21-114: Calculus for Architecture
Homework #2 Solutions

November 18, 2004
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§2.2 #12. Sketch the graph of the following function to determine the values of a for which $\lim_{x\to a} f(x)$ exists.

$$f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ (x-1)^2 & \text{if } x \ge 1 \end{cases}$$

Solution:

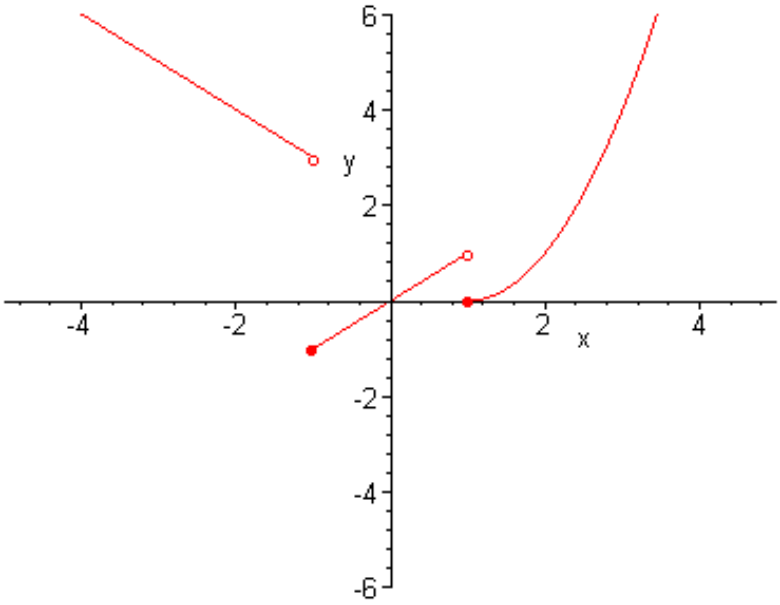


Figure 1: Graph of $f(x)$.

From the graph, we clearly see that f is continuous except at the points where the piecewise definition changes, specifically $a = -1, 1$, so we have continuity for a in $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

§2.2 #26. Determine the infinite limit of

$$\lim_{x\to 0} \frac{x-1}{x^2(x+2)}.$$

Solution: First, we consider the limits of the numerator and the denominator separately. For the numerator, $\lim_{x\to 0}(x-1) = 0 - 1 = -1$. For the denominator, since

$\lim_{x\to 0} x^2 = 0^+$ (since $x^2 \geq 0$ for all x) and $\lim_{x\to 0}(x+2) = 0+2 = 2$, we have that $\lim_{x\to 0}(x^2)(x+2) = (0^+)(2) = 0^+$.

Therefore

$$\lim_{x\to 0} \frac{x-1}{x^2(x+2)} = \frac{-1}{0^+} = -\infty.$$

§2.3 #40. Find the limit, if it exists:

$$\lim_{x\to -4^-} \frac{|x+4|}{x+4}.$$

Solution: Since $x < -4$, $|x+4| = -(x+4)$, so $\frac{|x+4|}{x+4} = \frac{-(x+4)}{x+4} = -1$, and

$$\lim_{x\to -4^-} \frac{|x+4|}{x+4} = -1.$$

§2.3 #46. Let

$$f(x) = \begin{cases} 4-x^2 & \text{if } x \leq 2 \\ x-1 & \text{if } x > 2 \end{cases}.$$

(a) Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.

Solution: For $x < 2$, $f(x) = 4-x^2$ and therefore $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4-x^2) = 4-(2)^2 = 0$.

For $x > 2$, $f(x) = x-1$, and therefore $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+}(x-1) = 2-1 = 1$.

(b) Does $\lim_{x\to 2} f(x)$ exist?

Solution: If $\lim_{x\to 2} f(x)$ exists, then $\lim_{x\to 2} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$. But that implies that $0 = 1$, which is clearly false, so the limit does not exist.

(c) Sketch the graph of f.

Solution:

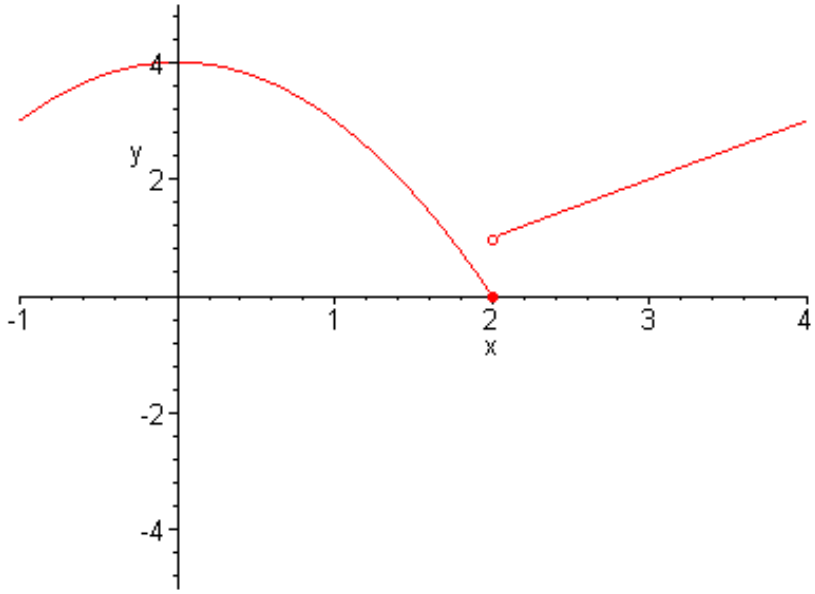


Figure 2: Graph of $f(x)$.

§2.3 #58. Evaluate

$$\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}.$$

Solution: First, we notice that

$$\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \frac{\sqrt{6-2}-2}{\sqrt{3-2}-1} = \frac{0}{0},$$

an indeterminate form.

The trick then is to use conjugates: for $x \neq 2$,

$$\begin{aligned}
\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\right)\left(\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\right)\left(\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right) \\
&= \frac{((6-x)-4)(\sqrt{3-x}+1)}{((3-x)-1)(\sqrt{6-x}+2)} \\
&= \frac{(2-x)(\sqrt{3-x}+1)}{(2-x)(\sqrt{6-x}+2)} \\
&= \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2},
\end{aligned}$$

so therefore

$$\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}=\lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}=\frac{\sqrt{3-2}+1}{\sqrt{6-2}+2}=\frac{2}{4}=\frac{1}{2}.$$

§2.6 #24. Find the limit:

$$\lim_{x\to-\infty}(x+\sqrt{x^2+2x}).$$

Solution: We first consider each term in the sum separately: $\lim_{x\to-\infty} x=-\infty$, and $\lim_{x\to-\infty}\sqrt{x^2+2x}=\lim_{x\to-\infty}\sqrt{x^2}=+\infty$. Combining the two, we have that

$$\lim_{x\to-\infty}(x+\sqrt{x^2+2x})=-\infty+\infty,$$

an indeterminate form.

Once again, we use conjugates. Also, since $x\to-\infty$, once $x\le 0$ we have the following:

$$\begin{aligned}
x+\sqrt{x^2+2x} &= (x+\sqrt{x^2+2x})\left(\frac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}\right)\\
&= \frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}}\\
&= \frac{-2x}{x-\sqrt{x^2+2x}}\\
&= \frac{\frac{-2x}{-x}}{\frac{x}{-x}-\frac{\sqrt{x^2+2x}}{-x}}\\
&= \frac{2}{-1-\sqrt{\frac{x^2+2x}{(-x)^2}}}\\
&= \frac{2}{-1-\sqrt{1+2/x}}.
\end{aligned}$$

(We needed $x\le 0$ in order for $(-x)=\sqrt{(-x)^2}$, which let us put it under the radical.) Noting that $\lim_{x\to-\infty}\frac{2}{x}=0$,

$$\lim_{x\to-\infty}(x+\sqrt{x^2+2x})=\lim_{x\to-\infty}\frac{2}{-1-\sqrt{1+2/x}}=-\frac{2}{2}=-1.$$

§2.6 #30. Find the limit:

$$\lim_{x\to\infty} \frac{x^3 - 2x + 3}{5 - 2x^2}.$$

Solution: Here we use the fact that the limit at ∞ or $-\infty$ of a rational function is the limit of the monomials of highest degree in the numerator and denominator:

$$\lim_{x\to\infty} \frac{x^3 - 2x + 3}{5 - 2x^2} = \lim_{x\to\infty} \frac{x^3}{-2x^2} = \lim_{x\to\infty} \frac{x}{-2} = \frac{\infty}{-2} = -\infty.$$

§2.6 #40. Find the horizontal and vertical asymptotes of

$$f(x) = \frac{x^3 + 1}{x^3 + x}.$$

Check your work by graphing the curve $y = f(x)$ and estimating the asymptotes.

Solution: We first find the vertical asymptotes: they are values of a where one of $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ is equal to one of $-\infty, +\infty$. Since f is a rational function, we're only interested in values of a where $f(a)$ is undefined, i.e. $a^3 + a = 0$. But $a^3 + a = a(a^2 + 1)$, and $a^2 + 1 > 0$ shows the only solution is $a = 0$:

$$\lim_{x\to 0^+} \frac{x^3 + 1}{x^3 + x} = \frac{(0^+) + 1}{(0^+) + (0^+)} = \frac{1}{0^+} = +\infty,$$

and $x = 0$ is the only vertical asymptote.

Horizontal asymptotes correspond to values of L such that $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to+\infty} f(x) = L$. Again, since the function is rational, we only need the monomials of highest degree:

$$\lim_{x\to-\infty} \frac{x^3 + 1}{x^3 + x} = \lim_{x\to-\infty} \frac{x^3}{x^3} = \lim_{x\to-\infty} 1 = 1,$$

and

$$\lim_{x\to\infty} \frac{x^3 + 1}{x^3 + x} = \lim_{x\to\infty} \frac{x^3}{x^3} = \lim_{x\to\infty} 1 = 1.$$

Hence, the only horizontal asymptote is $y = 1$.

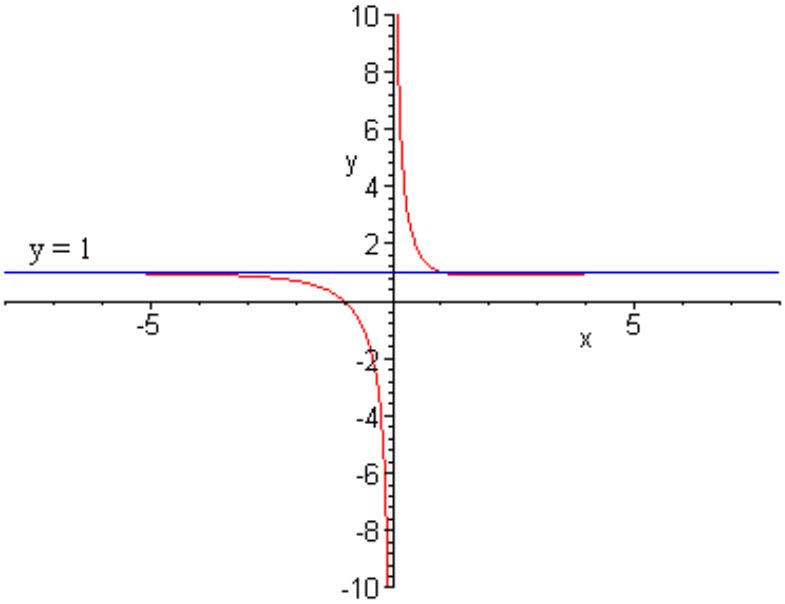


Figure 3: Graph of $f(x)$ (red) and line $y = 1$ (blue).