**21-241** **Matrix Algebra** **D. Handron**

**Exam #3 Review**
April 20, 2005 **Name:**_____

Closed book and notes; calculators not permitted.

1. Let
$$A = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$
   (a) Find the eigenvalues and eigenvectors of $A$.
   (b) Diagonalize $A$. That is, write $A$ as
$$A = S \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} S^{-1},$$
   for some matrix $S$.
   (c) Compute $\lim_{n\to\infty} A^n \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the plane given by the equation $x_3 = 0$.
   (a) Find the matrix $B$ of this transformation with respect to the basis $\mathfrak{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where
$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
   (b) $[\mathbf{u}]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $[T(\mathbf{u})]_{\mathfrak{B}}$ and $T(\mathbf{u})$.

3. Let
$$A = \begin{bmatrix} -7 & 6 \\ -15 & 12 \end{bmatrix}.$$
   (a) Find the eigenvalues of $A$.
   (b) Find an eigenbasis for $A$.

(c) Let $T(\mathbf{x}) = A\mathbf{x}$. Find the matrix of $T$ with respect to the eigenbasis found in (b).

4. Let

$$A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}.$$

(a) Find a basis for $\text{Nul}(A)$.

(b) Find a basis for $\text{Col}(A)$.

(c) What is the dimension of $\text{Nul}(A)$? $\text{Col}(A)$?
What is $\dim(\text{Nul}(A)) + \dim(\text{Col}(A))$? Why?

5. **(20 points)** $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ are bases for $\mathbb{R}^2$.

(a) If $\mathbf{x} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$.

(b) If $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, then what is $[\mathbf{y}]_{\mathcal{C}}$?

6. **(20 points)** Let

$$A = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

(a) Find the eigenvalues of $A$.

(b) Find a basis for each eigenspace of $A$.

(c) Find (if possible) a diagonalization of $A$, i.e. an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. If it is not possible, explain why is is not.

7. **(20 points)** Let $T : \mathcal{P}_2 \to \mathbb{R}^3$ be defined by

$$T(p(x)) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}.$$

(a) Find the matrix for this linear transformation with respect to the basis of Hermete polynomials $\mathcal{B} = \{1, 2x, -2 + 4x^2\}$ on $\mathcal{P}_2$ and the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ on $\mathbb{R}^3$.

(b) What is the rank of this linear transformation? Is this linear transformation invertible?

8. **(20 points)** Let $M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$.

   (a) Find a basis for $\mathrm{Col}(M)$.

   (b) Find a basis for $\mathrm{Nul}(M)$.

   (c) What is the dimension of $\mathrm{Nul}(M^T)$? the dimension of $\mathrm{Col}(M^T)$?

9. **(20 points)** Let $A$ and $B$ be $n \times n$ matrices, $S$ is an invertible $n \times n$ matrix. $B = S^{-1}AS$.

   (a) Show that $A$ and $B$ have the same eigenvalues.

   (b) What is the relation between the eigenvectors of $A$ and $B$?

10. **(20 points)** Let

$$A = \begin{bmatrix} 2 & 0 \\ 6 & -1 \end{bmatrix}$$

   (a) Find the eigenvalues and eigenvectors of $A$.

   (b) Plot several trajectories of the discrete dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

   (c) Let $\{\mathbf{x}_k\}$ be a solution of the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Compute $\{\mathbf{x}_1\}$ and find a formula for $\{\mathbf{x}_k\}$.