

Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2025: Test 1

Name: _____

Andrew ID: _____

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)

Find a recurrence for the number of sequences in $\{a, b\}^n$ that do not contain $aaaa$ as a subsequence.

DO NOT SOLVE THE RECURRENCE.

Solution: Let B_n be the set of sequences in question. Let $B_n^x, x = a, b$ be the sequences in B_n that begin with x . Then

$$\begin{aligned} |B_n| &= |B_n^{(a)}| + |B_n^{(b)}| = |B_n^{(a)}| + |B_{n-1}|. \\ |B_n^{(a)}| &= |B_n^{(aa)}| + |B_n^{(ab)}| = |B_n^{(aa)}| + |B_{n-2}|. \\ |B_n^{(aa)}| &= |B_n^{(aaa)}| + |B_n^{(aab)}| = |B_n^{(aaa)}| + |B_{n-3}|. \\ |B_n^{(aaa)}| &= |B_n^{(aaab)}| = |B_{n-4}|. \end{aligned}$$

So, the recurrence for $b_n = |B_n|$ is that

$$b_n = b_{n-1} + b_{n-2} + b_{n-3} + b_{n-4}.$$

Q2: (33pts)

The sequence $a_0, a_1, \dots, a_n, \dots$ satisfies the following: $a_0 = 0, a_1 = 1$ and

$$a_n - 4a_{n-1} + 4a_{n-2} = 2^{n-2}$$

for $n \geq 2$.

(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.

(b): Find an expression for $a_n, n \geq 0$.

Solution: Multiplying the equation by x^n and summing over $n \geq 2$ we obtain

$$(a(x) - x) - 4xa(x) + 4x^2a(x) = \frac{x^2}{1-2x} \text{ or } a(x) = \frac{x}{(1-2x)^2} + \frac{x^2}{(1-2x)^3}.$$

Now

$$\frac{x}{(1-2x)^2} = \sum_{n=1}^{\infty} n2^{n-1}x^n \text{ and } \frac{x^2}{(1-2x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} 2^{n-2}x^n.$$

So

$$a_n = n2^{n-1} + \frac{n(n-1)}{2} 2^{n-2}.$$

Q3: (34pts)

How many ways are there of putting $3n$ distinguishable balls $\{b_1, b_2, \dots, b_{3n}\}$ into n boxes $\{B_1, B_2, \dots, B_n\}$ so that there does not exist i such that B_i contains $\{b_{3(i-1)+1}, b_{3(i-1)+2}, b_{3i}\}$?

Solution: Let A_i be those allocations where B_i contains $\{b_{3(i-1)+1}, b_{3(i-1)+2}, b_{3i}\}$. Then if $|S| = s$,

$$|A_S| = \frac{(3(n-s))!}{6^{n-s}}.$$

Explanation: every allocation of the balls yields 6^n permutations of the $3n$ balls and so there are $(3n)!/6^n$ ways of allocating balls to boxes.

Thus, the number of possible seatings is

$$\sum_{s=0}^n (-1)^s \binom{n}{s} \frac{(3(n-s))!}{6^{n-s}}.$$