21-114: Calculus for Architecture
Homework #3 Solutions

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§3.1 #20. Differentiate

$$f(t) = \sqrt{t} - \frac{1}{\sqrt{t}}.$$

Solution: First, since $\sqrt{t} = t^{1/2}$, $f(t) = t^{1/2} - t^{-1/2}$, so by the Difference and Power rules, we have

$$\begin{aligned}
\frac{d}{dt}\left(f(t)\right) &= \frac{d}{dt}\left(t^{1/2} - t^{-1/2}\right) \\
&= \frac{d}{dt}\left(t^{1/2}\right) - \frac{d}{dt}\left(t^{-1/2}\right) \\
&= \frac{1}{2}t^{-1/2} - \left(-\frac{1}{2}\right)t^{-3/2} \\
&= \frac{t+1}{2t\sqrt{t}}.
\end{aligned}$$

§3.1 #30. Differentiate $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$.

Solution: First, we rewrite the radicals as powers: $\sqrt[3]{t^2} = t^{2/3}$ and $\sqrt{t^3} = t^{3/2}$. Then, we apply the Constant Multiple, Sum, and Power rules:

$$\begin{aligned}
\frac{d}{dt}(u) &= \frac{d}{dt}(t^{2/3} + 2t^{3/2}) \\
&= \frac{d}{dt}(t^{2/3}) + \frac{d}{dt}(2t^{3/2}) \\
&= \frac{d}{dt}(t^{2/3}) + 2\frac{d}{dt}(t^{3/2}) \\
&= \frac{2}{3}t^{-1/3} + 2\left(\frac{3}{2}\right)t^{1/2} \\
&= \frac{2}{3\sqrt[3]{t}} + 3\sqrt{t}.
\end{aligned}$$

§3.1 #46. For what values of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

Solution: Horizontal tangents occur wherever $f'(x) = 0$, so we first calculate $f'(x)$:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3 + 3x^2 + x + 3) \\ &= \frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(3) \\ &= 3x^2 + 3(2x) + 1 + 0 \\ &= 3x^2 + 6x + 1. \end{aligned}$$

Setting $3x^2 + 6x + 1 = 0$, we use the quadratic formula, which gives us

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(1)}}{2(3)} = -1 \pm \frac{\sqrt{7}}{3}.$$

Therefore, the graph of $f(x)$ has horizontal tangents at $x = -1 + \frac{\sqrt{7}}{3}$ and $x = -1 - \frac{\sqrt{7}}{3}$.

§3.2 #20. Differentiate

$$y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}.$$

Solution: Since y is clearly a quotient, we'll attempt to apply the Quotient Rule: let $f(x) = \sqrt{x} - 1$ and $g(x) = \sqrt{x} + 1$, so

$$y = \frac{f(x)}{g(x)}.$$

Then $f'(x) = (1/2)x^{-1/2} = 1/(2\sqrt{x})$ and $g'(x) = (1/2)x^{-1/2} = 1/(2\sqrt{x})$ by the Power and Sum/Difference rules. Therefore, by the Quotient Rule,

$$\begin{aligned} \frac{d}{dx}(y) &= \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{(\sqrt{x}+1)(1/(2\sqrt{x})) - (\sqrt{x}-1)(1/(2\sqrt{x}))}{(\sqrt{x}+1)^2} \\ &= \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}. \end{aligned}$$

§3.2 #24. Find an equation to the tangent line of

$$y = \frac{\sqrt{x}}{x+1}$$

at the point $(4, 0.4)$.

Solution: Since we already have the point, we need to find the slope of the tangent line, which we know is the derivative at $x = 4$. Since y is a quotient, we'll have to apply the Quotient Rule, and we first note that $\frac{d}{dx}(\sqrt{x}) = 1/(2\sqrt{x})$ and $\frac{d}{dx}(x+1) = 1 + 0 = 1$, so

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\sqrt{x}}{x+1}\right) \\
&= \frac{(x+1)(1/(2\sqrt{x})) - (\sqrt{x})(1)}{(x+1)^2} \text{ (Quotient Rule)} \\
&= \frac{1/(2\sqrt{x}) - \sqrt{x}/2}{(x+1)^2} \\
&= \frac{1-x}{2\sqrt{x}(x+1)^2}.
\end{aligned}$$

Therefore the slope of the line is

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{1-4}{2\sqrt{4}(4+1)^2} = -\frac{3}{100}.$$

Hence, by the point-slope form, the tangent line is

$$f(x) = -\frac{3}{100}(x-4) + 0.4.$$

§3.2 #32. If $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$, and $g'(3) = 5$, find the following numbers:

(a) $(f+g)'(3)$.

Solution: By the Sum Rule, $(f+g)'(3) = f'(3) + g'(3) = -6 + 5 = -1$.

(b) $(fg)'(3)$.

Solution: By the Product Rule,

$$(fg)'(3) = f(3)g'(3) + g(3)f'(3) = (4)(5) + (2)(-6) = 8.$$

(c) $(f/g)'(3)$.

Solution: By the Quotient Rule,

$$\left(\frac{f}{g}\right)'(3) = \frac{g(3)f'(3) - f(3)g'(3)}{g(3)^2} = \frac{(2)(-6) - (4)(5)}{(2)^2} = -8.$$

(d) $\left(\frac{f}{f-g}\right)'(3)$.

Solution: By the Quotient Rule,

$$\begin{aligned}\left(\frac{f}{f-g}\right)'(3) &= \frac{(f(3) - g(3))f'(3) - f(3)(f'(3) - g'(3))}{(f(3) - g(3))^2} \\ &= \frac{(4-2)(-6) - (4)(-6-5)}{(4-2)^2} \\ &= 8.\end{aligned}$$

§3.2 #42. Find the equations of the tangent lines to the curve

$$f(x) = \frac{x-1}{x+1}$$

that are parallel to the line $x - 2y = 2$.

Solution: First, we write the line $x - 2y = 2$ as $y = (1/2)x - 1$. The tangent lines parallel to this line have slope $1/2$, so we find the values of x where the derivative of $(x-1)/(x+1)$ is $1/2$:

$$\frac{1}{2} = \frac{d}{dx}\left(\frac{x-1}{x+1}\right) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}.$$

Solving, we get that $(x+1)^2 = 4$, and therefore $x = -1 \pm 2$.

For $x = -1 - 2 = -3$: $f(x) = f(-3) = 2$, so the tangent line is $y = (1/2)(x+3) + 2 = (1/2)x + 7/2$.

For $x = -1 + 2 = 1$: $f(x) = f(1) = 0$, and the tangent line is $y = (1/2)(x-1) = (1/2)x - 1/2$.