21-260 **Differential Equations** **D. Handron**

Final Exam Review
May, 2003

Closed book and notes; calculators not permitted.

1. Use Laplace transforms to solve the following non-homogenious equations:

   (a) $x'' + 9x = \cos(4t)$, where $x(0) = 0$ and $x'(0) = 0$. It may help to know that $\cos(A)\sin(B) = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$.

   (b) $x'' + 9x = 3\delta(t-3)$, with $x(0) = 1$ and $x'(0) = 0$.

2. Compute the Fourier Series of the periodic function f with period 4 such that

$$f(t) = t + 1 \quad \text{for} \quad -2 \leq t \leq 2.$$

3. The goal of this problem is to solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}$$

   subject to the conditions $y(0,t) = y(2,t) = 0$, $y(x,0) = \sin(\frac{\pi t}{2})$ and $\frac{\partial y}{\partial t}(x,0) = 0$. Begin by looking for a solution of the form $y(x,t) = X(x)T(t)$.

   (a) Solve $X'' + \lambda X = 0$, such that $X(0) = X(2) = 0$. For what values of λ does such a solution exist? Why did we require the endpoint conditions $X(0) = X(2) = 0$?

   (b) For each λ in part (a), find a solution to $T'' + 4\lambda T = 0$, such that $T'(0) = 0$. Why do we require that $T'(0) = 0$?

   (c) For each λ_n in (a) you get a solution $y_n(x,t) = X_n(t)T_n(t)$. write $y(x,t) = \sum_{n=1}^{\infty} A_n y_n(x,t)$. Find the coefficients A_n needed to satisfy the condition $y(x,0) = \sin(\frac{\pi t}{2})$.

4. Let $f(t)$ be the function with period 3 such that

$$f(t) = \begin{cases} 0 & 0 < t < 1 \\ -3 & 1 < t < 2 \\ 0 & 2 < t < 3. \end{cases}$$

   (a) Find the Laplace Transform for $f(t)$.

   (b) Find the Fourier Series for $f(t)$.

5. (a) Let F(x) be the odd periodic extension, with period 4, of the function defined on the interval $(0, 2)$ by

$$f(x) = \begin{cases} 3 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

Find the Fourier Series for $F(x)$.

(b) Solve the Heat equation problem

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$$
$$u(0,t) = u(2,t) = 0$$
$$u(x,0) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

A Short List of Laplace Transforms

Function	Transform
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}

Properties of the Laplace Transform

Translation on the s-axis: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

Translation on the t-axis: $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$.

Transforms of Integrals: $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$.

Integrals of Transforms:$\mathcal{L}\{\frac{f(t)}{t}\} = \int_\sigma^\infty F(\sigma)d\sigma$.

Transforms of Derivatives: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.

Derivatives of Transforms: $\mathcal{L}\{-tf(t)\} = F'(s)$.

Convolution Property: $\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$.

Transforms of Periodic Functions: $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt$.

Trigonometric Identities

$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$.
$\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$.
$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$.