21-301 Combinatorics
Homework 6
Due: Monday,November 10

1. There are n teams playing some sport. At some point in the season we have the following statistics: $w(i), i \in [n]$ denotes the number of games that team i has won so far. $p(i,j), i \neq j \in [n]$ denotes the number of times team i will play team j in the rest of the season. Set up the question: "Can team n still end up with the most wins?" as a network flow problem.

   Solution: The network has vertices s, t, $L = \{\{i,j\} : i,j \in [n-1] \text{ and } p(i,j) > 0\}$ and $R = [n-1]$. For each $\{i,j\} \in L$ add an edge $(s, \{i,j\})$ of capacity $p(i,j)$. For each $\{i,j\} \in L$ add edges $(\{i,j\}, i)$ and $(\{i,j\}), j)$ of capacity ∞. For each $i \in R$ add an edge (i,t) of capacity $W - w(i)$ where $W = w(n) + \sum_{j<n} p(j,n)$ is the maximum number of games that team n can win.

   We can assume that $W \geq w(i)$, else team n cannot end up with the most wins. Finally note that integral flows that saturate the edges leaving L are in 1-1 correspondence with outcomes of future matches. Also, the flow out of $i \in R$ into t is the number of wins $g(i)$ by team i in the associated schedule. Such a flow also represents an outcome where no team wins more than team n, since we would have $g(i) \leq W - w(i)$. So what we do is find a maximum flow in the network and see if it saturates the edges leaving s.

2. Find the set of P-positions for the take-away games with subtraction sets

   (a) $S = \{1, 3, 7\}$.
   (b) $S = \{1, 4, 6\}$.

   Suppose now that there are two piles and the rules for each pile are as above. Now find the P positions for the two pile game where in one pile S is as in (a) and the other pile is as in (b).

   Solution:

   (a) The first few numbers are

j	0	1	2	3	4	5	6	7	8	9	10
$g_1(j)$	0	1	0	1	0	1	0	1	0	1	0

   It is apparent that $g_1(j) = j \mod 2$ and this follows by an easy induction: If j is even then $j - x, x \in S$ is odd and if j is odd then $j - x, x \in S$ is even.

   (b) The first few numbers are

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$g_2(j)$	0	1	0	1	2	0	1	0	1	2	0	1	0	1	2

   So, we see that the pattern 0 1 0 1 2 repeats itself. Again, induction can be used to verify that this continues indefinitely.

   (c) The P-positions are those j, k for which $g_1(j) \oplus g_2(k) = 0$.
   Thus

   $$P = \{(j = 0 \mod 2 \text{ and } k = 0, 2 \mod 5) \text{ or } (j = 1 \mod 2 \text{ and } k = 1, 3 \mod 5\}.$$

3. In a take-away game, the set S of the possible numbers of chips to remove is finite. Show that the Sprague-Grundy numbers satisfy $g(n) \leq |S|$ where n is the number of chips remaining.

   Solution: Observe that for any finite set A, $mex(A) \leq |A|$ since $mex(A) > |A|$ implies that $A \supseteq \{0, 1, 2, \ldots, |A|\}$ which is obviously impossible. In the take-away game $g(n)$ is the mex of a set of size at most $|S|$ and the result follows.