

21-602 Intro to Set Theory I

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0 Introduction

Text: Kunen, Kenneth. *Set Theory: An Intro to Independence Proofs*

Major outline of course:

- read Chapter 1
- ZFC axioms
- Ordinal and cardinal arithmetic
- V : proper class of all sets

$$V = \bigcup_{\alpha \text{ an ordinal}} V_\alpha$$

where $V_0 = \emptyset$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) = \{X : X \subseteq V_\alpha\}$$
$$V_\beta = \bigcup_{\alpha < \beta} V_\alpha \text{ if } \beta \text{ is a limit ordinal}$$

Note: V_α is Kunen's $R(\alpha)$

- $WF = \bigcup_{\alpha \in OR} R(\alpha)$. The Foundation Axiom is equivalent to $V = WF$.

Main topics:

1. Relativization (the basic model theory of proper classes)
2. Reflection Theorem (set theorist's version of the Downward Lowenheim-Skolem Theorem)
3. Mostowski collapse theorem
4. Δ_1 absoluteness theorem
5. $H(\kappa)$ hierarchy

6. *OD* and *HOD*

Theorem 0.1 (Gödel). $Con(ZF) \Rightarrow Con(ZFC)$

One proof uses *HOD*.

Continuum Hypothesis (CH): $2^{\aleph_0} = \aleph_1$.

Equivalently: If $A \subseteq \mathbb{R}$ is uncountable then $|A| = |\mathbb{R}|$.

GCH: $2^\kappa = \kappa^+$ for all $\kappa \geq \aleph_0$.

Theorem 0.2 (Gödel). $Con(ZFC) \Rightarrow Con(ZFC+GCH)$

Theorem 0.3 (Cohen, 1963). $Con(ZFC) \Rightarrow Con(ZFC + \neg CH)$

Combined result: CH is independent of ZFC.

Methods:

- Gödel: constructibility.

$$\begin{aligned}
 L_0 &= \emptyset \\
 L_{\alpha+1} &= \mathcal{D}(L_\alpha) = \{X \subseteq L_\alpha : X \text{ is first order definable with parameters over } (L_\alpha, \in)\} \\
 L_\beta &= \bigcup_{\alpha < \beta} L_\alpha \text{ if } \beta \text{ is a limit ordinal} \\
 L &= \bigcup_{\alpha \text{ is an ordinal}} L_\alpha
 \end{aligned}$$

$L \subseteq V$. This says “If V is a model of ZFC then L is a model of ZFC+GCH.”

- Cohen: forcing. Add sets to V to get $V[G]$. (Add \aleph_2 many reals to kill CH.) (Don’t kill ZFC.) (Don’t kill $\aleph_0, \aleph_1, \aleph_2, \dots$)

Infinitary Combinatorics: Trees (Aronszajn and Suslin trees, Suslin lines), graphs, special kinds of combinatorics related to uncountable cardinals, “the club filter and non-stationary ideal”, “diamond”

Descriptive set theory: (not in Kunen). Definable sets of real numbers

1 Beginnings

Let

$$R(0) = \emptyset \quad R(\alpha + 1) = \mathcal{P}(R(\alpha)) \quad R(\beta) = \bigcup_{\alpha < \beta} R(\alpha) \text{ if } \beta \text{ limit ordinal}$$

Proposition 1.1. 1. $R(\alpha) \subseteq R(\beta)$ for $\alpha < \beta$

2. $R(\beta)$ is transitive

Definition 1.2. A is transitive (a transitive set) $\iff \forall Y \in A (Y \subseteq A) \iff \forall Y \in A \forall X \in Y (X \in A)$

Definition 1.3. α is an ordinal $\iff \alpha$ is transitive and (α, \in) is a well-ordering $\iff \alpha$ is transitive and (α, \in) is a linear ordering (assuming foundation).

Definition 1.4. (A, \in) means $(A, \{(X, Y) \in A \times A : X \in Y\})$

Recall:

$$0 = \emptyset \quad 1 = \{0\} \quad 2 = \{0, 1\} \quad \dots \quad \omega = \{0, 1, 2, \dots\}$$

Lemma 1.5. If A is transitive then so is $\mathcal{P}(A)$.

Proof. Let $Y \in \mathcal{P}(A)$. Then $Y \subseteq A$. Let $X \in Y$. Then $X \in A$. But A transitive so $X \subseteq A$. Thus $X \in \mathcal{P}(A)$. Done. \square

Lemma 1.6. *If $A = \bigcup_{\alpha < \beta} A_\alpha$ where every A_α is transitive then A is transitive.*

Proof. Let $Y \in A$. Pick $\alpha < \beta$ s.t. $Y \in A_\alpha$. A_α is transitive so $Y \subseteq A_\alpha \subseteq \bigcup_{\alpha' < \beta} A_{\alpha'} = A$. \square

This gives clause (2) in the proposition above. For (1) $R(\alpha) \subseteq R(\alpha)$ so $R(\alpha) \subseteq R(\alpha + 1)$. By (2) $R(\alpha) \subseteq R(\alpha + 1)$. This easily gives (1).

$$WF = \bigcup_{\alpha \in OR} R(\alpha)$$

Definition 1.7. *For $X \in WF$ let $\text{rank}(X) =$ the least α s.t. $X \in R(\alpha + 1) =$ the least α s.t. $X \subseteq R(\alpha)$.*

Clause (3): $R(\beta) = \{X \in WF : \text{rank}(X) < \beta\}$.

Clause (4): For every set X , $X \in WF \iff X \subseteq WF$.

Proof. (\Rightarrow) Obvious. Let $X \in WF$. Then $X \in R(\beta)$ for some β . Then $X \subseteq R(\alpha)$ for some $\alpha < \beta$. Then $X \subseteq WF$.

(\Leftarrow) Assume $X \subseteq WF$. For large enough α , $X \subseteq R(\alpha)$. Why? Consider $a \mapsto \text{rank}(a)$ as a “function” on X . By Replacement Axiom Scheme, f is a function, meaning $\text{range}(f)$ is a set, so there’s an ordinal, α , greater than $\text{rank}(a)$ for every $a \in X$. Thus $X \in R(\alpha + 1) \subseteq WF$. \square

Clause (5): If $X \in Y \in WF$ then $X \in WF$ ($Y \in R(\alpha) \Rightarrow Y \subseteq R(\alpha) \subseteq WF$) and $\text{rank}(X) < \text{rank}(Y)$.

Proof. $Y \subseteq R(\beta) \Rightarrow X \in R(\beta) \Rightarrow \exists \alpha < \beta$ s.t. $X \subseteq R(\alpha)$. \square

Clause (6): $\text{rank}(Y) = \sup_{X \in Y} (\text{rank}(X) + 1)$ for $Y \in WF$.

Proof. (\leq) Let $\sigma = \text{RHS}$. Then $Y \subseteq R(\sigma)$ so $\text{rank}(Y) \leq \sigma$.

(\geq) $X \in Y \Rightarrow \text{rank}(X) < \text{rank}(Y)$ by Clause (5) so $\sigma \leq \text{rank}(Y)$. \square