MATROIDS

Hereditary Families

Given a Ground Set E, a Hereditary Family $\mathcal{A}$ on E is collection of subsets $\mathcal{I} = \{I_1, I_2, \ldots, I_m\}$ (the independent sets) such that

$$I \in \mathcal{I} \text{ and } J \subseteq I \text{ implies that } J \in \mathcal{I}.$$

1. The set $\mathcal{M}$ of matchings of a graph $G = (V, E)$.
2. The set of (edge-sets of) forests of a graph $G = (V, E)$.
3. The set of stable sets of a graph $G = (V, E)$. We say that S is stable if it contains no edges.
4. If $G = (A, B, E)$ is a bipartite graph and $\mathcal{I} = \{S \subseteq B : \exists \text{ a matching } M \text{ that covers } S\}$.
5. Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ be the columns of an $m \times n$ matrix $\mathbf{A}$. Then $E = [n]$ and $\mathcal{I} = \{S \subseteq [n] : \{\mathbf{c}_i, i \in S\} \text{ are linearly independent}\}$.

Matroids

An independence system is a matroid if whenever $I, J \in \mathcal{I}$ with $|J| = |I| + 1$ there exists $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$. We call this the Independent Augmentation Axiom – IAA.

Matroid independence is a generalisation of linear independence in vector spaces. Only Examples 2,4 and 5 above are matroids.

To check Example 5, let $\mathbf{A}_I$ be the $m \times |I|$ sub-matrix of $\mathbf{A}$ consisting of the columns in I. If there is no $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$ then $\mathbf{A}_J = \mathbf{A}_I\mathbf{M}$ for some $|I| \times |J|$ matrix $\mathbf{M}$.

Matrix $\mathbf{M}$ has more columns than rows and so there exists $\mathbf{x} \neq 0$ such that $\mathbf{Mx} = \mathbf{0}$. But then $\mathbf{A}_J\mathbf{x} = \mathbf{0}$, implying that the columns of $\mathbf{A}_J$ are linearly dependent. Contradiction.

These are called Representable Matroids.

Cycle Matroids/Graphic Matroids

To check Example 2 we define the vertex-edge incidence matrix $\mathbf{A}_G$ of graph $G = (V, E)$ over GF_2.

$\mathbf{A}_G$ has a row for each vertex $v \in V$ and a column for each edge $e \in E$. There is a 1 in row v, column e iff $v \in e$.

We verify that a set of columns $\mathbf{c}_i, i \in I$ are linearly dependent iff the corresponding edges contain a cycle.

If the edges contain a cycle $(v_1, v_2, \ldots, v_k, v_1)$ then the sum of the columns corresponding to the vertices of the cycle is $\mathbf{0}$.

To show that a forest F defines a linearly independent set of columns I_F, we use induction on the number of edges in the forest. This is trivial if $|E(F)| = 1$.

Cycle Matroids/Graphic Matroids

Let $\mathbf{A}_F$ denote the submatrix of $\mathbf{A}$ made up of the columns corresponding F.

Now a forest F must contain a vertex v of degree one. This means that the row corresponding to v in $\mathbf{A}_F$ has a single one, in column e say.

Consider the forest $F' = F \setminus \{e\}$. Its corresponding columns $I_{F'}$ are linearly independent, by induction. Adding back e adds a row with a single one and preserves independence. Let $\mathbf{B}$ denote $\mathbf{A}_{F'}$ minus row e.

$$\mathbf{A}_F = \begin{bmatrix} 1 & \mathbf{0} \\ & \mathbf{B} \end{bmatrix}.$$

Transversal Matroids

We now check Example 4. These are called Transversal Matroids. If M_1, M_2 are two matchings in a graph G then $M_1 \oplus M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$ consists of alternating paths and cycles.

Suppose now that we have two matchings M_1, M_2 in bipartite graph $G = (A, B, E)$. Let $I_j, j = 1, 2$ be the vertices in B covered by M_j. Suppose that $|I_1| > |I_2|$.

Then $M_1 \oplus M_2$ must contain an alternating path P with end points $b \in I_1 \setminus I_2, a \in A$. Let E_1 be the M_1 edges in P and let E_2 be the M_2 edges of P. Then $(M_1 \cup E_1) \setminus E_2$ is a matching that covers $I_1 \cup \{b\}$.

Representable Matroids

A matroid is binary if is representable by a matrix over GF_2.

So a graphic matroid is binary.

A matroid is regular if it can be represented by a matrix of elements in $\{0, \pm 1\}$ for which every square sub-matrix has determinant $0, \pm 1$. These are called totally unimodular matrices

A matrix with 2 non-zeros in each column, one equal to +1 and the other equal to -1 is totally unimodular. This implies that graphic matroids are regular. (Take the vertex-edge incidence matrix and replace one of the ones in each column by a -1.)

Partition Matroids

Given a partition $E_1, E_2, \ldots, E_m$ of E and non-negative integers $k_1, k_2, \ldots, k_m$ we define the associated partition matroid as follows:

$$I \in \mathcal{I} \text{ iff } |I \cap E_i| \leq k_i, i = 1, 2, \ldots, m.$$

Partition matroids are representable.

Bases

A matroid basis is a maximal independent set i.e. B is a basis if there does **not** exist an independent set $I \neq B$ such that $I \supset B$.

So the bases of the cycle matroid of a graph G consist of the spanning trees of G.

Lemma

If B_1, B_2 are bases of a matroid $\mathcal{M}$, then $|B_1| = |B_2|$.

Proof: If $|B_1| > |B_2|$ then there exists $e \in B_1 \setminus B_2$ such that $B_2 \cup \{e\}$ is independent. Contradicting the fact that B_2 is maximal. $\square$

Bases

Theorem

A collection $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ *of subsets of* E *form the bases of a matroid on* E *iff for all* i, j *and* $e \in B_i \setminus B_j$ *there exists* $f \in B_j \setminus B_i$ *such that* $(B_i \cup \{f\}) \setminus \{e\} \in \mathcal{B}$*.*

Proof: Suppose first that $\mathcal{B}$ are the bases of a matroid with independent sets $\mathcal{I}$ and that $e \in B_i$ and $e \notin B_j$. Then $B_i' = B_i \setminus \{e\} \in \mathcal{I}$ and $|B_i'| < |B_j|$. So there exists $f \in B_j \setminus B_i'$ such that $B_i'' = B_i' \cup \{f\} \in \mathcal{I}$. Now $f \neq e$ since $e \notin B_j$ and $|B_i''| = |B_i|$. So B_i'' must be a basis.

Conversely, suppose that $\mathcal{B}$ satisfies the conditions of the theorem and that $\mathcal{I} = \{S : \exists i \text{ s.t. } S \subseteq B_i\}$. Clearly $\mathcal{I}$ is hereditary.

Bases

We first argue that all the sets in $\mathcal{B}$ are of the same size.

Suppose that $A = \{i : |B_i| = \max\{|B| : B \in \mathcal{B}\}\}$ and suppose that $A \neq [m]$. Suppose that

$$\min\{|B_i - B_j| : i \in A, j \notin A\} = |B_1 \setminus B_2|.$$

Let $x \in B_1 \setminus B_2$ and let $y \in B_2 \setminus B_1$ be such that $B' = ((B_1 \cup y) \setminus \{x\}) \in \mathcal{B}$.

Then we have $B' \in A$ and $|B' \setminus B_2| < |B_1 \setminus B_2|$, contradiction.

Bases

Suppose now that $I_1, I_2 \in \mathcal{I}$ with $|I_2| > |I_1|$ and there does not exist $e \in I_2 \setminus I_1$ for which $I_1 \cup \{e\} \in \mathcal{I}$.

Choose $B_j \supseteq I_j, j = 1, 2$ such that $|B_2 \setminus (I_2 \cup B_1)|$ is minimal.

We must have $I_2 \setminus B_1 = I_2 \setminus I_1$. If $x \in I_2 \cap B_1$ and $x \notin I_1$ then $I_1 \cup \{x\} \subseteq B_1$ and so $I_1 \cup \{x\} \in \mathcal{I}$.

Suppose there exists $x \in B_2 \setminus (I_2 \cup B_1)$. Then by assumption there is $y \in B_1 \setminus B_2$ such that $B' = (B_2 \cup \{y\}) \setminus \{x\} \in \mathcal{B}$. But then $B' \setminus (I_2 \cup B_1) = (B_2 \setminus (I_2 \cup B_1)) \setminus \{x\}$, contradicting the definition of B_2.

Bases

So $B_2 \subseteq (I_2 \cup B_1) = (I_2 \setminus B_1) \cup (B_1 \setminus I_2) = (I_2 \setminus I_1) \cup (B_1 \setminus I_2)$ and so

$$B_2 \setminus B_1 = I_2 \setminus I_1. \tag{1}$$

We show next that $B_1 \subseteq (I_1 \cup B_2)$. If there exists $x \in B_1 \setminus (I_1 \cup B_2)$ then there exists $y \in B_2 \setminus B_1$ such that $B' = (B_1 \cup \{y\}) \setminus \{x\} \in \mathcal{B}$. But $(I_1 \cup \{x\}) \subseteq B'$, contradiction.

So, $B_1 \setminus B_2 = I_1 \setminus B_2 \subseteq I_1 \setminus I_2$. Since $|B_1 \setminus B_2| = |B_2 \setminus B_1|$ we see from this and (1) that $|I_1 \setminus I_2| \geq |I_2 \setminus I_1|$ and so $|I_1| \geq |I_2|$, contradiction.

Rank

If $S \subseteq E$ then its rank

$$r(S) = \max |\{I \in \mathcal{I} : I \subseteq S\}| .$$

So $S \in \mathcal{I}$ iff $r(S) = |S|$. We show next that r is submodular.

Theorem

If $S, T \subseteq E$ then $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$.

Proof: Let I_1 be a maximal independent subset of $S \cap T$ and let I_2 be a maximal independent subset of $S \cup T$ that contains I_2. (Such a set exists because of the IAA.)

But then

$$r(S \cap T) + r(S \cup T) = |I_1| + |I_2| = |I_1 \cap S| + |I_2 \cap T| \leq r(S) + r(T).$$

Rank

For representable matroids this coresponds to the usual definition of rank.

For the cycle matroid of graph $G = (V, E)$, if $S \subseteq E$ is a set of edges and G_S is the graph (V, S) then $r(S) = |V| - \kappa(G_S)$, where $\kappa(G_S)$ is the number of components of G_S.

This clearly true for connected graphs and so if $C_1, C_2, \ldots, C_s$ are the components of G_S then $r(S) = \sum_{i=1}^{s} |C_i| - 1 = |V| - s$.

For a partition matroid as defined above,

$$r(S) = \sum_{i=1}^{m} \min\{k_i, |S \cap E_i|\}.$$

Circuits

A circuit of a matroid $\mathcal{M}$ is a minimal dependent set. If a set $S \subseteq E, S \notin \mathcal{I}$ then S contains a circuit.

So the circuits of the cycle matroid of a graph G are the cycles.

Theorem

If C_1, C_2 are circuits of $\mathcal{M}$ and $e \in C_1 \cap C_2$ then there is a circuit $C \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Proof: We have $r(C_i) = |C_i| - 1, i = 1, 2$. Also, $r(C_1 \cap C_2) = |C_1 \cap C_2|$ since $C_1 \cap C_2$ is a proper subgraph of C_1.

If $C' = (C_1 \cup C_2) \setminus \{e\}$ contains no circuit then $r(C_1 \cup C_2) \geq r(C') = |C_1 \cup C_2| - 1$. But then

$$|C_1 \cup C_2| - 1 \leq r(C_1 \cup C_2) \leq r(C_1) + r(C_2) - r(C_1 \cap C_2)$$
$$= (|C_1| - 1) + (|C_2| - 1) - |C_1 \cap C_2|.$$

Contradiction.

Circuits

Theorem

If B is a basis of $\mathcal{M}$ and $e \in E \setminus B$ then $B' = B \cup \{e\}$ contains a unique circuit $C(e, B)$. Furthermore, if $f \in C(e, B)$ then $(B \cup \{e\}) \setminus \{f\}$ is also a basis of $\mathcal{M}$.

Proof: $B' \notin \mathcal{I}$ because B is maximal. So B' must contain at least one circuit.

Suppose it contains distinct circuits C_1, C_2. Then $e \in C_1 \cap C_2$ and so B' contains a circuit $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

But then $C_3 \subseteq B$, contradiction. $\square$

Dual Matroid

Theorem

If $\mathcal{B}$ denotes the set of bases of a matroid $\mathcal{M}$ on ground set E then $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$ is the set of bases of a matroid $\mathcal{M}^*$, the dual matroid.

Proof: Suppose that $B_1^*, B_2^* \in \mathcal{B}^*$ and $e \in B_1^* \setminus B_2^*$.

Let $B_i = E \setminus B_i^*, i = 1, 2$. Then $e \in B_2 \setminus B_1$.

So there exists $f \in B_1 \setminus B_2$ such that $(B_2 \cup \{e\}) \setminus \{f\} \in \mathcal{B}$.

This implies that $(\mathcal{B}_2^* \cup \{f\}) \setminus \{e\} \in \mathcal{B}^*$. □

Suppose that each $e \in E$ is given a weight w_e and that the weight $w(I)$ of an independent set I is given by $w(I) = \sum_{e \in I} c_e$. The problem we discuss is

$$\text{Maximize } w(I) \text{ subject to } I \in \mathcal{I}.$$

Greedy Algorithm:
begin
  Sort $E = \{e_1, e_2, \ldots, e_m\}$ so $w(e_i) \geq w(e_{i+1})$ for $1 \leq i < m$;
  $S \leftarrow \emptyset$;
  for $i = 1, 2, \ldots, m$;
  begin
    if $S \cup \{e_i\} \in \mathcal{I}$ **then**;
    begin;
      $S \leftarrow S \cup \{e_i\}$;
    end;
  end;
end

Theorem

The greedy algorithm finds a maximum weight independent set **for all choices of** w *if and only if it is a matroid.*

Suppose first that the Greedy Algorithm always finds a maximum weight independent set. Suppose that $\emptyset \neq I, J \in \mathcal{I}$ with $|J| = |I| + 1$. Define

$$w(e) = \begin{cases} 1 + \frac{1}{2|I|} & e \in I. \\ 1 & e \in J \setminus I. \\ 0 & e \notin I \cup J. \end{cases}$$

If there does not exist $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$ then the Greedy Algorithm will choose the elements of I and stop. But I does not have maximum weight. Its weight is $|I| + 1/2 < |J|$. So if Greedy succeeds, then the IAA holds.

Conversely, suppose that our independence system is a matroid. We can assume that $w(e) > 0$ for all $e \in E$. Otherwise we can restrict ourselves to the matroid defined by $\mathcal{I}' = \{I \subseteq E^+\}$ where $E^+ = \{e \in E : w(e) > 0\}$.

Suppose now that Greedy chooses $I_G = e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ where $i_t < i_{t+1}$ for $1 \leq t < k$. Let $I = e_{j_1}, e_{j_2}, \ldots, e_{j_\ell}$ be any other independent set and assume that $j_t < j_{t+1}$ for $1 \leq t < \ell$. We can assume that $\ell \geq k$, for otherwise we can add something from I_G to I to give it larger weight.

We show next that $k = \ell$ and that $i_t \leq j_t$ for $1 \leq t \leq k$. This implies that $w(I_G) \geq w(I)$.

Suppose then that there exists t such that $i_t > j_t$ and let t be as small as possible for this to be true.

Now consider $I = \{e_{i_s} : s = 1, 2, \ldots, t-1\}$ and $J = \{e_{j_s} : s = 1, 2, \ldots, t\}$. Now there exists $e_{j_s} \in J \setminus I$ such that $I \cup \{e_{j_s}\} \in \mathcal{I}$.

But $j_s \leq j_t < i_t$ and Greedy should have chosen e_{j_s} before choosing $e_{i_{t+1}}$.

Also, $i_k \leq j_k$ implies that $k = \ell$. Otherwise Greedy can find another element from $I \setminus I_G$ to add.

Minors

Given a graph $G = (V, E)$ and an edge e we can get new graphs by deleting e or contracting e.

We describe a corresponding notion for matroids. Suppose that $F \subseteq E$ then we define the matroid $\mathcal{M}_{\setminus F}$ with independent sets $\mathcal{I}_{\setminus F}$ obtained by deleting F: $I \in \mathcal{I}_{\setminus F}$ if $I \in \mathcal{I}, \ I \cap F = \emptyset$.

It is clear that the IAA holds for $\mathcal{M}_{\setminus F}$ and so it is a matroid.

For contraction we will assume that $F \in \mathcal{I}$. Then contracting F defines $\mathcal{M}.F$with independent sets $\mathcal{I}.F = \{I \in \mathcal{I} : I \cap F = \emptyset, \ I \cup F \in \mathcal{I}\}$.

We argue next that $\mathcal{M}.F$ is also a matroid.

Lemma

$\mathcal{M}.F = (\mathcal{M}^*_{\setminus F})^*$ *and* $\mathcal{M}_{\setminus F} = (\mathcal{M}^*.F)^*$.

Proof:

$$\begin{aligned} I \in \mathcal{I}.F &\leftrightarrow \exists B \in \mathcal{B}_{\setminus F}, I \subseteq B \\ &\leftrightarrow \exists B^* \in \mathcal{B}^*_{\setminus F}, I \cap B^* = \emptyset \\ &\leftrightarrow I \in (\mathcal{I}^*_{\setminus F})^*. \end{aligned}$$

For the second claim we use

$$\mathcal{M}^*.F = (\mathcal{M}^{**}_{\setminus F})^* = (\mathcal{M}_{\setminus F})^*.$$

□

Matroid Intersection

Suppose we are given two matroids $\mathcal{M}_1, \mathcal{M}_2$ on the same ground set E with $\mathcal{I}_1, \mathcal{I}_2$ and r_1, r_2 etc. having there obvious meaning.

An intersection is a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. We give a min-max relation for the size of the largest independent intersection. Let $\mathcal{J}$ denote the set of intersections.

Theorem (Edmonds)

$$\max\{J \in \mathcal{J}\} = \min\{r_1(A) + r_2(E \setminus A) : A \subseteq E\}.$$

Matroid Intersection

Before proving the theorem let us see a couple of applications:

Hall's Theorem: suppose we are given a bipartite graph $G = (A, B, E)$. Let $\mathcal{M}_A, \mathcal{M}_B$ be the following two partition matroids.

For $\mathcal{M}_A$ we define the partition $E_a = \{e \in x : a \in e\}$, $a \in A$. We let $k_a = 1$ for $a \in A$. We define $\mathcal{M}_B$ similarly.

Intersections correspond to matchings and $r_1(A)$ is the number of vertices in A that are incident with an edge of A. Similarly $r_2(E \setminus A)$ is the number of vertices in B that are incident with an edge not in A.

Matroid Intersection

For $X \subseteq A$, let

$$A_X = \{v \in A : v \in e \text{ for some } e \in X\}.$$

Define B_X similarly.

So

$$\max\{|M|\} = \min\{|A_X| + |B_{E \setminus X}| : \ X \subseteq E\}.$$

Now we can assume that if $e \in E \setminus X$ then $e \cap A_X = \emptyset$, otherwise moving e to X does not increase the RHS of the above.

Let $S = A \setminus A_X$. Then $|B_{E \setminus X}| = |N(A)|$ and so

$$\max\{|M|\} = \min\{|A| - |S| + |N(S)| : S \subseteq A\}.$$

Matroid Intersection

Rainbow Spanning Trees: we are given a connected graph $G = (V, E)$ where each edge $e \in E$ is given a color $c(e) \in [m]$ where $m \geq n - 1$. Let $E_i = \{e : c(e) = i\}$ for $i \in [m]$.

A set of edges S is said to be rainbow colored if $e, f \in S$ implies that $c(e) \neq c(f)$.

For a set $A \subseteq E$, we let

$$r_1(A) = c(A) = |\{i \in [m] : \exists e \in A \; s.t. \; c(e) = i\}|$$
$$r_2(E \setminus A) = n - \kappa(G \setminus A).$$

So, G contains a rainbow spanning tree iff

$$c(A) + (n - \kappa(G \setminus A)) \geq n - 1 \text{ for all } A \subseteq E. \qquad (2)$$

Matroid Intersection

We simplify (2) to obtain

$$c(A) + 1 \geq \kappa(G \setminus A). \tag{3}$$

We can then further simplify (3) as follows: if we add to A all edges that use a color used by some edge of A then we do not change $c(A)$ but we do not decrease $\kappa(G \setminus A)$.

Thus we can restrict our sets A to $E_I = \bigcup_{i \in I} E_i$ for some $I \subseteq [m]$. Then (3) becomes

$$\kappa(E_{[m] \setminus I}) \leq |I| + 1 \text{ for all } I \subseteq [m]$$

or

$$\kappa(E_I) \leq m - |I| + 1 \text{ for all } I \subseteq [m]$$

If you think for a moment, you will see that this is obviously necessary.

Matroid Intersection

Proof of the matroid intersection theorem.

For the upper bound consider $J \in \mathcal{J}$ and $A \subseteq E$. Then

$$|J| = |J \cap A| + |J \setminus A| \leq r_1(A) + r_2(E \setminus A).$$

We assume that $e \in \mathcal{J}$ for all $e \in E$. (Loops can be “ignored”.)

We proceed by induction on $|E|$. Let

$$k = \min\{r_1(A) + r_2(E \setminus A) : A \subseteq E\}.$$

Suppose that $|J| < k$ for all $J \in \mathcal{J}$.

Matroid Intersection

Then $(\mathcal{M}_1)_{\setminus\{e\}}$ and $(\mathcal{M}_2)_{\setminus\{e\}}$ have no common independent set of size k. This implies that if $F = E \setminus \{e\}$ then

$$r_1(A) + r_2(F \setminus A) \leq k - 1 \text{ for some } A \subseteq F.$$

Similarly, $\mathcal{M}_1.\{e\}$ and $\mathcal{M}_2.\{e\}$ have no common independent set of size $k - 1$. This implies that

$$r_1(B) - 1 + r_2(E \setminus (B \setminus \{e\})) - 1 \leq k - 2 \text{ for some } e \in B \subseteq E.$$

This gives

$$r_1(A) + r_2(E \setminus (A \cup \{e\})) + r_1(B) + r_2(E \setminus (B \setminus \{e\})) \leq 2k - 1.$$

So, using submodularity and

$$(E \setminus (A \cup \{e\})) \cup (E \setminus (B \setminus \{e\})) = E \setminus (A \cap B)$$

and

$$(E \setminus (A \cup \{e\})) \cap (E \setminus (B \setminus \{e\})) = E \setminus (A \cup B).$$

We have used $e \notin A$ and $e \in B$ here. So,

$$r_1(A \cup B) + r_2(E \setminus (A \cup B)) + r_1(A \cap B) + r_2(E \setminus (A \cap B)) \\ \leq 2k - 1.$$

But, by assumption,

$$r_1(A \cup B) + r_2(E \setminus (A \cup B)) \geq k, \;\; r_1(A \cap B) + r_2(E \setminus (A \cap B)) \geq k,$$

contradiction. □