21-301 Combinatorics
Homework 5
Due: Wednesday, October 13

1. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i \geq 1$ for $i = 1, 2, \ldots, n$. Let J be any open interval of width 2. Show that of the 2^n sums $\sum_{i=1}^n \varepsilon_i x_i$, $(\varepsilon_i = \pm 1)$, at most $\binom{n}{\lfloor n/2 \rfloor}$ lie in J.
   (Hint: use Sperner's lemma.)

   Solution: For $A \subseteq [n]$ let $x_A = \sum_{i \in A} x_i - \sum_{i \notin A} x_i$. Let $\mathcal{A} = \{A : x_A \in J\}$. It is enough to show that $\mathcal{A}$ is a Sperner family. Indeed, if $A, B \in \mathcal{A}$ and $A \subset B$ then $x_B - x_A = 2 \sum_{i \in B \setminus A} x_i \geq 2$. Thus we cannot have both $x_A, x_B \in J$.

2. Suppose that we two-color the edges of K_6 Red and Blue. Show that there are at least two monochromatic triangles.

   Solution: Assume w.l.o.g. that triangle $(1, 2, 3)$ is Red and that $(4, 5, 6)$ is not Red and in particular that edge $(4, 5)$ is Blue. If $x = 4, 5$ or 6 then there can be at most one Red edge joining x to $1, 2, 3$, else we get a Red triangle. So we can assume that there are two Blue edges joining each of $4, 5$ to $1, 2, 3$. So there must be $x \in \{1, 2, 3\}$ such that both $(x, 4)$ and $(x, 5)$ are Blue. But then triangle $(x, 4, 5)$ is Blue.

3. Show that $r(C_4, C_4) = 6$ where C_4 denotes a cycle of length four.
   Solution: (a) Color the edges of the 5-cycle (1,2,3,4,5,1) Red and the edges of the remaining 5-cycle (1,3,5,2,4,1) Blue. There are no mono-chromatic 4-cycles.

   (b) From class, there is a monochromatic triangle; Assume {1, 3, 5} is a red triangle (odds are chosen on purpose here for clarity). Suppose some even vertex contains two odd red neighbors; say, {2, 1} and {2, 3} are red, then {1, 2, 3, 5, 1} is red. Next, suppose instead that two even vertices have two common odd blue neighbors, then the four edges involved form a blue cycle.

   The only way to avoid one of the previous two scenarios is to have, between evens and odds, only three disjoint red edges and the rest blue; Assume then that {1, 4},{2, 5}, and {3, 6} are red and let {1, 2, 3, 4, 5, 6, 1} be a blue 6-cycle. There are now only three edges to consider (evens to evens); any of these would complete a red 4-cycle if colored red (for example: making {2, 4} red gives {1, 4, 2, 5, 1}), so color all even-to-even edges blue. Now {2, 3, 4, 6, 2} is a blue cycle, so we are done.