21-301 Combinatorics
Homework 7
Due: Wednesday, November 24

1. How many ways are there to arrange 3 M's, 6 A's, 8 T's and 4 H's under the condition that any arrangement and its reversal are to be considered the same.

   Solution: The group G consists of $\{e, a\}$ where a is a reflection through the middle of the word. Now

   $$\begin{aligned} |Fix(e)| &= \frac{21!}{2!4!5!6!} = 12317633280000 \\ |Fix(a)| &= \frac{10!}{1!3!4!2!} = 12600 \end{aligned}$$

   A sequence is in $Fix(a)$ if it is a palindrome i.e. looks the same backwards as forwards. It must have middle letter M. Then we arrange 1 M, 3 A's, 4 T"s and 2 H's in any order and then complete the sequence uniquely to a palindrome.

   Thus by Burnside's theorem, the number of sequences is $\frac{12317633280000+12600}{2} = 6158816646300$.

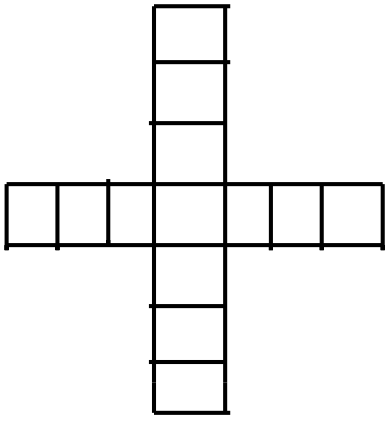
2. How many ways are there of k-coloring the squares of the cross below if the group acting is e_0, e_1, e_2, e_3 where e_j is rotation by $2\pi j/4$. Assume that instead of 13 squares there are $4n + 1$.

   Solution:

g	e_0	e_1	e_2	e_3
$\lvert Fix(g)\rvert$	k^{4n+1}	k^{n+1}	k^{2n+1}	k^{n+1}

   So the total number of colorings is

   $$\frac{k^{4n+1} + k^{n+1} + k^{2n+1} + k^{n+1}}{4}.$$



3. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\} \subseteq \binom{E}{k}$. Suppose also that $|H_i \cap H_j| \leq k - 2$ for $i \neq j$. Show that $\mathcal{B}_{\mathcal{H}} = \binom{E}{k} \setminus \mathcal{H}$ forms the set of bases of a matroid.
   ($\binom{E}{k} = \{S \subseteq E : |S| = k\}$.)

Solution: let $B_1 \neq B_2 \in \mathcal{B}_{\mathcal{H}}$ and suppose that there exists $x \in B_1 \setminus B_2$ such that for all $y \in B_2$, we have $(B_1 \cup \{y\}) \setminus \{x\} \in \mathcal{H}$.
Case 1: $|B_1 \setminus B_2| = |B_2 \setminus B_1| = 1$. In this case $B_1 \setminus B_2 = \{x\}$ and $B_2 \setminus B_1 = \{y\}$ and $(B_1 \cup \{y\}) \setminus \{x\} = B_2 \in \mathcal{B}_{\mathcal{H}}$, contradiction.

Case 2: $|B_1 \setminus B_2| = |B_2 \setminus B_1| \geq 2$. Choose $y_1, y_2 \in B_2 \setminus B_1$ and let $H_i = (B_1 \cup \{y_i\}) \setminus \{x\}, i = 1, 2$. But then $H_1, H_2 \in \mathcal{H}$ and $|H_1 \cap H_2| = k - 1$, contradiction.