21-260 **Differential Equations** **D. Handron**

Review #3
April, 2003 **Name:**_____

Closed book and notes; calculators not permitted.

1. An undamped, forced harmonic harmonic oscillator may be modelled by the equation $mx'' + kx = f(t)$, where m is the mass of the weight, k is the spring constant, and $f(t)$ is the force applied to the mass. Consider the case where $m = 2$, $k = 8$ and $f(t) = 3\cos(\omega t)$.

   (a) Find a particular solution, $x_p(t)$.

   (b) For what value of ω will resonance occur? How can this be determined from the particular solution found in part (a)?

2. Compute the Laplace transform of the function

$$f(t) = e^{at}g(t), \quad \text{where} \quad g(t) = \begin{cases} 2t & 0 \le t < 3 \\ 0 & t \ge 3 \end{cases}$$

3. Use Laplace transforms to solve the initial value problem

$$x'' + 4x = 1, \quad \text{where} \quad x(0) = 1, \text{ and } x'(0) = 0.$$

4. (a) Consider the second order, linear differential equation

$$y'' - 2y' - 8y = 0.$$

   Find the general solution, and the particular solution satisfying $y(0) = 0$ and $y'(0) = 0$.

   (b) The equation

$$x'' + 4x = F\sin(\omega t)$$

   models a forced, undamped mass-spring system. Find the general solution for this differential equation. (Assume that $\omega \neq 2$.)

   (c) What can you say about the behavior of the solutions to the differential equation in (4b) when $\omega = 2$? (You do *not* need to find any solutions, merely describe their behavior.)

5. Compute the Laplace transform of the function

$$g(t) = \begin{cases} e^{2t} & 0 \le t \le 3 \\ e^{-3t} & 3 < t < 5 \\ 0 & t \ge 5 \end{cases}$$

6. Determine the following inverse Laplace transforms:

   (a) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\}$

   (b) $\mathcal{L}^{-1}\left\{\frac{e^{-3t}}{(s-2)^2+1}\right\}$

7. Use the Laplace transform method to solve the initial value problem

$$x'' + 4x = h(t-2); \quad x(0) = 0, x'(0) = 1.$$

8. Use the Laplace transform method to solve the initial value problem

$$x'' + 2x' + 5x = 0; \quad x(0) = 1, x'(0) = 2.$$

A Short List of Laplace Transforms

Function	Transform
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$t\cos(kt)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t\sin(kt)$	$\frac{2ks}{(s^2+k^2)^2}$
$h(t-a)$	$\frac{e^{-as}}{s}$

Properties of the Laplace Transform

First Shift Theorem: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

Second Shift Theorem: $\mathcal{L}\{h(t-a)f(t-a)\} = e^{-as}F(s)$.

First Integration Theorem: $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$.

Second Integration Theorem: $\mathcal{L}\{\frac{f(t)}{t}\} = \int_\sigma^\infty F(\sigma)d\sigma$.

First Differentiation Theorem: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.

Second Differentiation Theorem: $\mathcal{L}\{t^n f(t)\} = (-1)^n \left(\frac{d}{ds}\right) F(s)$.