21-114: Calculus for Architecture
Homework #1 Solutions

November 9 , 2004

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§1.1 #26. Find the domain of $g(u) = \sqrt{u} + \sqrt{4-u}$.

Solution: We solve this by considering the terms in the sum separately: $\sqrt{u}$ is only defined when $u \geq 0$, and $\sqrt{4-u}$ is only defined when $4 - u \geq 0$, so $g(u) = \sqrt{u} + \sqrt{4-u}$ is only defined when both $u \geq 0$ and $4 - u \geq 0$. But we can rewrite $4 - u \geq 0$ as $u \leq 4$, and we get that the domain of g is $\{u \mid 0 \leq u \leq 4\}$, or, in interval notation, $[0, 4]$.

§1.1 #40. Find the domain and sketch the graph of

$$f(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ 3x + 2 & \text{if } |x| < 1 \\ 7 - 2x & \text{if } x \geq 1 \end{cases}.$$

Solution: We see that f is defined when $x \leq -1$, $|x| < 1$, and $x \geq 1$. But $|x| < 1$ can be written as $-1 < x < 1$, so f is defined for all x, and its domain is $(-\infty, \infty)$.

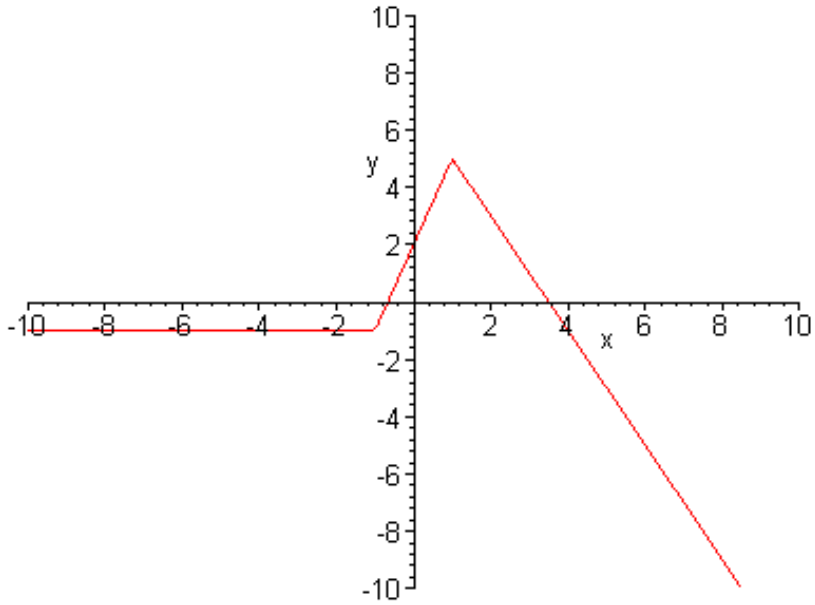


Figure 1: Graph of $f(x)$.

§1.1 #48. Find a formula and the domain for the following function: A rectangle has area 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

Solution: We know that a rectangle has four sides: two pairs of equal-length sides, call their lengths L and W. We can then write the area as $A = LW$, and the perimeter as $P = 2W + 2L$. Since we're given that the area is 16 m^2, substituting gives us $16 = LW$. We can then solve this equation for W, giving $W = 16/L$. Substituting this into the formula for the perimeter, we get

$$\begin{aligned} P(L) &= 2\left(\frac{16}{L}\right) + 2L \\ &= \frac{32}{L} + 2L. \end{aligned}$$

Since the length of a side of a rectangle can never be negative, $L \geq 0$, and since we can't divide by 0, the domain of P must be $\{L|L > 0\}$, or $(0, \infty)$.

§1.2 #6. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of this family.

Solution: We write $y = f(x)$ for convenience, so the formula becomes $y = 1+m(x+3)$. But this is almost the point-slope form of the line, so we subtract 1 from both sides and rewrite $x + 3$ as $x - (-3)$:

$$y - 1 = m(x - (-3)).$$

This is exactly the point-slope form of any line passing through the point $(-3, 1)$, or, equivalently, any linear function with $f(-3) = 1$, which is what they all have in common.

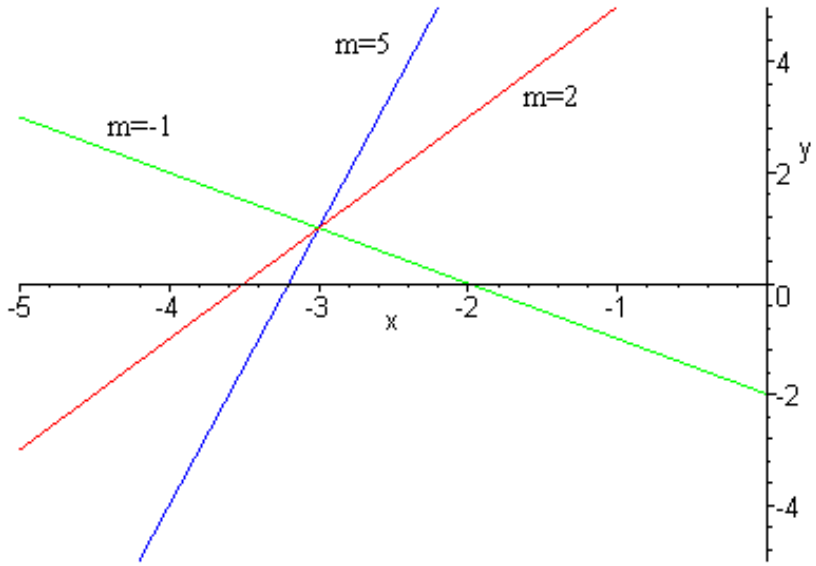


Figure 2: Graph of three members of the family.

§B #52. Sketch the following region in the xy-plane:

$$\left\{(x,y)\Big| -x \le y < \frac{1}{2}(x+3)\right\}.$$

Solution: This is the region above and including the line $y = -x$ (since $y \ge -x$) and below but excluding the line $y = \frac{1}{2}(x+3)$ (since $y < \frac{1}{2}(x+3)$):

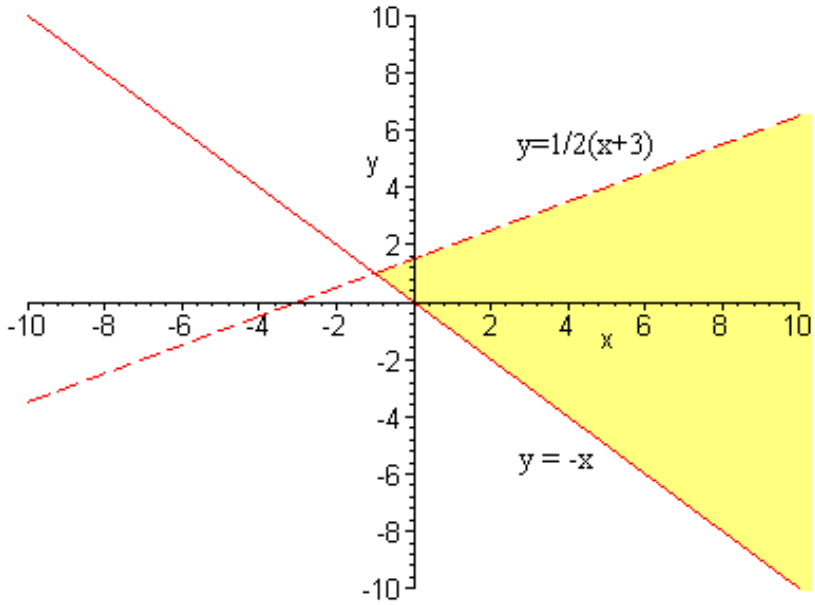


Figure 3: Graph of $\{(x,y)| -x \le y < \frac{1}{2}(x+3)\}$.

§B #58. Show that the lines $3x - 5y + 19 = 0$ and $10x + 6y - 50 = 0$ are perpendicular, and find their point of intersection.

Solution: We show the two lines are perpendicular by first computing their slopes, m_1 and m_2, and showing that $m_1 m_2 = -1$:

Since the first line is $3x - 5y + 19 = 0$, we write it as $-5y = -3x - 19$, and then divide both sides by -5 to get $y = \frac{3}{5}x + \frac{19}{5}$, and we see that its slope is $m_1 = \frac{3}{5}$.

The second line is $10x + 6y - 50 = 0$, which we rewrite as $6y = -10x + 50$, and dividing by 6 gives us $y = -\frac{5}{3}x + \frac{25}{3}$, so $m_2 = -\frac{5}{3}$.

From here, we easily see that $m_1 m_2 = (\frac{3}{5})(-\frac{5}{3}) = -1$, so the lines are perpendicular.

To find the point of intersection, we set the equations equal to each other:

$$\begin{aligned} \frac{3}{5}x + \frac{19}{5} &= -\frac{5}{3}x + \frac{25}{3}, \\ \frac{34}{15}x = \left(\frac{3}{5} + \frac{5}{3}\right)x &= \frac{25}{3} - \frac{19}{5} = \frac{68}{15}, \\ x &= \frac{68}{15}\left(\frac{15}{34}\right) = 2, \end{aligned}$$

and substituting 2 into either equation gives us $y = 5$, so they intersect at the point $(2, 5)$.

§C #6. Show that the equation represents a circle, and find the center and radius:

$$x^2 + y^2 + 6y + 2 = 0.$$

Solution: The first (and best) method you can use to solve this is the method of completing squares. First, group the x and y terms together:

$$(x^2) + (y^2 + 6y) + 2 = 0.$$

Then, complete the squares for the x-terms and y-terms separately, giving

$$(x - 0)^2 + ((y + 3)^2 - 9) + 2 = 0.$$

Finally, arrange the terms into the standard form of a circle, i.e. $(x - h)^2 + (y - k)^2 = r^2$:

$$(x - 0)^2 + (y + 3)^2 = 7.$$

From here we read off the center $(0, -3)$ and the radius $\sqrt{7}$.

A second method is to consider the standard form of a circle and expand it, giving

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2,$$

and rewriting gives

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

Subtracting our specific equation above from the general form we just found gives us:

$$(-2h)x + (-2k - 6)y + (h^2 + k^2 - r^2 - 2) = 0.$$

But the RHS is exactly 0, so the coefficients of all variables must be 0: $(-2h) = 0$, $(-2k - 6) = 0$, and $(h^2 + k^2 - r^2 - 2) = 0$. Solving gives us $h = 0$, $k = -3$, meaning the center must be the point $(0, -3)$. But then we have $(0)^2 + (-3)^2 - r^2 - 2 = 0$, and so $r^2 = 7$, and the radius is $r = \sqrt{7}$.

§C #34. Sketch the region bounded by the curves $y = 4 - x^2$, $x - 2y = 2$.

Solution: The region bounded by the curves $y = 4 - x^2$ and $x - 2y = 2$ is the region below the quadratic $y = 4 - x^2$, and above the line $x - 2y = 2$, which we can rewrite as $y = \frac{1}{2}x - 1$:

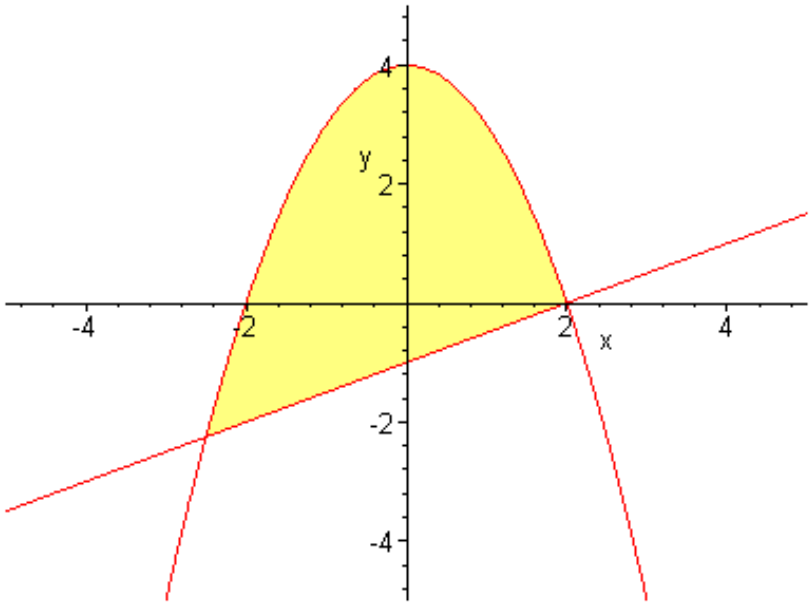


Figure 4: Graph of the region bounded by $y = 4 - x^2$ and $x - 2y = 2$.

§D #70. Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation $2\cos x + \sin 2x = 0$.

Solution: We first use the double-angle formula $\sin 2x = 2\sin x \cos x$, which gives us

$$0 = 2\cos x + 2\sin x \cos x = 2(\cos x)(1 + \sin x),$$

so therefore $\cos x = 0$ or $(1 + \sin x) = 0$. The only solutions to $\cos x = 0$ in $[0, 2\pi]$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}$. Since we can rewrite $1 + \sin x = 0$ to $\sin x = -1$, its only solution is $x = \frac{3\pi}{2}$. Therefore, the solutions to $2\cos x + \sin 2x = 0$ in $[0, 2\pi]$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

§D #74. Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality $2\cos x + 1 > 0$.

Solution: Rewriting, we wish to find the values of x for which $2\cos x > -1$, and hence $\cos x > -\frac{1}{2}$. To do this, we first find the values of x for which $\cos x = -\frac{1}{2}$: but, solving that, we see that $x = \frac{2\pi}{3}, \frac{4\pi}{3}$. Additionally (by considering the unit circle), we see that $\cos x > -\frac{1}{2}$ in $[0, 2\pi]$ only when $0 \leq x < \frac{2\pi}{3}$ or $\frac{4\pi}{3} < x \leq 2\pi$, or, in interval notation, $[0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi]$.

§D #82. Graph the function by starting with the graphs in Figures 13 and 14 and applying the transformation of Section 1.3 where appropriate:

$$y = 2 + \sin\left(x + \frac{\pi}{4}\right).$$

We do this by starting with $f(x) = \sin(x)$, then shifting the graph to the left by $\pi/4$ to get $g(x) = \sin(x + \frac{\pi}{4})$:

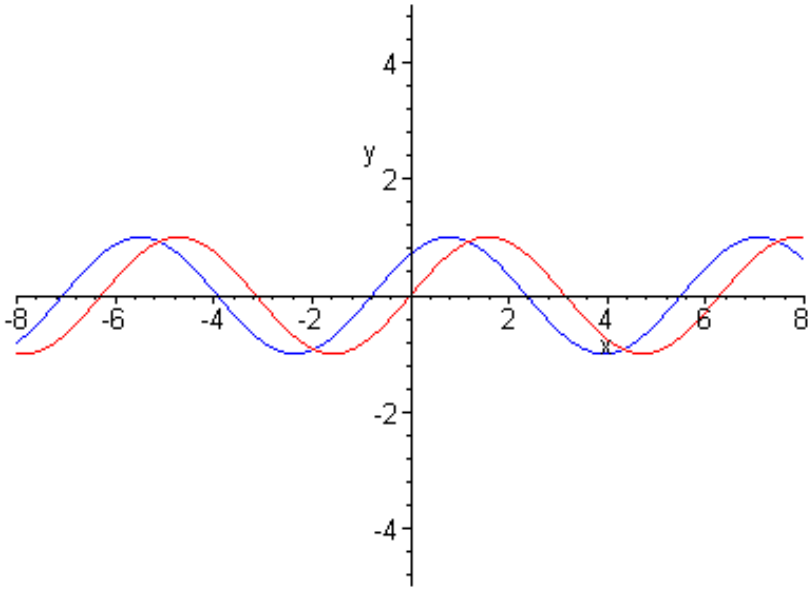


Figure 5: $f(x)$ is red, $g(x)$ is blue.

Then we shift the graph up by 2 to get $h(x) = 2 + \sin\left(x + \frac{\pi}{4}\right) = y$:

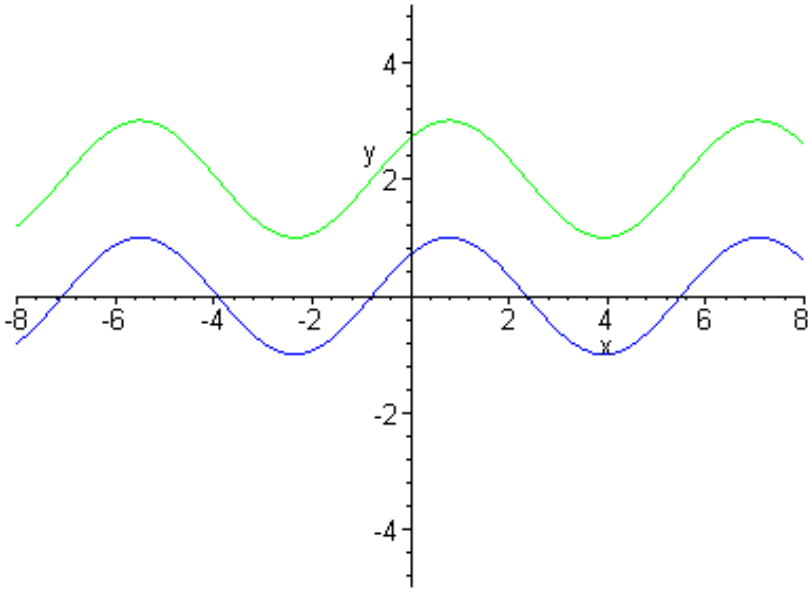


Figure 6: $g(x)$ is blue, $h(x)$ is green.