mathbb{C}^m)$ as a matrix composed of the scalar differential operators $q_{k\ell}(D) : H(\mathbb{C}; \mathbb{C}) \to H(\mathbb{C}; \mathbb{C})$ for $1 \leq k \leq m$ and $1 \leq \ell \leq n$. Note that often in our study of this problem we will shift the indexing to $0 \leq k \leq m-1$.

One of the advantages of this perspective is that it allows us to use tools associated to polynomials on each of the components in the matrix. In particular, we can consider the question of whether the rows or columns are linearly independent (mod p), where $p : \mathbb{C} \to \mathbb{C}$ is a given polynomial. For instance, the rows are linearly independent (mod p) if

$$\sum_{j=1}^{m} \alpha_j q_{jk} \equiv_p 0 \Rightarrow \alpha_1 = \cdots = \alpha_m = 0. \tag{3.3}$$

This is a condition that will play a key role in our analysis.

Given the above perspective, for any polynomial $p : \mathbb{C} \to \mathbb{C}^{N \times N}$ we can think of $\det p : \mathbb{C} \to \mathbb{C}$ as either the pointwise evaluation of the determinant of $p(z) \in \mathbb{C}^{N\times N}$, or else as appropriate linear combination of products of the polynomials $\{p_{jk}(z)\}_{1 \leq j,k \leq N}$. The following proposition records the basic properties of the determinant.

Proposition 3.1. *Let $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ be given by $p(z) = \sum_{k=0}^{n} A_k z^k$. Then the following hold.*

1. *$\det p : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $d \leq nN$.*

2. *Write* $\det p(z) = \sum_{k=0}^{nN} \alpha_k z^k$. *Then* $\alpha_{nN} = \det A_n$ *and* $\alpha_0 = \det A_0$.

3. $p(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ *is elliptic if and only if* $d = nN$.

Proof. Each component of the matrix $p(z) \in \mathbb{C}^{N\times N}$ is a polynomial of degree at most n. As such, $\det p(z)$ consists of linear combinations of products of N polynomials, each of degree at most n, and hence the degree of $\det p$ is at most nN. This proves the first item.

We now prove the second item. First note that $\alpha_0 = \det p(0) = \det(A_0)$. On the other hand, for $r \in (0,\infty)$ we have that

$$\alpha_{nN} = \lim_{r\to\infty} \frac{\det p(r)}{r^{nN}} = \lim_{r\to\infty} \det\left(r^{-n}\sum_{k=0}^{n} A_k r^k\right) = \lim_{r\to\infty} \det\left(\sum_{k=0}^{n} A_k r^{k-n}\right) = \det(A_n). \tag{3.4}$$

This proves the second item, and the third follows immediately from the second. □

Let's consider an example

Example 3.2. Suppose that $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ is a polynomial in upper-triangular form:

$$p(z) = \begin{pmatrix} p_{11}(z) & p_{12}(z) & \cdots & p_{1N}(z) \\ 0 & p_{22}(z) & \cdots & p_{2N}(z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{NN}(z) \end{pmatrix}. \tag{3.5}$$

Then the usual rules of the determinant show that

$$\det p(z) = \prod_{j=1}^{N} p_{jj}(z) \text{ and } \deg(\det p) = \sum_{j=1}^{N} \deg(p_{jj}). \tag{3.6}$$

Setting $p_{1N}(z) = z^n$, $p_{jj}(z) = a_j z^{n_j}$ for $a_j \in \mathbb{C}$ and $0 \le n_j \le n$, and all other entries to 0, we readily deduce that $\deg(\det p)$ can take on any value from -1 (when $\det p = 0$) to nN when p has degree n. △

This example shows one of the bizarre features of polynomial determinants, namely that the determinant can end up a polynomial of any degree $-1 \le d \le nN$, with the endpoints corresponding to degeneracy, i.e. $\det p = 0$, and ellipticity. This suggests a definition.

Definition 3.3. *Let* $p(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ *be a differential operator of order* $n \in \mathbb{N}$, *and consider the differential operator* $\det p(D) : H(\mathbb{C};\mathbb{C}) \to H(\mathbb{C};C)$. *We define the determinant-order of* $p(D)$ *to be the order of the operator* $\det p(D)$, *i.e. the degree of the polynomial* $\det p(z)$. *Note that Proposition 3.1 guarantees that the determinant-order of* $p(D)$ *is no more than* nN.

Let's consider some examples.

Example 3.4. Example 3.2 shows that for any $n, N \in \mathbb{N}$ with $N \ge 2$ and $-1 \le d \le nN$, we can construct a differential operator $p(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ of order n with determinant order d. △

Example 3.5. Define $p(D), q(D), r(D) : H(\mathbb{C};\mathbb{C}^2) \to H(\mathbb{C};\mathbb{C}^2)$ via

$$p(D) = \begin{pmatrix} D & 1 \\ 2D & -1 \end{pmatrix}, q(D) = \begin{pmatrix} D & 1 \\ -1 & 2D \end{pmatrix}, \text{ and } r(D) = \begin{pmatrix} 4D^3 & 2iD^2 \\ -2iD^2 & D \end{pmatrix} \tag{3.7}$$

Then

$$\det p(D) = -D - 2D = -3D, \det q(D) = 2D^2 + 1, \text{ and } \det r(D) = 4D^4 + 4i^2 D^4 = 0 \tag{3.8}$$

so $p(D)$ has determinant-order $1 < 2 = 1{\cdot}2$ and is thus not elliptic, while $q(D)$ has determinant-order 2 and is thus elliptic, and $r(D)$ has determinant order -1 and is not elliptic. △

Example 3.6. Fix $r \in \mathbb{R}$ and define $p(D) : H(\mathbb{C};\mathbb{C}^3) \to H(\mathbb{C};\mathbb{C}^3)$ via

$$p(D) = \begin{pmatrix} -D^2 + r^2 & 0 & ir \\ 0 & -D^2 + r^2 & D \\ ir & D & 0 \end{pmatrix}. \tag{3.9}$$

Then

$$\det p(D) = r^2(-D^2 + r^2) - D^2(-D^2 + r^2) = (D^2 - r^2)^2, \tag{3.10}$$

so $p(D)$ has determinant-order $4 < 6 = 2 \cdot 3$ and is thus not elliptic.

△

Example 3.7. Define $p(D) : H(\mathbb{C};\mathbb{C}^2) \to H(\mathbb{C};\mathbb{C}^2)$ via

$$p(D) = \begin{pmatrix} D & D^2 - 1 \\ 1 & D \end{pmatrix}. \tag{3.11}$$

Then

$$\det p(D) = D^2 - D^2 + 1 = 1, \tag{3.12}$$

so $p(D)$ has determinant-order 0 and is thus not elliptic.

Suppose that $f : \mathbb{C} \to \mathbb{C}^2$ is a given holomorphic function and we wish to find $x : \mathbb{C} \to \mathbb{C}^2$ holomorphic satisfying $p(D)x = f$. Note that

$$\begin{pmatrix} D & -D^2 + 1 \\ -1 & D \end{pmatrix} \begin{pmatrix} D & D^2 - 1 \\ 1 & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} D & D^2 - 1 \\ 1 & D \end{pmatrix} \begin{pmatrix} D & -D^2 + 1 \\ -1 & D \end{pmatrix}. \tag{3.13}$$

Hence, if there is a solution we must have that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} D & -D^2 + 1 \\ -1 & D \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} Df_1 - D^2 f_2 + f_2 \\ -f_1 + Df_2 \end{pmatrix}. \tag{3.14}$$

Conversely, if we define x in this manner a simple calculation shows that $p(D)x = f$. △

This example shows that the strange behavior of polynomial determinants leads to even stranger behavior in differential systems: in the case that $p(D)$ has determinant-order 0 it's possible that we can invert the differential operator $p(D)$ by applying another differential operator. This phenomenon is more general than what we observed in Example 3.7, as we now prove.

Theorem 3.8. *Let $N \geq 2$ and $p(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ be a differential operator. Then p is of determinant-order 0 if and only if there exists a differential operator $q(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ such that $p(D)q(D) = q(D)p(D) = I$. In either case, the following hold.*

1. $q(D)$ *is also of determinant-order* 0.

2. *For each* $f \in H(\mathbb{C};\mathbb{C}^N)$ *the unique solution to* $p(D)x = f$ *is* $x = q(D)f \in H(\mathbb{C};\mathbb{C}^N)$.

3. $p(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ *is a linear isomorphism.*

Proof. Suppose that p has determinant-order 0 and form the adjugate $A(z)$ from the matrix-valued polynomial $p(z)$ in the usual way. The adjugate matrix is constructed by computing the determinants of the minors of $p(z)$, and as such, $A(z)$ is itself a polynomial. Since $\det p(z)$ is a polynomial of degree 0, it is some nonzero constant, and hence the matrix $p(z) \in \mathbb{C}^{N\times N}$ is invertible for all $z \in \mathbb{C}$. Then the identity

$$(p(z))^{-1} = (\det p(z))^{-1} A(z) \text{ for } z \in \mathbb{C} \tag{3.15}$$

shows that $(p(z))^{-1}$ is also a polynomial. Setting $q = p^{-1}$, we then find that $p(D)q(D) = q(D)p(D) = I$. We then have that

$$\det q(z) = (\det p(z))^{-1}, \tag{3.16}$$

so $q(D)$ also has determinant-order 0.

We now prove the converse. The identity $pq = I$ implies that $\det p(z) \det q(z) = 1$ for all $z \in \mathbb{C}$, and the only way that two complex polynomials can be multiplied together to produce unity is if both are unity. Hence, $\det p = 1$, which means p has determinant-order 0.

The fact that the unique solution to $p(D)x = f$ is $x = q(D)f$ follows as in Example 3.7. In turn this readily implies that $p(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ is an isomorphism.

□

We have now seen that there is interesting information encoded in $\det p$ when $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ is a polynomial. The proof of Theorem 3.8 also suggests that the adjugate matrix associated to p may prove useful. We define it now.

Definition 3.9. *Let* $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ *be a polynomial. We define its adjugate* $\operatorname{adj} p : \mathbb{C} \to \mathbb{C}^{N\times N}$ *by setting* $\operatorname{adj} p(z) \in \mathbb{C}^{N\times N}$ *to be the adjugate matrix associated to the matrix* $p(z) \in \mathbb{C}^{N\times N}$*, with the convention that* $\operatorname{adj} p = 1$ *when* $N = 1$*. For* $N \geq 2$*, the adjugate is computed by taking determinants of the minors of* $p(z)$*, so* $\operatorname{adj} p$ *is also a polynomial. We then define the differential operator* $\operatorname{adj} p(D)$ *as per usual.*

Our next result records some basic properties of the adjugate.

Proposition 3.10. *Let* $p : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ *be a differential operator. Then the following hold.*

1. *We have that*
$$p(D) \operatorname{adj} p(D) = \operatorname{adj} p(D) p(D) = (\det p(D)) I. \tag{3.17}$$

2. *If* $p(D)$ *has determinant-order* 0*, then* $(p(D))^{-1} = (\det p(D))^{-1} \operatorname{adj} p(D)$.

3. *If* $x, f \in H(\mathbb{C};\mathbb{C}^N)$ *and* $p(D)x = f$*, then* $\det p(D) x_j = (\operatorname{adj} p(D) f)_j$ *for each* $1 \leq j \leq N$.

4. *If* $x \in \ker(p(D))$*, then* $x_j \in \ker(\det p(D))$ *for each* $1 \leq j \leq N$.

5. *If* $y, f \in H(\mathbb{C};\mathbb{C}^N)$ *satisfy* $\det p(D) y = f$ *and* $x = \operatorname{adj} p(D) y \in H(\mathbb{C};\mathbb{C}^N)$*, then* $p(D)x = f$.

6. $\operatorname{adj} p(D) : \ker(\det p(D) I) \to \ker(p(D))$ *is a linear map.*

Proof. The first two items follow from elementary linear algebra and the fact that if $p(D)$ has determinant-order 0 then it is an isomorphism on $H(\mathbb{C};\mathbb{C}^N)$. The third follows directly from the first, and the fourth follows from the third with $f = 0$. The fifth item also follows from the first, and the sixth follows from the fifth. $\square$

The fifth and sixth items of Proposition 3.10 give us a mechanism for producing solutions to $p(D)x = f$ for a given $f \in H(\mathbb{C};\mathbb{C}^N)$, provided that $p(D)$ has determinant-order $d \geq 1$. Indeed, we first find $y \in H(\mathbb{C};\mathbb{C}^N)$ such that $\det p(D)y = f$, which can be accomplished by working component-wise and using our previous scalar theory, i.e. we solve $\det p(D)y_j = f_j$ for $1 \leq j \leq N$ and then form $y = (y_1, \ldots, y_N) \in H(\mathbb{C};\mathbb{C}^N)$. Then we take any $h \in H(\mathbb{C};\mathbb{C}^N)$ such that $h_j \in \ker(\det p(D))$ for $1 \leq j \leq N$, which again we can construct using our previous analysis. Then $x = \operatorname{adj}(p(D))(y+h) \in H(\mathbb{C};\mathbb{C}^N)$ satisfies $p(D)x = f$. However, it's not clear yet that all solutions are of this form or that we have any uniqueness of such representations. We now turn to addressing these issues.

First, we need two technical results. The first is a version of Bézout's lemma for polynomials.

Lemma 3.11 (Bézout's lemma)**.** *Let $p, q : \mathbb{C} \to \mathbb{C}$ be nontrivial polynomials and suppose that $g : \mathbb{C} \to \mathbb{C}$ is the greatest common divisor of p and q, i.e. g divides both p and q and is the monic polynomial of maximal degree that does so. Then there exist polynomials $r, s : \mathbb{C} \to \mathbb{C}$ such that*

$$pr + qs = g. \tag{3.18}$$

Proof. Define the set

$$E = \{ap + bq \mid a, b : \mathbb{C} \to \mathbb{C} \text{ are polynomials}\}. \tag{3.19}$$

Since p and q are nontrivial, there are nontrivial polynomials in E, and so by the well-ordering principle there is a nontrivial polynomial $c = pR + qS \in E$ of minimal degree. Using the Euclidean algorithm, we can write $p = uc + a$, where $\deg(a) < \deg(c)$. Then

$$a = p - uc = (1 - uR)p - (uS)q \in E, \tag{3.20}$$

so we contradict the minimality of $\deg(c)$ unless $a = 0$. Thus $p = uc$. Arguing similarly, we write $q = vc + b$ with $\deg(b) < (c)$ and deduce that $b = 0$, and hence $q = vc$.

We now know that c divides both p and q. On the other hand, if d divides both p and q, then d divides everything in E, and in particular d divides c. Consequently, c has the maximal degree of all polynomial divisors of p and q, and so there exists $\alpha \in \mathbb{C}\backslash\{0\}$ such that $c = \alpha g$. Upon setting $r = R/\alpha$ and $s = S/\alpha$, we find that

$$g = pr + qs, \tag{3.21}$$

as desired. $\square$

The second technical result is a variant of Gaussian row reduction for matrix-valued polynomials.

Proposition 3.12. *Let $N \geq 2$ and $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ be a polynomial. Then there exist polynomials $q, u : \mathbb{C} \to \mathbb{C}^{N\times N}$ such that $qp = u$, $\det q = \pm 1$, and u is in upper-triangular form in the sense that $u_{ij} = 0$ for $j < i$.*

Proof. We essentially follow the procedure of Gaussian elimination with one technical complication caused by the fact that polynomials form a ring and not a field.

Suppose that p is such that p_{11} and p_{21} are nontrivial. Let g be the greatest common factor of p_{11} and p_{21}, and write

$$p_{11} = g\varphi \text{ and } p_{21} = g\psi. \tag{3.22}$$

Then φ and ψ have greatest common factor 1, and so Bézout's lemma allows us to choose polynomials r and s such that

$$s\psi + r\varphi = 1. \tag{3.23}$$

Then

$$\psi p_{11} - \varphi p_{21} = g(\varphi\psi - \psi\varphi) = 0 \text{ and } rp_{11} + sp_{21} = g(r\varphi + s\psi) = g. \tag{3.24}$$

Writing

$$\mu = \begin{pmatrix} r & s \\ \psi & -\varphi \end{pmatrix} : \mathbb{C} \to \mathbb{C}^{2\times 2} \text{ and } M = \begin{pmatrix} \mu & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & I_{N-2} \end{pmatrix} : \mathbb{C} \to \mathbb{C}^{N\times N}, \tag{3.25}$$

we then compute $\det M = -1$ and

$$Mp = \begin{pmatrix} g & * & \cdots & * \\ 0 & * & \cdots & * \\ p_{31} & p_{32} & \cdots & p_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{pmatrix}, \tag{3.26}$$

where $*$ denotes terms whose precise form we don't care about.

Now, if at most one entry in the first column of p is nontrivial, we only need to multiply by a row permutation matrix to move this entry into the top position. Otherwise, we use row permutations in conjunction with the above argument to remove all but the top nontrivial entries in the column. We may thus produce a polynomial $Q_1 : \mathbb{C} \to \mathbb{C}^{N\times N}$ such that $\det Q_1 = \pm 1$ and

$$Q_1 p = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ 0 & r_{22} & \cdots & r_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & r_{N2} & \cdots & r_{NN} \end{pmatrix}. \tag{3.27}$$

Iterating the above argument, we can produce polynomials $Q_k : \mathbb{C} \to \mathbb{C}^{N\times N}$ for $1 \le k \le N-1$ such that $\det Q_k = \pm 1$ and

$$Q_{N-1}Q_{N-2}\cdots Q_1 p = u = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ 0 & u_{22} & \cdots & u_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{NN} \end{pmatrix} \tag{3.28}$$

is in upper-triangular form. The conclusion then follows by setting $q = Q_{N-1}Q_{N-2}\cdots Q_1$.

□

We now have the tools needed to make a first attempt at characterizing all solutions to $p(D)x = f$ when $p(D)$ has positive determinant-order. In stating this theorem we recall the above discussion of the meaning of the linear independence of the rows of a matrix polynomials (mod q) for a given polynomial $q : \mathbb{C} \to \mathbb{C}$.

Theorem 3.13. *Let $N \ge 2$ and $p(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ be a differential operator of determinant-order $d \ge 1$. Let $q, u : \mathbb{C} \to \mathbb{C}^{N\times N}$ be the polynomials from Proposition 3.12, so that $q(D)p(D) = u(D)$ and $u(D)$ is in upper-triangular form. Write d_k for the degree of the polynomial $u_{kk}(z)$ for $1 \le k \le N$. Then the following hold.*

1. *For $1 \leq k \leq N$ we have that $d_k \geq 0$, so $u_{kk}(D)$ is a nontrivial differential operator. Also, if we define $s_0 = 0$ and $s_m = \sum_{j=1}^m d_j$ for $1 \leq m \leq N$, then $s_0 \leq s_1 \leq \cdots \leq s_N = \sum_{k=1}^N d_k = d$.*

2. *Define the polynomial $B : \mathbb{C} \to \mathbb{C}^{d \times N}$ via*

$$B(z)_{km} = \begin{cases} z^{k-s_{m-1}} & \text{if } s_{m-1} \leq k \leq s_m - 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.29}$$

   for $0 \leq k \leq d-1$ and $1 \leq m \leq N$. Then for every holomorphic $f : \mathbb{C} \to \mathbb{C}^N$ and $(\xi_0, \ldots, \xi_{d-1}) \in \mathbb{C}^d$ there exists a unique holomorphic $x : \mathbb{C} \to \mathbb{C}^N$ satisfying

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \ldots, \xi_{d-1}). \end{cases} \tag{3.30}$$

3. *The map $\Phi : H(\mathbb{C}; \mathbb{C}^N) \to H(\mathbb{C}; \mathbb{C}^N) \times \mathbb{C}^d$ given by*

$$\Phi x = (p(D)x, B(D)x(0)) \tag{3.31}$$

   is an isomorphism.

4. $\dim(\ker(p(D))) = d$, *and the map* $\ker(p(D)) \ni x \mapsto B(D)x(0) \in \mathbb{C}^d$ *is an isomorphism.*

5. *The rows of the matrix polynomial $B \operatorname{adj} p : \mathbb{C} \to \mathbb{C}^{d \times N}$ are linearly independent (mod $\det p$).*

Proof. The third item is a linear algebraic restatement of the second, and the fourth item follows easily from the third, so it suffices to only prove the first two and fifth items.

Proposition 3.12 guarantees that $\det q(D) = \pm 1$ and

$$q(D)p(D) = u(D) = \begin{pmatrix} u_{11}(D) & u_{12}(D) & \cdots & u_{1N}(D) \\ 0 & u_{22}(D) & \cdots & u_{2N}(D) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{NN}(D) \end{pmatrix}, \tag{3.32}$$

so upon taking the determinant of both sides, we find that

$$\pm \det p(D) = \prod_{m=1}^N u_{mm}(D), \tag{3.33}$$

and hence the order of $\det p(D)$ equals $\sum_{m=1}^N d_m$, and none of the operators $u_{mm}(D)$ vanishes. This proves the first item.

We now turn to the proof of the second item. For a holomorphic $x : \mathbb{C} \to \mathbb{C}^N$ we have that

$$p(D)x = f \Leftrightarrow u(D)x = q(D)p(D)x = q(D)f. \tag{3.34}$$

We thus reduce to solving the upper-triangular problem $u(D)x = q(D)f$, which reads

$$\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ 0 & u_{22} & \cdots & u_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{NN} \end{pmatrix} x = q(D)f. \tag{3.35}$$

The condition $B(D)x(0) = (\xi_0, \ldots, \xi_{d-1})$ is equivalent to

$$D^{k-s_{m-1}}x_m(0) = \xi_k \text{ if } s_{m-1} \le k \le s_m - 1 \text{ for } 0 \le k \le d-1 \text{ and } 1 \le m \le N, \tag{3.36}$$

but note that

$$s_{m-1} \le k \le s_m - 1 \Leftrightarrow 0 \le k - s_{m-1} \le s_m - s_{m-1} - 1 = d_m - 1, \tag{3.37}$$

so a condition on x_m is only enforced if $d_m \ge 1$. In other words, $B(D)x(0) = (\xi_0, \ldots, \xi_{d-1})$ is equivalent to

$$D^\ell x_m(0) = \xi_{\ell+s_{m-1}} \text{ if } d_m \ge 1 \text{ and } 0 \le \ell \le d_m - 1. \tag{3.38}$$

The advantage of this triangular form is that we can solve the system iteratively, starting from the final scalar equation, which reads

$$u_{NN}(D)x_N = (q(D)f)_N. \tag{3.39}$$

If $d_N \ge 1$ we use Theorem 1.7 to solve for a unique holomorphic function $x_N : \mathbb{C} \to \mathbb{C}$ satisfying

$$\begin{cases} u_{NN}(D)x_N = (q(D)f)_N \\ D^\ell x_N(0) = \xi_{\ell+s_{N-1}} \text{ for } 0 \le \ell \le d_N - 1, \end{cases} \tag{3.40}$$

and if $d_N = 0$ we use Theorem 3.8 to solve for a unique holomorphic function $x_N : \mathbb{C} \to \mathbb{C}$ satisfying $u_{NN}(D)x_N = (q(D)f)_N$. Once we have produced holomorphic $x_{m+1}, \ldots, x_N$ for $1 \le m \le N-1$, we then find a unique holomorphic $x_m : \mathbb{C} \to \mathbb{C}$ via the m^{th} equation, which reads

$$u_{mm}(D)x_m = (q(D)f)_m - \sum_{m+1 \le j \le N} u_{mj}(D)x_j, \tag{3.41}$$

subject to the extra condition that

$$D^\ell x_m(0) = \xi_{\ell+s_{m-1}} \text{ if } d_m \ge 1 \text{ and } 0 \le \ell \le d_m - 1. \tag{3.42}$$

Proceeding iteratively, we then find the unique holomorphic functions $x_1, \ldots, x_N$ such that $x = (x_1, \ldots, x_N) : \mathbb{C} \to \mathbb{C}^N$ satisfies

$$\begin{cases} u(D)x = q(D)f \\ B(D)x(0) = (\xi_0, \ldots, x_{d-1}). \end{cases} \tag{3.43}$$

This proves the second item.

Finally, we turn to the proof of the fifth item. We first claim that the rows of $B \operatorname{adj} p$ are linearly independent (mod $\det p$) if and only if the rows of $B \operatorname{adj} u$ are linearly independent (mod $\det p$). To see this we first use the basic properties of the adjugate to write

$$B \operatorname{adj} p = B \operatorname{adj}(q^{-1}u) = (B \operatorname{adj} u) \operatorname{adj}(q^{-1}) = (B \operatorname{adj} u)\frac{q}{\det q} = \pm(B \operatorname{adj} u)q. \tag{3.44}$$

From this it's clear that the rows of $B \operatorname{adj} p$ are are linearly independent (mod $\det p$) if and only if the rows of $(B \operatorname{adj} u)q$ are linearly independent (mod $\det p$), but since q is invertible we have that

$$\sum_{j=0}^{d-1} \sum_{k=1}^{N} \alpha_j (B \operatorname{adj} u)_{jk} q_{km} \equiv_{\det p} 0 \text{ for all } 1 \le m \le N, \tag{3.45}$$

if and only if

$$\sum_{j=0}^{d-1} \alpha_j (B \operatorname{adj} u)_{jk} \equiv_{\det p} 0 \text{ for all } 1 \le k \le N, \tag{3.46}$$

and so the claim follows.

We now claim that the rows of $B \operatorname{adj} u$ are linearly independent (mod $\det p$); once this is established, the fifth item is proved. The structure of B shows that if

$$\sum_{j=0}^{d-1} \alpha_j (B \operatorname{adj} u)_{jk} \equiv_{\det p} 0 \text{ for } 1 \le k \le N, \tag{3.47}$$

then for $1 \le j \le N$ there exist polynomials $\pi_j : \mathbb{C} \to \mathbb{C}$ with coefficients given by partitioning $\{\alpha_k\}_{k=0}^{d-1}$ according to the partitioning from B, with $\deg(\pi_j) < d_j = \deg(u_{jj})$ such that

$$\sum_{j=1}^{N} \pi_j (\operatorname{adj} u)_{jk} \equiv_{\det p} 0 \text{ for } 1 \le k \le N, \tag{3.48}$$

and hence there exist polynomials $\delta_k : \mathbb{C} \to \mathbb{C}$ for $1 \le k \le N$ such that

$$\sum_{j=1}^{N} \pi_j (\operatorname{adj} u)_{jk} = \delta_k \det p \text{ for } 1 \le k \le N. \tag{3.49}$$

Now, since u is upper triangular, $\operatorname{adj} u$ is as well, and so

$$\pi_1 (\operatorname{adj} u)_{11} = \delta_1 \det p \Rightarrow \pi_1 \prod_{j \ne 1} u_{jj} = \pm \delta_1 \prod_{j=1}^{N} u_{jj} \Rightarrow \pi_1 = \pm \delta_1 u_{11}, \tag{3.50}$$

but $\deg(\pi_1) < \deg(u_{11})$ so this implies that $\pi_1 = \delta_1 = 0$. Using this and arguing similarly, we find that $\pi_2 = 0$, and upon iterating we deduce that $\pi_1 = \cdots = \pi_N = 0$, and hence that $\alpha_1 = \cdots = \alpha_N = 0$. Thus, the rows of $B \operatorname{adj} u$ are linearly independent (mod $\det p$).

□

This theorem provides us with the key piece of information that the determinant-order of $p(D)$ is the dimension of $\ker(p(D))$, which in turn yields precisely the number of scalar conditions we can expect to impose on a solution to $p(D)x = f$ at 0 in order to uniquely determine x. However, the choice of $B(D) : H(\mathbb{C}; \mathbb{C}^N) \to H(\mathbb{C}; \mathbb{C}^d)$ from the theorem is somewhat awkward to work with, so it is natural to seek more general operators and to investigate when such an operator uniquely determines solutions.

To motivate our approach to this problem we first return to the context of Proposition 3.10, which gave us a mechanism for producing elements of $\ker(p(D))$, where $p : H(\mathbb{C}; \mathbb{C}^N) \to H(\mathbb{C}; \mathbb{C}^N)$ is a differential operator with determinant-order $d \ge 1$. Let $\{q_k\}_{k=0}^{d-1}$ be the polynomials associated to $\det p$ from Definition 2.3, and let $R > 0$ be such that $Z(\det p) \subset B(0, R)$. If $y \in \ker(\det p(D) I)$, then Theorem 2.5 shows that there exist $c_{jk} \in \mathbb{C}$ for $1 \le j \le N$ and $0 \le k \le d-1$ such that

$$y_j(\tau) = \frac{1}{2\pi i} \sum_{k=0}^{d-1} \int_{\partial B(0,R)} e^{\tau z} \frac{c_{jk} q_k(z)}{\det p(z)} dz. \tag{3.51}$$

Then we define $x = \operatorname{adj} p(D) y \in H(\mathbb{C};\mathbb{C}^N)$, and Proposition 3.10 shows that $p(D)x = 0$. However, we can use the above formula to compute

$$x_j(\tau) = \frac{1}{2\pi i} \sum_{\ell=1}^{N} \sum_{k=0}^{d-1} \int_{\partial B(0,R)} e^{\tau z} \frac{\operatorname{adj} p_{j\ell}(z) c_{\ell k} q_k(z)}{\det p(z)} dz. \tag{3.52}$$

Using this recipe, we now have a way of generating $x \in \ker(p(D))$ that is convenient for working with more general operators $B(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^d)$. To proceed further, we need a definition.

Definition 3.14. *Let $N, d \in \mathbb{N}$ with $N \geq 2$ and $d \geq 1$, and let $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ be a polynomial such that $p(D)$ has determinant-order d. Let $B : \mathbb{C} \to \mathbb{C}^{d\times N}$ be a polynomial with components $B_{jk} : \mathbb{C} \to \mathbb{C}$ for $0 \leq j \leq d-1$ and $1 \leq k \leq N$. Consider the polynomial $C : \mathbb{C} \to \mathbb{C}^{d\times N}$ given by $C(z) = B(z) \operatorname{adj} p(z)$. Let $R_{jk} : \mathbb{C} \to \mathbb{C}$ be the polynomial given by $R_{jk} \equiv_{\det p} C_{jk}$ and write*

$$R_{jk}(z) = \sum_{m=0}^{d-1} R_{jkm} z^m. \tag{3.53}$$

We then define $\mathcal{R}_B \in \mathcal{L}(\mathbb{C}^{N\times d}, \mathbb{C}^d)$ via

$$(\mathcal{R}_B c)_j = \sum_{k=1}^{N} \sum_{m=0}^{d-1} R_{jkm} c_{km}. \tag{3.54}$$

The purpose of $\mathcal{R}_B$ is seen in the following lemma.

Lemma 3.15. *Let $N \geq 2$ and $p(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ be a differential operator of determinant-order $d \geq 1$. Let $B(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^d)$ be a differential operator and let $\mathcal{R}_B \in \mathcal{L}(\mathbb{C}^{N\times d}, \mathbb{C}^d)$ be the associated map from Definition 3.14. If $x \in H(\mathbb{C};\mathbb{C}^N)$ is defined by (3.52) with $c \in \mathbb{C}^{N\times d}$, then*

$$B(D)x(0) = \mathcal{R}_B c. \tag{3.55}$$

Proof. By Cauchy-Goursat and Proposition 2.2, we may compute

$$\begin{aligned}
(B(D)x(0))_m &= \frac{1}{2\pi i} \sum_{\ell=1}^{N} \sum_{j,k=0}^{d-1} \int_{\partial B(0,r)} \frac{B_{mj}(z) \operatorname{adj} p_{j\ell}(z) c_{\ell k} q_k(z)}{\det p(z)} dz \\
&= \frac{1}{2\pi i} \sum_{\ell=1}^{N} \sum_{k=0}^{d-1} \int_{\partial B(0,r)} \frac{C_{m\ell}(z) c_{\ell k} q_k(z)}{\det p(z)} dz = \frac{1}{2\pi i} \sum_{\ell=1}^{N} \sum_{k=0}^{d-1} \int_{\partial B(0,r)} \frac{R_{m\ell}(z) c_{\ell k} q_k(z)}{\det p(z)} dz \\
&= \frac{1}{2\pi i} \sum_{\ell=1}^{N} \sum_{k,n=0}^{d-1} \int_{\partial B(0,r)} \frac{R_{m\ell n} z^n c_{\ell k} q_k(z)}{\det p(z)} dz = \sum_{\ell=1}^{N} \sum_{k=0}^{d-1} R_{m\ell k} c_{\ell k} = (\mathcal{R}_B c)_m,
\end{aligned} \tag{3.56}$$

and the result follows. $\square$

We now have all of the tools necessary to characterize the solvability of ordinary differential systems with general boundary operators.

Theorem 3.16. *Let $N \geq 2$ and $p(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^N)$ be a differential operator of determinant-order $d \geq 1$. Let $B(D) : H(\mathbb{C};\mathbb{C}^N) \to H(\mathbb{C};\mathbb{C}^d)$ be a differential operator and let $\mathcal{R}_B \in \mathcal{L}(\mathbb{C}^{N\times d}, \mathbb{C}^d)$ be the associated map from Definition 3.14. Then the following are equivalent.*

1. *The rows of the matrix polynomial* $B \operatorname{adj} p : \mathbb{C} \to \mathbb{C}^{d\times N}$ *are linearly independent (mod* $\det p$*).*

2. $\mathcal{R}_B$ *is surjective.*

3. *If* $f : \mathbb{C} \to \mathbb{C}^N$ *is holomorphic and* $\xi_0, \dots, \xi_{d-1} \in \mathbb{C}$*, then there exists a unique holomorphic* $x : \mathbb{C} \to \mathbb{C}^N$ *such that*
$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \dots, \xi_{d-1}). \end{cases} \tag{3.57}$$

4. *The map* $\Phi : H(\mathbb{C}; \mathbb{C}^N) \to H(\mathbb{C}; \mathbb{C}^N) \times \mathbb{C}^d$ *given by*
$$\Phi x = (p(D)x, B(D)x(0)) \tag{3.58}$$
is an isomorphism.

5. *For every* $(\xi_0, \dots, \xi_{d-1}) \in \mathbb{C}^d$ *there exists a holomorphic* $x : \mathbb{C} \to \mathbb{C}^N$ *such that*
$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = (\xi_0, \dots, \xi_{d-1}). \end{cases} \tag{3.59}$$

6. *If* $x : \mathbb{C} \to \mathbb{C}^N$ *is holomorphic and*
$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = 0, \end{cases} \tag{3.60}$$
then $x = 0$.

Moreover, if any of these conditions is satisfied, then the following hold.

(a) The operator $\mathcal{R}_B^* : \mathbb{C}^d \to \mathbb{C}^{N\times d}$ *given by*
$$(\mathcal{R}_B^* v)_{km} = \sum_{j=0}^{d-1} \overline{R_{jkm}} v_j. \tag{3.61}$$
is injective.

(b) $\mathcal{R}_B \mathcal{R}_B^* : \mathbb{C}^d \to \mathbb{C}^d$ *is invertible.*

(c) For any $f \in H(\mathbb{C}; \mathbb{C}^N)$ *and* $\xi = (\xi_0, \dots, \xi_{d-1}) \in \mathbb{C}^d$*, the unique solution to*
$$\begin{cases} p(D)x = f \\ B(D)x(0) = \xi \end{cases} \tag{3.62}$$
is the holomorphic map $x \in H(\mathbb{C}; \mathbb{C}^N)$ *given by*
$$x(\tau) = \operatorname{adj} p(D)y(\tau) + \frac{1}{2\pi i} \sum_{k=0}^{d-1} \int_{\partial B(0,r)} e^{\tau z} \left(p(z)^{-1} \mathcal{R}_B^* (\mathcal{R}_B \mathcal{R}_B^*)^{-1} (\xi - B(D) \operatorname{adj} p(D) y(0)) \right)_k q_k(z) dz, \tag{3.63}$$
where $Z(\det p) \subset B(0, r)$ *and* $y \in H(\mathbb{C}; \mathbb{C}^N)$ *is uniquely determined by*
$$\begin{cases} \det p(D) y_j = f_j \\ D^k y_j(0) = 0 \text{ for } 0 \le k \le d. \end{cases} \tag{3.64}$$

Proof. We begin by recalling from Definition 3.14 that $C = B \operatorname{adj} p$ and

$$C_{jk}(z) \equiv_{\det p} R_{jk}(z) = \sum_{m=0}^{d-1} R_{jkm} z^m. \tag{3.65}$$

Pick $r > 0$ such that $Z(\det p) \subset B(0, r)$. For a fixed $c \in \mathbb{C}^{N\times d}$ we define $x \in \ker(p(D))$ via

$$x_j(\tau) = \frac{1}{2\pi i} \sum_{\ell=1}^{N} \sum_{k=0}^{d-1} \int_{\partial B(0,r)} e^{\tau z} \frac{\operatorname{adj} p_{j\ell}(z) c_{\ell k} q_k(z)}{\det p(z)} dz. \tag{3.66}$$

Then Lemma 3.15 tells us that

$$B(D)x(0) = \mathcal{R}_B c. \tag{3.67}$$

With these tools in hand, we now turn to the proof of the various equivalences. The equivalence of (3) and (4) is trivial, as are the implications (3) $\Rightarrow$ (5) and (3) $\Rightarrow$ (6). To conclude, we will prove that (1) $\Leftrightarrow$ (2), (5) $\Leftrightarrow$ (6), (2) $\Rightarrow$ (5), (6) $\Rightarrow$ (2), and (5) and (6) $\Rightarrow$ (3).

Proof of (1) $\Leftrightarrow$ (2)**:** Forgetting the matrix structure of $\mathbb{C}^{N\times d}$, we may view $\mathcal{R}_B$ as a matrix in $\mathbb{C}^{d\times Nd}$ with components $(\mathcal{R}_B)_{j,km} = R_{jkm}$ with R_{jkm} as above. From this perspective, basic linear algebra shows that $\mathcal{R}_B$ is surjective if and only if the rows of $\mathcal{R}_B \in \mathbb{C}^{d\times Nd}$ are linearly independent, i.e. $\sum_{j=0}^{d-1} \alpha_j R_{jkm} = 0$ for $1 \le k \le N$ and $0 \le m \le d-1$ implies that $\alpha_0 = \cdots = \alpha_{d-1} = 0$. Thus, to prove the equivalence of (1) and (2) it suffices to prove that the rows of $B \operatorname{adj} p$ are linearly independent (mod $\det p$) if and only if $\sum_{j=0}^{d-1} \alpha_j R_{jkm} = 0$ for $1 \le k \le N$ and $0 \le m \le d-1$ implies that $\alpha_0 = \cdots = \alpha_{d-1} = 0$. To prove this equivalence we note that for $\alpha_0, \dots, \alpha_{d-1} \in \mathbb{C}$ we have that

$$\begin{aligned} &\sum_{j=0}^{d-1} \alpha_j R_{jkm} = 0 \text{ for } 1 \le k \le N, 0 \le m \le d-1 \Leftrightarrow \sum_{j,m=0}^{d-1} \alpha_j R_{jkm} z^m = 0 \text{ for } 1 \le k \le N \\ &\quad \Leftrightarrow \sum_{j,m=0}^{d-1} \alpha_j R_{jk}(z) = 0 \text{ for } 1 \le k \le N \Leftrightarrow \sum_{j,m=0}^{d-1} \alpha_j (B \operatorname{adj} p)_{jk}(z) \equiv_{\det p} 0 \text{ for } 1 \le k \le N, \end{aligned} \tag{3.68}$$

from which the desired equivalence directly follows.

Proof of (5) $\Leftrightarrow$ (6)**:** Define the linear map $T : \ker(p(D)) \to \mathbb{C}^d$ via $Tx = B(D)x(0)$. Then (5) is equivalent to the assertion that T is surjective, while (6) is equivalent to the assertion that T is injective. Theorem 3.13 tells us that $\dim(\ker(p(D))) = d = \dim(\mathbb{C}^d)$, so we know from linear algebra that T is injective if and only if it's surjective.

Proof of (2) $\Rightarrow$ (5)**:** Let $(\xi_0, \dots, \xi_{d-1}) \in \mathbb{C}^d$. Since $\mathcal{R}_B$ is surjective, we can find $c \in \mathbb{C}^{N\times d}$ such that $\mathcal{R}_B c = (\xi_0, \dots, \xi_{d-1})$. Using this c in (3.66) defines $x \in \ker(p(D))$, and (3.67) implies that

$$B(D)x(0) = \mathcal{R}_B c = (\xi_0, \dots, \xi_{d-1}). \tag{3.69}$$

Thus (5) holds.

Proof of (6) $\Rightarrow$ (2)**:** We will prove the contrapositive. Suppose that (2) is false, i.e. $\mathcal{R}_B : \mathbb{C}^{N\times d} \to \mathbb{C}^d$ is not surjective. We know that

$$\dim \operatorname{ran}(\mathcal{R}_B) + \dim \ker(\mathcal{R}_B) = Nd, \tag{3.70}$$

so

$$Nd - \dim \ker(\mathcal{R}_B) = \dim \operatorname{ran}(\mathcal{R}_B) \le d - 1 \tag{3.71}$$

and hence

$$Nd - d + 1 \leq \dim\ker(\mathcal{R}_B). \tag{3.72}$$

Let $\tilde{B} : \mathbb{C} \to \mathbb{C}^{d\times N}$ be the polynomial from Theorem 3.13 and let $\mathcal{R}_{\tilde{B}} : \mathbb{C}^{N\times d} \to \mathbb{C}^d$ be the map associated to $\tilde{B}$ as in Definition 3.14. The theorem shows that the rows of $\tilde{B}\operatorname{adj} p$ are linearly independent (mod $\det p$), so by the (1) $\Leftrightarrow$ (2) assertion above, applied with $\tilde{B}$, we know that $\mathcal{R}_{\tilde{B}}$ is surjective. Then

$$\dim\ker(\mathcal{R}_{\tilde{B}}) = Nd - \dim\operatorname{ran}(\mathcal{R}_{\tilde{B}}) = Nd - d, \tag{3.73}$$

so

$$\dim\ker(\mathcal{R}_{\tilde{B}})^{\perp} = Nd - \dim\ker(\mathcal{R}_{\tilde{B}}) = d, \tag{3.74}$$

where $\ker(\mathcal{R}_{\tilde{B}})^{\perp} \subseteq \mathbb{C}^{N\times d}$ denotes the orthogonal complement of $\ker(\mathcal{R}_{\tilde{B}})$.

Now, if $\ker(\mathcal{R}_{\tilde{B}})^{\perp} \cap \ker(\mathcal{R}_B) = \{0\}$, then

$$\dim\ker(\mathcal{R}_{\tilde{B}})^{\perp} + \dim\ker(\mathcal{R}_B) = \dim(\ker(\mathcal{R}_{\tilde{B}})^{\perp} + \ker(\mathcal{R}_B)) \leq Nd, \tag{3.75}$$

but by the above,

$$\dim\ker(\mathcal{R}_{\tilde{B}})^{\perp} + \dim\ker(\mathcal{R}_B) \geq d + (Nd - d + 1) = Nd + 1, \tag{3.76}$$

a contradiction. Hence, there exists $0 \neq c \in \ker(\mathcal{R}_{\tilde{B}})^{\perp} \cap \ker(\mathcal{R}_B)$. In other words, $c \in \mathbb{C}^{N\times d}$ satisfies

$$\mathcal{R}_B c = 0 \text{ and } \mathcal{R}_{\tilde{B}} c = \mu \in \mathbb{C}^d \backslash \{0\}. \tag{3.77}$$

We use this c in (3.66) to define $x \in \ker(p(D))$. Then (3.67), applied with both B and $\tilde{B}$, shows that

$$B(D)x(0) = \mathcal{R}_B c = 0 \text{ and } \tilde{B}(D)x(0) = \mathcal{R}_{\tilde{B}} c = \mu \neq 0, \tag{3.78}$$

the latter of which implies that $x \neq 0$. Thus, we have constructed $x \neq 0$, holomorphic, such that

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = 0, \end{cases} \tag{3.79}$$

which shows that (6) is false.

Proof of (5) **and** (6) $\Rightarrow$ (2)**:** First note that (6) implies that there exists at most one solution to (3.57), so it suffices to prove the existence of a solution. According to Theorem 3.13 we can find a holomorphic $y : \mathbb{C} \to \mathbb{C}^N$ such that $p(D)y = f$. According to (5), we can then find a holomorphic $h : \mathbb{C} \to \mathbb{C}^N$ such that

$$\begin{cases} p(D)h = 0 \\ B(D)h(0) = (\xi_0, \ldots, \xi_{d-1}) - B(D)y(0). \end{cases} \tag{3.80}$$

Then $x = h + y$ is holomorphic and satisfies (3.57), and (2) is proved.

It remains to prove that (a), (b), and (c) hold whenever any and hence all of these conditions holds. Note that $\mathcal{R}_B^*$ is the conjugate transpose associated to $\mathcal{R}_B$ and satisfies

$$\mathcal{R}_B c \cdot v = c : \mathcal{R}_B^* v \tag{3.81}$$

for all $c \in \mathbb{C}^{N\times d}$ and $v \in \mathbb{C}^d$, where $\cdot$ is the usual complex inner-product on $\mathbb{C}^d$ and $:$ is the same for $\mathbb{C}^{N\times d}$. Thus, we know from linear algebra shows that $\mathcal{R}_B$ is surjective if and only if $\mathcal{R}_B^*$ is injective, which proves (a).

The map $\mathcal{R}_B\mathcal{R}_B^*$ is Hermitian and satisfies

$$\mathcal{R}_B\mathcal{R}_B^* v \cdot v = \mathcal{R}_B^* v \cdot \mathcal{R}_B^* v = |\mathcal{R}_B^* v|^2, \tag{3.82}$$

which together with (a) proves that $\mathcal{R}_B\mathcal{R}_B^*$ is positive definite and hence invertible. This is (b).

Finally, Proposition 3.10 shows that $p(D) \operatorname{adj} p(D) y = f$, so if we set $c = \mathcal{R}_B^*(\mathcal{R}_B\mathcal{R}_B^*)^{-1}(\xi - B(D) \operatorname{adj} p(D) y(0)) \in \mathbb{C}^{N\times d}$ we compute

$$\mathcal{R}_B c = \mathcal{R}_B\mathcal{R}_B^*(\mathcal{R}_B\mathcal{R}_B^*)^{-1}(\xi - B(D) \operatorname{adj} p(D) y(0)) = \xi - B(D) \operatorname{adj} p(D) y(0). \tag{3.83}$$

This, (3.66), and (3.67) prove (c). □

Remark 3.17. *The condition from the theorem that the rows of the matrix polynomial* $B \operatorname{adj} p : \mathbb{C} \to \mathbb{C}^{d\times N}$ *are linearly independent (mod* $\det p$*) is called the Shapiro-Lopatinsky condition.*

Again, the power of the Shapiro-Lopatinsky condition is that it is a purely algebraic condition that can be readily checked. Let's consider some examples.

Example 3.18. Let $r \in \mathbb{R}$ and

$$p(D) = \begin{pmatrix} ir & D \\ D & -ir \end{pmatrix}. \tag{3.84}$$

Then $\det p(D) = -D^2 + r^2$, so the determinant-order is 2. We compute

$$\operatorname{adj} p(D) = \begin{pmatrix} -ir & -D \\ -D & ir \end{pmatrix}. \tag{3.85}$$

Consider the operators

$$B_1(D) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B_2(D) = \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}. \tag{3.86}$$

Then

$$B_1(z) \operatorname{adj} p(z) = \begin{pmatrix} -ir & -z \\ -z & ir \end{pmatrix} \text{ and } B_2(z) \operatorname{adj} p(z) = \begin{pmatrix} -z & ir \\ -irz & -z^2 \end{pmatrix}. \tag{3.87}$$

If

$$(\alpha(-ir) + \beta(-z), -\alpha z + \beta ir) \equiv_{-z^2+r^2} 0 \tag{3.88}$$

then there exist polynomials $q_1, q_2 : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned} \alpha(-ir) - \beta z &= q_1(z)(r^2 - z^2) \\ -\alpha z + \beta i z &= q_2(z)(r^2 - z^2), \end{aligned} \tag{3.89}$$

and by comparing degrees we readily deduce that $\alpha = \beta = 0$. Thus, the Shapiro-Lopatinsky condition is satisfied by B_1, p and we can solve

$$\begin{cases} p(D)x = f \\ B_1(D)x(0) = (\xi_0, \xi_1) \end{cases} \tag{3.90}$$

for arbitrary $f \in H(\mathbb{C}; \mathbb{C}^2)$ and $\xi_0, \xi_1 \in \mathbb{C}$.

On the other hand,

$$((-ir)(-z) - irz, (-ir)(ir) - z^2) = (0, -z^2 + r^2) \equiv_{-z^2+r^2} 0, \tag{3.91}$$

so the Shapiro-Lopatinsky condition is not satisfied by B_2, p, and we can't solve the problem $p(D)x = f$ while specifying $B(D)x(0)$ arbitrarily. △

Example 3.19. Let $w \in \mathbb{C}\backslash\{1\}$ and consider

$$p(D) = \begin{pmatrix} D & 1 \\ D & w \end{pmatrix}, \tag{3.92}$$

which satisfies $\det p(D) = (w-1)D$ and

$$\operatorname{adj} p(D) = \begin{pmatrix} w & -1 \\ -D & D \end{pmatrix}. \tag{3.93}$$

The determinant-order is 1. Consider the initial condition operators

$$B_1(D) = \begin{pmatrix} 1 & 0 \end{pmatrix} \text{ and } B_2(D) = \begin{pmatrix} 0 & 1 \end{pmatrix}. \tag{3.94}$$

Then

$$B_1(z) \operatorname{adj} p(z) = \begin{pmatrix} w & -1 \end{pmatrix} \text{ and } B_2(z) \operatorname{adj} p(z) = \begin{pmatrix} -z & z \end{pmatrix} \tag{3.95}$$

From these it's clear that B_1, p satisfy the Shapiro-Lopatinsky condition but B_2, p do not. △

Example 3.20. Let

$$p(D) = \begin{pmatrix} D & D^3 & D^2-1 \\ D & D^3+D & D^2+D \\ 0 & D & D-1 \end{pmatrix}, \tag{3.96}$$

which means that $\det p(D) = -2D^2$ and

$$\operatorname{adj} p(D) = \begin{pmatrix} D(D^4-2D^2-1) & D(-D^3+2D^2-1) & D(D^3+1) \\ D(-D+1) & D(D-1) & D(-D-1) \\ D^2 & -D^2 & D^2 \end{pmatrix}. \tag{3.97}$$

The determinant-order is 2. Consider initial condition operators

$$B_1(D) = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B_2(D) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.98}$$

Then

$$B_1(z) \operatorname{adj} p(z) = \begin{pmatrix} -2z^2+z & 2z^2-z & -2z^2-z \\ z^2 & -z^2 & z^2 \end{pmatrix}, \tag{3.99}$$

and every term in the second row is a multiple of $\det p(z)$, so the Shapiro-Lopatinsky condition is not satisfied by B_1, p. On the other hand,

$$B_2(z) \operatorname{adj} p(z) = \begin{pmatrix} -z^2+z & z^2-z & -z^2-z \\ z^5-2z^3-z & -z^4+2z^3-z & z^4+z \end{pmatrix}, \tag{3.100}$$

and if

$$\alpha(-z^2+z) + \beta(z^5-2z^3-z) = q_1(z)z^2 \text{ and } \alpha(z^2-z) + \beta(-z^4+2z^3-z) = q_2(z)z^2 \tag{3.101}$$

then by comparing the linear terms we must have that

$$\alpha - \beta = 0 \text{ and } -\alpha - \beta = 0 \tag{3.102}$$

and hence $\alpha = \beta = 0$. Thus B_2, p satisfy the Shapiro-Lopatinsky condition.

△

4 General systems on real intervals and with real coefficients and data

As mentioned above, in many applications of ordinary differential systems we think of the solution as describing the dynamics of some time-parameterized process, in which case we restrict to $\tau = t \in J \subseteq \mathbb{R}$, where J is a nontrivial interval containing 0. By an interval, we mean any nonempty connected (or equivalently, convex) subset of $\mathbb{R}$; note that we do not require the interval to be open or closed. By nontrivial we mean $J \neq \{0\}$, i.e. J is not a singleton, in which case $J^\circ \neq \varnothing$. Unlike above, when we restrict our attention to real intervals, we can no longer work entirely with holomorphic functions. Instead, we will substitute holomorphic functions with smooth ones belonging to the space

$$C^\infty(J;\mathbb{C}^N) = \{x : J \to \mathbb{C}^N \mid x \text{ is smooth}\}. \tag{4.1}$$

In the event that J contains one of its boundary points, for instance $J = [0,\infty)$, then we recall that differentiability is characterized at the boundary point by taking limits from one side only. Given a polynomial $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ we continue to view $p(D)$ as a differential operator mapping $C^\infty(J;\mathbb{C}^N)$ to itself the obvious way. We write

$$\ker_J(p(D)) = \{x \in C^\infty(J;\mathbb{C}^N) \mid p(D)x = 0\} \tag{4.2}$$

to distinguish the kernel in our new space from the holomorphic kernel.

Our first result establishes a basic solvability result when $p(D)$ is elliptic.

Theorem 4.1. *Let $N \geq 1$ and $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ be a polynomial of degree $n \geq 1$ such that $p(D)$ is elliptic. Let $0 \in J \subseteq \mathbb{R}$ be a nontrivial interval. Then for every $f \in C^\infty(J;\mathbb{C}^N)$ and $\xi_0, \dots, \xi_{n-1} \in \mathbb{C}^N$ there exists a unique $x \in C^\infty(J;\mathbb{C}^n)$ such that*

$$\begin{cases} p(D)x = f \\ D^k x(0) = \xi_k \text{ for } 0 \leq k \leq n-1. \end{cases} \tag{4.3}$$

Proof. The result follows by slightly modifying the proofs of Lemma 1.5 and Theorem 1.7, replacing $H(\mathbb{C};X)$ with $C^\infty(J;\mathbb{C}^N)$. The only novelty is that smoothness doesn't immediately follow from the existence of one derivative via holomorphicity. Rather, the smoothness of Ξ follows from the equation $\Xi' = \mathbb{A}\Xi + F$, the fact that F is smooth, and a simple induction argument. We leave it as an exercise to fill in the details. □

We next turn our attention to general systems with positive determinant order. The following is the analog of Theorem 3.13 for real intervals in place of $\mathbb{C}$.

Theorem 4.2. *Let $0 \in J \subseteq \mathbb{R}$ be a nontrivial interval, and let $N \geq 2$ and $p : \mathbb{C} \to \mathbb{C}^N$ be a polynomial such that $p(D)$ has determinant-order $d \geq 1$. There exists a polynomial $B : \mathbb{C} \to \mathbb{C}^{d\times N}$ such that the following hold.*

1. *For every $f \in C^\infty(J;\mathbb{C}^N)$ and $(\xi_0, \dots, \xi_{d-1}) \in \mathbb{C}^d$ there exists a unique $x \in C^\infty(J;\mathbb{C}^N)$ satisfying*

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \dots, \xi_{d-1}). \end{cases} \tag{4.4}$$

In other words, the map $\Phi : C^\infty(J;\mathbb{C}^N) \to C^\infty(J;\mathbb{C}^N) \times \mathbb{C}^d$ given by

$$\Phi x = (p(D)x, B(D)x(0)) \tag{4.5}$$

is an isomorphism.

2. *We have that* $\dim(\ker_J(p(D))) = d$*, and the map* $\ker_J(p(D)) \ni x \mapsto B(D)x(0) \in \mathbb{C}^d$ *is an isomorphism.*

3. *The rows of the matrix polynomial* $B \operatorname{adj} p : \mathbb{C} \to \mathbb{C}^{d\times N}$ *are linearly independent (mod* $\det p$*).*

Proof. The proof is essentially identical to that of Theorem 3.13, with the following exception. When solving the upper-triangular system, when $d_k \geq 1$ we use Theorem 4.1 to produce solutions in $C^\infty(J;\mathbb{C})$ rather than Theorem 1.7. When $d_k = 0$, producing solutions in $C^\infty(J;\mathbb{C})$ is trivial, as the scalar operator $q_{kk}(D)$ is simply multiplication by a nontrivial constant. We leave it as an exercise to verify the details. □

We can combine Theorems 4.1 and 4.2 to obtain a deeper understanding of $\ker_J(p(D))$.

Theorem 4.3. *Let* $0 \in J \subseteq \mathbb{R}$ *be a nontrivial interval,* $N \geq 1$*, and* $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ *be a polynomial such that* $p(D)$ *has determinant-order* $d \geq 1$*. Then*

$$\ker_J(p(D)) = \{x|J \mid x \in H(\mathbb{C};\mathbb{C}^N) \text{ and } p(D)x = 0\}. \tag{4.6}$$

In other words, $\ker_J(p(D))$ *consists of the restrictions to* J *of the elements of* $\ker(p(D)) \subset H(\mathbb{C};\mathbb{C}^N)$.

Proof. We have the obvious subspace inclusion

$$\{x|J \mid x \in H(\mathbb{C};\mathbb{C}^N) \text{ and } p(D)x = 0\} \subseteq \ker_J(p(D)). \tag{4.7}$$

In light of Theorems 4.1 and 4.2, to prove equality it suffices to show that the dimension of the space on the left is d. Let $\{x_0,\ldots,x_{d-1}\} \subset \ker(p(D))$ be a basis. Suppose that $\alpha_0,\ldots,\alpha_{d-1} \in \mathbb{C}$ and $\sum_{k=0}^{d-1} \alpha_k x_k = 0$ on J. Since the function on the left belongs to $H(\mathbb{C};\mathbb{C}^N)$ and $J^\circ \neq \varnothing$, the function is holomorphic and vanishes on an open set containing a limit point, and hence vanishes identically. Thus $\sum_{k=0}^{d-1} \alpha_k x_k = 0$ on $\mathbb{C}$ and since this collection is a basis, we deduce that $\alpha_0 = \cdots = \alpha_{d-1} = 0$. Hence, the set of restrictions of $\{x_0,\ldots,x_{d-1}\}$ to J is a basis. □

As before, the key insight provided by Theorem 4.2 is that the dimension of $\ker_J(p(D))$ is d, the determinant-order of p. This again suggests that we can specify d scalar general initial conditions. To study these we introduce an analog of $\mathcal{R}_B$ from our previous analysis.

Definition 4.4. *Let* $N, d \in \mathbb{N}\backslash\{0\}$*, and let* $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ *be a polynomial such that* $p(D)$ *has determinant-order* $d \geq 1$*. Let* $B : \mathbb{C} \to \mathbb{C}^{d\times N}$ *be a polynomial with components* $B_{jk} : \mathbb{C} \to \mathbb{C}$ *for* $0 \leq j \leq d-1$ *and* $1 \leq k \leq N$*. Consider the polynomial* $C : \mathbb{C} \to \mathbb{C}^{d\times N}$ *given by* $C(z) = B(z) \operatorname{adj} p(z)$*. Let* $R_{jk} : \mathbb{C} \to \mathbb{C}$ *be the polynomial given by* $R_{jk} \equiv_{\det p} C_{jk}$ *and write*

$$R_{jk}(z) = \sum_{m=0}^{d-1} R_{jkm} z^m. \tag{4.8}$$

We then define $\mathcal{R}_B \in \mathcal{L}(\mathbb{C}^{N\times d}, \mathbb{C}^d)$ *via*

$$(\mathcal{R}_B c)_j = \sum_{k=1}^{N} \sum_{m=0}^{d^- -1} R_{jkm} c_{km}. \tag{4.9}$$

The purpose of $\mathcal{R}_B$ is seen in the following lemma.

Lemma 4.5. *Let $0 \in J \subseteq \mathbb{R}$ be a nontrivial interval, and let $N \geq 1$ and $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ be a polynomial such that $p(D)$ has determinant-order $d \geq 1$. Let $B(D) : \mathbb{C} \to \mathbb{C}^{d\times N}$ be a polynomial and let $\mathcal{R}_B \in \mathcal{L}(\mathbb{C}^{N\times d}, \mathbb{C}^d)$ be the associated map from Definition 4.4. Let $Z(\det p) \subset B(0,r)$ and define $x \in H(\mathbb{C};\mathbb{C}^N)$ by*

$$x_j(\tau) = \frac{1}{2\pi i} \sum_{\ell=1}^{N} \sum_{k=0}^{d-1} \int_{\partial B(0,r)} e^{\tau z} \frac{\operatorname{adj} p_{j\ell}(z) c_{\ell k} q_k(z)}{\det p(z)} dz. \tag{4.10}$$

with $c \in \mathbb{C}^{N\times d}$ and $\{q_k\}_{k=0}^{d-1}$ associated to $\det p$. Then the restriction of x to J belongs to $C^\infty(J;\mathbb{C}^n)$, and

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = \mathcal{R}_B c. \end{cases} \tag{4.11}$$

Proof. Since the restriction $x \in H(\mathbb{C};\mathbb{C}^N)$ to J is smooth, the result follows immediately from (2.57) when $N = 1$ and Lemma 3.15 when $N \geq 2$.

□

We now have all of the tools needed to prove the analog of Theorems 2.10 and 3.16, which reduces the solvability question for the problem

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \ldots, \xi_{d-1}) \end{cases} \tag{4.12}$$

in $C^\infty(J;\mathbb{C}^N)$ to the Shapiro-Lopatinsky condition.

Theorem 4.6. *Let $0 \in J \subseteq \mathbb{R}$ be a nontrivial interval, and let $N \geq 1$ and $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ be a polynomial such that $p(D)$ has determinant-order $d \geq 1$. Let $B : \mathbb{C} \to \mathbb{C}^{d\times N}$ be a polynomial and let $\mathcal{R}_B \in \mathcal{L}(\mathbb{C}^{N\times d}, \mathbb{C}^d)$ be the associated map from Definition 4.4. Then the following are equivalent.*

1. *The rows of the matrix polynomial $B \operatorname{adj} p : \mathbb{C} \to \mathbb{C}^{d\times N}$ are linearly independent (mod $\det p$).*

2. *$\mathcal{R}_B$ is surjective.*

3. *If $f \in C^\infty(J;\mathbb{C}^N)$ and $\xi_0, \ldots, \xi_{d-1} \in \mathbb{C}$, then there exists a unique $x \in C^\infty(J;\mathbb{C}^N)$ such that*

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \ldots, \xi_{d-1}). \end{cases} \tag{4.13}$$

4. *The map $\Phi : C^\infty(J;\mathbb{C}^N) \to C^\infty(J;\mathbb{C}^N) \times \mathbb{C}^d$ given by*

$$\Phi x = (p(D)x, B(D)x(0)) \tag{4.14}$$

is an isomorphism.

5. *For every $(\xi_0, \ldots, \xi_{d-1}) \in \mathbb{C}^d$ there exists an $x \in C^\infty(J;\mathbb{C}^N)$ such that*

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = (\xi_0, \ldots, \xi_{d-1}). \end{cases} \tag{4.15}$$

6. *If* $x \in C^\infty(J;\mathbb{C}^N)$ *and*

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = 0, \end{cases} \tag{4.16}$$

then $x = 0$.

Moreover, if any of these conditions is satisfied, then the following hold.

(a) *The operator* $\mathcal{R}_B^* : \mathbb{C}^d \to \mathbb{C}^{N\times d}$ *given by*

$$(\mathcal{R}_B^* v)_{km} = \sum_{j=0}^{d-1} \overline{R_{jkm}} v_j. \tag{4.17}$$

is injective.

(b) $\mathcal{R}_B\mathcal{R}_B^* : \mathbb{C}^d \to \mathbb{C}^d$ *is invertible.*

(c) *For any* $f \in C^\infty(J;\mathbb{C}^N)$ *and* $\xi = (\xi_0, \ldots, \xi_{d-1}) \in \mathbb{C}^d$, *the unique solution to*

$$\begin{cases} p(D)x = f \\ B(D)x(0) = \xi \end{cases} \tag{4.18}$$

is the map $x \in C^\infty(J;\mathbb{C}^N)$ *given by*

$$\begin{aligned} x(t) &= \operatorname{adj} p(D)y(t) \\ &\quad + \frac{1}{2\pi i}\sum_{k=0}^{d-1}\int_{\partial B(0,r)} e^{tz}\left(p(z)^{-1}\mathcal{R}_B^*(\mathcal{R}_B\mathcal{R}_B^*)^{-1}(\xi - B(D)\operatorname{adj} p(D)y(0))\right)_k q_k(z)dz, \end{aligned} \tag{4.19}$$

where $Z(\det p) \subset B(0,r)$ *and* $y \in C^\infty(J;\mathbb{C}^N)$ *is uniquely determined by*

$$\begin{cases} \det p(D)y_j = f_j \\ D^k y_j(0) = 0 \text{ for } 0 \le k \le d. \end{cases} \tag{4.20}$$

Proof. The proof is the same as that of Theorem 3.16 except that we substitute the use of Theorem 3.13 with Theorem 4.2 when $N \ge 2$ and Theorem 4.1 when $N = 1$. We again leave it as an exercise to verify the details.

□

In switching to studying systems on J rather than $\mathbb{C}$ we were forced to switch from using holomorphic data to smooth data. It turns out that this is a feature rather than a bug, as $C^\infty(J;\mathbb{C}^N)$ is closed under complex conjugation, whereas $H(\mathbb{C};\mathbb{C}^N)$ is not. This is important, as it allows us to port our theory to the special case in which the coefficients of $p(D)$ and $B(D)$ are real. In this case, we might seek a theory of $\mathbb{R}^N-$valued solutions given $f \in C^\infty(J;\mathbb{R}^N)$, and this closure property facilitates this.

Theorem 4.7. *Let* $0 \in J \subseteq \mathbb{R}$ *be a nontrivial interval, and let* $N \ge 1$ *and* $p : \mathbb{C} \to \mathbb{R}^{N\times N}$ *be a polynomial such that* $p(D)$ *has determinant-order* $d \ge 1$. *Let* $B : \mathbb{C} \to \mathbb{R}^{d\times N}$ *be a polynomial and suppose that the rows of the matrix polynomial* $B \operatorname{adj} p : \mathbb{C} \to \mathbb{R}^{d\times N}$ *are linearly independent (mod*

$\det p$). *Then for every* $f \in C^\infty(J;\mathbb{R}^N)$ *and* $\xi_0,\ldots,\xi_{d-1} \in \mathbb{R}$ *there exists a unique* $x \in C^\infty(J;\mathbb{R}^N)$ *such that*

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0,\ldots,\xi_{d-1}). \end{cases} \tag{4.21}$$

In other words, the map $\Phi : C^\infty(J;\mathbb{R}^N) \to C^\infty(J;\mathbb{R}^N) \times \mathbb{R}^d$ *given by*

$$\Phi x = (p(D)x, B(D)x(0)) \tag{4.22}$$

is an isomorphism.

Proof. Note that $\xi_0,\ldots,\xi_{d-1} \in \mathbb{R} \subset \mathbb{C}$ and that since $\mathbb{R}^N \subset \mathbb{C}^N$ we can view $f \in C^\infty(J;\mathbb{C}^N)$. Then Theorem 4.6 provides us with a unique $x \in C^\infty(J;\mathbb{C}^N)$ solving (4.21). Applying the complex conjugate and utilizing the fact that the coefficients of p and B are real, we find that $\bar{x} \in C^\infty(J;\mathbb{C}^N)$ satisfies

$$\begin{cases} p(D)\bar{x} = f \\ B(D)\bar{x}(0) = (\xi_0,\ldots,\xi_{d-1}). \end{cases} \tag{4.23}$$

Thus, by the uniqueness assertion of Theorem 4.6 we have that $x = \bar{x}$ and hence that $x(t) \in \mathbb{R}^N$ for all $t \in J$. □

Remark 4.8. *The explicit formula for* x *given in part* (*c*) *of Theorem 4.6 is still valid in the case that* p *and* B *have real coefficients, and this formula determines the unique solution to* (4.21).

5 Systems on the half line with decay conditions at infinity

We now specialize to the case that our interval is $J = [0,\infty)$. In some applications of the theory developed on this interval, the need arises to impose the extra condition on solutions that they decay to zero as $t \to \infty$. Our goal now is to develop the theory needed to produce such solutions.

In the simplest setting of the scalar equation $x' = -x$ the solution is $x(t) = x(0)e^{-t}$, which shows that the solution and all of its derivatives decay exponentially as $t \to \infty$. This is the condition that we will impose on our solutions. We define the appropriate space now.

Definition 5.1. *For* $1 \le N \in \mathbb{N}$ *we define*

$$C_e^\infty([0,\infty);\mathbb{C}^N) = \{x \in C^\infty([0,\infty);\mathbb{C}^N) \mid \text{for each } k \in \mathbb{N} \text{ there exists } \lambda_k > 0 \text{ such that } \sup_{t\ge 0} e^{\lambda_k t} \left|x^{(k)}(t)\right| < \infty\}. \tag{5.1}$$

This is the space of smooth $\mathbb{C}^N-$*valued functions on* $[0,\infty)$ *with derivatives of every order decaying exponentially as* $t \to \infty$. *If* $p : \mathbb{C} \to \mathbb{C}^{d\times N}$ *is a polynomial we write*

$$\ker_e(p(D)) = \{x \in C_e^\infty([0,\infty);\mathbb{C}^N) \mid p(D)x = 0\} \tag{5.2}$$

to distinguish this kernel from the kernel in $H(\mathbb{C};\mathbb{C}^N)$.

The following records some basic properties of this space.

Proposition 5.2. *The following hold.*

1. $C_e^\infty([0,\infty);\mathbb{C}^N)$ *is a complex vector space.*

2. $C_e^\infty([0,\infty);\mathbb{C})$ *is a complex algebra.*

3. *If* $p:\mathbb{C}\to\mathbb{C}^{M\times N}$ *is a polynomial and* $x\in C_e^\infty([0,\infty);\mathbb{C}^N)$*, then* $p(D)x\in C_e^\infty([0,\infty);\mathbb{C}^M)$*.*

Proof. The first and third items are trivial. The second follows directly from the Leibniz rule. $\square$

Next we show that in the scalar case, all ordinary differential operators act surjectively on our space $C_e^\infty([0,\infty);\mathbb{C})$.

Proposition 5.3. *Let* $p:\mathbb{C}\to\mathbb{C}$ *be a polynomial of degree* $n\in\mathbb{N}$*. Then the linear map* $p(D):C_e^\infty([0,\infty);\mathbb{C})\to C_e^\infty([0,\infty);\mathbb{C})$ *is surjective.*

Proof. The result is trivial if $n=0$, as in this case the polynomial p is just a constant $c\in\mathbb{C}\backslash\{0\}$, so we can set $x=f/c\in C_e^\infty([0,\infty);\mathbb{C})$. We reduce, then, to the case $n\geq 1$. We can further reduce to the case that p is monic. Indeed, if $p(z)=a_nz^n+\cdots+a_0$, then the equation $p(D)x=f$ is equivalent to $(D^n+\cdots+a_0/a_n)x=f/a_n$, so if we can solve the problem with p monic, then we can solve the problem in general. We assume, then, that p is monic. Fix $f\in C_e^\infty([0,\infty);\mathbb{C})$ and let $\lambda>0$ be such that $\sup_{t\geq 0}e^{\lambda t}|x(t)|=M<\infty$.

Suppose initially that p has the special form $p(D)=D-r$ for $r\in\mathbb{R}$. We will break to cases based on the size of r relative to λ. In the first case we assume that $r+\lambda<0$, which means that $r<-\lambda<0$. We then define $x:[0,\infty)\to\mathbb{C}$ via

$$x(t)=e^{rt}\int_0^t e^{-rs}f(s)ds, \tag{5.3}$$

from which it easily follows that x is differentiable on $[0,\infty)$ and satisfies $x'=rx+f$. Using the assumed decay properties of f, we may estimate

$$|x(t)|\leq e^{rt}\int_0^t Me^{-(r+\lambda)s}ds=\frac{M}{-(r+\lambda)}e^{rt}(e^{-(r+\lambda)t}-1)\leq\frac{M}{-(r+\lambda)}e^{-\lambda t} \tag{5.4}$$

for $t\geq 0$. Combining this with the facts that $f\in C_e^\infty([0,\infty);\mathbb{C})$ and $x'=rx+f$, a simple induction argument then reveals that $x\in C_e^\infty([0,\infty);\mathbb{C})$.

In the second case we assume $r+\lambda=0$ and define $x\in C^\infty([0,\infty);\mathbb{C})$ exactly as above, but then we estimate

$$|x(t)|\leq e^{rt}\int_0^T Me^{0s}ds=Mte^{-\lambda t}\leq cMe^{-\lambda t/2} \tag{5.5}$$

for $t\geq 0$, where $c=\sup_{t\geq 0}te^{-\lambda t/2}<\infty$. Then again, this and the equation $x'=rx+f$ imply, via an induction argument, that $x\in C_e^\infty([0,\infty);\mathbb{C})$ in this case as well.

In the final case we assume that $r+\lambda>0$, in which case we have that

$$\int_0^\infty e^{-rs}|f(s)|\,ds\leq M\int_0^\infty e^{-(r+\lambda)s}ds<\infty. \tag{5.6}$$

This allows us to define $x:[0,\infty)\to\mathbb{C}$ via

$$x(t)=-e^{rt}\int_t^\infty e^{-rs}f(s)ds, \tag{5.7}$$

where here in the integral we employ the Lebesgue integral on the real and imaginary parts of the integrand to make sense of the expression. This is justified by the above calculation, which shows that the real and imaginary parts are integrable on $[0, \infty)$. It is then as simple matter to check that x is differentiable on $[0, \infty)$ and $x' = rx + f$. We then estimate

$$|x(t)| \leq e^{rt} \int_t^\infty M e^{-(r+\lambda)s} ds = \frac{M}{r+\lambda} e^{rt} e^{-(r+\lambda)t} = \frac{M}{r+\lambda} e^{-\lambda t} \tag{5.8}$$

for $t \geq 0$, from which can again deduce that $x \in C_e^\infty([0,\infty);\mathbb{C})$ by inducting with the equation $x' = rx + f$. This completes the proof in the special case $p(D) = D - r$ for $r \in \mathbb{R}$.

We now consider the slightly more general case of $p(D) = D - z$ for $z \in \mathbb{C}$ with $\mathrm{Im}(z) \neq 0$. We write $z = r + is$ for $r, s \in \mathbb{R}$. Then

$$p(D)x(t) = f(t) \Leftrightarrow x'(t) - rx(t) - isx(t) = f(t) \Leftrightarrow (e^{-ist}x(t))' - r(e^{-ist}x(t)) = e^{-ist}f(t). \tag{5.9}$$

Set $F : [0,\infty) \to \mathbb{C}$ via $F(t) = e^{-ist}f(t)$ and note that $F \in C_e^\infty([0,\infty);\mathbb{C})$ as well since

$$D^k F(t) = e^{-ist} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} D^j f(t)(-is)^{k-j}. \tag{5.10}$$

Using the above special case, we may find $y \in C_e^\infty([0,\infty);\mathbb{C})$ solving $(D - r)y = F$. Similarly, we define $x \in C_e^\infty([0,\infty);\mathbb{C})$ via $x(t) = e^{ist}y(t)$, and then the above equivalence shows that $p(D)x = f$. This proves the theorem in the special case $p(D) = D - z$ for $z \in \mathbb{C}$.

The theorem is now proved in the case that $p(D)$ has order $n = 0$ or $n = 1$, so assume that $n \geq 2$. Using the fundamental theorem of algebra, we may then factor the polynomial

$$p(z) = \prod_{j=1}^{n} (z - z_j) \tag{5.11}$$

with the understanding that repeated roots are listed multiple times. Using the above, we can find $y_1 \in C_e^\infty([0,\infty);\mathbb{C})$ such that $(D - z_1)y_1 = f$. Again using the above, we can find $y_2 \in C_e^\infty([0,\infty);\mathbb{C})$ such that $(D - z_1)y_2 = y_1$. If $n \geq 3$ we iterate, using the above to produce $y_j \in C_e^\infty([0,\infty);\mathbb{C})$ such that $(D - z_j)y_j = y_{j-1}$ for $j = 3, \ldots, n$. We then set $x = y_n \in C_e^\infty([0,\infty);\mathbb{C})$ and note that

$$\begin{aligned}(D - z_n)x = y_{n-1} &\Rightarrow (D - z_{n-1})(D - z_n)x = (D - z_{n-1})y_{n-1} = y_{n-2} \\ &\Rightarrow \cdots \Rightarrow p(D)x = (D - z_1)\cdots(D - z_n)x = (D - z_1)y_1 = f,\end{aligned} \tag{5.12}$$

which completes the proof in the general case.

□

Remark 5.4. *Proposition 5.3 is the main reason we need to work on $[0,\infty)$ in studying decaying solutions. In the complex setting, the integrals on $(0,t)$ can be replaced with road integrals along a line segment from 0 to $\tau \in \mathbb{C}$, but we run into trouble trying define the second type of integral when* $\mathrm{Re}(\tau) < 0$.

Next we introduce a trio of definitions needed to progress further. The first defines a special subset of zeros of a holomorphic function.

Definition 5.5. *Let X be a complex Banach space and $f : \mathbb{C} \to X$ be holomorphic. We define*

$$Z_-(f) = Z(f) \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\} \tag{5.13}$$

for the roots of f with negative real part.

The next definition factors polynomials into products of polynomials with only positive and negative roots.

Definition 5.6. *Let $p : \mathbb{C} \to \mathbb{C}$ be a nontrivial polynomial. Using the fundamental theorem of algebra, we write*

$$p(z) = a \prod_{w \in Z(p)} (z - w)^{\mathrm{ord}(p,w)}. \tag{5.14}$$

We define the polynomials $p^{\pm} : \mathbb{C} \to \mathbb{C}$ as follows. If $Z_-(p) = \varnothing$, we set $p^- = a$. On the other hand, if $Z_-(p) \neq \varnothing$ we set

$$p^-(z) = a \prod_{w \in Z_-(p)} (z - w)^{\mathrm{ord}(p,w)}. \tag{5.15}$$

In either case we define

$$p^+(z) = \frac{p(z)}{p^-(z)}, \tag{5.16}$$

which means that $\deg(p^-) + \deg(p^+) = \deg(p)$.

Finally, we define an analog of determinant-order that we call the decay-order.

Definition 5.7. *Let $N \geq 1$ and $p : \mathbb{C} \to \mathbb{C}^{N \times N}$ be a polynomial with determinant-order $d \geq 0$. We define $(\det p)^{\pm} : \mathbb{C} \to \mathbb{C}$ as in Definition 5.6 and write*

$$d^{\pm} = \deg(\det p)^{\pm} \in \{0, \ldots, d\}. \tag{5.17}$$

We call d^- the decay-order of p

We now show that in the scalar case the decay-order determines how many initial conditions can be specified when solving $p(D)x = f \in C_e^\infty([0,\infty);\mathbb{C})$.

Theorem 5.8. *Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $n \in \mathbb{N}$ with decay-order $d^- \in \mathbb{N}$. Then the following hold.*

1. *If $d^- = 0$, then for every $f \in C_e^\infty([0,\infty);\mathbb{C})$ there exists a unique $x \in C_e^\infty([0,\infty);\mathbb{C})$ such that $p(D)x = f$. In other words, the map $C_e^\infty([0,\infty);\mathbb{C}) \ni x \mapsto p(D)x \in C_e^\infty([0,\infty);\mathbb{C})$ is a linear isomorphism.*

2. *If $d^- \geq 1$, then for every $f \in C_e^\infty([0,\infty);\mathbb{C})$ and $\xi_0, \ldots, \xi_{d^- - 1} \in \mathbb{C}$ there exists a unique $x \in C_e^\infty([0,\infty);\mathbb{C})$ such that*

$$\begin{cases} p(D)x = f \\ D^k x(0) = \xi_k \text{ for } 0 \leq k \leq d^- - 1. \end{cases} \tag{5.18}$$

In other words, the map

$$C_e^\infty([0,\infty);\mathbb{C}) \ni x \mapsto (p(D)x, (x(0), \ldots, D^{d^- - 1}x(0))) \in C_e^\infty([0,\infty);\mathbb{C}) \times \mathbb{C}^{d^-} \tag{5.19}$$

is a linear isomorphism.

Proof. Suppose initially that $d^- = 0$. In light of Proposition 5.3, we only need to prove that $p(D) : C_e^\infty([0,\infty);\mathbb{C}) \to C_e^\infty([0,\infty);\mathbb{C})$ is injective. Suppose, then, that $x \in C_e^\infty([0,\infty);\mathbb{C})$ satisfies $p(D)x = 0$ and set $\xi_k = D^k x(0)$ for $0 \le k \le n-1$. According to Theorems 4.1 and 2.5, we then have that

$$x(t) = \sum_{k=0}^{n-1} \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{tz} \frac{\xi_k q_k(z)}{p(z)} dz \tag{5.20}$$

where $Z(p) \subset B(0,R)$. Proposition 2.1 then allows us to rewrite

$$x(t) = \sum_{z \in Z(p)} e^{tz} \pi_z(t) \tag{5.21}$$

where $\pi_z : \mathbb{C} \to \mathbb{C}$ is a polynomial, but since $d^- = 0$ we have that $Z(p) \subset \{z \in \mathbb{C} \mid \operatorname{Re}(z) \ge 0\}$, and so either $x = 0$ or else $|x(t)| \to \infty$ as $t \to \infty$. The latter possibility is excluded by the inclusion $x \in C_e^\infty([0,\infty);\mathbb{C})$, so we deduce that $x = 0$. Hence, $p(D)$ is injective and the first item is proved.

Now suppose that $1 \le d^- \le n$. Thanks to Proposition 5.3 and the uniqueness part of Theorem 4.1, it suffices to show that for $\xi_0, \dots, \xi_{d^- -1} \in \mathbb{C}$ there exists $x \in C_e^\infty([0,\infty);\mathbb{C})$ such that

$$\begin{cases} p(D)x = 0 \\ D^k x(0) = 0 \text{ for } 0 \le k \le d^- - 1. \end{cases} \tag{5.22}$$

Let $y \in H(\mathbb{C};\mathbb{C})$ be the unique solution to

$$\begin{cases} p^-(D)y = 0 \\ D^k y(0) = \xi_k \text{ for } 0 \le k \le d^- - 1. \end{cases} \tag{5.23}$$

Then Theorem 2.5 and Proposition 2.1 again show that

$$y(\tau) = \sum_{k=0}^{n-1} \frac{1}{2\pi i} \int_{\partial B(0,R)} e^{\tau z} \frac{\xi_k q_k(z)}{p^-(z)} dz = \sum_{z \in Z(p^-)} e^{\tau z} \pi_z(\tau) \tag{5.24}$$

for polynomials π_z. Since $z \in Z(p^-)$ implies $\operatorname{Re}(z) < 0$, we deduce that the restriction of y to $[0,\infty)$, which we denote by x, is such that $x \in C_e^\infty([0,\infty);\mathbb{C})$. To complete the proof of the second item, we then note that $p(D)x = p^+(D)p^-(D)x = p^+(D)0 = 0$. □

Next we prove that in the case of a system, the decay-order again determines how many scalar initial conditions can be imposed. This is a variant of Theorem 3.13. The proof largely follows that of Theorem 3.13 but differs in a few key ways, so we include the full details.

Theorem 5.9. *Let $N \ge 2$ and $p : \mathbb{C} \to \mathbb{C}^N$ be a polynomial such that $p(D)$ has decay-order $d^- \ge 1$. Let $q, u : \mathbb{C} \to \mathbb{C}^{N \times N}$ be the polynomials from Proposition 3.12, so that $q(D)p(D) = u(D)$ and $u(D)$ is in upper-triangular form. Write d_k^- for the degree of the polynomial $u_{kk}^-(z)$ for $1 \le k \le N$, which is nontrivial due to Theorem 3.13. Then the following hold.*

1. *If we define $s_0 = 0$ and $s_m = \sum_{j=1}^m d_j^-$ for $1 \le m \le N$, then $s_0 \le s_1 \le \cdots \le s_N = \sum_{k=1}^N d_k^- = d^-$.*

2. *Define the polynomial* $B : \mathbb{C} \to \mathbb{C}^{d^- \times N}$ *via*

$$B(z)_{km} = \begin{cases} z^{k-s_{m-1}} & \text{if } s_{m-1} \le k \le s_m - 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.25}$$

for $0 \le k \le d^- - 1$ *and* $1 \le m \le N$*. Then for every* $f \in C_e^\infty([0,\infty);\mathbb{C}^N)$ *and* $(\xi_0, \ldots, \xi_{d^- -1}) \in \mathbb{C}^{d^-}$ *there exists a unique* $x \in C_e^\infty([0,\infty);\mathbb{C}^N)$ *satisfying*

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \ldots, \xi_{d^- -1}). \end{cases} \tag{5.26}$$

3. *The map* $\Phi : C_e^\infty([0,\infty);\mathbb{C}^N) \to C_e^\infty([0,\infty);\mathbb{C}^N) \times \mathbb{C}^{d^-}$ *given by*

$$\Phi x = (p(D)x, B(D)x(0)) \tag{5.27}$$

is an isomorphism.

4. *We have that* $\dim(\ker_e(p(D))) = d^-$*, and the map* $\ker_e(p(D)) \ni x \mapsto B(D)x(0) \in \mathbb{C}^{d^-}$ *is an isomorphism.*

5. *The rows of the matrix polynomial* $B \operatorname{adj} p : \mathbb{C} \to \mathbb{C}^{d^- \times N}$ *are linearly independent (mod* $(\det p)^-$*).*

Proof. The third item is a linear algebraic restatement of the second, and the fourth item follows easily from the third, so it suffices to only prove the first two and fifth items.

Proposition 3.12 guarantees that $\det q(z) = \pm 1$ and

$$qp = u = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ 0 & u_{22} & \cdots & u_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{NN} \end{pmatrix}, \text{ so } \pm \det p = \prod_{m=1}^N u_{mm}. \tag{5.28}$$

It follows that

$$\pm(\det p)^- = \prod_{m=1}^N u_{mm}^- \text{ and } \pm(\det p)^+ = \prod_{m=1}^N u_{mm}^+, \tag{5.29}$$

and hence the degree of $(\det p)^-$ equals $\sum_{m=1}^N d_m^-$. This proves the first item.

We now turn to the proof of the second item. For $x \in C_e^\infty([0,\infty);\mathbb{C}^N)$ we have that

$$p(D)x = f \Leftrightarrow u(D)x = q(D)p(D)x = q(D)f \in C_e^\infty([0,\infty);\mathbb{C}^N). \tag{5.30}$$

We thus reduce to solving the upper-triangular problem $u(D)x = q(D)f$, which reads

$$\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ 0 & u_{22} & \cdots & u_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{NN} \end{pmatrix} x = q(D)f \in C_e^\infty([0,\infty);\mathbb{C}^N). \tag{5.31}$$

The condition $B(D)x(0) = (\xi_0, \ldots, \xi_{d^- -1})$ is equivalent to

$$D^{k-s_{m-1}} x_m(0) = \xi_k \text{ if } s_{m-1} \le k \le s_m - 1 \text{ for } 0 \le k \le d^- - 1 \text{ and } 1 \le m \le N, \tag{5.32}$$

but note that

$$s_{m-1} \le k \le s_m - 1 \Leftrightarrow 0 \le k - s_{m-1} \le s_m - s_{m-1} - 1 = d_m^- - 1, \tag{5.33}$$

so a condition on x_m is only enforced if $d_m^- \ge 1$. In other words, $B(D)x(0) = (\xi_0, \dots, \xi_{d^- - 1})$ is equivalent to

$$D^\ell x_m(0) = \xi_{\ell + s_{m-1}} \text{ if } d_m^- \ge 1 \text{ and } 0 \le \ell \le d_m^- - 1. \tag{5.34}$$

As in the proof of Theorem 3.13, the advantage of this triangular form is that we can solve the system iteratively, starting from the final scalar equation, which reads

$$u_{NN}(D)x_N = (q(D)f)_N. \tag{5.35}$$

If $d_N^- \ge 1$ we use the second item of Theorem 5.8 to solve for a unique $x_N \in C_e^\infty([0,\infty);\mathbb{C})$ satisfying

$$\begin{cases} u_{NN}(D)x_N = (q(D)f)_N \\ D^\ell x_N(0) = \xi_{\ell + s_{N-1}} \text{ for } 0 \le \ell \le d_N^- - 1, \end{cases} \tag{5.36}$$

and if $d_N^- = 0$ we use the first item of Theorem 5.8 to solve for a unique $x_N \in C_e^\infty([0,\infty);\mathbb{C})$ satisfying $u_{NN}(D)x_N = (q(D)f)_N$. Once we have produced $x_{m+1}, \dots, x_N \in C_e^\infty([0,\infty);\mathbb{C})$ for $1 \le m \le N-1$, we then find a unique $x_m \in C_e^\infty([0,\infty);\mathbb{C})$ via the m^{th} equation, which reads

$$u_{mm}(D)x_m = (q(D)f)_m - \sum_{m+1 \le j \le N} u_{mj}(D)x_j, \tag{5.37}$$

subject to the extra condition that

$$D^\ell x_m(0) = \xi_{\ell + s_{m-1}} \text{ if } d_m^- \ge 1 \text{ and } 0 \le \ell \le d_m^- - 1. \tag{5.38}$$

Proceeding iteratively, we then find the unique functions $x_1, \dots, x_N \in C_e^\infty([0,\infty);\mathbb{C})$ such that $x = (x_1, \dots, x_N) \in C_e^\infty([0,\infty);\mathbb{C}^N)$ satisfies

$$\begin{cases} u(D)x = q(D)f \\ B(D)x(0) = (\xi_0, \dots, x_{d^- - 1}). \end{cases} \tag{5.39}$$

This proves the second item.

Finally, we turn to the proof of the fifth item. We first claim that the rows of $B \operatorname{adj} p$ are linearly independent (mod $(\det p)^-$) if and only if the rows of $B \operatorname{adj} u$ are linearly independent (mod $(\det p)^-$). To see this we first use the basic properties of the adjugate to write

$$B \operatorname{adj} p = B \operatorname{adj}(q^{-1}u) = (B \operatorname{adj} u) \operatorname{adj}(q^{-1}) = (B \operatorname{adj} u)\frac{q}{\det q} = \pm (B \operatorname{adj} u) q. \tag{5.40}$$

From this it's clear that the rows of $B \operatorname{adj} p$ are are linearly independent (mod $(\det p)^-$) if and only if the rows of $(B \operatorname{adj} u)q$ are linearly independent (mod $(\det p)^-$), but since q is invertible we have that

$$\sum_{j=0}^{d^- -1} \sum_{k=1}^{N} \alpha_j (B \operatorname{adj} u)_{jk} q_{km} \equiv_{(\det p)^-} 0 \text{ for all } 1 \le m \le N, \tag{5.41}$$

if and only if

$$\sum_{j=0}^{d^- -1} \alpha_j (B \operatorname{adj} u)_{jk} \equiv_{(\det p)^-} 0 \text{ for all } 1 \le k \le N, \tag{5.42}$$

and so the claim follows.

We now claim that the rows of $B \operatorname{adj} u$ are linearly independent (mod $(\det p)^-$); once this is established, the fifth item is proved. The structure of B shows that if

$$\sum_{j=0}^{d^- -1} \alpha_j (B \operatorname{adj} u)_{jk} \equiv_{(\det p)^-} 0 \text{ for } 1 \le k \le N, \tag{5.43}$$

then for $1 \le j \le N$ there exist polynomials $\pi_j : \mathbb{C} \to \mathbb{C}$ with coefficients given by partitioning $\{\alpha_k\}_{k=0}^{d^- -1}$ according to the partitioning from B, with $\deg(\pi_j) < d_j^- = \deg(u_{jj}^-)$ such that

$$\sum_{j=1}^{N} \pi_j (\operatorname{adj} u)_{jk} \equiv_{(\det p)^-} 0 \text{ for } 1 \le K \le N, \tag{5.44}$$

and hence there exist polynomials $\delta_k : \mathbb{C} \to \mathbb{C}$ for $1 \le k \le N$ such that

$$\sum_{j=1}^{N} \pi_j (\operatorname{adj} u)_{jk} = \delta_k (\det p)^- \text{ for } 1 \le k \le N. \tag{5.45}$$

Now, since u is upper triangular, $\operatorname{adj} u$ is as well, and $(\operatorname{adj} u)_{kk} = \prod_{j \ne k} u_{jj} \ne 0$.

Suppose, by way of contradiction, that $\pi_1 \ne 0$. Then

$$\pi_1 (\operatorname{adj} u)_{11} = \delta_1 (\det p)^- \Rightarrow \pi_1 \prod_{j \ne 1} u_{jj} = \pm \delta_1 \prod_{j=1}^{N} u_{jj}^- \Rightarrow \pi_1 \prod_{j \ne 1} u_{jj}^+ = \pm \delta_1 u_{11}^-, \tag{5.46}$$

but any root of $\prod_{j \ne 1} u_{jj}^+$ must have nonnegative real part, so

$$\pi_1^- = \pm \delta_1^- u_{11}^-. \tag{5.47}$$

Since π_1 and $(\operatorname{adj} u)_{11}$ are nontrivial, we must have that δ_1 is nontrivial as well, but then

$$\deg(u_{11}^-) > \deg(\pi_1^-) = \deg(\delta_1^- u_{11}^-) \ge \deg(u_{11}^-), \tag{5.48}$$

a contradiction. Hence $\pi_1 = 0$.

Using this and arguing similarly, we find that $\pi_2 = 0$, and upon iterating we deduce that $\pi_1 = \cdots = \pi_N = 0$, and hence that $\alpha_1 = \cdots = \alpha_N = 0$. Thus, the rows of $B \operatorname{adj} u$ are linearly independent (mod $(\det p)^-$).

□

We now aim to study more general initial conditions. To this end we introduce an analog of $\mathcal{R}_B$ from our previous analysis.

Definition 5.10. *Let $N, d \in \mathbb{N} \backslash \{0\}$, and let $p : \mathbb{C} \to \mathbb{C}^{N \times N}$ be a polynomial such that $p(D)$ has decay-order $d^- \ge 1$. Let $B : \mathbb{C} \to \mathbb{C}^{d^- \times N}$ be a polynomial with components $B_{jk} : \mathbb{C} \to \mathbb{C}$ for $0 \le j \le d^- - 1$ and $1 \le k \le N$. Consider the polynomial $C : \mathbb{C} \to \mathbb{C}^{d^- \times N}$ given by $C(z) = B(z) \operatorname{adj} p(z)$. Let $R_{jk} : \mathbb{C} \to \mathbb{C}$ be the polynomial given by $R_{jk} \equiv_{(\det p)^-} C_{jk}$ and write*

$$R_{jk}(z) = \sum_{m=0}^{d^- -1} R_{jkm} z^m. \tag{5.49}$$

We then define $\mathcal{R}_B \in \mathcal{L}(\mathbb{C}^{N \times d^-}, \mathbb{C}^{d^-})$ via

$$(\mathcal{R}_B c)_j = \sum_{k=1}^{N} \sum_{m=0}^{d^- -1} R_{jkm} c_{km}. \tag{5.50}$$

The purpose of $\mathcal{R}_B$ is seen in the following lemma.

Lemma 5.11. *Let $N \geq 1$ and $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ be a polynomial such that $p(D)$ has decay-order $d^- \geq 1$. Let $B(D) : \mathbb{C} \to \mathbb{C}^{d^-\times N}$ be a polynomial and let $\mathcal{R}_B \in \mathcal{L}(\mathbb{C}^{N\times d^-}, \mathbb{C}^{d^-})$ be the associated map from Definition 5.10. Let $Z((\det p)^-) \subset B(0,r)$ and define $x \in H(\mathbb{C};\mathbb{C}^N)$ by*

$$x_j(\tau) = \frac{1}{2\pi i}\sum_{\ell=1}^{N}\sum_{k=0}^{d^--1}\int_{\partial B(0,r)} e^{\tau z}\frac{\operatorname{adj} p_{j\ell}(z) c_{\ell k} q_k(z)}{(\det p)^-(z)}dz. \tag{5.51}$$

with $c \in \mathbb{C}^{N\times d^-}$ and $\{q_k\}_{k=0}^{d^--1}$ associated to $(\det p)^-$. Then the restriction of x to $[0,\infty)$ belongs to $C_e^\infty([0,\infty);\mathbb{C}^n)$, and

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = \mathcal{R}_B c. \end{cases} \tag{5.52}$$

Proof. First note that

$$\begin{aligned} p(D)x_m(\tau) &= \frac{1}{2\pi i}\sum_{\ell,m=1}^{N}\sum_{k=0}^{d^--1}\int_{\partial B(0,r)} e^{\tau z}\frac{p_{mj}(z)\operatorname{adj} p_{j\ell}(z) c_{\ell k} q_k(z)}{(\det p)^-(z)}dz \\ &= \frac{1}{2\pi i}\sum_{k=0}^{d^--1}\int_{\partial B(0,r)} e^{\tau z}\frac{\det p(z) c_{mk} q_k(z)}{(\det p)^-(z)}dz = \frac{1}{2\pi i}\sum_{k=0}^{d^--1}\int_{\partial B(0,r)} e^{\tau z}(\det p)^+(z) c_{mk} q_k(z) dz = 0 \end{aligned} \tag{5.53}$$

by Cauchy-Goursat. Also, by Cauchy-Goursat and Proposition 2.2, we may compute

$$\begin{aligned} (B(D)x(0))_m &= \frac{1}{2\pi i}\sum_{\ell=1}^{N}\sum_{j,k=0}^{d^--1}\int_{\partial B(0,r)} \frac{B_{mj}(z)\operatorname{adj} p_{j\ell}(z) c_{\ell k} q_k(z)}{(\det p)^-(z)}dz \\ &= \frac{1}{2\pi i}\sum_{\ell=1}^{N}\sum_{k=0}^{d^--1}\int_{\partial B(0,r)} \frac{C_{m\ell}(z) c_{\ell k} q_k(z)}{(\det p)^-(z)}dz = \frac{1}{2\pi i}\sum_{\ell=1}^{N}\sum_{k=0}^{d^--1}\int_{\partial B(0,r)} \frac{R_{m\ell}(z) c_{\ell k} q_k(z)}{(\det p)^-(z)}dz \\ &= \frac{1}{2\pi i}\sum_{\ell=1}^{N}\sum_{k,n=0}^{d^--1}\int_{\partial B(0,r)} \frac{R_{m\ell n} z^n c_{\ell k} q_k(z)}{(\det p)^-(z)}dz = \sum_{\ell=1}^{N}\sum_{k=0}^{d^--1} R_{m\ell k} c_{\ell k} = (\mathcal{R}_B c)_m. \end{aligned} \tag{5.54}$$

To conclude it suffices to show that the restriction of x to $[0,\infty)$ belongs to $C_e^\infty([0,\infty);\mathbb{C}^N)$, but this follows from the expression for x and Proposition 2.1 since $Z((\det p)^-) \subset \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$. □

We now have all of the tools needed to prove our main result on the decay problem, which establishes a number of equivalent ways to guarantee the solvability of $p(D)x = f$ subject to general initial conditions. The result is the analog of Theorem 3.16. While the proof is similar, it differs in some key places, so we provide the full details.

Theorem 5.12. *Let $N \geq 1$ and $p : \mathbb{C} \to \mathbb{C}^{N\times N}$ be a polynomial such that $p(D)$ has decay-order $d^- \geq 1$. Let $B : \mathbb{C} \to \mathbb{C}^{d^-\times N}$ be a polynomial and let $\mathcal{R}_B \in \mathcal{L}(\mathbb{C}^{N\times d^-}, \mathbb{C}^{d^-})$ be the associated map from Definition 5.10. Then the following are equivalent.*

1. *The rows of the matrix polynomial $B \operatorname{adj} p : \mathbb{C} \to \mathbb{C}^{d^-\times N}$ are linearly independent (mod $(\det p)^-$).*

2. $\mathcal{R}_B$ *is surjective.*

3. *If* $f \in C_e^\infty([0,\infty);\mathbb{C}^N)$ *and* $\xi_0, \ldots, \xi_{d^- -1} \in \mathbb{C}$, *then there exists a unique* $x \in C_e^\infty([0,\infty);\mathbb{C}^N)$ *such that*

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \ldots, \xi_{d^- -1}). \end{cases} \tag{5.55}$$

4. *The map* $\Phi : C_e^\infty([0,\infty);\mathbb{C}^N) \to C_e^\infty([0,\infty);\mathbb{C}^N) \times \mathbb{C}^{d^-}$ *given by*

$$\Phi x = (p(D)x, B(D)x(0)) \tag{5.56}$$

is an isomorphism.

5. *For every* $(\xi_0, \ldots, \xi_{d^- -1}) \in \mathbb{C}^{d^-}$ *there exists* $x \in C_e^\infty([0,\infty);\mathbb{C}^N)$ *such that*

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = (\xi_0, \ldots, \xi_{d^- -1}). \end{cases} \tag{5.57}$$

6. *If* $x \in C_e^\infty([0,\infty);\mathbb{C}^N)$ *and*

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = 0, \end{cases} \tag{5.58}$$

then $x = 0$.

Moreover, if any of these conditions is satisfied, then the following hold.

(a) The operator $\mathcal{R}_B^* : \mathbb{C}^{d^-} \to \mathbb{C}^{N \times d^-}$ *given by*

$$(\mathcal{R}_B^* v)_{km} = \sum_{j=0}^{d^- -1} \overline{R_{jkm}} v_j. \tag{5.59}$$

is injective.

(b) $\mathcal{R}_B \mathcal{R}_B^* : \mathbb{C}^{d^-} \to \mathbb{C}^{d^-}$ *is invertible.*

(c) For any $f \in C_e^\infty([0,\infty);\mathbb{C}^N)$ *and* $\xi = (\xi_0, \ldots, \xi_{d^- -1}) \in \mathbb{C}^{d^-}$, *the unique solution to*

$$\begin{cases} p(D)x = f \\ B(D)x(0) = \xi \end{cases} \tag{5.60}$$

is given by

$$x_j(t) = y_j(t) + \frac{1}{2\pi i} \sum_{\ell=1}^{N} \sum_{k=0}^{d^- -1} \int_{\partial B(0,r)} e^{tz} \frac{\operatorname{adj} p_{j\ell}(z) c_{\ell k} q_k(z)}{(\det p)^-(z)} dz, \tag{5.61}$$

where $Z((\det p)^-) \subset B(0,r)$, $y \in C_e^\infty([0,\infty);\mathbb{C}^N)$ *satisfies* $p(D)y = f$, *and*

$$c = \mathcal{R}_B^* (\mathcal{R}_B \mathcal{R}_B^*)^{-1} (\xi - B(D)y(0)) \in \mathbb{C}^{N \times d^-}. \tag{5.62}$$

Proof. We begin by recalling from Definition 5.10 that $C = B \operatorname{adj} p$ and

$$C_{jk}(z) \equiv_{(\det p)^-} R_{jk}(z) = \sum_{m=0}^{d^- -1} R_{jkm} z^m. \tag{5.63}$$

Pick $r > 0$ such that $Z((\det p)^-) \subset B(0,r)$. For a fixed $c \in \mathbb{C}^{N\times d^-}$ we use Lemma 3.15 to define $x \in \ker_e(p(D))$ via

$$x_j(\tau) = \frac{1}{2\pi i} \sum_{\ell=1}^{N} \sum_{k=0}^{d^- -1} \int_{\partial B(0,r)} e^{\tau z} \frac{\operatorname{adj} p_{j\ell}(z) c_{\ell k} q_k(z)}{(\det p)^-(z)} dz, \tag{5.64}$$

and the lemma implies that

$$B(D)x(0) = \mathcal{R}_B c. \tag{5.65}$$

With these tools in hand, we now turn to the proof of the various equivalences. The equivalence of (3) and (4) is trivial, as are the implications (3) $\Rightarrow$ (5) and (3) $\Rightarrow$ (6). To conclude, we will prove that (1) $\Leftrightarrow$ (2), (5) $\Leftrightarrow$ (6), (2) $\Rightarrow$ (5), (6) $\Rightarrow$ (2), and (5) and (6) $\Rightarrow$ (3).

Proof of (1) $\Leftrightarrow$ (2)**:** Forgetting the matrix structure of $\mathbb{C}^{N\times d^-}$, we may view $\mathcal{R}_B$ as a matrix in $\mathbb{C}^{d^-\times Nd^-}$ with components $(\mathcal{R}_B)_{j,km} = R_{jkm}$ with R_{jkm} as above. From this perspective, basic linear algebra shows that $\mathcal{R}_B$ is surjective if and only if the rows of $\mathcal{R}_B \in \mathbb{C}^{d^-\times Nd^-}$ are linearly independent, i.e. $\sum_{j=0}^{d^- -1} \alpha_j R_{jkm} = 0$ for $1 \le k \le N$ and $0 \le m \le d^- - 1$ implies that $\alpha_0 = \cdots = \alpha_{d^- -1} = 0$. Thus, to prove the equivalence of (1) and (2) it suffices to prove that the rows of $B \operatorname{adj} p$ are linearly independent (mod $(\det p)^-$) if and only if $\sum_{j=0}^{d^- -1} \alpha_j R_{jkm} = 0$ for $1 \le k \le N$ and $0 \le m \le d^- - 1$ implies that $\alpha_0 = \cdots = \alpha_{d^- -1} = 0$. To prove this equivalence we note that for $\alpha_0, \ldots, \alpha_{d^- -1} \in \mathbb{C}$ we have that

$$\begin{aligned} &\sum_{j=0}^{d^- -1} \alpha_j R_{jkm} = 0 \text{ for } 1 \le k \le N, 0 \le m \le d^- - 1 \Leftrightarrow \sum_{j,m=0}^{d^- -1} \alpha_j R_{jkm} z^m = 0 \text{ for } 1 \le k \le N \\ &\Leftrightarrow \sum_{j,m=0}^{d^- -1} \alpha_j R_{jk}(z) = 0 \text{ for } 1 \le k \le N \Leftrightarrow \sum_{j,m=0}^{d^- -1} \alpha_j (B \operatorname{adj} p)_{jk}(z) \equiv_{\det p} 0 \text{ for } 1 \le k \le N, \end{aligned} \tag{5.66}$$

from which the desired equivalence directly follows.

Proof of (5) $\Leftrightarrow$ (6)**:** Define the linear map $T : \ker_e(p(D)) \to \mathbb{C}^{d^-}$ via $Tx = B(D)x(0)$. Then (5) is equivalent to the assertion that T is surjective, while (6) is equivalent to the assertion that T is injective. Theorem 5.9 tells us that $\dim(\ker_e(p(D))) = d^- = \dim(\mathbb{C}^{d^-})$, so we know from linear algebra that T is injective if and only if it's surjective.

Proof of (2) $\Rightarrow$ (5)**:** Let $(\xi_0, \ldots, \xi_{d^- -1}) \in \mathbb{C}^{d^-}$. Since $\mathcal{R}_B$ is surjective, we can find $c \in \mathbb{C}^{N\times d^-}$ such that $\mathcal{R}_B c = (\xi_0, \ldots, \xi_{d^- -1})$. Using this c in (5.64) defines $x \in \ker_e(p(D))$, and (5.65) implies that

$$B(D)x(0) = \mathcal{R}_B c = (\xi_0, \ldots, \xi_{d^- -1}). \tag{5.67}$$

Thus (5) holds.

Proof of (6) $\Rightarrow$ (2)**:** We will prove the contrapositive. Suppose that (2) is false, i.e. $\mathcal{R}_B : \mathbb{C}^{N\times d^-} \to \mathbb{C}^{d^-}$ is not surjective. We know that

$$\dim \operatorname{ran}(\mathcal{R}_B) + \dim \ker(\mathcal{R}_B) = Nd^-, \tag{5.68}$$

so

$$Nd^- - \dim \ker(\mathcal{R}_B) = \dim \operatorname{ran}(\mathcal{R}_B) \le d^- - 1 \tag{5.69}$$

and hence

$$Nd^- - d^- + 1 \leq \dim\ker(\mathcal{R}_B). \tag{5.70}$$

Let $\tilde{B} : \mathbb{C} \to \mathbb{C}^{d^- \times N}$ be the polynomial from Theorem 5.9 if $N \geq 2$ and be given by $\tilde{B}_{k1}(z) = z^k$ if $N = 1$. Let $\mathcal{R}_{\tilde{B}} : \mathbb{C}^{N \times d^-} \to \mathbb{C}^-$ be the map associated to $\tilde{B}$ as in Definition 5.10. When $N \geq 2$, Theorem 5.9 shows that the rows of $\tilde{B}\,\mathrm{adj}\,p$ are linearly independent (mod $(\det p)^-$) so by the (1) $\Leftrightarrow$ (2) assertion above, applied with $\tilde{B}$, we know that $\mathcal{R}_{\tilde{B}}$ is surjective in this case. On the other hand, if $N = 1$, then the surjectivity of $\mathcal{R}_{\tilde{B}}$ follows from (5.65) (applied with $\tilde{B}$ in place of B) and the second item of Theorem 5.8. Then

$$\dim\ker(\mathcal{R}_{\tilde{B}}) = Nd^- - \dim\mathrm{ran}(\mathcal{R}_{\tilde{B}}) = Nd^- - d^-, \tag{5.71}$$

so

$$\dim\ker(\mathcal{R}_{\tilde{B}})^\perp = Nd^- - \dim\ker(\mathcal{R}_{\tilde{B}}) = d^-, \tag{5.72}$$

where $\ker(\mathcal{R}_{\tilde{B}})^\perp \subseteq \mathbb{C}^{N \times d^-}$ denotes the orthogonal complement of $\ker(\mathcal{R}_{\tilde{B}})$.

Now, if $\ker(\mathcal{R}_{\tilde{B}})^\perp \cap \ker(\mathcal{R}_B) = \{0\}$, then

$$\dim\ker(\mathcal{R}_{\tilde{B}})^\perp + \dim\ker(\mathcal{R}_B) = \dim(\ker(\mathcal{R}_{\tilde{B}})^\perp + \ker(\mathcal{R}_B)) \leq Nd^-, \tag{5.73}$$

but by the above,

$$\dim\ker(\mathcal{R}_{\tilde{B}})^\perp + \dim\ker(\mathcal{R}_B) \geq d^- + (Nd^- - d^- + 1) = Nd^- + 1, \tag{5.74}$$

a contradiction. Hence, there exists $0 \neq c \in \ker(\mathcal{R}_{\tilde{B}})^\perp \cap \ker(\mathcal{R}_B)$. In other words, $c \in \mathbb{C}^{N \times d^-}$ satisfies

$$\mathcal{R}_B c = 0 \text{ and } \mathcal{R}_{\tilde{B}} c = \mu \in \mathbb{C}^{d^-} \backslash \{0\}. \tag{5.75}$$

We use this c in (5.64) to define $x \in \ker_e(p(D))$. Then (5.65), applied with both B and $\tilde{B}$, shows that

$$B(D)x(0) = \mathcal{R}_B c = 0 \text{ and } \tilde{B}(D)x(0) = \mathcal{R}_{\tilde{B}} c = \mu \neq 0, \tag{5.76}$$

the latter of which implies that $x \neq 0$. Thus, we have constructed $0 \neq x \in C_e^\infty([0,\infty); \mathbb{C}^N)$ such that

$$\begin{cases} p(D)x = 0 \\ B(D)x(0) = 0, \end{cases} \tag{5.77}$$

which shows that (6) is false.

Proof of (5) **and** (6) $\Rightarrow$ (2)**:** First note that (6) implies that there exists at most one solution to (5.55), so it suffices to prove the existence of a solution. According to Theorem 5.9 we can find $y \in C_e^\infty([0,\infty); \mathbb{C}^N)$ such that $p(D)y = f$. According to (5), we can then find $h \in C_e^\infty([0,\infty); \mathbb{C}^N)$ such that

$$\begin{cases} p(D)h = 0 \\ B(D)h(0) = (\xi_0, \ldots, \xi_{d^- - 1}) - B(D)y(0). \end{cases} \tag{5.78}$$

Then $x = h + y \in C_e^\infty([0,\infty); \mathbb{C}^N)$ satisfies (5.55), and (2) is proved.

It remains to prove that (a), (b), and (c) hold whenever any and hence all of these conditions holds. Note that $\mathcal{R}_B^*$ is the conjugate transpose associated to $\mathcal{R}_B$ and satisfies

$$\mathcal{R}_B c \cdot v = c : \mathcal{R}_B^* v \tag{5.79}$$

for all $c \in \mathbb{C}^{N\times d}$ and $v \in \mathbb{C}^d$, where $\cdot$ is the usual complex inner-product on $\mathbb{C}^d$ and $:$ is the same for $\mathbb{C}^{N\times d}$. Thus, we know from linear algebra shows that $\mathcal{R}_B$ is surjective if and only if $\mathcal{R}_B^*$ is injective, which proves (a).

The map $\mathcal{R}_B\mathcal{R}_B^*$ is Hermitian and satisfies

$$\mathcal{R}_B\mathcal{R}_B^* v \cdot v = \mathcal{R}_B^* v \cdot \mathcal{R}_B^* v = |\mathcal{R}_B^* v|^2, \tag{5.80}$$

which together with (a) proves that $\mathcal{R}_B\mathcal{R}_B^*$ is positive definite and hence invertible. This is (b).

Finally, let $y \in C_e^\infty([0,\infty);\mathbb{C}^N)$ be such that $p(D)y = f$. If we set $c = \mathcal{R}_B^*(\mathcal{R}_B\mathcal{R}_B^*)^{-1}(\xi - B(D)y(0)) \in \mathbb{C}^{N\times d^-}$, then we compute

$$\mathcal{R}_B c = \mathcal{R}_B\mathcal{R}_B^*(\mathcal{R}_B\mathcal{R}_B^*)^{-1}(\xi - B(D)y(0)) = \xi - B(D)y(0). \tag{5.81}$$

This, (5.64), and (5.65) prove (c). □

Remark 5.13. *Again, the condition that the rows of the matrix polynomial* $B \operatorname{adj} p : \mathbb{C} \to \mathbb{C}^{d^-\times N}$ *are linearly independent (mod* $(\det p)^-$*) is called the Shapiro-Lopatinsky condition. Recall that when* $N = 1$ *we set* $\operatorname{adj} p = 1$*, so the Shapiro-Lopatinsky condition is only concerned with the operator* B*.*

Let's consider some examples.

Example 5.14. Consider the scalar operator $p(D) = -D^2 + r^2 = (-D + r)(D + r)$ for $r \in (0,\infty)$. Then $p^-(z) = z + r$, so p has decay-order 1. Consider the operators $B_1(D) = 1$ and $B_2(D) = D$. Neither 1 nor z are equal to 0 (mod $z + r$), so B_1, p and B_2, p satisfy the Shapiro-Lopatinsky condition.

△

Example 5.15. Let $r \in (0,\infty)$ and consider

$$p(D) = \begin{pmatrix} -D^2 + r^2 & 0 & ir \\ 0 & -D^2 + r^2 & D \\ ir & D & 0 \end{pmatrix}. \tag{5.82}$$

We saw in Example 3.6 that $\det p(D) = (D^2 - r^2)^2 = (D - r)^2(D + r)^2$, which means $p(D)$ has determinant-order 4 and decay-order 2. We compute

$$\operatorname{adj} p(D) = \begin{pmatrix} -D^2 & irD & ir(D^2 - r^2) \\ irD & r^2 & (D^2 - r^2)D \\ ir(D^2 - r^2) & (D^2 - r^2)D & (D^2 - r^2)^2 \end{pmatrix} \tag{5.83}$$

and $(\det p(z))^- = (z + r)^2$. Consider the operators

$$B_1(D) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } B_2(D) = \begin{pmatrix} D & ir & 0 \\ 0 & -2D & 1 \end{pmatrix}. \tag{5.84}$$

Then

$$B_1(z) \operatorname{adj} p(z) = \begin{pmatrix} -z^2 & irz & ir(z^2 - r^2) \\ irz & r^2 & (z^2 - r^2)z \end{pmatrix}, \tag{5.85}$$

and it's easy to deduce from the second column that the Shapiro-Lopatinsky condition is satisfied. On the other hand,

$$B_2(z) \operatorname{adj} p(z) = \begin{pmatrix} -z^3 - r^2 z & ir(z^2 + r^2) & 2ir(z^3 - r^2 z) \\ -ir(z^2 + r^2) & z^3 - 3r^2 z & -z^4 + r^4 \end{pmatrix} =: \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} = q(z) \tag{5.86}$$

satisfies

$$q(-r) = \begin{pmatrix} 2r^3 & 2ir^3 & 0 \\ -2ir^3 & 2r^3 & 0 \end{pmatrix} \text{ and } q'(-r) = \begin{pmatrix} -4r^2 & -2ir^2 & 4ir^3 \\ 2ir^2 & 0 & 4r^3 \end{pmatrix}, \tag{5.87}$$

so if

$$\alpha q_1(z) + \beta q_2(z) \equiv_{(z+r)^2} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \tag{5.88}$$

then

$$\alpha q_1(-r) + \beta q_2(-r) = 0 \text{ and } \alpha q_1'(z)(-r) + \beta q_2'(-r) = 0, \tag{5.89}$$

which implies that

$$\alpha(2r^3) + \beta(-2ir^3) = 0 \text{ and } \alpha(-2ir^2) + \beta 0 = 0, \tag{5.90}$$

and since $r \neq 0$ we deduce that $\alpha = \beta = 0$. Thus B_2, p also satisfy the Shapiro-Lopatinsky condition. △

We can also consider the special case of real coefficients in p and B in the decay problem. This leads us to an analog of Theorem 4.7, in which we write

$$C_e^\infty([0,\infty);\mathbb{R}^N) = \{f \in C_e^\infty([0,\infty);\mathbb{C}^N) \mid f(t) \in \mathbb{R}^N \text{ for all } t \in [0,\infty)\}. \tag{5.91}$$

Theorem 5.16. *Let $N \geq 1$ and $p : \mathbb{C} \to \mathbb{R}^{N\times N}$ be a polynomial such that $p(D)$ has decay-order $d^- \geq 1$. Let $B : \mathbb{C} \to \mathbb{R}^{d^-\times N}$ be a polynomial and suppose that the rows of the matrix polynomial $B \operatorname{adj} p : \mathbb{C} \to \mathbb{R}^{d^-\times N}$ are linearly independent (mod $(\det p)^-$). Then for every $f \in C_e^\infty([0,\infty);\mathbb{R}^N)$ and $\xi_0, \dots, \xi_{d^--1} \in \mathbb{R}$ there exists a unique $x \in C_e^\infty([0,\infty);\mathbb{R}^N)$ such that*

$$\begin{cases} p(D)x = f \\ B(D)x(0) = (\xi_0, \dots, \xi_{d^--1}). \end{cases} \tag{5.92}$$

In other words, the map $\Phi : C_e^\infty([0,\infty);\mathbb{R}^N) \to C_e^\infty([0,\infty);\mathbb{R}^N) \times \mathbb{R}^{d^-}$ given by

$$\Phi x = (p(D)x, B(D)x(0)) \tag{5.93}$$

is an isomorphism.

Proof. The proof is the same as that of Theorem 5.16 except that we use Theorem 5.12 in place of Theorem 4.6. □

A Some reminders from complex analysis

Here we record three results from complex analysis. The first is a standard formula for computing residues.

Proposition A.1. *Let $\varnothing \neq U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C} \cup \{\infty\}$ be meromorphic. For $z_0 \in P(f)$ we have that*

$$\operatorname{Res}(f, z_0) = \frac{1}{(\operatorname{ord}(f,z_0)-1)!} \lim_{z\to z_0} \left(\frac{d}{dz}\right)^{\operatorname{ord}(f,z_0)-1} \left((z-z_0)^{\operatorname{ord}(f,z_0)} f(z)\right) \tag{A.1}$$

In particular, if z_0 is a simple pole, then

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} (z-z_0) f(z). \tag{A.2}$$

The residue theorem also allows us to easily compute integrals involving certain rational functions.

Proposition A.2. *Let X be a complex Banach space, and suppose that $q : \mathbb{C} \to X$ and $p : \mathbb{C} \to \mathbb{C}$ are polynomials such that* $\deg(p) \geq \deg(q) + 2$*. If $Z(p) \subset B(0, R)$, then*

$$\int_{\partial B(0,R)} \frac{q}{p} = 0. \tag{A.3}$$

Proof. Define $I : [R, \infty) \to \mathbb{C}$ via

$$I(r) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{q}{p}. \tag{A.4}$$

Since p and q are polynomials, we can pick constants $C_0, C_1, C_2 \in (0, \infty)$ such that

$$|p(z)| \geq C_0 |z|^{\deg(p)} - C_1 \text{ and } \|q(z)\|_X \leq C_2(1 + |z|^{\deg(q)}). \tag{A.5}$$

Hence, the condition $\deg(p) \geq \deg(q) + 2$ implies that

$$|I(r)| \leq \frac{2\pi r}{2\pi} \max_{|z|=r} \frac{\|q(z)\|_X}{|p(z)|} \leq \frac{C_2 r(1 + r^{\deg(q)})}{C_0 r^{\deg(p)} - C_1} \to 0 \text{ as } r \to \infty. \tag{A.6}$$

On the other hand, the rational function q/p is meromorphic on $\mathbb{C}$, so the residue theorem implies that

$$I(r) = \sum_{z \in Z(p)} \operatorname{Res}(q/p, z). \tag{A.7}$$

This means that I is a constant function that vanishes at infinity, and hence $I(r) = 0$ for all $r \geq R$. □

The third result shows the continuity of the roots of a polynomial with respect to the coefficients. To formulate the continuity assertion we will use the Hausdorff metric h on the set of nonempty compact subsets of $\mathbb{C}$, $\mathfrak{K}(\mathbb{C})$. Recall that, given a metric space X, the Hausdorff metric space $\mathfrak{H}(X)$ consists of all nonempty closed subsets of X and that

$$\mathfrak{K}(X) = \{\varnothing \neq K \subseteq X \mid K \text{ is compact}\} \subseteq \mathfrak{H}(X) \tag{A.8}$$

is the subspace of nonempty compact sets in X.

Theorem A.3. *Let $1 \leq n \in \mathbb{N}$ and define the open set $U = \{a = (a_0, \dots, a_n) \in \mathbb{C}^{n+1} \mid a_n \neq 0\}$. For $a \in U$ let $p_a : \mathbb{C} \to \mathbb{C}$ be the polynomial $p_a(z) = \sum_{k=0}^{n} a_k z^k$. Define the map $\Phi : U \to \mathfrak{K}(\mathbb{C})$ via $\Phi(a) = Z(p_a) = p_a^{-1}(\{0\})$. Then Φ is continuous.*

Proof. See, for instance, Theorem 3.25 in [5]. □

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